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Topics in Tropical and Analytic Geometry

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Mathematics

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An abstract of
A dissertation submitted to the Faculty of the
James T. Laney School of Graduate Studies of Emory University
in partial fulfillment of the requirements for the degree of
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Abstract

Topics in Tropical and Analytic Geometry
Charles Morrissey

In this thesis, the author proves theorems on the existence and mapping properties of tropical stacks that arise from the theory of toric artin stacks. The author also provides a generalization, using the same ideas from toric artin stacks, of recent work involving analytic stacks and their tropicalizations. The author also proves a result comparing the tropicalization of the jet bundle to the jet bundle of the tropicalization.

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Chapter 1

Introduction

There are many ways that tropical geometry has cropped up and gained attention over its history. The most common being that if one plots the solutions to algebraic equations logarithmically, the solutions look like polyhebral regions. More recently in the work of the Giansiracusa's [GG16a] a theory of tropical schemes has been developed that more closely resembles the traditional scheme theory. While not important for this paper, there is also the complex analytic picture, in which $\log_t(x + y)$ behaves like a valuation as $t \to \infty$. This view, however, has fallen out of favor largely due to the work of Berkovich.

Chapter 2 discusses the background for all of these different view points. In particular we provide the main results of tropicalization, which can be found in [MS15] and [Gat06] when the target space is the tropical numbers $\mathbb{T}$.

In Chapter 3 we are interested in questions involving the tropicalizations of the jet spaces of a regular variety. Following ideas found in [Yag16] we arrive at the following theorem.

Theorem 1.1 (Theorem A)**.** *Let X be a variety defined by a single equation f and let J^n_X be the jet bundle for X. Then the tropical variety* Trop X *has a "tropical jet bundle"* Trop J^n_X *coming from the tropicalization of J^n_X. Moreover, these equations can be visualized as degree n polyhedral complexes that have boundaries arising from* Trop X*.*

In chapter 4 involves new tropical structures that come from different considerations of stacks. With the ideas from Gerashenko and Satriano [GS15] on toric Artin stacks we prove two of the main properties of these geometric objects on the tropical side.

Theorem 1.2 (Theorem B)**.** *Let $(\Sigma, \beta \colon L \to N)$ and $(\Sigma', \beta \colon L' \to N')$ be stacky fans, and suppose $f \colon X_\Sigma \to X_{\Sigma'}$ is a toric morphism. Then the following diagram commutes.*

$$
\begin{array}{ccc}
\text{Trop } \mathcal{X}_{\Sigma_0,\beta_0} & \xrightarrow{\text{Trop } \mathcal{X}_{(\Phi',\phi')}} & \\
\downarrow {\scriptstyle \text{Trop } \mathcal{X}_{(\Phi,\phi)}} & & \text{Trop } \mathcal{X}_{\Sigma',\beta'} \\
\text{Trop } \mathcal{X}_{\Sigma,\beta} & \xrightarrow{\text{Trop } f} &
\end{array}
$$

Theorem 1.3 (Theorem C)**.** *Suppose $\mathcal{X}_{\Sigma',\beta'} \to \mathcal{X}_{\Sigma,\beta}$ is a toric morphism from a smooth toric stack, which restricts to an isomorphism of tori, and which restricts to a canonical stack morphism over every torus-invariant cohomologically affine open substack of $\mathcal{X}_{\Sigma,\beta}$. Then* Trop $\mathcal{X}_{\Sigma',\beta'} \to$ Trop $\mathcal{X}_{\Sigma,\beta}$ *is a canonical stack morphism.*

The second part of chapter 4 is devoted to analytic stacks and tropicalizations as developed by Ulrisch in [Uli16]. In particular using ideas from [GS15] we show that Ulrisch's work is the special case of when a map between two lattices is the zero map.

Theorem 1.4 (Theorem D)**.** *Let $\beta\colon \Delta \to \Lambda$ be a map of lattices with finite cokernel. Then the following diagram commutes.*

$$\begin{array}{ccc} & X^{an} & \\ {}^{\mu}\swarrow & & \searrow^{\mathrm{Trop}_\Lambda} \\ |[X^{an}/G_\beta^\circ]| & \xrightarrow[\mu_\Lambda]{\sim} & N_{\mathbb{R}}(\Lambda) \end{array}$$

Chapter 2

Background

2.1 Classical Setting

This section is the general setup for tropical geometry and the various lenses that it can be viewed through. We will be largely following [MS15]; [Gat06] is also a very good introductory resource.

2.1.1 General construction

These results can be found in chapters 2 and 3 of [MS15].

Definition 2.1. *By a* ***valuation*** *on a field* K *we mean a map* $\operatorname{val}\colon K \to \mathbb{R} \cup \{\infty\}$ *that satifies the following*

1. $\operatorname{val}(a) = \infty$ *if and only if* $a = 0$*,*

2. $\operatorname{val}(ab) = \operatorname{val}(a) + \operatorname{val}(b)$*, and*

3. $\operatorname{val}(a + b) \geq \min\{\operatorname{val}(a), \operatorname{val}(b)\}$ *for all* $a, b \in K^*$*.*

We also denote the image of the valuation map as $\Gamma_{\operatorname{val}}$*.*

Definition 2.2. *Let $K = \mathbb{C}\{\{t\}\}$ be the field of Puiseux Series over $\mathbb{C}$. The elements in this field are formal power series*

$$c(t) = c_1 t^{a_1} + c_2 t^{a_2} + \cdots$$

where $c_i \in \mathbb{C}$ and $a_1 < a_2 < a_3 < \cdots$ are rational numbers with common denominator. Alternatively we have that

$$\mathbb{C}\{\{t\}\} = \bigcup_{n \geq 1} \mathbb{C}\left(\left(t^{1/n}\right)\right).$$

where $\mathbb{C}\left(\left(t^{1/n}\right)\right)$ is the Laurent series field with variable $t^{1/n}$. Note that we have a valuation map $\operatorname{val}\colon \mathbb{C}\{\{t\}\} \to \mathbb{R}$ given by taking a non-zero scalar $c(t) \in \mathbb{C}\{\{t\}\}^*$ to the lowest exponent a_1 in the series expansion of $c(t)$.

The above definition can be extended to any field in an analogous way. In particular we have the following theorem.

Theorem 2.3. *If k is field that is algebraically closed of characteristic zero then $k\{\{t\}\}$ is algebraically closed.*

Remark. Note that we do need characteristic zero, as the Artin–Schreier polynomial $x^p - x - t^{-1}$ has roots $\left(t^{-1/p} + t^{-1/p^2} + \cdots\right) + c$ where c runs over $\mathbb{F}_p$.

Definition 2.4. *Let K be a field and $f = \sum_{u \in \mathbb{Z}^n} c_u x^u \in K[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$. We define $\operatorname{Trop}(f)$ by the following,*

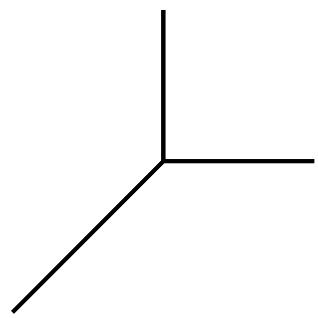
$$\operatorname{Trop}(f)(w) = \min_{u \in \mathbb{Z}^n} \left(\operatorname{val}(c_u) + \sum_{i=1}^{n} u_i w_i \right) = \min_{u \in \mathbb{Z}^n} \left(\operatorname{val}(c_u) + u \cdot w \right).$$

Definition 2.5. *The **tropical hypersurface** $\operatorname{Trop}(V(f))$ is the set*

$$\{w \in \mathbb{R}^n \mid \text{the minimum in } \operatorname{Trop}(f) \text{ is achieved at least twice}\}.$$

Example 2.6. Let $K = \mathbb{C}\{\{t\}\}$ and $f = x + y + 1$. Then $\mathrm{Trop}(f) = \min(x, y, 0)$ so

$$\mathrm{Trop}(V(f)) = \{w_1 = w_2 \leq 0\} \cup \{w_1 = 0 \leq w_2\} \cup \{w_2 = 0 \leq w_1\}.$$



Theorem 2.7 (Kapranov's Theorem)**.** *Let $f \in K[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$. Then the following sets coincide:*

1. *the tropical hypersurface* $\mathrm{Trop}\,(V\,(f))$ *in* $\mathbb{R}^n$*, and*
2. *the closure in* $\mathbb{R}^n$ *of* $\{(\mathrm{val}\,(y_1), \ldots, \mathrm{val}\,(y_n)) \mid (y_1, \ldots, y_n) \in V\,(f)\}$.

Theorem 2.8. *Let I be an ideal in the Laurent polynomial ring $K[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$ and let $X = V\,(I)$ in an algebraic torus T^n, then*

$$\mathrm{Trop}\,(X) = \bigcap_{f \in I} \mathrm{Trop}\,(V\,(f)) \subset \mathbb{R}^n.$$

Remark. One has to consider all elements of the ideal, not just a generating set. There is, however a notion of a tropical basis that can be used to make the above intersection finite.

Example 2.9. Let $n = 2$, $K = \mathbb{C}\{\{t\}\}$ and $I = \langle x + y + 1, x + 2y \rangle$. Then $X = V(I) = \{(-2, 1)\}$ and hence $\mathrm{Trop}(X) = \{(0, 0)\}$. However, the intersection of the two tropical lines given by the ideal generators equals

$$\mathrm{Trop}(V(x + y + z)) \cap \mathrm{Trop}(V(x + 2y)) = \left\{(w_1, w_2) \in \mathbb{R}^2 \mid w_1 = w_2 \leq 0\right\}.$$

This halfray is not a tropical variety. It is just a tropical prevariety.

