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On Spanning Trees with few Branch Vertices

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Doctor of Philosophy

Mathematics

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Abstract

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Hamiltonian paths, which are a special kind of spanning tree, have long been of interest in graph theory and are notoriously hard to compute. One notable feature of a Hamiltonian path is that all its vertices have degree at most two in the path. In a tree, we call vertices of degree at least three *branch vertices*. If a connected graph has no Hamiltonian path, we can still look for spanning trees that come "close," in particular by having few branch vertices (since a Hamiltonian path would have none).

A conjecture of Matsuda, Ozeki, and Yamashita posits that, for any positive integer $k$, a connected claw-free $n$-vertex graph $G$ must contain either a spanning tree with at most $k$ branch vertices or an independent set of $2k+3$ vertices whose degrees add up to at most $n-3$. In other words, $G$ has this spanning tree whenever $\sigma_{2k+3}(G) \geq n-2$. We prove this conjecture, which was known to be sharp.

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# Contents

# Chapter 1

# Introduction

For this thesis we assume a basic knowledge of graph theory; for terms and concepts not defined see [2]. Also, we consider only simple graphs. For a graph $G$ the graph $H$ is a *subgraph*, denoted $H \subseteq G$, if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. We call $H$ a *spanning* subgraph if $V(H) = V(G)$, and we call $H$ an *induced* subgraph if $E(H) = \{xy : x \in H, y \in H, xy \in E(G)\}$. The set of neighbors in $G$ of a vertex $v$ is called the *neighborhood* of $v$ and is denoted $N(v)$. The degree of $v$ is $|N(v)|$, denoted $\deg(v)$. For two vertices $u$ and $v$ in a graph $G$, a $u - v$ *path* $P$ is a sequence of vertices in $G$ beginning with $u$ and ending at $v$ such that consecutive vertices in $P$ are adjacent in $G$ and no vertex is repeated. A graph $G$ is *connected* if there is a $u - v$ path for every pair of vertices $u, v \in V(G)$. The graph in which every two distinct vertices are adjacent is the *complete graph* of order $n$, denoted $K_n$, having $\binom{n}{2}$ edges. The path $P_n$ is a graph of order $n$ and size $n - 1$ whose vertices can be labeled by $v_1, v_2, \ldots, v_n$ and whose edges are $v_i v_{i+1}$ for $i = 1, 2, \ldots, n - 1$. The cycle $C_n$ is a graph of order $n$ and size $n$, for integer $n \geq 3$, whose vertices can be labeled by $v_1, v_2, \ldots, v_n$ and whose edges are $v_1 v_n$ and $v_i v_{i+1}$ for $i = 1, 2, \ldots, n - 1$.

If a graph $T$ is connected, and no subgraph of $T$ is a cycle, we say $T$ is a *tree.* If $T$ is a spanning subgraph of some other graph, we call it a *spanning tree.* We invite the reader to verify that every connected graph has a spanning tree. Note that paths are a special kind of tree; if a spanning tree is a path, we call it a *Hamiltonian path*. The problem of checking a graph for Hamiltonian paths is well known to be NP-complete. Consequently, sufficient conditions for the existence of such a path are widely sought. One condition that has helped repeatedly is if a graph is *claw-free*, meaning it has no claw as an induced subgraph. (A claw consists of four vertices $a, b, c, d$ with edges $ab, ac, ad$.)

In a tree, vertices of degree one and vertices of degree at least three are called *leaves* and *branch vertices*, respectively. A Hamiltonian path can be regarded as a spanning tree with maximum degree at most two, a spanning tree with at most two leaves, or a spanning tree with no branch vertex. Sufficient conditions for a Hamiltonian path may, therefore, be extendable to sufficient conditions for a spanning tree that is "almost" a Hamiltonian path in one or more of these ways.

We denote by $\sigma_m(G)$ the smallest possible sum of degrees of an independent set of $m$ vertices in $G$. If there is no such independent set, we say $\sigma_m(G) = \infty$. This parameter will be central to our own results and those leading up to them. We also denote by $G[V] = G[v_1, v_2, \ldots, v_t]$ the subgraph induced by $V = \{v_1, v_2, \ldots, v_t\}$ for any $V \subseteq V(G)$, as will be helpful in our proofs.

Many researchers have investigated conditions for spanning trees with low maximum degree [4, 11, 13, 15, 17]; we give a good example below.

**Theorem 1.** [17] *Let $k \geq 2$ and let $G$ be a connected graph. If $G - S$ has at most $(k-2)|S|+2$ components for all $S \subset V(G)$, then $G$ has a spanning tree with maximum degree at most $k$.*

Spanning trees with few leaves have also been widely sought [1, 10, 14, 16]. The following such instance is particularly useful to us.

**Theorem 2.** [10] *Let $k$ be a non-negative integer and let $G$ be a connected claw-free graph. If $\sigma_{k+3}(G) \leq n - k - 2$, then $G$ has a spanning tree with at most $k+2$ leaves.*

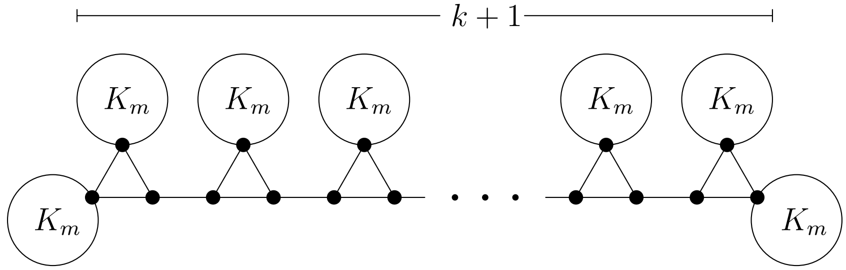


Figure 1.1: Any spanning tree of this graph $G$ must contain more than $k$ branch vertices, while a maximum independent set contains $2k+3$ vertices with degrees adding up to at least $|V(G)|-3$.

From this point forward, we turn our attention to bounds on the number of branch vertices in a graph. Examples can be found in [3, 5, 6, 7, 12]. In particular, a paper of Matsuda, Ozeki, and Yamashita [12] conjectures a condition on connected claw-free graphs which ensures the existence of a spanning tree with at most $k$ branch vertices.

**Conjecture 1.** [12] *Let $k$ be a non-negative integer and let $G$ be a connected claw-free graph of order $n$. If $\sigma_{2k+3}(G) \geq n-2$, then $G$ has a spanning tree with at most $k$ branch vertices.*

A weaker version of this result, which requires just as large an independent set ($\alpha$ denotes independence number) but ignores its degree sum, was shown in the same paper:

**Theorem 3.** [12] *Let $k$ be a non-negative integer. A connected claw-free graph $G$ has a spanning tree with at most $k$ branch vertices if $\alpha(G) \leq 2k+2$.*

Both of the above statements are shown to be best possible by the example in Figure 1.1.

The $k=0$ case of Conjecture 1 follows from Theorem 2. The conjecture's authors prove the $k=1$ case in the same paper.

**Theorem 4.** [12] *Suppose that a connected claw-free graph $G$ of order $n$ satisfies $\sigma_5(G) \geq n-2$. Then $G$ has a spanning tree with at most one branch vertex.*

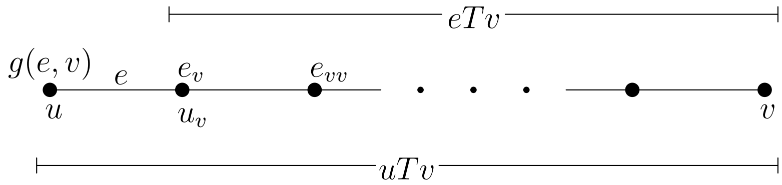


Figure 1.2: A path between vertices $u$ and $v$ within some tree $T$, as described in Definition 2, showing $g(e, v)$ as described in Defintion 1. If $T$ is a spanning tree of some graph $G$, note that $v$ is an oblique neighbor of $e$ with respect to $T$ if and only if $vg(e, v) \in E(G)$.

In Chapter 2, we prove the $k = 2$ case.

**Theorem 5.** [8] *Suppose that a connected claw-free graph $G$ of order $n$ satisfies $\sigma_7(G) \geq n - 2$. Then $G$ has a spanning tree with at most two branch vertices.*

The proofs of Theorem 4 and Theorem 5 make use of Theorem 2. It was not, however, needed for our proof of the entire conjecture.

**Theorem 6.** [9] *Let $G$ be a connected, claw-free graph on $n$ vertices, and let $k$ be a non-negative integer. If $\sigma_{2k+3} \geq n - 2$, then $G$ has a spanning tree with at most $k$ branch vertices.*

An essential concept for the above result is that of *pseudoadjacency* and *pseudoindependence*, which mean something very particular in this context. These require a concept of *oblique neighbors*; the three terms are defined below along with some useful notation. These concepts are new, to our knowledge, and only make sense with respect to a fixed spanning tree.

**Definition 1.** *Let $T$ be a spanning tree of a graph $G$ and let $v \in V(G)$ and $e \in E(T)$. Denote $g(e, v)$ as the vertex incident to $e$ farthest away from $v$ in $T$. We say $v$ and $e$ are* ***oblique neighbors with respect to T*** *if $vg(e, v) \in E(G)$. See Figure 1.2.*

**Definition 2.** *Any two vertices of a tree $T$, say $u$ and $v$, are joined by a unique path, denoted $uTv$, and we denote $d_T(u,v) = |E(uTv)|$. Now if $e \in E(T)$, then $eTv$ denotes the unique shortest path containing $v$ and one of the vertices incident to $e$, but not the edge $e$ itself. We also denote $u_v := V(uTv) \cap N_T(u)$ and $e_v$ as the vertex incident to $e$ in the direction toward $v$. If $e_v \neq v$, then we denote $e_{vv} := V(e_vTv) \cap N_T(e_v)$, similar to the $u_v$ notation. See Figure 1.2.*

Note that both vertices incident to a given edge of $T$ are among its oblique neighbors.

**Definition 3.** *Let $T$ be a spanning tree of a graph. Two vertices are* ***pseudoadjacent with respect to T*** *if there is some $e \in E(T)$ which has them both as oblique neighbors. Similarly, a vertex set is* ***pseudoindependent*** *with respect to $T$ if no two vertices in the set are pseudoadjacent with respect to $T$.*

Note that pseudoadjacency (with respect to any tree) is implied by adjacency and is thus a weaker condition, while pseudoindependence is a stronger condition than independence. We also include an equally useful, but less novel, set of notations for trees:

**Definition 4.** *Let $B = B(T)$ denote the set of branch vertices of a tree $T$, and let $L(T)$ denote the set of leaves. Let $B_n(T)$ denote the set of branch vertices of $T$ with degree exactly $n$, and let $B_{\leq n}(T)$ $\big(B_{\geq n}(T)\big)$ denote the set of branch vertices of $T$ with degree at most (at least) $n$. Lastly, we call the set $S_T = \bigcup_{u,v \in B} uTv$ the* ***internal subtree*** *of $T$.*

# Chapter 2

# Proof for $k = 2$

In this chapter, we prove Theorem 5 which proves the $k = 2$ case of Conjecture 1. We restate the theorem below.

**Theorem 5** [8] *Suppose that a connected claw-free graph $G$ of order $n$ satisfies $\sigma_7(G) \geq n - 2$. Then $G$ has a spanning tree with at most two branch vertices.*

We separate this result into more specific ones based on the structure of a carefully chosen "minimal" spanning tree, as we explain below. All notations given in Definition 4 apply here. Also, in this proof, $[t]$ refers to the set of all positive integers less than or equal to $t$. Some additional notation will be helpful.

**Definition 5.** *Let $v \in V(T) \setminus V(S_T)$. The induced subgraph of $T$ given by those vertices in the same component of $T[V(T) \setminus V(S_T)]$ as $v$ must form a path, which we call $M_v$. We denote the end of this path which is a leaf in $T$ as $l_v$, and the other end as $u_v$. We define $b_v = N_T(u_v) \cap V(S_T)$. Furthermore, we define $v^+ = N_T(v) \cap vTb_v$, and if $v$ is not a leaf we define $v^- = N_T(v) \cap vTl_v$. See Figure 2.1.*

To prove Theorem 5, let $G$ be a connected claw-free graph. Assume $\sigma_7(G) \geq n-2$. By way of contradiction, assume every spanning tree of $G$ has at least 3 branch vertices.

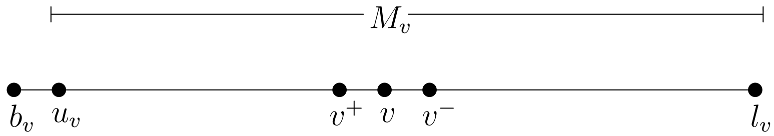


Figure 2.1: A vertex $v$ outside the internal subtree $S_T$ and some nearby vertices. In this diagram, only $\deg_T(b_v) \geq 3$ while $\deg_T(l_v) = 1$, and all other vertices in the diagram have degree 2. The only vertex of $S_T$ shown in this diagram is $b_v$.

By Theorem 2 with $k = 4$, $G$ has a spanning tree with at most 6 leaves. Among all spanning trees of $G$ with at most 6 leaves, choose a spanning tree $T$ also satisfying:

(T1) $T$ has as few branch vertices as possible.

(T2) $T$ has as few leaves as possible, subject to (T1).

Given that $T$ has at most six leaves, it must have at most four branch vertices. Define the *derived tree* $\tau = \tau(T)$ by homeomorphically reducing $T$ (so there are no more degree two vertices) and deleting all leaves. It is not hard to show that $\tau$ is also a tree, as any cycle in $\tau$ would correspond to a cycle in $T$, of which there are none. Now since $T$ has at most six leaves, it can have either three or four branch vertices. If $T$ has only three branch vertices, then necessarily $\tau \cong P_3$, and at most one of the branch vertices of $T$ has degree four in $T$. If one vertex of $T$ has degree 4, it can correspond to either the middle vertex of $\tau(T)$ or an end vertex. We can thus impose two more conditions (the second of which applies regardless of the structure of $T$):

(T3) Suppose two trees $A$ and $B$ exist satisfying (T2), each with exactly one vertex of degree 4, and suppose the middle vertex of $\tau(A)$ corresponds to the degree 4 vertex of $A$, while an end vertex of $\tau(B)$ corresponds to the degree 4 vertex of $B$. We select $A$ over $B$.

(T4) $S_T$ is as small as possible, subject to (T3) if applicable or (T2) otherwise.

Once this $T$ is chosen, several lemmas follow.

