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Half Covering, Half Coloring

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Doctor of Philosophy

Mathematics

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S.B., Massachusetts Institute of Technology, 2017

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An abstract of
A dissertation submitted to the Faculty of the
James T. Laney School of Graduate Studies of Emory University
in partial fulfillment of the requirements for the degree of
Doctor of Philosophy
in Mathematics
2022

Abstract

Half Covering, Half Coloring

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Alon and Füredi determined the minimum number of affine hyperplanes needed to cover all but one point of an n-dimensional rectangular grid. It is natural to extend this to higher covering multiplicities and ask for the minimum number of affine hyperplanes needed to cover every grid point at least k times each, except for one point that is not covered at all. In the special case of an n-cube, we use a Punctured Combinatorial Nullstellensatz of Ball and Serra to exactly determine the minimum for $k = 3$ and to formulate a lower bound for $k \geq 4$. We also treat the problem as an integer program to determine an asymptotic answer for fixed n as $k \to \infty$. Again using the Punctured Combinatorial Nullstellensatz, we answer the question for a general grid when $k = 2$.

Another generalization we address for the n-cube is the minimum number of affine subspaces of codimension d needed to cover all but one vertex at least once. We also consider the minimum number of hyperplanes needed to cover all points of a triangular grid.

In the second half of this dissertation, we consider arithmetic Ramsey theory problems in the spirit of van der Waerden's theorem. For a set $D \subset \mathbb{Z}_{>0}$, Landman and Robertson introduced the notion of a D-diffsequence, which is an increasing sequence $a_1 < \cdots < a_k$ such that all the consecutive differences $a_i - a_{i-1}$ are in D for $i = 2, \cdots, k$. We say that the set D is r-accessible if every r-coloring of the positive integers contains arbitrarily long monochromatic D-diffsequences. For an r-accessible set D, we define $\Delta(D, k; r)$ to be the minimum n such that every r-coloring of $[n] := \{1, 2, \cdots, n\}$ contains a monochromatic D-diffsequence of length k.

By considering the case where D consists of all powers of 2, we provide an example

of a 2-accessible set where $\Delta(D, k; 2)$ grows faster than any polynomial. The proof relies on a series of periodic colorings based on the Thue–Morse sequence. We also use Beatty sequences to classify which sets of the form $D = \{d_1, d_2, \cdots\}$ with $d_i \mid d_{i+1}$ for all i are 2-accessible.

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Acknowledgments

To my parents, Anne Renée and M. Bradford Clifton, the unstoppable team who made everything possible.

I am lucky to be the grandson of Merc and Rod Clifton who have cultivated a loving and supportive environment for our whole family. I am thankful for every moment I got to spend with my granddad Alfred Zarouni and my aunt Jane "Jane the Brain" Simmons.

I want to thank everyone who influenced my mathematical journey. Alyce Doehner introduced me to math competitions, which is the best gift I've ever received. Kevin Wang and Math Zoom Summer Program provided an opportunity for me to meet likeminded peers. Richard Rusczyk and Art of Problem Solving created the single greatest resource I could have asked for. Ashwath Rabindranath provided the encouragement and mentorship I needed to believe in myself.

Michael Damron taught me in my first-ever college math classes while I was in high school, and it was so meaningful that he came to attend my defense in-person ten years later. When I was at MIT and not sure I wanted to pursue math anymore, Péter Csikvári introduced me to Combinatorial Nullstellensatz and other fascinating combinatorics which convinced me that I wanted to stay on this path. Abdollah Khodkar, Bruce Landman, and the University of West Georgia REU introduced me to math research. I am particularly indebted to Bruce Landman for introducing me to arithmetic Ramsey theory, which figures prominently in this dissertation.

I would like to thank Steve Butler, Florian Pfender, and the organizers of the 2021 Graduate Research Workshop in Combinatorics for running an excellent program which has provided me with many research opportunities. In particular, I would like to thank my collaborators Abdul Basit and Paul Horn for their help in covering triangular grids and in covering hypercubes with affine subspaces of codimension more than 1. I would also like to thank Xiaonan Liu, Reem Mahmoud, and Abhinav Shantanam for many interesting discussions.

I want to thank my advisor Hao Huang for his endless supply of interesting problems, but mostly for his patience and steadfast support. Whether I was applying for jobs, refereeing papers, or writing my dissertation, Hao was an advisor to me in the truest sense of the word. I would also like to thank him for his collaboration on covering hypercubes which kickstarted my graduate research program.

I am extremely grateful to my committee members. Dwight Duffus served as my teaching mentor and provided valuable feedback on this dissertation. Vojtěch Rödl ran an excellent student seminar. Liana Yepremyan did a great job cultivating a sense of community within the combinatorics group.

I also would like to thank the other professors who influenced my mathematical development in graduate school, especially my teaching mentors David Borthwick and Mike Carr, as well as R. Parimala. While teaching Math 116 during a pandemic, I was very fortunate to work with Longmei Shu. I am also appreciative of the efforts of everyone who taught for Emory Math Circle alongside me, especially Brad Elliott and Tomer Reiter who prepared me for the Director role.

Over the past several years, I have been lucky to work with many fantastic students in the undergraduate classes I taught, at Emory Math Circle, and in contest coaching. Particular thanks goes to my Directed Reading Program mentee Michael Liu.

I would like to thank my friends at Emory who made the graduate school experience more fun, including Ayush Basu, Lea Beneish, Brad Elliott, Xiangxi Gao, Harris Greenwood, Jayanth Guhan, Kai Hess, Kenny Jones, Chris Keyes, Jackson Morrow, Tomer Reiter, Marcelo Sales, and Juvaria Tariq.

I also appreciate the longtime support of my non-Emory friends, especially Neil Chitrao, Krishna Murugesan, Theron Nipson, Shruti Sharma, Rishabh Singh, and Eli

Wang.

Special consideration goes to my roommate Dylanger Pittman who introduced me to lots of new things, while putting up with five years of me talking about chess players and singing “I Want It That Way”.

I also want to thank my great friend Ahaan Rungta for our numerous mathematical, musical, athletic, and oneiric adventures, from MIT to NYC to YouTube.

Tack så mycket to Maryam Khaqan for teaching me just enough Swedish to write this, but mostly for years of math, puzzles, encouragement, and friendship. Look at us!

I have immense gratitude for my BFF Apoorva Shah. It was so special to go through this math journey together from age 13. We no longer have to deal with Lemons, so it’s time to enjoy the lemonade we’ve worked to create.

I am incredibly thankful to be the brother of Kimberly Clifton who has spent more time listening to math problems, chess memes, and “Auf uns” than she ever wanted to, but is still the most loyal big sibling I could have had looking out for me every step of the way.

If we’re truly the average of the five people we spend the most time with, it’s pretty safe to say that I got the better deal.

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Chapter 1

Introduction

1.1 Covering and Coloring

This dissertation considers various problems in extremal combinatorics. We primarily consider Covering questions and Coloring questions. In the Covering questions, our goal is to find the minimum number of affine hyperplanes that cover a given set of points (possibly with multiplicity) while avoiding a forbidden point. In the Coloring questions, our goal is to find the smallest positive integer n such that every coloring of $\{1, \cdots, n\}$ with r colors contains a monochromatic copy of a certain arithmetic structure, or to show that no such coloring exists. The Covering questions here exist within a wider context of covering problems, particularly graph covering, while the Coloring questions exist within the wider context of Ramsey theory.

The common thread between these topics is that they are both optimization problems where the goal is to find the minimum size of a set satisfying certain conditions. Furthermore, making sure each point is covered enough times or making sure a coloring of $\{1, \cdots, n\}$ avoids a monochromatic copy of some pattern are both linear constraints. Thus, both of these problem types, at their core, are integer linear programs. This means they can in part be addressed by a common set of tools, such as using software like Gurobi to compute data for small cases.

While viewing these questions as integer programs allows us to make some progress, we will also utilize other techniques. For lower bounds in Covering problems, we will lean heavily on algebraic methods such as the Punctured Combinatorial Nullstellensatz [BS09]. For lower bounds in Coloring problems, we will rely on a creative choice of ad hoc colorings.

1.2 Background for Covering

1.2.1 Covering Problems

Covering problems are a broad class of combinatorial problems where the goal is often to find the smallest collection of sets such that every object in some family is contained in at least one set of the collection. Many covering problems take place on graphs. A classic example is the *vertex cover problem* where the goal is to find the smallest set of vertices of a graph G such that each edge of G is incident to (covered by) at least one vertex in the collection.

An equivalent formulation of the vertex cover problem is to find the smallest collection of star subgraphs such that each edge of G is contained in at least one such subgraph. More generally, one could ask for the smallest number of complete bipartite graphs such that each edge of G is contained in at least one of these graphs. There are further ways to modify the question such as requiring that no edge is contained in more than one graph of the collection. The celebrated Graham-Pollak theorem [GP72] states that the smallest collection of complete bipartite graphs which contains every edge of the complete graph K_n exactly once, is of size $n-1$. One way to obtain this optimum is to take a collection of stars centered at the first $n-1$ vertices. Another modification we have considered [BCC$^+$22] is to determine the smallest number of complete bipartite graphs needed to cover each edge of G an odd number of times and each non-edge of G an even number of times. In this problem, we are likely to incidentally cover objects outside the family we aim to cover.

The covering problems we consider in Chapters 2 and 3 will generally not take place on graphs. Instead, we endeavour to cover sets of points using collections of hyperplanes (or other affine subspaces). In particular, we will have some points we wish to cover, possibly more than once, and potentially other points we wish to not cover at all.

These constraints on how many times each relevant point should be covered allow us to formulate an integer program where the variables are how many times each hyperplane is used. This allows us compute the optimum for many examples using software such as Gurobi (see Appendix C). Another advantage of formulating a covering problem as an integer program is that it can be easier to solve its linear relaxation, which immediately gives a lower bound. Our other predominant means of computing lower bounds, particularly when the points we wish to cover lie on some rectangular grid in $\mathbb{R}^n$, is via algebraic methods, in particular Combinatorial Nullstellensatz and its generalizations.

1.2.2 Combinatorial Nullstellensatz

One of the most fundamental facts about polynomials is that a nonzero single-variable polynomial of degree n has at most n zeros. Thus, if a polynomial of degree $\leq n$ vanishes at $n+1$ different values, it is necessarily the zero polynomial. A multivariable polynomial such as $f(x_1, x_2) = x_1x_2$ can vanish at infinitely many points. However, we can still use its degree to restrict where it vanishes if we focus our attention on just a rectangular grid.

The Combinatorial Nullstellensatz, introduced by Alon [Alo99] states that a multivariable polynomial cannot vanish on the entirety of some grid. In particular, for any nonzero term of maximal degree, the polynomial cannot vanish on a grid whose

i^{th} dimension is larger than the x_i degree of that term for all $i = 1, \cdots, n$.

Theorem 1.2.1. *[Alon, Combinatorial Nullstellensatz] Let $\mathbb{F}$ be an arbitrary field, and let $f = f(x_1, \cdots, x_n)$ be a polynomial in $\mathbb{F}[x_1, \cdots, x_n]$. Suppose the degree* $\deg f$ *of f is $\sum_{i=1}^{n} t_i$ where each t_i is a nonnegative integer, and suppose the coefficient of $\prod_{i=1}^{n} x_i^{t_i}$ in f is nonzero. Then, if $S_1, \cdots, S_n$ are subsets of $\mathbb{F}$ with $|S_i| > t_i$, there are $s_1 \in S_1, s_2 \in S_2, \cdots, s_n \in S_n$ so that*

$$f(s_1, \cdots, s_n) \neq 0.$$

Example 1. $f(x_1, x_2) = -2x_1^3 + 3x_1^2x_2 + 4x_1x_2^2 + x_1x_2 - 4x_4 + 5$ cannot vanish at all points of $\{0, 5, 6\} \times \{3, 5\}$ since it has degree 3 and has a nonzero $x_1^2x_2$ coefficient. Likewise, f cannot vanish at all points of $\{-6, 3\} \times \{0, 2, 100\}$ since it also has a nonzero $x_1x_2^2$ coefficient.

Combinatorial Nullstellensatz has a stunning array of uses, often providing new proofs for classical results. Among other applications, it has been used to find upper bounds on the list coloring number of graphs, show the existence of subgraphs with certain degree conditions, and to provide another proof of the well-known Cauchy–Davenport Theorem in Additive Number Theory.

Here, we will focus on one particular application of Combinatorial Nullstellensatz. Komjáth [Kom94] asked for the minimum number, $m(n)$, of affine hyperplanes needed to cover all but one vertex of $Q^n := \{0, 1\}^n$ while leaving the last vertex uncovered. (The question in this form is actually due to Imre Bárány (see [AF93]) as Komjáth's original question only concerned showing that $m(n) \to \infty$.) It is important to specify that exactly one vertex is left uncovered since otherwise, two hyperplanes, such as $x_1 = 0$ and $x_1 = 1$, will be sufficient for any $n \geq 1$. For Komjáth's question, it is easy to see that n hyperplanes suffice. One option is to use $x_i = 1$ for $i = 1, \cdots, n$, while another is to use $x_1 + \cdots + x_n = i$ for $i = 1, \cdots, n$.

Alon and Füredi showed that indeed, n is also the minimum. While their original proof [AF93] does not make use of Combinatorial Nullstellensatz, the proof we include here does.

Theorem 1.2.2. *[AF93] If $H_1, \cdots, H_m$ are a collection of hyperplanes such that none contains $\vec{0}$ and every point of $\{0,1\}^n \setminus \{\vec{0}\}$ is contained in at least one hyperplane of the collection, then $m \geq n$.*

Proof. For the sake of contradiction, assume that there exists a collection of $n-1$ hyperplanes meeting the conditions. Without loss of generality, we may write the equation of each hyperplane H_i as $a_{i,1}x_1 + \cdots + a_{i,n}x_n = 1$. For $i = 1, \cdots, n-1$, we can then define the polynomials $P_i = a_{i,1}x_1 + \cdots + a_{i,n}x_n - 1$ and let $P := \prod_{i=1}^{n-1} P_i$. The polynomial P has degree $n-1$ and vanishes on $\{0,1\}^n \setminus \{\vec{0}\}$. The premise of Combinatorial Nullstellensatz is that a nonzero polynomial cannot vanish on the entirety of a grid that is too large. A natural choice for our grid is to take $S_i = \{0,1\}$ for $i = 1, \cdots, n$. However, the polynomial P does not vanish on the entirety of this grid so we are not yet primed for a contradiction.

Instead, we will tweak the polynomial P so that it also vanishes at $\vec{0}$. Note that $P(\vec{0}) = (-1)^{n-1}$. Therefore, if we let

$$Q(x_1, \cdots, x_n) := P(x_1, \cdots, x_n) + \prod_{i=1}^{n}(x_i - 1),$$

we have that $Q(\vec{0}) = (-1)^{n-1} + (-1)^n = 0$. Furthermore, Q still vanishes on all of $\{0,1\}^n \setminus \{\vec{0}\}$.

Q is a degree n polynomial which vanishes on $\{0,1\}^n$. The only degree n term used in the sum defining Q is $\prod_{i=1}^{n} x_i$ so Q has a nonzero $\prod_{i=1}^{n} x_i$ coefficient. In the setting of Combinatorial Nullstellensatz, this is $t_1 = \cdots = t_n = 1$, so Q cannot vanish on the entirety of any grid $S_1 \times \cdots S_n$ with $|S_1|, \cdots, |S_n| \geq 2$. However, Q vanishes on $\{0,1\}^n$, giving a contradiction. Thus, our original collection of $n-1$ hyperplanes with the correct covering properties cannot exist. □

In Chapter 2, we will consider generalizations of Komjáth's question to higher multiplicity. In particular, we will ask for the smallest number of affine hyperplanes $f(n,k)$ such that every point of $\{0,1\}^n \setminus \{\vec{0}\}$ is contained in at least k of the hyperplanes while $\vec{0}$ is contained in none of them. In Chapter 3, we will address this higher multiplicity covering question for general rectangular grids $S_1 \times \cdots \times S_n$. For these multiplicity questions, it will help to consider the following modified version of Combinatorial Nullstellensatz [BS09].

Theorem 1.2.3. *[Ball–Serra, Punctured Combinatorial Nullstellensatz]*
For $i = 1, \cdots, n$, let $D_i \subset S_i \subset \mathbb{F}$ and $g_i = \prod_{s \in S_i}(x_i - s)$ and $l_i = \prod_{d \in D_i}(x_i - d)$. If f has a zero of multiplicity at least t at all the common zeros of $g_1, \cdots, g_n$, except at at least one point of $D_1 \times \cdots \times D_n$ where it has a zero of multiplicity less than t: Then, there are polynomials h_τ satisfying $\deg h_\tau \leq \deg f - \sum_{i \in \tau} \deg g_i$, and a non-zero polynomial u satisfying $\deg u \leq \deg f - \sum_{i=1}^n (\deg g_i - \deg l_i)$ such that

$$f = \sum_{\tau \in T(n,t)} g_{\tau(1)} \cdots g_{\tau(t)} h_\tau + u \prod_{i=1}^n \frac{g_i}{l_i}.$$

Here, $T(n,t)$ indicates the set of all non-decreasing sequences of length t on $[n]$.

The crux of this theorem is that if a multivariable polynomial vanishes to multiplicity at least t at every point of some rectangular grid, except for on some rectangular subgrid, we can write it as a combination of auxiliary polynomials satisfying certain degree conditions. For our covering questions, a collection of m hyperplanes which covers all but one grid point at least t times without covering the last point, can be used to define a degree m polynomial, as in the beginning of the proof of Theorem 1.2.2. This polynomial then vanishes to multiplicity at least t on the entirety of the original grid, except for one point, which can be viewed as a $1 \times \cdots \times 1$ rectangular subgrid. Thus, the degree m polynomial corresponding to a collection of hyperplanes with the correct covering properties will satisfy the criteria needed to apply Theorem 1.2.3.

1.3 Background for Coloring

1.3.1 Ramsey Theory

For any k, a sufficiently large group of people will have either k people who all know each other or k people who all don't know each other. Alternatively, if we color the edges of the complete graph K_n either red or blue, for sufficiently large n, we will end up with either a red copy of K_k or a blue copy of K_k as a subgraph. A common slogan used in the field of related questions, known as Ramsey theory, is that complete randomness is impossible.

Another way to view a Ramsey theory problem is to ask the following question: If the whole contains some special structure and the whole is then divided into parts, is some part guaranteed to retain that structure?

Arithmetic Ramsey theory concerns whether coloring positive integers (or tuples of positive integers) with r colors will guarantee the presence of a monochromatic copy of some arithmetic structure. An early example is Schur's theorem [Sch17] which states that for any finite number of colors r, an r-coloring of $\mathbb{Z}_{>0}$ contains x, y, z of the same color such that $x + y = z$.

Another classical result is van der Waerden's theorem [Wae27]. It states that any r-coloring of $\mathbb{Z}_{>0}$ contains arbitrarily long monochromatic arithmetic progressions. Equivalently, for every k, there is some n such that every r-coloring of $[n] := \{1, \cdots, n\}$ contains a monochromatic arithmetic progression of length k.

For a given r and k, the smallest such n is the van der Waerden number $W(r, k)$. Outside of a few trivial infinite families, van der Waerden numbers are only known exactly in a few small cases. The best known general upper bound [Gow01] is

$$2^{2^{r^{2^{2^{k+9}}}}}.$$

1.3.2 Diffsequences

Besides arithmetic progressions, one can ask what other monochromatic structures are guaranteed to show up in an arbitrary r-coloring of the positive integers. A tightening of van der Waerden's result is to look at only arithmetic progressions with certain gaps. A set $D \subset \mathbb{Z}_{>0}$ is called *r-large* if every r-coloring of the positive integers contains arbitrarily long arithmetic progressions whose common difference lies in D. The content of van der Waerden's Theorem is that $\mathbb{Z}_{>0}$ is r-large for all r.

In [BGL99], Brown, Graham, and Landman determined some conditions that D must satisfy to be r-large for all r. In particular, D must contain an infinite number of multiples of every positive integer and $D = \{d_1, d_2, \cdots\}$ cannot have $\liminf_{n\to\infty} \frac{d_{n+1}}{d_n} > 1$.

A relaxation of considering arithmetic progressions with a fixed difference is to consider sequences whose consecutive differences lie in a given set but are not necessarily the same as each other. This is the notion of a *diffsequence*, introduced by Landman and Robertson in [LR03].

Definition 1. For a set D of positive integers, a *D-diffsequence* of length k is a sequence of positive integers $a_1 < a_2 \cdots < a_k$ such that

$$a_i - a_{i-1} \in D$$

for $i = 2, 3, \cdots, k$.

We can then ask for which sets D and which numbers of colors r a van der Waerden-like theorem exists.

Definition 2. A set D is called *r-accessible* if every r-coloring of the positive integers contains arbitrarily long monochromatic D-diffsequences.

Any r-accessible set D is automatically t-accessible for $t < r$. The notion of r-accessibility is analogous to van der Waerden's theorem with r colors. Note that r-accessibility refers to the presence of arbitrarily long finite monochromatic D-diffsequences and we do not consider questions about infinite D-diffsequences.

Example 2. The set, D, of even numbers is r-accessible for all r.

Proof. If the positive integers are partitioned into finitely many colors, at least one color will contain infinitely many integers. Thus, it contains infinitely many integers of the same parity, from which we can take arbitrarily large subsets which are D-diffsequences. $\square$

Example 3. The set, D, of odd numbers is not r-accessible for $r \geq 2$.

Proof. In general, to show that a set D is not r-accessible, we just need to demonstrate one r-coloring of the positive integers which avoids arbitrarily long monochromatic D-diffsequences. For $r = 2$, we can color all the even numbers with color 0 and all the odd numbers with color 1. Thus, a monochromatic sequence only contains numbers of the same parity and we do not even have a monochromatic D-diffsequence of length 2. Since D is not 2-accessible, it is also not r-accessible for any larger number of colors. $\square$

The set of Fibonacci numbers is known to be 2-accessible [LR03] but not 6-accessible [AGJ$^+$08]. The set of primes is not 3-accessible [LR03] but the question of whether it is 2-accessible remains unresolved. Results on accessibility for fixed translates of the set of primes are given in [LR03] and [LV10]. The set of powers of n is only 2-accessible when $n = 2$ since otherwise there is a periodic coloring modulo $n - 1$ which avoids arbitrarily long monochromatic D-diffsequences.

In Chapter 4, we primarily address two types of questions. First, for a special 2-accessible choice of D, we study the smallest $n := n(k)$ such that any 2-coloring of $\{1, \cdots, n\}$ contains a monochromatic D-diffsequence of length k. This is analogous to the van der Waerden number $W(2, k)$. Next we consider sets which are generalizations of the set of powers of n and determine precisely when they are 2-accessible. We conclude with a brief exploration of 2-accessibility for randomly generated sets D.

Remark 1. The question of whether every 2-coloring of $\{1, \cdots, n\}$ contains a monochromatic D-diffsequence of length k can be conveniently reframed as an integer program. We have n binary variables representing whether each number is assigned color 0 or 1. For each possible D-diffsequence of length k, we have a constraint that the sum of the variables corresponding to elements of that diffsequence is at least 1 and at most $k-1$. We are guaranteed a monochromatic D-diffsequence of length k precisely when this integer program is infeasible.

1.4 Dissertation Synopsis

We generalize Komjáth's question to higher covering multiplicity, and ask for the smallest size, $f(n, k)$, of a collection of hyperplanes which covers each point of $\{0, 1\}^n \setminus \{\vec{0}\}$ at least k times without covering $\vec{0}$ at all. In Section 2.2, we use the Punctured Combinatorial Nullstellensatz of Ball and Serra [BS09] to resolve this question for $k = 3$ and to improve further the lower bound for $k \geq 4$. In Section 2.3, we examine the linear relaxation of this integer program and utilize it to determine the asymptotic behavior for $f(n, k)$ as $k \to \infty$ for fixed n. In the process, we prove a generalization of the Lubell-Yamamoto-Meshalkin inequality [Lub66, Yam54, Mes63].

In Section 2.4, we consider the smallest number of affine hyperplanes of codimension d needed to cover every point of $\{0, 1\}^n \setminus \{\vec{0}\}$ without covering $\vec{0}$. This question has been previously addressed over finite fields [Jam77, BBDM21], but we consider it over $\mathbb{R}$. We then consider, in Section 2.5, the question of finding the smallest collection of affine hyperplanes over $\mathbb{F}_2$ that do not contain the origin but do contain every point of $\{0,1\}^n$ with a fixed number of 1's as coordinates. We note some connections between this problem, traditional graph covering, and Sidon sets.

We next study higher multiplicity generalizations of Alon and Füredi's result that $\sum_{i=1}^{n}(|S_i|-1)$ affine hyperplanes are the minimum needed to cover all but one point of a general rectangular grid $S_1 \times \cdots \times S_n$. We demonstrate a construction in Section 3.1 that resolves this question when each covered point is covered at least $k = 2$ times and contrast this with the difficulties encountered when $k = 3$ (Subsection 3.1.1). In the process, we generalize a question solved for $\{0,1\}^n$ by Sauermann and Wigderson [SW20] and ask for the lowest possible degree of a polynomial which vanishes to multiplicity at least k on all but one point of $S_1 \times \cdots \times S_n$ while not vanishing at the last point.

In Section 3.2, we consider covering questions for triangular lattices. Most of our results concern determining the smallest number of lines needed to cover *every* point of $T_1(d,2) := \{(x_1,x_2) \in \mathbb{Z}^2_{\geq 0} \mid x_1 + x_2 \leq d-1\}$. We solve the fractional version of this problem when $d \equiv 1 \pmod 3$ and conjecture a corresponding result for all d. We also solve the integer programs of determining the smallest number of lines needed to cover every point of $T_1(d,2)$ at least k times, when $k \in \{1,2,4\}$. Using data from Gurobi (see Appendix A), we formulate a conjecture for general k and highlight a surprising connection to our answer for the fractional problem.

We then shift to the Coloring half. The questions we consider will be, in some sense, two steps away from van der Waerden's theorem [Wae27]. We focus on the notion of a D-diffsequence, introduced by Landman and Robertson [LR03], which is a sequence, $a_1, \cdots, a_k$, of positive integers where every consecutive difference, $a_{i+1} - a_i$, lies in some fixed set $D \subset \mathbb{Z}_{>0}$. In Section 4.1, we consider a question that can be thought of as finding analogues of van der Waerden numbers: finding the smallest

n such that any r-coloring of $\{1, \cdots, n\}$ contains a monochromatic D-diffsequence of length k. We demonstrate the first known example of a set D such that the smallest $n := n(k)$ where any 2-coloring of $\{1, \cdots, n\}$ contains a monochromatic D-diffsequence of length k grows faster than polynomial in k. In Section 4.2, we generalize the notion of a periodic coloring to show, for a wide class of D's, that there exist 2-colorings of $\mathbb{Z}_{>0}$ which avoid monochromatic D-diffsequences. Lastly, in Subsection 4.2.3, we consider what happens when the elements of D are chosen at random.

