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Asymptotics of Resonances for Radial Potential Scattering in Hyperbolic Space

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An abstract of
A dissertation submitted to the Faculty of the
James T. Laney School of Graduate Studies of Emory University
in partial fulfillment of the requirements for the degree of
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Abstract

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It is shown that for scattering by a spherically symmetric potential with compact support in $\mathbb{H}^{n+1}$, the resonance counting function $N(r)$ is asymptotic to Cr^{n+1}. C is found explicitly, and is shown to depend only on the dimension and the radius of support of the potential.

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Acknowledgments

My sincere thanks go to my advisor, Dr. David Borthwick. Without his encouragement to study spectral theory and his time, help, and endless patience, this dissertation would not have been possible.

I also want to especially thank Drs. Karen Brakke and Alan Koch for all their support and encouragement of my mathematical aspirations. Without them I would never even have considered attending graduate school.

My thanks go my undergraduate professors Drs. Myrtle Lewin and Larry Riddle, who showed me just how breathtakingly amazing mathematics can be.

Thanks to all my friends at Emory: you inspire me to study hard, to go new places and to try new things.

Finally and always, I want to thank my family for their unfailing love.

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Chapter 1

Introduction

For a compact d-dimensional Riemannian manifold M the Weyl law shows the connection between the asymptotic values of eigenvalues and global geometric quantities of the manifold: if $N(t)$ is the number of eigenvalues less than or equal to t, then

$$N(t) \sim C(n)\mathrm{Vol}(M)t^d.$$

Weyl-type asymptotics are still unknown for the most general manifolds. We do have upper and lower bounds in many cases. There is the well-known result of Zworski [18], where he showed that for potential scattering in $\mathbb{R}^d$, d odd, the number of poles in a disc of radius t, $N(t)$, is bounded by

$$N(t) \leq Ct^d,$$

and this bound is sharp. Roughly speaking, Zworski wrote a solution u of

$$\left(\Delta - \lambda^2 + V\right) u = 0$$

in terms of the free resolvent of Δ. Lax and Phillips [13] showed that the resolvent is related to the kernel of the scattering matrix. Zworski estimated this kernel, using the explicit formulas known for u and the resolvent. We will follow this approach, adapted for hyperbolic manifolds.

Stefanov [16] refined Zworski's result by proving a sharp constant with geometric significance. For compact hyperbolic manifolds Weyl-type asymptotics can be obtained from the Selberg trace formula, see e.g., [14]. This approach can also be used for non-compact hyperbolic surfaces of finite area [17].

Once we move to the infinite-area case, however, we need another interpretation of spectral counting. One can either supplement the counting function for the discrete spectrum by a term related to the scattering phase, or else use the counting function for resonances instead of eigenvalues.

For manifolds hyperbolic near infinity it is already known (see [5]) that

$$N(t) = O(t^n).$$

For infinite-area hyperbolic surfaces, Guillopé-Zworski [11] proved that $N(t) \asymp t^2$, although the lower bound is proportional to the 0-volume, which might be zero. Borthwick [5, 6] proved sharp upper bounds for compactly supported perturbations of $\mathbb{H}^{n+1}$ with constants related to geometric information. In this paper we show the bound from [5] is sharp for compactly suppported, spherically symmetric potential scattering in $\mathbb{H}^{n+1}$.

Let Δ be the positive Laplacian on $\mathbb{H}^{n+1}$. Denote the kernel of the resolvent by $R := (\Delta + s(n-s))^{-1}$. It is well known that in this setting

$$R(s; z, z') = \frac{2^{-2s-1}\pi^{-\frac{n}{2}}\Gamma(s)}{\Gamma(s-\frac{n}{2}+1)}\sigma^{-s}F(s, s-\frac{n-1}{2}; 2s-n+1; \sigma^{-1}), \quad (1.0.1)$$

where F is the Gauss hypergeometric function and $\sigma := \cosh^2\left(\frac{1}{2}d(z, z')\right)$. From this we can deduce that $R(s)$ admits an analytic extension to $s \in \mathbb{C}$ if n is even, and a meromorphic extension with poles at $s = -k$ for $k = 0, 1, 2, \ldots$ if n is odd. The multiplicities of the poles in the second case are given by

$$m(-k) = (2k+1)\frac{(k+1)\cdots(k+n-1)}{n!}. \quad (1.0.2)$$

Let $\mathcal{R}$ be the resonance set for $\mathbb{H}^{n+1}$ with resonances repeated according to multiplicity. ($\mathcal{R}$ is empty for n even.) Then the resonance counting function is defined by

$$N(t) := \#\{\zeta \in \mathcal{R} : \left|\zeta - \frac{n}{2}\right| \leq t\}.$$

For n odd, we can find an asymptotic for $N(t)$ by integrating 1.0.2. For later use, define the constant

$$B_n^{(0)} := \begin{cases} \frac{2}{(n+1)!} & n \text{ odd,} \\ 0 & n \text{ even.} \end{cases}$$

Then asymptotics for the resonance counting function in $\mathbb{H}^{n+1}$ are given by

$$N(t) \sim B_n^{(0)} t^{n+1} \quad (1.0.3)$$

as $t \to \infty$. Let $V \in L^\infty_c(\mathbb{H}^{n+1})$ be spherically symmetric with support contained in a ball of radius r_V. The resonance set of the perturbed operator $\Delta + V$ is $\mathcal{R}_V$, with resonances repeated according to multiplicity, and the associated counting function is

$$N_V(t) := \#\{\zeta \in \mathcal{R}_V : \left|\zeta - \frac{n}{2}\right| \le t\}.$$

Let

$$\tilde{N}_V = (n+1)\int_0^a \frac{N_V(t)}{t}\,dt. \tag{1.0.4}$$

Stefanov [16] showed that

$$N_V(t) \sim C(t)t^{n+1} \;\Leftrightarrow\; \tilde{N}_V(t) \sim C(t)t^{n+1}. \tag{1.0.5}$$

Therefore as the basis of the estimate of $N_V(t)$ we use the following relative counting formula:

$$\begin{aligned}(n+1)&\int_0^a \frac{N_V(t)-N(t)}{t}\,dt \\ &= 2\int_0^a \frac{\sigma(t)}{t}\,dt + \frac{1}{2\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \log\left|\tau\left(\frac{n}{2}+ae^{i\theta}\right)\right|d\theta + O(\log a),\end{aligned} \tag{1.0.6}$$

where $\tau(s)$ is the relative scattering determinant for $\Delta + V$ and $\sigma(t)$ is the corresponding relative scattering phase. The contribution from the scattering phase will be of lower order in this case, as shown in appendix A.

To estimate the relative scattering determinant we will use a combination of series solutions, asymptotic analysis of Legendre functions, and singular value techniques to produce an asymptotic

$$\frac{n+1}{2\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \log\left|\tau\left(\frac{n}{2}+ae^{i\theta}\right)\right|d\theta \sim a^{n+1}B_V^{(1)} + o(a^{n+1}) \tag{1.0.7}$$

with

$$B_V^{(1)} := \frac{2(n+1)}{\pi\Gamma(n)}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_0^\infty \frac{[H(x,e^{i\theta},r_V)]_+}{x^{n+2}}\,dx\,d\theta \tag{1.0.8}$$

where $[\cdot]_+$ denotes the positive part and

$$H(\alpha, r) := \mathrm{Re}\left[2\alpha \log\left(\alpha \cosh r + \sqrt{1 + \alpha^2 \sinh^2 r}\right) - \alpha \log\left(\alpha^2 - 1\right)\right] + \log\left|\frac{\cosh r - \sqrt{1 + \alpha^2 \sinh^2 r}}{\cosh r + \sqrt{1 + \alpha^2 \sinh^2 r}}\right|. \tag{1.0.9}$$

Using Stefanov's result and the estimate (1.0.7) leads directly to our main result:

Theorem 1.0.1. *Let* $V \in L^\infty_c(\mathbb{H}^{n+1})$ *be a radial potential with* $\operatorname{supp} V \subseteq B(0, r_V)$. *Assume* V *is continuous near* r_V *and satisfies* $V(r) \sim c(r_v - r)^{\omega-1}$ *as* $r \to r_V$ *for some constant* c *and* $\omega \geq 1$. *Assume that* $\mathrm{Re}\, s > \frac{n}{2}$ *and* $\left|s - \frac{n}{2}\right| \in \mathbb{N}$. *Then*

$$(n+1)\int_0^a \frac{N_V(t)}{t}\, dt = B_n^{(0)} a^{n+1} + B_V^{(1)} a^{n+1} + o(a^{n+1}) \tag{1.0.10}$$

In chapter 2 sections 2.1 and 2.2 we give some background results for the relative scattering determinant $\tau(s)$. In 2.3 we obtain estimates on the scattering matrix. In chapter 3 we apply these estimates to obtain the claimed bound on $\tau(s)$.

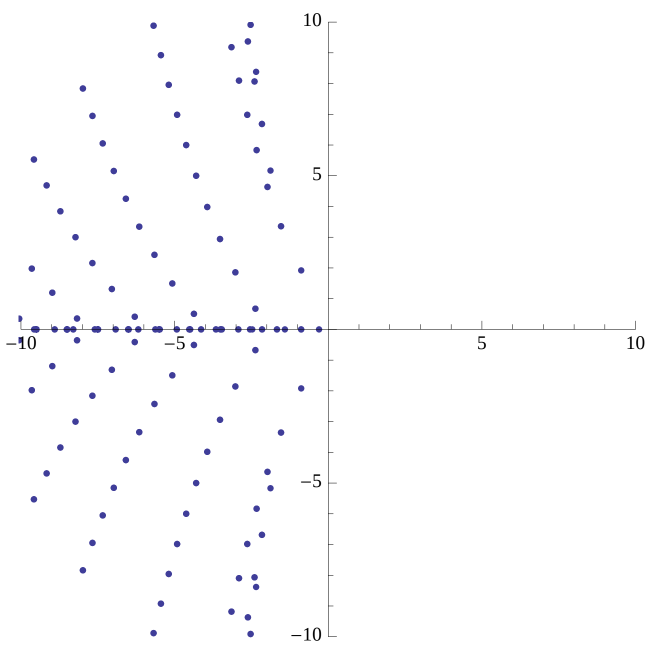

Figure 1.1: Resonances for the step potential $V = \chi_{r\leq 1}$ in $\mathbb{H}^2$.

Chapter 2

Estimates on the Scattering Matrix

2.1 Relative Scattering Theory Background

The scattering matrix arises when we try to use generalized eigenfunctions to associate solutions of $(\Delta - s(n-s))\,u = 0$ to functions on $\partial\mathbb{H}^{n+1}$. Let

$$E_0(s; z, x') := \lim_{\rho' \to 0} \rho'^{-s} R(s; z, z')$$

be the Poisson kernel for $(\Delta - s(n-s))$. For $\operatorname{Re} s > -N + \frac{n}{2}$ this kernel defines the Poisson operator

$$E_0(s) : L^2(\partial\overline{\mathbb{H}^{n+1}}, dh) \to \rho^{-N} L^2(\mathbb{H}^{n+1}, dg)$$

where h is the metric induced on $\partial\overline{\mathbb{H}^{n+1}}$ by $\rho^2 g$ and g is the usual metric on $\mathbb{H}^{n+1}$.

Let $f \in C^\infty(S^n)$. Then the equation $(\Delta - s(n-s))\,u = 0$ has a solution given by setting $u = E_0(s)f$. Further, if $\operatorname{Re} s \geq \frac{n}{2}$ with $s(n-s)$ not in the discrete spectrum of Δ, then u has a two-part asymptotic expansion as $\rho \to 0$:

$$(2s-n)E_0(s)f \sim \rho^{n-s} f + \rho^s f'. \tag{2.1.1}$$

The scattering matrix is defined as the map $S_0(s) : f \mapsto f'$. For general f, this expansion uniquely determines the scattering matrix through meromorphic continuation.

The same construction gives the perturbed scattering matrix $S_V(s)$ associated to the operator $\Delta + V$. In this case the scattering "matrix" is actually a pseudo-differential operator. However we can see from (2.1.1) that the scattering matrix still in some sense takes us from the incoming to the outgoing solution of the differential equation $(\Delta + V - s(n-s))u = 0$.

Given $V \in L^\infty_c(\mathbb{H}^{n+1})$ we associate the relative scattering determinant

$$\tau(s) := \det S_V(s) S_0(s)^{-1}.$$

$\tau(s)$ admits a factorization into an exponential part and quotients of Hadamard products, as follows. Let $H_*(s)$ denote the Hadamard product over the resonance set $\mathcal{R}_*$:

$$H_*(s) := \prod_{\zeta \in \mathcal{R}_*} E\left(\frac{s}{\zeta} + 1\right), \tag{2.1.2}$$

where

$$E(z,p) := (1-z)\exp z + \frac{z^2}{2} + \cdots + \frac{z^p}{p}$$

and $* = V$ or 0. Then τ admits a factorization in terms of these Hadamard products:

Proposition 2.1.1.

$$\tau(s) = e^{q(s)} \frac{H_V(n-s)}{H_V(s)} \frac{H_0(s)}{H_0(n-s)}, \tag{2.1.3}$$

where $q(s)$ is a polynomial of degree at most $n+1$.