Theorem 2.10 (Fundamental Theorem of Tropical Geometry)**.** *Let I be an ideal in $K[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$ and $X = V(I)$ be the associated variety in the torus $(K^*)^n$. Then the following sets coincide*

1. *the tropical variety* $\mathrm{Trop}(X)$*, and*

2. *the closure of the set of coordinatewise valuations of points in X,*

$$\mathrm{val}(X) = \{(\mathrm{val}(u_1), \dots, \mathrm{val}(u_n)) \mid (u_1, \dots, u_n) \in X\}.$$

2.1.2 Polyhedral connections

Here we recall the basic constructions and ideas of polyhedral geometry. These can be found in chapter 2 of [MS15].

Definition 2.11. *A set $X \subset \mathbb{R}^n$ is **convex** if, for all $u, v \in X$ and all $0 \leq \lambda \leq 1$ we have that $\lambda u + (1-\lambda) v \in X$. The **convex hull** $\mathrm{conv}(U)$ of a set $U \subseteq \mathbb{R}^n$ is the smallest convex set containing U; in particular if $U = \{u_1, \dots, u_r\}$ then $\mathrm{conv}(U) = \Big\{ \sum_{i=1}^r \lambda_i u_i \mid 0 \leq \lambda \leq 1, \sum_{i=1}^r \lambda_i = 1 \Big\}$. Such an object is called a **polytope**.*
*A **polyhedral cone** in $\mathbb{R}^n$ is the positive hull of a finite set in $\mathbb{R}^n$, namely*

$$C = \mathrm{pos}(v_1, \dots, v_r) = \Big\{ \sum_{i=1}^r \lambda_i v_i \mid \lambda_i \geq 0 \Big\}.$$

Alternatively we have that

$$C = \{x \in \mathbb{R}^n \mid Ax \leq 0\}$$

where A is a $d \times n$ matrix.
*A **face of a cone** is determined by a linear functional $w \in (\mathbb{R}^n)^\vee$; in other words*

$$\mathrm{face}_w(C) = \{x \in C \mid w \cdot x \leq w \cdot y, \textit{for all } y \in C\}.$$

*A **polyhedral fan** is a collection of polyhedral cones.*

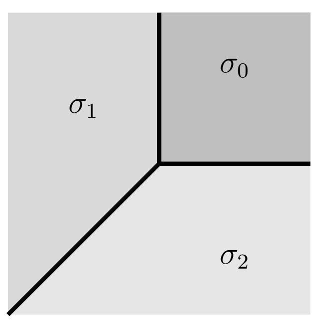
Example 2.12 (Fan of $\mathbb{P}^2$). Consider the fan, below, Σ in $N_{\mathbb{R}} = \mathbb{R}^2$ where $N = \mathbb{Z}^2$ with standard basis e_1, e_2. We have the three, two-dimensional, cones $\sigma_0 = \text{cone}(e_1, e_2)$, $\sigma_1 = \text{cone}(-e_1 - e_2, e_2)$, $\sigma_2 = \text{cone}(e_1, -e_1 - e_2)$, together with the three rays $\tau_{ij} = \sigma_i \cap \sigma_j$ for $i \neq j$, and the origin. The toric variety is covered by the affine opens

$$U_{\sigma_0} = \text{Spec}\,(\mathbb{C}[S_{\sigma_0}]) \simeq \text{Spec}(\mathbb{C}[x, y])$$
$$U_{\sigma_1} = \text{Spec}\,(\mathbb{C}[S_{\sigma_1}]) \simeq \text{Spec}(\mathbb{C}[x^{-1}, x^{-1}y]))$$
$$U_{\sigma_2} = \text{Spec}\,(\mathbb{C}[S_{\sigma_2}]) \simeq \text{Spec}(\mathbb{C}[xy^{-1}, y^{-1}]).$$

Moreover the gluing data on the coordinate rings is given by

$$g_{10}^*\,\mathbb{C}[x, y]_x \simeq \mathbb{C}[x^{-1}, x^{-1}y]_{x^{-1}}$$
$$g_{20}^*\,\mathbb{C}[x, y]_y \simeq \mathbb{C}[xy^{-1}, y^{-1}]_{y^{-1}}$$
$$g_{21}^*\,\mathbb{C}[x^{-1}, x^{-1}y]_{x^{-1}y} \simeq \mathbb{C}[xy^{-1}, y^{-1}]_{xy^{-1}}.$$

Now if $\mathbb{P}^2$ has the standard homogeneous coordinates (x_0, x_1, x_2) then the assignment $x \mapsto \dfrac{x_1}{x_0}$ and $y \mapsto \dfrac{x_2}{x_0}$ identifies the standard affine open $U_i \subset \mathbb{P}^2$ with $U_{\sigma_i} \subset X_\Sigma$.



Remark. Note that the rays τ_{ij} above look surprising similar to the tropical line from the previous example.

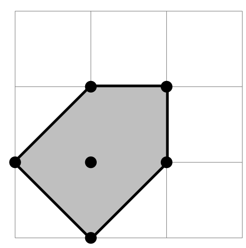
Definition 2.13. *Let $S = K[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$ be the Laurent polynomial ring over a field K and let $f = \sum_{u \in \mathbb{Z}^n} c_u x^u \in S$ the* ***Newton polytope*** *of f is the polytope*

$$\text{Newt}\,(f) = \text{conv}\,(u \mid c_u \neq 0) \subset \mathbb{R}^n.$$

Example 2.14. Let $S = \mathbb{C}\left[x^{\pm 1}, y^{\pm 1}\right]$ and consider the polynomial

$$f = 7x + 8y - 3xy + 4x^2y - 17xy^2 + x^2y^2.$$

The Newton polygon of f can be seen below.



Definition 2.15. *Let $v_1, \ldots, v_r$ be an ordered list of vectors in $\mathbb{R}^{n+1}$ and fix $w = (w_1, \ldots, w_r) \in \mathbb{R}^r$. The **regular subdivision** of $v_1, \ldots, v_r$ induced by w is the polyhedral fan with support* $\operatorname{pos}(v_1, \ldots, v_r)$ *whose cones are* $\operatorname{pos}(v_i \mid i \in \sigma)$ *for all subsets $\sigma \subset \{1, \ldots, r\}$ such that there exists $c \in \mathbb{R}^{n+1}$ with $c \cdot v_i = w_i$ for $i \in \sigma$ and $c \cdot v_i < w_i$ for $i \notin \sigma$. When the fan is **simplicial**, that is, all the cones are spanned by linearly independent vectors, then the subdivision is called a **regular triangulation**.*

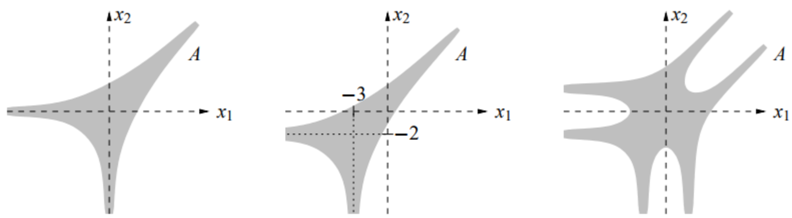
Theorem 2.16 (Structure theorem)**.** *Let X be an irreducible d-dimensional subvariety of T^n. Then* $\operatorname{Trop}(X)$ *is the support of a balanced weighted Γ_{val}-rational polyhedral complex pure of dimension d. Moreover, that polyhedral complex is connected through codimension one.*

Proposition 2.17 (Converse to the structure theorem)**.** *Let Σ be a balanced weighted Γ_{val}-rational polyhedral complex in $\mathbb{R}^n$ that is pure of dimension $n-1$. Then there exists a tropical polynomial F with coefficients in Γ_{val} such that $\Sigma = V(F)$. This ensures that $\Sigma =$* $\operatorname{Trop}(V(f))$ *for some Laurent polynomial $f \in K\left[x_1^{\pm 1}, \ldots, x_n^{\pm 1}\right]$.*

Remark. Note that this only works for hypersurfaces; see Example 2.28.

2.1.3 Analytic picture

The following picture can be found in [Gat06, §1.1]. This is the illustration of what happens one maps from $(\mathbb{C}^*)^2 \to \mathbb{R}^2$ via the log map and takes an appropriate intersection. The "tropical curves" arise as the limits of these "amoebas" . This can be made explicit by the description below when $n = 2$.



(a) $C = \{z; z_1 + z_2 = 1\}$ (b) $C = \{z; e^3 z_1 + e^2 z_2 = 1\}$ (c) $C =$ a generic conic

.

As we will not need the need the full analytic picture for the work we develop here, we cite some of the main properties from [MS15, §1.4].
Consider an ideal $I \in \mathbb{C}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$ with variety $V(I) = \{z \in \mathbb{C}^* \mid f(z) = 0, f \in I\}$ the amoeba of the ideal I is defined as

$$\mathcal{A}(I) = \{(\log(|z_1|), \log(|z_2|), \ldots, \log(|z_n|)) \in \mathbb{R}^n \mid z = (z_1, \ldots, z_n) \in V(I)\}.$$

Let $M > 0$ be any real number and consider the set $\mathcal{A}_M(I) = \frac{1}{M}\mathcal{A}(I) \cap \mathbb{S}^{n-1}$ where $\mathbb{S}^{n-1}$ is the unit sphere in dimension n.