Sections 2.2 and 2.3 are joint work with Hao Huang and originally appeared in [CH20]. Sections 2.4 and 3.2 are part of an ongoing project with Abdul Basit and Paul Horn. Most of Chapter 4 originally appeared in [Cli21]. Patrick Schnider suggested thinking about D-diffsequences for a randomly chosen D.

Chapter 2

Covering for Hypercubes

2.1 Preliminary Results

In this chapter, we will primarily consider higher multiplicity generalizations of the Alon-Füredi theorem 1.2.2. Let $f(n,k)$ be the minimum number of affine hyperplanes needed to cover every vertex of $Q^n := \{0,1\}^n$ at least k times except for $\vec{0} = (0,\cdots,0)$ which is not covered at all. We call such a cover an *almost k-cover* of the n-cube. The Alon-Füredi theorem gives $f(n,1) = n$.

No hyperplane can cover a single point with multiplicity more than one. Thus, removing a hyperplane from an almost k-cover still leaves an almost $(k-1)$-cover. This gives a recursive lower bound of $f(n,k) \geq f(n,k-1)+1$ for $k \geq 2$. By induction on k, this yields $f(n,k) \geq n+k-1$.

For an upper bound, we can restrict our attention to two types of hyperplanes: those of the form $x_i = 1$ for some $i = 1,\cdots,n$ and those of the form $\sum_{i=1}^n x_i = t$ for some $t = 1,\cdots,n$. One construction is to use $x_i = 1$ for $i = 1,\cdots,n$, together with $k-t$ copies of $\sum_{i=1}^n x_i = t$, for $t = 1,\cdots,k-1$. In this construction, every binary vector with $t \geq 1$ coordinates equal to 1 is covered t times by $\{x_i = 1\}$, and $k-t$ times by $x_1 + \cdots + x_n = t$. The total number of hyperplanes is $n + \sum_{t=1}^{k-1}(k-t) = n + \binom{k}{2}$.

This establishes $f(n,k) \leq n + \binom{k}{2}$. The upper and lower bound match for $k = 2$, giving $f(n,2) = n+1$ for all n. The first case we consider is $k = 3$, where these basic observations have given us $n + 2 \leq f(n,3) \leq n + 3$.

In the next section, we will consider the problem of finding the smallest almost k-cover of the n-cube for fixed k. In the following section, we will consider what happens when the dimension n is fixed and $k \to \infty$.

2.2 Fixed Covering Multiplicity

We determine $f(n,k)$ for $k = 3$ and improve the lower bound for $k \geq 4$.

Theorem 2.2.1. *For $n \geq 2$,*

$$f(n,3) = n + 3.$$

Theorem 2.2.2. *For $n \geq 3$, $f(n,4) \geq n + 5$.*

Because of our recursive lower bound, $f(n,k) \geq f(n,k-1)+1$, we get that $f(n,k) \geq n+k+1$ for $k \geq 4$ as an immediate consequence of Theorem 2.2.2.

The primary tool we will use in the proofs of these results is Theorem 1.2.3. If we have an almost k-cover consisting of the hyperplanes $H_1, H_2, \cdots, H_{f(n,k)}$, then each H_i does not contain $\vec{0}$ and so can be written as $a_{i,1}x_1+a_{i,2}x_2+\cdots+a_{i,n}x_n = 1$ without loss of generality. We can then define the polynomials $P_i := a_{i,1}x_1+a_{i,2}x_2+\cdots+a_{i,n}x_n - 1$ and $P = \prod_{i=1}^{f(n,k)} P_i$. The polynomial P vanishes to multiplicity at least k on all vertices of Q^n except for $\vec{0}$ where it does not vanish at all. Theorem 1.2.3 provides conditions that must be satisfied by any polynomial vanishing to higher multiplicity on all of a grid $S_1 \times \cdots \times S_n$ except for on some subgrid $D_1 \times \cdots \times D_n$. Taking each S_i to be $\{0,1\}$ and each D_i to be $\{0\}$, these are exactly the vanishing conditions on P. If we succeed in using Theorem 1.2.3 to give a lower bound on the degree of P, then this immediately gives a lower bound on the size of an almost k-cover. In fact, once we construct the polynomial P, we never make use of the fact that it splits completely into linear factors and actually corresponds to a a set of hyperplanes.

Proof of Theorem 2.2.1. To show that $f(n,3) = n+3$, it suffices to establish the lower bound. We will prove this by contradiction. Suppose $H_1, \cdots, H_{n+2}$ are $n+2$ affine hyperplanes that form an almost 3-cover of $\{0,1\}^n$. Without loss of generality, assume the equation defining H_i is $\langle \vec{b}_i, \vec{x} \rangle = 1$, for some nonzero vector $\vec{b}_i \in \mathbb{R}^n$. Define $P_i = \langle \vec{b}_i, \vec{x} \rangle - 1$, and let

$$P = P_1 P_2 \cdots P_{n+2}.$$

Since $H_1, \cdots, H_{n+2}$ form an almost 3-cover of Q^n, every binary vector $\vec{x} \in Q^n \setminus \{\vec{0}\}$ is a zero of multiplicity at least 3 of the polynomial P. We apply Theorem 1.2.3 with

$$S_i = \{0,1\}, \quad D_i = \{0\}, \quad g_i = x_i(x_i - 1), \quad \ell_i = x_i,$$

and write P in the following form:

$$P = \sum_{1 \le i \le j \le k \le n} x_i(x_i-1)x_j(x_j-1)x_k(x_k-1)h_{ijk} + u \prod_{i=1}^{n}(x_i - 1),$$

with $\deg(u) \le \deg(P) - n = 2$.

Note that $P = 0$ on $Q^n \setminus \{\vec{0}\}$. Moreover,

$$\frac{\partial P}{\partial x_i} = \sum_{j=1}^{n+2} P_1 \cdots P_{j-1} \cdot \frac{\partial P_j}{\partial x_i} \cdot P_{j+1} \cdots P_{n+2}.$$

Since each P_j is a polynomial of degree 1, each first order partial derivative $\partial P / \partial x_i$ is just a linear combination of $P_1 \cdots \hat{P}_j \cdots P_{n+2}$ terms. Note that removing a single hyperplane still gives an almost 2-cover, so $\partial P / \partial x_i$ vanishes on $Q^n \setminus \{\vec{0}\}$. Similarly, each second order partial derivative of P is a linear combination of terms of the form

$P_1 \cdots \hat{P}_i \cdots \hat{P}_j \cdots P_{n+2}$. Removing two hyperplanes still gives an almost 1-cover, so we can show that all the second order partial derivatives of P vanish on $Q^n \setminus \{\vec{0}\}$ as well. More generally, if P is the product of equations of the affine hyperplanes from an almost k-cover, then all the j^{th} order partial derivatives of P vanish on $Q^n \setminus \{\vec{0}\}$, for $j = 0, \cdots, k-1$. It is not hard to observe that $x_i(x_i-1)x_j(x_j-1)x_k(x_k-1)h_{ijk} = g_i g_j g_k h_{ijk}$ also has its t^{th} order partial derivatives vanishing on the entire cube Q^n, for $t \in \{0, 1, 2\}$, since every term in the product rule expansion will contain at least one of $x_i(x_i-1), x_j(x_j-1)$, or $x_k(x_k-1)$, which each vanish on Q^n. Therefore the following polynomial

$$h = u \prod_{i=1}^{n} (x_i - 1)$$

has j^{th} order partial derivatives vanishing on $Q^n \setminus \{\vec{0}\}$, for $j = 0, 1, 2$.

We denote by e_i the n-dimensional unit vector with the i^{th} coordinate being 1. By calculations,

$$\frac{\partial h}{\partial x_i} = \frac{\partial u}{\partial x_i} \prod_{j=1}^{n} (x_j - 1) + u \prod_{j \neq i} (x_j - 1).$$

Therefore

$$0 = \frac{\partial h}{\partial x_i}(e_i) = (-1)^{n-1} u(e_i),$$

and this implies

$$u(e_i) = 0 \quad \text{for } i = 1, \cdots, n.$$

Furthermore,

$$\frac{\partial^2 h}{\partial x_i^2} = \frac{\partial^2 u}{\partial x_i^2} \prod_{j=1}^{n} (x_j - 1) + 2 \frac{\partial u}{\partial x_i} \prod_{j \neq i} (x_j - 1).$$

Therefore

$$0 = \frac{\partial^2 h}{\partial x_i^2}(e_i) = (-1)^{n-1} \cdot 2 \frac{\partial u}{\partial x_i}(e_i),$$

and this implies

$$\frac{\partial u}{\partial x_i}(e_i) = 0 \quad \text{for } i = 1, \cdots, n.$$

Finally,

$$\frac{\partial^2 h}{\partial x_i x_j} = \frac{\partial^2 u}{\partial x_i x_j} \prod_{k=1}^{n} (x_k - 1) + \frac{\partial u}{\partial x_i} \prod_{k \neq j} (x_k - 1) + \frac{\partial u}{\partial x_j} \prod_{k \neq i} (x_k - 1) + u \prod_{k \neq i,j} (x_k - 1).$$

By evaluating at e_i and $e_i + e_j$, we have

$$\frac{\partial u}{\partial x_j}(e_i) = u(e_i) = 0, \quad \text{and} \quad u(e_i + e_j) = 0.$$

Summarizing the above results, u is a nonzero polynomial of degree at most 2, satisfying: (i) $u = 0$ at e_i and $e_i + e_j$; (ii) $\partial u / \partial x_i = 0$ at e_j (possible to have $i = j$). We define a new single-variable polynomial w,

$$w(x) = u(x \cdot e_i + e_j).$$

Then $\deg(w) \leq 2$, and $w(0) = w(1) = w'(0) = 0$. Thus, w has a zero of multiplicity at least two at 0, as well as a zero at 1, despite only being degree at most 2, so $w \equiv 0$. We write out u as a generic degree at most 2 polynomial in n variables:

$$u = \sum_i a_{ii} x_i^2 + \sum_{i<j} a_{ij} x_i x_j + \sum_i b_i x_i + c.$$

Evaluating u at $x \cdot e_i + e_j$ yields $a_{ii} x^2 + (a_{ij} + b_i) x + (a_{jj} + b_j + c)$. Each coefficient of this must be 0, so this gives for all $i \neq j$,

$$a_{ii} = 0, \quad a_{ij} + b_i = 0, \quad a_{ii} + b_i + c = 0.$$

On other other hand $\partial u/\partial x_i = 0$ at e_i gives

$$2a_{ii} + b_i = 0.$$

It is not hard to derive from these equalities that

$$a_{ii} = a_{ij} = b_i = c = 0$$

for all i, j. Therefore, $u \equiv 0$. Since $\sum_{1\le i\le j\le k\le n} x_i(x_i-1)x_j(x_j-1)x_k(x_k-1)h_{ijk}$ vanishes at $\vec{0}$, we have $f(\vec{0}) = 0$, which contradicts the assumption that $\vec{0}$ is not covered by any of the $n+2$ affine hyperplanes. Therefore, $f(n,3) = n+3$ for $n \geq 2$. $\square$

Note that $f(1,3) = 3$ and the proof does not work for $n = 1$ because $e_i + e_j$ does not exist in a 1-dimensional space.

Having $f(n,3) = n+3$ for $n \geq 2$ immediately gives that $f(n,4) \geq n+4$ for $n \geq 2$. For $n = 2$, it is straightforward to check that $f(2,4) = 6$, with an optimal almost 4-cover $x_1 = 1$ (twice), $x_2 = 1$ (twice), and $x_1 + x_2 = 1$ (twice). However for $n \geq 3$, we can improve this lower bound 1 further.

Proof of Theorem 2.2.2. For $n \geq 3$, we would like to prove by contradiction that $n+4$ affine hyperplanes cannot form an almost 4-cover of Q^n. Following the notations in the previous proof, we have

$$P_1 \cdots P_{n+4} =: P = \sum_{1\le i\le j\le k\le l\le n} g_i g_j g_k g_l h_{ijkl} + u\prod_{i=1}^{n}(x_i - 1),$$

with $\deg(u) \leq 4$. Following the proof of Theorem 2.2.2, we are able to ignore the terms $g_i g_j g_k g_l h_{ijkl}$ and conclude that $u\prod_{i=1}^{n}(x_i-1)$ and its partial derivatives of order less than or equal to 3 vanish on $Q^n \setminus \{\vec{0}\}$. Using product rule to expand out these derivatives, we see that u satisfies the following relations: (i) $u = 0$ at e_i, $e_i + e_j$ and $e_i + e_j + e_k$ for distinct i, j, k; (ii) $\partial u/\partial x_i = 0$ at e_j and $e_j + e_k$ for distinct j, k ($i = j$ or $i = k$ possible); (iii) $\partial^2 u/\partial x_i^2 = 0$ at e_j ($i = j$ possible); (iv) $\partial^2 u/\partial x_i \partial x_j = 0$ at e_k ($i = k$ or $j = k$ possible). The polynomial u is nonzero and has degree at most 4, so we write

$$u = \sum a_{iiii} x_i^4 + \sum a_{iiij} x_i^3 x_j + \cdots + \sum b_{iii} x_i^3 + \cdots + \sum c_{ii} x_i^2 + \cdots + \sum d_i x_i + e.$$

Since $P(\vec{0}) = (-1)^{n+4} = (-1)^n$, we know that $u(\vec{0}) = 1$ and thus $e = 1$.

Let $w(x) = u(x \cdot e_i + e_j)$. Then $w(0) = w(1) = w'(0) = w'(1) = w''(0) = 0$. Since $w(x)$ has degree at most 4, we immediately have $w \equiv 0$. Recognizing that each coefficient of $w(x)$ must be 0, this gives

$$a_{iiii} = 0, \tag{2.1}$$

$$a_{iiij} + b_{iii} = 0. \tag{2.2}$$

$$a_{iijj} + b_{iij} + c_{ii} = 0. \tag{2.3}$$

$$a_{ijjj} + b_{ijj} + c_{ij} + d_i = 0. \tag{2.4}$$

$$a_{jjjj} + b_{jjj} + c_{jj} + d_j + 1 = 0 \tag{2.5}$$

Using $\partial u/\partial x_i(e_i) = 0$ and $\partial^2 u/\partial x_i^2(e_i) = 0$, we have

$$4a_{iiii} + 3b_{iii} + 2c_{ii} + d_i = 0, \tag{2.6}$$

$$12a_{iiii} + 6b_{iii} + 2c_{ii} = 0. \tag{2.7}$$

Equation 2.5 can be rewritten with i's instead of j's. Using $a_{iiii} = 0$, we can then solve the system of linear equations given by equations 2.5, 2.6, and 2.7 to get

$b_{iii} = -1$, $c_{ii} = 3$, $d_i = -3$. This implies $a_{iiij} = 1$. Plugging into the equations (2.3) and (2.4), we have:

$$a_{iijj} + b_{iij} = -3,$$

$$b_{iij} + c_{ij} = 2.$$

Now using $\partial^2 u/\partial x_i \partial x_j(e_i) = 0$, we have $3a_{iiij} + 2b_{iij} + c_{ij} = 0$, which gives

$$2b_{iij} + c_{ij} = -3.$$

The three linear equations above give $b_{iij} = -5$, $c_{ij} = 7$, $a_{iijj} = 2$.

For $n \geq 3$, we can also utilize the relation $\partial^2 u/(\partial x_i \partial x_j) = 0$ at e_k with $k \neq i, j$. This gives $a_{ijkk} + b_{ijk} + c_{ij} = 0$, hence

$$a_{ijkk} + b_{ijk} = -7.$$

Also $\partial u/(\partial x_i) = 0$ at $e_j + e_k$ simplifies to

$$a_{ijkk} + a_{ijjk} + b_{ijk} + 3 = 0.$$

Together they give $b_{ijk} = -11$ and $a_{ijkk} = 4$. Finally, by calculations

$$\begin{aligned} u(e_i + e_j + e_k) &= 3a_{iiii} + 6a_{iiij} + 3a_{iijj} + 3a_{iijk} + 3b_{iii} + 6b_{iij} + b_{ijk} + 3c_{ii} + 3c_{ij} + 3d_i + e \\ &= 2 \neq 0. \end{aligned}$$

This gives a contradiction. Therefore for $n \geq 3$, there is no u of degree at most 4 satisfying the aforementioned relations. This shows for $n \geq 3$, $f(n, 4) \geq n + 5$. $\square$

The proof does not work for $n < 3$ because $e_i + e_j + e_k$ does not exist in a 1-dimensional or 2-dimensional space. Since $f(n, 4) \leq n + \binom{4}{2} = n + 6$, it can only be either $n+5$ or $n+6$.

We note that for $n \in \{3,4,5\}$, we have $f(n,4) = n+5$. For Q^3, note that $x_1 = 1$, $x_2 = 1$, $x_3 = 1$, and $x_1 + x_2 + x_3 = 1$ form an almost 2-cover. Doubling it gives an almost 4-cover of Q^3 with 8 affine hyperplanes. For Q^4, the following 9 affine hyperplanes form an almost 4-cover: $x_1 = 1$, $x_2 = 1$, $x_3 = 1$, $x_4 = 1$, $x_1 + x_4 = 1$, $x_2 + x_4 = 1$, $x_3 + x_4 = 1$, $x_1 + x_2 + x_3 = 1$, $x_1 + x_2 + x_3 + x_4 = 1$. For Q^5, one can take $x_i = 1$ for $i = 1, \cdots 5$, together with $x_i + x_{i+1} + x_{i+2} = 1$ for $i = 1, \cdots, 5$, where the addition is in $\mathbb{Z}_5$.

We have determined the minimum size of an almost k-cover of Q^n, for $k \leq 3$. Note that $f(n,1) = n$ for $n \geq 1$, $f(n,2) = n+1$ for $n \geq 1$, and $f(n,3) = n+3$ for $n \geq 2$. All of these attain the upper bound $f(n,k) \leq n + \binom{k}{2}$ whenever n is sufficiently large. For larger k, the following conjecture seems plausible.

Conjecture 2.2.3. *For an arbitrary fixed integer $k \geq 1$ and sufficiently large n,*

$$f(n,k) = n + \binom{k}{2}.$$

In other words, for large n, an almost k-cover of Q^n contains at least $n + \binom{k}{2}$ affine hyperplanes.

The following result of Noga Alon provides some evidence toward Conjecture 2.2.3.

Remark 2. Alon (see [CH20]) showed that the upper bound of $n + \binom{k}{2}$ is tight for sufficiently large dimension n for any covering which makes use of the standard set of hyperplanes $x_1 = 1, x_2 = 1, \cdots, x_n = 1$. That is, the following holds for sufficiently large n: Suppose $H_1, \cdots, H_m$ are affine hyperplanes in $\mathbb{R}^n$ not containing $\vec{0}$, and they cover all the vectors with t ones as coordinates at least $k-t$ times, for $t = 1, \cdots, k-1$. Then, $m \geq \binom{k}{2}$.

In practice, the affine hyperplanes $x_1 = 1, \cdots, x_n = 1$ have been useful in minimal constructions, so for Conjecture 2.2.3 to be false, there would have to be situations where the minimal almost k-cover misses at least one of these hyperplanes. We revisit this idea in Subsection 3.1.2 in the context of general polynomial vanishing problems.

Remark 3. Lisa Sauermann and Yuval Wigderson [SW20] showed a lower bound of $f(n,k) \geq n + 2k - 3$ in the case of $k \geq 2$ and $n \geq 2k - 3$. This offers an improvement over the stated bound of $n + k + 1$ whenever $k \geq 5$. Notably, they showed that for these choices of n and k, there exists a polynomial of degree exactly $n + 2k - 3$ which vanishes to multiplicity at least k on $Q^n \setminus \{\vec{0}\}$ without vanishing at $\vec{0}$. This means any potential further improvement to the lower bound for the almost k-cover question would have to make use of the fact that the polynomial P splits completely into linear factors. For each $k \geq 4$, $n + 2k - 3$ is the best possible lower bound for sufficiently large n if we ignore that the polynomial we work with actually corresponds to an almost k-cover with hyperplanes.

2.3 Fixed Dimension

In this section, we study the growth of $f(n,k)$ as $k \to \infty$ for n fixed. To do so, it helps to view $f(n,k)$ as the optimum of an integer program where the variables correspond to the hyperplanes not passing through the origin and the constraints are that each remaining point of Q^n is covered the requisite number of times. (If one wishes to avoid having infinitely many variables, it suffices to consider the possible intersection patterns of a hyperplane in $\mathbb{R}^n$ with Q^n.) We can then obtain a lower bound on $f(n,k)$ by considering the linear relaxation of this integer program.

We will assign to every affine hyperplane H in $\mathbb{R}^n$ a nonnegative weight $w(H)$, with the constraints

$$\sum_{\vec{v} \in H} w(H) \geq k, \quad \text{for every } \vec{v} \in Q^n \setminus \{\vec{0}\},$$

and

$$\sum_{\vec{0}\in H} w(H) = 0.$$

Such an assignment w of weights is called a *fractional almost k-cover* of Q^n. We would like to minimize the sum of the weights, $\sum_H w(H)$. Denote by $f^*(n,k)$ the minimum of $\sum_H w(H)$, i.e. the minimum size of a fractional almost k-cover. We are able to determine the precise value of $f^*(n,k)$ for every value of n and k.

Theorem 2.3.1. *For every n and k,*

$$f^*(n,k) = \left(1 + \frac{1}{2} + \cdots + \frac{1}{n}\right) k.$$

This implies that for fixed n and $k \to \infty$,

$$f(n,k) = \left(1 + \frac{1}{2} + \cdots + \frac{1}{n} + o(1)\right) k,$$

which grows linearly in k.

Note that this linear growth in k is in contrast to our upper bound of $n + \binom{k}{2}$.

As an intermediate step of proving Theorem 2.3.1, we prove the following theorem, which can be viewed as an analogue of the well-known Lubell-Yamamoto-Meshalkin inequality [Bol65, Lub66, Mes63, Yam54] for subset sums.

Theorem 2.3.2. *Given n real numbers $a_1, \cdots, a_n$, let*

$$\mathcal{A} = \{S : \emptyset \neq S \subset [n], \sum_{i\in S} a_i = 1\}.$$

Then,

$$\sum_{S\in\mathcal{A}} \frac{1}{|S|\binom{n}{|S|}} \leq 1.$$

Equivalently, let $\mathcal{A}_t = \{S : S \in \mathcal{A}, |S| = t\}$*. Then,*

$$\sum_{t=1}^{n} \frac{|\mathcal{A}_t|}{t\binom{n}{t}} \le 1.$$

Moreover, the inequality is tight whenever $\vec{a} = (a_1, \cdots, a_n)$ *is a nonzero binary vector.*

To prove Theorem 2.3.1, we first establish an upper bound via an explicit construction of almost k-covers.

Lemma 2.3.3. *(i) For every* n, k*,*

$$f^*(n,k) \le \left(1 + \frac{1}{2} + \cdots + \frac{1}{n}\right) k.$$

(ii) When k *is divisible by* nx*, with* $x = lcm(\binom{n-1}{0}, \binom{n-1}{1}, \cdots, \binom{n-1}{n-1})$*, we have*

$$f(n,k) \le \left(1 + \frac{1}{2} + \cdots + \frac{1}{n}\right) k.$$

Proof. For (ii), it suffices to show that when $k = nx$, we can find an almost k-cover of Q^n, using $k(1+1/2+\cdots+1/n)$ hyperplanes. We can then replicate these hyperplanes to upper bound $f(n,k)$ where k is any multiple of nx.

For $j = 1, \cdots, n$, we will use every affine hyperplane of the form $x_{i_1} + x_{i_2} + \cdots + x_{i_j} = 1$ a total of $\frac{nx}{j\binom{n}{j}}$ times. This number is actually an integer since it is equal to $\frac{x}{\binom{n-1}{j-1}}$, and by definition, x is divisible by all $\binom{n-1}{j-1}$.

There are $\binom{n}{j}$ affine hyperplanes in this form, so the total number used is

$$\sum_{j=1}^{n} \frac{nx}{j\binom{n}{j}} \cdot \binom{n}{j} = \sum_{j=1}^{n} \frac{nx}{j} = \left(1 + \frac{1}{2} + \cdots + \frac{1}{n}\right) k.$$

This is the number of hyperplanes claimed. If we can show that they form an almost nx-cover of Q^n, then we can scale the weights by a constant factor to obtain a fractional almost k-cover of Q^n for every k and (i) will follow immediately.