## 2.1 Lemmas

**Lemma 1.** *If $N_T(v) = \{a, b, c\}$ and $a, b \notin S_T$, then $ab \in E(G)$.*

*Proof.* Let $v \in V(G)$ such that $N_T(v) = \{a, b, c\}$, and assume $a, b \notin S_T$. Since $T$ has more than one branch vertex, $c \in S_T$. Now if $ac \in E(G)$, then $T' := T - \{va\} + \{ac\}$ either has fewer branch vertices than $T$ (if $c \in B(T)$) or else it has the same number of branch vertices and leaves as $T$, with the same structure, but a smaller internal subtree. Thus either (T1) or (T4) is violated. □

**Lemma 2.** *If $v \in V(T) \setminus V(S_T)$ and $v^+l_v \in E(G)$, then $vl \notin E(G)$ if $l$ is any leaf of $T$ other than $l_v$. In particular, $L(T)$ is an independent set.*

*Proof.* Let $v \in V(T) \setminus V(S_T)$, and assume $v^+l_v \in E(G)$. Let $l$ be a leaf of $T$ other than $l_v$. Then $T' := T - \{vv^+, b_vu_v\} + \{vl, l_vv^+\}$ has no more branch vertices than $T$ and fewer leaves, violating either (T1) or (T2). □

**Lemma 3.** *If $v \in V(T) \setminus V(S_T)$, $v^+l_v \in E(G)$, and $\deg_T(b_v) = 3$, then $vb \notin E(G)$ if $b$ is any branch vertex of $T$ other than $b_v$. In particular, if $b \in B(T)$ and $l \in L(T)$ such that $\deg_T(b_l) = 3$ and $b \neq b_l$, then $lb \notin E(G)$.*

*Proof.* Let $v \in V(T) \setminus V(S_T)$, and assume $v^+l_v \in E(G)$ and $\deg_T(b_v) = 3$. Let $b$ be a branch vertex of $T$ other than $b_v$. Then $T' := T - \{vv^+, b_vu_v\} + \{vb, l_vv^+\}$ has fewer branch vertices than $T$, violating (T1). □

**Lemma 4.** *Let $v \in V(T) \setminus V(S_T)$ such that $\deg_T(b_v) = 3$, $vb_v \in E(G)$, and $|N_T(b_v) \cap S_T| = 1$. Then $v^+l_v \notin E(G)$. In particular, if $l \in L(T)$ such that $\deg_T(b_l) = 3$ and $|N_T(b_l) \cap S_T| = 1$, then $lb_l \notin E(G)$.*

*Proof.* Suppose $v^+l_v \in E(G)$. Define $u' = N_T(b_v) \setminus (S_T \cup \{u_v\})$, so Lemma 1 gives that $u_vu' \in E(G)$. It follows that $T' := T - \{vv^+, b_vu_v, b_vu'\} + \{vb_v, v^+l_v, u_vu'\}$ violates (T1). □

**Lemma 5.** *If $a, c \in L(T)$ and $v \in V(T) \setminus V(S_T)$ and $c \neq l_v \neq a$, then $v \notin N_G(a) \cap N_G(c)$.*

*Proof.* Suppose $av, cv \in E(G)$ for some $a, c, v$ as above. Since $v$ is not a leaf (by Lemma 2), there exists $v^-$. Since $G[v, v^-, a, c]$ is not a claw and Lemma 2 ensures that $ac \notin E(G)$, it follows that either $av^- \in E(G)$ or $cv^- \in E(G)$. Without loss of generality, assume $av^- \in E(G)$. Then $T' := T - \{vv^-, u_v b_v\} + \{av^-, cv\}$ has no more branch vertices than $T$ and fewer leaves, violating either (T1) or (T2). □

**Lemma 6.** *Let $l \in L(T)$, $b \in B(T)$, and $v \in V(T) \setminus V(S_T)$ such that $l \neq l_v$, $b_l \neq b \neq b_v$, $lb \notin E(G)$, and $\deg_T(b_v) = 3$. Then $v \notin N_G(l) \cap N_G(b)$.*

*Proof.* Assume $lv, bv \in E(G)$ for some $l, b, v$ as above. Lemma 2 ensures that $v$ is not a leaf, so there exists $v^-$. Since $G[v, v^-, l, b]$ is not a claw and $lb \notin E(G)$, either $lv^- \in E(G)$ or $bv^- \in E(G)$. If $lv^- \in E(G)$, then $T' := T - \{vv^-, b_v u_v\} + \{lv^-, bv\}$ has fewer branch vertices than $T$, violating (T1). Otherwise $bv^- \in E(G)$, so $T' := T - \{vv^-, b_v u_v\} + \{lv, bv^-\}$ has fewer branch vertices than $T$, still violating (T1). □

**Lemma 7.** *Let $u \in V(T) \setminus V(S_T)$ such that $ub_u \in E(T)$, and let $l_u \neq l \in L(T)$. Then $ul \notin E(G)$.*

*Proof.* Suppose $ul \in E(G)$ for some $u, l$ as above. Then $T' := T - \{ub_u\} + \{ul\}$ has no more branch vertices than $T$ and fewer leaves, violating either (T1) or (T2). □

**Lemma 8.** *Let $u \in V(T) \setminus V(S_T)$ such that $ub_u \in E(T)$ and $\deg_T(b_u) = 3$, and let $b_u \neq b \in B(T)$. Then $ub \notin E(G)$.*

*Proof.* Suppose $ub \in E(G)$. Then $T' := T - \{ub_u\} + \{ub\}$ has fewer branch vertices than $T$, violating (T1). □

We now prove several results about $T$, ruling out one at a time the possible structures it could have.

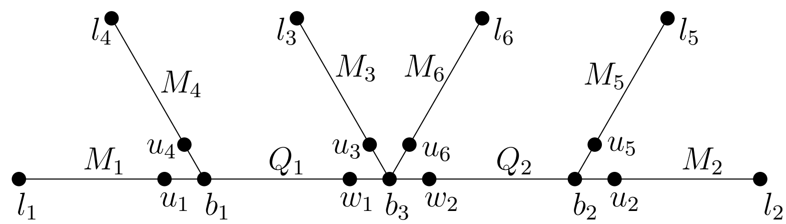


Figure 2.2: If $\tau \cong P_3$, $T$ may have a degree 4 vertex corresponding to the middle vertex of $\tau$. Each vertex labeled $b_i$ is also called $b_{i+3}$.

## 2.2 First Structure

**Proposition 1.** *It is not the case that $\tau(T) \cong P_3$ with its middle vertex corresponding to a degree 4 vertex of $T$.*

*Proof.* By contradiction, suppose $\tau(T) \cong P_3$ with its middle vertex corresponding to a degree 4 vertex of $T$. Then we may represent $T$ with Figure 2.2. As shown in Figure 2.2, we select two leaves with the same nearest branch vertex, which has degree three, and call them $l_1$ and $l_4$. We then call the other two such $l_2$ and $l_5$. We also call the two leaves whose nearest branch vertex has degree four $l_3$ and $l_6$, and we then abbreviate $u_{l_i}$ as $u_i$, and $b_{l_i}$ as $b_i$, and $M_{l_i}$ as $M_i$, for each $i \in [6]$. We also define $w_j = N_T(b_3) \cap V(b_3Tb_j)$ and $Q_j = w_jTb_j$ for each $j \in [2]$. Note that $b_3 = b_6$ is in none of the labeled paths.

Since $G$ is claw-free, there can be no induced claw centered at $b_3$. Among the four vertices of $N_T(b_3)$, therefore, there must be two disjoint cliques whose union is all of $N_T(b_3)$. If these are a singleton and a triplet, the singleton cannot be $u_{3i}$ for any $i \in [2]$, since otherwise $T' := T - \{u_{9-3i}b_3, b_3w_2\} + \{u_{9-3i}w_1, w_1w_2\}$ violates either (T1) or (T4). Therefore either $u_3u_6 \in E(G)$ or $u_3w_1, u_6w_2 \in E(G)$ or $u_3w_2, u_6w_1 \in E(G)$. Also, $u_1, u_2, u_4, u_5 \notin S_T$ are neighbors of $b_1$ and $b_2$, so Lemma 1 gives that $u_1u_4, u_2u_5 \in E(G)$.

**Claim 1.** *The vertex set $X := \{l_1, l_2, l_3, l_4, l_5, l_6, b_3\}$ is independent.*

*Proof.* By Lemmas 2 and 3 and symmetry, we need only show that $l_3b_3 \notin E(G)$, so suppose $l_3b_3 \in E(G)$. If $u_3u_6 \in E(G)$, then $T' := T - \{b_3u_3, b_3u_6\} + \{u_3u_6, b_3l_3\}$ has the same number of branch vertices as $T$ but fewer leaves, violating (T2). On the other hand, if $u_3u_6 \notin E(G)$, then without loss of generality we may assume $u_3w_1 \in E(G)$, so $T' := T - \{b_3u_3, b_3w_1\} + \{u_3w_1, b_3l_3\}$ has the same number of branch vertices as $T$ but fewer leaves, still violating (T2). □

**Claim 2.** *For every* $h \in [6]$, $(N_G(l_h) \cap V(M_h))^- \cap N_G(b_3) = \emptyset$.

*Proof.* Suppose some $v \in (N_G(l_h) \cap V(M_h))^- \cap N_G(b_3)$. By Lemma 3, we may assume $3 \mid h$. Now if $u_3u_6 \in E(G)$, then we may assume $h = 3$ without loss of generality, so $T' := T - \{vv^+, u_3b_3, u_6b_3\} + \{vb_3, v^+l_3, u_3u_6\}$ has the same number of branch vertices as $T$ and one less leaf, violating (T2). Otherwise, either $u_3w_1, u_6w_2 \in E(G)$ or $u_3w_2, u_6w_1 \in E(G)$. Without loss of generality, we may assume $h = 3$ and $u_3w_1 \in E(G)$. Then $T' := T - \{vv^+, b_3u_3, b_3w_1\} + \{b_3v, l_3v^+, u_3w_1\}$ has the same number of branch vertices as $T$ and one less leaf, violating (T2). □

**Claim 3.** *If* $i \neq h$, *then* $N_G(l_i) \cap V(M_h) \cap N_G(b_3) = \emptyset$.

*Proof.* Suppose $v \in N_G(l_i) \cap V(M_h) \cap N_G(b_3)$. Lemma 6 ensures that either $3 \mid h$ or $3 \mid i$. Consider cases:

Case 1: Suppose $3 \nmid h$. Then $3 \mid i$, and since $v \neq l_h$ by Lemma 2, there exists $v^-$. Since $G[v, v^-, b_3, l_i]$ is not a claw and $b_3l_i \notin E(G)$ by Claim 1, either $b_3v^- \in E(G)$ or $l_iv^- \in E(G)$. If $b_3v^- \in E(G)$, then $T' := T - \{vv^-, b_hu_h\} + \{vl_i, b_3v^-\}$ has fewer branch vertices than $T$, violating (T1). Otherwise $l_iv^- \in E(G)$, so $T' := T - \{vv^-, b_hu_h\} + \{vb_3, l_iv^-\}$ has fewer branch vertices than $T$, still violating (T1).

Case 2: Suppose $3 \nmid i$. Then $3 \mid h$, and since $v \neq l_h$ by Lemma 2, there exists $v^-$. Since $G[v, v^-, l_i, b_3]$ is not a claw and $l_ib_3 \notin E(G)$, it follows that either $l_iv^- \in E(G)$ or $b_3v^- \in E(G)$. If $b_3v^- \in E(G)$, then $T' := T - \{vv^-, u_ib_i\} + \{b_3v^-, l_iv\}$ has fewer branch vertices than $T$, contradicting (T1). On the other hand, if $l_i v^- \in E(G)$, we consider whether or not $u_3 u_6 \in E(G)$. If $u_3 u_6 \in E(G)$, then $T' := T - \{vv^-, b_3 u_3, b_3 u_6\} + \{l_i v^-, b_3 v, u_3 u_6\}$ has the same number of branch vertices as $T$ but fewer leaves, contradicting (T2). If $u_3 u_6 \notin E(G)$, then $u_h w_j \in E(G)$ for some $j \in [2]$, and $T' := T - \{b_3 u_h, b_3 w_j, vv^-\} + \{u_h w_j, b_3 v, l_i v^-\}$ has the same number of branch vertices as $T$ but fewer leaves, contradicting (T2).

Case 3: Suppose both $3 \mid i$ and $3 \mid h$. Without loss of generality, assume $h = 3$ and $i = 6$, so $v \in V(M_3)$ and $vb_3, vl_6 \in E(G)$ (and there exists $v^-$, as before). Consider cases:

Case 3a: Suppose $w_i u_3 \in E(G)$ for some $i \in [2]$. Since $G[v, v^-, b_3, l_6]$ is not a claw and $b_3 l_6 \notin E(G)$, either $l_6 v^- \in E(G)$ or $b_3 v^- \in E(G)$. If $l_6 v^- \in E(G)$, then $T' := T - \{vv^-, b_3 u_3, b_3 w_i\} + \{l_6 v^-, b_3 v, u_3 w_i\}$ has the same number of branch vertices as $T$ and fewer leaves, contradicting (T2). On the other hand, if $b_3 v^- \in E(G)$, then $T' := T - \{vv^-, b_3 u_3, b_3 w_i\} + \{b_3 v^-, l_6 v, u_3 w_i\}$ has the same number of branch vertices as $T$ but fewer leaves, still contradicting (T2).

Case 3b: Suppose $w_i u_6 \in E(G)$ for some $i \in [2]$. Since $G[v, v^-, b_3, l_6]$ is not a claw and $b_3 l_6 \notin E(G)$, either $l_6 v^- \in E(G)$ or $b_3 v^- \in E(G)$. If $l_6 v^- \in E(G)$, then $T' := T - \{vv^-, b_3 u_6, b_3 w_i\} + \{l_6 v^-, l_6 v, u_6 w_i\}$ has the same number of branch vertices as $T$ and fewer leaves, contradicting (T2). On the other hand, if $b_3 v^- \in E(G)$, then $T' := T - \{vv^-, b_3 u_6, b_3 w_i\} + \{b_3 v^-, l_6 v, u_6 w_i\}$ has the same number of branch vertices as $T$ but fewer leaves, still contradicting (T2).

Case 3c: Suppose $w_1 u_3, w_1 u_6, w_2 u_3, w_2 u_6 \notin E(G)$. In this case, since $G[b_3, w_1, u_3, u_6]$ is not a claw and $u_3 w_1, u_6 w_1 \notin E(G)$, it follows that $u_3 u_6 \in E(G)$. Also, since $G[b_3, w_1, w_2, u_3]$ is not a claw and $w_1 u_3, w_2 u_3 \notin E(G)$, it follows that $w_1 w_2 \in E(G)$. As before, since $G[v, v^-, b_3, l_6]$ is not a claw and $b_3 l_6 \notin E(G)$, either $l_6 v^- \in E(G)$ or $b_3 v^- \in E(G)$. If $l_6 v^- \in E(G)$, then $T' := T - \{b_3 u_3, b_3 u_6, vv^-\} + \{u_3 u_6, b_3 v, l_6 v^-\}$ has the same number of branch vertices as $T$ but one less leaf, contradicting (T2). We consider separately the case where $b_3v^- \in E(G)$:

Case 3c′: Suppose $u_3u_6, w_1w_2, b_3v^- \in E(G)$. For each $i \equiv 0 \pmod 3$, $j \in [2]$, since $G[b_3, v^-, u_i, w_j]$ is not a claw and $u_iw_j \notin E(G)$, it follows that either $v^-u_i \in E(G)$ or $v^-w_j \in E(G)$. In other words, there does not exist a pair $(i, j)$ such that $v^-u_i, v^-w_j \notin E(G)$. Therefore either $v^-w_1, v^-w_2 \in E(G)$, or else $v^-u_3, v^-u_6 \in E(G)$. If $v^-w_1, v^-w_2 \in E(G)$, then $T' := T - \{vv^-, b_3w_2, b_3u_3, b_3u_6\} + \{w_1v^-, w_1w_2, b_3v, u_3u_6\}$ is a tree with the same number of branch vertices (barring $w_1 = b_1$, which would violate (T1)) and leaves, with the same structure, but $|V(S_{T'})| < |V(S_T)|$, contradicting (T4). On the other hand, if $v^-u_3, v^-u_6 \in E(G)$, then $T' := T - \{vv^-, b_3u_3\} + \{l_6v, v^-u_3\}$ has the same number of branch vertices at $T$ and fewer leaves, contradicting (T2) and completing the proof of Claim 3. □

**Claim 4.** *If $i \equiv j \pmod 3$, then $N_G(l_i) \cap V(Q_j) = \emptyset$.*

*Proof.* Suppose $v \in N_G(l_i) \cap V(Q_j)$. Then $v \neq b_i$ by Lemma 4, so $T' := T - \{b_iu_i\} + \{vl_i\}$ has the same number of branch vertices and leaves as $T$, still with the same structure, but $|V(S_{T'})| < |V(S_T)|$, violating (T4). □

**Claim 5.** *If $i + j \equiv h \equiv 0 \pmod 3$, then $N_G(l_i) \cap V(Q_j) \cap N_G(l_h) = \emptyset$.*

*Proof.* Suppose some $v \in N_G(l_i) \cap V(Q_j) \cap N_G(l_h)$. Lemma 3 ensures that $v \neq b_j$, so $T' := T - \{b_iu_i, b_3u_h\} + \{l_iv, l_hv\}$ matches the structure of $T$ but $|V(S_{T'})| < |V(S_T)|$, violating (T4). □

**Claim 6.** *If $i + j \equiv 0 \pmod 3$, then $(N_G(b_3) \cap V(Q_j))^- \cap N_G(l_i) = \emptyset$.*

*Proof.* Suppose some $v \in (N_G(b_3) \cap V(Q_j))^- \cap N_G(l_i)$. Then $v^+b_3 \in E(G)$, so $T' := T - \{vv^+, b_iu_i\} + \{v^+b_3, l_iv\}$ violates (T1). □

**Claim 7.** *If $i + j = 3$, then $N_G(l_i) \cap V(Q_j) \cap N_G(l_{i+3}) = \emptyset$.*

*Proof.* Suppose some $v \in N_G(l_i) \cap V(Q_j) \cap N_G(l_{i+3})$. Then $T' := T - \{b_iu_i, b_3w_i\} + \{l_iv, l_{i+3}v\}$ violates (T4) since $|V(S_{T'})| < |V(S_T)|$. □

**Claim 8.** *For every $j \in [2]$, $N_G(l_3) \cap V(Q_j) \cap N_G(l_6) = \emptyset$.*

*Proof.* Suppose $v \in N_G(l_3) \cap V(Q_j) \cap N_G(l_6)$. Now if $u_3u_6 \in E(G)$, then $T' := T - \{b_3u_3, b_3u_6\} + \{vl_3, u_3u_6\}$ has no more branch vertices than $T$ and fewer leaves, violating either (T1) or (T2). Otherwise, either $u_3w_1, u_6w_2 \in E(G)$ or $u_3w_2, u_6w_1 \in E(G)$. Without loss of generality, assume $u_3w_1, u_6w_2 \in E(G)$ and $j = 1$. Then $T' := T - \{b_3u_6, b_3w_2\} + \{vl_6, u_6w_2\}$ has at most as many branch vertices as $T$ and fewer leaves, again violating (T1) or (T2). □