Theorem 2.18. *The tropical variety of I coincides with the cone over the logarithmic set $\mathcal{A}_\infty(I)$, which is defined to be all points $v \in \mathbb{S}^{n-1}$ such that there is a sequence $v_M \in \mathcal{A}_M(I)$ converging to v, e.g., $\lim_{M\to\infty} v_m = v$. That is to say, $w \in \mathbb{R}^n$ is in $\mathrm{Trop}(V(I))$ if and only if the corresponding unit vector $\frac{1}{||w||}w$ lies in $\mathcal{A}_\infty(I)$.*

2.2 Scheme Theoretic

We review the basics laid out in the papers [GG16a] and [GG16b] to enrich the previous theory to allow for the more flexible structure of schemes as opposed to varieties.

2.2.1 $\mathbb{F}_1$ and semiring algebra

The algebraic set up for this type of theory can be thought of as extending the standard picture from rings to semirings and finding a correct notion of "Spec".

Definition 2.19. *An $\mathbb{F}_1$-**module** M will be a pointed set with a distinguished base point denoted 0_M. An $\mathbb{F}_1$-**algebra** will be a commutative monoid with zero. Taking an initial object in the category of $\mathbb{F}_1$-modules gives $\mathbb{F}_1 \simeq \{0, 1\}$.*

Definition 2.20. *A **semiring** is a monoidal object in the monoidal category of commutative monoids. That is, it is an object that satisfies all axioms of a ring except the existence of additive inverses.*

Proposition 2.21. *Given a semiring S there is an adjoint pair of functors*

$$\mathbb{F}_1 - \mathrm{Mod} \rightleftarrows S - \mathrm{Mod}.$$

Namely, extension of scalars $- \otimes S$ and the forgetful functor, which sends a semiring to its underlying set with zero as the base-point.

Definition 2.22. *Let A be either an $\mathbb{F}_1$ module or semiring, then an **ideal** I of A is any submodule of A (when A is regarded as an A-module).*

Definition 2.23. *Let S and M be a semiring and an S-module. A **semiring congruence** on S is an equivalence relation $J \subset S \times S$ that is a sub-semiring, and a **module congruence** on M is an S-submodule $J \subset M \times M$ that is also an equivalence relation.*

2.2.2 Tropical schemes

Definition 2.24. *Let A be either a semiring algebra or an $\mathbb{F}_1$-algebra. A proper ideal $\mathfrak{p} \subset A$ is* ***prime*** *if its complement is closed under multiplication. Given a prime ideal $\mathfrak{p}$ we can form the localization $A_{\mathfrak{p}}$ via equivalence classes of fractions in the same way as rings. As a topological space $|\mathrm{Spec}\ A|$ is the set of prime ideals of A together with a basis given by the* ***principal opens****, $D(f) = \{\mathfrak{p} \mid f \notin \mathfrak{p}\}$. Any A-module M determines a sheaf $\widetilde{M}$ that sends a principal open set $D(f)$ to the localization $M_f = A_f \otimes M$. In particular A itself gives a sheaf Q-algebras; this sheaf is the* ***structure sheaf*** *$\mathcal{O}_A$. An* ***affine scheme*** *is a pair $(|X|, \mathcal{O})$ that is isomorphic to $(|\mathrm{Spec}\ A|, \mathcal{O}_A)$.*

Proposition 2.25. *Any locally integral $\mathbb{F}_1$-scheme has a topological basis given by the spectra of integral $\mathbb{F}_1$-algebras.*

Definition 2.26. *A* ***congruence sheaf*** *$\mathscr{J}$ on X is a subsheaf of $\mathcal{O}_X \times \mathcal{O}_X$ such that $\mathscr{J}(U)$ is a congruence on $\mathcal{O}_X$ for each open $U \subset X$. A congruence sheaf is* ***quasi-coherent*** *if it is quasi-coherent when regarded as a sub-$\mathcal{O}_X$-module of $\mathcal{O}_X \times \mathcal{O}_X$ (analogous to the case of modules over rings).*

Proposition 2.27. *Let S be a semiring.*

1. *Let $X = \mathrm{Spec}\ A$ be an affine S-scheme. The global sections functor induces a bijection between quasi-coherent congruence sheaves on $\mathrm{Spec}\ A$ and congruences on A.*

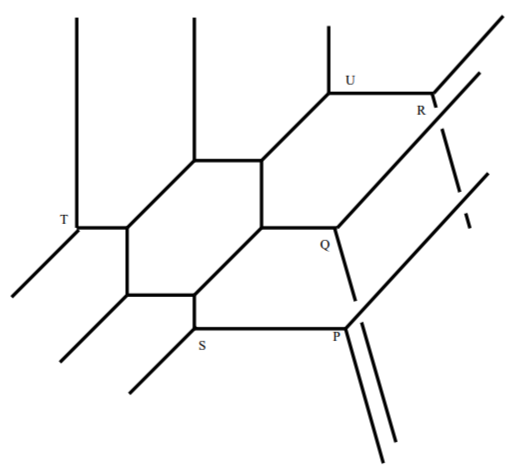
2. *For X an arbitrary S-scheme, there is a bijection between closed subschemes of X and quasi-coherent congruences sheaves on X.*

Remark. While it will be stated explicitly in chapter 4 (see Proposition 4.7) there exists a Cox construction for toric varieties that allows one to work over $\mathbb{F}_1$. We do not need it for this paper, but it is something to consider for generalizations of the theory of toric stacks developed.

2.2.3 Examples of the differences between using rings and semirings

Here we list some issues with the theory and provided examples to illustrate these issues.

Example 2.28 (An example of Mikhalkin)*.* This can be found in [Spe05, §5.1]. This example should be thought of as a "tropical curve" in space. While the curve looks tropical, it can be shown that it does not arise from the tropicalization of an algebraic curve. In particular, this shows why we need the condition "pure of dimension $n-1$" in Proposition 2.17.



Proposition 2.29. *Closed immersions.*

1. *Over a ring a closed immersion is a morphism* $\Phi\colon Y \to X$ *such that* $\Phi(Y)$ *is topologically a closed subspace of* X*, the induced map* $Y \to \Phi(Y)$ *is a homeomorphism and the sheaf map* $\Phi^\sharp\colon \mathcal{O}_X \to \Phi_*\mathcal{O}_Y$ *is surjective. These conditions on* Φ *are equivalent to requiring that* Φ *is an affine morphism and* $\Phi^\sharp$ *surjective. In particular there is a bijection between closed subschemes of* X *and quasi-coherent ideal sheaves on* X*.*

2. *For a semiring a closed immersion is an affine morphism* $\Phi\colon Y \to X$ *such that* $\Phi^\sharp\colon \mathcal{O}_X \to \Phi_*\mathcal{O}_Y$ *is surjective.*

However the equivalence breaks down as with the following example.

Example 2.30. Consider $\Phi\colon \operatorname{Spec} \mathbb{T} \to \mathbb{A}^n_{\mathbb{T}}$ corresponding to a $\mathbb{T}$-algebra morphism $\phi\colon \mathbb{T}[x_1, \dots, x_n] \twoheadrightarrow \mathbb{T}$ sending each x_i to a finite value. This is a closed immersion but the image is not Zariski closed; in fact it is a dense point, as $\phi^{-1}(\infty) = \{\infty\}$ which is contained in all the primes.

Definition 2.31. *Let* X *be a scheme over a semiring* S*.*

1. *Let* $U \hookrightarrow X$ *be an open immersion. Then the* S*-points* $X(S)$ *induce a Zariski topology on the set of* X *via* $U(S) \subset X(S)$*.*

2. *Similarly, if* $Z \hookrightarrow X$ *is a close immersion. Then we get a strong Zariski topology whose closed subsets are of the form* $Z(S) \subset X(S)$*.*

Example 2.32. Zariski vs strong Zariski topologies.

1. For a ring, the above two topologies agree, [GG16b, §3.4.2].

2. For a semiring the strong Zariski topology is finer than the Zariski topology. Consider $\mathbb{A}^1$ over $\mathbb{N}$, the strong Zariski topology is the finite complement topology; whereas, the only nontrivial Zariski closed subset is the singleton $\{0\}$.

Chapter 3

Jet Bundles of Tropical Hypersurfaces

In this section we consider a simple question, loosely put, "What happens when we tropicalize the equations for the jet bundle of a hypersurface?".

3.1 Review

In this section we recall the standard defintions of algebraic geometry for tangent spaces in algebraic geometry.

Definition 3.1. *Let X be a variety and $p \in X$ be a point of X. The Zariski Tangent Space of X at p is $T_pX := \mathfrak{m}/\mathfrak{m}^2$ where $\mathfrak{m}$ is the maximal ideal of $\mathcal{O}_{X,p}$.*

Definition 3.2. *A quasi-projective variety X over an algebraically closed field is smooth at p if the local dimension of X at p is equal to the dimension of the Zariski tangent space at p.*

Definition 3.3. *A tropical variety X is smooth if its Newton subdivision consists of only triangles of area 1/2.*

3.2 Tropical Connections

In [Yag16] the author outlines the proper formalism to "linearize" tropical monomials. In particular the polyhedral behavior of these linearizations is formalized.