It is apparent that $(0, \cdots, 0)$ is never covered. Because of the symmetric nature of our construction, we just need to check, for $t = 1, \cdots, n$, how many times we have covered any particular vertex that has t ones as coordinates. It gets covered by $t\binom{n-t}{j-1}$ distinct hyperplanes of the form $x_{i_1} + x_{i_2} + \cdots + x_{i_j} = 1$, each of which appears $\frac{nx}{j\binom{n}{j}}$ times. Thus, the total number of times a point with t ones is covered is given by:

$$
\begin{aligned}
\sum_{j=1}^{n} \frac{nx}{j\binom{n}{j}} \cdot t\binom{n-t}{j-1} &= nxt \sum_{j=1}^{n-t+1} \frac{\binom{n-t}{j-1}}{j\binom{n}{j}} = nxt \sum_{j=1}^{n-t+1} \frac{(n-t)!(n-j)!}{(n-t-j+1)!n!} \\
&= nxt \cdot \frac{(n-t)!}{n!} \cdot \sum_{j=1}^{n-t+1} \frac{(n-j)!}{(n-t-j+1)!} \\
&= \frac{nx}{(t-1)!\binom{n}{t}} \sum_{j=1}^{n-t+1} \frac{(n-j)!}{(n-t-j+1)!} = \frac{nx}{\binom{n}{t}} \sum_{j=1}^{n-t+1} \binom{n-j}{t-1} \\
&= \frac{nx}{\binom{n}{t}} \binom{n}{t} = nx = k.
\end{aligned}
$$

□

To establish the lower bound in Theorem 2.3.1, first we assign weights to each vertex of Q^n we wish to cover. A vertex with t ones as coordinates is given weight $\frac{1}{t\binom{n}{t}}$. Then the sum of the weights of all the vertices is:

$$
\sum_{t=1}^{n} \binom{n}{t} \cdot \frac{1}{t\binom{n}{t}} = \sum_{t=1}^{n} \frac{1}{t}.
$$

If we cover each vertex k times, the sum over all affine hyperplanes of the weights of the vertices they cover is $k(1 + 1/2 + \cdots + 1/n)$. Thus, if we can show that no hyperplane can cover a set of vertices whose weights sum to more than 1, we will have proven the lower bound. Given an affine hyperplane H not containing $\vec{0}$, denote by

$\mathcal{A}_t$ the set of vertices with t ones covered by H. We wish to prove Theorem 2.3.2, i.e.

$$\sum_{t=1}^{n} \frac{|\mathcal{A}_t|}{t\binom{n}{t}} \leq 1.$$

In general, vertices of $Q^n \setminus \{\vec{0}\}$ correspond to nonempty subsets of $[n]$. It is worth noting that if the equation of H is $a_1x_1 + \cdots + a_nx_n = 1$, and all coefficients a_i are strictly positive, the subsets corresponding to the vertices it covers will form an antichain. By the Lubell-Yamamoto-Meshalkin inequality,

$$\sum_{t=1}^{n} \frac{|\mathcal{A}_t|}{t\binom{n}{t}} \leq \sum_{t=1}^{n} \frac{|\mathcal{A}_t|}{\binom{n}{t}} \leq 1.$$

However, some coefficients a_i may be non-positive. In order to consider a more general hyperplane, we will associate each vertex it covers to some permutations of $[n]$. Consider the vertex $(c_1, c_2, \cdots, c_n) \in Q^n$ where the coordinates which are ones are $c_{i_1}, \cdots, c_{i_t}$. We will associate this vertex to the permutations, $(d_1, d_2, \cdots, d_n)$ of $[n]$ which begin with $\{i_1, i_2, \cdots, i_t\}$ in some order and also have $\sum_{k=1}^{j} a_{d_k} < 1$ for $1 \leq j < t$. We will make use of the following lemmas.

Lemma 2.3.4. *No permutation of $[n]$ is associated to more than one vertex on the same hyperplane.*

Lemma 2.3.5. *The total number of permutations associated to a vertex with t ones as coordinates is at least $(t-1)!(n-t)!$.*

Essentially, each vertex covered by a given hyperplane is associated to a lot of permutations but there are a limited number of permutations available, since the same permutation cannot be repeated. Thus, we will be able to establish an upper bound on the total weight of the covered vertices. Conditional on the lemmas, we give a proof of Theorem 2.3.2.

Proof of Theorem 2.3.2. By definition, sets in $\mathcal{A}$ correspond to vertices of Q^n covered by the hyperplane H with equation $a_1x_1 + \cdots + a_nx_n = 1$. From Lemma 2.3.4 and 2.3.5, these vertices define disjoint collections of permutations of length n. Moreover if $S \in \mathcal{A}$ has size t then there are at least $(t-1)!(n-t)!$ permutations associated to it. Since in total there are at most $n!$ permutations, we get

$$\sum_{S \in \mathcal{A}} (|S|-1)!(n-|S|)! \leq n!,$$

which implies

$$\sum_{S \in \mathcal{A}} \frac{1}{|S|\binom{n}{|S|}} \leq 1$$

as desired.

Note that when $\vec{a} = (a_1, \cdots, a_n)$ is a nonzero binary vector with j ones, we have that $\mathcal{A}_t \subset Q^n$ consists of all vertices with a 1 in one of the coordinates where $\vec{a}$ has a 1 and $t-1$ 1's in coordinates where $\vec{a}$ has a 0. Thus, $|\mathcal{A}_t| = j\binom{n-j}{t-1}$, and we have

$$\sum_{t=1}^{n} \frac{|\mathcal{A}_t|}{t\binom{n}{t}} = \sum_{t=1}^{n} \frac{j\binom{n-j}{t-1}}{t\binom{n}{t}} = 1,$$

where the second equality was previously shown in the proof of Lemma 2.3.3. Thus, the inequality is tight when $(a_1, \cdots, a_n)$ is a nonzero binary vector. $\square$

We now prove the lemmas used.

Proof of Lemma 2.3.4. Suppose for the sake of contradiction that a permutation is associated to two vertices, v and w, of the same hyperplanes. They may have either the same or a different number of ones as coordinates.

Suppose that v and w both have a ones as coordinates. The permutations associated to v have the a indices where v has a 1 as their first a entries and the permutations associated to w will have the a indices where w has a 1 as their first a entries. However, v and w do not have their ones in the exact same places so the set of the first a entries is not the same for any pair of a permutation associated to v and a permutation associated to w.

We are left to consider the case where v and w do not have the same number of ones as coordinates. Without loss of generality, v has a ones as coordinates and w has b ones as coordinates where $a > b$. Suppose the permutation associated to both of them begins with $(d_1, d_2, \cdots, d_b)$. By the restrictions on permutations associated to v, we have that $\sum_{j=1}^{b} a_{d_j} < 1$. However, the conditions on permutations associated to w tell us that $(d_1, d_2, \cdots, d_b)$ are precisely the indices where w has a 1 coordinate. This implies $\sum_{j=1}^{b} a_{d_j} = 1$, giving a contradiction.

□

Proof of Lemma 2.3.5. There are $(n-t)!$ ways to arrange the indices other than $\{i_1, i_2, \cdots, i_t\}$, so it suffices to show that there exist at least $(t-1)!$ ways to order $\{i_1, i_2, \cdots, i_t\}$ as $(d_1, d_2, \cdots, d_t)$ such that we have $\sum_{k=1}^{j} a_{d_k} < 1$ for $1 \leq j < t$. We notice that $(t-1)!$ is the number of ways to order $\{i_1, i_2, \cdots, i_t\}$ around a circle (up to rotations, but not reflections). Thus it suffices to show that for each circular ordering of $\{i_1, i_2, \cdots, i_t\}$, we can choose a starting place from which we may continue clockwise and label the elements as $(d_1, d_2, \cdots, d_t)$ in such a way that $\sum_{k=1}^{j} a_{d_k} < 1$ for all $1 \leq j < t$.

Equivalently, the values of a_{i_k} for $1 \leq k \leq t$, which happen to sum to 1, have been listed around a circle. We wish to find some starting point from which all the partial sums from that point of up to $t-1$ terms are less than 1. We can subtract $1/t$ from each to give the equivalent problem of t numbers, which sum to 0, written around a circle and needing to find a starting point from which all the partial sums of $1 \leq j \leq t-1$ terms are less than $1 - \frac{j}{t}$. It suffices to find a starting point for which the aforementioned partial sums are at most 0.

Consider all possible sums of any number of consecutive terms along the circle and choose the largest. We will label the terms in this sum as $e_1, e_2, \cdots, e_m$ and continue to order clockwise around the circle $e_{m+1}, e_{m+2}, \cdots, e_t$. Choose the starting point to be e_{m+1}. If any of the partial sums $e_{m+1}+e_{m+2}+\cdots+e_{m+j}$ exceeds 0, for $m+j \leq t$, we could simply have chosen $e_1, e_2, \cdots, e_{m+j}$ to get a larger sum than $e_1+e_2+\cdots+e_m$. Similarly, if $e_{m+1}+e_{m+2}+\cdots+e_t+e_1+e_2+\cdots+e_j > 0$ for some $1 \leq j < m$, then we can note that $(e_1+e_2+\cdots+e_t)+(e_1+e_2+\cdots+e_j)$ exceeds $e_1+e_2+\cdots+e_m$, and since $e_1+e_2+\cdots+e_t = 0$, we have that $e_1+e_2+\cdots+e_j > e_1+e_2+\cdots+e_m$, a contradiction. Thus, if we start at e_{m+1} and move clockwise around the circle, the first $t-1$ partial sums will be at most 0, as desired.

□

Now we are ready to prove our main theorem in this section.

Proof of Theorem 2.3.1. As mentioned before, we assign weight $\frac{1}{t\binom{n}{t}}$ to each vertex of $Q^n \setminus \{\vec{0}\}$ with t ones as coordinates. By Lemma 2.3.2, every affine hyperplane covers a set of vertices whose weights sum to at most 1. Therefore in an optimal fractional almost k-cover $\{w(H)\}$,

$$f^*(n,k) = \sum_H w(H) \geq k \cdot \sum_{t=1}^{n} \frac{\binom{n}{t}}{t\binom{n}{t}} = \left(\sum_{i=1}^{n} \frac{1}{i}\right) k.$$

With the upper bound proved in Lemma 2.3.3, we have

$$f^*(n,k) = \left(\sum_{i=1}^{n} \frac{1}{i}\right) k.$$

For integral almost k-covers, note that $f(n,k) \geq f^*(n,k)$. Using Lemma 2.3.3 again,

$$f(n,k) = f^*(n,k) = \left(\sum_{i=1}^{n} \frac{1}{i}\right) k,$$

whenever nx divides k. For fixed n and $k \to \infty$, note that $f(n,k)$ is monotone in k, which immediately implies

$$f(n,k) = \left(1 + \frac{1}{2} + \cdots + \frac{1}{n} + o(1)\right) k.$$

$\square$

For small values of n, we can actually determine the value of $f(n,k)$ for every k. It seems that for large k, $f(n,k)$ is not far from its lower bound $\lceil f^*(n,k) \rceil$. Trivially $f(1,k) = k$.

Theorem 2.3.6. *The following statements are true:*
(i) $f(2,k) = \lceil \frac{3k}{2} \rceil$ *for* $k \geq 1$.
(ii) $f(3,k) = \lceil \frac{11k}{6} \rceil$ *for* $k \geq 2$ *and* $f(3,1) = 3$.

Proof. (i) The lines $x = 1, y = 1$, and $x + y = 1$ give an almost 2-cover of Q^2 using 3 affine hyperplanes. Therefore $f(2, k+2) \leq f(2,k) + 3$, and it suffices to check $f(2,1) = 2$ and $f(2,2) = 3$ which are both obvious.
(ii) There exists an almost 6-cover of Q^3 using 11 affine hyperplanes: each of $x_i = 1$ twice, each of $x_i + x_j = 1$ once, and $x_1 + x_2 + x_3 = 1$ twice. Therefore $f(3, k+6) \leq f(3,k) + 11$. It suffices to check $f(3,k) \leq \lceil \frac{11k}{6} \rceil$ for $k = 2, \cdots, 5$ and $k = 7$. From $f(n,2) = n+1$, we have $f(3,2) = 4$. $f(3,3) \leq 6$ follows from Theorem 2.2.1. $f(3,4) \leq 8$ since $f(3,4) \leq 2f(3,2)$. $f(3,5) \leq 10$ by taking each of $x_i = 1$ twice, $x_1 + x_2 + x_3 = 1$ three times, and $x_1 + x_2 + x_3 = 2$ once. $f(3,7) \leq 13$ follows from taking each of $x_1 = 1$, $x_2 = 1$, $x_3 = 1$, $x_1 + x_2 = 1$, $x_1 + x_3 = 1$ twice, and $x_2 + x_3 = 1$, $x_2 + x_3 - x_1 = 1$, $x_1 + x_2 + x_3 = 1$ once. $\square$

With the assistance of a computer program, we also checked that $f(4,k) = \lceil \frac{25k}{12} \rceil$ for $k \geq 2$. $f(5,k) = \lceil \frac{137}{60} k \rceil$ for $k \geq 15$ except when $k \equiv 7 \pmod{60}$ where $f(5,k) = \lceil \frac{137}{60} k \rceil + 1$. The following question is natural.

Question 2.3.7. *Does there exist an absolute constant $C > 0$ which does not depend on n, such that for a fixed integer n, there exists M_n, so that whenever $k \geq M_n$,*

$$f(n,k) \leq \left(1 + \frac{1}{2} + \cdots + \frac{1}{n}\right) k + C?$$

If so, this would show that $f(n,k)$ and $f^*(n,k)$ differ by at most a constant when k is large. Specifying k large is necessary since when $k = 1$, $f(n,k) - f^*(n,k) = n - (1 + \frac{1}{2} + \cdots + \frac{1}{n})$ can be arbitrarily large.

2.4 Higher Codimension

We can also consider analogous almost-covering problems where the goal is to cover all but one point of $\{0,1\}^n$ using affine subspaces of codimension d. Jamison [Jam77] considered the question of covering all the points except $\vec{0}$ in the vector space $\mathbb{F}_q^n$ at least once without covering $\vec{0}$ while Bishnoi et al. [BBDM21] considered this question for $\mathbb{F}_2^n$ and higher covering multiplicity.

We will also consider the n-cube, except we will regard it as a proper subset of $\mathbb{R}^n$, rather than as $\mathbb{F}_2^n$. Working over characteristic 0 differs from the finite field case.

Working over $\mathbb{F}_2$, all codimension d affine subspaces will contain 2^{n-d} points. Over $\mathbb{R}$, a codimension d subspace may contain up to 2^{n-d} points of $\{0,1\}^n$ but could also contain fewer. Groenland and Johnston [GJ20] determined which numbers between 2^{n-d-1} and 2^{n-d} are possible cardinalities of the intersection $H \cap \{0,1\}^n$ for a codimension d affine subspace H of $\mathbb{R}^n$.

The fact that a codimension d affine subspace over $\mathbb{F}_2$ will cover at least as many points of $\{0,1\}^n$ as one over $\mathbb{R}$ might suggest that the minimum number of subspaces needed is higher over $\mathbb{R}$. To the contrary, this is never the case for covering all but the forbidden point at least once. We also establish some absolute bounds for the minimum number of codimension d subspaces needed over $\mathbb{R}$.

Theorem 2.4.1. *For $n \geq d \geq 2$, let $h(n,d)$ represent the minimum number of affine hyperplanes needed to cover all but one point of $\mathbb{F}_2^n$ while leaving the last point uncovered. Let $g(n,d)$ represent the minimum number of affine hyperplanes over $\mathbb{R}$ needed to cover all but one point of $\{0,1\}^n$ while leaving the last point uncovered. Then, the following statements are true.*

(i) $n \leq g(n,d) \leq h(n,d)$,

(ii) $g(n,2) = n$ for $n \geq 4$ and $g(n,3) \leq n+2$ for $n \geq 7$, and

(iii) $g(n,d) \geq 2^d + 1$ for $n \geq d+3$.

Proof. (i) For $n = d$, codimension d affine subspaces are just points, so $g(d,d) = h(d,d) = 2^d - 1$. Lemma 3.2 of [BBDM21] demonstrates that $h(n+1,d) \geq h(n,d) + 1$. To show that $g(n,d) \leq h(n,d)$ for all $n \geq d$, it suffices to show that $g(n+1,d) \leq g(n,d) + 1$. To see this, we consider a minimal covering $H_1, H_2, \cdots, H_{g(n,d)}$ of $\{0,1\}^n \setminus \{\vec{0}\}$ in $\mathbb{R}^n$. Then, $H_1 \times \{0,1\}, H_2 \times \{0,1\}, \cdots, H_{g(n,d)} \times \{0,1\}$ cover all points of $\{0,1\}^{n+1}$ except for $\vec{0}$ and $(0,\cdots,0,1)$. It is easy to find a subspace of codimension d which contains $(0,\cdots,0,1)$ but not $\vec{0}$ in order to complete a covering of size $g(n,d)+1$. From the later bounds, we will see that $h(n,d)$ and $g(n,d)$ are not always equal.

Suppose we have a cover of $\{0,1\}^n \setminus \{\vec{0}\}$ consisting of $g(n,d)$ codimension d affine subspaces over $\mathbb{R}$. Each subspace is of the form $\{v + \sum_{i=1}^{n-d} c_i v_i : c_1, \cdots, c_{n-d} \in \mathbb{R}\}$ for some vectors $v, v_1, v_2, \cdots, v_{n-d} \in \mathbb{R}^n$. We can extend these subspaces to affine subspaces of codimension 1 by including $d-1$ vectors with "irrelevant" directions such as $(1,\cdots,1,\pi,1,\cdots,1)$ and taking spans. Thus, we obtain a list of $g(n,d)$ affine hyperplanes which cover the same subset of $\{0,1\}^n$ as before. By Alon and Füredi's result [AF93], this list must consist of at least n hyperplanes so $g(n,d) \geq n$.

(ii) The reason we sometimes obtain $g(n,d) < h(n,d)$ is because there are affine subspaces containing the origin over $\mathbb{F}_2$ whose counterparts over $\mathbb{R}$ do not contain the origin. For $n = 4, d = 2$, we are able to obtain the following cover consisting of four affine subspaces over $\mathbb{R}$:

- $x_4 = 0, x_1 + x_2 + x_3 = 2$,
- $x_3 = x_4 = 1$,
- $x_3 + x_4 = 1, x_1 = x_2$, and
- $x_3 = 0, x_1 + x_2 = 1$.

Note that the 2 in the equation of the first subspace is the reason no analogous construction exists over $\mathbb{F}_2$. Since we have that $g(n+1,d) \leq g(n,d)+1$, obtaining $g(n,2) = n$ for $n = 4$ ensures $g(n,2) \leq n$ for $n \geq 4$, which combined with the lower bound gives $g(n,2) = n$ for $n \geq 4$.

Similarly, we can show that $g(n,3) \leq n+2$ for $n \geq 7$ by demonstrating that $g(7,3) \leq 9$. We present the following construction:

- $x_1 = x_3 + x_4 = 1, x_4 = x_5$,
- $x_3 = x_1 - x_4 = 0, x_2 + x_6 = 1$,
- $x_1 = 0, x_3 = x_7 = 1$,
- $x_7 = x_1 - x_5 = 0, x_4 + x_5 = 1$,
- $x_7 = x_1 - x_4 = 0, x_3 = 1$,
- $x_7 = 1, x_1 = x_3 = x_5$,
- $x_3 = x_2 - x_6 = 0, x_1 + x_5 = 1$,
- $x_3 = x_4 - x_5 = 0, x_1 + x_2 + x_4 + x_6 = 2$,
- $x_1 - x_7 = x_3 - x_4 = 0, x_1 + x_3 + x_5 = 2$.

Note that seven of these subspaces cover 16 points of $\{0,1\}^7$, while the other two cover 12 points. It is impossible to find a construction using nine subspaces which each cover 16 points of $\{0,1\}^7$ since using the same equations over $\mathbb{F}_2^7$ would avoid the origin and thus imply $h(7,3) \leq 9$, whereas we know that $h(7,3) \geq 4 + h(3,3) = 4+(2^3-1) = 11$. Similarly, it is impossible to find a construction using eight subspaces which each cover 16 points and one that covers 12 since the ones that cover 16 points will still cover the same points over $\mathbb{F}_2^7$ and the one that covers 12 points can be replaced by two codimension 3 subspaces over $\mathbb{F}_2^7$ which cover those 12 points without covering $\vec{0}$. (For example, the last subspace in our construction could be replaced by the subspaces defined by $x_1 - x_7 = x_3 - x_4 = 0, x_1 = 1$ and by $x_1 - x_7 = x_3 - x_4 = 0, x_3 = 1$.) This implies that $h(7,3) \leq 8 + 2 = 10$, a contradiction.

(iii) Lastly, we will show that $g(n,d) \geq 2^d + 1$ for $n - d \geq 3$. There are $2^n - 1$ points to cover and we can only cover 2^{n-d} at a time, so this gives an immediate lower bound of 2^d for $d \leq n-1$. Further, 2^d can only be obtained if one of the following scenarios holds:

- Each subspace used covers 2^{n-d} points of Q^n.
- $2^d - 1$ subspaces used cover 2^{n-d} points, while the last covers $2^{n-d} - 1$ points.

In the first scenario, since $\vec{0}$ is uncovered, exactly one vertex of $\{0,1\}^n$ is covered twice.

For a particular affine subspace of codimension d which covers the maximum possible 2^{n-d} points of Q^n, the vertices covered look like $a + \sum_{i=1}^{n-d} c_i v_i$ where each $c_i \in \{0,1\}$, $a \in Q^n$, and $v_1, \cdots, v_{n-d} \in \{-1,0,1\}^n$. We can partition these into pairs $\{a + \sum_{i=1}^{n-d-1} c_i v_i + v_{n-d}, a + \sum_{i=1}^{n-d-1} c_i v_i\}$, for fixed choices of $c_1, \cdots, c_{n-d-1}$. If v_{n-d} has a 0 for its x_i coordinate, then both members of each pair will have the same x_i coordinate, meaning that an even number of points covered by this subspace will have an x_i coordinate of 1. If v_{n-d} has a 1 or -1 as its x_i coordinate, then half of the vertices covered by this subspace have a 1 as their x_i coordinate, for a total of 2^{n-d-1}, which is again even.

This means that for each $i = 1, \cdots, n$, if we add up the x_i coordinates of the points covered (with multiplicity), we will always get $0 \in \mathbb{F}_2$. However, the points covered are all the vertices of $\{0,1\}^n$ except $\vec{0}$, and then one vertex covered a second time. The sum of the x_i coordinates of the points in $\{0,1\}^n \setminus \{\vec{0}\}$ is always $0 \in \mathbb{F}_2$ for $n \geq 2$ so the vertex covered twice must have an even x_i coordinate for all i. But that means $\vec{0}$ is covered twice when it's not supposed to be covered at all. Thus, the first scenario is impossible.

For $k \geq 6$, Groenland and Johnston [GJ20] determined the possible large sizes of an intersection of a dimension k affine subspace over $\mathbb{R}$ with Q^n.

Theorem 2.4.2. *[GJ20] If H is a dimension k affine subspace of $\mathbb{R}^n$ with $k \geq 6$ and $|H \cap Q^n| > 2^{k-1}$, then $|H \cap Q^n| \in \{2^{k-1} + 2^i \mid i = 0, \cdots, k-1\} \cup \{2^{k-1} + 2^{k-5} + 2^{k-6}\}$.*

Note that for $k \geq 6$, none of these possible sizes is equal to $2^k - 1$. Therefore, we conclude that a codimension d affine subspace cannot contain exactly $2^{n-d} - 1$ points of Q^n when $d \leq n - 6$.

We will now show that such a subspace cannot exist for $d \in \{n-5, n-4, n-3\}$ either. Note that when $d = n - 2$, it is possible to have an affine subspace of codimension d which contains exactly $2^{n-d} - 1 = 3$ points of Q^n and that we obtain $g(n, n-2) = 2^{n-2}$ for $n \geq 4$.

Let $k := n - d \in \{3, 4, 5\}$. Suppose that n is the smallest possible dimension such that there is a dimension k affine subspace, H, which contains exactly $2^k - 1$ points from Q^n. Without loss of generality, at least 2^{k-1} of these points have $x_1 = 0$. Consider the points which lie in the intersection of $H \cap Q^n$ and the hyperplane $x_1 = 0$. They either lie in a dimension $k - 1$ affine subspace of $\mathbb{R}^n$ or they span a dimension k affine subspace.

If they span a dimension k affine subspace, then there are no other points of $H \cap Q^n$ with $x_1 = 1$, since then the points of $H \cap Q^n$ would span a subspace of dimension at least $k + 1$. Thus, all $2^k - 1$ points of $H \cap Q^n$ lie in the intersection of Q^n and $x_1 = 0$. This is isomorphic to a copy of $\{0,1\}^{n-1}$ embedded into $\mathbb{R}^n$, contradicting the minimality of n.

In the remaining case, the intersection of $H \cap Q^n$ and $x_1 = 0$ lies in a dimension $k-1$ affine subspace of $\mathbb{R}^n$. Therefore, this intersection, which contains at least 2^{k-1} points, must contain exactly 2^{k-1} points. The remaining $2^{k-1}-1$ points of $H \cap Q^n$ have $x_1 = 1$. In particular, the points of $H \cap Q^n$ with $x_1 = 1$ can be written as $v + v_k$ where v lies in the intersection of $H \cap Q^n$ with $x_1 = 0$ and v_k is a fixed vector in $\{-1, 0, 1\}^n$ with 1 as its x_1 coordinate.

As before, we can partition the elements in the intersection of $H \cap Q^n$ with $x_1 = 0$ into pairs $\{a + \sum_{i=1}^{k-2} c_i v_i, a + \sum_{i=1}^{k-2} c_i v_i + v_{k-1}\}$ for some vectors $a \in Q^n$, $v_1 \cdots, v_{k-1} \in \{-1, 0, 1\}^n$, and with $c_1, \cdots, c_{k-2} \in \{0, 1\}$. If the x_j coordinate of v_{k-1} is 0, we conclude that an even number of these vectors have $x_j = 1$. If the x_j coordinate of v_{k-1} is 1 or -1, we conclude that half of these vectors, a total of 2^{k-2}, have $x_j = 1$. In either case, we get that for every $j = 1, \cdots, n$, an even number of vectors in the intersection of $H \cap Q^n$ with $x_1 = 0$ have $x_j = 1$, and thus that an even number of vectors in the intersection of $H \cap Q^n$ with $x_1 = 0$ have $x_j = 0$.

We know that v_k has $x_1 = 1$ and also that $v_k \neq e_1$ since $|H \cap Q^n| < 2^k$. Therefore, there is some $j \in \{2, 3, \cdots, n\}$ where $x_j \in \{-1, 1\}$ for v_k. If the x_j coordinate of v_k is 1, it means that for every v in the intersection of $H \cap Q^n$ with $x_1 = 0$ that has $x_j = 1$, we have that $v + v_k$ is not in Q^n. There are an even number of such v's, so if the number of such v's is nonzero, then we have $|H \cap Q^n| \leq 2^k - 2$. This means that for every j where $x_j = 1$ for v_k, every single vector in the intersection of $H \cap Q^n$ with $x_1 = 0$ must have $x_j = 0$. Similarly, if the x_j coordinate of v_k is -1, it means that for every v in the intersection of $H \cap Q^n$ with $x_1 = 0$ that has $x_j = 0$, we have that $v + v_k$ is not in Q^n. There are an even number of such v's, so if the number of such v's is nonzero, then we have $|H \cap Q^n| \leq 2^k - 2$. This means that for every j where $x_j = -1$ for v_k, every single vector in the intersection of $H \cap Q^n$ with $x_1 = 0$ must have $x_j = 1$.

Since every v in the intersection of $H \cap Q^n$ with $x_1 = 0$ has $x_j = 0$ when v_k has $x_j = 1$ and has $x_j = 1$ when v_k has $x_j = -1$, we can add v_k to all 2^{k-1} of those vectors and still be in Q^n which would give $|H \cap Q^n| = 2^k$.