**Claim 9.** *If $3|i$, then $(N_G(b_3) \cap V(Q_j))^- \cap N_G(l_i) = \emptyset$.*

*Proof.* Suppose $v \in (N_G(b_3) \cap V(Q_j))^- \cap N_G(l_i)$. Then $v^+b_3 \in E(G)$, so $T' := T - \{vv^+, b_3w_j\} + \{l_iv, v^+b_3\}$ violates (T4) since $|V(S_T')| < |V(S_T)|$. □

Claim 1 gives an independent 7-vertex set $X := \{l_1, l_2, l_3, l_4, l_5, l_6, b_3\}$. For every $h, i \in [6]$ with $i \neq h$, $(N_G(l_h) \cap V(M_h))^-$ is disjoint from both $N_G(l_i) \cap V(M_h)$ and $N_G(b_3) \cap V(M_h)$, by Lemma 2 and Claim 2, respectively. Lemma 5 gives that the five sets $N_G(l_i) \cap V(M_h)$ are disjoint from each other, and Claim 3 ensures that $N_G(b_3) \cap V(M_h)$ is disjoint from any of them. Therefore, for every $h \in [6]$, the seven sets $(N_G(l_h) \cap V(M_h))^-$, $N_G(b_3) \cap V(M_h)$, and $N_G(l_i) \cap V(M_h)$ for each $i \neq h$ are all disjoint. Furthermore, Lemmas 7 and 8 show that $u_h$ is in none of these sets if $3 \nmid h$. Therefore:

$$\begin{aligned}
&\sum_{v\in X}|N_G(v)\cap V(M_h)|\\
=\ &|N_G(b_3)\cap V(M_h)|+|N_G(l_h)\cap V(M_h)|+\sum_{i\neq h}|N_G(l_i)\cap V(M_h)|\\
=\ &|N_G(b_3)\cap V(M_h)|+|(N_G(l_h)\cap V(M_h))^-|+\sum_{i\neq h}|N_G(l_i)\cap V(M_h)|\\
\leq\ &\begin{cases}|V(M_h)| & h\equiv 0(\text{mod } 3)\\ |V(M_h)|-1 & h\not\equiv 0(\text{mod } 3).\end{cases}
\end{aligned}$$

Claim 4, meanwhile, shows that for each $j\in[2]$ the only possible neighbors of vertices in $V(Q_j)$ in $X$ are $l_{3-j}$, $l_{6-j}$, $l_3$, $l_6$, and $y_3$; Claims 5-9 show that for each $j\in[2]$, the five sets $N_G(l_{3-j})\cap V(Q_j)$, $N_G(l_{6-j})\cap V(Q_j)$, $N_G(l_3)\cap V(Q_j)$, $N_G(l_6)\cap V(Q_j)$, and $(N_G(b_3)\cap V(Q_j))^-$ are all disjoint. Therefore, for each $j\in[2]$:

$$\sum_{v\in X}|N_G(v)\cap V(Q_j)|=|N_G(l_{3-j})\cap V(Q_j)|+|N_G(l_{6-j})\cap V(Q_j)|+|N_G(l_3)\cap V(Q_j)|+|N_G(l_6)\cap V(Q_j)|+|(N_G(b_3)\cap V(Q_j))^-|\leq|V(Q_j)|.$$

Since $b_3\in X$, no vertex of $X$ is adjacent to $b_3$ in $G$, so we can sum these inequalities to $\sum_{v\in X}\deg_G(v)\leq n-4$, contradicting the assumption that $\sigma_7(G)\geq n-2$.

□

## 2.3 Second Structure

**Proposition 2.** *There is no degree 4 vertex in $T$.*

*Proof.* Suppose there is a degree 4 vertex in $T$. Proposition 1 gives that it cannot correspond to the middle vertex of $\tau(T)$, so it must correspond to an end vertex. We call the degree 4 vertex $b$, and we call the middle branch vertex $x$ and the remaining one $y$. The three leaves whose nearest branch vertex is $b$ shall be called $l_1$, $l_2$, and

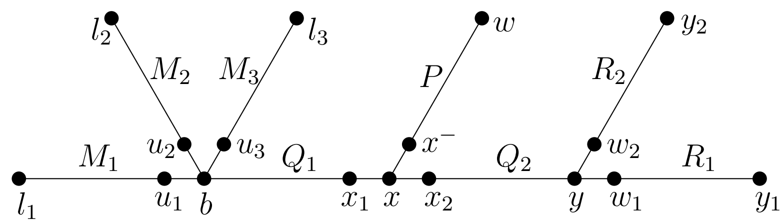


Figure 2.3: If $\tau \cong P_3$, $T$ may have a degree 4 vertex corresponding to an end vertex of $\tau$.

$l_3$, and we abbreviate $u_{l_i}$ as $u_i$ and $M_{l_i}$ as $M_i$ for each $i \in [3]$. The other leaves and branch vertex neighbors are labeled as shown in Figure 2.3, with the labeled paths running only between nearest labeled vertices, similar to Figure 2.2 (for example, $Q_1 = bTx_1$), with one important exception: $P = wTx$.

Recall condition (T3), which prefers trees whose middle branch vertex has degree 4 over trees with an "end" branch vertex of degree 4. This condition, together with our choice of $T$, rules out the existence of any spanning tree of $G$ whose middle branch vertex (of three) has degree 4.

Once this $T$ is chosen, since $G$ is claw-free, there can be no induced claw centered at $b$. Define $b^+ := N_T(b) \cap V(bTx)$. If there are two distinct $i, j \in [3]$ such that $u_ib^+, u_jb^+ \in E(G)$, then consider $T' := T - \{bu_i, bu_j\} + \{u_ib^+, u_jb^+\}$. If $b^+ = x$, then $T'$ has fewer branch vertices than $T$, violating (T1). Otherwise $T'$ has the same number of branch vertices and leaves as $T$, with the same structure, but $|V(S_{T'})| < |V(S_T)|$, violating (T4). Therefore there is at most one $i \in [3]$ such that $u_ib^+ \in E(G)$. If such an $i$ exists, let $\{j, k\} = [3] \setminus \{i\}$, so it is easily seen that $u_ju_k \in E(G)$. Otherwise, it is easily seen that $\{u_1, u_2, u_3\}$ is a clique. Also, Lemma 1 gives that $w_1w_2 \in E(G)$.

**Claim 1.** *The vertex set $X := \{l_1, l_2, l_3, w, y_1, y_2, b\}$ is independent.*

*Proof.* By Lemmas 2 and 3, we need only show that $l_ib \notin E(G)$ for each $i \in [3]$. Assume $l_ib \in E(G)$. Then either $u_ib^+ \in E(G)$ or $u_iu_j \in E(G)$ for some $j \neq i$. If

$u_ib^+ \in E(G)$, then $T' := T - \{bb^+, bu_i\} + \{b^+u_i, l_ib\}$ has the same number of branch vertices as $T$ but fewer leaves, violating either (T1) or (T2). Otherwise $u_iu_j \in E(G)$ for some $j \neq i$, so $T' := T - \{bu_i, bu_j\} + \{bl_i, u_iu_j\}$ has the same number of branch vertices as $T$ but fewer leaves, violating (T2). □

**Claim 2.** *For every $h \in [3]$, $(N_G(l_h) \cap V(M_h))^- \cap N_G(b) = \emptyset$.*

*Proof.* Suppose $v \in (N_G(l_h) \cap V(M_h))^- \cap N_G(b)$. Then $v^+ \in N_G(l_h) \cap V(M_h)$, and either $u_hb^+ \in E(G)$ or $u_hu_i \in E(G)$ for some $i \neq h$. If $u_hb^+ \in E(G)$, then $T' := T - \{bb^+, bu_h, vv^+\} + \{v^+l_h, vb, b^+u_h\}$ has the same number of branch vertices as $T$ and fewer leaves, violating (T2). Otherwise $u_hu_i \in E(G)$ for some $i \neq h$, so $T' := T - \{vv^+, bu_h, bu_i\} + \{vb, v^+l_h, u_hu_i\}$ has the same number of branch vertices as $T$ and fewer leaves, violating (T2). □

**Claim 3.** *For every $i \in [3]$, $N_G(l_i) \cap V(P) \cap N_G(b) = \emptyset$.*

*Proof.* Suppose $v \in N_G(l_i) \cap V(P) \cap N_G(b)$. Now if $v = x$, then consider $G[x, x^-, b, l_i]$. We have $bl_i \notin E(G)$ by Claim 1, $x^-l_i \notin E(G)$ by Lemma 7, and $x^-b \notin E(G)$ by Lemma 8. This makes $G[x, x^-, b, l_i]$ an induced claw, which is a contradiction. On the other hand, if $v \neq x$, then since $v \neq w$, there exists $v^-$. Since $G[v, v^-, b, l_i]$ is not a claw and $bl_i \notin E(G)$, it follows that either $v^-b \in E(G)$ or $v^-l_i \in E(G)$. If $v^-b \in E(G)$, then $T' := T - \{vv^-, xx^-\} + \{v^-b, vl_i\}$ has fewer branch vertices than $T$; otherwise $v^-l_i \in E(G)$, so $T' := T - \{vv^-, xx^-\} + \{bv, l_iv^-\}$ has fewer branch vertices than $T$. Either way (T1) is still violated. □

**Claim 4.** *For every $i \in [3]$ and $h \in [2]$, $N_G(l_i) \cap V(R_h) \cap N_G(b) = \emptyset$.*

*Proof.* Suppose $v \in N_G(l_i) \cap V(R_h) \cap N_G(b)$. Since $v \neq y_h$, there exists $v^-$. Since $G[v, v^-, b, l_i]$ is not a claw and $bl_i \notin E(G)$, either $bv^- \in E(G)$ or $l_iv^- \in E(G)$. If $bv^- \in E(G)$, then $T' := T - \{vv^-, yw_h\} + \{bv^-, l_iv\}$ has fewer branch vertices than

$T$; otherwise $l_iv^- \in E(G)$, so $T' := T - \{vv^-, yw_h\} + \{bv, l_iv^-\}$ has fewer branch vertices than $T$. Either way (T1) is violated. □

**Claim 5.** *For every $h \in [3]$, $N_G(w) \cap V(M_h) \cap N_G(b) = \emptyset$.*

*Proof.* Suppose $v \in N_G(w) \cap V(M_h) \cap N_G(b)$. Now either $u_hb^+ \in E(G)$ or there exists some $i \in [3] \setminus \{h\}$ such that $u_hu_i \in E(G)$. Consider two cases:

Case 1: Suppose $u_hb^+ \in E(G)$. Now if $b^+ = x$, then $T' := T - \{bu_h\} + \{xu_h\}$ corresponds to Figure 2.2, violating (T3). If $b^+ \neq x$, then $T' := T - \{bu_h, bb^+, xx_1\} + \{vw, vb, b^+u_h\}$ corresponds to Figure 2.2, violating (T3).

Case 2: Suppose $u_hu_i \in E(G)$. Since $v \neq l_h$, there exists $v^-$, and since $G[v, v^-, b, w]$ is not a claw and $bw \notin E(G)$, it follows that either $bv^- \in E(G)$ or $wv^- \in E(G)$. Now if $bv^- \in E(G)$, then $T' := T - \{vv^-, bu_h, bu_i\} + \{bv^-, wv, u_hu_i\}$ has the same number of branch vertices as $T$ but fewer leaves, violating (T2). Otherwise $wv^- \in E(G)$, so $T' := T - \{vv^-, bu_h, bu_i\} + \{u_hu_i, bv, wv^-\}$ has the same number of branch vertices as $T$ but fewer leaves, violating (T2). □

**Claim 6.** *For every $h \in [3]$ and $i \in [2]$, $N_G(y_i) \cap V(M_h) \cap N_G(b) = \emptyset$.*

*Proof.* Suppose $v \in N_G(y_i) \cap V(M_h) \cap N_G(b)$. Now either $u_hb^+ \in E(G)$ or there exists some $j \in [3] \setminus \{h\}$ such that $u_hu_j \in E(G)$. Consider two cases:

Case 1: Suppose $u_hb^+ \in E(G)$. Now if $b^+ = x$, then $T' := T - \{bu_h\} + \{xu_h\}$ corresponds to Figure 2.2, violating (T3). Otherwise, $T' := T - \{bu_h, bb^+, xx_2\} + \{vy_i, vb, b^+u_h\}$ corresponds to Figure 2.2, again violating (T3).

Case 2: Suppose $u_hu_j \in E(G)$. Since $v \neq l_h$, there exists $v^-$, and since $G[v, v^-, b, y_i]$ is not a claw and $by_i \notin E(G)$, it follows that either $bv^- \in E(G)$ or $y_iv^- \in E(G)$. Now if $bv^- \in E(G)$, then $T' := T - \{vv^-, bu_h, bu_j\} + \{bv^-, y_iv, u_hu_j\}$ has the same number of branch vertices as $T$ but fewer leaves, violating (T2). Otherwise, $y_iv^- \in E(G)$, so

$T' := T - \{vv^-, bu_h, bu_j\} + \{u_hu_j, bv, y_iv^-\}$ has the same number of branch vertices as $T$ but fewer leaves, again violating (T2). □

**Claim 7.** *If $h \neq i$, then $N_G(l_i) \cap V(M_h) \cap N_G(b) = \emptyset$.*

*Proof.* Suppose $v \in N_G(l_i) \cap V(M_h) \cap N_G(b)$. Choose $j \in [3] \setminus \{h, i\}$ and consider two cases:

Case 1: Suppose $u_jb^+ \notin E(G)$. Then either $u_ju_i \in E(G)$ or $u_ju_h \in E(G)$. If $u_ju_i \in E(G)$, then $T' := T - \{bu_i, bu_j\} + \{u_ju_i, vl_i\}$ has the same number of branch vertices as $T$ but fewer leaves, violating (T2). Otherwise $u_ju_h \in E(G)$, so $T' := T - \{bu_h, bu_j\} + \{u_hu_j, vl_i\}$ has the same number of branch vertices as $T$ but fewer leaves, still violating (T2).