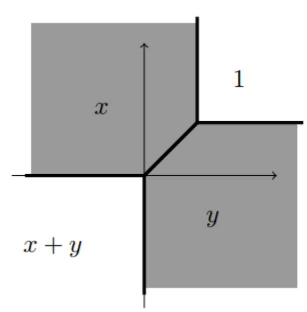
Definition 3.4. *The semiring of tropical dual numbers* $\widetilde{\mathbb{T}}$ *is the quotient of* $\mathbb{T}[\epsilon]$ *by the congruence* $\langle \epsilon^2 \sim 1_{\mathbb{T}} \rangle$.

This definition has the obvious generalization for any non-negative integral power of ϵ. The following definition is almost correct for what we want to do with tangent bundles.

Definition 3.5. *A simple monomial term of* $f \in \widetilde{\mathbb{T}}[x]$ *is a monomial term in the* $\mathbb{T}$*-algebra* $\mathbb{T}[x, \epsilon]$ *of the form* cx^n *or* $cx^n\epsilon$.

While the results in [Yag16] are concerned with simple monomial terms as defined above, we will consider products of simple monomial terms. The motivation being that we want a formalism consistent with the jet bundle calculations. Below is a picture, that can be found in [Yag16], of the tropical variety,

$$V(f) = xy + (0 + \epsilon)x + (0 + \epsilon)y + 1 \in \widetilde{\mathbb{T}}[x].$$



3.3 Results

This section is devoted to providing a generalization of Yaghmayi's work to include products of monomials and to get the correct graphical interpretation. Namely, in the above picture, we want to find a way to "fill in" the area labeled with "$x + y$" as this also appears when jet bundle calculations are done.

Theorem 3.6 (Theorem A)**.** *Let X be a variety defined by a single equation f and let J_X^n be the jet bundle for X. Then for the tropical variety* Trop X *has a "tropical jet bundle"* Trop J_X^n *coming from the tropicalization of J_X^n. Moreover, these equations can be visualized as degree n polyhedral complexes that have boundaries arising from* Trop X*.*

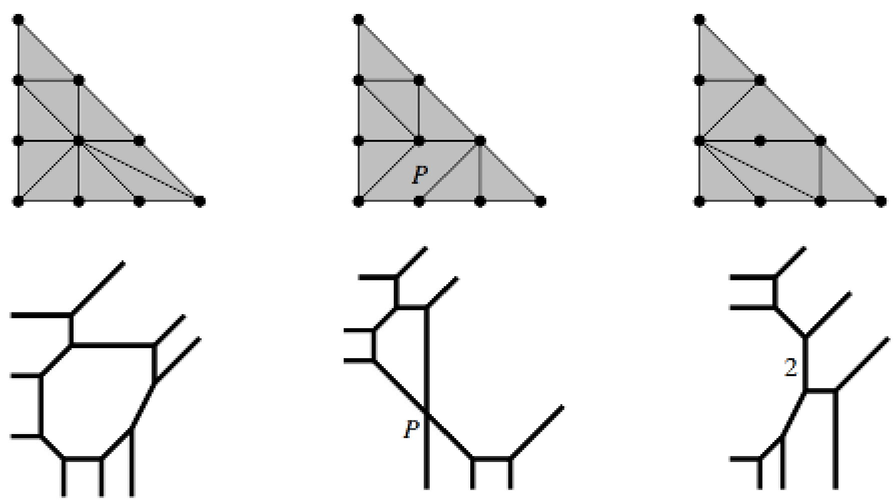
Proof. The proof largely follows that of [Yag16]. First we need to work in the appropriate semiring for jet bundles, instead of the dual numbers we have

$$\widetilde{\mathbb{T}} = \mathbb{T}[\epsilon]/\langle \epsilon^{n+1} \sim 1_{\mathbb{T}} \rangle.$$

Fix a polynomial $f \in \mathbb{T}[x_1, \ldots, x_n]$. The graph of f is the lower envelope of the union of hyperplanes in $\mathbb{T}^{k+1}$, and its projection onto $\mathbb{T}^k$ induces a finite decomposition of $\mathbb{T}^k$ into r closed k-dimensional polyhedra. Note that $V(f)$ is the boundary of the polyhedra where at least two monomials attain the same minimum. A more explicit description of this process can be found in [MS15] Section 3.1.

Now consider a subset $S \subset \{1, \ldots, r\}$ and a polynomial $F \in \widetilde{\mathbb{T}}[x_1, \ldots, x_n]$ obtained from f by replacing *all* monomials $c_j x^{m_j}$ with $(c_j + c_j\epsilon + c_j\epsilon^2 + \cdots + c_j\epsilon^n)x^{m_j}$. Then $V(F)$ will be the boundary of those polyhedra, say P_j, as well as the interior of the P_j where $j \in S$. □

Remark. While it is a nice picture it does not solve the "traditional problem" of whether or not a variety is smooth or singular, as the notion of "tropically singular" does not exist. However, Definition 3.3 provides a notion of "tropically smooth", there still no clear indication of how to make use of this idea in such a setting. Below are three examples of cubic tropical curves, the first of which is tropically smooth, where the other two are not. These can be found in [Gat06].



Chapter 4

All Things Stacks

The goal of this chapter is to define tropical toric stacks, which loosely speaking, are stacks that arise as tropicalizations of toric stacks in the sense of [GS15]. Given the work of [Spe05] we do not expect this to be the full description of a “tropical stack”. After the toric case we turn our attention to analytic stacks and their tropicalizations in the sense of [Uli16] and prove a generalization where we can take quotients by subgroups of the big affinoid torus T°.

4.1 All Things Stacks

The majority of this section can be found in [Sta17]. As we will not be needing the full generality of stacks we direct the readers to [Sta17] for the proofs for a majority of the theorems.

4.1.1 Main definitions

In this section we cite the major definitions and properties of stacks, the proofs of which can be found in [Sta17].

Definition 4.1. *Let $\mathcal{C}$ be a site. A stack over $\mathcal{C}$ is a category $p\colon \mathcal{S} \to \mathcal{C}$ over $\mathcal{C}$ which satisfies the following conditions:*

1. *$p\colon \mathcal{S} \to \mathcal{C}$ is a fibered category,*

2. *for any $U \in \mathrm{Ob}\,(\mathcal{C})$ and any $x, y \in S_U$ the presheaf $\mathrm{Mor}\,(x, y)$ is a sheaf on the site $\mathcal{C}/U$, and*

3. *for any covering $\mathcal{U} = \{f_i \mid U_i \to U\}_{i \in I}$ of the site $\mathcal{C}$, any descent datum in $\mathcal{S}$ relative to $\mathcal{U}$ is effective.*

A descent datum $(X_i, \varphi_{i,j})$ in $\mathcal{S}$ relative to the family $\{f_i \mid U_i \to U\}$ is given by an object X_I of $\mathcal{S}_{U_i}$ for each $i \in I$ an isomorphism $\varphi_{i,j}\colon \mathrm{pr}_0^* X_i \to \mathrm{pr}_1^* X_j$ in $\mathcal{S}_{U_i \times_U U_j}$ for each pair $(i, j) \in I^2$ such that for every triple of indices $(i, j, k) \in I^3$ (the diagram below) in the category $\mathcal{S}_{U_i \times_U U_j \times_U U_k}$ commutes (this is a cocycle condition).

$$\begin{array}{ccc} pr_0^* X_i & \xrightarrow{pr_{02}^* \phi_{ik}} & pr_2^* X_k \\ & {\scriptstyle pr_{01}^* \phi_{ij}} \searrow \quad \nearrow {\scriptstyle pr_{12}^* \phi_{jk}} & \\ & pr_1^* X_j & \end{array}$$

An important collection of stacks are those that are fibered in groupoids. They will ultimately allow us to take quotients so we define them here.

Definition 4.2. *A stack in groupoids over a site $\mathcal{C}$ is a category $p\colon \mathcal{S} \to \mathcal{C}$ such that:*

1. *$p\colon \mathcal{S} \to \mathcal{C}$ is fibered in groupoids over $\mathcal{C}$,*

2. *for all $U \in \mathrm{Ob}\,(\mathcal{C})$ and for all $x, y \in \mathrm{Ob}\,(\mathcal{S}_U)$ the presheaf $\mathrm{Isom}\,(x, y)$ is a sheaf on the site $\mathcal{C}/\mathcal{S}$, and*

3. *for any covering $\mathcal{U} = \{f_i \mid U_i \to U\}_{i \in I}$ of the site $\mathcal{C}$, any descent datum in $\mathcal{S}$ (relative to $\mathcal{U}$) is effective.*

Definition 4.3. *Let S be a scheme and B an algebraic space. Consider a map $B \to S$.*

1. *Let (U, R, s, t, c) be a groupoid in algebraic spaces over B. The quotient stack,*
$$p\colon [U/R] \to (\mathrm{Sch}/S)_{fppf}$$
of (U, R, s, t, c) is the stackification of the category fibered in groupoids $[U/_pR]$ over $(\mathrm{Sch}/S)_{fppf}$.

2. *Let (G, m) be a group algebraic space over B. Let $a\colon G \times_B X \to X$ be an action of G on an algebraic space over B. The quotient stack*
$$p\colon [X/G] \to (\mathrm{Sch}/S)_{fppf}$$
is the qutoent stack associated to the gorupoid in algebraic spaces $(X, G \times_B X, s, t, c)$ over B.

4.2 All Things Toric

This section will follow [MS15] chapter 6 for the basic connections between toric geometry and tropicalizations. After which we turn to [GS15], primarily sections 2, 3, and 4, for the theory of toric stacks which will serve as the foundation for our theory of tropical stacks.