Therefore, we conclude that this scenario is impossible as well, forcing $g(n, d) \geq 2^d + 1$ for $n \geq d + 3$.

$\square$

2.5 Connections to Graph Covering and Sidon Sets

One consideration which arises when determining how many points of $\{0, 1\}^n$ each subspace can contain in an optimal cover over $\mathbb{R}$ is how many affine subspaces of the same codimension, not passing through the origin, are needed to cover the same set of points over $\mathbb{F}_2$. For example, the points of $\{0, 1\}^7$ covered by the subspace $x_3 = x_4 - x_5 = 0, x_1 + x_2 + x_4 + x_6 = 2$ over $\mathbb{R}$ cannot be covered by a single codimension 3 subspace over $\mathbb{F}_2$ without covering $\vec{0}$, but can be covered by the subspaces $x_3 = x_4 - x_5 = 0, x_1 + x_2 = 1$ and $x_3 = x_4 - x_5 = 0, x_1 + x_4 = 1$ of $\mathbb{F}_2^7$, neither of which contains $\vec{0} \in \mathbb{F}_2^7$.

We will focus our attention on subspaces of codimension 1. For an intersection pattern of an affine hyperplane and $\{0, 1\}^n$ that does not contain the origin over $\mathbb{R}$, how many affine hyperplanes over $\mathbb{F}_2$ not passing through the origin are needed to cover the same set of points? We will consider the cases where the normal vector of the hyperplane is an element of $\{0, 1\}^n$.

If the hyperplane is of the form $x_{i_1} + \cdots + x_{i_k} = c$ where c is odd, we can use the same equation over $\mathbb{F}_2$ without passing through the origin. If c is even, we will need more than one hyperplane over $\mathbb{F}_2$ to cover the same set of points.

Consider $x_{i_1} + \cdots + x_{i_k} = 2d$ for $d \in \mathbb{Z}_{>0}$. Over $\mathbb{F}_2$, we can replace it with a collection of hyperplanes of the form $\sum_{i \in S} x_i = 1$ for various subsets $S \subset [n]$. First we make the case that the minimum number of such hyperplanes needed is the same as the number needed when $k = n$.

Without loss of generality, we can consider $x_1 + \cdots + x_k = 2d$. For any set of $2d$ distinct indices, $\{j_1, j_2, \cdots, j_{2d}\} \subset [k]$, this collection must cover all points that have 1's in each of those coordinates and 0's in the remaining coordinates in $[k]$. For dimension k, this is equivalent to making sure that for each $\{j_1, j_2, \cdots, j_{2d}\} \subset [k]$, some hyperplane corresponds to a set $S \subset [k]$ which contains an odd number of elements from $\{j_1, j_2, \cdots, j_{2d}\}$. Taking the equations of a minimum cover over dimension k and regarding them instead as hyperplanes of $\mathbb{F}_2^n$ will still cover all points of $\mathbb{F}_2^n$ with 1's in $2d$ of their first k coordinates. Thus, the number of hyperplanes needed over dimension $n > k$ is at most the number needed for dimension k.

However, for any choice of $2d$ indices $\{j_1, j_2, \cdots, j_{2d}\} \subset [k]$, a cover over dimension n needs to cover, in particular the points with 1's in those coordinates and 0's in the remaining $n - 2d$ coordinates. That is equivalent to making sure that for each $\{j_1, j_2, \cdots, j_{2d}\} \subset [k]$ some hyperplane corresponds to a set S which contains an odd number of elements from $\{j_1, j_2, \cdots, j_{2d}\}$. Note that satisfying this criterion is unaffected by which elements of $[n] \setminus [k]$ are in which S's. Thus, this necessary condition corresponds to choosing subsets of $[k]$ such that for each choice of $2d$ indices in $[k]$, there is some subset where an odd number of those indices appear. This was exactly the condition for finding a cover over dimension k, so the number of hyperplanes needed over dimension $n > k$ is at least the number needed for dimension k.

In $\mathbb{R}^n$, consider the hyperplane $x_1 + x_2 + \cdots + x_n = 2$. The collection of hyperplanes chosen over $\mathbb{F}_2$ to cover these points without covering $\vec{0}$ must, for any $1 \leq p < q \leq n$, cover all points with $x_p = x_q = 1$ and $x_i = 0$ for $i \in \setminus \{p, q\}$. That is equivalent to making sure some hyperplane corresponds to a set S which contains exactly one of p and q.

Another way of viewing this is to regard each hyperplane, $\sum_{i \in S} x_i = 1$, in the cover as a complete bipartite graph with partite sets S and $[n] \setminus S$. For these hyperplanes to form a valid cover, there must be, for each pair $1 \leq p < q \leq n$, some complete bipartite graph in the list where p and q are in different partite sets. This corresponds to using complete bipartite graphs to cover the edges of a complete graph on $[n]$. Thus, the minimum number of affine hyperplanes of $\mathbb{F}_2^n$ needed to cover all points with exactly two 1's as coordinates, while avoiding the origin, is $\lceil \log_2 n \rceil$.

Next we can consider the case of covering all points of $\mathbb{F}_2^n$ with four 1's as coordinates. For any subset $D \subset [n]$ of size 4, we must choose some hyperplane corresponding to an $S \subset [n]$ which contains either exactly one or exactly three elements of D.

Let $hc(n, 4)$ denote the minimum number of affine hyperplanes needed to cover all the points of $\mathbb{F}_2^n$ with exactly four 1's as coordinates, without covering $\vec{0}$. $hc(n, 4)$ is non-decreasing in n since taking a cover for dimension n and removing any appearance of x_n from the equations of hyperplanes in the collection will still produce a cover over dimension $n - 1$. We establish an asymptotic upper bound by showing that $hc(3k, 4) \leq 2k$.

To do this, we will make use of the hyperplanes $x_{3i-2}+x_{3i-1} = 1$ and $x_{3i-2}+x_{3i} = 1$ for $i = 1, \cdots, k$. Consider $D \subset [n] := [3k]$ of size 4. There is some $i \in [k]$ where the set $\{3i - 2, 3i - 1, 3i\}$ contains an element of D. In fact, if it contains three elements of D, some other set of the form $\{3j - 2, 3j - 1, 3j\}$ for $j \in [k]$ must contain exactly one element of D. Thus, some set of the form $\{3i - 2, 3i - 1, 3i\}$ contains exactly one or two elements of D. If it contains exactly one or just $3i - 1$ and $3i$, then either of the hyperplanes $x_{3i-2} + x_{3i-1} = 1$ or $x_{3i-2} + x_{3i} = 1$ will include the point with 1's in coordinates corresponding to elements of D and 0's elsewhere. If it contains $3i - 2$ and $3i - 1$, we can use $x_{3i-2} + x_{3i} = 1$. If it contains $3i - 2$ and $3i$, we can use $x_{3i-2} + x_{3i-1} = 1$.

Note that unlike in the case of covering the points of $\mathbb{F}_2^n$ with two 1's as coordinates, it is now possible for some indices to appear in exactly the same set of S's. In particular, we could use the hyperplanes $x_1 = 1, x_2 = 1, \cdots, x_{n-3} = 1$ and the last three indices would not appear at all. However, it is impossible to choose $\{a, b, c, d\} \subset [n]$ where a and b appear in exactly the same S's and c and d appear in exactly the same S's. The reason for this is the point with 1's in coordinates a, b, c, and d and 0's elsewhere will not lie on any of the hyperplanes $\sum_{i \in S} x_i = 1$.

Finding a minimum cover consists of picking $hc(n, 4)$ subsets, S, of $[n]$ that serve as index sets for each hyperplane $\sum_{i \in S} x_i = 1$. For each $i \in [n]$, picking which hyperplanes use it as an index is equivalent to picking a vector, v_i, in $\mathbb{F}_2^{hc(n,4)}$. The condition that for each set $D \subset [n]$ of size 4, there is some hyperplane that uses exactly one or exactly three elements of D as indices is equivalent to $v_a + v_b + v_c + v_d \neq \vec{0}$ for $\{a, b, c, d\} \subset [n]$. This can be rewritten as $v_a + v_b \neq v_c + v_d$. While the v_i's do not need to be unique for every $i \in [n]$, we can discard at most two of them to get a unique set of vectors. Thus, determining $hc(n, 4)$ is closely related to the problem of finding the smallest dimension, k, for which there is a set, $T \subset \mathbb{F}_2^k$, with $|T| = n$ for which the pairwise sums $v_a + v_b$ for distinct $v_a, v_b \in T$ are unique.

This T can be thought of as analogous to a Sidon set. A Sidon set, $A \subset \mathbb{Z}_{>0}$, is a set where $a + b = c + d$ for $a, b, c, d \in A$ implies $\{a, b\} = \{c, d\}$ as multisets. Extending this definition to $\mathbb{F}_2^k$, there is no such thing as a nontrivial Sidon set since we always have $v_a + v_a = 0 = v_b + v_b$, but the condition imposed on T is the closest notion in the sense that the pairwise sums will be distinct unless they a priori agree.

More generally, we can consider the question of finding the smallest dimension, $k := s(n, d)$, such that we can find a collection of n vectors in $\mathbb{F}_2^k$ such that no d distinct vectors in the collection sum to $\vec{0}$. When d is even, $s(n, d)$ is related to the question of finding the smallest number of hyperplanes over $\mathbb{F}_2^n$ which cover all points with exactly d 1's as coordinates, without covering $\vec{0}$. For d odd, we expect the behavior of $s(n,d)$ to be significantly different since it is possible to choose all vectors whose first coordinate is 1 in our collection.

In fact, $s(2^{m-1}+1,3) = m < s(2^{m-1}+2,3)$ for $m \geq 2$, which can be rearranged to give $s(n,3) = \lceil \log_2(n-1) \rceil + 1$ for $n \geq 3$. To see this, we show that $2^{m-1}+1$ is the largest possible size of a subset of $\mathbb{F}_2^m$ such that no three distinct elements sum to $\vec{0}$. An optimal construction consists of taking $\vec{0}$ along with every vector in $\mathbb{F}_2^m$ with a 1 in the first coordinate. To see that this is optimal, suppose we have a set $T \subset \mathbb{F}_2^m$ with $|T| = 2^{m-1}+2$ such that no three distinct elements of T sum to $\vec{0}$. Consider some nonzero $t \in T$ and let $W := T \setminus \{t\}$. Both W and $t - W$ are of size $2^{m-1}+1$ so they share a common element. Thus, there exist $a, b \in W$ with $a = t - b$. Note that $a \neq b$ since otherwise $t = a + b = \vec{0}$. Therefore, we have three distinct elements $a, b, t \in T$ such that $a + b - t = a + b + t = \vec{0}$, a contradiction.

Chapter 3

Covering for General Grids

3.1 Rectangular Grids

Alon and Füredi [AF93] also showed a more general version of their hypercube almost-covering result which holds for any rectangular grid:

Theorem 3.1.1 (Alon-Füredi, 1993)**.** *For subsets $S_1, \cdots, S_n$ of a field $\mathbb{F}$, the smallest number of affine hyperplanes needed to cover every point of the grid $S_1 \times \cdots \times S_n$ except for one point that must remain uncovered is:*

$$\sum_{i=1}^{n}(|S_i| - 1).$$

Note that the n-cube is the case where $|S_1| = \cdots = |S_n| = 2$. An easy way of interpreting this summation is that the natural construction using only hyperplanes orthogonal to the standard unit vectors is optimal. For example, in two dimensions, the optimum number can be obtained just from taking all the horizontal and vertical lines of the grid which do not pass through the excluded point.

It is natural to extend this question to higher multiplicity and ask for the smallest number of affine hyperplanes needed to cover every point of $S_1 \times \cdots \times S_n$ at least k times except for one point that must remain uncovered. A related question is to extend the work of Sauermann and Wigderson [SW20] and ask for the lowest degree of a polynomial which vanishes to multiplicity at least k at every point of $S_1 \times \cdots \times S_n$ except for one point where it does not vanish. The hyperplane covering question is the same as the polynomial vanishing question with the added condition that the polynomial splits completely into linear factors. In fact, this condition is not used at all in Alon and Füredi's lower bound so the answer to these questions is the same for $k = 1$.

In this section, we will solve both questions for $k = 2$ and explain why they are more difficult for $k \geq 3$.

Claim 3.1.2. *For subsets $S_1, \cdots, S_n$ of a field $\mathbb{F}$, the smallest number of affine hyperplanes needed to cover every point of the grid $S_1 \times \cdots \times S_n$ twice except for one point that must remain uncovered is:*

$$\max_i |S_i| - 1 + \sum_{i=1}^{n} (|S_i| - 1).$$

Furthermore, the minimum degree of a polynomial in $\mathbb{F}[x_1, \cdots, x_n]$ which vanishes to multiplicity two on all but one point (where it does not vanish) of $S_1 \times \cdots \times S_n$, is again

$$\max_i |S_i| - 1 + \sum_{i=1}^{n} (|S_i| - 1).$$

Proof. Ball and Serra [BS09] used the Punctured Combinatorial Nullstellensatz (Theorem 1.2.3) to prove a lower bound of $(t-1)(\max_i |S_i| - 1) + \sum_{i=1}^{n}(|S_i| - 1)$ on the degree of a polynomial which vanishes to multiplicity t on all but one point of $S_1 \times \cdots \times S_n$ and does not vanish at the remaining point. We will include a proof of this, specifically for the case $t = 2$. This also serves as a lower bound for the hyperplane covering problem. We will construct a collection of $\max_i |S_i| - 1 + \sum_{i=1}^{n}(|S_i| - 1)$ hyperplanes with the desired covering properties. Each hyperplane can be written as $f_i(x_1, \cdots, x_n) = 0$ for some linear polynomial f_i. The product of the f_i's in turn provides a matching upper bound for the polynomial vanishing question.

For the lower bound, we may write each hyperplane, H_i, in our cover as $a_{i,1}x_1 + a_{i,2}x_2 + \cdots + a_{i,n}x_n - c_i = 0$. Writing $f_i := a_{i,1}x_1 + a_{i,2}x_2 + \cdots + a_{i,n}x_n - c_i$, we see that the product of the f_i's vanishes to multiplicity at least 2 everywhere on $S_1 \times \cdots \times S_n$ except for the uncovered point $(a_1, \cdots, a_n)$, with $a_i \in S_i$ for $i = 1, \cdots, n$. Thus, the lower bound for the polynomial vanishing problem also serves as a lower bound for the hyperplane covering problem. Now we provide a proof of the lower bound for the polynomial vanishing problem.

In the context of Theorem 1.2.3, we take $D_i = \{a_i\}$ for $i = 1, \cdots, n$. Thus, $g_i = \prod_{j \in S_i}(x_i - j)$ and $l_i = x_i - a_i$. Note that throughout this proof, we will treat $\frac{g_i}{l_i}$ and similar expressions as polynomials defined everywhere rather than as rational functions defined only on the domain $\mathbb{F} \setminus \{a_i\}$. Without loss of generality, we assume that $\max_i |S_i| = |S_1|$. Assume that a polynomial f of degree less than $|S_1| - 1 + \sum_{i=1}^n (|S_i| - 1)$ vanishes with multiplicity at least 2 at every point of $S_1 \times \cdots \times S_n$ except for $(a_1, \cdots, a_n)$ where it does not vanish.

Then, we have polynomials h_τ with $\tau \in T(n, 2)$ and a nonzero polynomial u with $\deg u \leq \deg f - \sum_{i=1}^n (|S_i| - 1) < |S_1| - 1$ such that:

$$f = \sum_{\tau \in T(n,t)} g_{\tau(1)} g_{\tau(2)} h_\tau + u \prod_{i=1}^n \frac{g_i}{l_i}$$

Any first-order partial derivative $\frac{\partial f}{\partial x_i}$ vanishes everywhere on $S_1 \times \cdots \times S_n \setminus \{(a_1, \cdots, a_n)\}$. Note that

$$\frac{\partial (g_{\tau(1)} g_{\tau(2)} h_\tau)}{\partial x_i} = g_{\tau(1)} \left(g_{\tau(2)} \frac{\partial h_\tau}{\partial x_i} + \frac{\partial g_{\tau(2)}}{\partial x_i} h_\tau \right) + g_{\tau(2)} h_\tau \frac{\partial g_{\tau(1)}}{\partial x_i}$$

so terms of this form will vanish on $S_1 \times \cdots \times S_n$ for all i and τ. Therefore, all the first order partial derivatives of $u \prod_{i=1}^n \frac{g_i}{l_i}$ will vanish everywhere on $S_1 \times \cdots \times S_n \setminus \{(a_1, \cdots, a_n)\}$.

In particular, consider $\frac{\partial}{\partial x_1}\left(u\prod_{i=1}^{n}\frac{g_i}{l_i}\right)$. This is equal to

$$\left(\prod_{i=2}^{n}\frac{g_i}{l_i}\right)\left(\frac{\partial u}{\partial x_1}\frac{g_1}{l_1}+u\frac{\partial}{\partial x_1}\prod_{j\in S_1\setminus\{a_1\}}(x_1-j)\right)$$

We consider grid points that agree with the uncovered point everywhere but the first coordinate. Since $x_i = a_i$ for $i \geq 2$, then $\prod_{i=2}^{n}\frac{g_i}{l_i}$ is nonzero. Therefore, for $r \in S_1 \setminus \{a_1\}$, the following vanishes at $(r, a_2, \cdots, a_n)$:

$$\frac{\partial u}{\partial x_1}\frac{g_1}{l_1}+u\frac{\partial}{\partial x_1}\prod_{j\in S_1\setminus\{a_1\}}(x_1-j).$$

$\frac{g_1}{l_1} = \prod_{j\in S_1\setminus\{a_1\}}(x_1-j)$ must vanish at $(r, a_2, \cdots, a_n)$, so we conclude that the following expression must as well:

$$u(x_1,x_2,\cdots,x_n)\frac{\partial}{\partial x_1}\prod_{j\in S_1\setminus\{a_1\}}(x_1-j).$$

Note that

$$\frac{\partial}{\partial x_1}\prod_{j\in S_1\setminus\{a_1\}}(x_1-j)=\sum_{j_2\in S_1\setminus\{a_1\}}\frac{\prod_{j\in S_1\setminus\{a_1\}}(x_1-j)}{(x_1-j_2)}.$$

When $x_1 = r \in S_1 \setminus \{a_1\}$, this is equal to $\prod_{j\in S_1\setminus\{a_1,r\}}(r-j)$.

Therefore, for $r \in S_1 \setminus \{a_1\}$, we have that:

$$u(r,a_2,\cdots,a_n)\prod_{j\in S_1\setminus\{a_1,r\}}(r-j)=0$$

$$u(r,a_2,\cdots,a_n)=0.$$

Recall that $u(x_1, \cdots, x_n)$ is a multivariable polynomial of degree less than $|S_1|-1$. Thus, $u(x_1, a_2, \cdots, a_n)$ is a single-variable polynomial of degree less than $|S_1| - 1$.

However, it vanishes for all values of x_1 in $S_1 \setminus \{a_1\}$, which is a total of $|S_1|-1$ values. This necessitates that $u(x_1, a_2, \cdots, a_n)$ is the zero polynomial. However, that would mean that $u(a_1, a_2, \cdots, a_n) = 0$. Also note that any g_i vanishes at $(a_1, \cdots, a_n)$. This means $f = \sum_{\tau \in T(n,t)} g_{\tau(1)} g_{\tau(2)} h_\tau + u \prod_{i=1}^{n} \frac{g_i}{l_i}$ will vanish at $(a_1, a_2, \cdots, a_n)$, which gives a contradiction because our original function f did not vanish at $(a_1, \cdots, a_n)$.

Thus, we conclude that any polynomial vanishing with multiplicity at least 2 at every point of $S_1 \times \cdots \times S_n$ except for one point where it doesn't vanish, must have degree at least $|S_1| - 1 + \sum_{i=1}^{n}(|S_i| - 1)$.

We now construct an optimal covering. Suppose that the point in $S_1 \times S_2 \cdots \times S_n$ that we do not wish to cover is $(a_1, \cdots, a_n)$. Utilizing the hyperplanes $x_i = s_i$ for $i = 1, \cdots, n$ and $s_i \in S_i \setminus \{a_i\}$ will cover every point at least twice except for the forbidden point $(a_1, \cdots, a_n)$ and the grid points that share all but one coordinate with $(a_1, \cdots, a_n)$. Then, the remaining points that need to be covered an additional time lie on a set of n pairwise orthogonal lines. A hyperplane is a dimension $n - 1$ affine subspace so there is at least one hyperplane passing through a given set of n points. Thus, we can take hyperplanes which cover one point each from the n lines containing points that still need to be covered. The most points that still needed to be covered on any given line was $\max_i |S_i| - 1$, so we need a total of at most $(\max_i |S_i| - 1) + \sum_{i=1}^{n}(|S_i| - 1)$ hyperplanes.

□

3.1.1 Complications for $k = 3$

Unlike for $k = 1$ and $k = 2$, we will see that the minimum number of affine hyperplanes needed to cover all but one grid point at least 3 times depends on the point being removed.

Example 4. Consider the two dimensional grid $\{0, 1, 2\} \times \{0, 1, 2\}$. Covering all points but $(1, 1)$ at least 3 times and leaving $(1, 1)$ uncovered requires ten lines while covering all points but $(0,0)$ (or another corner) at least 3 times and leaving $(0,0)$ (that same corner) uncovered requires only nine lines. Note that $(1,0)$ behaves similarly to $(0,0)$.

Proof. An optimal construction for leaving $(1,1)$ uncovered is to use $x = 0$ and $x = 2$ twice each, along with $y = 0$, $y = 2$, $x+y = 1$, $x+y = 3$, $y = x+1$ and $y = x-1$. An optimal construction for leaving $(0,0)$ uncovered is to use $x = 2$, $y = 2$ and $x+y = 1$ twice each, along with $x + y = 2, x = 1$, and $y = 1$. Now we show that using ten and nine lines respectively cannot be beaten.

Suppose it is possible to use nine or fewer lines to cover all grid points three times except for $(1,1)$ which is left uncovered. It is always possible to find a cover with the same number of lines in which each line covers at least two grid points. For simplicity, refer to $(0,0), (2,0), (0,2)$, and $(2,2)$ as *corners* and refer to $(1,0), (0,1), (2,1)$, and $(1,2)$ as edges. For a line to cover at least two grid points without covering $(1,1)$, the only options are to cover two corners and an edge (Type A), to cover two edges (Type B), or to cover one corner and one edge (Type C). Suppose that a covering with nine lines uses a of type A, b of type B, and c of type C. This means $a+b+c = 9$. Also, the four corners are covered a total of at least $3(4) = 12$ times, as are the four edges. This gives that $2a + c \geq 12$ and $a + 2b + c \geq 12$. This last inequality simplifies to $b \geq 3$.

However, since the total covering multiplicity is at least $3(8) = 24$, we also have that $3a + 2b + 2c \geq 24$ which simplifies to $a + 2(9) \geq 24$. Since $a \geq 6$, we have that $b+c \leq 3$. This is only possible if $b = 3, c = 0$, and $a = 6$. Note that the total covering multiplicity is $3(6)+2(3)+2(0) = 24$, so every grid point except $(1,1)$ must be covered exactly three times. There are only four lines of Type A: $x = 0, x = 2, y = 0$, and $y = 2$. Without loss of generality, $x = 0$ is used at least two times. If it is used three times, then $y = 0$ and $y = 2$ cannot be used at all without covering $(0,0)$ or $(0,2)$ too many times. Then, since $a = 6$, $x = 2$ would have to also be used three times. This would leave us with needing to cover both $(1,0)$ and $(1,2)$ three times each using just three lines. This cannot be done since a line through both $(1,0)$ and $(1,2)$ would also contain the forbidden point $(1,1)$.

Thus, $x = 0$ is used exactly twice (and no other line of Type A is used more than twice). Again, $(0,0)$ and $(0,2)$ can only be covered exactly three times so $y = 0$ and $y = 2$ are used at most once each. Thus, $x = 2$ is used at least twice. This can only happen if $x = 2$ is used exactly twice and $y = 0$ and $y = 2$ are used exactly once. However, we then need to cover $(1,0)$ and $(1,2)$ an additional two times each using only three more lines. This cannot be done since no valid line contains both $(1,0)$ and $(1,2)$.

It is quicker to show that nine lines is optimal for the case of leaving a corner uncovered. Suppose it is possible to use eight lines to cover all points three times except for $(0,0)$ which is left uncovered. No line can cover more than three grid points at a time and the total covering multiplicity for all the points is at least $8(3) = 24$. Thus, this can only be done if every point is covered exactly three times and every line used covers exactly three grid points. However, the only way to cover $(1,0)$ with a line through three grid points and not cover $(0,0)$ is to use $x = 1$ and the only way to cover $(0,1)$ with a line through three grid points and not cover $(0,0)$ is to use $y = 1$. This means $x = 1$ and $y = 1$ are each used at least three times, so $(1,1)$ is covered at least six times instead of exactly three times. $\square$

A further, and perhaps the most important, complication with $k = 3$ is that the polynomial vanishing question no longer has the same answer as the hyperplane covering question:

Example 5. Covering all points of $\{0,1,2\} \times \{0,1,2\}$ but $(1,1)$ at least 3 times and leaving $(1,1)$ uncovered requires ten lines. However, there is a degree 8 polynomial which vanishes to multiplicity at least 3 at all points of $\{0,1,2\} \times \{0,1,2\}$ except for $(1,1)$ where it doesn't vanish. To see this, we may use the polynomial $(x^2 - xy + y^2 -$

$x-y)(x^2+xy+y^2-3x-3y+2)xy(x-2)(y-2)$.

Similarly, the answers to the hyperplane covering and polynomial vanishing questions also differ when covering all points of $\{0,1,2,3\}\times\{0,1,2,3\}$ or $\{0,1,2,3,4\}\times\{0,1,2,3,4\}$ at least 3 times except for $(1,1)$, which is left uncovered. We know from Gurobi computations that 14 lines are necessary for $\{0,1,2,3\}\times\{0,1,2,3\}$ while 18 are necessary for $\{0,1,2,3,4\}\times\{0,1,2,3,4\}$. However, there is a degree 13 polynomial, $xy(x-2)(x-3)^2(y-2)(y-3)^2(x+y-3)(x^2-xy+y^2-x-y)(x^2+y^2-3x-3y+2)$, that has the correct vanishing on $\{0,1,2,3\}\times\{0,1,2,3\}$.