The arguments of Guillarmou [9] and Borthwick [3] show that (2.1.3) holds with $q(s)$ a polynomial of unknown degree. For the proof that the degree of $q(s)$ is at most $n+1$, see the argument in [5]. This factorization is particularly useful in that it gives a Jenson-type formula connecting the resonance counting founctions to a contour integral involving $\tau(s)$. Froese [8] developed such a counting formula for Schrödinger operators in the Euclidean setting. The following version, by Borthwick [5], is the asymptotically hyperbolic analog.

Proposition 2.1.2. *Assume that $V \in L^\infty_c(\mathbb{H}^{n+1})$. As $a \to \infty$,*

$$\int_0^a \frac{N_V(t) - N_0(t)}{t}\, dt$$
$$= 2\int_0^a \frac{\sigma(t)}{t}\, dt + \frac{1}{2\pi}\int_{-\frac{\pi}{s}}^{\frac{\pi}{2}} \log\left|\tau(\frac{n}{2} + ae^{i\theta})\right| d\theta + O(\log a), \quad (2.1.4)$$

where $\sigma(\xi)$ is the relative scattering phase of V,

$$\sigma(\xi) := \frac{i}{2\pi}\log\tau\left(\frac{n}{2} + i\xi\right).$$

Proof. According to (2.1.3), for $\operatorname{Re} s > \frac{n}{2}$, $\tau(s)$ has zeros when $n - s \in \mathcal{R}_V$ or $s \in \mathcal{R}_0$ and the latter case occurs only if $s(n-s)$ is in the discrete spectrum of Δ_0. Likewise poles of $\tau(s)$ for $\operatorname{Re} s > \frac{n}{2}$ occur when either $n - s \in \mathcal{R}_0$ or $s \in \mathcal{R}_V$, and the latter case only if $s(n-s)$ is in the discrete spectrum of $\Delta + V$. All of these are counted with multiplicity.

For $t > 0$ let η be the contour $(\frac{n}{2} + t\exp(i[-\pi/2, \pi/2]) \cup [\frac{n}{2} + it, \frac{n}{2} - it]$, as shown in Figure 2.1. Assuming that η does not contain a resonance in $\mathcal{R}_V$ or $\mathcal{R}_0$ we have

$$\frac{1}{2\pi i}\oint_\eta \frac{\tau'}{\tau}(s)\, ds = N_V(t) - N_0(t) - 2d_V(t) + 2d_0(t),$$

where $d_*(u)$ is the counting function for the (finite) set $\mathcal{R}_* \cap (\frac{n}{2}, \infty)$. Evaluating the contour integral yields

$$\frac{1}{2\pi i}\oint_\eta \frac{\tau'}{\tau}(s)\, ds = \operatorname{Im}\frac{1}{2\pi}\oint_\eta \frac{\tau'}{\tau}(s)\, ds$$
$$= \int_{-t}^t \sigma'(\xi)\, d\xi + \operatorname{Im}\frac{1}{2\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{\tau'}{\tau}\left(\frac{n}{2} + te^{i\theta}\right) ite^{i\theta}\, d\theta$$
$$= 2\sigma(t) + \frac{1}{2\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} t\frac{\partial}{\partial t}\log\left|\tau\left(\frac{n}{2} + te^{i\theta}\right)\right| d\theta.$$

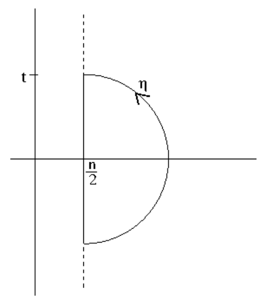

Figure 2.1: The contour η.

Divide by t and integrate to obtain the claimed formula with remainder term given by

$$2\int_0^a \frac{d_V(t)-d_0(t)}{t}\,dt = O(\log a).$$

□

The proposition can be stated for a more general space and perturbation, but we will not need it in this paper.

In [5] the relative scattering phase had the Weyl-like asymptotic

$$\sigma(\xi) = C(n)[\mathrm{vol}(K,g)-\mathrm{vol}(K_0,g_0)]\xi^{n+1} + O(\xi^n)$$

where K was a compact set within which the manifold might differ from $\mathbb{H}^{n+1}$. Our perturbation of the Laplacian does not affect the topology of $\mathbb{H}^{n+1}$, so the relative scattering phase is of lower order in this case. See Appendix A for a full proof of this for a non-smooth potential.

From [6] we have the following bound on $\log|\tau(s)|$, which we will need in the proof of the main theorem.

Lemma 2.1.3. *Let $\mathcal{Q}$ denote the joint set of zeros and poles of $\tau\left(\frac{n}{2}+z\right)$ and $\tau\left(\frac{n}{2}+e^{i\pi/(n+1)}z\right)$. If $|z|\geq 1$ and $\mathrm{dist}(z,\mathcal{Q})>|z|^{-\beta}$ with $\beta>2$, then*

$$-c(\beta)|z|^{n+1} \leq \log\left|\tau\left(\frac{n}{2}+z\right)\right| \leq C(\beta)|z|^{n+1}.$$

Proof. Since $\tau\left(\frac{n}{2}-z\right) = 1/\tau\left(\frac{n}{2}+z\right)$ and $\tau\left(\frac{n}{2}+\overline{z}\right) = \overline{\tau\left(\frac{n}{2}+z\right)}$, it suffices to prove the bounds for z in the first quadrant. The upper bound

was given in Borthwick [5, Prop. 5.4]. For the lower bound, consider the Hadamard products appearing in the factorization of $\tau(s)$ given in 2.1.3. These products are of order 2 but not finite type, so applying the Minimum Modulus Theorem Directly would give $-\log\left|\tau\left(\frac{n}{2}+z\right)\right| = O(|z|^{2+\eta})$, away from the zeros. However, Lindelöf's Theorem (see e.g. [2, Thm. 2.10.1]) shows that products of the form
$H_*\left(\frac{n}{2}+z\right) H_*\left(\frac{n}{2} \pm e^{i\pi/(n+1)}z\right)$ are of finite type. That is,

$$\log\left|H_*\left(\frac{n}{2}+z\right) H_*\left(\frac{n}{2} \pm e^{i\pi/(n+1)}z\right)\right| \leq C|z|^{n+1}$$

as $|z| \to \infty$. Using these estimates and their implications via the Minimum Modulus Theorem [2, Thm. 3.7.4], we can show that

$$\log\left|\tau\left(\frac{n}{2}+z\right)\right| \geq -c(\beta)|z|^{n+1} - \log\left|\tau\left(\frac{n}{2} \pm e^{i\pi/(n+1)}z\right)\right|, \qquad (2.1.5)$$

provided $\frac{n}{2}+z$ and $\frac{n}{2} \pm e^{i\pi/(n+1)}z$ stay at least a distance $|z|^{-\beta}$ away from the set $\mathcal{Q}$.

Assuming $\arg z \in [0, \frac{\pi}{2}]$ and again that $\frac{n}{2}+z$ and $\frac{n}{2} \pm e^{i\pi/(n+1)}z$ stay at least a distance $|z|^{-\beta}$ away from the set $\mathcal{Q}$, we know

$$\log\left|\tau\left(\frac{n}{2} \pm e^{i\pi/(n+1)}z\right)\right| \leq C(\beta)|z|^{n+1}$$

from above. The lower bound in the first quadrant then follows from 2.1.5.

□

2.2 Form of the Relative Scattering Matrix

In this section we find an explicit formula for $\tau(s)$. Let $V \in L^\infty_c[0, r_V]$ be continuous near r_V and satisfy

$$V(r) \sim c(r_V - r)^{\omega-1} \qquad as \quad r \to r_V,$$

for some constant c and $\omega \geq 1$. We want to solve the equation

$$(\Delta + V(r))\,\varphi = s(n-s)\varphi \qquad (2.2.1)$$

using geodesic polar coordinates on $\mathbb{H}^{n+1}$. In geodesic polar coordinates $\mathbb{H}^{n+1} \cong \mathbb{R}_+ \times S^n$ and the hyperbolic metric is given by

$$g_0 = dr^2 + \sinh^2 r d\omega^2$$

where $d\omega^2$ is the standard sphere metric on S^n. The Laplacian on $\mathbb{H}^{n+1}$ is

$$\Delta = -\frac{1}{\sinh^n r}\partial_r(\sinh^n r\partial_r) + \frac{1}{\sinh^2 r}\Delta_{S^n}.$$

The eigenfunctions of Δ_{S^n} are spherical harmonics Y_l^m with eigenvalues given by

$$\Delta_{S^n} Y_l^m = l(l+n-1)Y_l^m.$$

Here $l = 0, 1, 2, \ldots$ and $m = 0, 1, \ldots, h_n(l)$ with

$$h_n(l) := \frac{2l+n-1}{n-1}\binom{l+n-2}{n-2}.$$

Plugging this expression for Δ into (2.2.1) and separating the radial part gives

$$-\varphi''(r) - n\coth r\varphi'(r) + \left[\frac{l(l+n-1)}{\sinh^2 r} + V(r) - s(n-s)\right]\varphi(r) = 0 \tag{2.2.2}$$

After changing variables this is the inhomogeneous associated Legendre equation. Define

$$k := l + \tfrac{n-1}{2} \quad \text{and} \quad \nu := s - \tfrac{n+1}{2}$$

Using the P and $\mathbf{Q}$ solutions to the associated Legendre equation, we take

$$u_0^k(s,r) = \sinh^{-\frac{n-1}{2}}(r) P_\nu^{-k}(\cosh r)$$

and

$$v_0^k(s,r) = \sinh^{-\frac{n-1}{2}}(r) \mathbf{Q}_\nu^k(\cosh r)$$

as a pair of satisfactory homogeneous solutions. Note

$$\mathbf{Q}_\nu^k(z) := \frac{e^{-k\pi i}}{\Gamma(\nu+k+1)} Q_\nu^k(z). \tag{2.2.3}$$

This convention is due to Olver, and has the advantage that $\mathbf{Q}_\nu^k(z) = \mathbf{Q}_\nu^{-k}(z)$ and $\mathbf{Q}_\nu^k(z)$ is an entire function of either k or ν. Define a pair of general solutions by

$$u^k(s,r) \sim u_0^k(s,r)\,\text{at}\,r=0$$
$$v^k(s,r) = v_0^k(s,r)\,\text{at}\,r=\infty$$

By the method of variation of parameters, $u^k(s,r)$ and $v^k(s,r)$ satisfy the following integral equations:

$$u^k(s,r) = u_0^k(s,r) + \int_r^{r_V} J^k(s;r,t)V(t)u^k(s,t)\,dt \tag{2.2.4}$$

$$v^k(s,r) = v_0^k(s,r) + \int_r^{r_V} J^k(s;r,t)V(t)v^k(s,t)\,dt \tag{2.2.5}$$

where

$$J^k(s;r,t) := \frac{\Gamma(k+\nu+1)}{\sinh^n(r)}\left[u_0^k(s,r)v_0^k(s,t) - u_0^k(s,t)v_0^k(s,r)\right]. \tag{2.2.6}$$

Note $J^k(s;r,t) = J^k(n-s;r,t)$. Define

$$M^k(s) = \lim_{r\to 0} R^k(s)\sinh^{l+n-1}(r)v^k(s,r),$$

where $R^k(s)$ is to chosen so that $\lim_{r\to 0} R^k(s)\sinh^{l+n-1}(r)v_0^k(s,r) = 1$. For $s\in\mathbb{C}$ we define $M^k(n-s)$ in exactly the same way as $M^k(s)$. In [5] it was shown that $u^k(s,r)$ has a two-part asymptotic expansion:

$$u^k(s,r) = \mathcal{A}(s)v^k(s,r) + \mathcal{B}(s)v^k(n-s,r) \tag{2.2.7}$$

for $r > r_V$. The scattering matrix can then be read off:

$$[S_V]_l = -2^{n-2s}\frac{\Gamma(\frac{n}{2}-s)}{\Gamma(s-\frac{n}{2})}\frac{\mathcal{A}(s)}{\mathcal{B}(s)}.$$

Since $v^k(s,r)$ and $v^k(n-s,r)$ go as $\sinh^{-l-n+1} r$ at $r=0$, but $u^k(s,r)$ is regular at $r=0$, we must have that

$$\frac{\mathcal{A}(s)}{\mathcal{B}(s)} = -\frac{\lim_{r\to 0}\sinh^{-l-n+1}(r)\,v^k(n-s,r)}{\lim_{r\to 0}\sinh^{-l-n+1}(r)\,v^k(s,r)}.$$

We get rid of the extraneous poles appearing in the gamma factors of (2.2.7) by considering the relative scattering matrix $[S_V(s)S_0(s)^{-1}]_k$. Therefore the relative scattering matrix is

$$[S_V(s)S_0(s)^{-1}]_k = \frac{M^k(n-s)}{M^k(s)}.$$

We want to derive bounds for the scattering matrix. We begin by following Zworski's approach in [19]. The integral equation (2.2.5) can be solved by iteration using the method of Neumann series solutions, and we obtain formally at least

$$v^k = \sum_{j=0}^{\infty} v_j^k \tag{2.2.8}$$

where

$$v_{j+1}^k(s,r) = \int_r^{r_V} J^k(s,r,t)V(t)v_j^k(s,t)\,dt \tag{2.2.9}$$

We will show in the next section that the series (2.2.8) converges provided k is sufficiently large and α is bounded away from the imaginary axis. Therefore if we denote by M_j^k the contribution to M^k coming from v_j^k, we can write formally at least that

$$M^k = \sum_j M_j^k \tag{2.2.10}$$

We then use our estimates for $v^k(s,r)$ and $v^k(n-s)$ to derive bounds for $M^k(s)$ and $M^k(n-s)$.