4.2.1 Toric Geometry

Here we review the basics of chapter 6 of [MS15] with the goal of showing that tropicalization commutes with the Cox construcion.

Definition 4.4. *A Toric Variety X_Σ is defined by a rational fan Σ in $N_\mathbb{R} = N \otimes R \simeq \mathbb{R}^n$ for a lattice $N \simeq \mathbb{Z}^N$. It has dual lattice $M = \mathrm{Hom}\,(N, \mathbb{Z})$ and $M_\mathbb{R}$ is the vector space $M \otimes R \simeq \mathbb{R}^n$ with torus $T^n = N \otimes K \simeq \mathrm{Hom}\,(M, K^*) \simeq (K^*)^n$.*

We can say more: given $\sigma \in \Sigma$ determines a local chart $U_\sigma = \text{Spec }(K[\sigma^\vee \cap M])$ where $\sigma^\vee = \{u \in M \mid u \cdot v \geq 0 \; \forall \; v \in \sigma\}$.

Proposition 4.5 (Cox Construction, [MS15, §6.1])**.** *Let X_Σ be a simplicial toric variety. Then*

$$X_\Sigma = \left(\mathbb{A}^N \setminus V(B)\right)/H$$

where B is the irrelevant ideal and $H = \text{Hom}\left(A_{n-1}(X_\Sigma), K^*\right)$.

We will outline the argument as in [MS15]. Since X_Σ is simplicial we can number the rays of Σ from 1 to N and set $S = K[x_1, \ldots, x_N]$. We have that S is graded by the class group $A_{n-1}(X_\Sigma)$ which can be fit into the following exact sequence,

$$0 \to M \simeq \mathbb{Z}^n \xrightarrow{V} \mathbb{Z}^N \xrightarrow{deg} A_{n-1}(X_\Sigma) \to 0,$$

where V is the $N \times n$ matrix whose ith row is v_i which is the first lattice point of the ith ray. If we take $\text{Hom}(-, K^*)$ we obtain the following:

$$\text{Hom}(M, K^*) \simeq T^n \xleftarrow{V^t} \text{Hom}\left(\mathbb{Z}^N, K^*\right) \simeq (K^*)^N \leftarrow H \leftarrow 0$$

Note that this H is the H in the above proposition, and has an action on $\mathbb{A}^N$. We also define the **irrelevant ideal** as

$$B = \left\langle \prod_{v_i \notin \sigma} x_i \mid \sigma \in \Sigma \right\rangle.$$

Since this is now in place we can move onto the tropical aspects of toric geometry namely to describe the following two results.

Proposition 4.6. *If X_Σ is a toric variety as above then as a set,*

$$X_\Sigma^{\text{Trop}} = \coprod_{\sigma \in \Sigma} N(\sigma),$$

where $N(\sigma) = N_\mathbb{R}/\text{span}(\sigma)$.

With Proposition 4.6 in mind, we can place the point-wise convergence topology on each U_σ^{Trop} by realizing them as $\text{Hom}(\sigma^\vee \cap M, \mathbb{T})$. Gluing these spaces along common faces will give us the space X_Σ^{Trop}.

Proposition 4.7 (Tropical Cox construction, [MS15, Proposition 6.2.5])**.** *If X_Σ is a simplicial toric variety then*

$$X_\Sigma^{\text{Trop}} = \left(\text{Trop } \mathbb{A}^N \setminus \text{Trop } V(B)\right) \;/\; \text{Trop } H.$$

Proposition 4.8. *Let Y be a subvariety of a smooth toric variety X_Σ and let I be its B-saturated ideal in the Cox ring $K[x_1, \ldots, x_N]$ of X_Σ. Then*

$$\text{Trop } Y = \left(\bigcap_{f \in I} \text{Trop } V(f) \setminus \text{Trop } V(B)\right) / \text{Trop } H = \bigcup_{\sigma \in \Sigma} \text{Trop } \left(\bar{Y} \cap \mathcal{O}_\sigma\right)$$

where $\mathcal{O}_\sigma$ is a toris orbit.

Proposition 4.9. *Let $\pi \colon X_\Sigma \to X_\Delta$ be a map of toric varieties given by a map of fans $\pi \colon \Sigma \to \Delta$ and let $\text{Trop}(\pi) : \text{Trop } X_\Sigma \to \text{Trop } X_\Delta$ be the induced map on tropical varieties. Then for a subvariety Y of X_Σ $\text{Trop } (\pi(Y)) = \text{Trop}(\pi)(\text{Trop } Y)$.*

4.2.2 GS toric stacks

In this section we recall the import constructions and definitions from [GS15]. These can be found in sections 2, 3 and 4 of [GS15].

Definition 4.10. *A **toric stack** is an Artin stack of the form $[X/G]$ together with an action of the torus $T = T_0/G$, where T_0 is the torus of the normal toric variety X and G is a subgroup of T_0. A non-strict toric stack is an Artin stack of the form $[Z/G]$ together with an action of the 'stacky' torus $[T'/G]$ where Z is an integral T_0 -invariant subvariety of X with torus T'.*

Definition 4.11. *If L is a lattice put T_L to be the torus*

$$D\left(L^*\right) = \operatorname{Hom}_{gp}\left(\operatorname{Hom}_{gp}\left(L, \mathbb{Z}\right), \mathbb{G}_m\right)$$

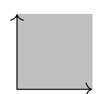
whose lattice of 1*-parameter subgroups is naturally isomorphic to L.*

Remark. A fact that comes up quite a bit is that a morphism $\beta\colon L \to N$ of arbitrary finitely generated abelian groups has finite cokernel if and only if $\beta^*\colon N^* \to L^*$ is injective.

Definition 4.12. *A **stacky fan** is a pair (Σ, β) where Σ is a fan on the lattice L and $\beta\colon L \to N$ is a homomorphism to a lattice N such that* cok β *is finite.*

Definition 4.13. *If (Σ, β) is a stacky fan then the **toric stack** $\mathcal{X}_{\Sigma,\beta}$ is defined to be $[X_\Sigma/G_\beta]$ with torus $T_N = T_L/G_\beta$ where $G_\beta = \ker\left(T_\beta\colon T_L \to T_N\right)$.*

Example 4.14. Consider the toric variety with the following fan below. Then we have that $X_\Sigma = \mathbb{A}^2$. Consider the map $\beta\colon \mathbb{Z}^2 \to \mathbb{Z}^2$ given by the matrix $\left[\begin{smallmatrix} 1 & 0 \\ 1 & 2 \end{smallmatrix}\right]$. We then have that $\beta^*\colon \mathbb{Z}^2 \to \mathbb{Z}^2$ is given by $\left[\begin{smallmatrix} 1 & 1 \\ 0 & 2 \end{smallmatrix}\right]$. Now we get a map on tori $\mathbb{G}_m^2 \to \mathbb{G}_m^2$ given by $(s,t) \mapsto (st, t^2)$. The kernel of this action is $G_\beta = \mu_2 = \{(\zeta,\zeta) \mid \zeta^2 = 1\} \subset \mathbb{G}_m^2$ So our toric stack is $\mathcal{X}_{\Sigma,\beta} = [\mathbb{A}^2/\mu_2]$ with action $\zeta \cdot (x,y) = (\zeta \cdot x, \zeta \cdot y)$.



Definition 4.15. *A **toric morphism** is a morphism which restricts to a homomorphism of (stacky) tori and is equivariant with respect to that homomorphism.*

Definition 4.16. *A **morphism of stacky fans** $(\Sigma, \beta\colon L \to N) \to (\Sigma', \beta'\colon L' \to N')$ is a pair of group morphisms $\Phi\colon L \to L'$ and $\phi\colon N \to N'$ so that $\beta' \circ \Phi = \phi \circ \beta$ and so that for every cone $\sigma \in \Sigma$ $\Phi(\sigma)$ is contained in a cone of Σ'. We can represent this as the following diagram.*

$$\Sigma \longrightarrow \Sigma'$$

$$\begin{array}{ccc} L & \xrightarrow{\Phi} & L' \\ \big\downarrow{\scriptstyle \beta} & & \big\downarrow{\scriptstyle \beta'} \\ N & \xrightarrow{\phi} & N' \end{array}$$

Theorem 4.17. *Let $(\Sigma, \beta \colon L \to N)$ and $(\Sigma', \beta \colon L' \to N')$ be stacky fans, and suppose $f \colon X_\Sigma \to X_{\Sigma'}$ is a toric morphism. Then there exists a stacky fan (Σ_0, β_0) and morphisms $(\Phi, \phi) : (\Sigma_0, \beta_0) \to (\Sigma, \beta)$ and $(\Phi', \phi') : (\Sigma_0, \beta_0) \to (\Sigma', \beta')$ such that the following triangle commutes and $\mathcal{X}_{\Phi,\phi}$ is an isomorphism.*

$$\begin{array}{ccc} \mathcal{X}_{\Sigma_0,\beta_0} & & \\ & \searrow^{\mathcal{X}_{(\Phi',\phi')}} & \\ {\scriptstyle \mathcal{X}_{(\Phi,\phi)}}\big\downarrow & & \mathcal{X}_{\Sigma',\beta'} \\ & \nearrow_{f} & \\ \mathcal{X}_{\Sigma,\beta} & & \end{array}$$