There is also a degree 17 polynomial with the correct vanishing for $\{0,1,2,3,4\}\times\{0,1,2,3,4\}$. Note that the degree 15 polynomial, $xy(x-2)(x-3)(x-4)^2(y-2)(y-3)(y-4)^2(x+y-4)(x^2-xy+y^2-x-y)(x^2+y^2-3x-3y+2)$, does not vanish at $(1,1)$, vanishes twice at $(1,3),(1,4),(3,1),(3,3)$, and $(4,1)$, and vanishes at least thrice at the remaining grid points. Then, since there is a conic through every five points, there exists a degree 2 polynomial which vanishes at $(1,3),(1,4),(3,1),(3,3)$, and $(4,1)$. Note that such a polynomial does not vanish at $(1,1)$, since if it did, there would be three intersections between the conic and each of the lines $x=1$ and $y=1$. That could only happen if the conic was reducible and was simply the union of lines, $x=1$ and $y=1$. However, it is not, since it must contain $(3,3)$. Thus, multiplying our degree 15 polynomial by this degree 2 polynomial will yield a degree 17 polynomial with the proper vanishing.

In general, we can ask if this discrepancy between the answers to the polynomial vanishing and hyperplane covering questions persists for larger grids.

Question 3.1.3. *For $m\geq n\geq 2$, is the minimum degree of a polynomial which vanishes to multiplicity at least 3 at every point of $\{0,1,\cdots,m\}\times\{0,1,\cdots,n\}$ except $(1,1)$ and does not vanish at $(1,1)$ strictly less than the minimum number of lines needed to cover every point of $\{0,1,\cdots,m\}\times\{0,1,\cdots,n\}$ at least three times except for $(1,1)$, which is left uncovered?*

Question 3.1.4. *When $|S_1|, |S_2|, \cdots, |S_n| \geq 3$ and $k \geq 3$, does there exist some point $p \in S_1 \times \cdots \times S_n$ such that the minimum degree of a polynomial which vanishes to multiplicity at least k on every point of $S_1 \times \cdots \times S_n \setminus \{p\}$ without vanishing at p is strictly less than the minimum number of affine hyperplanes needed to cover every point of $S_1 \times \cdots \times S_n \setminus \{p\}$ at least k times without covering p?*

Remark 4. The answer to the polynomial vanishing question again depends on the point removed. It is impossible to find a degree 8 polynomial which vanishes to multiplicity at least 3 at all points of $\{0, 1, 2\} \times \{0, 1, 2\}$ except for $(1, 0)$ where it doesn't vanish. For the sake of contradiction, assume that f is a degree 8 polynomial satisfying these vanishing conditions, and consider the irreducible factors of f. By Bézout's theorem, a degree $d > 1$ irreducible factor will have at most d intersections with each of the lines $x = 0$, $x = 1$, and $x = 2$, so it contributes no more than $3d$ to the total vanishing. Also, a linear factor contributes at most 3 to the total vanishing. f must vanish to multiplicity at least 3 at eight points, so the total vanishing multiplicity is at least $3(8) = 24$, requiring that each degree d irreducible factor of f have total vanishing multiplicity of $3d$ on $\{0, 1, 2\} \times \{0, 1, 2\} \setminus \{(1, 0)\}$. We know from Example 4 that all linear factors will not work. An irreducible conic ($d = 2$) cannot vanish to multiplicity 2 at any given point so it necessarily would vanish at two grid points on each of the lines $x = 0, x = 1, x = 2, y = 0, y = 1$, and $y = 2$. As it cannot vanish at $(1, 0)$, this would require it to vanish at $(0, 0), (2, 0), (1, 1)$, and $(1, 2)$. Such a factor also must vanish with total multiplicity at most 2 along the lines $x - y = 0$ and $x + y = 2$, so it cannot vanish at $(2, 2)$ nor $(0, 2)$. This would then necessitate that it vanishes at $(0, 1)$ and $(2, 1)$, which is a contradiction since it cannot vanish three times along $y = 1$. To potentially achieve degree 8, we are then forced to consider irreducible factors of degree $d \geq 3$. Such a factor vanishes to total multiplicity d on each of the lines $x = 0, x = 1, x = 2, y = 0, y = 1$, and $y = 2$. Since it does not vanish at $(1, 0)$, it must vanish with total multiplicity d on $\{(0, 0), (2, 0)\}$ and on $\{(1,1),(1,2)\}$. At one point in each pair, the vanishing multiplicity is at least $\lceil d/2 \rceil$. However, the total vanishing multiplicity cannot exceed d along any line, so there are no two points for which the vanishing multiplicity sums to more than d. For odd d, we have that $2\lceil d/2 \rceil > d$, so the only remaining possibility is that d is even and the irreducible factor of degree d vanishes with multiplicity exactly $d/2$ at each of $(0,0),(2,0),(1,1)$, and $(1,2)$. Again, it can only vanish to multiplicity at most d along each of $x-y=0$ and $x+y=2$, so it cannot vanish at $(2,2)$ nor $(0,2)$. This necessitates vanishing to multiplicity $d/2$ at $(0,1)$ and $(2,1)$, which is a contradiction since it now vanishes to multiplicity $3d/2$ along $y=1$.

Similarly, a polynomial f which vanishes to multiplicity at least 3 everywhere on $\{0,1,2\} \times \{0,1,2\}$ except for $(0,0)$, where it does not vanish, must have multiplicity at least 9. Using linear factors is not enough to achieve degree 8 and we get that the only potentially helpful higher degree irreducible factors must have even degree d and vanish with multiplicity $d/2$ at $(0,1),(0,2),(1,0)$, and $(2,0)$. In turn, such a factor cannot vanish at $(1,1)$, and thus vanishes with multiplicity $d/2$ at $(1,2)$ and $(2,1)$. Notably, it cannot vanish at $(2,2)$ either. This means only linear factors of f can vanish at $(1,1)$ and $(2,2)$, but any line through those points also contains the forbidden point $(0,0)$. Thus, we must use a total of six linear factors to vanish thrice at both $(1,1)$ and $(2,2)$ and can then only use a degree 2 polynomial vanishing at $(0,1),(0,2),(1,0),(1,2),(2,0)$, and $(2,1)$. This will not work since every linear factor must vanish at three of the grid points, so $(2,2)$ can only be handled by $x-2$ or $y-2$. Additionally, the total vanishing multiplicity of a degree 8 polynomial on this grid is at most 24, so f cannot vanish to multiplicity more than 3 anywhere. Thus, we must utilize $(y-2)^2(x-2)$ or $(y-2)(x-2)^2$ in order to vanish thrice at $(2,2)$ without vanishing more than thrice elsewhere. By symmetry along $y=x$, we may assume that we must use $(y-2)^2(x-2)$. There are three possible linear factors which vanish at $(1,1)$ and two additional grid points, without vanishing at $(0,0)$: $x-1$, $y-1$, and $x+y-2$. However, of these, only $y-1$ vanishes at three points where we don't already have vanishing multiplicity three, so we must use $(y-1)^3$. However, now f vanishes to multiplicity greater than 3 at $(0,1)$ and $(2,1)$.

3.1.2 Restrictions on the Minimum Degree Polynomial

Proposition 3.1.5. *Let $S_1, \cdots, S_n$ be subsets of a field $\mathbb{F}$ and $2 \leq k \leq n$. Suppose f is a polynomial of minimal degree which vanishes to multiplicity at least k on all but one point, $(a_1, \cdots, a_n)$, of $S_1 \times \cdots \times S_n$, and does not vanish at $(a_1, \cdots, a_n)$. Then, there exists a polynomial g with $\deg g = \deg f$ and $\prod_{i \in [n]} \prod_{s \in S_i \setminus \{a_i\}} (x_i - s) \mid g$ such that g vanishes to multiplicity at least k on $S_1 \times \cdots \times S_n \setminus \{(a_1, \cdots, a_n)\}$ and does not vanish at $(a_1, \cdots, a_n)$.*

A consequence of this is that to determine the minimum degree polynomial with the desired vanishing properties, we only need to consider the minimum degree polynomial among those divisible by $\prod_{i \in [n]} \prod_{s_i \in S_i \setminus \{a_i\}} (x_i - s_i)$. In the hypercube case, where $|S_i| = 2$ for $i = 1, \cdots, n$, this means that we only need to consider polynomials divisible by $\prod_{i \in [n]} (x_i - 1)$. Each of these linear factors corresponds to a hyperplane $x_i = 1$. In Remark 2, we mention that the $n + \binom{k}{2}$ upper bound on the number of affine hyperplanes needed to cover $\{0,1\}^n \setminus \{\vec{0}\}$ k times without covering $\vec{0}$ is tight for sufficiently large n if we assume the almost k-cover includes $x_i = 1$ for $i = 1, \cdots, n$. If we could show that the minimal almost k-cover utilizing all of these hyperplanes cannot be beaten for sufficiently large n, we would be able to establish that the $n + \binom{k}{2}$ upper bound is tight for sufficiently large n, proving Conjecture 2.2.3. The fact that the optimal answer to the related polynomial question must include factors of $x_i - 1$ corresponding to each of these hyperplanes provides some circumstantial evidence supporting Conjecture 2.2.3.

Proof of Proposition 3.1.5. Suppose that f vanishes to multiplicity at least k on $S_1 \times$

$\cdots \times S_n \setminus \{(a_1, \cdots, a_n)\}$ and does not vanish on $(a_1, \cdots, a_n)$. We can use Theorem 1.2.3 with $g_i := \prod_{s \in S_i}(x_i - s)$, $l_i = x_i - a_i$, and $t = k$ to express f as:

$$\sum_{\tau \in T(n,k)} g_{\tau(1)} \cdots g_{\tau(k)} h_\tau + u \prod_{i=1}^{n} \frac{g_i}{l_i}$$

where $T(n, k)$ is the set of all non-decreasing sequences of length k on $[n]$ and u is a nonzero polynomial of degree at most $\deg f - \sum_{i=1}^{n}(|S_i| - 1)$.

Let M_i denote the set of points of $S_1 \times \cdots \times S_n$ which disagree with $(a_1, \cdots, a_n)$ on exactly i coordinates. Generalizing part of the proof of Theorem 2.2.1, we will demonstrate that for $i = 1, 2, \cdots, k-1$, u and its partial derivatives of order up to $k - i - 1$ vanish on M_i. Alternatively, we may say that u vanishes to multiplicity at least $k - i$ on M_i.

Each term of the summation $\sum_{\tau \in T(n,k)} g_{\tau(1)} \cdots g_{\tau(k)} h_\tau$ vanishes at $(a_1, \cdots, a_n)$ but f does not vanish at $(a_1, \cdots, a_n)$, so $u \prod_{i=1}^{n} \frac{g_i}{l_i}$ does not vanish at $(a_1, \cdots, a_n)$. Thus, u does not vanish at $(a_1, \cdots, a_n)$. Let h be a minimum degree polynomial which vanishes to multiplicity at least $k - i$ on M_i for $i = 1, 2, \cdots, k-1$ but does not vanish at $(a_1, \cdots, a_n)$. We observe that u shares these vanishing properties so $\deg u \geq \deg h$. Then, the inequality $\deg u \leq \deg f - \sum_{i=1}^{n}(|S_i| - 1)$ can be rearranged to yield that

$$\begin{aligned} \deg f &\geq \deg u + \sum_{i=1}^{n}(|S_i| - 1) \\ &\geq \deg h + \sum_{i=1}^{n}(|S_i| - 1). \end{aligned}$$

Also, we can construct a polynomial of degree $\deg h + \sum_{i=1}^{n}(|S_i| - 1)$ with the correct vanishing by taking $h \prod_{i=1}^{n} \prod_{s \in S_i \setminus \{a_i\}}(x_i - s)$. Therefore, among polynomials with the correct vanishing on $S_1 \times \cdots \times S_n$ and minimal degree, there exists one which is divisible by $\prod_{i=1}^{n} \prod_{s \in S_i \setminus \{a_i\}}(x_i - s)$.

What is left is to verify our claim that for $i = 1, 2, \cdots, k-1$, u and its partial derivatives of order up to $k-i-1$ vanish on M_i.

Since f vanishes to multiplicity at least k on $S_1 \times \cdots \times S_n \setminus \{(a_1, \cdots, a_n)\}$, we know that f and its partial derivatives of order at most $k-1$ vanish on $S_1 \times \cdots \times S_n \setminus \{(a_1, \cdots, a_n)\}$. First we note that for all $\tau \in T(n,t)$, $g_{\tau(1)} \cdots g_{\tau(k)} h_\tau$ and any of its partial derivatives of order up to $k-1$ will vanish on $S_1 \times \cdots \times S_n \setminus \{(a_1, \cdots, a_n)\}$. Thus we observe that $u \prod_{i=1}^n \frac{g_i}{l_i}$ and its partial derivatives of order up to $k-1$ will vanish on $S_1 \times \cdots \times S_n \setminus \{(a_1, \cdots, a_n)\}$.

First, we will consider a point, $p := (p_1, \cdots, p_n)$, in M_1 with $p_i \neq a_i$. Points in M_1 only disagree with $(a_1, \cdots, a_n)$ in one coordinate so we have $p_j = a_j$ for $j \in [n] \setminus \{i\}$. We consider the product rule expansion of $\frac{\partial}{\partial x_i}$ applied to $u \prod_{i=1}^n \frac{g_i}{l_i}$. All the summands contain a factor $x_i - p_i$ except for one. Therefore, all the terms but one automatically vanish at p, so the last term must as well.

This last term is

$$u \left(\prod_{j \in [n] \setminus \{i\}} \frac{g_j}{l_j} \right) \left(\prod_{s_i \in S_i \setminus \{a_i, p_i\}} (x_i - s_i) \right).$$

All these factors besides u cannot vanish at p, so we get that u vanishes at p. Similarly, u vanishes on all of M_1. We now proceed by induction.

For $2 \leq d \leq k-1$, suppose that by looking at the partial derivatives of $u \prod_{i=1}^n \frac{g_i}{l_i}$ of order less than or equal to $d-1$, we have shown that u vanishes on M_i for $i = 1, \cdots, d-1$ and that the partial derivatives of u of order less than or equal to $d-i-1$ vanish on M_i for $i = 1, \cdots, d-2$.

Now consider a point, $p := (p_1, \cdots, p_n)$, in M_q with $1 \leq q \leq d$ such that p disagrees with $(a_1, \cdots, a_n)$ in coordinates $i_1, i_2, \cdots, i_q$. We apply a partial derivative

$$\frac{\partial}{\partial x_{i_1} \partial x_{i_2} \cdots \partial x_{i_q} \partial x_J}$$

to $u \prod_{i=1}^n \frac{g_i}{l_i}$ where ∂x_J signifies any combination of $d-q$ ∂x_j's with $j \in [n]$. We know this function will vanish at p since any partial derivative of $u\prod_{i=1}^{n}\frac{g_i}{l_i}$ of order at most $k-1$ will vanish there.

Every term in the product rule expansion looks like the the product of some c^{th} order partial derivative of u and some $(d-c)^{th}$ order partial derivative of $\prod_{i=1}^{n}\frac{g_i}{l_i}$. For $c \leq d-q-1$, we already know that all c^{th} order partial derivatives of u vanish at p. If $d-c<q$, then every term in the product rule expansion of a $(d-c)^{th}$ order partial derivative of $\prod_{i=1}^{n}\frac{g_i}{l_i}$ contains some factor (x_i-p_i) for some index i where p disagrees with $(a_1,\cdots,a_n)$. Thus, all the terms automatically vanish except for those with $c=d-q$. Furthermore, the product rule expansion of a $(d-c)^{th}$ order partial derivative of $\prod_{i=1}^{n}\frac{g_i}{l_i}$ contains all but $d-c$ of the original factors x_i-s_i. This means when $d-c=q$, a term in a $(d-c)^{th}$ order partial derivative of $\prod_{i=1}^{n}\frac{g_i}{l_i}$ will contain at least one factor of the form x_i-p_i except in the special case where the q missing factors are precisely those of the form x_i-p_i for each i at which p disagrees with $(a_1,\cdots,a_n)$. This means that every term but one in the product rule expansion of

$$\frac{\partial}{\partial x_{i_1}\partial x_{i_2}\cdots\partial x_{i_q}\partial x_J}\left(u\prod_{i=1}^{n}\frac{g_i}{l_i}\right)$$

automatically vanishes at p. The only term that does not automatically vanish at p is

$$\frac{\partial u}{\partial x_J}\frac{\partial}{\partial x_{i_1}\partial x_{i_2}\cdots\partial x_{i_q}}\prod_{i=1}^{n}\frac{g_i}{l_i}.$$

In fact, $\frac{\partial}{\partial x_{i_1}\partial x_{i_2}\cdots\partial x_{i_q}}\prod_{i=1}^{n}\frac{g_i}{l_i}$ is equal to

$$\left(\prod_{i\in[n]\setminus\{i_1,i_2,\cdots,i_q\}}\frac{g_i}{l_i}\right)\left(\sum_{(b_1,\cdots,b_q)\in S_{i_1}\setminus\{a_{i_1}\}\times\cdots\times S_{i_q}\setminus\{a_{i_q}\}}\prod_{j=1}^{q}\frac{g_{i_j}}{(x_{i_j}-a_{i_j})(x_{i_j}-b_j)}\right).$$

Furthermore, the only summand that does not automatically vanish at p is when $b_j=p_{i_j}$ for $j=1,\cdots,q$. Thus, in order for $\frac{\partial}{\partial x_{i_1}\partial x_{i_2}\cdots\partial x_{i_q}\partial x_J}\left(u\prod_{i=1}^{n}\frac{g_i}{l_i}\right)$ to vanish at

p, we must have

$$\frac{\partial u}{\partial x_J}\left(\prod_{i\in[n]\setminus\{i_1,i_2,\cdots,i_q\}}\frac{g_i}{l_i}\right)\left(\prod_{j=1}^{q}\frac{g_{i_j}}{(x_{i_j}-a_{i_j})(x_{i_j}-p_{i_j})}\right)$$

vanish at p. However, $\left(\prod_{i\in[n]\setminus\{i_1,i_2,\cdots,i_q\}}\frac{g_i}{l_i}\right)\left(\prod_{j=1}^{q}\frac{g_{i_j}}{(x_{i_j}-a_{i_j})(x_{i_j}-p_{i_j})}\right)$ is nonzero at p, so we get that $\frac{\partial u}{\partial x_J}$ vanishes, where $\frac{\partial}{\partial x_J}$ can be any order $d-q$ partial derivative. (Note that in the case of $q=d$, this is simply telling us that u vanishes, so we now have that u vanishes on M_i for $i=1,\cdots,d$.)

From the induction hypothesis, we had that looking at partial derivatives of $u\prod_{i=1}^{n}\frac{g_i}{l_i}$ of order up to $d-1$ revealed that partial derivatives of u of order less than or equal to $d-q-1$ vanish on M_q for $q=1,\cdots,d-2$. Now we have that the partial derivatives of u of order less than or equal to $d-q$ vanish on M_q for $q=1,\cdots,d-1$. Continuing until $d=k-1$, we get that the partial derivatives of u of order less than or equal to $k-1-q$ vanish on M_q for $q=1,\cdots,k-2$. We also have that u vanishes on M_i for $i=1,\cdots,k-1$. This completes the proof.

□

3.2 Triangular Grids

We need not restrict our attention to rectangular grids. In particular, in n dimensions, we can consider a triangular grid consisting of lattice points $(x_1,\cdots,x_n)$ with nonnegative coordinates satisfying $x_1+\cdots+x_n\le c$ for some fixed constant c. In the case of $c=1$, this is just a simplex and any choice of all but one of the points lie on a common hyperplane.

We define the set $T_1(d,n)$ as $\{(x_1,\cdots,x_n)\in\mathbb{Z}^n_{\ge 0}\mid x_1+\cdots+x_n\le d-1\}$ so that each outer edge of $T_1(d,n)$ has d points. In this section, we let $t_1(d,n)$ represent the minimum number of affine hyperplanes needed to cover *every* point of T_1, $t_1^*(d,n)$

represent the minimum needed for the fractional version of this covering question, and $m_1(d,n)$ represent the minimum needed to leave exactly one point uncovered. We note that $m_1(d,n)$ is well-defined as the number of affine hyperplanes needed will not depend on the choice of which point is left uncovered. Note that for the covering problems we study, any set of points that can be obtained from such a $T_1(d,n)$ via an affine transformation will behave the same way.

In Subsection 3.2.1, we determine $t_1(d,n)$ as a step toward finding $m_1(d,n)$. In Subsection 3.2.2, we note that questions about covering every point of a triangular grid are interesting in their own right. We address the problem of computing $t_1^*(d,2)$ and also consider questions involving higher covering multiplicity for $T_1(d,2)$ and for a related grid.

A fundamental limitation of triangular grids is that unlike rectangular grids, they cannot be thought of as a product of sets. This limits our ability to use Combinatorial Nullstellensatz.

3.2.1 With a Point Uncovered

Proposition 3.2.1. *$t_1(d,n) = d$.*

Proof. We can cover $T_1(d,n)$ using the d hyperplanes $x_1 = 0, x_1 = 1, \cdots, x_1 = d-1$ so $t_1(d,n) \leq d$.

Note that $t_1(d,1) = d$ since each of the d points needs to be covered but the hyperplanes are single points. $t_1(1,n) = 1$ since the single point needs one hyperplane to cover it. Now we proceed by induction on n. One "face" of the lattice $T_1(d,n)$ is an $n-1$ dimensional triangular lattice with d points along a side. A face is either the intersection of $T_1(d,n)$ with the hyperplane $x_i = 0$ for some i or the intersection of $T_1(d,n)$ with $x_1 + \cdots + x_n = d-1$. Each face is a copy of $T_1(d,n-1)$ embedded in n-dimensional space.

Choose a face of $T_1(d,n)$ and suppose we do not use the hyperplane, H, containing this entire face. Then the $(n-1)$-dimensional hyperplanes we do use will have $(n-2)$-dimensional intersections with this face. The fewest number of hyperplanes needed to cover this face is then $t_1(d, n-1)$, which is d by induction. Thus, to have any chance of getting less than d, we have to use this face.

Now suppose that we do use H. It does not cover any other grid points outside of this face so we still need to cover an n-dimensional triangular lattice with $d-1$ points along the longest side. That is, the set of grid points not covered by H is a copy of $T_1(d-1, n)$. By induction on d, this requires $d-1$ hyperplanes to cover and in conjunction with H, that gives a total of at least d. □

Proposition 3.2.2. $m_1(d, n) = d-1$ *regardless of which grid point is left uncovered.*

Proof. Note that $d-1$ hyperplanes are sufficient. If $(a_1, \cdots, a_n)$ is the forbidden point, we can cover everything else using hyperplanes of the form $x_i = k$ for $k = 0, 1, \cdots, a_i - 1$ for every $i = 1, \cdots, n$, along with $x_1 + \cdots + x_n = k$ for $k = a_1 + \cdots + a_n + 1, \cdots, d-1$. Any grid point $(b_1, \cdots, b_n)$ with $b_i < a_i$ for some $i = 1, \cdots, n$ is covered by a hyperplane of the first type and any remaining grid point besides $(a_1, \cdots, a_n)$ is covered by a hyperplane of the second type. This construction uses a total of $a_1 + \cdots + a_n + (d - 1 - (a_1 + \cdots + a_n)) = d - 1$ hyperplanes.

If $d = 2$, the lattice $T_1(d, n)$ is a simplex and once one point is removed, the remainder can be covered with one hyperplane, so $m_1(1, n) = 1$. Now we induct on d.

For $d \geq 3$, regardless of which point is left uncovered, there is some face of $T_1(d, n)$, isomorphic to a copy of $T_1(d, n-1)$, that does not contain it. If we use the hyperplane, H, containing that entire face, then we are left with a n-dimensional lattice with $d-1$ points along the longest side with one forbidden point. By induction, that requires an additional $m_1(d-1, n) = d-2$ hyperplanes to almost cover, so we could not use fewer than $1 + (d-2) = d-1$ affine hyperplanes when H is included.

If we do not use H, we need to find some other list of hyperplanes that cover all grid points contained in H. Since the face of $T_1(d,n)$ contained in H is isomorphic to a copy of $T_1(d,n-1)$ and the intersection of each such hyperplane with that face is $(n-2)$-dimensional, the minimum number of hyperplanes needed is $t_1(d,n-1)=d$, which is already too many. Therefore, there is no way to beat the construction which gave $d-1$. □

3.2.2 With All Points Covered

For a rectangular grid, the question of finding the minimum number of affine hyperplanes which cover all points is not interesting. Assuming that S_n is the smallest of the sets $S_1,\cdots,S_n$, we can cover all points by using the hyperplanes $x_n=s$ for each $s\in S_n$. No hyperplane contains more grid points than these and every grid point lies on some hyperplane which covers this maximum possible number of points. For triangular grids, this is not the case. For example, there exist lines that cover four points from $T_1(4,2)$, yet the grid point $(1,1)$ is not covered by any of those lines. Since we can no longer simply use the hyperplanes which each cover the most points, the question of finding the minimum number of affine hyperplanes which cover all points is interesting in its own right.

In the previous section, we already determined the minimum number needed to cover every point of $T_1(d,n)$ at least once. In the 2-dimensional setting, we will consider the fractional version of this problem to compute $t_1^*(d,2)$. We will also examine the problem of covering each point of $T_1(d,n)$ more than once. We define $t_1(d,n,k)$ as the minimum number of affine hyperplanes needed to cover every point of $T_1(d,n)$ at least k times. Note that $t_1(d,n,1)=t_1(d,n)=d$ and that $t_1(d,n,k)\geq t_1^*(d,n,k)=kt_1^*(d,n)$.

Theorem 3.2.3. $t_1^*(3j+1,2)=2j+1$ *for all integers* $j\geq 0$.

Proof. For $j=0$, there is a single point to cover which requires a single line. Otherwise, $j\geq 1$. $T_1(3j+1,2)$ is $\{(x,y)\mid x,y\geq 0, x+y\leq 3j\}$. We can cover all these points with the following lines:

- $x = i$ for $i = 0, \cdots, 2j-1$ with weight $\frac{2j-i}{3j}$,
- $y = i$ from $i = 0, \cdots, 2j-1$ with weight $\frac{2j-i}{3j}$, and
- $x + y = 3j - i$ from $i = 0, \cdots, 2j-1$ with weight $\frac{2j-i}{3j}$.