Case 2: Suppose $u_jb^+ \in E(G)$. Then $u_hu_i \in E(G)$, and since $v \neq l_h$, there exists $v^-$. Since $G[v, v^-, b, l_i]$ is not a claw and $bl_i \notin E(G)$, it follows that either $l_iv^- \in E(G)$ or $bv^- \in E(G)$. If $l_iv^- \in E(G)$, then $T' := T - \{bu_h, bu_i, vv^-\} + \{bv, u_hu_i, l_iv^-\}$ has the same number of branch vertices as $T$ but fewer leaves, violating (T2). Otherwise $bv^- \in E(G)$, and since $G[b, u_h, v^-, b^+]$ is not a claw and $u_hb^+ \notin E(G)$, it follows that either $u_hv^- \in E(G)$ or $b^+v^- \in E(G)$. If $u_hv^- \in E(G)$, then $T' := T - \{vv^-, bu_h\} + \{l_iv, u_hv^-\}$ has the same number of branch vertices as $T$ but fewer leaves, violating (T2). Otherwise $b^+v^- \in E(G)$, so consider $T' := T - \{vv^-, bu_h, bu_j\} + \{b^+v^-, b^+u_j, l_iv\}$. If $b^+ = x$, then $T'$ has fewer branch vertices than $T$, violating (T1). Otherwise, $T'$ has the same number of branch vertices and leaves as $T$, with the same structure, but $|V(S_{T'})| < |V(S_T)|$, violating (T4). □

**Claim 8.** *If $h, i \in [2]$, then $N_G(y_i) \cap V(Q_h) = \emptyset$.*

*Proof.* Suppose $v \in N_G(y_i) \cap V(Q_h)$. Choose $j \in [2] \setminus \{i\}$ and consider $T' := T - \{yw_i, yw_j\} + \{vy_i, w_iw_j\}$. If $v = b$ or $v = y$, then $T'$ has fewer branch vertices than

$T$, violating (T1). Otherwise, $T'$ has the same number of branch vertices and leaves as $T$, both matching Figure 2.3, but $|V(S_{T'})| < |V(S_T)|$, violating (T4). □

**Claim 9.** *If $i \neq j$, then $N_G(l_i) \cap V(Q_1) \cap N_G(l_j) = \emptyset$.*

*Proof.* Suppose $v \in N_G(l_i) \cap V(Q_1) \cap N_G(l_j)$. Then $v \neq b$, so $T' := T - \{bu_i, bu_j\} + \{vl_i, vl_j\}$ has the same number of branch vertices and leaves as $T$, with the same structure, but $|V(S_{T'})| < |V(S_T)|$, violating (T4). □

**Claim 10.** *If $i \neq j$, then $N_G(l_i) \cap V(Q_2) \cap N_G(l_j) = \emptyset$.*

*Proof.* Suppose $v \in N_G(l_i) \cap V(Q_2) \cap N_G(l_j)$. Then consider $T' := T - \{bu_i, bu_j\} + \{vl_i, vl_j\}$. If $v = y$, then $T'$ has fewer branch vertices than $T$, violating (T1). Otherwise $T'$ corresponds to Figure 2.2, violating (T3). □

**Claim 11.** *If $i \in [3]$ and $h \in [2]$, then $N_G(l_i) \cap V(Q_h) \cap N_G(w) = \emptyset$.*

*Proof.* Suppose $v \in N_G(l_i) \cap V(Q_h) \cap N_G(w)$. Then $v \neq b$, and it is easily verified that $v \neq y$, so $T' := T - \{bu_i, xx^-\} + \{vw, vl_i\}$ has corresponds to Figure 2.2, violating (T3). □

**Claim 12.** *If $i \in [3]$, then $(N_G(b) \cap V(Q_1))^- \cap N_G(l_i) = \emptyset$.*

*Proof.* Suppose $v \in (N_G(b) \cap V(Q_1))^- \cap N_G(l_i)$. Then $v^+ \in N_G(b) \cap V(Q_1)$, so $T' := T - \{vv^+, bb^+\} + \{l_iv, bv^+\}$ has the same number of branch vertices and leaves as $T$, with the same structure, but $|V(S_{T'})| < |V(S_T)|$, violating (T4). □

**Claim 13.** *We have $(N_G(b) \cap V(Q_1))^- \cap N_G(w) = \emptyset$.*

*Proof.* Suppose $v \in (N_G(b) \cap V(Q_1))^- \cap N_G(w)$. Then $v^+ \in N_G(b) \cap V(Q_1)$, so $T' := T - \{vv^+, xx^-\} + \{vw, v^+b\}$ has fewer branch vertices than $T$, violating (T1). □

**Claim 14.** *If $i \in [3]$, then $(N_G(b) \cap V(Q_2))^- \cap N_G(l_i) = \emptyset$.*

*Proof.* Suppose $v \in (N_G(b) \cap V(Q_2))^- \cap N_G(l_i)$. Then $v^+ \in N_G(b) \cap V(Q_2)$, so $T' := T - \{vv^+, xx_2\} + \{bv^+, l_iv\}$ has fewer branch vertices than $T$, violating (T1). □

**Claim 15.** *We have* $(N_G(b) \cap V(Q_2))^- \cap N_G(w) = \emptyset$*.*

*Proof.* Suppose $v \in (N_G(b) \cap V(Q_2))^- \cap N_G(w)$. Then $v^+ \in N_G(b) \cap V(Q_2)$, so $T' := T - \{vv^+, xx_2\} + \{wv, bv^+\}$ has fewer branch vertices than $T$, violating (T1). □

**Claim 16.** *We have* $wx \notin E(G)$*.*

*Proof.* Suppose $wx \in E(G)$. Since $G[x, x^-, x_1, x_2]$ is not a claw, either $x^-x_1 \in E(G)$ or $x^-x_2 \in E(G)$ or $x_1x_2 \in E(G)$. If $x^-x_1 \in E(G)$, then $T' := T - \{xx^-, xx_1\} + \{wx, x^-x_1\}$ violates (T1). If $x^-x_2 \in E(G)$, then $T' := T - \{xx^-, xx_2\} + \{wx, x^-x_2\}$ violates (T1). Otherwise $x_1x_2 \in E(G)$, so $T' := T - \{xx_1\} + \{x_1x_2\}$ violates (T4). □

Lemma 2 ensures that $(N_G(w) \cap V(P))^-$ is disjoint from $N_G(y_i) \cap V(P)$ for each $i \in [2]$ and from $N_G(l_j) \cap V(P)$ for each $j \in [3]$. Lemma 3 ensures that $(N_G(w) \cap V(P))^-$ is disjoint from $N_G(b) \cap V(P)$. Lemma 5 ensures that the five sets $N_G(y_i) \cap V(P)$ for each $i \in [2]$ and $N_G(l_j) \cap V(P)$ for each $j \in [3]$ are all disjoint. Lemma 6 ensures that $N_G(b) \cap V(P)$ is disjoint from $N_G(y_i) \cap V(P)$ for each $i \in [2]$, and Claim 3 ensures that $N_G(l_j) \cap V(P)$ is disjoint from $N_G(b) \cap V(P)$ for each $j \in [3]$. Therefore the seven sets $(N_G(w) \cap V(P))^-$, $N_G(y_i) \cap V(P)$ for $i \in [2]$, $N_G(l_j) \cap V(P)$ for $j \in [3]$, and $N_G(b) \cap V(P)$ are all disjoint. Furthermore, Lemmas 7 and 8 and Claim 16 ensure that none of them contain $x^-$, so the sum of their cardinalities is at most $|V(P)| - 1$.

Similarly, for each $h \in [2]$, Lemma 2 ensures that $(N_G(y_h) \cap V(R_h))^-$ is disjoint from any of $N_G(y_{3-h}) \cap V(R_h)$, $N_G(w) \cap V(R_h)$, and $N_G(l_j) \cap V(R_h)$ (for $j \in [3]$), and Lemma 3 ensures that $(N_G(y_h) \cap V(R_h))^-$ is disjoint from $N_G(b) \cap V(R_h)$. Lemma 5 ensures that the five sets $N_G(y_{3-h}) \cap V(R_h)$, $N_G(w) \cap V(R_h)$, and $N_G(l_j) \cap V(R_h)$ are all disjoint. Lemma 6 ensures that $N_G(b) \cap V(R_h)$ is disjoint from both $N_G(y_{3-h}) \cap V(R_h)$ and $N_G(w) \cap V(R_h)$, while Claim 4 ensures that $N_G(b) \cap V(R_h)$ is disjoint from

$N_G(l_j) \cap V(R_h)$. Therefore the seven sets $(N_G(y_h) \cap V(R_h))^-$, $N_G(y_{3-h}) \cap V(R_h)$, $N_G(w) \cap V(R_h)$, $N_G(l_j) \cap V(R_h)$ for $j \in [3]$, and $N_G(b) \cap V(R_h)$ are all disjoint. Now Lemmas 7 and 8 ensure that none of them contain $w_h$, so the sum of their cardinalities is at most $|V(R_h)| - 1$.

Similarly, for each $h \in [3]$, Lemma 2 ensures that $(N_G(l_h) \cap V(M_h))^-$ is disjoint from any of $N_G(l_i) \cap V(M_h)$ (for $i \neq h$), $N_G(w) \cap V(M_h)$, and $N_G(y_j) \cap V(M_h)$ (for $j \in [2]$), and Claim 2 ensures that $(N_G(l_h) \cap V(M_h))^-$ is disjoint from $N_G(b) \cap V(M_h)$. Meanwhile, Lemma 5 ensures that the five sets $N_G(l_i) \cap V(M_h)$ for $i \neq h$, $N_G(w) \cap V(M_h)$, and $N_G(y_j) \cap V(M_h)$ are all disjoint. Now $N_G(b) \cap V(M_h)$ is disjoint from $N_G(y_j) \cap V(M_h)$ (by Claim 6), $N_G(w) \cap V(M_h)$ (by Claim 5), and $N_G(l_i) \cap V(M_h)$ (by Claim 7). Therefore the seven sets $(N_G(l_h) \cap V(M_h))^-$, $N_G(l_i) \cap V(M_h)$ for $i \neq h$, $N_G(w) \cap V(M_h)$, $N_G(y_j) \cap V(M_h)$ for $j \in [2]$, and $N_G(b) \cap V(M_h)$ are all disjoint, so the sum of their cardinalities is at most $|V(M_h)|$.

Finally, for each $h \in [2]$, Claim 8 gives that the two sets $N_G(y_i) \cap V(Q_h)$ are empty, and Claims 9-15 give that the five sets $N_G(l_i) \cap V(Q_h)$, $N_G(w) \cap V(Q_h)$, and $(N_G(b) \cap V(Q_h))^-$ are all disjoint, so the sum of their cardinalities is at most $|V(Q_h)|$.

Summing these inequalities gives $\sum_{v \in X} \deg_G(v) \leq n - 3$, contradicting the assumption of the theorem. □

We now know that $T$ has no degree 4 vertices.

## 2.4 Third Structure

**Proposition 3.** *Our tree $T$ has at least four branch vertices.*

*Proof.* By contradiction, suppose $T$ has only three branch vertices. Since Proposition 2 requires that they all have degree 3, we label vertices and paths as shown in Figure

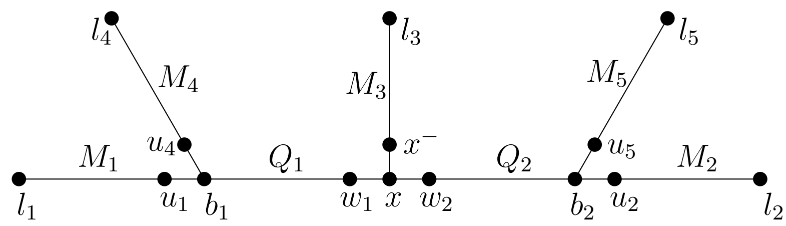


Figure 2.4: If $\tau \cong P_3$, $T$ may have no degree 4 vertices. Each vertex labeled $b_i$ is also called $b_{i+3}$.

2.4, with each labeled path connecting only the nearest labeled vertices, as with the other figures, with one important exception: $M_3 = xTl_3$. Lemma 1 gives that $u_1u_4, u_2u_5 \in E(G)$. Furthermore, (T4) gives that $w_1w_2 \notin E(G)$, so either $w_1x^- \in E(G)$ or $w_2x^- \in E(G)$.

**Claim 1.** *The vertex set $X := \{l_1, l_2, l_3, l_4, l_5, b_1, b_2\}$ is independent.*

*Proof.* By Lemmas 2, 3, and 4, we need only show that $b_1b_2 \notin E(G)$. If $b_1b_2 \in E(G)$, then $T' := T - \{w_1x\} + \{b_1b_2\}$ has fewer branch vertices than $T$, violating (T1). □

**Claim 2.** *If $h \neq i$ and $j \in [2]$, then $N_G(l_i) \cap V(M_h) \cap N_G(b_j) = \emptyset$.*

*Proof.* Suppose $v \in N_G(l_i) \cap V(M_h) \cap N_G(b_j)$. Lemma 6 requires that either $h \equiv j (\text{mod } 3)$ or $i \equiv j (\text{mod } 3)$, and since $v \neq l_h$, there exists $v^-$. Consider cases:

Case 1: Suppose $h \equiv i \equiv j (\text{mod } 3)$. Since $G[v, v^-, l_i, b_j]$ is not a claw and $l_ib_j \notin E(G)$, it follows that either $l_iv^- \in E(G)$ or $b_jv^- \in E(G)$. If $l_iv^- \in E(G)$, then $T' := T - \{b_ju_h, b_ju_i, vv^-\} + \{l_iv^-, b_jv, u_hu_i\}$ has fewer branch vertices than $T$, violating (T1). Otherwise $b_jv^- \in E(G)$, so since $G[b_j, v^-, u_h, b_j^+]$ is not a claw and $u_hb_j^+ \notin E(G)$, it follows that either $b_j^+v^- \in E(G)$ or $u_hv^- \in E(G)$. If $b_j^+v^- \in E(G)$, then $T' := T - \{vv^-, b_ju_h\} + \{l_iv, b_j^+v^-\}$ either has fewer branch vertices than $T$ (if $b_j^+ = x$) or else the same number of branch vertices and leaves, with the same structure, but with a smaller internal subtree. Otherwise $u_hv^- \in E(G)$, so $T' :=$

$T - \{vv^-, b_ju_h\} + \{l_iv, u_hv^-\}$ has fewer branch vertices than $T$. In each case, (T1) or (T4) is violated.

Case 2: Suppose $h \equiv j \not\equiv i \pmod 3$. If $i = 3$, then $T' := T - \{xx^-, b_ju_j, b_ju_{j+3}\} + \{u_ju_{j+3}, vl_i, vb_j\}$ has fewer branch vertices than $T$. If $i \neq 3$, then $T' := T - \{b_iu_i, b_ju_j, b_ju_{j+3}\} + \{u_ju_{j+3}, vl_i, vb_j\}$ has fewer branch vertices than $T$. Either way (T1) is violated.

Case 3: Suppose $h = 3$. Then $i \equiv j \pmod 3$, so $T' := T - \{xx^-, b_ju_j, b_ju_{j+3}\} + \{b_jv, l_iv, u_ju_{j+3}\}$ has fewer branch vertices than $T$, violating (T1).

Case 4: Suppose $3 \neq h \not\equiv j \equiv i \pmod 3$. Then $T' := T - \{b_ju_j, b_ju_{j+3}, xw_j\} + \{vb_j, vl_i, u_ju_{j+3}\}$ has fewer branch vertices than $T$, violating (T1) and proving the claim. □

**Claim 3.** *For every $h \in [5]$, $N_G(b_1) \cap V(M_h) \cap N_G(b_2) = \emptyset$.*

*Proof.* Suppose $v \in N_G(b_1) \cap V(M_h) \cap N_G(b_2)$. Since $v \neq l_h$ by Claim 1, there exists $v^-$. Consider cases:

Case 1: Suppose $h \neq 3$. Without loss of generality, suppose $h = 1$. Since $G[v, v^-, b_1, b_2]$ is not a claw, either $v^-b_1 \in E(G)$ or $v^-b_2 \in E(G)$. If $v^-b_1 \in E(G)$, then $T' := T - \{vv^-, b_1u_1, b_1u_4\} + \{b_2v, b_1v^-, u_1u_4\}$ has fewer branch vertices than $T$, violating (T1). Otherwise $v^-b_2 \in E(G)$, so $T' := T - \{vv^-, b_1u_1, b_1u_4\} + \{b_1v, b_2v^-, u_1u_4\}$ similarly violates (T1).