Definition 4.18. *Let Σ be a fan on a lattice N and $\beta \colon \mathbb{Z}^n \to N$ be a homomorphism with finite cokernel so that every ray of Σ contains some $\beta(e_i)$ lies in the support of Σ. For a cone $\sigma \in \Sigma$ let $\widehat{\sigma} = \operatorname{cone}(\{e_i \mid \beta(e_i) \in \sigma\})$ and let $\widehat{\Sigma}$ be the fan on $\mathbb{Z}^n$ that is generated by $\widehat{\sigma}$. Define $\mathcal{F}_{\Sigma,\beta} = \mathcal{X}_{\widehat{\Sigma},\beta}$. Any toric stack isomorphic to some $\mathcal{F}_{\Sigma,\beta}$ is called a **fantastack**.*

Remark. Fantastacks have a description similar to the Cox Construction for toric varieties which can be described as follows. The cones of $\widehat{\Sigma}$ are indexed by the sets $\{e_{i_1}, \ldots, e_{i_n}\}$ such that $\{\beta(e_{i_1}), \ldots, \beta(e_{i_n})\}$ is contained in a single cone of Σ. It then becomes easy to identify the subvariety of $\mathbb{A}^n$ that is represented by $\widehat{\Sigma}$. Explicitly we have an ideal

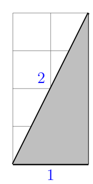
$$J_\Sigma = \left(\prod_{\beta(e_i) \notin \sigma} x_i \mid \sigma \in \Sigma \right)$$

so that $X_{\widehat{\Sigma}} = \mathbb{A}^n \backslash V(J_\Sigma)$ and we get that $\mathcal{F}_{\Sigma,\beta} = [(\mathbb{A}^n \setminus V(J_\Sigma))/D(\text{cok } \beta^*)]$.

Example 4.19. Consider the toric variety that has the fan below. Since a single cone contains all of the $\beta(e_i)$ we have that $X_{\widehat{\Sigma}} = \mathbb{A}^2$. The cokernel of β^* is

$$\mathbb{Z}^2 \xrightarrow{\beta^*} \mathbb{Z}^2 \xrightarrow{f} \mathbb{Z}/2$$

where $\beta^* = \left[\begin{smallmatrix} 1 & 0 \\ 1 & 2 \end{smallmatrix}\right]$ and $f = \left[\begin{smallmatrix} 1 & 1 \end{smallmatrix}\right]$. Note that the rows of β^* are infact the coordinates of $\beta(e_i)$ marked on the picture. So the fantastack is $\mathcal{F}_{\Sigma,\beta} = [\mathbb{A}^2/\mu_2]$



Before giving the next set of definitions consider the following. Let (Σ, β) be a stacky fan and put $\Sigma(1)$ to be the set of rays of Σ. Let $M \subset L$ be the saturated sublattice spanned by Σ and $M' \subset L$ be the direct complement to M. For each ray $\rho \in \Sigma(1)$ put u_ρ to be the first element of M along ρ and e_ρ to be the generator in $\mathbb{Z}^{\Sigma(1)}$ corresponding to ρ. We then have a morphism $\Phi\colon \mathbb{Z}^{\Sigma(1)} \times M' \to L$ given by $(e_\rho, m) \mapsto u_\rho + m$. We can now define a fan $\widetilde{\Sigma}$ on $\mathbb{Z}^{\Sigma(1)} \times M'$. For each $\sigma \in \Sigma$ we put $\widetilde{\sigma} \in \widetilde{\Sigma}$ as the cone generated by $\{e_\rho \mid \rho \in \sigma\}$ The morphism of stacky fans is then,

$$\widetilde{\Sigma} \longrightarrow \Sigma$$

$$\begin{array}{ccc} \mathbb{Z}^{\Sigma(1) \times M'} & \xrightarrow{\Phi} & L \\ \downarrow{\scriptstyle \widetilde{\beta}} & & \downarrow{\scriptstyle \beta} \\ N & = & N. \end{array}$$

This construction leads us to make the following definition that captures the construction just stated.

Definition 4.20. *With the above, we call* $\widetilde{\mathcal{X}}_{\widetilde{\Sigma},\widetilde{\beta}}$ *the* ***canonical stack*** *over* $\mathcal{X}_{\Sigma,\beta}$ *and the morphism* $\widetilde{\mathcal{X}}_{\widetilde{\Sigma},\widetilde{\beta}} \to \mathcal{X}_{\Sigma,\beta}$ *is a* ***canonical stack morphism****.*

Definition 4.21. *We say that* $\mathcal{X}_{\Sigma,\beta}$ *is* ***cohomologically affine*** *if* X_Σ *is affine.*

It can be easily shown that this only depends on $\mathcal{X}_{\Sigma,\beta}$ and not on the stacky fan (Σ, β). With these two definitions we get the following universal property.

Theorem 4.22. *Suppose* $\mathcal{X}_{\Sigma',\beta'} \to \mathcal{X}_{\Sigma,\beta}$ *is a toric morphism from a smooth toric stack, which restricts to an isomorphism of tori, and which restricts to a canonical stack morphism over every torus-invariant cohomologically affine open substack of* $\mathcal{X}_{\Sigma,\beta}$*. Then* $\mathcal{X}_{\Sigma',\beta'} \to \mathcal{X}_{\Sigma,\beta}$ *is a canonical stack morphism.*

4.2.3 Tropical Toric Stacks

The goal of this section is to give a tropical version of the work done in [GS15], namely to write down what a tropical toric stack is (at least when the ambient space is $\mathbb{T}$) and to describe the properties of maps between them.

Before starting, consider the previous example of the stack $[\mathbb{A}^2/\mu_2]$. In terms of tropicalization we know what each of the pieces is individually, that Trop $\mathbb{A}^2 = \mathbb{T}^2$ and Trop μ_2 is just a translation (with the condition that $2\zeta = 0$) with the action given by $\zeta \cdot (x, y) \mapsto (x + \zeta, y + \zeta)$. It is also worth noting that this is a subgroup of Trop $\mathbb{G}_m^2$ (which is equal to $\mathbb{G}_a^2$ since multiplication becomes addition under tropicalization). Now that we know what each of the individual pieces are, and we know that they live in $\mathbb{T}^2$ we can take the stack quorient in the point-wise convergence topology. With this in mind we can make the following definition.

Definition 4.23. *A tropical toric stack is a (real) stack of the form* Trop $\mathcal{X}_{\Sigma,\beta} =$ [Trop X_Σ/Trop G_β].

We need one last definition before stating the next theorem.

Definition 4.24. *If* $\widetilde{\mathcal{X}}_{\widetilde{\Sigma},\widetilde{\beta}}$ *is a canonical stack over* $\mathcal{X}_{\Sigma,\beta}$ *(Definition 4.20) then* Trop $\widetilde{\mathcal{X}}_{\widetilde{\Sigma},\widetilde{\beta}}$ *is a canonical tropical stack over* Trop $\mathcal{X}_{\Sigma,\beta}$*.*

Theorem 4.25 (Theorem B)**.** *Let* $(\Sigma, \beta\colon L \to N)$ *and* $(\Sigma', \beta\colon L' \to N')$ *be stacky fans, and suppose* $f\colon X_\Sigma \to X_{\Sigma'}$ *is a toric morphism. Then the following diagram commutes.*

$$\begin{array}{ccc} \operatorname{Trop}\,\mathcal{X}_{\Sigma_0,\beta_0} & \xrightarrow{\operatorname{Trop}\,\mathcal{X}_{(\Phi',\phi')}} & \operatorname{Trop}\,\mathcal{X}_{\Sigma',\beta'} \\ {\scriptstyle \operatorname{Trop}\,\mathcal{X}_{(\Phi,\phi)}}\big\downarrow & \nearrow{\scriptstyle \operatorname{Trop}\, f} & \\ \operatorname{Trop}\,\mathcal{X}_{\Sigma,\beta} & & \end{array}$$

Proof. This is a fairly straight forward application of Proposition 4.9 and the original proof of Theorem 4.17 found in [GS15]. We will recall the important parts of the original proof. Since f restricts to a homomorphism of tori $T_N \to T_{N'}$ it also induces a homomorphism of lattices of 1-parameter subgroups $\phi\colon N \to N'$. Now, since this is a map of tori by Proposition 4.9 we get a map Trop $f\colon$ Trop $T_N \to$ Trop $T_{N'}$ as well as a map on the subgroups N and N'. In [GS15] the authors construct a variety Y_0 with torus T_0 that has a fan Σ_0 and 1-parameter subgroup L_0. The importance of this Y_0 is that there are morphisms of toric varieties $Y_0 \to X_\Sigma$ and $Y_0 \to X_{\Sigma'}$ and we get the following diagram.

$$\Sigma \longleftarrow \Sigma_0 \longrightarrow \Sigma'$$

$$\begin{array}{ccccc} L & \xleftarrow{\Phi} & L_0 & \longrightarrow & L' \\ {\scriptstyle \beta}\big\downarrow & & {\scriptstyle \beta_0}\big\downarrow & & \big\downarrow{\scriptstyle \beta'} \\ N & = & N & \xrightarrow[\phi]{} & N' \end{array}$$

This then gives us that Y_0 is a G_Φ-torsor over X_Σ, where G_Φ is the kernel of the surjection $T_0 \to T_L$.
Since all of the above are arising from toric varieties or toric morphisms, once again, we can tropicalize to get the following diagram.

$$\text{Trop } \Sigma \longleftarrow \text{Trop } \Sigma_0 \longrightarrow \text{Trop } \Sigma'$$

$$\begin{array}{ccccc} \text{Trop } L & \xleftarrow{\text{Trop } \Phi} & \text{Trop } L_0 & \longrightarrow & \text{Trop } L' \\ {\scriptstyle \text{Trop } \beta}\downarrow & & {\scriptstyle \text{Trop } \beta_0}\downarrow & & \downarrow{\scriptstyle \text{Trop } \beta'} \\ \text{Trop } N & = & \text{Trop } N & \xrightarrow[\text{Trop } \phi]{} & \text{Trop } N' \end{array}$$

The final set of equalities and maps in [GS15] give that $\mathcal{X}_{\Sigma_0,\beta_0} \simeq [X_\Sigma/G_\beta] \to [(Y/G_\Phi)/G_\beta] = \mathcal{X}_{\Sigma,\beta}$ and $[Y_0/G_{\beta_0}] \simeq [X_\Sigma/G_\beta] \to X_{\Sigma',\beta'}$. Then by our definition of tropical stack we get the necessary diagram stated above. $\square$

The next big result that needs to be addressed is that of canonical stacks. Funnily enough there is no real cohomomology theory for tropical geometry (yet) so the cohomologically affine definition seems out of place, but we will still use it as it is makes sense on the toric side.