2.3 Scattering Matrix Estimates

Define

$$\lambda_l := [S_V S_0^{-1}]_l - 1 \tag{2.3.1}$$

The goal of this section is the

Proposition 2.3.1. *Let $V \in L_c^\infty[0, r_V]$ be a spherically symmetric potential that is continuous near r_V and satisfies*

$$V(r) \sim c(r_V - r)^{\omega-1} \qquad as \quad r \to r_V,$$

for some constant c *and* $\omega \geq 1$*. Assume that* $\operatorname{Re} s > \frac{n}{2}$ *and* $\left|s - \frac{n}{2}\right| \in \mathbb{N}$*. For* $\varepsilon > 0$*,* $\arg \alpha \in \left[0, \frac{\pi}{2} - \varepsilon\right]$*, and* $r \in [0, \infty]$*, we have*

$$\lambda_l \left(\frac{n}{2} + ae^{i\theta}\right) = (k^2 + a^2)^{\frac{\omega}{2}} e^{kH\left(\frac{ae^{i\theta}}{k}, r_V\right)} + O(1). \tag{2.3.2}$$

We defer the proof of Proposition 2.3.1 until the end of this chapter. After building up the needed estimates on $v_j^k(s,r)$ and $v_j^k(n-s,r)$, we will show a bound of the form

$$\log |M_1^k(n-s)| = kH(\alpha, r_1) + O(-\omega[\log k + \log \alpha]). \tag{2.3.3}$$

2.3.1 Background Estimates

In this section we will present some estimates on the growth of the Legendre functions $P_\nu^{-k}(\cosh r)$ and $\mathbb{Q}_\nu^k(\cosh r)$ as k, $|\nu| \to \infty$ simultaneously. We want to have leading asymptotic behaviour with error bounds uniform in $\alpha := (\nu + \frac{1}{2})/k$ for $\operatorname{Re} \alpha \geq 0$. For proofs and more detailed derivations, see [5, Appendix A].

Throughout this section we let $z = \cosh r$ and switch feely between the two variables. Let

$$w(z) = \sinh r \begin{cases} P_\nu^{-k}(\cosh r) & \text{or} \\ \mathbf{Q}_\nu^k(\cosh r) \end{cases}$$

The Legendre equation then reduces to

$$\partial_z^2 w = (k^2 f + g)w, \tag{2.3.4}$$

where

$$f(r) := \frac{1 + \alpha^2 \sinh^2 r}{\sinh^4 r}, \qquad g(r) := -\frac{\sinh^2 r + 4}{4 \sinh^4 r} \tag{2.3.5}$$

If $\operatorname{Re} \alpha = 0$ then the equation (2.3.4) has turning points (points where f vanishes to first order) when $\alpha = \pm i / \sinh r$. By conjugation it suffices to assume $\operatorname{Im} \alpha \geq 0$ and so we focus on the upper turning point. To obtain uniform estimates near this point, we introduce the complex variable ζ defined by integrating

$$\sqrt{\zeta} d\zeta = \sqrt{f} dz, \tag{2.3.6}$$

starting from $\zeta = 0$ on the left and from $z_0 = \sqrt{1 - 1/\alpha^2}$, the turning point, on the right. Throughout this section we assume principal branches for the logs and square roots, under the restriction that $\arg \alpha \in [0, \pi/2]$.

Integrating both sides of (2.3.6) gives

$$\frac{2}{3}\zeta^{\frac{3}{2}} = \phi, \tag{2.3.7}$$

where

$$\begin{aligned} \phi(\alpha, r) &:= \int_{\cosh^{-1} z_0}^{r} \frac{\sqrt{1+\alpha^2 \sinh^2 t}}{\sinh t}\, dt \\ &= \alpha \log\left(\frac{\alpha \cosh r + \sqrt{1+\alpha^2 \sinh^2 r}}{\sqrt{\alpha^2 - 1}}\right) \\ &\quad + \frac{1}{2}\log\left[\frac{\cosh r - \sqrt{1+\alpha^2 \sinh^2 r}}{\cosh r + \sqrt{1+\alpha^2 \sinh^2 r}}\right]. \end{aligned} \tag{2.3.8}$$

The expression (2.3.8) is well-defined by principal branches for $\arg \alpha \in (0, \pi/2]$, and we extend the definition to the positive real axis by continuity. At $\alpha = 1$, this extension gives $\phi(1, r) = \log \sinh r$.

The region we are interested in, where $\arg \alpha \in [0, \pi/2]$ and $r \geq 0$, corresponds to the sector $\arg \phi \in [-\pi, \pi/2]$. For future reference we note that ϕ satisfies the equation

$$\partial_r \phi = \sqrt{f} \sinh r \tag{2.3.9}$$

In particular note this implies that $\operatorname{Re} \partial_r \phi \geq 0$.

The asymptotics of $\phi(\alpha, \cdot)$ will be important in our estimates. As $r \to 0$,

$$\phi(\alpha, r) = \log\left(\frac{r}{2}\right) + p(\alpha) + O(r^2), \tag{2.3.10}$$

where

$$p(\alpha) := \frac{\alpha}{2}\log\left(\frac{\alpha+1}{\alpha-1}\right) + \frac{1}{2}\log\left(1 - \alpha^2\right). \tag{2.3.11}$$

As $r \to \infty$,

$$\phi(\alpha, r) = \alpha r + q(\alpha) + O(r^{-2}), \tag{2.3.12}$$

where

$$q(\alpha) := \alpha \log\left(\frac{\alpha}{\sqrt{\alpha^2 - 1}}\right) + \frac{1}{2}\log\left(\frac{1-\alpha}{1+\alpha}\right). \tag{2.3.13}$$

Proposition 2.3.2. *Assuming $k > 0$, arg $\alpha \in [0, \frac{\pi}{2}]$ and $r \in [0, \infty)$,*

$$P_\nu^{-k}(\cosh r) = \frac{2\pi^{1/2}}{\Gamma(k+1)} \frac{k^{1/6}\zeta^{1/4} e^{\pi i/6}}{[1+\alpha^2 \sinh^2 r]^{1/4}} e^{-kp(\alpha)} \left[\mathrm{Ai}\,(k^{2/3} e^{2\pi i/3}\zeta) + h_1(k,\alpha,r)\right] \tag{2.3.14}$$

and

$$\mathbf{Q}_\nu^{k}(\cosh r) = \frac{2\pi}{\Gamma(k\alpha+1)} \frac{k^{1/6}\zeta^{1/4} \left|\frac{\alpha}{2}\right|^{1/2}}{[1+\alpha^2 \sinh^2 r]^{1/4}} e^{kq(\alpha)} \left[\mathrm{Ai}\,(k^{2/3}\zeta) + h_0(k,\alpha,r)\right] \tag{2.3.15}$$

where ζ is defined by (2.3.7) and (2.3.8), and $p(\alpha)$ and $q(\alpha)$ are defined in (2.3.11) and (2.3.13) respectively. The error terms satisfy

$$|k^{\frac{1}{6}}\zeta^{\frac{1}{4}} h_1(k,\alpha,r)| \leq C e^{k \operatorname{Re}\phi} k^{-1} \left(1+|\alpha|^{-\frac{2}{3}}\right)$$
$$|k^{\frac{1}{6}}\zeta^{\frac{1}{4}} h_0(k,\alpha,r)| \leq C e^{-k \operatorname{Re}\phi} k^{-1} \left(1+|\alpha|^{-\frac{2}{3}}\right)$$

with C independent of both α and r.

As long as we do not let α come too close to the imaginary axis and $|k\alpha|$ is not too small, we have the easier estimate:

Corollary 2.3.3. *Assuming that $|k\alpha| \geq 1$, $\operatorname{Re}\alpha > 0$, $\arg\alpha \in [0, \frac{\pi}{2} - \varepsilon]$ for some $\varepsilon > 0$, and $r \in [r_0, r_1]$:*

$$\left|P_\nu^{-k}(\cosh r)\right| \leq \frac{C}{\Gamma(k+1)} e^{k \operatorname{Re}[\phi(\alpha,r) - p(\alpha)]}$$

and

$$\left|\mathbf{Q}_\nu^{-k}(\cosh r)\right| \le \frac{C\left|\frac{\alpha}{2}\right|^{\frac{1}{2}}}{|\Gamma(k\alpha+1)|} e^{-k\mathrm{Re}[\phi(\alpha,r)-q(\alpha)]}$$

where C depends only on r_0 and r_1.

The behavior of the Legendre functions near $r = 0$ will become important in later sections. From [1]

$$P_\nu^{-k}(\cosh r) = C_1(k)\sinh^k r + O(r) \tag{2.3.16}$$

$$Q_\nu^{-k}(\cosh r) = C_2(k)\sinh^{-k} r + O(r) \tag{2.3.17}$$

where the constants C_1, C_2 do not depend on r. Recall Q_ν^{-k} is related to $\mathbf{Q}_\nu^k$ by

$$\mathbf{Q}_\nu^k = \mathbf{Q}_\nu^{-k} = \frac{e^{k\pi i}}{\Gamma(\nu-k+1)} Q_\nu^{-k}.$$

The connection formula

$$P_\nu^{-k} = \frac{1}{\cos(\nu\pi)\Gamma(\nu+k+1)}\mathbf{Q}_{-\nu-1}^k - \frac{1}{\cos(\nu\pi)\Gamma(k-\nu)}\mathbf{Q}_\nu^k \tag{2.3.18}$$

gives us the way to move between the s and $(n-s)$ terms.

Throughout this paper we will abbreviate $P_\nu^{-k}(\cosh r)$ and $\mathbf{Q}_\nu^k(\cosh r)$ as P_ν^{-k} and $\mathbf{Q}_\nu^k$ as long as it remains clear which variable is meant.

2.3.2 Estimating the Exponentially Decaying Terms

Define the constant

$$A = \frac{2|\Gamma(\nu+k+1)|}{\Gamma(k+1)|\Gamma(k\alpha+1)|}\left|\frac{\alpha}{2}\right|^{1/2} e^{k[q(\alpha)-p(\alpha)]}.$$

Lemma 2.3.4. *For $\mathrm{Re}\,\alpha > 0$, as $k \to \infty$,*

$$A = \frac{C(r_V)}{k} + O(1) \tag{2.3.19}$$

where $C(r_V)$ is a constant independent of k and α.

Proof. Using $\nu = k\alpha - 1/2$, we have $\Gamma(k+\nu+1) = \Gamma(k(1+\alpha)+1/2)$. Assuming $\operatorname{Re}\alpha \geq 0$, we use Stirling's formula to estimate

$$\begin{aligned}\log&\left|\frac{\Gamma(k(1+\alpha)+1/2)}{\Gamma(k+1)\Gamma(k\alpha+1)}(2\alpha)^{\frac{1}{2}}\right| \\ &= k\mathrm{Re}\left[(1+\alpha)\log(1+\alpha) - \alpha\log(\alpha)\right] + \log\frac{1}{k} + \frac{1}{2}\log(2) \\ &\qquad + \frac{3}{2} + O(1)\end{aligned} \tag{2.3.20}$$

for large k. From the definitions of $q(\alpha)$ and $p(\alpha)$,

$$\log e^{jk[q(\alpha)-p(\alpha)]} = jk[-(1+\alpha)\log(1+\alpha) + \alpha\log(\alpha)],$$

leaving

$$\log A = \log\left(\frac{1}{k}\right) + \frac{1}{2}\log(2) + \frac{3}{2} + O(1).$$

Thus $A = \frac{C(r_V)}{k} + O(1)$ for k sufficiently large. □

Because of the recursive relation (2.2.9) and the connection formula (2.3.18) we will need to analyze terms of the form

$$\int_r^{r_V} J^k(s,r,t)V(t)f_*(s,t)\,dt$$

where $f_* = v_0^k$ or u_0^k.

Lemma 2.3.5. *Assume that* $\operatorname{Re}s > \frac{n}{2}$ *and* $\left|s - \frac{n}{2}\right| \in \mathbb{N}$. *For* $\operatorname{Re}\phi \geq 0$ *and* k *sufficiently large,*

$$\left|\int_r^{r_V} J^k(s,r,t)V(t)v_0^k(s,t)\,dt\right| \leq \frac{C(r_V)}{k}\left|v_0^k(s,r)\right| \tag{2.3.21}$$

where $C(r_V)$ *is a constant depending only on* r_V.