Case 2: Suppose $h = 3$. If $v = x$, then without loss of generality, assume $x^-w_1 \in E(G)$, so it is easily seen that $b_1 \neq w_1$, so $T' := T - \{xx^-, xw_1\} + \{xb_1, x^-w_1\}$ has fewer branch vertices than $T$, violating (T1). If $v \neq x$, then since $G[v, v^-, b_1, b_2]$ is not a claw, and $b_1b_2 \notin E(G)$, either $v^-b_1 \in E(G)$ or $v^-b_2 \in E(G)$. Without loss of generality, assume $v^-b_1 \in E(G)$, so $T' := T - \{vv^-, xx^-\} + \{v^-b_1, vb_2\}$ has fewer vertices than $T$, violating (T1) and proving the claim. □

**Claim 4.** *If $i \neq 3$, then $N_G(l_i) \cap V(Q_j) = \emptyset$.*

*Proof.* Suppose $v \in N_G(l_i) \cap V(Q_j)$ for some $i \neq 3$. For $T' := T - \{b_i u_i\} + \{vl_i\}$, we have $|V(S_{T'})| < |V(S_T)|$, violating (T4). □

**Claim 5.** *For every $j \in [2]$, $(N_G(b_j) \cap V(Q_j))^- \cap N_G(l_3)$.*

*Proof.* Suppose $v \in (N_G(b_j) \cap V(Q_j))^- \cap N_G(l_3)$. Then $v^+ \in N_G(b_j) \cap V(Q_j)$, so $T' := T - \{vv^+, xx^-\} + \{v^+ b_j, vl_3\}$ has fewer branch vertices than $T$, violating (T1). □

**Claim 6.** *If $\{i, j\} = \{1, 2\}$, then $(N_G(b_j) \cap V(Q_j))^- \cap N_G(b_i) = \emptyset$.*

*Proof.* Suppose $v \in (N_G(b_j) \cap V(Q_j))^- \cap N_G(b_i)$. Then $v^+ \in N_G(b_j) \cap V(Q_j)$, so $T' := T - \{vv^+, xw_i\} + \{v^+ b_j, vb_i\}$ has fewer branch vertices than $T$, violating (T1). □

**Claim 7.** *If $\{i, j\} = \{1, 2\}$, then $N_G(b_i) \cap V(Q_j) \cap N_G(l_3) = \emptyset$.*

*Proof.* Suppose $v \in N_G(b_i) \cap V(Q_j) \cap N_G(l_3)$. Then since $b_j l_3 \notin E(G)$, $v \neq b_j$ so there exists $v^-$. Since $G[v, v^-, b_i, l_3]$ is not a claw and $b_i l_3 \notin E(G)$, either $v^- b_i \in E(G)$ or $v^- l_3 \in E(G)$. If $v^- l_3 \in E(G)$, then $T' := T - \{vv^-, xw_j\} + \{b_i v, l_3 v^-\}$ has fewer branch vertices than $T$, violating (T1). Otherwise $v^- b_i \in E(G)$, so $T' := T - \{vv^-, xw_j\} + \{l_3 v, b_i v^-\}$ has fewer branch vertices than $T$, again violating (T1). □

**Claim 8.** *We have $xl_3 \notin E(G)$.*

*Proof.* We already know $x^- w_i \in E(G)$ for some $i \in [2]$, so if $xl_3 \in E(G)$, then $T' := T - \{xx^-, xw_i\} + \{x^- w_i, xl_3\}$ has fewer branch vertices than $T$, violating (T1). □

**Claim 9.** *If $\{i, j\} = [2]$, then $w_j \notin N_G(b_i) \cup N_G(l_3)$.*

*Proof.* Suppose $w_j \in N_G(b_i) \cup N_G(l_3)$. Then either $w_j \in N_G(b_i)$ (in which case $T' := T - \{xw_j\} + \{b_i w_j\}$ violates (T1)) or else $w_j \in N_G(l_3)$ (in which case $T' := T - \{xw_j\} + \{l_3 w_j\}$ violates (T1)). □

For every $i \neq h \in [5]$, Lemma 2 ensures that $(N_G(l_h) \cap V(M_h))^-$ is disjoint from $N_G(l_i) \cap V(M_h)$. Lemma 3 ensures that $(N_G(l_h) \cap V(M_h))^-$ is disjoint from $N_G(b_j) \cap V(M_h)$ when $h \not\equiv j (\text{mod } 3)$, and Lemma 4 ensures the same when $h \equiv j (\text{mod } 3)$. Lemma 5 ensures that the four sets $N_G(l_i) \cap V(M_h)$ are all disjoint, and Claim 2 ensures that each $N_G(l_i) \cap V(M_h)$ with $i \neq h$ is disjoint from each $N_G(b_j) \cap V(M_h)$. Finally, Claim 3 ensures that $N_G(b_1) \cap V(M_h)$ does not intersect $N_G(b_2) \cap V(M_h)$, so the seven sets $(N_G(l_h) \cap V(M_h))^-$, $N_G(l_i) \cap V(M_h)$ (for each $i \neq h$), and $N_G(b_j) \cap V(M_h)$ (for $j \in [2]$) are disjoint, so the sum of their cardinalities equals the cardinality of their union, which cannot exceed the cardinality of $V(M_h)$. Furthermore, none of these contain $x^-$ by Lemmas 7 and 8 and Claim 8, so:

$$
\begin{aligned}
&\sum_{v \in X} |N_G(v) \cap V(M_h)| \\
&= \sum_{i=1}^{5} |N_G(l_i) \cap V(M_h)| + \sum_{j=1}^{2} |N_G(b_j) \cap V(M_h)| \\
&= |N_G(l_h) \cap V(M_h)| + \sum_{i \neq h} |N_G(l_i) \cap V(M_h)| + \sum_{j=1}^{2} |N_G(b_j) \cap V(M_h)| \\
&= \left|(N_G(l_h) \cap V(M_h))^-\right| + \sum_{i \neq h} |N_G(l_i) \cap V(M_h)| + \sum_{j=1}^{2} |N_G(b_j) \cap V(M_h)| \\
&\leq |V(M_h) \setminus \{x^-\}| = \begin{cases} |V(M_h)| & h \neq 3 \\ |V(M_h)| - 1 & h = 3. \end{cases}
\end{aligned}
$$

Meanwhile, for each $j \in [2]$ (and $\{i\} = [2] \setminus \{j\}$), Claim 4 gives that $b_1$, $b_2$, and $l_3$ are the only vertices in $X$ that can be adjacent to any vertex of $V(Q_j)$, and Claims 5, 6, and 7 give that the three sets $(N_G(b_j) \cap V(Q_j))^-$, $N_G(l_3) \cap V(Q_j)$, and $N_G(b_i) \cap V(Q_j)$ are disjoint, and none of them contain $w_j$ by Claim 9, so the sum of

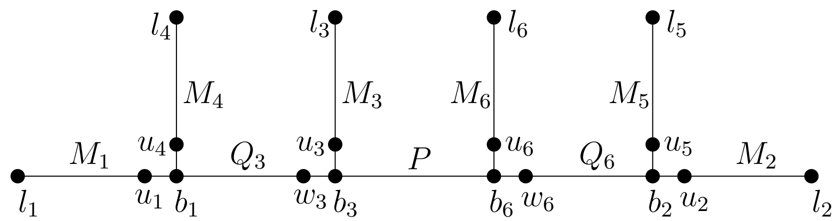


Figure 2.5: If $T$ has 4 branch vertices, we may have $\tau \cong P_4$. Here, $b_1$ is also called $b_4$, while $b_2$ is also called $b_5$.

their cardinalities is at most $|V(Q_j) \setminus \{w_j\}| = |V(Q_j)| - 1$, so

$$\begin{aligned}
& \sum_{v\in X} |N_G(v) \cap V(Q_j)| \\
= \; & \sum_{h=1}^{5} |N_G(l_h) \cap V(Q_j)| + \sum_{k=1}^{2} |N_G(b_k) \cap V(Q_j)| \\
= \; & |N_G(l_3) \cap V(Q_j)| + |N_G(b_i) \cap V(Q_j)| + |N_G(b_j) \cap V(Q_j)| \\
= \; & |N_G(l_3) \cap V(Q_j)| + |N_G(b_i) \cap V(Q_j)| + \left|(N_G(b_j) \cap V(Q_j))^-\right| \\
\leq \; & |V(Q_j)| - 1.
\end{aligned}$$

Summing these inequalities gives $\sum_{v\in X} \deg_G(v) \leq n - 3$, contradicting the assumption of the theorem. □

Therefore $T$ must have at least 4 branch vertices (all with degree 3 of course), so either $\tau \cong P_4$ or $\tau$ is a claw.

## 2.5 Fourth Structure

**Proposition 4.** *The derived tree $\tau(T) \not\cong P_4$.*

*Proof.* By contradiction, suppose $\tau(T) \cong P_4$. We then label vertices and paths as shown in Figure 2.5. Note that each vertex is in exactly one labeled path. Once this $T$ is chosen, we choose a (potentially different, but still with $\tau \cong P_4$) $T$ such that

(T5) $P$ is as short as possible.

By Lemma 1, $u_1u_4 \in E(G)$ and $u_2u_5 \in E(G)$. Similarly, since no induced claw is centered at $b_{3i}$ for any $i \in [2]$, (T4) and (T5) give that $u_{3i}w_{3i} \in E(G)$. Meanwhile, Lemmas 2, 3, and 4 ensure that $X := \{l_1, l_2, l_3, l_4, l_5, l_6, b_1\}$ is an independent set. Define $b_1^+ = N_T(b_1) \cap V(S_T)$.

**Claim 1.** *If $h \neq i$, then $N_G(l_i) \cap V(M_h) \cap N_G(b_1) = \emptyset$.*

*Proof.* Suppose $v \in N_G(l_i) \cap V(M_h) \cap N_G(b_1)$. By Lemma 6, we may assume $h \equiv 1 \pmod 3$ or $i \equiv 1 \pmod 3$. Consider several cases:

Case 1: Suppose $i \not\equiv 1 \equiv h \pmod 3$. Then $T' := T - \{b_iu_i, b_1u_1, b_1u_4\} + \{u_1u_4, vb_1, vl_i\}$ has fewer branch vertices than $T$, violating (T1).

Case 2: Suppose $i \equiv 1 \not\equiv h \pmod 3$. Then $T' := T - \{b_hu_h, b_1u_1, b_1u_4\} + \{u_1u_4, vb_1, vl_i\}$ has fewer branch vertices than $T$, violating (T1).

Case 3: Suppose $i \equiv 1 \equiv h \pmod 3$. Since $v \neq l_h$, there exists $v^-$. Since $G[v, b_1, l_i, v^-]$ is not a claw and $b_1l_i \notin E(G)$, either $v^-l_i \in E(G)$ or $v^-b_1 \in E(G)$. Now if $v^-l_i \in E(G)$, then $T' := T - \{vv^-, b_1u_1, b_1u_4\} + \{u_1u_4, vb_1, v^-l_i\}$ has fewer branch vertices than $T$, violating (T1). Otherwise $v^-b_1 \in E(G)$, then since $G[b_1, b_1^+, v^-, u_h]$ is not a claw and $b_1^+u_h \notin E(G)$, it follows that either $b_1^+v^- \in E(G)$ or $u_hv^- \in E(G)$. If $b_1^+v^- \in E(G)$, then $T' := T - \{vv^-, b_1u_h\} + \{b_1^+v^-, l_iv\}$ either has fewer branch vertices than $T$ (if $b_1^+ = b_3$) or else has the same number of branch vertices and leaves as $T$ with $|V(S_{T'})| < |V(S_T)|$, so either (T1) or (T4) is violated. On the other hand, if $u_hv^- \in E(G)$, then $T' := T - \{b_1u_h, vv^-\} + \{l_iv, u_hv^-\}$ has fewer branch vertices than $T$, violating (T1). □

**Claim 2.** *The following statements hold:*

**Part 1.** *If $i \not\equiv 0 (\mathrm{mod}\ 3)$, then $N_G(l_i) \cap V(Q_j) = \emptyset$.*

**Part 2.** *We have $N_G(b_1) \cap V(Q_6) \cap N_G(l_3) = \emptyset$.*

**Part 3.** *We have $N_G(b_1) \cap V(Q_6) \cap N_G(l_6) = \emptyset$.*

**Part 4.** *If $i \in [2]$, then $N_G(l_3) \cap V(Q_{3i}) \cap N_G(l_6) = \emptyset$.*

**Part 5.** *We have $(N_G(b_1) \cap V(Q_3))^- \cap N_G(l_3) = \emptyset$.*

**Part 6.** *We have $(N_G(b_1) \cap V(Q_3))^- \cap N_G(l_6) = \emptyset$.*

**Part 7.** *We have $N_G(l_i) \cap V(P) = \emptyset$ for each $i \in [6]$.*

*Proof.* To prove Part 1, suppose $v \in N_G(l_i) \cap V(Q_j)$. By symmetry, $v \neq b_{j/3}$, so $T' := T - \{b_i u_i\} + \{l_i v\}$ has the same number of branch vertices and leaves as $T$ with $|V(S_{T'})| < |V(S_T)|$, violating (T4). To prove Part 2, suppose $v \in N_G(b_1) \cap V(Q_6) \cap N_G(l_3)$. By symmetry, $v \neq b_2$, so $T' := T - \{w_3 b_3, w_6 b_6\} + \{v l_3, v b_1\}$ has fewer branch vertices than $T'$, violating (T1). To prove Part 3, suppose $v \in N_G(b_1) \cap V(Q_6) \cap N_G(l_6)$. By symmetry, $v \neq b_2$, so $T' := T - \{w_3 b_3, w_6 b_6\} + \{v l_6, v b_1\}$ has fewer branch vertices than $T'$, violating (T1). To prove Part 4, let $i \in [2]$ and suppose $v \in N_G(l_3) \cap V(Q_{3i}) \cap N_G(l_6)$. Then $T' := T - \{u_3 b_3, u_6 b_6\} + \{v l_3, v l_6\}$ has fewer branch vertices than $T$, violating (T1). To prove Part 5, suppose $v \in (N_G(b_1) \cap V(Q_3))^- \cap N_G(l_3)$. Then $v^+ \in N_G(b_1) \cap V(Q_3)$, so $T' := T - \{v v^+, b_3 u_3\} + \{l_3 v, b_1 v^+\}$ has fewer branch vertices than $T$, violating (T1). To prove Part 6, suppose $v \in (N_G(b_1) \cap V(Q_3))^- \cap N_G(l_6)$. Then $v^+ \in N_G(b_1) \cap V(Q_3)$, so $T' := T - \{v v^+, b_6 u_6\} + \{l_6 v, b_1 v^+\}$ has fewer branch vertices than $T$, violating (T1). To prove Part 7, suppose $v \in N_G(l_i) \cap V(P)$. Now if $v \in \{b_3, b_6\}$, Lemma 3 ensures that $i \equiv 0 (\mathrm{mod}\ 3)$ and $v = b_i$, so $T' := T - \{b_i w_i, b_i u_i\} + \{b_i l_i, u_i w_i\}$ has fewer branch vertices than $T$, violating (T1). Otherwise, $b_3 \neq v \neq b_6$, so $T' := T - \{b_i u_i\} + \{v l_i\}$ has the same number of branch vertices and leaves as $T$, and $|S_T| = |S_{T'}|$, but $P$ is shorter for $T'$ than it is for $T$, violating (T5) and proving the claim. □

Lemma 2 ensures that $(N_G(l_h) \cap V(M_h))^-$ is disjoint from $N_G(l_i) \cap V(M_h)$ (for each $i \neq h$). Lemma 3 ensures that $(N_G(l_h) \cap V(M_h))^-$ is disjoint from $N_G(b_1) \cap V(M_h)$ for $h \not\equiv 1 (\text{mod } 3)$. Lemma 4 ensures the latter for $h \equiv 1 (\text{mod } 3)$. Lemma 5 and Claim 1 ensure that the five sets $N_G(l_i) \cap V(M_h)$ are disjoint from each other and $N_G(b_1) \cap V(M_h)$, respectively. Therefore the seven sets $(N_G(l_h) \cap V(M_h))^-$, $N_G(b_1) \cap V(M_h)$, and $N_G(l_i) \cap V(M_h)$ for $i \neq h$ are all disjoint, and by Lemmas 7 and 8, none of them contain $u_h$ if $h \not\equiv 1$ (mod 3). Therefore:

$$
\begin{aligned}
& \sum_{v \in X} |N_G(v) \cap V(M_h)| \\
= \quad & |N_G(b_1) \cap V(M_h)| + \sum_{i=1}^{6} |N_G(l_i) \cap V(M_h)| \\
= \quad & |N_G(b_1) \cap V(M_h)| + |N_G(l_h) \cap V(M_h)| + \sum_{i \neq h} |N_G(l_i) \cap V(M_h)| \\
= \quad & |N_G(b_1) \cap V(M_h)| + |(N_G(l_h) \cap V(M_h))^-| + \sum_{i \neq h} |N_G(l_i) \cap V(M_h)| \\
\leq \quad & \begin{cases} |V(M_h)| - 1 & h \not\equiv 1 (\text{mod } 3) \\ |V(M_h)| & h \equiv 1 (\text{mod } 3). \end{cases}
\end{aligned}
$$

By Claim 2 Part 1, for $i \in [2]$, the only vertices of $X$ that can be adjacent to $Q_{3i}$ are $l_3$, $l_6$, and $b_1$. By Parts 2, 3, and 4, the three sets $N_G(l_3) \cap V(Q_6)$, $N_G(l_6) \cap V(Q_6)$, and $N_G(b_1) \cap V(Q_6)$ are disjoint. By Parts 4, 5, and 6, the three sets $N_G(l_3) \cap V(Q_3)$, $N_G(l_6) \cap V(Q_3)$, and $(N_G(b_1) \cap V(Q_3))^-$ are disjoint. Therefore:

$$
\begin{aligned}
& \sum_{v \in X} |N_G(v) \cap V(Q_6)| \\
= \; & |N_G(l_3) \cap V(Q_6)| + |N_G(l_6) \cap V(Q_6)| + |N_G(b_1) \cap V(Q_6)| \\
\leq \; & |V(Q_6)|
\end{aligned}
$$

and:

$$
\begin{aligned}
& \sum_{v \in X} |N_G(v) \cap V(Q_3)| \\
= \; & |N_G(l_3) \cap V(Q_3)| + |N_G(l_6) \cap V(Q_3)| + |N_G(b_1) \cap V(Q_3)| \\
= \; & |N_G(l_3) \cap V(Q_3)| + |N_G(l_6) \cap V(Q_3)| + |(N_G(b_1) \cap V(Q_3))^-| \\
\leq \; & |V(Q_3)|
\end{aligned}
$$

By Claim 2 Part 7, $b_1$ is the only vertex of $X$ that can be adjacent to any of $P$, so

$$\sum_{v \in X} |N_G(v) \cap V(P)| = |N_G(b_1) \cap V(P)| \leq |V(P)|.$$

Summing these inequalities gives $\sum_{v \in X} \deg_G(v) \leq n - 4$, contradicting the assumption of the theorem. □

## 2.6 Fifth Structure

**Proposition 5.** *The derived tree $\tau$ is not a claw.*

*Proof.* By contradiction, suppose $\tau$ is a claw. We label vertices and paths as shown in Figure 2.6. Since $u_i b_i \in E(T)$ and $u_i \notin S_T$ for every $i \in [6]$, Lemma 1 gives that

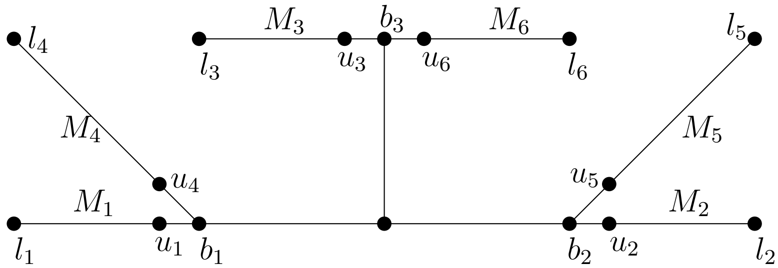


Figure 2.6: If $T$ has 4 branch vertices, $\tau$ may be a claw. Each vertex labeled $b_i$ is also called $b_{i+3}$.