If $i_1, i_2 \leq 2j-1$, then $(i_1, i_2) \in T_1(3j+1, 2)$ is covered with weight $\frac{2j-i_1}{3j}$ by a vertical line and weight $\frac{2j-i_2}{3j}$ by a horizontal line for a total weight of at least $\frac{4j-i_1-i_2}{3j}$. If $i_1 + i_2 \leq j$, this is already at least 1. If instead $i_1 + i_2 > j$, we have that (i_1, i_2) is covered by the line $x + y = i_1 + i_2$ with weight $\frac{i_1+i_2-j}{3j}$, for a total weight of $\frac{4j-i_1-i_2}{3j} + \frac{i_1+i_2-j}{3j} = 1$.

It is not possible for both i_1 and i_2 to exceed $2j-1$ for a point (i_1, i_2) in $T_1(3j+1, 2)$. Without loss of generality, we may now assume that $i_1 \leq 2j-1$ and $i_2 \geq 2j$. Then (i_1, i_2) is covered with weight $\frac{2j-i_1}{3j}$ by a vertical line and weight $\frac{i_1+i_2-j}{3j}$ by a diagonal line for a total weight of at least $\frac{2j-i_1+i_1+i_2-j}{3j} = \frac{j+i_2}{3j} \geq 1$.

Thus, this is a valid covering and the total weight of all of these lines is

$$\begin{aligned} \frac{3}{3j} \sum_{i=0}^{2j-1} (2j - i) &= \frac{1}{j} \left(\frac{(2j)(2j+1)}{2} \right) \\ &= 2j + 1, \end{aligned}$$

which establishes the upper bound.

To determine the lower bound, we will assign weights to all of the points of $T_1(3j+1, 2)$ in such a way that no line covers points of total weight more than 1. Thus $t_1^*(3j+1, 2)$ will be at least the sum of the weights assigned to the points.

In fact, many points will be assigned a weight of 0. The nonzero weights are assigned to points belonging to the hexagons $X_1, X_2, \cdots, X_j$ defined in the following manner:

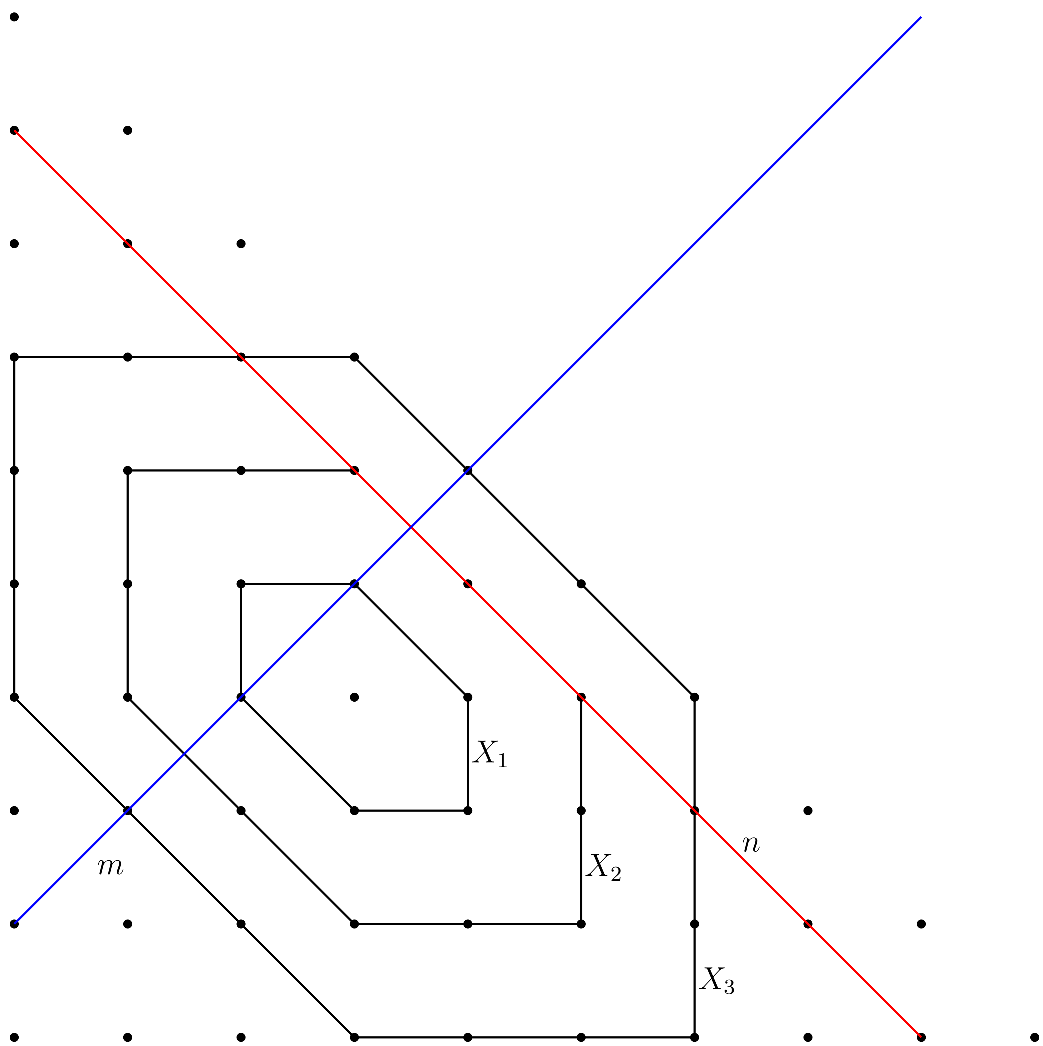

Figure 3.1: Weights for $T_1(10, 2)$

The grid points in X_1 are assigned weight $\frac{1}{12}$, those in X_2 are assigned weight $\frac{2}{12}$, and those in X_3 are assigned weight $\frac{3}{12}$. The remaining points receive weight 0.

The line m is not parallel to any sides of a hexagon, so it intersects each X_i in at most two points. The line n has the same slope as some hexagon sides. In this case, it contains an entire side of X_2, which means it cannot contain any points of X_1 nor more than two points of X_3.

X_i consists of the grid points lying on the boundary of the hexagon formed by the lines $x = j - i, x + y = 2j - i, y = j - i, x = j + i, x + y = 2j + i$, and $y = j + i$. Note that grid points that lie on one or more of these lines but not on the sides of the hexagon are not included in X_i. Figure 3.1 shows an example for $j = 3$.

For $i = 1, \cdots, j-1$, the hexagon X_i is nested inside X_{i+1}. Since no grid point belongs to multiple hexagons, we can unambigously assign a weight of $\frac{i}{j(j+1)}$ to each grid point of X_i and a weight of 0 to any grid point that does not belong to any hexagon.

The hexagon X_i has $i + 1$ grid points along a side for a total of $6(i + 1) - 6 = 6i$ points. Thus, the total weight of all the points in the grid is

$$\begin{aligned} \sum_{i=1}^{j} 6i\left(\frac{i}{j(j+1)}\right) &= \frac{1}{j(j+1)} \sum_{i=1}^{j} 6i^2 \\ &= \frac{1}{j(j+1)} \left(j(j+1)(2j+1)\right) = 2j + 1. \end{aligned}$$

It now remains to show that with this weighting, no line passes through points of total weight more than 1. The sides of the hexagons X_i all are vertical, horizontal, or have slope -1. A line with any other slope will intersect each hexagon in at most two points. It will thus cover points of total weight at most

$$\begin{aligned} 2\sum_{i=1}^{j} \frac{i}{j(j+1)} &= \frac{2}{j(j+1)} \sum_{i=1}^{j} i \\ &= \frac{2}{j(j+1)} \left(\frac{j(j+1)}{2}\right) = 1. \end{aligned}$$

For a line that is vertical, horizontal, or has slope -1, the only way it can potentially cover more than two points from a single hexagon is if it contains an entire side of some hexagon X_a. It would then contain no points of X_i for $i < a$ but could contain up to two points for each of $X_{a+1}, \cdots, X_j$. Thus, it would cover points of total weight at most

$$(a+1)\left(\frac{a}{j(j+1)}\right)+2\sum_{i=a+1}^{j}\frac{i}{j(j+1)}=\frac{1}{j(j+1)}\left(a(a+1)+2(j-a)(\frac{a+1+j}{2})\right)$$
$$=\frac{1}{j(j+1)}\left(a^2+a+(j-a)(a+1+j)\right)$$
$$=\frac{1}{j(j+1)}\left(a^2+a+ja+j+j^2-a^2-a-aj\right)=1.$$

Thus, no line can cover points of total weight more than 1 and our lower bound matches the upper bound. □

In fact, we conjecture a general formula for $t_1^*(d,2)$:

Conjecture 3.2.4. *For integers $j \geq 0$, $t_1^*(3j+2,2)=2j+1+\frac{2j+1}{3j+2}$ and $t_1^*(3j+3,2)=2j+2+\frac{j+1}{3j+4}$.*

We revisit this later as it appears to have a non-obvious connection to the problem of covering every point of $T_1(d,2)$ with multiplicity at least k.

We can demonstrate the upper bound of Conjecture 3.2.4, via the following constructions. For $d=3j+2$, we can use the lines:

- $x=i$ for $i=0,\cdots,2j$ with weight $\frac{2j+1-i}{3j+2}$,
- $y=i$ for $i=0,\cdots,2j$ with weight $\frac{2j+1-i}{3j+2}$, and
- $x+y=3j+1-i$ for $i=0,\cdots,2j$ with weight $\frac{2j+1-i}{3j+2}$.

For $d=3j+3$, we can use the lines:

- $x=i$ for $i=0,\cdots,2j+1$ with weight $\frac{2j+2-i}{3j+4}$,
- $y=i$ for $i=0,\cdots,2j+1$ with weight $\frac{2j+2-i}{3j+4}$, and
- $x+y=3j+2-i$ for $i=0,\cdots,2j+1$ with weight $\frac{2j+2-i}{3j+4}$.

We can also prove some exact results for the integral covering problem and low multiplicity.

Theorem 3.2.5. *For $d \geq 2$, we have that*

$$t_1(d,2,k) = \begin{cases} \lfloor \frac{3d+1}{2} \rfloor, & \text{if } k = 2, \\ 3d, & \text{if } k = 4. \end{cases}$$

Proof. First we consider the case of covering twice. We begin by demonstrating a construction to show that $\lfloor \frac{3d+1}{2} \rfloor$ lines are sufficient to cover every point of $T_1(d,2)$ at least twice. $T_1(d,2)$ consists of the points $\{(x,y) \in \mathbb{Z}^2_{\geq 0} \mid x+y \leq d-1\}$, so we can use the following lines:

For even d:

- $x = i$ for $i = 0, 1, \cdots, d/2 - 1$
- $y = i$ for $i = 0, 1, \cdots, d/2 - 1$
- $x + y = i$ for $i = d/2, d/2 + 1, \cdots, d - 1$

For odd d:

- $x = i$ for $i = 0, 1, \cdots, \frac{d-1}{2}$
- $y = i$ for $i = 0, 1, \cdots, \frac{d-1}{2}$
- $x + y = i$ for $i = \frac{d-1}{2} + 1, \frac{d-1}{2} + 2, \cdots, d - 1$

To verify that these are valid constructions, we first note that these constructions contain $\lfloor \frac{3d+1}{2} \rfloor$ lines as claimed. We must check that each point of $T_1(d,2)$ is in fact covered the requisite number of times. Any point of $T_1(d,2)$ has either $x < d/2$ or $y < d/2$. If both of these hold, it is covered twice using vertical and horizontal lines. If only one of these inequalities holds, then $x + y \geq d/2$, so (x,y) is also covered by a diagonal line and still covered at least twice.

For the lower bound, note that for $d = 2$, we have three points that need to be covered twice each and we can only cover two at a time. This necessitates the use of three lines, which will suffice. We now induct on d to establish the lower bound.

Suppose that $t_1(m, 2, 2) = \lfloor \frac{3m+1}{2} \rfloor$ for some $m \geq 2$. Let us consider the case of $d = m + 1$. If some line corresponding to a side of the triangle ($x = 0$, $y = 0$, or $x + y = d - 1$) is used with multiplicity two, then there is still a triangular grid with m points along each side, isomorphic to $T_1(m, 2)$, where no points have been covered. This requires an additional $t_1(m, 2, 2)$ lines to cover for a total of $\lfloor \frac{3m+1}{2} \rfloor + 2$. If no line corresponding to a side of a triangle is used more than once, there are $3m - 3$ (all the non-corners) points remaining on the sides of the triangle that need to be covered an additional time. No remaining line covers more than two of these at a time, so this forces us to use at least $\lceil \frac{3m-3}{2} \rceil$ additional lines, so at least $\lceil \frac{3m-3}{2} \rceil + 3$ in total, if each of $x = 0$, $y = 0$, and $x + y = d - 1$ is used exactly once. If one of those three lines is never used, then separate lines are needed to cover the $m + 1$ points along the corresponding side of the triangle twice each, for a total of at least $2(m + 1)$, which is at least $\frac{3m}{2} + 3$.

Thus, comparing across all cases,

$$t_1(m + 1, 2, 2) \geq \min\{\lfloor \frac{3m + 1}{2} \rfloor + 2, \lceil \frac{3m - 3}{2} \rceil + 3\}$$

This gives a lower bound of $\frac{3m}{2} + 2$ when m is even and $\frac{3m+3}{2}$ when m is odd. Thus, $t_1(m + 1, 2, 2) \geq \lfloor \frac{3m}{2} \rfloor + 2 = \lfloor \frac{3(m+1)+1}{2} \rfloor$, completing the induction.

Next we do the case of covering four times. We begin with a construction involving $3d$ lines which works for $d \geq 2$. Note that for $d = 2$ or $d = 4$, we have that $\lfloor \frac{d-1}{3} \rfloor + 1 > \lfloor \frac{2d}{3} \rfloor - 1$, so three of these classes will be empty.

- $x = i$ twice for $i = 0, 1, \cdots, \lfloor \frac{d-1}{3} \rfloor$
- $x = i$ once for $i = \lfloor \frac{d-1}{3} \rfloor + 1, \lfloor \frac{d-1}{3} \rfloor + 2, \cdots, \lfloor \frac{2d}{3} \rfloor - 1$

- $y = i$ twice for $i = 0, 1, \cdots, \lfloor \frac{d-1}{3} \rfloor$
- $y = i$ once for $i = \lfloor \frac{d-1}{3} \rfloor + 1, \lfloor \frac{d-1}{3} \rfloor + 2, \cdots, \lfloor \frac{2d}{3} \rfloor - 1$
- $x + y = i$ twice for $i = d - 1 - \lfloor \frac{d-1}{3} \rfloor, d - 1 - \lfloor \frac{d-1}{3} \rfloor + 1, \cdots, d - 1$
- $x + y = i$ once for $i = d - \lfloor \frac{2d}{3} \rfloor, d - \lfloor \frac{2d}{3} \rfloor + 1 \cdots, d - 2 - \lfloor \frac{d-1}{3} \rfloor$

This construction contains $3\left(2(\lfloor \frac{d-1}{3} \rfloor + 1) + \lfloor \frac{2d}{3} \rfloor - \lfloor \frac{d-1}{3} \rfloor - 1\right) = 3d$ lines. We can now follow a process similar to that used in the partial proof of Conjecture 3.2.6 to verify that each point of $T_1(d, 2)$ is actually covered at least four times.

For $d = 2$, note that $3d = 6$ lines are necessary since there are three points to cover four times each and no line can cover more than two at a time. Now, we will establish the lower bound by inducting on d. Suppose that there exists some value of $m \geq 2$ for which $t_1(m, 2, 4) = 3m$. We now examine $T_1(m + 1, 2)$.

Consider some line that coincides with a side of $T_1(m + 1, 2)$, either $x = 0$, $y = 0$, or $x + y = m$. There is a triangular grid with m points along the each side, and thus isomorphic to $T_1(m, 2)$, for which none of the points are covered by this line. By induction, it takes $3m$ lines to cover this smaller triangular grid, so to have any chance of beating $3m + 3$, we would not be able to use the same line incident with a side of the triangle three times. Since each of $x = 0, y = 0$, and $x + y = m$ is used at most twice, there are $3m - 3$ points (non-corners) on the sides of the triangle which still need to be covered at least twice more. The remaining lines can only cover up to two of these points at a time. This requires at least $\frac{(3m-3)(2)}{2} = 3m - 3$ lines beyond the six coinciding with sides of a triangle. This again gives a lower bound of $(3m - 3) + 6 = 3m + 3$. (Using some line incident with a side of the triangle less than twice would require even more lines since we would be committing to covering two boundary points at a time even when there was still a chance to cover $m + 1$.) $\square$

Based on data for $d \leq 21$ (see Appendix A), we conjecture that $t_1(d, 2, 3) = \lfloor \frac{9d+3}{4} \rfloor$ as well as asymptotic behavior for general k.

Conjecture 3.2.6. *For $k \geq 1$, $t_1(d,2,k) = (t_1^*(k,2))d + O_d(1)$.*

The presence of $t_1^*(k,2)$ suggests a connection between the fractional problem on a grid of size k and the k-covering problem on a grid of any size. Conditional on the veracity of Conjecture 3.2.4, we can account for one direction of this correspondence.

Proof. For large d, we can cover every point k times by "lifting" the constructions given in the upper bounds for $t_1^*(k,2)$. As in those constructions, we will only use lines of the form $x = i$, $y = i$, and $x+y = i$ for $i = 0, \cdots, d-1$. In particular, among such lines, the ones which cover the most grid points will get used more times and the ones that cover the fewest grid points will not get used at all.

We will split the lines $x = 0, \cdots, x = d-1$ we utilize into sets, B_r, that are equal in size and assign the same weight to each line in the set, in proportion to the weights used in our fractional cover construction. We then make an analogous partition of the lines we use from $y = 0, \cdots, y = d-1$ and from $x+y = 0, \cdots, x+y = d-1$, adding those to the correct B_r. The construction varies slightly based on $k \pmod 3$.

For $k = 3j+1$, the first set B_1 consists of the lines $x = i, y = i$, and $x+y = d-1-i$ for $i = 0, \cdots, \lceil \frac{d}{3j} \rceil - 1$. Each subsequent B_r for $r = 2, \cdots, 2j$ consists of the lines $x = i, y = i$, and $x+y = d-1-i$ for $i = (r-1)\lceil \frac{d}{3j} \rceil, \cdots, r\lceil \frac{d}{3j} \rceil - 1$. The remaining lines parallel to $x = 0, y = 0$, and $x+y = 0$ will receive weight 0 so they do not need to be classified. For $r = 1, \cdots, 2j$, every line in B_r will be used $2j+1-r$ times. In total, we use $3(2j)\lceil \frac{d}{3j} \rceil$ lines an average of $\frac{2j+1}{2}$ times each for a total of at most $6j(\frac{d}{3j}+1)(\frac{2j+1}{2}) = (2j+1)d + 3j(2j+1)$ lines.

For $k = 3j+2$, the first set B_1 consists of the lines $x = i, y = i$, and $x+y = d-1-i$ for $i = 0, \cdots, \lceil \frac{d}{3j+2} \rceil - 1$. Each subsequent B_r for $r = 2, \cdots, 2j+1$ consists of the lines $x = i, y = i$, and $x+y = d-1-i$ for $i = (r-1)\lceil \frac{d}{3j+2} \rceil, \cdots, r\lceil \frac{d}{3j+2} \rceil - 1$. The remaining lines parallel to $x = 0, y = 0$, and $x+y = 0$ will receive weight 0 so they do not need to be classified. For $r = 1, \cdots, 2j+1$, every line in B_r will be used $2j+2-r$

times. In total, we use $3(2j+1)\lceil\frac{d}{3j+2}\rceil$ lines an average of $j+1$ times each for a total of at most $3(2j+1)(\frac{d}{3j+2}+1)(j+1) = (2j+1+\frac{2j+1}{3j+2})d + 3(2j+1)(j+1)$ lines.

For $k = 3j+3$, the first set B_1 consists of the lines $x = i, y = i$, and $x+y = d-1-i$ for $i = 0, \cdots, \lceil\frac{d}{3j+4}\rceil - 1$. Each subsequent B_r for $r = 2, \cdots, 2j+2$ consists of the lines $x = i, y = i$, and $x+y = d-1-i$ for $i = (r-1)\lceil\frac{d}{3j+4}\rceil, \cdots, r\lceil\frac{d}{3j+4}\rceil - 1$. The remaining lines parallel to $x = 0, y = 0$, and $x+y = 0$ will receive weight 0 so they do not need to be classified. For $r = 1, \cdots, 2j+2$, every line in B_r will be used $2j+3-r$ times. In total, we use $3(2j+2)\lceil\frac{d}{3j+4}\rceil$ lines an average of $\frac{2j+3}{2}$ times each for a total of at most $3(2j+2)(\frac{d}{3j+4}+1)(\frac{2j+3}{2}) = (2j+2+\frac{j+1}{3j+4})d + 3(j+1)(2j+3)$ lines.

We have shown that these constructions are the right size and it remains to show that they actually cover every point of $T_1(d,2)$ at least k times. Consider a point (i_1, i_2) where $x = i_1$ is in the set B_{r_1} and $y = i_2$ is in the set B_{r_2}. Then horizontal and vertical lines cover this point $2k - 2j - (r_1 + r_2)$ times. This is already enough times unless $r_1 + r_2 > k - 2j$. In that case, $i_1 + i_2$ is at least $(r_1 + r_2 - 2)\lceil\frac{d}{2k-3j-2}\rceil$ so we have that $0 \leq d - 1 - (i_1 + i_2) \leq d - 1 - (r_1 + r_2 - 2)\lceil\frac{d}{2k-3j-2}\rceil \leq (2k - 3j - r_1 - r_2)\lceil\frac{d}{2k-3j-2}\rceil - 1$. This means that the line $x+y = i_1+i_2$ is in B_r for some $r \leq 2k - 3j - r_1 - r_2 \leq k - j - 1$ and is thus used at least $k - j - (2k - 3j - r_1 - r_2) = 2j - k + r_1 + r_2$ times. Thus, (i_1, i_2) is covered at least k times.

By symmetry, this also covers the cases where $x + y = i_1 + i_2$ and at least one of $x = i_1$ and $y = i_2$ are used a nonzero number of times. However, it turns out that every point of $T_1(d,2)$ is included by these cases. It is impossible for $x = i_1$ and $y = i_2$ to both be used zero times since this would give $x + y = i_1 + i_2 \geq (2k - 2j - 2)\lceil\frac{d}{2k-3j-2}\rceil \geq d$. If exactly one of $x = i_1$ or $y = i_2$ is used zero times, we have $\max\{i_1, i_2\} \geq (k - j - 1)\lceil\frac{d}{2k-3j-2}\rceil$. Then $d - 1 - (i_1 + i_2) \leq d - 1 - (k - j - 1)\lceil\frac{d}{2k-3j-2}\rceil \leq (2k - 3j - 2 - (k - j - 1))\lceil\frac{d}{2k-3j-2}\rceil - 1 = (k - 2j - 1)\lceil\frac{d}{2k-3j-2}\rceil - 1 \leq (k - j - 1)\lceil\frac{d}{2k-3j-2}\rceil - 1$, meaning that $x + y = i_1 + i_2$ is used a nonzero number of times.

□

Additionally, we can consider a slightly different triangular grid. We define the set $T_2(d,n)$ as $\{(x_1,\cdots,x_n)\in\mathbb{Z}^n_{\geq 0}\mid \frac{x_1}{2}+x_2+x_3+\cdots+x_n\leq d-1\}$. In two dimensions, this is the set of points $\{(x,y)\in\mathbb{Z}^2_{\geq 0}\mid \frac{x}{2}+y\leq d-1\}$, which is a triangular grid of d rows such that the i^{th} row has $2i-1$ grid points. While the lines that contained the most points in $T_1(d,2)$ could be of the form $x=i$, $y=i$, or $x+y=i$, there are some horizontal lines that contain substantially more points of $T_2(d,2)$ than any other line. This suggests that a minimal set of lines which covers the points of $T_2(d,2)$ at least k times each might consist mainly of k copies of each line $y=i$ for $i=0,\cdots,d-1$ (with some slight modifications such as accounting for $y=d-1$ being a useless inclusion which only covers one point). Indeed, if we define $t_2(d,n,k)$ to be the smallest number of affine hyperplanes needed to cover every point of $T_2(d,n)$ at least k times, the data (see Appendix A) appears to bear this out.

Based on the values of $t_2(d,2,k)$ computed for $k\leq 6$ and $d\leq 21$, we make the following conjecture.

Conjecture 3.2.7. *For $k\geq 1$, $t_2(d,2,k)=kd+O_d(1)$.*

We verify this conjecture for $k=1,2,3$.

Theorem 3.2.8.

$$t_2(d,2,k)=\begin{cases} d, & \text{if } k=1,\\ 2d, & \text{if } k=2,\\ 3d-1, & \text{if } k=3,\ \text{for } d\geq 2.\end{cases}$$

Proof. Covering each row with a horizontal line requires d lines. We see that $t_2(1,2,1)=1$ and can show $t_2(d,2,1)=d$ by induction. If we don't cover the bottom row, where $y=0$, with a horizontal line, then we need $2d-1$ separate lines to cover each of those points, so it is more efficient to use a horizontal line which gives $t_2(d,2,1) \geq 1 + t_2(d-1,2,1) = 1 + d - 1 = d$.

For covering twice, using each horizontal line $y = i$ twice for $i = 0, \cdots, d-1$ gives a construction with $2d$ lines. We see that $t_2(1,2,2) = 2$ and can show $t_2(d,2,1) = 2d$ by induction. If we don't cover the bottom row twice with horizontal lines, then we need 1 horizontal line and $2d-1$ additional separate lines which is already $2d$ or $2(2d-1)$ separate lines, which is at least $2d$. Thus, there is no way to do better than $2 + t_2(d-1,2,2) = 2 + 2(d-1) = 2d$.

We can cover 3 times using all the horizontal lines $y = i$ three times for $i = 0, \cdots, d-3$. We can then use $y = d-2$ only twice. We can then cover each point with $y = d-2$ an additional time by using a line that passes through that point and the last point $(0, d-1)$. This gives an upper bound of $3d-1$.

$t_2(2,2,3) = 5$ since using the only line, $y = 0$, which covers three points forces us to use three more lines to cover the top point, $(0,1)$. Thus, to potentially beat 5, we can only use $y = 0$ once and have to cover a total of nine more points (counting multiplicity) but can cover only two at a time. We now proceed by induction on d.