Proof. Using the estimates from (2.3.3) and Lemma (2.3.4):

$$\begin{aligned}
&\left|\int_r^{r_V} J^k(s,r,t)V(t)v_0^k(s,t)\,dt\right| \\
&\qquad \le C\frac{|\Gamma(k+\nu+1)|}{\Gamma(k+1)|\Gamma(k\alpha+1)|^2}\left|\frac{\alpha}{2}\right|^{2/2}\sinh^{-\frac{n-1}{2}}(r)\,e^{k\mathrm{Re}[\phi(\alpha,r)-p(\alpha)]} \\
&\qquad\qquad \times\int_r^{r_V}\sinh(t)V(t)\left(e^{-2k\mathrm{Re}[\phi(\alpha,t)-q(\alpha)]}+e^{k[q(\alpha)-p(\alpha)]}\right)\,dt \\
&\qquad \le C(r_V)A\left|v_0^k(s,r)\right| \qquad (2.3.22) \\
&\qquad \le \frac{C(r_V)}{k}\left|v_0^k(s,r)\right|
\end{aligned}$$

where (2.3.22) comes from the fact that $\mathrm{Re}\,\phi$ is increasing in r. □

Lemma 2.3.6. *Assume that* $\mathrm{Re}\,s>\frac{n}{2}$ *and* $\left|s-\frac{n}{2}\right|\in\mathbb{N}$. *For* $\mathrm{Re}\,\phi\ge 0$ *and* k *sufficiently large,*

$$\left|M_j^k(s)\right|\le\left[\frac{C(r_V)}{k}\right]^j.$$

Proof. Lemma 2.3.5 gives us immediately that

$$\left|v_1^k(s,r)\right|\le\frac{C(r_V)}{k}\left|v_0^k(s,r)\right|.$$

Assume $\left|v_j^k(s,r)\right|\le\left[\frac{C(r_V)}{k}\right]^j\left|v_0^k(s,r)\right|$. Then

$$\begin{aligned}
\left|v_{j+1}^k(s,r)\right| &\le\int_r^{r_V}\left|J^k(s,r,t)V(t)v_j^k(s,t)\right|\,dt \\
&\le\int_r^{r_V}\left|J^k(s,r,t)V(t)\left[\frac{C(r_V)}{k}\right]^j v_0^k(s,t)\right|\,dt \\
&\le\left[\frac{C(r_V)}{k}\right]^{j+1}\left|v_0^k(s,r)\right|
\end{aligned}$$

and by induction

$$\left|v_j^k(s,r)\right| \leq \left[\frac{C(r_V)}{k}\right]^j \left|v_0^k(s,r)\right|$$

for all j. Then

$$\begin{aligned}\left|M_j^k(s)\right| &= \lim_{r\to 0}\left|R^k(s)\sinh^{l+n-1}(r)v_j^k(s)\right| \\ &\leq \limsup_{r\to 0}\left|R^k(s)\sinh^{l+n-1}(r)\right|\left[\frac{C(r_V)}{k}\right]^j\left|v_0^k(s)\right| \\ &\leq \left[\frac{C(r_V)}{k}\right]^j.\end{aligned}$$

□

2.3.3 Estimating the Exponentially Growing Terms

Using the connection formula (2.3.18) and the definitions of $u_0^k(s,r)$ and $v_0^k(s,r)$, we can write

$$\begin{aligned}v_0^k(n-s) &= \sinh(r)^{-\frac{n-1}{2}}\mathbf{Q}^k_{-\nu-1}(\cosh(r)) \\ &= \Gamma(\nu+k+1)\cos(\nu\pi)u_0^k(s,r) + \frac{\Gamma(\nu+k+1)}{\Gamma(k-\nu)}v_0^k(s,r).\end{aligned} \tag{2.3.23}$$

It will be useful notationally to let

$$L = \frac{C_V\Gamma(\omega)}{(2\phi(\alpha,r_V))^\omega}$$

where C_V is defined by

$$V(t)\sinh t \sim C_V(r_V-t)^{\omega-1}$$

as $t \to r_V$.

Lemma 2.3.7. *Assume that* $\operatorname{Re} s > \frac{n}{2}$ *and* $\left|s - \frac{n}{2}\right| \in \mathbb{N}$*. For* $\operatorname{Re}\phi \geq 0$ *and* k *sufficiently large,*

$$\left|\int_r^{r_V} J^k(s,r,t)V(t)u_0^k(s,r)\right| \leq \left|AC(r_V)u_0^k(s,r)\right| + \left|L\frac{\Gamma(\nu+k+1)}{\Gamma(k+1)^2}\frac{e^{2k\operatorname{Re}\phi(\alpha,r_V)-p(\alpha)}}{k^\omega}\right|\left|v_0^k(s,r)\right| \tag{2.3.24}$$

Proof. First we estimate the $u_0^k(s,t)$ term against the first term in $J^k(s,r,t)$:

$$\begin{aligned}&\int_r^{r_V} \left|u_0^k(s,r)v_0^k(s,t)V(t)\Gamma(\nu+k+1)u_0^k(s,t)\right| \, dt \\ &\leq \frac{|\Gamma(\nu+k+1)|}{\Gamma(k+1)|\Gamma(k\alpha+1)|}\left|\frac{\alpha}{2}\right|^{\frac{1}{2}} e^{k\operatorname{Re}[q(\alpha)-p(\alpha)]}\left|u_0^k(s,r)\right|\int_r^{r_V}|V(t)|\sinh t\, dt \\ &\leq \left|AC(r_V)u_0^k(s,r)\right| \end{aligned} \tag{2.3.25}$$

We use Laplace's method (see Appendix A) to estimate the $u_0^k(s,t)$ term against the second term in $J^k(s,r,t)$:

$$\begin{aligned}&\int_r^{r_V} \left|-v_0^k(s,r)u_0^k(s,t)V(t)\Gamma(\nu+k+1)u_0^k(s,t)\right| \, dt \\ &\quad\leq \frac{|\Gamma(\nu+k+1)|}{\Gamma(k+1)^2}e^{-2k\operatorname{Re}p(\alpha)}\left|v_0^k(s,r)\right|\int_r^{r_V}|V(t)|\sinh t\; e^{2k\operatorname{Re}\phi(\alpha,r)}\, dt \\ &\quad\leq |L|\frac{|\Gamma(\nu+k+1)|}{\Gamma(k+1)^2}e^{2k\operatorname{Re}[\phi(\alpha,r_V)-p(\alpha)]}k^{-\omega}\left|v_0^k(s,r)\right|. \end{aligned} \tag{2.3.26}$$

□

Lemma 2.3.8. *Let* $V \in L_c^\infty[0, r_V]$ *be a spherically symmetric potential that is continuous near* r_V *and satisfies*

$$V(r) \sim c(r_V - r)^{\omega-1} \qquad as \quad r \to r_V,$$

for some constant c *and* $\omega \geq 1$. *Assume that* $\operatorname{Re} s > \frac{n}{2}$ *and* $\left|s - \frac{n}{2}\right| \in \mathbb{N}$. *For* $\operatorname{Re} \phi \geq 0$ *and* k *sufficiently large,*

$$|M^k(n-s)| = |1 + M_1^k(n-s) + O(k^{-1-\omega})|.$$

Proof. Lemma 2.3.7 together with our estimates for the kernel $J^k(s,r,t)V(t)v_0^k(s,r)$ and the connection formula give

$$\begin{aligned}\left|v_1^k(n-s,r)\right| \leq & \left|\Gamma(\nu+k+1)\cos(\nu\pi)AC(r_V)u_0^k(s,r)\right| \\ & + \left|L\frac{\Gamma(\nu+k+1)^2}{\Gamma(k+1)^2}\frac{e^{2k\operatorname{Re}[\phi(\alpha,r_V)-p(\alpha)]}}{k^\omega}v_0^k(s,r)\right| \\ & + \left|AC(r_V)\frac{\Gamma(\nu+k+1)}{\Gamma(k-\nu)}v_0^k(s,r)\right|\end{aligned}$$

Induction on j then gives

$$\begin{aligned}\left|v_j^k(n-s,r)\right| \leq & \left|\Gamma(\nu+k+1)\cos(\nu\pi)AC(r_V))^j u_0^k(s,r)\right| \\ & + \left|\left[L\frac{\Gamma(\nu+k+1)^2}{\Gamma(k+1)^2}e^{2k\operatorname{Re}\phi(\alpha,r_V)-p(\alpha)}k^{-\omega}\right]j(AC(r_V))^{j-1}v_0^k(s,r)\right| \\ & + \left|[AC(r_V)]^j\frac{\Gamma(\nu+k+1)}{\Gamma(k-\nu)}v_0^k(s,r)\right|.\end{aligned}$$

Since $AC(r_V)$ decays as $\frac{1}{k}$ for $k \geq N$, the series

$$\sum_{j=0}^{\infty} v_j^k(n-s,r)$$

converges for $k \geq N$. We can therefore consider $M^k(n-s)$ as the sum (2.2.10).

Define

$$B := \frac{|\Gamma(\nu+k+1)\Gamma(k-\nu)|}{\Gamma(k+1)^2}\left|\cos(\nu\pi)\right|\left|\alpha\right|^{\frac{1}{2}} \tag{2.3.27}$$

Then

$$\begin{aligned}\left|M_j^k(n-s)\right| &= \lim_{r\to 0}\left|\frac{\Gamma(k-\nu)}{\Gamma(\nu+k+1)v_0^k(s)}v_j^k(n-s)\right| \\ &\le |\Gamma(k-\nu)\cos(\nu\pi)|\,[AC(r_V)]^j\left|\frac{u_0^k(s)}{v_0^k(s)}\right| \\ &\quad + LB\left|\frac{2}{\alpha}\right|^{\frac{1}{2}} e^{2k\operatorname{Re}\phi(\alpha,r_V)-p(\alpha)}k^{-\omega}j(AC(r_V))^{j-1} \\ &\quad + [AC(r_V)]^j.\end{aligned}$$

From (2.3.16) and (2.3.17), $\frac{u_0^k(s)}{v_0^k(s)} \to 0$ as $r \to 0$. A^j decays as k^{-j} for $k \ge N$ from the analysis of $v_j^k(s)$. Thus

$$\left|M^k(n-s)\right| = \left|1 + M_1^k(n-s) + O(k^{-1-\omega})\right|$$

for k sufficiently large. □

2.3.4 Estimating the Leading Exponential Growth Term

By Lemmas 2.3.6 and 2.3.8, to prove Proposition 2.3.1 we need to prove the asymptotic for $\left|M_1^k(n-s)\right|$. By our definition of $M^k(n-s)$ and the connection formula,

$$\begin{aligned}M_1^k(n-s) &= \lim_{r\to 0}\frac{\Gamma(k-\nu)}{\Gamma(\nu+k+1)v_0^k(r,s)}v_1^k(n-s) \\ &= \lim_{r\to 0}\frac{\Gamma(k-\nu)}{v_0^k(r,s)}u_0^k(r,s)\int_r^{r_V} V(t)\sinh^n(t)\, v_0^k(s,t)v_0^k(n-s,t)\,dt \\ &\quad - \lim_{r\to 0}\Gamma(k-\nu)\int_r^{r_V} V(t)\sinh^n(t)\, u_0^k(s,t)v_0^k(n-s,t)\,dt\end{aligned} \tag{2.3.28}$$

We show the first term in $M_1^k(n-s)$ goes to zero as $r \to 0$.