$u_iu_{i+3} \in E(G)$ for each $i \in [3]$. Furthermore, the vertex set $X := \{l_1, l_2, l_3, l_4, l_5, l_6, b_3\}$ is independent by Lemmas 2, 3, and 4.

**Claim 1.** *If $i \neq h$, then $N_G(l_i) \cap V(M_h) \cap N_G(b_3) = \emptyset$.*

*Proof.* Suppose $v \in N_G(l_i) \cap V(M_h) \cap N_G(b_3)$. By Lemma 6, we may assume that either $3|i$ or $3|h$. Now if $i \equiv 0 \not\equiv h \pmod 3$, then $T' := T - \{b_3u_3, b_3u_6, b_hu_h\} + \{vl_i, vb_3, u_3u_6\}$ has fewer branch vertices than $T$, violating (T1). On the other hand, if $h \equiv 0 \not\equiv i \pmod 3$, then $T' := T - \{b_3u_3, b_3u_6, b_iu_i\} + \{vb_3, vl_i, u_3u_6\}$ has fewer branch vertices than $T$, violating (T1). Otherwise, $h \equiv i \equiv 0 \pmod 3$, so since $v \neq l_h$, there exists $v^-$. Since $G[v, v^-, b_3, l_i]$ is not a claw and $b_3l_i \notin E(G)$, it follows that either $v^-l_i \in E(G)$ or $v^-b_3 \in E(G)$. If $v^-l_i \in E(G)$, then $T' := T - \{vv^-, b_3u_h, b_3u_i\} + \{vb_3, u_hu_i, l_iv^-\}$ has fewer branch vertices than $T$, violating (T1). Otherwise, $v^-b_3 \in E(G)$, and since $G[b_3, b_3^+, u_h, v^-]$ is not a claw and $u_hb_3^+ \notin E(G)$, it follows that either $v^-u_h \in E(G)$ or $v^-b_3^+ \in E(G)$. If $v^-u_h \in E(G)$, then $T' := T - \{b_3u_h, vv^-\} + \{vl_i, v^-u_h\}$ has fewer branch vertices than $T$, violating (T1). Otherwise, $v^-b_3^+ \in E(G)$, so $T' := T - \{vv^-, b_3u_i\} + \{v^-b_3^+, vl_i\}$ either has fewer branch vertices than $T$ (if $b_3^+ = x$) or else has the same number of branch vertices and leaves as $T$, but $|V(S_{T'})| < |V(S_T)|$, so either (T1) or (T4) is violated, so we have proven our claim. □

**Claim 2.** *If $i \in [6]$, then $N_G(l_i) \cap V(S_T) = \emptyset$.*

*Proof.* Suppose $v \in N_G(l_i) \cap V(S_T)$. By Lemma 4, $v \neq b_j$, so $T' := T - \{b_i u_i\} + \{v l_i\}$ may have fewer branch vertices than $T$, violating (T1), or the same number of branch vertices and leaves, violating (T4) since $|V(S_{T'})| < |V(S_T)|$. □

For any $h \in [6]$, Lemma 2 ensures that $(N_G(l_h) \cap V(M_h))^-$ is disjoint from $N_G(l_i) \cap V(M_h)$ for $i \neq h$. Lemma 3 ensures that $(N_G(l_h) \cap V(M_h))^-$ is disjoint from $N_G(b_3) \cap V(M_h)$ for $h \not\equiv 0 (\text{mod } 3)$. Lemma 4 ensures that the latter are disjoint for $h \equiv 0 (\text{mod } 3)$. Lemma 5 and Claim 1 ensure that the five sets $N_G(l_i) \cap V(M_h)$ with $i \neq h$ are disjoint from each other and from $N_G(b_3) \cap V(M_h)$ respectively. Therefore the seven sets $(N_G(l_h) \cap V(M_h))^-$, $N_G(b_3) \cap V(M_h)$, and $N_G(l_i) \cap V(M_h)$ for $i \neq h$ are all disjoint. Furthermore, if $3 \nmid h$, Lemmas 7 and 8 give that $u_h$ is not in any of these sets. Therefore:

$$\begin{aligned}
&\sum_{v \in X} |N_G(v) \cap V(M_h)| \\
= \;& |N_G(b_3) \cap V(M_h)| + \sum_{i=1}^{6} |N_G(l_i) \cap V(M_h)| \\
= \;& |N_G(b_3) \cap V(M_h)| + |N_G(l_h) \cap V(M_h)| + \sum_{i \neq h} |N_G(l_i) \cap V(M_h)| \\
= \;& |N_G(b_3) \cap V(M_h)| + |(N_G(l_h) \cap V(M_h))^-| + \sum_{i \neq h} |N_G(l_i) \cap V(M_h)| \\
\leq \;& \begin{cases} |V(M_h)| & 3 | h \\ |V(M_h) \setminus \{u_h\}| = |V(M_h)| - 1 & 3 \nmid h. \end{cases}
\end{aligned}$$

Meanwhile, Claim 2 gives that $b_3$ is the only vertex of $X$ that can be adjacent to any vertex of $S_T$. Therefore

$$\sum_{v \in X} |N_G(v) \cap V(S_T)| = |N_G(b_3) \cap V(S_T)| \leq |V(S_T) \setminus \{b_3\}| = |V(S_T)| - 1$$

Summing these inequalities gives $\sum_{v \in X} \deg_G(v) \leq n - 5$, contradicting the assumption of the theorem. □

By Propositions 3, 4, and 5, the $T$ we have chosen must have four branch vertices but cannot have any of the possible structures on four branch vertices, and therefore cannot exist. This is a contradiction, so Theorem 5 is proven. Thus Conjecture 1 holds when $k = 2$.

# Chapter 3

# General Case

In this chapter, we prove Conjecture 1 in full as Theorem 6, which we now restate:

**Theorem 6** [9] *Let $G$ be a connected, claw-free graph on $n$ vertices, and let $k$ be a non-negative integer. If $\sigma_{2k+3} \geq n-2$, then $G$ has a spanning tree with at most $k$ branch vertices.*

Our proof uses the concept of pseudoadjacency mentioned in the introduction. We also make use of definition 4.

Suppose some $G$ as described in the theorem has no spanning tree with at most $k$ branch vertices. Choose some spanning tree $T$ of $G$ such that:

(T1) $B(T)$ is as small as possible.

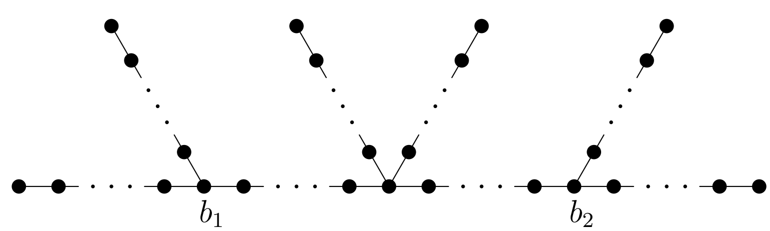


Figure 3.1: An example of a tree $T$. Its internal subtree, in this case, is the path $b_1Tb_2$.

(T2) We select trees with at least one degree 3 vertex over those with none, subject to (T1).

(T3) If (T2) allows no trees with a degree 3 vertex, $L(T)$ is as small as possible.

(T4) If (T2) allows a tree with at least one degree 3 vertex, the sum total of the degrees in $T$ of the vertices of $B_{\geq 5}(T)$ is as small as possible. That is,

$$\sum_{v \in B_{\geq 5}(T)} \deg_T(v)$$

is as small as possible.

We begin by showing that $T$ must have at least one vertex of degree 3. Suppose $T$ has no vertices of degree 3. The number of leaves in $T$ is therefore:

$$|L(T)| = 2 + \sum_{b \in B(T)} (\deg_T(b) - 2) \geq 2 + \sum_{b \in B(T)} (2) \geq 2 + (k+1)(2) = 2k + 4.$$

We will first establish that $L(T)$ is independent, and then that it is pseudoindependent.

Suppose two leaves $s$ and $t$ are adjacent in $G$. Then $s$ has some nearest branch vertex $b$, so $T' := T - \{bb_s\} + \{st\}$ has fewer leaves than $T$, violating either (T2) or (T3) depending on $\deg_T(b)$. Therefore $L(T)$ must be independent in $G$.

Suppose two leaves $s$ and $t$ are pseudoadjacent with respect to $T$. Then there is some edge $e \in E(T)$ such that $sg(e,s), tg(e,t) \in E(G)$. Consider two cases.

Case 1: Suppose $g(e,s) = g(e,t)$. Define $a = g(e,s) = g(e,t)$, so $V(sTt) \cap V(sTa) \cap V(tTa) =: w \notin \{s,t,a\}$. Since $G[a, e_w, s, t]$ is not a claw, either $se_w \in E(G)$ or $te_w \in E(G)$ (we know $st \notin E(G)$ since $L(T)$ is independent). We may assume the first by symmetry, so $T' := T - \{e, ww_s\} + \{se_w, ta\}$ violates either (T2) or (T3) since two leaves are lost ($s$ and $t$) while at most one is gained ($w_s$).

Case 2: Suppose $g(e,s) \neq g(e,t)$. The $e_s = g(e,t)$ and $e_t = g(e,s)$, so $se_t, te_s \in E(G)$. This implies that $e_s, e_t \in V(sTt)$. Choose an arbitrary branch vertex $b \in V(sTt)$; assume $b \in V(eTt)$ by symmetry. Then $T' := T - \{e, bb_t\} + \{se_t, te_s\}$ violates either (T2) or (T3) since two leaves are lost ($s$ and $t$) while at most one is gained ($b_t$).

Therefore $L(T)$ is pseudoindependent with respect to $T$, so no edge of $T$ has more than one leaf of $T$ as an oblique neighbor. We next find two edges of $T$ that have no leaves of $T$ as oblique neighbors. Choose a leaf of $S_T$ (note that it is a branch vertex of $T$) and call it $b$. As $T$ has no vertices of degree exactly 3, then $\deg_T(b) \geq 4$ and $|N_T(b) \cap S_T| = 1$, so $|N_T(b) \setminus S_T| \geq 3$. Choose three of these vertices and call them $u, v, w$. Since $G[b, u, v, w]$ is not a claw, $\{u, v, w\}$ cannot be independent in $G$. By symmetry, assume $uv \in E(G)$. We will show that $bu$ and $bv$ have no leaves as oblique neighbors.

Since $u \notin S_T$, there is some $z \in L(T)$ such that $u = b_z$. If some leaf $l \neq z$ is an oblique neighbor of $bu$, then $lu \in E(G)$, so $T' := T - \{bu\} + \{lu\}$ violates either (T2) or (T3) via $l$. If $z$ is an oblique neighbor of $bu$, then $bz \in E(G)$, so $T' := T - \{bu, bv\} + \{bz, uv\}$ violates either (T2) or (T3) via $z$. Therefore $bu$ has no leaves as oblique neighbors, and by the same argument, neither does $bv$.

For any $v, x \in V(G)$, we have $vx \in E(G)$ if and only if $v$ is an oblique neighbor of $xx_v$. Therefore the number of edges with $v$ as an oblique neighbor equals the degree of $v$. Since no edge has more than one leaf as an oblique neighbor, and two of them have no leaves as oblique neighbors, the degrees of the leaves can add up to at most $|E(T)| - 2 = (n-1) - 2 = n - 3$, contradicting the assumption of the theorem.

Therefore $T$ must have at least one vertex of degree 3, so we can choose a root $r \in B_3(T)$, denoting $N_T(r) =: \{r_1, r_2, r_3\}$. Since no claw can be centered at $r$, we may assume by symmetry that $r_1 r_2 \in E(G)$. We denote the branch vertex closest to any $e \in E(T)$ toward the root as $p = p(e)$, and denote the branch vertex or leaf closest to $e$ the opposite direction as $x = x(e)$. For each $i \in [3]$, define $x_i = x(rr_i)$. We will need one more definition.

**Definition 6.** *For any rooted spanning tree $T$ with root $r \in B_3(T)$, denoted $(T, r)$, each branch vertex $x \in B(T) \setminus \{r\}$ has a* ***distance-degree pair*** $(d(x, r), \deg_T(x))$*. We define a* ***pair sequence*** *on the entire set $B(T)$, which contains the distance-degree pairs of all vertices of $B(T)$ in lexicographically increasing order (shortest distance first, and smallest degree first given equal distance).*

Since such an $r$ must exist, choose $(T, r)$ such that:

(T5) The sequence of distance-degree pairs of $B(T) \setminus \{r\}$, as defined above, is lexicographically as small as possible. That is, given a tree $T_A$ with its root $r_A$, and another tree $T_B$ with its root $r_B$, we select $(T_A, r_A)$ over $(T_B, r_B)$ if and only if the earliest entry that differs in their pair sequences is "smaller" (lexicographically, as described in Definition 6) for $(T_A, r_A)$ than it is for $(T_B, r_B)$.

Before completing the proof of Conjecture 1, we introduce three useful lemmas.

**Lemma 9.** *If $a$ is a child of $b \in B(T)$, then $a$ is adjacent in $G$ to some $c \in N_T(b) \setminus \{a\}$.*

*Proof.* Suppose there is no such $c$. To avoid claws centered at $b$, $N_T(b) \setminus \{a\}$ must be a clique in $G$, so $T' := T - \{bd : b = d_r, d \neq a\} + \{b_r d : b = d_r, d \neq a\}$ violates (T1) if $b_r \in B(T)$, or (T5) otherwise since $d(b_r, r) < d(b, r)$. □

**Lemma 10.** *Let $a, x, y \in V(G)$. If $\deg_T(x) = 3$, $\deg_T(y) \neq 2$, $a \in V(rTx)$, and $x \in V(rTy)$, then $ya \notin E(G)$.*

*Proof.* If $ya \in E(G)$, then $T' := T - \{xx_y\} + \{ya\}$ violates (T1) if $a \in B(T)$ or $x_y \in B(T)$, or (T5) otherwise. □

**Corollary 1.** *If $x \in B_3(T) \setminus \{r\}$, then the two children of $x$ are adjacent in $G$.*

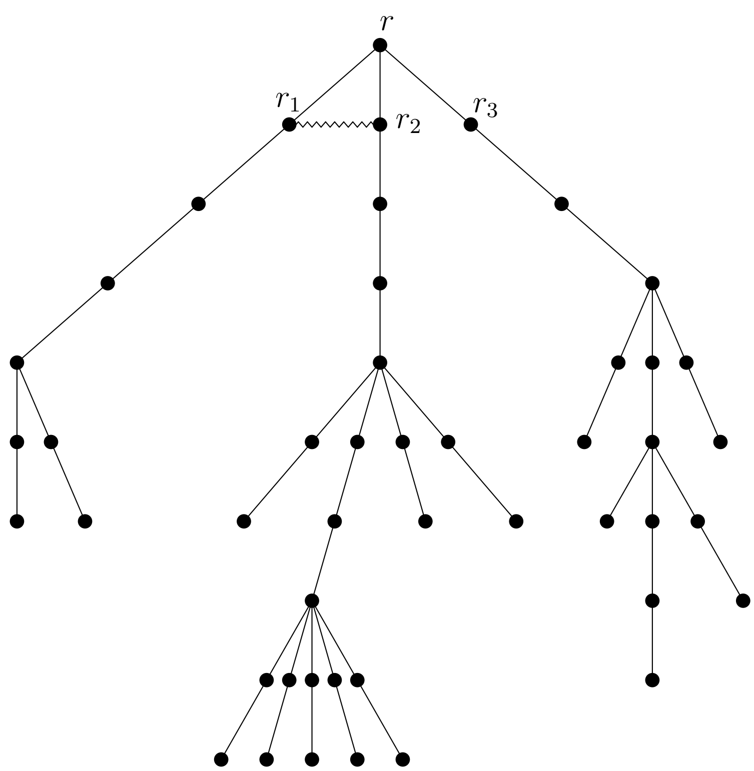


Figure 3.2: An example of a rooted spanning tree $(T, r)$ of a connected claw-free graph $G$ with pair sequence $((3,4),(4,3),(4,5),(5,4),(7,6))$. Since $G[r, r_1, r_2, r_3]$ cannot be a claw, we assume by symmetry that $r_1r_2 \in E(G)$ (shown as a squiggly line segment). Note that $\displaystyle\sum_{v \in B_{\geq 5}(T)} \deg_T(v) = 11$.