Theorem 4.26 (Theorem C)**.** *Suppose $\mathcal{X}_{\Sigma',\beta'} \to \mathcal{X}_{\Sigma,\beta}$ is a toric morphism from a smooth toric stack, which restricts to an isomorphism of tori, and which restricts to a canonical stack morphism over every torus-invariant cohomologically affine open substack of $\mathcal{X}_{\Sigma,\beta}$. Then* Trop $\mathcal{X}_{\Sigma',\beta'} \to$ Trop $\mathcal{X}_{\Sigma,\beta}$ *is a canonical stack morphism.*

Proof. Since Theorem 4.22 is a corollary of Lemma 5.3 in [GS15] we will prove a tropical version of that Lemma. The main points brought up by the authors are that if $f\colon \mathcal{X}_{\Sigma',\beta'} \to \mathcal{X}_{\Sigma,\beta}$ is a toric surjection from a smooth cohomologically affine toric stack with n torus-invariant divisors which restricts to an isomorphism on tori, then we have that for $(\Phi, \phi)\colon (\Sigma'.\beta') \to (\Sigma, \beta)$ we have that ϕ must also be an isomorphism. The second observation is that

since we are interested in torus-equivariant morphisms we can identify $\mathcal{X}_{\Sigma',\beta'}$ as $[\mathbb{A}^n/\mathbb{G}_m^n]$. The final point is that as $f\colon \Sigma' \to \Sigma$ is surjective, every ray of Σ is the image of a unique ray in Σ', namely $\Phi(e_i) = k_i\rho_i$ where e_i is the first lattice point of the rays in Σ'. This factors uniquely through the canonical stack by sending $e_i \mapsto k_i e_{\rho_i}$.
Now with the above we note that we can work with

$$\text{Trop } \mathcal{X}_{\Sigma',\beta'} = [\text{Trop } \mathbb{A}^n/\text{Trop } \mathbb{G}_m^n].$$

Moreover, since we know what the generators are for the fan of Σ' are the e_i's we have that $\text{Trop}(\Phi(e_i)) = \text{Trop}(k_i\rho_i)$ as it is a map of lattices. We also have that Trop Φ factors uniquely through the canonical stack through the morphism of fans $\Sigma' \to \widetilde{\Sigma}$ by sending the generators to their tropicalizations, $\text{Trop}(e_i) \mapsto \text{Trop}(k_i e_{\rho_i})$. □

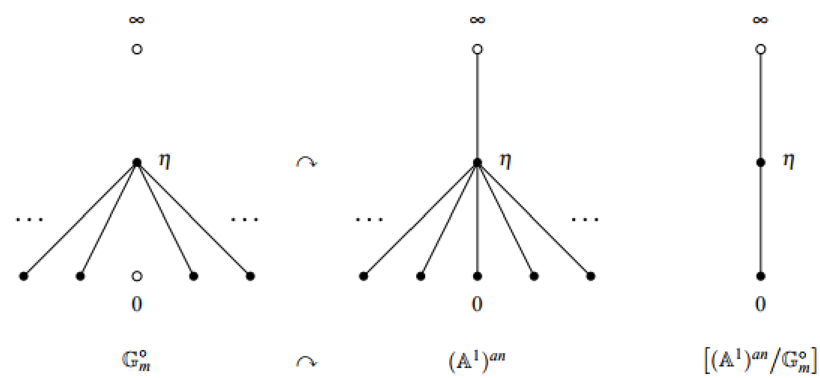
Remark. While there has been progress in extending tropicalizations to larger categories, locally integral $\mathbb{F}_1$ schemes in [GG16a], which are supposed to be a generalization of the notion of "toric", it remains unclear as to how to proceed with similar ideas for stacks at least in the sense of [GS15]. Namely the biggest missing piece is the object playing the role of a lattice on the $\mathbb{F}_1$ side; maybe something clever can be done with the monoids. We also note that we were extensively working with $\mathbb{T}$, in particular we were making use of the point-wise convergence topology. It seems possible to generalize the above (to arbitrary semirings) if instead one works with the strong Zariski topology defined in [GG16a].

4.3 All Things Analytic

4.3.1 Analytic Geometry

Consider the following example which can be found in [Uli16].

Example 4.27. Consider the affine line $\mathbb{A}^1$ over a trivially valued field k. The non-Archimedean unit circle $\mathbb{G}_m^\circ$ is the subset of elements $x \in (\mathbb{A}^1)^{an}$ with $|t|_x = 1$ where t is the coordinate on $\mathbb{A}^1$. The skeleton $\mathfrak{S}(\mathbb{A}^1)$ of $(\mathbb{A}^1)^{an}$ is the line connecting 0 to ∞. It is precisely the set of "$\mathbb{G}_m^\circ$-invariant" points in $(\mathbb{A}^1)^{an}$ and therefore naturally homeomorphic to the topological space underlying $[(\mathbb{A}^1)^{an}/\mathbb{G}_m^\circ]$. Below is a picture of this situation.



With this example in mind we give the background necessary for such an observation. This material can be found in full in sections 2 and 3 of [Uli16]. Given an analytic space X its associated functor of points is given by,

$$h_X\colon\ (\mathrm{An}_k)^{op} \to \mathrm{Set}$$

where $T \mapsto X(T) = \mathrm{Hom}(T, X)$. A presheaf on An_k is representable by an analytic space X if it is isomorphic to h_X. A morphism $X \to Y$ of presheaves on An_k is representable if for every morphism $T \to X$ from an analytic space T the base change $X \times_Y T$ is representable by an analytic space S.

Definition 4.28. *An étale analytic space is a sheaf*

$$X\colon\ (\mathrm{An}_k)^{op}_{\acute{e}t} \to \mathrm{Set},$$

such that there is an analytic space U with a representable morphism $U \to X$ that is surjective and étale.

Lemma 4.29. *Let $\mathcal{X}$ be a category fibered in groupoids over* An_k*. The following properties are equivalent.*

1. *The diagonal morphism $\Delta_{\mathcal{X}} \colon \mathcal{X} \to \mathcal{X} \times \mathcal{X}$ is representable by étale analytic spaces,*

2. *or every analytic space T and any two objects $x, y \in \mathcal{X}(T)$ the presheaf* $\mathrm{Isom}_{\mathcal{X}}(x, y)$ *is representable by an étale analytic space, and*

3. *every morphism $U \to \mathcal{X}$ from an analytic space is representable by an étale analytic space.*

Definition 4.30. *A stack $\mathcal{X}$ over $(\mathrm{An}_k)_{\acute{e}t}$ is said to be analytic if the following two axioms hold,*

1. *the diagonal morphism $\Delta_{\mathcal{X}} \colon \mathcal{X} \to \mathcal{X} \times \mathcal{X}$ is representable by étale analytic spaces, and*

2. *there is an analytic space U and a morphism $U \to \mathcal{X}$ that is G-smooth, surjective, and universally submersive.*

Definition 4.31. *Let U be an analytic space. An étale equivalence relation on U consists of a monomorphism $R \hookrightarrow U \times U$ such that*

1. *for all analytic spaces T the subset $R(T) \subseteq U(T) \times U(T)$ defines an equivalence relation, and*

2. *the compositions $R \hookrightarrow U \times U \rightrightarrows U$ are étale.*

Given an étale equivalence relation R on an analytic space U the association $T \mapsto U(T)/R(T)$ defines a presheaf on An_k and the sheafification U/R is referred to as the quotient of U by R.

Proposition 4.32. *Let R be an étale equivalence relation on an analytic space U. Then the quotient U/R is an étale analytic space.*

Lemma 4.33. *Let $R \hookrightarrow U \times_S U$ be an étale equivalence relation in the category of analytic spaces over S. If $U \to S$ is étale then the quotient sheaf U/R over $(\mathrm{An}_k/S)_{ét}$ is representable by an analytic space X, which is étale over S.*

Definition 4.34. *An analytic groupoid is a groupoid object in the category of étale analytic spaces, that is a septuple (U, R, s, t, c, i, e) consisting of two étale analytic spaces U, R as well as*

1. *a source morphism $s\colon R \to S$,*
2. *a target morphism $t\colon R \to S$,*
3. *a composition morphism $c\colon R \times_{s,U,t} R \to R$,*
4. *an inverse morphism $i\colon R \to R$,*
5. *a unit morphism $e\colon U \to R$, and*

such that for all analytic spaces T over k the septuple $(U(T), R(T), s, t, c, i, e)$ is a groupoid category.