Lemma 2.3.9.

$$\lim_{r\to 0}\frac{\Gamma(k-\nu)}{v_0^k(s,r)}u_0^k(s,r)\int_r^b V(t)\sinh^n(t)\, v_0^k(s,t)v_0^k(n-s,t)\,dt = 0.$$

Proof. The proof depends on the behavior of the associated Legendre functions as $r \to 0$ and the relation (2.3.18). From (2.3.16) and (2.3.17) and the relation

$$\mathbf{Q}_\nu^k = \mathbf{Q}_\nu^{-k} = \frac{e^{k\pi i}}{\Gamma(\nu-k+1)}Q_\nu^{-k}$$

we have $\dfrac{u_0^k(s,r)}{v_0^k(s,r)} \to 0$ as $\sinh^{2k} r$ as $r \to 0$. Next we use (2.3.18) to write

$$\begin{aligned}&\lim_{r\to 0}\frac{\Gamma(k-\nu)}{v_0^k(s,r)}u_0^k(s,r)\int_r^{r_V} V(t)\sinh^n(t)\, v_0^k(s,t)v_0^k(n-s,t)\,dt\\ &=\lim_{r\to 0}\frac{\Gamma(k-\nu)}{v_0^k(s,r)}u_0^k(s,r)\int_r^{r_V} V(t)\sinh(t)\,\Gamma(\nu+k+1)\cos(\nu\pi)P_\nu^{-k}\mathbf{Q}_\nu^k\\ &\qquad + V(t)\sinh(t)\frac{\Gamma(\nu+k+1)}{\Gamma(k-\nu)}(\mathbf{Q}_\nu^k)^2\,dt \qquad (2.3.29)\end{aligned}$$

We use (2.3.3) to estimate the first term in of the integral:

$$\begin{aligned}&\lim_{r\to 0}\left|\frac{\Gamma(k-\nu)}{v_0^k(s,r)}u_0^k(s,r)\int_r^{r_V} V(t)\sinh(t)\,\Gamma(\nu+k+1)\cos(\nu\pi)P_\nu^{-k}\mathbf{Q}_\nu^k\,dt\right|\\ &\le \lim_{r\to 0} C(k,\nu)\left|\frac{u_0^k(r,s)}{v_0^k(r,s)}e^{k\mathrm{Re}[q(\alpha)-p(\alpha)]}\int_r^{r_V} V(t)\sinh t\,dt\right|\end{aligned}$$

which goes to zero in the limit $r \to 0$. For the second term

$$\lim_{r\to 0} C(k,\nu)\frac{u_0^k(s,r)}{v_0^k(s,r)}\int_r^{r_V} V(t)\sinh t\,\left(Q_\nu^{-k}\right)^2\,dt$$

the Q-Legendre function blows up as $r \to 0$, so we use L'Hospital's rule to compute the limit:

$$
\begin{aligned}
&C(k,\nu)\lim_{r\to 0}\frac{u_0^k(s,r)}{v_0^k(s,r)}\int_r^{r_V} V(t)\sinh t\left(Q_\nu^{-k}\right)^2\,dt\\
&=C(k,\nu)\lim_{r\to 0}\frac{-V(r)\sinh^2 r\,\left(Q_\nu^{-k}\right)^2}{\frac{P_\nu^{-k}\partial_r Q_\nu^{-k}-Q_\nu^{-k}\partial_r P_\nu^{-k}}{\left(P_\nu^{-k}\right)^2}}\\
&=C(k,\nu)\lim_{r\to 0}\frac{-V(r)\sinh^2 r\,\left(Q_\nu^{-k}\right)^2\left(P_\nu^{-k}\right)^2}{2\nu\cosh r\,Q_\nu^{-k}P_\nu^{-k}-(\nu-k)[Q_{\nu-1}^{-k}P_\nu^{-k}-Q_\nu^{-k}P_{\nu-1}^{-k}]}\\
&=C(k,\nu)\lim_{r\to 0}V(r)\sinh^2 r+O(r) \qquad (2.3.30)\\
&=0
\end{aligned}
$$

where (2.3.30) follows from the fact that the behavior of the Legendre functions near $r=0$ depends only on the order k. Notice we switched from Olver's $\mathbf{Q}$-function to Q_ν^k in order to use known formulas for the derivative. □

To estimate the second term in (2.3.28) we use (2.3.14), (2.3.15), and (2.3.18):

$$
\begin{aligned}
&-\lim_{r\to 0}\Gamma(k-\nu)\int_r^{r_V} V(t)\sinh^n(t)\,u_0^k(s,t)v_0^k(n-s,t)\,dt\\
&=-\lim_{r\to 0}\Gamma(k-\nu)\int_r^{r_V} V(t)\sinh(t)\,P_\nu^{-k}\Big[\cos(\nu\pi)\Gamma(\nu+k+1)P_\nu^{-k}\\
&\qquad+\frac{\Gamma(\nu+k+1)}{\Gamma(k-\nu)}\mathbf{Q}_\nu^k\Big]\,dt \qquad (2.3.31)
\end{aligned}
$$

It is easier to deal with the second term of 2.3.31 separately first. Using

(2.3.3) we have

$$\lim_{r\to 0}|\Gamma(\nu+k+1)|\int_r^{r_V}\left|V(t)\sinh t\, P_\nu^{-k}\mathbf{Q}_\nu^k\right|\,dt$$
$$\le \frac{|C(r_V)\Gamma(\nu+k+1)||\alpha|^{\frac{1}{2}}}{\Gamma(k+1)|\Gamma(k\alpha+1)|}e^{k\mathrm{Re}[q(\alpha)-p(\alpha)]}$$
$$\times \max_{t\in[0,r_V]}V(t)\int_0^{r_V}\sinh t\left(1+O(|k\phi|^{-1})\right)\,dt \tag{2.3.32}$$

Lemma 2.3.10. *The error term* $|k\phi|^{-1}\le C(r_V)|k|^{-1}$ *where the constant depends only on* r_V*.*

Proof. We break up the estimate into zones so as to use the estimate from [5, Appendix A] on $|\phi|$.

Zone 1: Assume that $|1+\alpha^2\sinh^2 r|\ge c$, and $|\alpha|\ge 1$. Then

$$|\phi|\asymp\begin{cases}-\log(\alpha r) & |\alpha|\sinh r\le\frac{1}{2}\\ |\alpha|r & |\alpha|\sinh r\ge\frac{1}{2}\end{cases}$$

Therefore in zone 1

$$\int_0^{r_V}\frac{|\sinh t|}{|\phi(\alpha,t)|}\,dt\le r_V\sinh r_V.$$

Zone 2: Assume that $|1+\alpha^2\sinh^2 r|\ge c$, and $|\alpha|\le 1$. Then

$$|\phi|\asymp\begin{cases}|\log(1-e^{-r})| & |\alpha|\sinh r\le\frac{1}{2}\\ |\alpha|(r+\log 2|\alpha|) & |\alpha|\sinh r\ge\frac{1}{2}\end{cases} \tag{2.3.33}$$

If $|\alpha|=0$, then

$$\lim_{r\to 0}\int_r^{r_V}\frac{|\sinh t|}{|\phi(0,t)|}\,dt\to 0.$$

Assume $|\alpha|>\delta$ for some small $\delta>0$. Then

$$\int_0^{r_V}\frac{|\sinh t|}{|\phi(\alpha,t)|}\,dt\le r_V\frac{\sinh r_V}{\log(1-e^{-r_V})}$$

in case 1 We subdivide case 2 as follows. If $|\alpha| \geq 1$,

$$\int_0^{r_V} \frac{|\sinh t|}{|\phi(\alpha,t)|}\,dt \leq r_V \sinh r_V.$$

If $\delta < |\alpha| < 1$,

$$\int_0^{r_V} \frac{|\sinh t|}{|\phi(\alpha,t)|}\,dt < \delta r_V \sinh r_V$$

Zone 3 is the region near the turning point, $\theta \in [\frac{\pi}{2} - \varepsilon a^{-2}, \frac{\pi}{2}]$. We will use the estimate from Lemma 2.1.3 when we estimate this region for the proof of the main theorem. □

We have already seen that the combination

$$C(r_V)\frac{|\Gamma(\nu+k+1)|}{\Gamma(k+1)|\Gamma(k\alpha+1)|}|\alpha|^{\frac{1}{2}}e^{k\mathrm{Re}[q(\alpha)-p(\alpha)]} \to 0 \text{ as } k \to \infty$$

in the analysis of $M_j^k(s)$. Therefore the term

$$\lim_{r\to 0}|\Gamma(\nu+k+1)|\int_r^{r_V}\left|V(t)\sinh(t)P_\nu^{-k}\mathbf{Q}_\nu^k\right|\,dt$$

is negligible for k sufficiently large.

2.4 Proof of the Bound on the Scattering Matrix

Proof of Proposition 2.3.1. Recall we want to show

$$\lambda_l\left(\frac{n}{2}+ae^{i\theta}\right) = (k^2+a^2)^{\frac{\omega}{2}}e^{kH\left(\frac{ae^{i\theta}}{k},r_V\right)} + O(1). \tag{2.4.1}$$

assuming $V \in L_c^\infty[0,r_V]$ be a spherically symmetric potential that is continuous near r_V and satisfies

$$V(r) \sim c(r_V - r)^{\omega-1} \qquad as \quad r \to r_V,$$

for some constant c and $\omega \geq 1$. Assume that $\mathrm{Re}\, s > \frac{n}{2}$ and $\left|s - \frac{n}{2}\right| \in \mathbb{N}$. Let $\varepsilon > 0$, $\arg\alpha \in \left[0, \frac{\pi}{2} - \varepsilon\right]$, and $r \in [0,\infty]$.

By Lemmas 2.3.4 and 2.3.9 we need to bound

$$\lim_{r\to 0}\left|\Gamma(\nu+k+1)\Gamma(k-\nu)\cos(\nu\pi)\int_r^{r_V} V(t)\sinh t\,(P_\nu^{-k}(\cosh t))^2\,dt\right|.$$

We show the proof for $k \geq N$, $N > 0$ sufficiently large, here. The proof for small k is sketched in the next section. Using (2.3.14) and the asymptotics for the Airy functions,

$$\begin{aligned}
&\lim_{r\to 0} -\Gamma(\nu+k+1)\Gamma(k-\nu)\cos(\nu\pi)\int_r^{r_V} V(t)\sinh t(P_\nu^{-k}(\cosh t))^2\,dt\\
&= \frac{-\Gamma(\nu+k+1)\Gamma(k-\nu)}{\Gamma(k+1)^2}\cos(\nu\pi)|\alpha|^{\frac{1}{2}}\\
&\quad\times\int_0^{r_V}\frac{V(t)\sinh t}{\sqrt{1+\alpha^2\sinh^2 t}}e^{2k[\phi(\alpha,t)-p(\alpha)]}\left(1+O(|k\phi|^{-1})+O(|k\phi|^{-2})\right)\,dt
\end{aligned} \tag{2.4.2}$$

Recall

$$B := \frac{|\Gamma(\nu+k+1)\Gamma(k-\nu)|}{\Gamma(k+1)^2}\,|\cos(\nu\pi)|\,|\alpha|^{\frac{1}{2}}.$$

Using the extension of Laplace's Method from Appendix A, (2.4.2) is asymptotic to

$$\begin{aligned}
&-B\frac{V(r_V)\sinh(r_V)e^{2k[\phi(\alpha,r_V)-p(\alpha)]}}{[1+\alpha^2\sinh^2 r_V]^{\frac{1}{2}}\,(2\phi'(\alpha,r_V))^{\omega}}k^{-\omega}\\
&\quad-BC\int_0^{r_V}\frac{V(t)\sinh t}{\sqrt{1+\alpha^2\sinh^2 t}}\left(\frac{1}{|k\phi|}+\frac{1}{|k\phi|^2}\right)\,dt
\end{aligned} \tag{2.4.3}$$

for $k \geq N$. Away from the turning point, we may assume $1+\alpha^2\sinh^2 t > c$ for some $c > 0$. The factor $1/|\phi|$ we estimated previously. Therefore for $k \geq N$,

$$M_1^k(n-s) = -B\frac{V(r_V)\sinh(r_V)e^{2k[\phi(\alpha,r_V)-p(\alpha)]}}{[1+\alpha^2\sinh^2 r_V]^{\frac{1}{2}}\,(2\phi'(\alpha,r_V))^{\omega}}k^{-\omega}+O(|k|^{-1-\omega}). \tag{2.4.4}$$

Let $\delta > 0$ and recall $\nu = k\alpha - 1/2$. Assuming $\alpha \notin [1, \infty)$, we can estimate $\log B$ using Stirling's formula:

$$\log B = \log \left| \frac{\sin(\pi k\alpha)\,\Gamma(k(1+\alpha)+\frac{1}{2})\Gamma(k(1-\alpha)+\frac{1}{2})}{\Gamma(k+1)^2} \right|$$
$$= \pi k|\mathrm{Im}\,\alpha| + k\mathrm{Re}\left[(\alpha+1)\log(\alpha+1) + (1-\alpha)\log(1-\alpha)\right]$$
$$+ O(\log k) \tag{2.4.5}$$

as $k \to \infty$, uniformly for $\arg(\alpha - 1) > \delta$. The same estimate can be extended to $\arg(\alpha - 1) \leq \delta$ using

$$\sin(\pi k\alpha)\,\Gamma\left(k(1-\alpha) + \frac{1}{2}\right) = \frac{-\pi \tan \pi k\alpha}{\Gamma\left(k(\alpha-1)+\frac{1}{2}\right)}$$

and the assumption $|k\alpha| \in \mathbb{N}$, which implies $|\tan \pi k\alpha| \leq 1$. Note that

$$H(\alpha, r_V) = \mathrm{Re}\left[2\phi(\alpha, r_V) - 2p(\alpha) + (\alpha+1)\log(\alpha+1)\right.$$
$$\left. -(\alpha-1)\log(\alpha-1)\right].$$

Also in the sector $|\theta| < \pi/2 - \epsilon$ and with r_V fixed, $\mathrm{Re}\,\phi'(\alpha)$ is comparable to $|\alpha|$, so that we have the estimate

$$\log|M_1^k(n-s)| = kH(\alpha, r_V) + O(-\omega[\log k + \log \alpha]) \tag{2.4.6}$$

for k sufficiently large. Exponentiating gives the desired result. □

2.4.1 Estimating the Scattering Matrix for small k

The method for small k is essentially the same as that for $k \geq N$. However, instead of asymptotics for the associated Legendre functions, we use Olver's expansion for the associated Legendre functions in terms of associated Bessel functions and then the Bessel function asymptotics. The induction is exactly analogous, and even slightly easier because of the simpler asymptotics for the Bessel functions. We give an outline of the proof here.