*Proof.* By Lemma 10, neither child of $x$ is adjacent to $x_r$. Since no claw can be centered at $x$, this requires that the two children are adjacent. □

**Lemma 11.** *If $y, z \in L(T) \cup B_3(T) \setminus \{r\}$ are both oblique neighbors of some $e \in E(T)$, then $p = p(e) \neq r$.*

*Proof.* Suppose $p = r$, implying $x = x_i$ for some $i \in [3]$. If both $y$ and $z$ are separated from $x$ by $r$, we consider whether or not $e = rr_i$. If so, then $T' := T - \{e\} + \{yr_i\}$ violates (T1) via $r$. Otherwise, since $G[e_x, e_r, y, z]$ is not a claw, either $ye_r \in E(G)$ or $ze_r \in E(G)$. We may assume the first by symmetry, so $T' := T - \{e, rr_x\} + \{ye_r, ze_x\}$ violates (T1) via $r$. Now if exactly one of the two (say $y$) is separated from $x$ by $r$, then we again consider whether or not $e = rr_i$. If so, then $T' := T - \{e\} + \{yr_i\}$ violates (T1) via $r$. Otherwise $T' := T - \{e, rr_x\} + \{ye_x, ze_r\}$ violates (T1) via $r$. The remaining possibility is that neither $y$ nor $z$ is separated from $x$ by $r$. Then $r \notin V(yTz)$, and Lemma 10 ensures that $y \notin V(rTz)$ and $z \notin V(rTy)$. We may therefore denote $V(rTy) \cap V(rTz) \cap V(yTz) =: w \notin \{r, y, z\}$. Now Lemma 10 also requires that $\deg_T(w) \geq 4$. If $\deg_T(w) = 4$, then $T' := T - \{xx_y, xx_z\} + \{ye_r, ze_r\}$ violates (T1) if $e = rr_i$, or (T5) if not. Otherwise $\deg_T(w) \geq 5$, and since $\{e_r, e_x, y, z\}$ is not a claw, either $ye_x \in E(G)$ or $ze_x \in E(G)$. We assume the first by symmetry, so $T' := T - \{e, ww_y\} + \{ye_x, ze_r\}$ violates (T4). □

**Lemma 12.** *If $y, z \in L(T) \cup B_3(T) \setminus \{r\}$ are both oblique neighbors of $e \in E(T)$, then $e_p = e_y = e_z$ (where $p = p(e)$ as described above).*

*Proof.* If this is not the case, then either $e_x = e_y = e_z$ (where $x = x(e)$ as described above), or $\{e_y, e_z\} = \{e_p, e_x\}$. Note that $x \in B_{\geq 4}(T)$ by Lemma 10 in both cases. Consider both these cases.

**Case 1**: Suppose $e_x = e_y = e_z$. Then either $y \in V(xTz)$, or $z \in V(xTy)$, or neither, so consider both cases.

*Subcase* 1a: Suppose neither inclusion is true, meaning $V(xTy) \cap V(xTz) \cap V(yTz) =: w \notin \{y, z\}$ (though it could be that $w = x$), and $\deg_T(w) \geq 4$ by Lemma 10. If $w \in B_4(T)$, then $T' := T - \{ww_y, ww_z\} + \{ye_p, ze_p\}$ violates (T1) if $e_p = p$ or (T5) if not. Otherwise $w \in B_{\geq 5}(T)$, and then we note that $G[e_p, e_x, y, z]$ is not a claw, so either $ye_x \in E(G)$ or $ze_x \in E(G)$. We may assume the first by symmetry, so $T' := T - \{e, xx_y\} + \{ye_x, ze_p\}$ violates (T4) via $w$.

*Subcase* 1b: Suppose one of the two inclusions is true, say $y \in V(xTz)$. We define $y^* = N_T(y) \setminus \{y_r, y_z\}$. Corollary 1 requires that $y_z y^* \in E(G)$, and then $T' := T - \{xx_y, yy_z, yy^*\} + \{ye_p, ze_p, y_z y^*\}$ violates (T1) if $e_p = p$, or else it violates (T4) if $x \in B_{\geq 5}(T)$, or (T5) if $x \in B_4(T)$.

**Case 2**: Suppose $e_y = e_p$ but $e_z = e_x$ (or vice versa, by symmetry). Depending on the location of $y$, we may have $r \in V(yTp)$, or $p \in V(rTy)$, or $y \in V(rTp)$, or none of the above. If $r \in V(yTp)$, then $T' := T - \{e, rr_p\} + \{ye_x, ze_p\}$ violates (T1) via $r$. If $p \in V(rTy)$, then $T' := T - \{e, pp_y\} + \{ye_x, ze_p\}$ violates (T1) if $p \in B_3(T)$, or (T4) if $p \in B_{\geq 5}(T)$, or (T5) if $p \in B_4(T)$. If $y \in V(rTp)$, we can define $y^* = N_T(y) \setminus \{y_r, y_p\}$; we then have from Corollary 1 that $y_p y^* \in E(G)$, implying that $T' := T - \{e, yy_p, yy^*\} + \{ye_x, ze_p, y_p y^*\}$ violates (T1) via $y$. Since we've ruled out all three inclusions, we may denote $V(rTp) \cap V(rTy) \cap V(pTy) =: w \notin \{r, p, y\}$, and then $T' := T - \{e, ww_p\} + \{ye_x, ze_p\}$ violates (T1) if $w \in B_3(T)$, or (T4) if $w \in B_{\geq 5}(T)$, or (T5) if $w \in B_4(T)$. □

**Lemma 13.** *If $y, z \in L(T) \cup B_3(T) \setminus \{r\}$ are both oblique neighbors of some $e \in E(T)$, then neither $y$ nor $z$ is separated from $p = p(e)$ by $r$.*

*Proof.* Suppose at least one of $y$ and $z$ is separated from $p$ by $r$. If they both are, then to avoid a claw centered at $e_x$, we must have either $ye_p \in E(G)$ or $ze_p \in E(G)$. We may assume the first by symmetry; therefore $T' := T - \{e, rr_p\} + \{ye_p, ze_x\}$ violates (T1) via $r$. Therefore only one of them is separated from $p$ by $r$ (say $r \in$

$V(pTz)\setminus V(pTy)$), and we note that $e_x \neq x$ (otherwise $T' := T - \{rr_p\} + \{zx\}$ violates (T1)), so $e_{xx}$ exists. We will categorize the location of $y$ by its relation to $p$ and $r$ (noting that $r \notin V(pTy)$).

Case 1: Suppose $V(rTp) \cap V(rTy) \cap V(pTy) =: w \notin \{r, p, y\}$. Since $G[e_x, e_{xx}, y, z]$ is not a claw, either $ye_{xx} \in E(G)$ or $ze_{xx} \in E(G)$. If $ye_{xx} \in E(G)$, then $T' := T - \{e_x e_{xx}, rr_p\} + \{ye_{xx}, ze_x\}$ violates (T1) via $r$. Otherwise $ze_{xx} \in E(G)$, and either $z \in L(T)$ or $z \in B_3(T)$. If $z \in B_3(T)$, then $T' := T - \{e_x e_{xx}, rr_p\} + \{ze_x, ze_{xx}\}$ violates (T1) via $r$. Otherwise $z \in L(T)$ and then $T' := T - \{e_x e_{xx}, ww_y\} + \{ye_x, ze_{xx}\}$ violates either (T1) if $w \in B_3(T)$, or (T4) if $w \in B_{\geq 5}(T)$, or (T5) if $w \in B_4(T)$.

Case 2: Suppose $y \in V(rTp_r)$. Define $y^* = N_T(y) \setminus \{y_r, y_p\}$, so Corollary 1 requires that $y_p y^* \in E(G)$, so $T' := T - \{yy_p, yy^*, rr_y\} + \{ye_x, ze_x, y_p y^*\}$ violates (T1) since at least two branch vertices are lost ($r$ and $y$) while only one is gained ($e_x$).

Case 3: Suppose $y = p$ (ensuring $p \in B_3(T)$). Define $p^* = N_T(p) \setminus \{p_r, p_x\}$, so Corollary 1 ensures that $p_x p^* \in E(G)$, so $T' := T - \{pp_x, pp^*, rr_p\} + \{pe_x, ze_x, p_x p^*\}$ violates (T1) since at least two branch vertices are lost ($r$ and $p$) while only one is gained ($e_x$).

Case 4: Suppose $p \in V(rTy)$. Note that Lemma 12 guarantees that $p \in V(xTy)$. Since $G[e_x, e_{xx}, y, z]$ is not a claw, either $ye_{xx} \in E(G)$ or $ze_{xx} \in E(G)$. If $ye_{xx} \in E(G)$, then $T' := T - \{e_x e_{xx}, rr_p\} + \{ye_{xx}, ze_x\}$ violates (T1) via $r$. Otherwise $ze_{xx} \in E(G)$, and either $z \in L(T)$ or $z \in B_3(T)$. If $z \in B_3(T)$, then $T' := T - \{e_x e_{xx}, rr_p\} + \{ze_x, ze_{xx}\}$ violates (T1) via $r$. Otherwise $z \in L(T)$, and then $T' := T - \{e_x e_{xx}, pp_x\} + \{ye_x, ze_{xx}\}$ violates (T1) if $p \in B_3(T)$, or (T4) if $p \in B_{\geq 5}(T)$, or (T5) if $p \in B_4(T)$.

□

Define $X = L(T) \cup B_3(T) \setminus \{r\}$. We first show that $|X| \geq 2k + 3$. Define $m = |B_3(T)|$, so $|B_{\geq 4}(T)| \geq k + 1 - m$. Now:

$$|L(T)| = 2 + \sum_{b \in B(T)} (\deg_T(b) - 2) \geq 2 + m + 2(k+1-m) = 2 + m + 2k + 2 - 2m = 2k + 4 - m$$

hence:

$$|X| = |L(T)| + |B_3(T) \setminus \{r\}| \geq (2k + 4 - m) + (m - 1) = 2k + 3.$$

We next show that $X$ is independent. Let $u, v \in X$ and assume $uv \in E(G)$. Now if $r \in V(uTv)$, then $T' := T - \{rr_u\} + \{uv\}$ violates (T1). If $u \in V(rTv)$ (or, symmetrically, $v \in V(rTu)$), then $u \in B_3(T)$, so define $u^* := N_T(u) \setminus \{u_r, u_v\}$. Now Corollary 1 gives that $u_v u^* \in E(G)$, so $T' := T - \{uu_v, uu^*\} + \{uv, u_v u^*\}$ violates (T1). The remaining possibility is that $V(rTu) \cap V(rTv) \cap V(uTv) =: w \notin \{r, u, v\}$. Now consider $T' := T - \{ww_u\} + \{uv\}$. If $w \in B_3(T)$, then $T'$ violates (T1) since $w$ is no longer a branch vertex. If $w \in B_{\geq 5}(T)$, then $T'$ violates (T4) since $w$ decreases the sum total but neither $u$ nor $v$ increase it (their degrees were originally at most 3 and are now at most 4). The remaining case is that $w \in B_4(T)$, in which case $T'$ violates (T5) since $w$, which is closer to $r$ than either $u$ or $v$, has its distance-degree pair decreased.

To limit the degree sum of $X$, we will show that $X$ is pseudoindependent, and then find two edges of $T$ with no oblique neighbors in $X$, as we did for the case $B_3(T) = \emptyset$. Suppose some $y, z \in X$ are pseudoadjacent with respect to $T$, so they are both oblique neighbors of some $e \in E(T)$. As before, we denote $p = p(e)$ and $x = x(e)$. Now either both $y$ and $z$ are on the path $rTp$, or exactly one of them is, or neither of them is, so consider all three cases.

**Case A**: Suppose $y, z \in V(rTp)$. Then $y, z \in B_3(T)$. By symmetry, we may assume $y \in V(rTz)$. Define $y^* = N_T(y) \setminus \{y_r, y_p\}$ and $z^* = N_T(z) \setminus \{z_r, z_p\}$, so Corollary 1 requires that $y_p y^*, z_p z^* \in E(G)$. Now $T' := T - \{yy_p, yy^*, zz_p, zz^*\} + \{ye_x, ze_x, y_p y^*, z_p z^*\}$ violates (T1) since two branch vertices are lost ($y$ and $z$) while at most one is gained ($e_x$). (See Figure 3.3.)

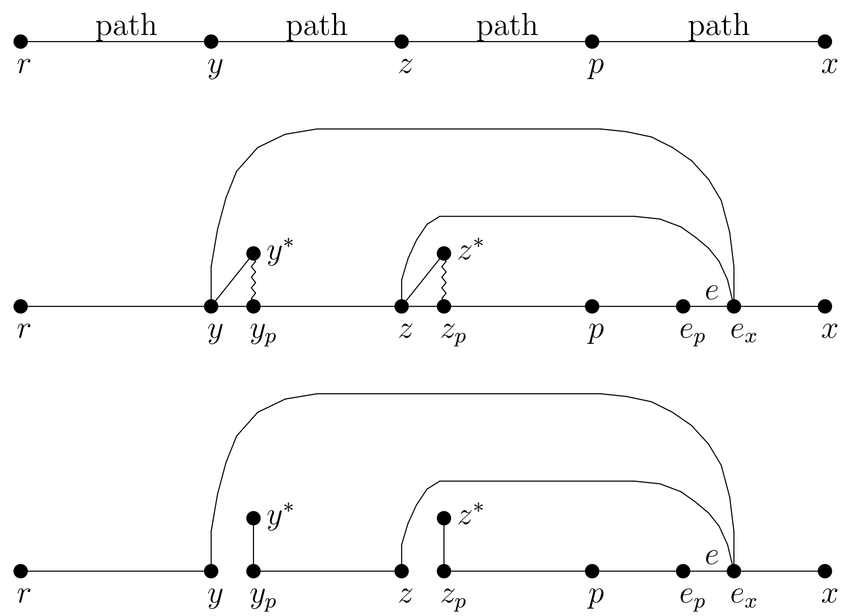


Figure 3.3: These pictures show how one might visualize Case A. The first picture shows the relative positions of important vertices, as they are assumed in this case. In the second picture, the straight-line edges are part of the tree, while the curved and jagged edges are known to exist in the graph. The third picture shows $T'$, which has one less branch vertex than $T$.

**Case B**: Suppose $y \in V(rTp)$ but $z \notin V(rTp)$. Either $y = p$ or $y \neq p$, so consider both cases.

*Subcase* B1: Suppose $y = p$ (ensuring $p \in B_3(T)$). Define $p^* = N_T(p) \setminus \{p_r, p_x\}$, so Corollary 1 requires that $p_x p^* \in E(G)$. Now either $p \in V(rTz)$ or $p \notin V(rTz)$, so consider both cases.

*Subcase* B1a: Suppose $p \in V(rTz)$, so $p_z = p^*$. We can see that $e \neq pp_x$ (otherwise $T' := T - \{pp_x\} + \{zp_x\}$ violates (T1)), and $e_p \neq p_x$ (otherwise $T' := T - \{pp_z, e\} + \{p_x p_z, ze_x\}$ violates (T1)). Since $G[e_x, e_p, p, z]$ is not a claw, either $pe_p \in E(G)$ or $ze_p \in E(G)$. If $pe_p \in E(G)$, then $T' := T - \{e, pp_x, pp_z\} + \{pe_p, ze_x, p_x p_z\}$ violates (T1) via $p$. Otherwise $ze_p \in E(G)$. Since $G[p, p_r, p_x, e_x]$ is not a claw and Lemma 10 implies $p_r p_x \notin E(G)$, either $p_r e_x \in E(G)$ or $p_x e_x \in E(G)$. If $p_r e_x \in E(G)$, then $T' := T - \{e, pp_x\} + \{p_r e_x, ze_p\}$ violates (T1) if $p_r \in B(T)$, or (T5) if not. Otherwise $p_x e_x \in E(G)$, so $T' := T - \{e, pp_x\} + \{p_x e_x, ze_p\}$ violates (T1) via $p$.