In an optimal cover, the lines $y = i$ for $i = 0, \cdots, t-1$ are used at least twice for some $d-1 \geq t \geq 0$ and the line $y = t$ is used at most once. Such a t exists since there is no incentive to use the top row $y = d-1$ which would only cover one point. With the lines $y = i$ for $i = 0, \cdots, t-1$, the top $d-t$ rows are not yet covered at all. The lowest of these rows has $2d-2t-1$ points that each need to be covered an additional two times by lines other than $y = t$. Thus, at least $2(2d-2t-1)+1$ more lines are used in additional to the $2t$ lines $y = i$ for $i = 0, \cdots, t-1$. We are trying to use less than $3d-1$ lines so we require that

$$\begin{aligned} 2(2d-2t-1)+1+2t &\leq 3d-2 \\ 4d-2t-1 &\leq 3d-2 \\ d+1 &\leq 2t \end{aligned}$$

This means at least $d+1$ horizontal lines are used. By induction, we cannot beat $3d-1$ if we use the line, $y=0$, containing the bottom row three times, so there are $2d-1$ points in the bottom row that need to be covered separately by non-horizontal lines. This gives at least $(d+1)+(2d-1)=3d$ lines total. Therefore, we cannot beat $3d-1$. □

Chapter 4

Results in Arithmetic Ramsey Theory

4.1 Superpolynomial Growth for $\Delta(D, k; 2)$

When a set $D \subset \mathbb{Z}_{>0}$ is r-accessible, we may study the function $\Delta(D, k; r)$ which denotes the smallest n for which every r-coloring of $[n]$ contains a monochromatic D-diffsequence of length k. There are numerous examples where $\Delta(D, k; r)$ grows polynomially in k.

Example 6. Suppose that D consists of all multiples of m for some fixed $m \in \mathbb{Z}_{>0}$. Then, D is r-accessible for all $r \in \mathbb{Z}_{>0}$, and

$$\Delta(D, k; r) \leq rm(k-1) + 1.$$

Proof. In $[rm(k-1)+1]$, there are $r(k-1)+1$ numbers that are $1 \pmod m$. By Pigeonhole Principle, at least k of these are the same color. Any sequence contained within a residue class $\pmod m$ is such that the consecutive gaps are multiples of m and thus lie in D. $\square$

Example 7. [CCLS18] For a fixed $m \geq 3$, suppose that D consists of all non-multiples of m. Then, D is 2-accessible and for $k \geq 2$,

$$\Delta(D, k; 2) = 2k - 1 + 2\lfloor \frac{k-2}{m-2} \rfloor.$$

Example 8. [Lan97] Let S be the set of odd positive integers and let $D = \{2\} \cup S$. Then, D is 3-accessible and

$$\Delta(D, k; 3) \leq 6k^2 - 13k + 6.$$

The key result of this section is to demonstrate that $\Delta(D, k; r)$ does not always need to be polynomial in terms of k. In particular, we will show that when $r = 2$, and D consists of all powers of 2, we have $\Delta(D, k; 2) = 2^{\Omega(\sqrt{k})}$.

Landman and Robertson [LR03] previously demonstrated that this choice of D is 2-accessible and determined an upper bound of $\Delta(D, k; 2) \leq 2^k - 1$, as well as a linear lower bound. Chokshi, Clifton, Landman, and Sawin [CCLS18] improved the lower bound to quadratic and conjectured the existence of a superpolynomial lower bound, which we confirm [Cli21].

Theorem 4.1.1. *When* $D = \{2^i \mid i \in \mathbb{Z}_{\geq 0}\}$,

$$\Delta(D, k; 2) \geq 2^{\sqrt{2k}} \left(\frac{(\sqrt{2}-1)k}{8} - \frac{\sqrt{k}}{8} \right) + \frac{\sqrt{k}}{2}.$$

In order for Theorem 4.1.1 to be meaningful, we must demonstrate that $D = \{2^i \mid i \in \mathbb{Z}_{\geq 0}\}$ is actually 2-accessible so that the function $\Delta(D, k; 2)$ is defined for all $k \in \mathbb{Z}_{>0}$. For completeness, we reproduce the induction argument of Landman and Robertson [LR03].

Claim 4.1.2. *Let $D = \{2^i \mid i \in \mathbb{Z}_{\geq 0}\}$. Then D is 2-accessible; in particular,*

$$\Delta(D, k; 2) \leq 2^k - 1.$$

Proof. $\Delta(D, 1; 2) = 1$ so the claim is true for $k = 1$. We now assume that it holds for $k = n$ and proceed by induction.

Consider an arbitrary 2-coloring of $\mathbb{Z}_{>0}$ using colors 0 and 1. Suppose that $a_1, \cdots, a_n$ is a monochromatic (in color 0) D-diffsequence with $a_n \leq 2^n - 1$. Then, $a_n + 1, a_n + 2, \cdots, a_n + 2^i, \cdots, a_n + 2^n$ are all at most $2^{n+1} - 1$. In order to avoid extending the monochromatic color 0 diffsequence of length n to one of length $n + 1$ in $[2^{n+1} - 1]$, we require that $a_n + 2^i$ is color 1 for $i = 0, 1, \cdots, n$.

However, $(a_n + 2^{i+1}) - (a_n + 2^i) = 2^i$ for $i = 0, 1, \cdots, n - 1$. Since the consecutive gaps are all powers of 2, these form a monochromatic D-diffsequence (in color 1) contained in $[2^{n+1} - 1]$. Thus, we are guaranteed a monochromatic D-diffsequence of length at least $n + 1$ in $[2^{n+1} - 1]$, completing the induction. □

We will now prove Theorem 4.1.1 by utilizing a series of periodic colorings to obtain the lower bound. The first set of colorings considered does not quite yield the desired bound but inspires a series of more refined colorings which do.

Let P_1 be a periodic coloring modulo two with a repeating block of “10”. This means that odd numbers are assigned color 1 and even numbers are assigned color 0.

Let P_2 be a periodic coloring modulo four with a repeating block of “1001”. This means numbers that are 1 or 4 (mod 4) are color 1 while numbers that are 2 or 3 (mod 4) are color 0.

P_3 will be a periodic coloring modulo eight with a repeating block of “10010110”.

Continuing in this manner, we define P_t as a periodic coloring with a repeating block of size 2^t. The first half of the repeating block is the repeating block of P_{t-1} while the second half is the complement of the first half. Thus, by construction, any two numbers x and $x + 2^{t-1}$ have different colors with respect to P_t.

The repeating block of P_t can be thought of as the first 2^t bits of the Thue-Morse sequence. (Note that P_3 is the periodic coloring that Landman and Robertson used to obtain their initial linear lower bound in [LR03].)

A monochromatic sequence $a_1 < a_2 < \cdots < a_k$ with respect to the coloring P_t has no gaps, $a_{i+1} - a_i$, of size 2^{t-1}. Furthermore, we can obtain an upper bound for the number of gaps of each size less than 2^{t-1} that depends only on the coloring used and not on the length, k, of the D-diffsequence:

Lemma 4.1.3. *For $m = 0, 1, \cdots, t-2$, there are at most $m+1$ gaps of size 2^m in a monochromatic (with respect to P_t) D-diffsequence.*

Proof. Consider a monochromatic D-diffsequence with respect to P_t. For each $m = 0, 1, \cdots, t-2$, we can split each repeating block of size 2^t into 2^{t-m-1} sub-blocks of size 2^{m+1}. Each sub-block resembles a repeating block for P_{m+1}, possibly with the colors swapped.

If $a_{i+1} - a_i = 2^m$, then by construction, a_i cannot be in the first half of a sub-block of size 2^{m+1}. So for any gap $a_{i+1} - a_i$ of size 2^m, $a_i \in \{2^m + 1, 2^m + 2, \cdots, 2^{m+1}\}$ (mod 2^{m+1}). We say that a_i is in the *second half* mod 2^{m+1} while a_{i+1} is in the *first half.*

If the gaps $a_{i+2} - a_{i+1}, a_{i+3} - a_{i+2}, \cdots, a_j - a_{j-1}$ are all larger than 2^m, then a_j is in the first half mod 2^{m+1} and thus $a_{j+1} - a_j$ cannot be a gap of size 2^m. This means that in between every two gaps of the same size, there is at least one gap of a smaller size. Since there is no way to have a gap of size smaller than 1, there is at most one gap of size 1.

We will now prove the lemma by induction. Suppose that there exists some $1 \leq q \leq t-2$ such that there are at most $m+1$ gaps of size 2^m for $m = 0, 1, 2, \cdots, q-1$. We will now show that there are at most $q+1$ gaps of size 2^q.

Assume for the sake of contradiction that there are at least $q+2$ gaps of size 2^q.

The first $q+2$ of these are the gaps between a_{i_j} and a_{i_j+1} for some indices i_j with $j = 1, 2, \cdots, q+2$. For $j = 1, 2, \cdots, q+1$, the sequence of gaps $a_{i_j+2} - a_{i_j+1}, a_{i_j+3} - a_{i_j+2}, \cdots, a_{i_{j+1}} - a_{i_{j+1}-1}$, consists of at least one power of 2 smaller than 2^q, as well as possibly some powers of 2 larger than 2^q, which we can ignore, as they are 0 mod 2^{q+1}. For convenience, let A_j denote the sum of the gaps $a_{i_j+2} - a_{i_j+1}, a_{i_j+3} - a_{i_j+2}, \cdots, a_{i_{j+1}} - a_{i_{j+1}-1}$ which are not divisible by 2^q for $j = 1, 2, \cdots, q+1$. For each $j = 1, 2, \cdots, q+1$, we have that $(a_{i_1} + 2^q) + (A_1 + 2^q) + (A_2 + 2^q) + \cdots + (A_{j-1} + 2^q) + A_j = a_{i_1} + (A_1 + A_2 + \cdots + A_j) + j2^q$ is in the second half mod 2^{q+1}. This means that $a_{i_1} + (A_1 + A_2 + \cdots + A_j)$ is in the second half mod 2^{q+1} if j is even (this includes when $j = 0$ and we are just considering a_{i_1}) and is in the first half mod 2^{q+1} if j is odd.

Beginning with a_{i_1}, every time we add the next A_j, we switch halves mod 2^{q+1}. This means that $A_1 + A_2 + \cdots + A_r > (r-1)2^q$ for any $r = 1, 2, \cdots, q+1$. In particular, $A_1 + A_2 + \cdots + A_{q+1} > q2^q$. Thus, there exist gaps of sizes $1, 2, 4, \cdots, 2^{q-1}$ which collectively sum to at least $q2^q + 1$. By the inductive hypothesis, we have at most $m+1$ gaps of size 2^m for $m = 0, 1, \cdots, q-1$. Thus the gaps of size $1, 2, 4, \cdots, 2^{q-1}$ collectively sum to at most:

$$\begin{aligned}
\sum_{m=0}^{q-1} (m+1)2^m &= \sum_{m=0}^{q-1} (2^q - 2^m) \\
&= q2^q - \sum_{m=0}^{q-1} 2^m \\
&= q2^q - (2^q - 1) \\
&= (q-1)2^q + 1.
\end{aligned}$$

This contradicts that these must sum to at least $q2^q + 1$, so it's impossible for a monochromatic D-diffsequence with respect to P_t to have $q+2$ gaps of size 2^q. Thus, there are at most $m+1$ gaps of size 2^m for $m = 0, 1, 2, \cdots, q$ and the induction is complete. □

Lemma 4.1.3 gives a constant upper bound, $1+2+\cdots+(t-1)=\frac{t(t-1)}{2}$ on the number of gaps smaller than 2^t in a monochromatic (with respect to the coloring P_t) D-diffsequence and thus a lower bound, $k-1-\frac{t(t-1)}{2}$ for the number of gaps of size at least 2^t. A lower bound on $\Delta(D,k;2)$ comes from summing the sizes of these longer gaps:

$$\Delta(D,k;2)\geq 2^t(k-1-\frac{t(t-1)}{2}).$$

This bound is linear in terms of k and has a slope which depends on t.

Picking $t=\lfloor\sqrt{2k}\rfloor$ for a given k is in fact enough to obtain an exponential lower bound of

$$\Delta(D,k;2)\geq 2^{\lfloor\sqrt{2k}\rfloor}(\frac{\sqrt{2k}}{2}-1).$$

This bound has the same exponent as in Theorem 4.1.1, but differs by a factor of $\Theta(\sqrt{k})$. However, we will instead demonstrate a more refined series of colorings which yield a slightly better lower bound.

For $u\in\mathbb{Z}_{\geq 0}$, let $P_{t,u}$ be the periodic coloring with a repeating block of size 2^{t+u} obtained by replacing each bit of P_t by 2^u copies of itself. We call these 2^u consecutive 1's or 2^u consecutive 0's a *sub-block*. Note that $P_{t,0}$ is simply P_t.

Example 9. $P_{3,1}$ has a repeating block of "1100001100111100", containing 8 sub-blocks of length 2.

Example 10. $P_{2,2}$ has a repeating block of "1111000000001111", containing 4 sub-blocks of length 4.

If a_i and a_{i+1} are consecutive entries in a monochromatic D-diffsequence with respect to the coloring $P_{t,u}$, there are three possibilities:

i) a_i and a_{i+1} are in different positions within the same sub-block, or

ii) a_i and a_{i+1} are in different sub-blocks (possibly in the same block) but the same position within their respective sub-blocks, or

iii) a_i and a_{i+1} are in different sub-blocks and different positions within their respective sub-blocks.

In the second case, $a_{i+1} - a_i$ is a multiple of 2^u, the length of a sub-block, so a_{i+1} appears exactly 2^l sub-blocks after a_i for some l. In the third case, we note that $a_{i+1} - a_i$ cannot be divisible by the sub-block size 2^u. Therefore, it is at most 2^{u-1}, so a_i and a_{i+1} are in consecutive sub-blocks.

If we consider what sub-blocks $a_1, a_2, \cdots, a_k$ lie in and ignore those that are not the first a_i in their respective sub-block, this corresponds to a monochromatic D-diffsequence with respect to the coloring P_t.

Example 11. $5, 6, 10, 11, 12, 28$ is a monochromatic sequence in color 0 with respect to the coloring $P_{2,2}$. If we look at the sub-blocks these are in, we have $2, 2, 3, 3, 3, 7$. Ignoring repeated occurrences of the same sub-block leaves us with $2, 3, 7$ which is a monochromatic D-diffsequence of color 0 with respect to P_2.

Applying Lemma 4.1.3 to this D-diffsequence gives that for $m = 0, 1, \cdots, t-2$, there are at most $m+1$ occurrences of a_i and a_{i+1} in the original sequence at a distance of 2^m sub-blocks apart. In particular, there is at most one i for which a_i and a_{i+1} are in consecutive sub-blocks. For $m = 1, \cdots, t-2$, there at most $m+1$ values of i with $a_{i+1} - a_i = 2^{m+u}$.

In the first case, a_{i+1} is in a later position within the sub-block than a_i is. There is at most one time (the third case), where a_{i+1} can be earlier in its sub-block than a_i is. Thus, if we keep track of changes in the position of a_i within its sub-block, it can increase within the range 1 to 2^u, decrease once, then increase again, potentially up to 2^u. This means the gaps accounted for by the first and third cases sum to at most $2(2^u) - 1$. For $m = 1, \cdots, t-2$, we have an upper bound on the number of gaps of size 2^{m+u}. We can then obtain a lower bound on the number of gaps of size at least 2^{t+u}. (Note that gaps of size 2^{t-1+u} cannot occur in a monochromatic D-diffsequence with respect to $P_{t,u}$.)

Although it is possible for gaps of size 2^u to occur in a monochromatic D-diffsequence with respect to $P_{t,u}$, there is never any reason to use them if the goal is to minimize a_k. This is because a gap of size 2^u can only occur if two sub-blocks of all 1's or two sub-blocks of all 0's are adjacent. In that case, it is better to use 2^u consecutive gaps of size 1 in place of one gap of size 2^u. This means there are at most $\sum_{m=1}^{t-2}(m+1)$ gaps of size less than 2^{t+u} accounted for by the second case, in addition to at most $2^{u+1}-1$ gaps of size less than 2^{t+u} accounted for by the first and third cases. There are a total of $k-1$ gaps $a_{i+1}-a_i$, so at least

$$(k-1)-\sum_{m=1}^{t-2}(m+1)-(2^{u+1}-1)$$

have size at least 2^{t+u}. Replacing a smaller gap by another gap of size at least 2^{t+u} would increase the sum of the gap sizes, so to minimize the total size of the gaps, we will take as many as possible of each size less than 2^{t+u}.

Thus, the sum of the $k-1$ gaps is at least:

$$\begin{aligned}
&(2^{u+1}-1)+\sum_{m=1}^{t-2}(m+1)2^{m+u}+[k-1-(2^{u+1}-1)-\sum_{m=1}^{t-2}(m+1)]2^{t+u}\\
=&2^{t+u}(k-2^{u+1}-\frac{(t-2)(t+1)}{2})+(2^{u+1}-1)+(2^{u+t-1}-2^{u+1})+\sum_{m=1}^{t-2}(2^{u+t-1}-2^{u+m})\\
=&2^{t+u}(k-2^{u+1}-\frac{t^2}{2}+\frac{t}{2}+1)+(t-2)2^{u+t-1}+2^{u+1}-1\\
=&2^{t+u}(k-2^{u+1}-\frac{t^2}{2}+t)+2^{u+1}-1.
\end{aligned}$$

This sum is a lower bound for $a_k - a_1$ so

$$\Delta(D,k;2) \geq 2^{t+u}(k - 2^{u+1} - \frac{t^2}{2} + t) + 2^{u+1}.$$

We can now strategically choose t and u to maximize the lower bound. Suppose $t = \lfloor\sqrt{2k}\rfloor$ and $u = \lfloor\log_2 \frac{\sqrt{k}}{2}\rfloor$. Then we have:

$$\begin{aligned}
\Delta(D,k;2) &\geq 2^{\lfloor\sqrt{2k}\rfloor + \lfloor\log_2 \frac{\sqrt{k}}{2}\rfloor}\left(k - 2^{\lfloor\log_2 \frac{\sqrt{k}}{2}\rfloor+1} - \frac{\lfloor\sqrt{2k}\rfloor^2}{2} + \lfloor\sqrt{2k}\rfloor\right) + 2^{\lfloor\log_2 \frac{\sqrt{k}}{2}\rfloor+1} \\
&\geq \frac{2^{\sqrt{2k}}(\frac{\sqrt{k}}{2})}{4}\left(k - \sqrt{k} - k + \sqrt{2k} - 1\right) + \frac{\sqrt{k}}{2} \\
&= 2^{\sqrt{2k}}\left(\frac{(\sqrt{2}-1)k}{8} - \frac{\sqrt{k}}{8}\right) + \frac{\sqrt{k}}{2}.
\end{aligned}$$

This completes the proof of Theorem 4.1.1.

Note that there is still a considerable gap between the upper bound of $2^{\Theta(k)}$ and the lower bound of $2^{\Theta(\sqrt{k})}$. The values of $\Delta(D,k;2)$ computed for $k \leq 12$ (see Appendix B) are more closely in line with the lower bound, suggesting that perhaps $\Delta(D,k;2) = 2^{\Theta(\sqrt{k})}$.

While we have shown that superpolynomial growth is possible for $\Delta(D,k;r)$, it remains to be seen whether there are any restrictions on how fast $\Delta(D,k;r)$ can grow.

Question 4.1.4. *For which $t \in \mathbb{R}^+$ does there exist an r-accessible set $D \subset \mathbb{Z}_{>0}$ for which*

$$\Delta(D,k;r) = e^{\Omega(k^t)} \quad ?$$

One possible mode of attack is as follows. For any infinite set $T \in \mathbb{Z}_{>0}$, the set $D = \{i - j \mid i,j \in T, i > j\}$ is r-accessible for any r [LR03]. This means we can construct arbitrarily sparse sets which are still r-accessible. In a heuristic sense, we expect a sparser set to have higher values for $\Delta(D,k;r)$. However, this could manifest as an increase in the size of the constants involved in a lower bound and not affect the asymptotic nature of the growth.

4.2 Inaccessibility of Certain Sets

In the previous section, we examined $\Delta(D, k; 2)$ where D was the range of an exponential function defined on $\mathbb{Z}_{>0}$. Other exponential functions would not yield similar results. In fact $D_t := \{t^i \mid i \in \mathbb{Z}_{\geq 0}\}$ is not 2-accessible for $t \in \mathbb{Z}_{\geq 3}$.

To see this, we can color multiples of $t-1$ with color 0 and everything else with color 1. Since $t^i \equiv 1 \pmod{t-1}$ for all i, a D_t-diffsequence of length at least $t-1$ would contain values in all residue classes $\pmod{t-1}$ and thus contain both colors.

We can generalize these periodic colorings to show that a wider class of D's are not 2-accessible. We will consider sets $D = \{d_1, d_2, d_3 \cdots\}$ with $d_i \mid d_{i+1}$ for all i. We call such a set a *dividing set* since $d_1 \mid d_2 \mid d_3, \cdots$ is a dividing sequence. Alternatively, we may view a dividing set as

$$D_{\{a_n\}} := \{\prod_{i=1}^{k} a_i \mid k \in \mathbb{Z}_{>0}\}$$

for a fixed sequence of positive integers $a_1, a_2, a_3, \cdots$ with $a_i \geq 2$ when $i \geq 2$.

The set of powers of t is always a dividing set, so we observe that it is possible for a dividing set to be 2-accessible or to not be 2-accessible. We classify [Cli21] precisely when each happens.

Theorem 4.2.1. *$D_{\{a_n\}}$ is 2-accessible if and only if $\{a_n\}$ contains arbitrarily long strings of consecutive 2's.*

First we will demonstrate that arbitrarily long strings of consecutive 2's is enough to guarantee 2-accessibility.

Claim 4.2.2. *If $\{a_n\}$ contains arbitrarily long strings of consecutive 2's, then $D_{\{a_n\}}$ is 2-accessible.*

Proof. Suppose for the sake of contradiction that some 2-coloring of $\mathbb{Z}_{>0}$ avoids arbitrarily long monochromatic $D_{\{a_n\}}$-diffsequences. Suppose that the longest such diffsequence is of length k and given by $b_1 < b_2 < \cdots < b_k$.

Because $\{a_n\}$ contains arbitrarily long strings of consecutive 2's, there exists some index j such that $a_j, a_{j+1}, \cdots, a_{j+k-1}$ all equal 2. Letting $C = \prod_{i=1}^{j-1} a_i$, we have that $2^t C \in D_{\{a_n\}}$ for $t = 0, 1, \cdots, k$. To avoid extending our monochromatic diffsequence $b_1 < b_2 < \cdots < b_k$ to a longer one, $b_k + 2^t C$ must be the other color for $t = 0, 1, \cdots, k$. However, this gives a monochromatic $D_{\{a_n\}}$-diffsequence in the other color, as the consecutive differences are

$$(b_k + 2^t C) - (b_k + 2^{t-1} C) = 2^{t-1} C$$

for $t = 1, 2, \cdots, k$. Thus, we have a monochromatic $D_{\{a_n\}}$-diffsequence of length $k + 1$, giving a contradiction.

□

The previous examples (powers of $t \in \mathbb{Z}_{\geq 3}$) of showing that a dividing set is not 2-accessible relied on periodic colorings. In many cases, a periodic coloring with two colors can be reinterpreted as coloring each positive integer n according to the parity of $\lfloor \alpha n \rfloor$ for some fixed rational number α which depends on the coloring. For the remaining cases, we will utilize a similar coloring, except where α is irrational. For every sequence $\{a_n\}$ that does not have arbitrarily long strings of consecutive 2's, we will demonstrate the existence of an α such that coloring the positive integers according to the parity of $\lfloor \alpha n \rfloor$ will avoid arbitrarily long monochromatic $D_{\{a_n\}}$-diffsequences.

The one case where we explicitly write α is when $a_n = n$ for all n. That is, when $D_{\{a_n\}}$ is the set of factorials.

4.2.1 Factorials

Claim 4.2.3. *The set, $D := \{i! \mid i \in \mathbb{Z}_{>0}\}$, of factorials is not 2-accessible.*

We will make use of the following coloring. The integer n is assigned color 0 or 1 depending on the parity of $\lfloor n\alpha \rfloor$ where

$$\alpha := 2 - \frac{e}{2} - \frac{1}{2e} = 1 - \sum_{i=1}^{\infty} \frac{1}{(2i)!}.$$

Here we show that this coloring avoids monochromatic D-diffsequences of length 4. That is, there are no $a_1 < a_2 < a_3 < a_4$ in $\mathbb{Z}_{>0}$ with $a_2 - a_1, a_3 - a_2, a_4 - a_3 \in D$ such that $\lfloor a_i\alpha \rfloor$ has the same parity for $i = 1, 2, 3, 4$.