From [15] we have

$$P_\nu^{-k}(\cosh r) = \left(\frac{1}{k\alpha}\right)^k \left(\frac{r}{\sinh r}\right)^{\frac{1}{2}} \mathrm{I}_k(k\alpha r)\left(1 + O(\frac{1}{k\alpha})\right) \tag{2.4.7}$$

$$\mathbf{Q}_\nu^k(\cosh r) = \frac{e^{-k\pi i}}{\Gamma(k+\nu+1)}(k\alpha)^k \left(\frac{r}{\sinh r}\right)^{\frac{1}{2}} \mathrm{K}_k(k\alpha r)\left(1 + O\left(\frac{1}{k\alpha}\right)\right) \tag{2.4.8}$$

where I_k and K_k are the associated Bessel functions. The asymptotics for the I_k and K_k are well known:

$$\mathrm{I}_k(z) \asymp \begin{cases} \frac{1}{\Gamma(k+1)}\left(\frac{z}{2}\right)^k & |z| < 1 \\ \frac{e^z}{\sqrt{2\pi z}} & |z| \geq 1 \end{cases}$$

$$\mathrm{K}_k(z) \asymp \begin{cases} \frac{\Gamma(k)}{2}\left(\frac{z}{2}\right)^{-k} & |z| < 1 \\ \sqrt{\frac{\pi}{2z}}e^{-z} & |z| \geq 1 \end{cases}$$

Using the substitution $z = ae^{i\theta}r$ we split the integrals appearing in the series solution of $v_j^k(s)$ and $v_j^k(n-s)$ at $1/a$. The induction is analogous to that done in for the case of large k, only now we consider large a and bounded k. The integral over $[0, r_V] = [0, 1/a]$ decays as $1/a$. The integral over $[1/a, r_V]$ is comparable to $e^{ar_1\cos\theta}/(ka)$. For a large and k fixed,

$$kH\left(\frac{ae^{i\theta}}{k}, r_V\right) \asymp ar_V\cos\theta.$$

Thus we have a bound corresponding to that in Proposition 2.3.1 for $k > 0$. In the case that $k = 0$, we take the limiting value of $H\left(\frac{ae^{i\theta}}{k}, r_V\right)$ to get

$$\lambda_k\left(\frac{n}{2} + ae^{i\theta}\right) = (k^2 + a^2)^{\frac{\omega}{2}} e^{ar_v\cos\theta} + O(1/a).$$

This together with (2.3.3) proves Proposition 2.3.1 for all k.

Chapter 3

Asymptotics for the Number of Scattering Poles

Since the relative scattering phase is of lower order, we only need to estimate $\log|\tau(s)|$ in the half-plane $\operatorname{Re} s > \frac{n}{2}$. Recall

$$\lambda_l := [S_V S_0^{-1}]_l - 1 \tag{3.0.1}$$

Thus

$$\begin{aligned}\log|\tau(s)| &= \sum_l h_n(l) \log\left|[S_V(s)S_0(s)^{-1}]_l\right| \\ &= \sum_l h_n(l) \log|1+\lambda_l(s)| \end{aligned} \tag{3.0.2}$$

We will need different estimates for the regions where H is positive, negative, and close to zero. Hence we introduce the notation

$$\sum_l h_n(l) \log|1+\lambda_l(s)| = \Sigma_+ + \Sigma_0 + \Sigma_- \tag{3.0.3}$$

Σ_+ will be the dominant term, for which we will use our estimates on Legendre function asymptotics. Σ_0 will be over a narrow region where H is roughly zero and will have a number of terms controlled by the width of the region. For Σ_- we will use the Poisson kernel estimates from [5]. In the region where $H \approx 0$ we will also need the following rough estimate on the matrix elements.

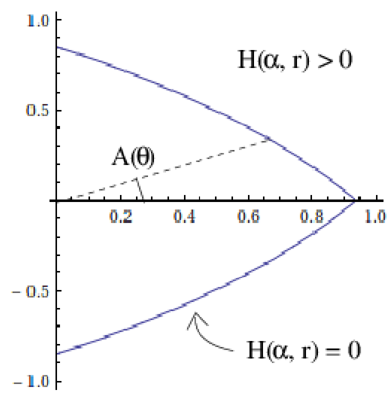

Figure 3.1: The positive region for $H(\alpha, r)$, shown for $r = 1$.

Lemma 3.0.1. *For* $0 \le k \le c\sqrt{a}$ *and* $\operatorname{Re} s \ge \frac{1}{2}$ *and assuming* $\operatorname{dist}(1 - s, \mathcal{R}_{\mathcal{V}}) \ge |s|^{-\beta}$ *with* $\beta > 2$*,*

$$\log \left| [S_V(s) S_0(s)^{-1}]_l \right| \le C(r_V, \beta, k)(k + |s|) \log |s|$$

and

$$\log \left| [S_V(s) S_0(s)^{-1}]_l \right| \ge -c(r_V, \beta)(k + |s|) \log |s|.$$

Proof. We give the proof for the case $k > N$ for some large N. The proof for $k \le N$ is analogous, but uses the estimates on Bessel functions from section 2.3.4 instead of those on Legendre functions. Recall

$$k := l + \tfrac{n-1}{2} \quad \text{and} \quad \nu := s - \tfrac{n+1}{2}.$$

Using the known behavior of $\mathbf{Q}^k_\nu(\cosh r)$ as $r \to 0$ and the definitions of $M^k(s)$, we note that $M^k(s)\frac{\Gamma(k)}{\Gamma(l+s)}$ is an entire function of s. We will base our (very rough) estimate on the version of Stirling's formula

$$\log \frac{1}{\Gamma(z)} \le \langle z \rangle \log \langle z \rangle.$$

Using the estimate for $v_j(s)$ and recalling

$$A := \frac{\Gamma(l+s)}{\Gamma(k+1)\Gamma(s - \frac{n}{2} + 1)} \left[\frac{\alpha}{2}\right]^{\frac{1}{2}} e^{k[q(\alpha) - p(\alpha)]}$$

we can estimate

$$\log|M_j^k(s)| \leq C(k, r_V)\left[\frac{s}{k - \frac{n}{2} + s}\right]^j$$

for large $|s|$. Applying Stirling to estimate

$$\log\left|\frac{\Gamma(k)}{\Gamma(l+s)}\right| \leq C(r_V, \beta)(k + |s|)\log|s|$$

and combining that with the estimate on $M_j^k(s)$ from Chapter 2 gives

$$\log|M^k(s)| \leq C_1(r_V, \beta, k)(k + |s|)\log|s| \tag{3.0.4}$$

By the Minimum Modulus Theorem (see, e.g., [2, Thm. 3.7.4]) assuming $\operatorname{dist}(1 - s, \mathcal{R}_\mathcal{V}) \geq |s|^{-\beta}$ with $\beta > 2$,

$$\log|M^k(s)| \geq -c_1(r_V, \beta, k)(k + |s|)\log|s| \tag{3.0.5}$$

for large $|s|$. Similarly for $M^k(n-s)$ we note that $M^k(n-s)\frac{\Gamma(k)}{\Gamma(l+n-s)}$ is an entire function of s and use our estimates for $v_j^k(n-s)$ to get

$$\log|M^k(n-s)| \leq C_2(r_V, \beta, k)(k + |s|)\log|s| \tag{3.0.6}$$

$$\log|M^k(n-s)| \geq -c_2(r_V, \beta, k)(k + |s|)\log|s| \tag{3.0.7}$$

for $\operatorname{dist}(1 - s, \mathcal{R}_\mathcal{V}) \geq |s|^{-\beta}$ with $\beta > 2$. The results follow from applying these estimates to

$$[S_V(s, r_V)S_0(s, r_V)^{-1}]_l = \frac{M^k(n-s, r_V)}{M^k(s, r_V)}.$$

□

Recall we want to show asymptotics for

$$\begin{aligned}\log\left|\tau\left(\frac{n}{2} + ae^{i\theta}\right)\right| &= \sum_l h_n(l)\log\left|\left[S_V\left(\frac{n}{2} + ae^{i\theta}\right)S_0\left(\frac{n}{2} + ae^{i\theta}\right)^{-1}\right]_l\right| \\ &= \sum_l h_n(l)\log\left|1 + \lambda_l\left(\frac{n}{2} + ae^{i\theta}\right)\right| \end{aligned} \tag{3.0.8}$$

Let $x = A(\theta)$ be the implicit solution of the equation $H(xe^{i\theta}, r_V) = 0$, so that $H(A(\theta)e^{i\theta}, r_V) = 0$. Assume $\theta \leq \frac{\pi}{2} - \varepsilon$. For $\delta > 0$, we will split the sum

$$\sum_l h_n(l) \log \left| 1 + \lambda_l \left(\frac{n}{2} + ae^{i\theta} \right) \right|$$

at

$$k = \frac{a}{A(\theta)(1 \pm a^{-1/2})}.$$

Using the inequality

$$\log |1 + x| \geq \log |x| - \log 2 \quad \text{for} \quad |x| \geq 2,$$

for a sufficiently large we have

$$H\left(\frac{ae^{i\theta}}{k}; r_V \right) \geq H\left(A(\theta)e^{i\theta}(1 + a^{-1/2}), r_V \right) \geq ca^{-1/2}.$$

Thus for $k \geq c\sqrt{a}$ we have from Lemma 3.0.1 that

$$\log \left| 1 + \lambda_l \left(\frac{1}{2} + ae^{i\theta} \right) \right| \geq kH\left(\frac{ae^{i\theta}}{k}; r_V \right) - O(\log k). \tag{3.0.9}$$

Together with the asymptotic

$$h_n(l) = \frac{2l^{n-1}}{\Gamma(n)} \left(1 + O(l^{-1}) \right)$$

we have

$$\Sigma_+ \geq \sum_{c\sqrt{a} \leq k \leq \frac{a}{A(\theta)(1+a^{-1/2})}} \left(\frac{2k^{n-1}}{\Gamma(n)} + Ck^{n-2} \right) \left(kH\left(\frac{ae^{i\theta}}{k}, r_V \right) - C \log k \right). \tag{3.0.10}$$

Since $H(\alpha, r_V) = O(|\alpha|)$ with a constant depending only on r_V,

$$\sum_{c\sqrt{a} \leq k \leq \frac{a}{A(\theta)(1+a^{-1/2})}} k^{n-1} H\left(\frac{ae^{i\theta}}{k}, r_V \right) = O(a^n)$$

Therefore

$$\sum_{c\sqrt{a}\le k\le \frac{a}{A(\theta)(1+a^{-1/2})}} h_n(l)\log|1+\lambda_l(s)| \\ \ge \frac{2}{\Gamma(n)}\sum_{k\le a/A(\theta)} k^n H\left(\frac{ae^{i\theta}}{k}, r_V\right) - Ca^n\log a \tag{3.0.11}$$

with C depending only on ε and r_V. Since $H(xe^{i\theta})$ is an increasing function of x, we can bound it with the corresponding integral. Thus

$$\sum_{c\sqrt{a}\le k\le \frac{a}{A(\theta)(1+a^{-1/2})}} h_n(l)\log|1+\lambda_l(s)| \\ \ge \frac{2}{\Gamma(n)}\int_{c\sqrt{a}}^{\frac{a}{A(\theta)(1+a^{-1/2})}} k^n H\left(\frac{ae^{i\theta}}{k}, r_V\right)\, dk - Ca^n\log a$$

Substituting $x = a/k$ gives

$$\sum_{c\sqrt{a}\le k\le \frac{a}{A(\theta)(1+a^{-1/2})}} h_n(l)\log|1+\lambda_l(s)| \\ \ge \frac{2a^{n+1}}{\Gamma(n)}\int_{A(\theta)(1+a^{-1/2})}^{c\sqrt{a}} \frac{H(x, e^{i\theta}, r_V)}{x^{n+2}}\, dx - Ca^n\log a$$

where C depends only on ε and r_V. For $k\le c\sqrt{a}$ we use the estimates from Lemma (3.0.1) to get

$$\sum_{\frac{n-1}{2}\le k\le c\sqrt{a}} \log\left|\left[S_V\left(\frac{n}{2}+ae^{i\theta}\right)S_0\left(\frac{n}{2}+ae^{i\theta}\right)^{-1}\right]_l\right| \ge -O(a^{\frac{n+1}{2}}\log a). \tag{3.0.12}$$

Since $H(\alpha; r_V)\asymp|\alpha r_V|$ for large $|\alpha|$, we also have

$$\frac{2a^{n+1}}{\Gamma(n)}\int_{c\sqrt{a}}^{\infty}\frac{H(x, e^{i\theta}, r_V)}{x^{n+2}}\, dx = O(a^{\frac{n+1}{2}})$$

Over the range $[A(\theta), A(\theta)(1+a^{-1/2})]$ $H(\alpha; r_V) = O(\delta)$ for some small $\delta > 0$, so we can also estimate

$$\frac{2a^{n+1}}{\Gamma(n)} \int_{A(\theta)}^{A(\theta)(1+a^{-1/2})} \frac{H(x, e^{i\theta}, r_V)}{x^{n+2}}\, dx = O(\delta a^{n+1})$$