*Subcase* B1b: Suppose $p \notin V(rTz)$. Lemma 13 implies that $r \notin V(pTz)$ and we began Case B by assuming $z \notin V(rTp)$. We may therefore denote $V(rTp) \cap V(rTz) \cap V(pTz) =: w \notin \{r, p, z\}$. If $e = pp_x$, then $T' := T - \{e\} + \{zp_x\}$ violates (T1) via $p$. Otherwise, since $G[e_x, e_p, p, z]$ is not a claw, either $pe_p \in E(G)$ or $ze_p \in E(G)$. If $pe_p \in E(G)$, then $T' := T - \{e, pp_x, pp^*\} + \{pe_p, ze_x, p_x p^*\}$ violates (T1) via $p$. Otherwise $ze_p \in E(G)$, so $T' := T - \{e, pp_x, pp^*\} + \{pe_x, ze_p, p_x p^*\}$ violates (T1) via $p$.

*Subcase* B2: Suppose $y \neq p$. Define $y^* = N_T(y) \setminus \{y_r, y_p\}$, so Corollary 1 requires that $y_p y^* \in E(G)$. If $e_x = x$, then $T' := T - \{yy_p, yy^*\} + \{xy, y_p y^*\}$ violates (T1) via $y$, so we may assume $e_{xx}$ exists. Recalling that $r \notin V(pTz)$ by Lemma 13, and that $x \notin V(pTz)$ by Lemma 12, consider two cases for the location of $z$.

*Subcase* B2a: Suppose $p \in V(rTz)$. By Lemma 12, $p_z \neq p_x$. Since $G[e_x, e_{xx}, y, z]$

is not a claw, either $ye_{xx} \in E(G)$ or $ze_{xx} \in E(G)$. If $ze_{xx} \in E(G)$, then $T' := T - \{e_xe_{xx}, yy_p, yy^*\} + \{ye_x, ze_{xx}, y_py^*\}$ violates (T1) via $y$. Otherwise $ye_{xx} \in E(G)$, and we consider $\deg_T(p)$. If $p \in B_{\geq 5}(T)$, then $T' := T - \{e_xe_{xx}, pp_x\} + \{ye_{xx}, ze_x\}$. Otherwise $p \in B_{\leq 4}(T)$, and then Lemma 9 requires that $p_x$ has some neighbor in $G$ among the remaining vertices of $N_T(p)$. If this neighbor is $p_r$, then $T' := T - \{e_xe_{xx}, pp_x, pp_r\} + \{ye_{xx}, ze_x, p_xp_r\}$ violates (T1) if $p \in B_{\leq 4}(T)$, or (T4) if $p \in B_{\geq 5}(T)$. If this neighbor is $p_z$, then $T' := T - \{e_xe_{xx}, pp_x, pp_z\} + \{ye_x, ye_{xx}, p_xp_z\}$ violates (T1) via $p$. Otherwise this neighbor must be $p^*$, where $N_T(p) =: \{p_r, p_x, p_z, p^*\}$, and then $T' := T - \{e_xe_{xx}, pp_x, pp^*\} + \{ye_{xx}, ze_x, p_xp^*\}$ violates (T1) via $p$.

*Subcase* B2b: Suppose $p \notin V(rTz)$. Lemma 13 implies that $r \notin V(pTz)$, and we began Case B by assuming $z \notin V(rTp)$. We may therefore denote $V(rTp) \cap V(rTz) \cap V(pTz) =: w \notin \{r, p, z\}$. Consider three cases for the location of $w$ relative to $y$.

*Subcase* B2b (i): Suppose $w \in V(rTy_r)$. Then $T' := T - \{ww_p, yy_p, yy^*\} + \{ye_x, ze_x, y_py^*\}$ violates (T1) if $w \in B_3(T)$, or (T4) if $w \in B_{\geq 5}(T)$, or (T5) if $w \in B_4(T)$, since at least one branch vertex is lost ($y$) while exactly one is gained ($e_x$).

*Subcase* B2b (ii): Suppose $w = y$. Note that $y^* = y_z$. Since $G[e_x, e_{xx}, y, z]$ is not a claw, either $ye_{xx} \in E(G)$ or $ze_{xx} \in E(G)$. If $ze_{xx} \in E(G)$, then $T' := T - \{e_xe_{xx}, yy_p, yy_z\} + \{ye_x, ze_{xx}, y_py_z\}$ violates (T1) via $y$. Otherwise $ye_{xx} \in E(G)$, so since $G[y, y_r, y_p, e_{xx}]$ is not a claw, either $y_re_{xx} \in E(G)$ or $y_pe_{xx} \in E(G)$. If $y_pe_{xx} \in E(G)$, then $T' := T - \{e_xe_{xx}, yy_p\} + \{y_pe_{xx}, ze_x\}$ violates (T1) via $y$. Otherwise $y_re_{xx} \in E(G)$, and then $T' := T - \{e_xe_{xx}, yy_z\} + \{y_re_{xx}, ze_x\}$ violates (T1) if $y_r \in B(T)$, or (T5) otherwise.

*Subcase* B2b (iii): Suppose $w \in V(y_pTp)$. Since $G[e_x, e_{xx}, y, z]$ is not a claw, either $ye_{xx} \in E(G)$ or $ze_{xx} \in E(G)$. If $ze_{xx} \in E(G)$, then $T' := T - \{e_xe_{xx}, yy_p, yy^*\} + \{ye_x, ze_{xx}, y_py^*\}$ violates (T1) via $y$. Otherwise $ye_{xx} \in E(G)$, and then we consider $\deg_T(w)$. If $w \in B_{\geq 5}(T)$, then $T' := T - \{e_xe_{xx}, ww_z\} + \{ye_{xx}, ze_x\}$ violates (T4) via

$w$. Otherwise $w \in B_{\leq 4}(T)$, so Lemma 9 requires that $w_z$ must have some neighbor in $G$ among the remaining vertices of $N_T(w)$. If this neighbor is $w_r$, then $T' := T - \{e_x e_{xx}, ww_r, ww_z\} + \{ye_{xx}, ze_x, w_r w_z\}$ violates (T1) via $w$. If, instead, this neighbor is $w_p$, then $T' := T - \{e_x e_{xx}, ww_p, ww_z\} + \{ye_x, ye_{xx}, w_p w_z\}$ violates (T1) via $w$. If this neighbor is neither $w_r$ nor $w_p$, then it must be $w^*$, where $N_T(w) =: \{w_r, w_p, w_z, w^*\}$, and then $T' := T - \{e_x e_{xx}, ww_z, ww^*\} + \{ye_{xx}, ze_x, w_z w^*\}$ violates (T1) via $p$.

**Case C**: Suppose $y, z \notin V(rTp)$. Recall that $p \neq r$ by Lemma 11, and that $r \notin V(pTy) \cup V(pTz)$ by Lemma 13. Now either both $y$ and $z$ are separated from $r$ by $p$, or one of them is, or neither of them is, so consider all three cases.

*Subcase* C1: Suppose both $y$ and $z$ are separated from $r$ by $p$, so $p \in V(rTy) \cap V(rTz)$. Since $G[e_p, e_x, y, z]$ is not a claw, either $ye_p \in E(G)$ or $ze_p \in E(G)$. We may assume the first by symmetry, so $T' := T - \{e, pp_x\} + \{ye_p, ze_x\}$ violates (T1) if $p \in B_3(T)$, or (T4) if $p \in B_{\geq 5}(T)$, or (T5) if $p \in B_4(T)$.

*Subcase* C2: Suppose exactly one of $y$ and $z$ is separated from $r$ by $p$. By symmetry, we may assume $p \in V(rTz)$ but $p \notin V(rTy)$. Note that Lemma 13 implies that $r \notin V(pTy)$, while in Case C we began by assuming $y \notin V(rTp)$. We may therefore denote $V(rTp) \cap V(rTy) \cap V(pTy) =: w \notin \{r, p, y\}$. If $e_x = x$, then $T' := T - \{ww_y\} + \{xy\}$ violates (T1) if $w \in B_3(T)$, or (T4) if $w \in B_{\geq 5}(T)$, or (T5) if $w \in B_4(T)$. We may therefore assume $e_{xx}$ exists. Since $G[e_x, e_{xx}, y, z]$ is not a claw, either $ye_{xx} \in E(G)$ or $ze_{xx} \in E(G)$. If $ze_{xx} \in E(G)$, then $T' := T - \{e_x e_{xx}, ww_p\} + \{ye_x, ze_{xx}\}$ violates (T1) if $w \in B_3(T)$, or (T4) if $w \in B_{\geq 5}(T)$, or (T5) if $w \in B_4(T)$. Otherwise $ye_{xx} \in E(G)$, and then we consider $\deg_T(p)$. If $p \in B_{\geq 5}(T)$, then $T' := T - \{e_x e_{xx}, pp_x\} + \{ye_{xx}, ze_x\}$ violates (T4) via $p$. Otherwise $p \in B_{\leq 4}(T)$, and then Lemma 9 ensures that $p_x$ is adjacent in $G$ to at least one other vertex of $N_T(p)$. If $p_r p_x \in E(G)$, then $T' := T - \{e_x e_{xx}, pp_r, pp_x\} + \{ye_{xx}, ze_x, p_r p_x\}$ violates (T1) via $p$. If $p_x p^* \in E(G)$ for some $p^* \in N_T(p) \setminus \{p_r, p_x, p_z\}$, then $T' := T - \{e_x e_{xx}, pp_x, pp^*\} + \{ye_{xx}, ze_x, p_x p^*\}$ violates

(T1) via $p$. Otherwise $p_xp_z \in E(G)$. Now if $y \in L(T)$, then $T' := T - \{e_xe_{xx}, pp_x\} + \{ye_{xx}, ze_x\}$ violates (T1) if $p \in B_3(T)$, or (T5) if $p \in B_4(T)$. Otherwise $y \in B_3(T)$, and then $T' := T - \{e_xe_{xx}, pp_x, pp_z\} + \{ye_x, ye_{xx}, p_xp_z\}$ violates (T1) via $p$.

*Subcase* C3: Suppose neither $y$ nor $z$ is separated from $r$ by $p$. This means $p \notin V(rTy), V(rTz)$, while Lemma 13 implies that $r \notin V(pTy), V(pTz)$. Furthermore, we began Case C by assuming $y, z \notin V(rTp)$. We may therefore denote $V(rTp) \cap V(rTy) \cap V(pTy) =: w \notin \{r, p, y\}$ and $V(rTp) \cap V(rTz) \cap V(pTz) =: u \notin \{r, p, z\}$. Suppose $u = w$. Since $G[e_x, e_p, y, z]$ is not a claw, either $ye_p \in E(G)$ or $ze_p \in E(G)$. We may assume the first by symmetry, so $T' := T - \{e, ww_p\} + \{ye_p, ze_x\}$ violates either (T1), (T4), or (T5), depending on $\deg_T(w)$. Otherwise $u \neq w$, and we may assume $u \in V(rTw)$ by symmetry. If $e_x = x$, then $T' := T - \{uu_z\} + \{xz\}$ violates (T1) if $u \in B_3(T)$, or (T4) if $u \in B_{\geq 5}(T)$, or (T5) if $u \in B_4(T)$. We may thus assume $e_{xx}$ exists, and since $G[e_x, e_{xx}, y, z]$ is not a claw, either $ye_{xx} \in E(G)$ or $ze_{xx} \in E(G)$. If $ye_{xx} \in E(G)$, then $T' := T - \{e_xe_{xx}, uu_p\} + \{ye_{xx}, ze_x\}$ violates either (T1), (T4), or (T5), depending on $\deg_T(u)$ as before. Otherwise $ze_{xx} \in E(G)$, and then we consider $\deg_T(w)$. If $w \in B_{\geq 5}(T)$, then $T' := T - \{e_xe_{xx}, ww_y\} + \{ye_x, ze_{xx}\}$ violates (T4) via $w$. Otherwise $w \in B_{\leq 4}(T)$, and then Lemma 9 ensures that $w_y$ is adjacent in $G$ to at least one other vertex of $N_T(w)$. If $w_rw_y \in E(G)$, then $T' := T - \{e_xe_{xx}, ww_r, ww_y\} + \{ye_x, ze_{xx}, w_rw_y\}$ violates (T1) via $w$. If $w_pw_y \in E(G)$, we consider $\deg_T(z)$. If $z \in L(T)$, then $T' := T - \{e_xe_{xx}, ww_y\} + \{ye_x, ze_{xx}\}$ violates (T1) if $w \in B_3(T)$, (T4) if $w \in B_{\geq 5}(T)$, or (T5) if $w \in B_4(T)$. Otherwise $z \in B_3(T)$, and then $T' := T - \{e_xe_{xx}, ww_p, ww_y\} + \{ze_x, ze_{xx}, w_pw_y\}$ violates (T1) via $w$. Suppose $w_yw^*$, where $N_T(w) = \{w_r, w_p, w_y, w^*\}$. Then $T' := T - \{e_xe_{xx}, ww_y, ww^*\} + \{ye_x, ze_{xx}, w_yw^*\}$ violates (T1) via $w$.

Therefore $X$ is a pseudoindependent set. We will now show that $rr_1$ (and $rr_2$, by symmetry) has no oblique neighbors in $X$. Suppose some $x \in X$ is an oblique neighbor of $rr_1$. Now either $r \in V(r_1Tx)$ or $r_1 \in V(rTx)$. If $r \in V(r_1Tx)$, then $xr_1 \in E(G)$, so $T' := T - \{rr_1\} + \{xr_1\}$ violates (T1) via $r$. Otherwise $r_1 \in V(rTx)$, and then $xr \in E(G)$, so $T' := T - \{rr_1, rr_2\} + \{xr, r_1r_2\}$ violates (T1) via $r$.

Therefore $rr_1$ and $rr_2$ have no oblique neighbors in $X$. As before, the number of edges with any $v \in X$ as an oblique neighbor equals the degree of $v$, so the degrees of $X$ add up to at most $|E(T)| - 2 = (n-1) - 2 = n - 3$, contradicting the assumption of the theorem. Therefore the theorem is proven.

# Chapter 4

# Future Work

Looking at the sharpness example in Figure 1.1, one might notice that many of its vertices have degree 3 (in the whole graph). A natural question is: how much stronger is our result if we require the graph to have a minimum degree of 4 or larger? If our graph must have minimum degree at least $t$, then Figure 4.1 shows that we cannot guarantee an independent set any larger than before, though we might be able to make their degrees add up to a smaller number.

Given the above example, the following corollary to Theorem 3 and new conjecture are sharp, no matter how high a minimum degree we require:

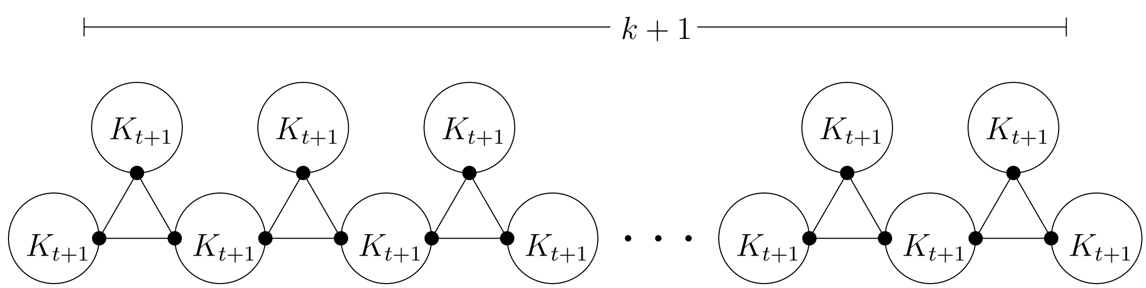


Figure 4.1: This graph has minimum degree $t$ and contains no spanning trees with at most $k$ branch vertices. A maximum independent set contains $2k+3$ vertices as before, and their degrees must add up to at least $|V(G)| - 2k - 3$.

**Corollary 2.** *Let $G$ be a connected claw-free graph with minimum degree at least 4. Then $G$ contains either a spanning tree with at most $k$ branch vertices or an independent set of $2k+3$ vertices.*

**Conjecture 2.** *Let $G$ be a connected claw-free $n$-vertex graph with minimum degree at least 4. Then $G$ contains either a spanning tree with at most $k$ branch vertices or an independent set of $2k+3$ vertices whose degrees add up to at most $n-2k-3$.*

Going forward, I will be considering ways to modify our argument so as to reduce this sum of degrees from its current level at $n-3$, or to prove Conjecture 2 for small values of $k$. An instrumental tool for the small cases of Conjecture 1 was Theorem 2. This result, either in its current form or improved for graphs of larger minimum degree, is likely to be helpful toward Conjecture 2 at least for small values of $k$.

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