Lemma 4.2.4. *For all $k \geq 1$, there exists some integer n such that*

$$2n + \frac{1}{3} \leq k!\alpha < 2n + 1.$$

Proof. We split into the cases where k is even and when k is odd:

When $k = 2m$ for $m \geq 1$, we have

$$\begin{aligned} k!\alpha &= (2m)! - \sum_{i=1}^{\infty} \frac{(2m)!}{(2i)!} = (2m)! - \sum_{i=1}^{m} \frac{(2m)!}{(2i)!} - \sum_{i=m+1}^{\infty} \frac{(2m)!}{(2i)!} \\ &= (2m)! - \sum_{i=1}^{m-1} [(2m)(2m-1)\cdots(2i+1)] - 1 - \sum_{i=m+1}^{\infty} \frac{1}{(2m+1)(2m+2)\cdots(2i)} \\ &= 1 - \sum_{i=m+1}^{\infty} \frac{1}{(2m+1)(2m+2)\cdots(2i)} \pmod 2. \end{aligned}$$

To reach the desired conclusion, it suffices to demonstrate that

$$\frac{2}{3} \geq \sum_{i=m+1}^{\infty} \frac{1}{(2m+1)(2m+2)\cdots(2i)} > 0.$$

It is clear that the sum is positive. It can also be bounded above by the following convergent geometric series:

$$\sum_{i=m+1}^{\infty} \frac{1}{(2m+1)^{2i-2m}} = \frac{\frac{1}{(2m+1)^2}}{1-\frac{1}{(2m+1)^2}} = \frac{1}{(2m+1)^2-1}$$
$$\leq \frac{1}{3^2-1} = \frac{1}{8}.$$

When $k = 2m+1$ for $m \geq 1$, we have

$$\begin{aligned} k!\alpha &= (2m+1)! - \sum_{i=1}^{\infty} \frac{(2m+1)!}{(2i)!} = (2m+1)! - \sum_{i=1}^{m} \frac{(2m+1)!}{(2i)!} - \sum_{i=m+1}^{\infty} \frac{(2m+1)!}{(2i)!} \\ &= (2m+1)! - \sum_{i=1}^{m-1} \frac{(2m+1)!}{(2i)!} - (2m+1) - \sum_{i=m+1}^{\infty} \frac{1}{(2m+2)(2m+3)\cdots(2i)} \\ &= 1 - \sum_{i=m+1}^{\infty} \frac{1}{(2m+2)(2m+3)\cdots(2i)} \pmod 2. \end{aligned}$$

To reach the desired conclusion, it suffices to demonstrate that

$$\frac{2}{3} \geq \sum_{i=m+1}^{\infty} \frac{1}{(2m+2)(2m+3)\cdots(2i)} > 0.$$

It is clear that the sum is positive. It can also be bounded above by the following convergent geometric series:

$$\sum_{i=m+1}^{\infty} \frac{1}{(2m+2)^{2i-2m-1}} = \frac{\frac{1}{2m+2}}{1-\frac{1}{(2m+2)^2}} = \frac{2m+2}{(2m+2)^2-1}$$
$$\leq \frac{4}{4^2-1} = \frac{4}{15}.$$

Only $k = 1$ remains to be checked. We have established that $1 - \frac{1}{8} \leq 2\alpha < 1 \pmod 2$. This means that either $\frac{7}{16} \leq \alpha < \frac{1}{2} \pmod 2$ or $1 + \frac{7}{16} \leq \alpha < 1 + \frac{1}{2} \pmod 2$. If the latter statement is true, then $\alpha > 1$ or $\alpha < 0$ but this is false because $\alpha < 2 - \frac{e}{2} < 2 - \frac{2}{2} = 1$ and $\alpha > 2 - \frac{3}{2} - \frac{1}{2(2)} = \frac{1}{4}$. Therefore, $\frac{7}{16} \leq \alpha < \frac{1}{2} \pmod 2$, giving us $\frac{1}{3} \leq \alpha < 1 \pmod 2$ as desired. $\square$

Proof of Claim 4.2.3. Now suppose that there exist a_1, a_2, a_3, a_4 all the same color such that the consecutive gaps lie in $D = \{i! \mid i \in \mathbb{Z}_{>0}\}$. Since this sequence is monochromatic, $\lfloor a_{i+1}\alpha \rfloor - \lfloor a_i \alpha \rfloor$ is even for $i = 1, 2, 3$. By Lemma 4.2.4, we know that for $i = 1, 2, 3$,

$$2n_i + \frac{1}{3} \leq (a_{i+1} - a_i)\alpha < 2n_i + 1$$

for some integers n_i.

If $a_i\alpha \in [\frac{2}{3}, 1) \pmod 2$, then $a_{i+1}\alpha \in [1, 2) \pmod 2$. Similarly, if $a_i\alpha \in [\frac{5}{3}, 2) \pmod 2$, then $a_{i+1}\alpha \in [0, 1) \pmod 2$. Thus, in order for a_{i+1} and a_i to be the same color, we need $0 \leq a_i\alpha < \frac{2}{3} \pmod 1$. In particular, we need $0 \leq a_2\alpha < \frac{2}{3} \pmod 1$ and $0 \leq a_3\alpha < \frac{2}{3} \pmod 1$.

By similar reasoning, these inequalities force $0 \leq a_1\alpha < \frac{1}{3} \pmod 1$ and $0 \leq a_2\alpha < \frac{1}{3} \pmod 1$, respectively. However, since $\lfloor a_2\alpha \rfloor$ and $\lfloor a_1\alpha \rfloor$ have the same parity, this gives $\frac{-1}{3} < a_2\alpha - a_1\alpha < \frac{1}{3} \pmod 2$, contradicting the restriction that $2n_1 + \frac{1}{3} \leq (a_2 - a_1)\alpha < 2n_1 + 1$. Therefore, no such a_1, a_2, a_3, a_4 exist with respect to this coloring. We avoid arbitrarily long monochromatic D-diffsequences, in particular by avoiding monochromatic D-diffsequences of length at least four. □

4.2.2 Existence Proof for General Dividing Sets

We are left to consider the general case of $D_{\{a_n\}}$ where $\{a_n\}$ does not have arbitrarily long strings of consecutive 2's. All elements of $D_{\{a_n\}}$ are multiples of a_1, so any monochromatic $D_{\{a_n\}}$-diffsequence stays within a residue class modulo a_1. Considering each residue class $\pmod{a_1}$ separately, we can avoid the presence of arbitrarily long monochromatic $D_{\{a_n\}}$-diffsequences in a residue class as long as it is possible to find a coloring which avoids arbitrarily long monochromatic $T_{\{a_n\}}$-diffsequences in $\mathbb{Z}_{>0}$ where $T_{\{a_n\}} = \{\frac{d}{a_1} \mid d \in D_{\{a_n\}}\}$. Thus, it suffices to only consider when $a_1 = 1$.

If there is no string of k consecutive 2's in $\{a_n\}$, we will show that there exists a coloring of $\mathbb{Z}_{>0}$ which avoids monochromatic $D_{\{a_n\}}$-diffsequences of length 2^k+1. As with the set of factorials, we will use Beatty sequences to make such a coloring. Every $n \in \mathbb{Z}_{>0}$ will be colored according to the parity of $\lfloor n\alpha \rfloor$ where our choice of α depends on $\{a_n\}$. For convenience, let $d_t := \prod_{i=1}^{t} a_i$ so that $D_{\{a_n\}} = \{d_t \mid t \in \mathbb{Z}_{>0}\}$.

Lemma 4.2.5. *If the sequence $\{a_n\}$ contains no string of k consecutive 2's, then there exists some $\alpha > 0$ such that for all $t \in \mathbb{Z}_{>0}$, there exists some integer n_t such that*

$$2n_t + \frac{1}{2^k} \le d_t\alpha \le 2n_t + 1.$$

Using this lemma, we can finish the proof of Theorem 4.2.1.

Proof of Theorem 4.2.1. Suppose for the sake of contradiction that we have a monochromatic $D_{\{a_n\}}$-diffsequence $b_1 < b_2 < \cdots < b_{2^k+1}$. Since this sequence is monochromatic, $\lfloor b_{i+1}\alpha \rfloor - \lfloor b_i\alpha \rfloor$ is even for $i = 1, 2, \cdots, 2^k$. By Lemma 4.2.5, we know that for $i = 1, 2, \cdots, 2^k$,

$$2m_i + \frac{1}{2^k} \le (b_{i+1} - b_i)\alpha \le 2m_i + 1 \tag{4.1}$$

for some integers m_i. In order for b_{i+1} and b_i to be the same color, we need $0 \le b_i\alpha < 1 - \frac{1}{2^k} \pmod 1$, so $0 \le b_{2^k}\alpha < 1 - \frac{1}{2^k} \pmod 1$. Since $\lfloor b_{2^k}\alpha \rfloor$ and $\lfloor b_{2^k-1}\alpha \rfloor$ have the same parity, this in turn necessitates

$$0 \le (b_{2^k-1})\alpha < 1 - \frac{2}{2^k} \pmod 1,$$

which necessitates

$$0 \le (b_{2^k-2})\alpha < 1 - \frac{3}{2^k} \pmod 1.$$

Continuing in this manner, we get that

$$0 \le b_2\alpha < 1 - \frac{2^k-1}{2^k} = \frac{1}{2^k} \pmod 1.$$

However, $\lfloor b_1\alpha\rfloor$ and $\lfloor b_2\alpha\rfloor$ have the same parity so either $0 \le b_2\alpha - b_1\alpha < \frac{1}{2^k} \pmod 2$ or $1 < b_2\alpha - b_1\alpha < 2 \pmod 2$. This directly contradicts Equation 4.1. $\square$

Proof of Lemma 4.2.5. For each $t \in \mathbb{Z}_{>0}$, we will construct closed intervals $I_b^t := [C_b^t, D_b^t]$ for $b = 1, \cdots, t$ such that if $d_b\gamma \in I_b^t \pmod 2$, then $d_h\gamma \in I_h^t \pmod 2$ for $h = b+1, b+2, \cdots, t$. These will be constructed in such a way that $\frac{1}{2^k} \le C_b^t < D_b^t \le 1$ for $b = 1, \cdots, t$. Then, for any $\gamma \in I_1^t$, we have $d_1\gamma \in I_1^t \pmod 2$ and thus $d_h\gamma \in I_h^t \pmod 2 \subset [\frac{1}{2^k}, 1] \pmod 2$ for $h = 1, 2, \cdots, t$.

We then define $J_t := [C_t, D_t] := \bigcap_{b=1}^{t} I_1^b$. Thus, for $\gamma \in J_t$, we have $d_b\gamma \in [\frac{1}{2^k}, 1] \pmod 2$ for $b = 1, 2, \cdots, t$. Also we have $J_{t+1} \subset J_t$ for all $t \in \mathbb{Z}$, so $C_t \le C_{t+1} < D_{t+1} \le D_t$ for $t \in \mathbb{Z}$.

Consider the sequence $\{C_t\}$ for $t = 1, 2, \cdots$. This is a non-decreasing sequence and it is bounded above by 1. Therefore, it converges to some limit α. Similarly, the sequence $\{D_t\}$ for $t = 1, 2, \cdots$ is non-increasing and bounded below by 0 so it also converges to some limit $\beta \ge \alpha$. For each $t \in \mathbb{Z}_{>0}$, we have $C_t \le \alpha \le \beta \le D_t$. Thus, this α lies in J_t for all $t \in \mathbb{Z}_{>0}$, meaning that $d_t\alpha \in [\frac{1}{2^k}, 1] \pmod 2$ for all $t \in \mathbb{Z}_{>0}$.

It now suffices to construct each I_b^t. We begin with $I_t^t := [\frac{1}{2}, 1]$. Then for $b \ge 2$, we recursively construct I_{b-1}^t from I_b^t as follows:

- If a_b is odd:

$$C_{b-1}^t = \frac{a_b - 1 + C_b^t}{a_b} \qquad\qquad D_{b-1}^t = \frac{a_b - 1 + D_b^t}{a_b}.$$

- If a_b is even:

$$C_{b-1}^t = \frac{a_b - 2 + C_b^t}{a_b} \qquad\qquad D_{b-1}^t = \frac{a_b - 2 + D_b^t}{a_b}.$$

We claim this construction satisfies the requisite properties:

Claim 4.2.6. *For* $b = 1, 2, \cdots, t$,

$$\frac{1}{2^k} \leq C_b^t < D_b^t \leq 1.$$

Claim 4.2.7. *For* $b = 1, \cdots, t$, *if* $d_b\gamma \in I_b^t \pmod 2$, *then* $d_h\gamma \in I_h^t \pmod 2$ *for* $h = b+1, b+2, \cdots, t$.

Proof of Claim 4.2.6. We set $C_t^t := \frac{1}{2}$ and $D_t^t := 1$. We have $D_b^t > C_b^t$ for $t = b$ and whenever $D_b^t > C_b^t$, we get that $a_b - 2 + D_b^t > a_b - 2 + C_b^t$ and $a_b - 1 + D_b^t > a_b - 1 + C_b^t$, so by backwards induction, $D_b^t > C_b^t$ for $b = 1, \cdots, t$.

Now we need to show that $C_b^t \geq \frac{1}{2^k}$ and $D_b^t \leq 1$ for $b = 1, \cdots, t-1$. We will proceed by induction by showing that for $b = 2, \cdots, t$, $C_h^t \geq \frac{1}{2^k}$ for $h = b, b+1, \cdots, t$ implies $C_{b-1}^t \geq \frac{1}{2^k}$ and that $D_b^t \leq 1$ implies $D_{b-1}^t \leq 1$. There are two cases to consider:

i) $a_b > 2$ or

ii) $a_b = a_{b+1} = a_{b+2} = \cdots = a_{b+q-1} = 2$ with $b + q - 1 \leq t$ but either $a_{b+q} > 2$ or $b + q = t + 1$. Note that that the sequence $\{a_n\}$ has no string of k consecutive 2's so $q < k$.

In either case,

$$D_{b-1}^t \leq \frac{a_b - 1 + D_b^t}{a_b} \leq \frac{a_b - 1 + 1}{a_b} = 1.$$

Since $C_b^t < D_b^t \leq 1$, we know that $1 \geq 1 - C_b^t > 0$. In Case i) and with a_b odd,

$$C_{b-1}^t = 1 - \frac{1 - C_b^t}{a_b}.$$

This is minimized when $a_b = 3$, in which case,

$$C_{b-1}^t = 1 - \frac{1 - C_b^t}{3} \geq \frac{2}{3}.$$

For Case i) and with even a_b,

$$C_{b-1}^t = 1 - \frac{2 - C_b^t}{a_b},$$

which is minimized when $a_b = 4$, yielding,

$$C_{b-1}^t = 1 - \frac{2 - C_b^t}{4} \geq \frac{1}{2}.$$

This means that when $a_b > 2$, we not only get $C_{b-1}^t \geq \frac{1}{2^k}$, but we get the potentially stronger claim of $C_{b-1}^t \geq \frac{1}{2}$.

Now we consider Case ii). If $b + q \leq t$ and $a_{b+q} > 2$, then a consequence of our analysis of Case i) is that $C_{b+q-1}^t \geq \frac{1}{2}$. If $b+q = t+1$, our initial definition of C_t^t also gives $C_{b+q-1}^t \geq \frac{1}{2}$.

Now since $a_b = a_{b+1} = \cdots = a_{b+q-1} = 2$, we have for $h = b-1, b, \cdots, b+q-2$,

$$\begin{aligned} C_h^t &= \frac{a_{h+1} - 2 + C_{h+1}^t}{a_{h+1}} \\ &= \frac{2 - 2 + C_{h+1}^t}{2} = \frac{C_{h+1}^t}{2}. \end{aligned}$$

This means

$$C_{b-1}^t = \frac{C_{b+q-1}^t}{2^q} \geq \frac{1/2}{2^q} = \frac{1}{2^{q+1}}.$$

Recall that $q < k$, so this quantity is at least $\frac{1}{2^k}$ and the proof is complete. □

Proof of Claim 4.2.7. It suffices to show that if $d_b\gamma \in I_b^t \pmod 2$, then $d_{b+1}\gamma \in I_{b+1}^t \pmod 2$ for $b = 1, \cdots, t-1$.

If $d_b\gamma \in I_b^t \pmod 2$ and a_{b+1} is odd,

$$
\begin{gathered}
C_b^t \leq d_b\gamma \leq D_b^t \pmod 2 \\
\frac{a_{b+1} - 1 + C_{b+1}^t}{a_{b+1}} \leq d_b\gamma \leq \frac{a_{b+1} - 1 + D_{b+1}^t}{a_{b+1}} \pmod 2 \\
a_{b+1} - 1 + C_{b+1}^t \leq a_{b+1}d_b\gamma \leq a_{b+1} - 1 + D_{b+1}^t \pmod 2 \\
C_{b+1}^t \leq d_{b+1}\gamma \leq D_{b+1}^t \pmod 2.
\end{gathered}
$$

Note that to go from the second pair of inequalities to the third pair, we use that $0 \leq C_{b+1}^t \leq D_{b+1}^t \leq 1$ and that $a_{b+1} - 1$ is even. In particular, we have some integer n_b where $d_b\gamma \in [2n_b + \frac{a_{b+1}-1+C_{b+1}^t}{a_{b+1}}, 2n_b + \frac{a_{b+1}-1+D_{b+1}^t}{a_{b+1}}]$. Multiplying by a_{b+1} gives $a_{b+1}d_b\gamma \in [2n_b a_{b+1} + (a_{b+1} - 1) + C_{b+1}^t, 2n_b a_{b+1} + (a_{b+1} - 1) + D_{b+1}^t]$.

Similarly if $d_b\gamma \in I_b^t \pmod 2$ and a_{b+1} is even,

$$
\begin{gathered}
C_b^t \leq d_b\gamma \leq D_b^t \pmod 2 \\
\frac{a_{b+1} - 2 + C_{b+1}^t}{a_{b+1}} \leq d_b\gamma \leq \frac{a_{b+1} - 2 + D_{b+1}^t}{a_{b+1}} \pmod 2 \\
a_{b+1} - 2 + C_{b+1}^t \leq a_{b+1}d_b\gamma \leq a_{b+1} - 2 + D_{b+1}^t \pmod 2 \\
C_{b+1}^t \leq d_{b+1}\gamma \leq D_{b+1}^t \pmod 2.
\end{gathered}
$$

Going from the second pair of inequalities to the third pair uses that $0 \leq C_{b+1}^t \leq D_{b+1}^t \leq 1$ and that $a_{b+1} - 2$ is even.

$\square$

$\square$

While we have demonstrated a wide class of sets that are not 2-accessible, this occurs for modular arithmetic reasons, as opposed to sparsity reasons. It would be interesting to determine whether there exists a growth condition on the elements of the set D which can disqualify it from being 2-accessible. For example, we propose the following question.

Question 4.2.8. *Does there exist an absolute constant $C \geq 2$ and a 2-accessible set $D = \{d_1, d_2, d_3, \cdots\}$ such that $d_{i+1} > Cd_i$ for all $i \in \mathbb{Z}_{>0}$? If yes, can C be arbitrarily large?*

We can also look specifically at the case where $\frac{d_{i+1}}{d_i}$ does not fluctuate much. We can interpolate between sets of the form $\{t^i \mid i \in \mathbb{Z}_{\geq 0}\}$ by defining $D^f_\alpha := \{\lfloor \alpha^i \rfloor \mid i \in \mathbb{Z}_{\geq 0}\}$ or $D^c_\alpha = \{\lceil \alpha^i \rceil \mid i \in \mathbb{Z}_{\geq 0}\}$ for $\alpha > 1$.

When $\alpha = 2^{\frac{1}{n}}$ for some $n \in \mathbb{Z}_{\geq 0}$, then both D^f_α and D^c_α contain all powers of 2 and therefore are 2-accessible. More generally, we can consider $D^f_{\delta,\alpha} := \{\lfloor \delta\alpha^i \rfloor \mid i \in \mathbb{Z}_{\geq 0}\} \cap \mathbb{Z}_{>0}$ and $D^c_\alpha = \{\lceil \delta\alpha^i \rceil \mid i \in \mathbb{Z}_{\geq 0}\}$ for $\alpha > 1, \delta > 0$. We could also consider the set consisting of the values (excluding 0) of $\delta\alpha^i$ rounded to the nearest integer, though that will sometimes be ambiguous if $\delta\alpha^i$ ever has fractional part $\frac{1}{2}$. When $\alpha = \frac{1+\sqrt{5}}{2}$ and $\delta = \frac{1}{\sqrt{5}}$, the rounding is never ambiguous and the set we obtain in this manner is the set of Fibonacci numbers, which is known to be 2-accessible [AGJ+08]. These pieces of evidence motivate the next two questions.

Question 4.2.9. *Is there any $\alpha \in (1, 2)$ for which $D^f_{\delta,\alpha}$ or $D^c_{\delta,\alpha}$ is not 2-accessible for some choice of $\delta > 0$? (for $\delta = 1$?)*

Question 4.2.10. *Is there any $\alpha > 2$ for which $D^f_{\delta,\alpha}$ or $D^c_{\delta,\alpha}$ is 2-accessible for some choice of $\delta > 0$? (for δ=1?)*

If the answer to any of these questions is yes, we can also consider whether the set of such α has nonzero measure.

4.2.3 Random Diffsequences

We may also consider what happens when D is a random subset of the positive integers.

Claim 4.2.11. *If we select the elements of D from the positive integers, independently and uniformly at random with probability $0 < \alpha < 1$, then D is 2-accessible with probability 1.*

Proof. For each integer $k \geq 1$ and each prime p, consider the set $\{p, 2p, 4p, \cdots, 2^{k-1}p\}$. Note that for different choices of p, these sets are disjoint. The probability that D includes all the numbers in $\{p, 2p, \cdots, 2^{k-1}p\}$ is a constant $\alpha^k > 0$. Since these sets are disjoint, there is probability 1 that there exists some prime p with $\{p, 2p, 4p, \cdots, 2^{k-1}p\} \subset D$. This is homothetic to $\{1, 2, 4, \cdots, 2^{k-1}\}$ so whatever we can say about integer sequences with gaps in $\{1, 2, 4, \cdots, 2^{k-1}\}$, we will be able to say about sequences within a fixed residue class mod p with gaps in $\{p, 2p, 4p, \cdots, 2^{k-1}p\}$.

We know that the set, T, of powers of 2 is 2-accessible. In particular, coloring $\{1, 2, \cdots, 2^k - 1\}$ with two colors guarantees the presence of a monochromatic T-diffsequence of length k. However, the numbers are small enough that there could be no gap of size more than 2^{k-1} in such a diffsequence. Thus, the presence of $\{1, 2, \cdots, 2^{k-1}\}$ in D ensures that any 2-coloring of $\mathbb{Z}_{>0}$ has monochromatic D-diffsequences of length k. Similarly, the presence of $\{p, 2p, 4p, \cdots, 2^{k-1}p\}$ in D will as well.

Thus, for any k, we have probability 1 that every 2-coloring of the positive integers contains a monochromatic D-diffsequence of length k. $\square$

It remains to be seen whether a similar argument can be used for r-accessibility by showing that with probability 1, D will contain homothetic copies of certain subsets whose union is a known r-accessible set.

If we include each positive integer x in our set with probability $\frac{1}{x^2}$, then the expected size of our set D is finite, so D is finite with probability 1 and thus 2-accessible with probability 0. There are countless other probability functions which result in a set that is 2-accessible with probability 0 because D is finite with probability 1. It is still possible for D to be 2-accessible with probability 0 despite being infinite with probability 1 since we could select D to be a random subset of a set like the odd numbers which is known to not be 2-accessible.

If neither of these criteria to make D 2-accessible with probability 0 is satisfied, we ask if there are still other obstructions to D being 2-accessible.

Question 4.2.12. *Suppose that D is a random subset of $\mathbb{Z}_{>0}$ where each $x \in \mathbb{Z}_{>0}$ is included in D independently with some positive probability $f(x)$. If $\sum_{x=1}^{\infty} f(x)$ diverges, is it possible for D to be 2-accessible with probability less than 1?*

One interesting case to examine might be when the probability function used is $f(x) = \frac{1}{x}$.

Lastly, as we range over all possible distributions for D as a random subset of $\mathbb{Z}_{>0}$, we ask what are the possible probabilities that D is 2-accessible.

Question 4.2.13. *Suppose that D is a random subset of $\mathbb{Z}_{>0}$ where each $x \in \mathbb{Z}_{>0}$ is included in D independently with some positive probability $f(x)$. Can the probability that D is 2-accessible be anything other than 0 or 1?*

We restrict our attention to when the elements of D are chosen independently, since otherwise, we can for any $0 < p < 1$, choose a distribution so that the random set D is equal to $\mathbb{Z}_{>0}$ with probability p and empty otherwise.

Appendix A

Number of Lines Needed to Cover Triangular Grids

d	$t_1(d,2,2)$	$t_1(d,2,3)$	$t_1(d,2,4)$	$t_1(d,2,5)$	$t_1(d,2,6)$	$t_1(d,2,7)$
1	2	3	4	5	6	7
2	3	5	6	8	9	11
3	5	7	9	12	14	16
4	6	9	12	15	18	21
5	8	12	15	18	22	26
6	9	14	18	22	26	30
7	11	16	21	26	30	35
8	12	18	24	30	35	40
9	14	21	27	33	39	45
10	15	23	30	36	44	51
11	17	25	33	40	48	56
12	18	27	36	44	52	60
13	20	30	39	48	56	65
14	21	32	42	51	60	70
15	23	34	45	54	65	75
16	24	36	48	58	69	80
17	26	39	51	62	74	85
18	27	41	54	66	78	90
19	29	43	57	69	82	95
20	30	45	60	72	86	100
21	32	48	63	76	90	105

d	$t_2(d,2,1)$	$t_2(d,2,2)$	$t_2(d,2,3)$	$t_2(d,2,4)$	$t_2(d,2,5)$	$t_2(d,2,6)$
1	1	2	3	4	5	6
2	2	4	5	7	9	10
3	3	6	8	11	13	16
4	4	8	11	15	18	22
5	5	10	14	19	23	28
6	6	12	17	23	28	34
7	7	14	20	27	33	40
8	8	16	23	31	38	46
9	9	18	26	35	43	52
10	10	20	29	39	48	58
11	11	22	32	43	53	64
12	12	24	35	47	58	70
13	13	26	38	51	63	76
14	14	28	41	55	68	82
15	15	30	44	59	73	88
16	16	32	47	63	78	94
17	17	34	50	67	83	100
18	18	36	53	71	88	106
19	19	38	56	75	93	112
20	20	40	59	79	98	118
21	21	42	62	83	103	124

Appendix B

Values of $\Delta(D, k; 2)$ when $D = \{2^i \mid i \in \mathbb{Z}_{\geq 0}\}$

k	$\Delta(D, k; 2)$
1	1
2	3
3	7
4	11
5	17
6	25
7	35
8	51
9	67
10	83
11	115
12	147

[CCLS18]

Appendix C

Gurobi Code

This Python implementation of Gurobi was used to compute $t_2(d,2,2)$. It can easily be modified to compute $t_1(d,2,k)$ or $t_2(d,2,k)$ for other values of k.

```
import math
import gurobipy as gp
from gurobipy import GRB
for n in range(2,21):
    points=[]
    for i in range(0,n+1):
        for j in range(0,2*i+1):
            points.append([i,j])
    lines=[]
    for i in range(0,n+1):
        line=[]
        for j in range(0,2*i+1):
            line.append([i,j])
        if len(line)>=2:
            lines.append(line)
```

```
for i in range(n+1):
    for i2 in range(i+1,n+1):
        for j in range(0,2*i+1):
            for j2 in range(0,2*i2+1):
                m=(j2-j)/((i2-i))
                line=[]
                for k in range(0,n+1):
                    if ((m*(k-i))%1==0 and 2*k>=m*(k-i)+j>=0):
                        line.append([k,m*(k-i)+j])
                if len(line)>=2:
                    if line not in lines:
                        lines.append(line)
m=gp.Model()
x=[m.addVar(vtype=GRB.INTEGER) for l in lines]
indices=[]
for p in points:
    count=[]
    for l in range(len(lines)):
        if p in lines[l]:
            count.append(l)
    indices.append(count)
for p in range(len(points)):
    m.addConstr(gp.quicksum(1*x[j] for j in indices[p])>=2)
m.update()
m.setObjective(gp.quicksum(1*x[i] for i in range(len(lines))))
m.optimize()
title='twice'+str(n)+'by'+str(n)+'full2triangle.sol'
```

```
m.write(title)
```

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