Combining these estimates, we have

$$\Sigma_+ \geq \frac{2a^{n+1}}{\Gamma(n)} \int_{A(\theta)}^{\infty} \frac{H(x, e^{i\theta}, r_V)}{x^{n+2}}\, dx - O(a^{\frac{n+1}{2}} \log a). \tag{3.0.13}$$

For $A(\theta)(1-a^{-1/2}) < a/k < A(\theta)(1+a^{-1/2})$, since there are $O(a^{1/2})$ values of k in this range, Lemma 3.0.1 gives the estimate

$$\Sigma_0 \geq -O(a^{\frac{n+1}{2}} \log a). \tag{3.0.14}$$

Last we have the region where $a/k \leq A(\theta)$.Fix some small $\varepsilon > 0$ and let $\eta = a^{-1/2}/2$. Define $r_j := r_V + j\eta$. Let $\chi_1, \chi_2 \in C_0^\infty(\mathbb{H}^{n+1})$ be cutoff functions such that $\chi_j = 1$ for $r \leq r_j$ and $\chi_j = 0$ for $r \geq r_{j+1}$. From [5] we have

$$\begin{aligned} \lambda_k(s) &:= [S_V(s)S_0(s)^{-1}]_l - 1 \\ &= (2s-n)\langle \mathbf{1}_{[r_2,r_3]} E_0(s) Y_l^m, [\Delta, \chi_2] R_V(s) [\Delta, \chi_1] \mathbf{1}_{[r_1,r_2]} E_0(n-s) Y_l^m \rangle \end{aligned} \tag{3.0.15}$$

where $E_0(s)$ is the Poisson kernel of the unperturbed Laplacian, Y_l^m are the spherical harmonics, and $\mathbf{1}_i$ denotes the multiplication operator of the characteristic function $\chi_{[r_i, r_{i+1}]}(r)$. Assuming $\operatorname{Re} s \geq \frac{n}{2}$, $|\arg(s - \frac{n}{2})| < \frac{\pi}{2} - \varepsilon$, and the distance from $s(n-s)$ to any points in the discrete spectrum is at least ε, we can apply the spectral theorem and standard elliptic estimates to obtain

$$\Big\| [\Delta, \chi_2] R_V(s) [\Delta, \chi_1] \Big\| \leq C(r_V, \varepsilon)\eta^{-4}. \tag{3.0.16}$$

Next we know

$$\mathbf{1}_2 E_0(s) Y_l^m = \chi_{[r_2,r_3]} a_l(s; r) Y_l^m,$$

where $a_l(s; r)$ was computed explicitly in [5]:

$$a_l(s; r) = 2^{\frac{n-1}{2} - s} \sqrt{\pi} \frac{\Gamma(l+s)}{\Gamma(s - \frac{n}{2} + 1)} (\sinh r)^{-\frac{n-1}{2}} P_\nu^{-k}(\cosh r).$$

Using the identity

$$E_0(s) = -E_0(n-s)S_0(s)$$

and that

$$[S_0(s)]_l = 2^{n-2s}\frac{\Gamma(\frac{n}{2}-s)}{\Gamma(s-\frac{n}{2})}\frac{\Gamma(l+s)}{\Gamma(l+n-s)}$$

we use Schwarz's inequality to estimate

$$\begin{aligned}
&\langle \mathbf{1}_{[r_2,r_3]}E_0(s)Y_l^m, [\Delta,\chi_2]R_V(s)[\Delta,\chi_1]\mathbf{1}_{[r_1,r_2]}E_0(n-s)Y_l^m\rangle \\
&\quad\leq \left|\sin\pi(s-\frac{n}{2})\Gamma(l+s)\Gamma(l+n-s)\right| \left[\int_{r_1}^{r_2} \left|P_\nu^{-k}(\cosh r)\right|^2 \sinh r\right]^{\frac{1}{2}} \\
&\qquad \times \left[\int_{r_2}^{r_3} \left|P_\nu^{-k}(\cosh r)\right|^2 \sinh r\right]^{\frac{1}{2}} \\
&\quad\leq C(\varepsilon, r_V)ke^{kH\left(\frac{ae^{i\theta}}{k};r_V+\eta\right)} \qquad (3.0.17)
\end{aligned}$$

The details of the calculation for 3.0.17 can be found in [5, Prop.5.3]. In the range $a/k \leq A(\theta)$ where $H(\alpha, r)$ is negative, setting $\eta = a^{-1/2}$ gives

$$\begin{aligned}
kH\left(\frac{ae^{i\theta}}{k}; r_V + a^{-1/2}\right) &= k\left[H\left(\frac{ae^{i\theta}}{k}; r_V\right) + O(a^{-1/2})\right] \\
&\leq -ca^{1/2} \qquad (3.0.18)
\end{aligned}$$

so that we have

$$\left|\lambda_k\left(\frac{n}{2}+ae^{i\theta}\right)\right| \leq C(\varepsilon, r_V)a^2ke^{-ca^{1/2}} \qquad (3.0.19)$$

Finally we apply this to $\log|\tau(s)|$ using

$$\log|1+x| \geq -|x|\log 4 \qquad \text{for} \quad |x| \leq \frac{1}{2}.$$

Thus for large a we have

$$\log|\tau(s)| \geq -C(\varepsilon, r_V)a^2ke^{-ca^{1/2}}$$

and hence

$$\Sigma_- \geq -C(\varepsilon, r_V)a^2 k e^{-ca^{1/2}}. \tag{3.0.20}$$

This leads to the proof of Theorem 1.0.1, which we restate here for convenience.

Theorem 3.0.2. *Let* $V \in L^\infty_c(\mathbb{H}^{n+1})$ *be a radial potential with* $\operatorname{supp} V \subseteq B(0, r_V)$, V *is continuous near* r_V *and satisfies* $V(r) \sim c(r_v - r)^{\omega-1}$ *as* $r \to r_V$ *for some constant* c *and* $\omega \geq 1$. *Assume that* $\operatorname{Re} s > \frac{n}{2}$ *and* $\left|s - \frac{n}{2}\right| \in \mathbb{N}$. *Then*

$$(n+1)\int_0^a \frac{N_V(t)}{t}\, dt = B_n^{(0)} a^{n+1} + B_V^{(1)} a^{n+1} + o(a^{n+1}) \tag{3.0.21}$$

where

$$B_V^{(1)} := \frac{2(n+1)}{\pi\Gamma(n)} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_0^\infty \frac{[H(x, e^{i\theta}, r_V)]_+}{x^{n+2}}\, dx\, d\theta. \tag{3.0.22}$$

Proof of Theorem 1.0.1. From Proposition 2.1.2 we can estimate the number of scattering poles by bounding the relative scattering phase and the relative scattering determinant:

$$\begin{aligned} &\int_0^a \frac{N_V(t) - N_0(t)}{t}\, dt \\ &\quad = 2\int_0^a \frac{\sigma(t)}{t}\, dt + \frac{1}{2\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \log\left|\tau(\frac{n}{2} + ae^{i\theta})\right| d\theta + O(\log a). \end{aligned} \tag{3.0.23}$$

In this case the relative scattering phase is of lower order, so we are concerned only with the relative scattering determinant. Borthwick in [5] showed that

$$\begin{aligned} &\log\left|\tau\left(\frac{n}{2} + ae^{i\theta}\right)\right| \\ &\qquad \leq b(\theta, r_V)a^{n+1} - C(\varepsilon, r_V)a^{\frac{n+1}{2}}\log a - C(\varepsilon, r_V)a^2 k e^{-ca^{1/2}} \end{aligned} \tag{3.0.24}$$

where

$$b(\theta, r_V) := \frac{2}{\Gamma(n)} \int_0^\infty \frac{\left[H(xe^{i\theta, r_V}\right]_+}{x^{n+2}}\, dx.$$

Applying the estimates (3.0.13), (3.0.14), (3.0.20) we have the corresponding lower bound

$$\begin{aligned} &\log\left|\tau\left(\frac{n}{2}+ae^{i\theta}\right)\right| \\ &\qquad \geq b(\theta, r_V)a^{n+1} - C(\varepsilon, r_V)a^{\frac{n+1}{2}}\log a - C(\varepsilon, r_V)a^2ke^{-ca^{1/2}} \end{aligned} \tag{3.0.25}$$

Then for any $\varepsilon > 0$, we can integrate 3.0.25 over $|\theta| \leq \frac{\pi}{2} - \varepsilon a^{-2}$, which proves the lower bound

$$\begin{aligned} &\frac{n+1}{2\pi}\int_{|\theta|\leq\frac{\pi}{2}-\varepsilon a^{-2}} \log\left|\tau\left(\frac{n}{2}+ae^{i\theta}|\right)\right. d\theta \\ &\qquad \geq a^{n+1}\frac{n+1}{\pi\Gamma(n)}\int_{|\theta|\leq\frac{\pi}{2}-\varepsilon a^{-2}}\int_0^\infty \frac{[H(xe^{i\theta, r_V})_+]}{x^{n+2}}\, dx\, d\theta + o(a^{n+1}). \end{aligned}$$

For the missing sectors, we use Lemma 2.1.3 to see that

$$\frac{n+1}{2\pi}\int_{\frac{\pi}{2}-\varepsilon}^{\frac{\pi}{2}} \log\left|\tau\left(\frac{n}{2}+ae^{i\theta}\right)\right| d\theta \geq -c\varepsilon a^{n+1}.$$

We let $\varepsilon \to 0$ to complete the proof of theorem (1.0.1). $\square$

Appendix

by David Borthwick

4.1 Scattering determinant estimate

The relative scattering determinant associated to $V \in L^\infty_c(\mathbb{H}^{n+1})$ is the meromorphic function

$$\tau(s) := \det S_V(s)S_0(s)^{-1}.$$

Sharp upper bounds for $|\tau(s)|$ were provided in Borthwick [5, Prop. 4.5]:

Proposition 4.1.1. *Assume that the support of V is contained within a ball of radius r_0. For $a \in \frac{n}{2} + \mathbb{N}$ and $\theta \leq \pi/2$ we have*

$$\log\left|\tau(\tfrac{n}{2} + ae^{i\theta})\right| \leq b(\theta, r_0)a^{n+1} + o(a^{n+1}),$$

uniformly for $|\theta| \leq \pi/2 - \varepsilon a^{-2}$, with

$$b(\theta, r_0) := \frac{2}{\Gamma(n)} \int_0^\infty \frac{\left[H(xe^{i\theta}, r_0)\right]_+}{x^{n+2}}\, dx.$$

4.2 Scattering phase estimate

The relative scattering phase is

$$\sigma(t) := \frac{i}{2\pi} \log \tau(n/2 + it).$$

For a metric perturbation we'd have $\sigma(t)$ of order t^{n+1}, with leading coefficient proportional to the difference in volumes. For potential scattering the following shows that this leading term is absent.

Theorem 4.2.1. *The relative scattering phase associated to* $V \in L^\infty_c(\mathbb{H}^{n+1})$ *satisfies*

$$\sigma(t) = O(t^n).$$

The resolvent $R_V(s)$ is related to $R_0(s)$ by

$$R_0(s) - R_V(s) = R_V(s) V R_0(s). \tag{4.2.1}$$

If we let $\Omega := \operatorname{supp} V \subset \mathbb{H}^{n+1}$, then this implies the relation

$$(1 - V R_V(s) \mathbf{1}_\Omega)\,(1 + V R_0(s) \mathbf{1}_\Omega) = 1. \tag{4.2.2}$$

Proposition 4.2.3 implies that $\|V R_0(s) \mathbf{1}_\Omega\| < 1$ for $|s - n/2|$ sufficiently large, in which case we can write

$$1 - V R_V(s) \mathbf{1}_\Omega = (1 + V R_0(s) \mathbf{1}_\Omega)^{-1}. \tag{4.2.3}$$

Lemma 4.2.2. *The scattering matrices satisfy a relative scattering formula*

$$S_V(s) S_0(s)^{-1} = 1 + (2s - n) E_0(s)^t \mathbf{1}_\Omega\,(1 + V R_0(s) \mathbf{1}_\Omega)^{-1}\, V E_0(n - s),$$

valid for $\operatorname{Re} s \geq n/2$ *with* $|s - n/2|$ *sufficiently large.*

Proof. Using equation (4.2.1) and its transpose we have

$$\begin{aligned} R_V(s) &= R_0(s) - R_0(s) V R_V(s) \\ &= R_0(s) - R_0(s) V R_0(s) + R_0(s) V R_V(s) V R_0(s) \\ &= R_0(s) - R_0(s) \mathbf{1}_\Omega\,(1 + V R_V \mathbf{1}_\Omega)\, V R_0(s) \end{aligned}$$

The formulas for the scattering matrix can then be derived by multiplying the kernels by $(2s - n)(\rho\rho')^{-s}$ and taking the limit as $\rho, \rho' \to 0$. This gives

$$S_V(s) = S_0(s) - (2s - n) E_0(s)^t \mathbf{1}_\Omega\,(1 - V R_V \mathbf{1}_\Omega)\, V E_0(s).$$

The result follows after applying $S_0(s)^{-1}$ on the right and using (4.2.3). □

In order to apply Lemma 4.2.1 we need some estimates on the model terms. From [10, Prop. 3.2] we have the following:

Proposition 4.2.3 (Guillarmou)**.** *For* $\operatorname{Re} s > n/2 - 1/8$*,* $s \neq n/2$*, the weighted resolvent satisfies estimates*

$$\left\| \rho^{\frac{1}{2}} R_0(s) \rho^{\frac{1}{2}} \right\| \leq C \left| s - n/2 \right|^{-1},$$

and

$$\left\| \rho^{\frac{1}{2}} R_0'(s) \rho^{\frac{1}{2}} \right\| \leq C \left| s - n/2 \right|^{-1}.$$

We will also need Hilbert-Schmidt norms of the Poisson operator. For this estimate it is easiest to write the Poisson kernel in the $\mathbb{B}^{n+1}$ model. We specify the boundary defining function $\rho = 2e^{-r}$, where r is hyperbolic distance from the origin. The normalizing factor is included so that the induced metric on $\partial\mathbb{B}^{n+1} = S^n$ is the standard sphere metric. For this boundary defining function, the Poisson kernel is given by

$$E_0(s; u, \omega) = 2^{-s-1} \pi^{-n/2} \frac{\Gamma(s)}{\Gamma(s - n/2 + 1)} \left(\frac{1 - |u|^2}{|u - \omega|^2} \right)^s,$$

where $u \in \mathbb{B}^{n+1}$, $\omega \in S^n$.