An analytic group gives rise to a presheaf via,

$$(\mathrm{An}_k)^{op} \to \mathrm{Groupoids}$$

given by $T \mapsto (U(T) \rightrightarrows R(T))$.

Definition 4.35. *For an analytic groupoid $(R \rightrightarrows U)$ the quotient stack $[U/R]$ is defined to be the stackification of the prestack $[U/_{pre}R]$.*

Proposition 4.36. *Let $(R \rightrightarrows U)$ be a G-smooth, surjective, and universally submersive analytic groupoid. Then the following are true.*

1. *The quotient stack $\mathcal{X} = [U/R]$ is an analytic stack, and*

2. *if the groupoid* $(R \rightrightarrows U)$ *is étale, the quotient* $[X/R]$ *is an analytic Deligne–Mumford stack.*

In Berkovich's work there is an analytification functor,

$$(.)^{an} : \mathrm{Sch}_{loc.f.t./k} \to \mathrm{An}_k$$

which sends X to X^{an}.

Proposition 4.37 (Proposition 2.19, [Uli16])**.** *Let* $\mathcal{X}$ *be an algebraic stack locally of finite type over* k *and* $[U/R] \simeq \mathcal{X}$ *a groupoid presenation of* $\mathcal{X}$ *of by algebraic spaces locally finite type over* k*. Then there is a natural equivalence* $\mathcal{X}^{an} \simeq [U^{an}/R^{an}]$*.*

Here we will list the important properties about the map of underlying topological spaces, namely we look at the functor

$$|.|\colon \mathrm{An.Stacks}_k \to \mathrm{Top}.$$

Definition 4.38. *The set of points* $|\mathcal{X}|$ *of* $\mathcal{X}$ *is the set of equivalence classes of pairs* (K, p) *where* K *is a non-Archimedean field extension of* k *and* $p\colon \mathcal{M}(K) \to \mathcal{X}$ *is a morphism. Two pairs are equivalent if the following diagram is 2-commutative. We denote a common extension of* (K, p) *and* (L, q) *as* Ω*.*

$$\begin{array}{ccc} \mathcal{M}(\Omega) & \longrightarrow & \mathcal{M}(L) \\ \downarrow & & \downarrow{\scriptstyle q} \\ \mathcal{M}(K) & \xrightarrow{p} & \mathcal{X} \end{array}$$

Proposition 4.39. *Let* $\mathcal{X}$ *be an analytic stack.*

1. *For every universally submersive surjective morphism* $U' \to \mathcal{X}$ *from an analytic space* U' *onto* $\mathcal{X}$ *the induced surjective map* $|U'| \to |\mathcal{X}|$ *is a topological quotient map. If* $U' \to \mathcal{X}$ *is étale, the quotent map is open.*

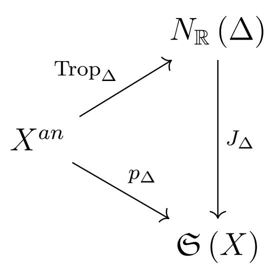
2. *Let $[U/R] \simeq \mathcal{X}$ be a groupoid presentation of an analytic stack $\mathcal{X}$. Then the image of $|R| \rightrightarrows |U| \times |U|$ defines an equivalence relation on $|U|$ and $|\mathcal{X}|$ is a topological quotient of $|U|$ by this equivalence relation.*

3. *For every morphism $f\colon \mathcal{X} \to \mathcal{Y}$ of analytic stacks the induced map $|f|\colon |\mathcal{X}| \to |\mathcal{Y}|$ is continuous.*

4.3.2 Ulrisch's Work

Throughout this section let $X = X(\Delta)$ be a toric variety with fan Δ. Let $T \simeq \mathbb{G}_m^n$ be a split algebraic torus with character lattice M and dual N. Finally let $T^\circ = \{x \in T^{an} \mid |\chi^m|_x = 1 \text{ for all } m \in M\}$.

Lemma 4.40 (Lemma 4.1, [Uli16])**.** *There is a strong deformation retraction $p_\Delta\colon X^{an} \to X^{an}$ as well as a homeomorphism $J_\Delta\colon N_{\mathbb{R}}(\Delta) \overset{homeo}{\to} \mathfrak{S}(X)$ making the following diagram commute, where $\mathfrak{S}(X)$ is the non-archimedean skeleton.*



Before stating the next lemma we need to two important morphisms. First we consider the action of the torus on our variety defined by $\mu\, T \times X \to X$. On a T-invariant open affince subset U_σ for a cone $\sigma in \Delta$ this morphism is induced by the homomorphism

$$\begin{aligned} \mu^\sharp\colon k[S_\sigma] &\to k[M] \otimes k[S_\sigma] \\ \chi^s &\mapsto \chi^s \otimes \chi^s. \end{aligned}$$

Moreover, we consider the projection morphism $\pi\colon T \times X \to X$, which is induced by the homomorphism

$$\begin{aligned} \pi^\sharp\colon k[S_\sigma] &\to k[M] \otimes k[S_\sigma] \\ \chi^s &\mapsto 1 \otimes \chi^s. \end{aligned}$$

Lemma 4.41 (Lemma 4.2, [Uli16])**.** *For a point $x \in U_\sigma^{an}$ consider the point $\eta \widehat{\otimes} x \in T^\circ \times U_\sigma^{an}$ given by the seminorm*

$$|f|_{\eta\widehat{\otimes}x} = \max_{m \in M} |a_m||f_m|_x$$

for an element $f = \Sigma_{m \in M}\left(a_m \chi^m \otimes f_m\right) \in k[M] \otimes_k k[S_\sigma]$ with unique regular functions $f_m \in k[S_\sigma]$. Then we have

$$\pi^{an}\left(\eta\widehat{\otimes}x\right) = x, \text{ and}$$

$$\mu^{an}\left(\eta\widehat{\otimes}x\right) = p_\sigma\left(x\right).$$

Theorem 4.42 (Theorem 1.1, [Uli16])**.** *There is a natural homeomorphism $\mu_\Delta\colon |[X^{an}/T^\circ]| \overset{homeo}{\to} N_{\mathbb{R}}\left(\Delta\right)$ that makes the following diagram commute.*

$$\begin{array}{ccc} & X^{an} & \\ \swarrow^{\mu} & & \searrow^{\mathrm{Trop}_\Delta} \\ |[X^{an}/T^\circ]| & \xrightarrow[\mu_\Delta]{\sim} & N_{\mathbb{R}}(\Delta) \end{array}$$

4.3.3 Extending Ulrisch's work

The goal of this section is to combine the works of [GS15] and [Uli16], namely to prove the following theorem.

Theorem 4.43 (Theorem D)**.** *Let $\beta\colon \Delta \to \Lambda$ be a map of lattices with finite cokernel. Then the following diagram commutes.*

$$\begin{array}{ccc} & X^{an} & \\ \swarrow^{\mu} & & \searrow^{\mathrm{Trop}_\Lambda} \\ |[X^{an}/G_\beta^\circ]| & \xrightarrow[\mu_\Lambda]{\sim} & N_{\mathbb{R}}(\Lambda) \end{array}$$

Proof. This proof largely follows the proof laid out in [Uli16]. The major difference is that we have the set up outlined in [GS15] so instead of working with the affinoid torus T° coming from Δ we can work with the affinoid torus $T^{\circ}_{\Lambda} = (T_L/G_{\beta})^{\circ}$.

Now by the work of [Uli16] we know that $|[X^{an}/T^{\circ}_{\Lambda}]|$ is the topological colimit of the maps

$$(\pi^{an}, \mu^{an} \colon T^{\circ}_{\Lambda} \times X^{an} \rightrightarrows X^{an}).$$

So now we only need to show that the deformation retraction $X^{an} \to \mathfrak{S}(X)$ makes $\mathfrak{S}(X)$ into a colimit as well. Since the retraction map is determined by torus invariant subsets U_{σ} we only need to check it on these. Let $x, x' \in U_{\sigma}^{an}$ such that $\pi^{an}(y) = x$ and $\mu^{an}(y) = x'$. Then we have that $p_{\sigma}(x) = p_{\sigma}(x')$ since

$$\begin{aligned} |\chi^s|_{x'} &= |\chi^s|_{\mu^{an}(y)} = |\chi^s \otimes \chi^s| y \\ &= |\chi^s \otimes 1|_y \cdot |1 \otimes \chi^s|_y = |1 \otimes \chi^s|_y \\ &= |\chi^s|_{\pi^{an}(y)} = |\chi^s|_x. \end{aligned}$$

Given $x \in U_{\sigma}^{an}$ there is a point $y = \eta \hat{\otimes} x \in T^{\circ}_{\Lambda} \otimes U_{\sigma}^{an}$ such that $\pi^{an}(y) = x$ and $\mu^{an}(y) = p_{\sigma}(x)$. Given two points $x, x' \in U_{\sigma}^{\beth}$ such that $p_{\sigma}(x) = p_{\sigma}(x')$, their images in $|U_{\sigma}^{an}/T^{\circ}_{\Lambda}|$ are equal. Since p_{σ} is continuous and proper we get that the skeleton $\mathfrak{S}(X)$ is a topological colimit as desired. □

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