Lemma 4.2.4. *Let* $\chi \in L^\infty_c(\mathbb{B}^{n+1})$*. For* $t \in \mathbb{R}$*, the Poisson operator* $E_0(n/2 + it) : L^2(S^n) \to L^2(\mathbb{B}^{n+1})$ *satisfies*

$$\left\| \chi E_0(n/2 + it) \right\|_2 \leq C \left| t \right|^{n/2-1},$$

and

$$\left\| \chi E_0'(n/2 + it) \right\|_2 \leq C \left| t \right|^{n/2-1}.$$

Proof. The Hilbert-Schmidt norm is calculated directly:

$$\begin{aligned} &\left\| \chi E_0(n/2 + it) \right\|_2 \\ &\quad = c_n \left| \frac{\Gamma(n/2 + it)}{\Gamma(1 + it)} \right| \left[\int_{S^n} \int_{\mathbb{B}^{n+1}} \chi(u)^2 \left(\frac{1 - |u|^2}{|u - \omega|^2} \right)^n \, dV(u)\, d\omega \right]^{1/2} \end{aligned}$$

Because χ is compactly supported, there is no convergence issue and the term in brackets is just a constant. The result follows from

$$\left|\frac{\Gamma(n/2+it)}{\Gamma(1+it)}\right| \leq C\,|t|^{n/2-1}\,,$$

which is easily deduced from Stirling's formula. The derivative estimate is similar. $\square$

Proof of Theorem 3.1. By virtue of Lemma 4.2.2 we can write this as

$$\tau(s) := \det(1+T(s)),$$

where

$$T(s) := (2s-n)E_0(s)^t\mathbf{1}_\Omega\,(1+VR_0(s)\mathbf{1}_\Omega)^{-1}\,VE_0(n-s).$$

Following the argument from Froese [8, Lemma 3.3], we will estimate the derivative

$$\sigma'(t) = -\frac{1}{2\pi}\,\mathrm{tr}\Big[(1+T(n/2+it))^{-1}T'(n/2+it)\Big]$$

Note first that $\det S_V(s)S_0(s)^{-1}$ is unitary for $\operatorname{Re} s = n/2$, so that

$$\left\|(1+T(n/2+it))^{-1}\right\| = 1.$$

Thus we can bound the derivative of the scattering phase by a trace norm,

$$|\sigma'(t)| \leq C\,\|T'(n/2+it)\|_1\,.$$

To control the trace norm, we have the Hilbert-Schmidt estimates on $E_0(n/2\pm it)$ and derivatives from Lemma 4.2.4. From Proposition 4.2.3 we also can estimate

$$\left\|(1+VR_0(s)\mathbf{1}_\Omega)^{-1}\right\| = O(1),$$

for $\operatorname{Re} s \geq n/2 - 1/8$ with $|s - n/2|$ sufficiently large, and same for the derivative. Putting these together (and noting the extra factor of $(2s - n)$) we obtain

$$|\sigma'(t)| = O(|t|^{n-1}).$$

The result follows by integrating this estimate. □

Let $\mathcal{R}_V$ denote the set of poles of $R_V(s)$, repeated according to multiplicity. The resonance counting function is

$$N_V(t) := \#\{\zeta \in \mathcal{R}_V : \ |\zeta - n/2| \leq t\}.$$

From the proof of [5, Thm. 1.1], we obtain the following:

Corollary 4.2.5. *Suppose that $V \in L^\infty_c(\mathbb{H}^{n+1})$ has support contained in a ball of radius r_0. Then*

$$(n+1)\int_0^a \frac{N_V(t)}{t}\, dt \leq \left[B_n^{(0)} + B^{(1)}(r_0)\right] a^{n+1} + o(a^{n+1}),$$

with

$$B_n^{(0)} := \begin{cases} \frac{2}{(n+1)!} & n \text{ odd}, \\ 0 & n \text{ even}, \end{cases}$$

and

$$B^{(1)}(r_0) := \frac{n+1}{\pi\Gamma(n)} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_0^\infty \frac{[H(xe^{i\theta}, r_0)]_+}{x^{n+2}}\, dx\, d\theta,$$

where $[\cdot]_+$ denotes the positive part and

$$H(\alpha, r) := \operatorname{Re}\left[2\alpha \log\Big(\alpha \cosh r + \sqrt{1 + \alpha^2 \sinh^2 r}\Big) - \alpha \log(\alpha^2 - 1)\right] + \log\left|\frac{\cosh r - \sqrt{1 + \alpha^2 \sinh^2 r}}{\cosh r + \sqrt{1 + \alpha^2 \sinh^2 r}}\right|.$$

4.3 Laplace's method

Consider the function $\phi(\alpha; r)$ defined in 2.3.8. Since

$$\phi(\alpha; r)' = \frac{\sqrt{1 + \alpha^2 \sinh^2 r}}{\sinh r},$$

where the prime denotes an r derivative, we see that $\operatorname{Re}\phi' > 0$ for $\operatorname{Re}\alpha > 0$. If $u(t)$ is smooth and non-vanishing at b, then Laplace's method gives the asymptotic

$$\int_0^b e^{2k\phi(t)} u(t)\, dt \sim \frac{u(b) e^{2k\phi(b)}}{2k\phi'(b)},$$

as $k \to \infty$.

It is not difficult to extend the classical result to include rougher assumptions on u, e.g. u is continuous near b with a finite order of vanishing as $t \to b$. However, we also require some uniformity with respect to the parameter α. In order to trace how the rate of convergence depends on this extra parameter, we will include the details of the proof.

Proposition 4.3.1. *Suppose that $u \in L^\infty[0, b]$ is continuous near b and satisfies*

$$u(t) \sim c(b - t)^{\sigma - 1} \quad \textit{as } t \to b,$$

for some $\sigma \geq 1$. Then, for

$$I(\alpha; k) := \int_0^b e^{2k\phi(\alpha;t)} u(t)\, dt, \qquad f(\alpha; k) := \frac{c\Gamma(\sigma)}{(2\phi'(\alpha; b))^\sigma}\, k^{-\sigma} e^{2k\phi(\alpha;b)},$$

given $\varepsilon, \kappa > 0$ there exists N (independent of α) such that

$$\left| \frac{I(\alpha; k)}{f(\alpha; k)} - 1 \right| \leq \kappa,$$

provided $|\arg \alpha| \leq \pi/2 - \varepsilon$.

Proof. Let us write

$$\int_a^b e^{2k\phi(t)} u(t)\, dt = I_1 + I_2 + I_3,$$

where for some $\eta > 0$

$$I_1 = c\int_{b-k^{-\eta}}^{b} e^{2k\phi(t)}(b-t)^{\sigma-1}\,dt,$$

$$I_2 = \int_{b-k^{-\eta}}^{b} e^{2k\phi(t)}\left(u(t) - c(b-t)^{\sigma-1}\right)\,dt.$$

$$I_3 = \int_{0}^{b-k^{-\eta}} e^{2k\phi(t)}u(t)\,dt.$$

For the first integral, we substitute $x = b - t$ and define

$$h(x) = \psi(b-x) - \psi(b) + \psi'(b)x,$$

so that

$$I_1 = ce^{2k\phi(b)}\int_0^{k^{-\eta}} e^{-2k\phi'(b)x}e^{kh(x)}x^{\sigma-1}\,dx. \tag{4.3.1}$$

This can be expressed as a Gamma integral plus some error terms:

$$I_1 = ce^{2k\phi(b)}\left[\frac{\Gamma(\sigma)}{(2k\phi'(b))^{\sigma}} + J_1 + J_2\right],$$

where

$$J_1 = \int_0^{k^{-\eta}} e^{-2k\phi'(b)x}\left(e^{kh(x)} - 1\right)x^{\sigma-1}\,dx,$$

and

$$J_2 = \int_{k^{-\eta}}^{\infty} e^{-2k\phi'(b)x}x^{\sigma-1}\,dx.$$

To estimate J_1 note that

$$\phi''(\alpha;t) = -\frac{\coth r}{\sqrt{1+\alpha^2\sinh^2 r}},$$

Thus, for $|\arg\alpha| \le \pi/2 - \varepsilon$ and $r \in [a,b]$, we have

$$|\phi''(\alpha,r)| \le \frac{\coth a}{2\sin\varepsilon}.$$

Taylor's approximation then gives $|h(x)| \le C_\varepsilon x^2$ for x near 0, and hence for large k,

$$\sup_{x\in[0,k^{-\eta}]} \left| e^{kh(x)} - 1 \right| = O_\varepsilon(k^{1-2\eta}).$$

After removing this term, we can replace the upper limit $k^{-\eta}$ by ∞ and estimate

$$|J_1| \le \frac{\Gamma(\sigma)}{(2k \operatorname{Re}\phi'(b))^\sigma} O_\varepsilon(k^{1-2\eta}).$$

The other error term in (4.3.1) can be estimated as an incomplete Gamma function:

$$|J_2| \le \frac{1}{2k \operatorname{Re}\phi'(b)} e^{-2k^{1-\eta}\operatorname{Re}\phi'(b)}.$$

Now consider the second integral. Given any $\delta > 0$, we will have

$$\left| u(t) - c(b-t)^{\sigma-1} \right| \le \delta(b-t)^{\sigma-1},$$

for all t sufficiently close to b. Hence, for k sufficiently large

$$|I_2| \le \delta \int_{b-k^{-\eta}}^{b} e^{2k\operatorname{Re}\phi(t)}(b-t)^{\sigma-1}\, dt,$$

Then, by the same analysis we used on J_2 we find

$$\begin{aligned} |I_2| &\le \delta e^{2k\operatorname{Re}\phi(b)} \\ &\times \left[\frac{\Gamma(\sigma)}{(2k\operatorname{Re}\phi'(b))^\sigma}(1 + O_\varepsilon(k^{1-2\eta})) + \frac{1}{2k\operatorname{Re}\phi'(b)} e^{-2k^{1-\eta}\operatorname{Re}\phi'(b)} \right], \end{aligned}$$

for sufficiently large k.

The third integral is estimated by

$$\begin{aligned} |I_3| &\le b \left\| u \right\|_\infty \exp\left[2k \operatorname{Re}\phi(b - k^{-\eta}) \right] \\ &\le C_\varepsilon b \left\| u \right\|_\infty e^{2k\operatorname{Re}\phi(b)} e^{-2k^{1-\eta}\operatorname{Re}\phi'(b)}. \end{aligned}$$

Collecting these estimates, we find that

$$\left|\frac{I(\alpha;r)}{f(\alpha;r)}-1\right| \leq \delta + C_\varepsilon \left(\frac{|\phi'(\alpha;b)|}{\operatorname{Re}\phi'(\alpha;b)}\right)^\sigma k^{1-2\eta} + C_\varepsilon k^\sigma \left|\phi'(\alpha;b)\right|^\sigma e^{-2k^{1-\eta}\operatorname{Re}\phi'(\alpha;b)},$$

which can be made arbitrarily small provided $\eta \in (1/2, 1)$. For $|\arg\alpha| \leq \pi/2 - \varepsilon$ we have

$$\frac{\operatorname{Re}\phi'(\alpha;b)}{|\phi'(\alpha;b)|} \geq \sin\varepsilon,$$

so the $k^{1-2\eta}$ term is controlled by choosing k large relative to $1/\varepsilon$. For the final error term we also need to note the lower bound

$$|\phi'(\alpha;b)| \geq \frac{\sqrt{\sin 2\varepsilon}}{\sinh b}.$$

□

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