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Heuristics Employed by Problem Solvers Engaged in a Robotics-Based Task

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Abstract

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Although Polya(1945) provided a list of heuristics that have proven fruitful in solving nonroutine problems, the mathematics education community has not reached agreement on how to apply this knowledge in mathematics classrooms. The purpose of this dissertation was to identify what heuristics were employed and how by six high school and six undergraduate students, who enjoyed solving mathematics problems, during their attempt to solve a robotic arm task designed to stimulate the applications of trigonometry in its solution. Structured, task-based interviews, audio and video recorded, were used for documenting the problem-solving episodes and were coded using a modified version of Kilpatrick's (1967) problem-solving coding scheme. The results suggest that access to the robotic arm together with heuristics identified by Polya enabled the mathematically proficient students to engage with the mathematical task. In particular, the heuristics Changes Condition and Successive Approximations promoted scientific engagement with mathematics – enabling students to experience mathematics as a discipline that includes exploration and reformulation of the original problem. The findings also suggest that when using the heuristic Drawing a Figure, identifying the unknown in the diagram is not an easy endeavour. This has implications for the teaching of mathematics because it suggests problem solvers should be given ample thinking time to recognize the salient features of the problem and, thus, be able to distinguish the unknowns from the knowns. Lastly, the study also finds that using the search engine Google (i.e. "Googling"), expands Polya's Recalling a Related Problem heuristic to now include Finding a Related Problem. Problem solvers are no longer limited to the information they know or can recall. Students, today, may need to be presented with more problems that challenge them to identify what information they need to help solve the problem at hand and then, be encouraged to search the web to find the specific missing knowledge needed for a successful resolution to the problem.

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“The end of all education should surely be service to others.”

- Cesar Chavez

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CHAPTER 1:

INTRODUCTION

In an ideal educational system, all students have access to good mathematics teachers and rich mathematical tasks to become proficient problem solvers. Unfortunately, the reality is far from the ideal. Despite efforts to improve students' mathematical problem solving, the majority of students, including college students who are majoring in fields associated with mathematics, still face great difficulty with problems that require analysis or creative mathematical thinking –nonroutine problem solving (Carpenter, Lindquist, Matthews, & Silver, 1984; Clement, 1982; Lindquist, Carpenter, Silver, Matthews, 1983; NAEP, 2013; Schoenfeld, 1985).

To inform the field of mathematics education, the highly regarded mathematician George Polya wrote a sequence of three books on solving nonroutine problems (Polya, 1945/1957, 1954, 1962-1981). In his first book, *How to Solve It*, he provides a list of heuristics that often prove fruitful in solving nonroutine problems (Polya, 1945/1957). Researchers have found that expert problem solvers typically employ a subset of these same heuristics when solving nonroutine problems (Schoenfeld, 1985). As of today, however, researchers have not reached consensus regarding whether students can be taught to use heuristics and how much these learned heuristics improve the problem-solving skills of novices – nor have they agreed on which heuristics should be taught to students or how they should be taught (Begle, 1979; Schoenfeld, 1985, 1992).

Thus far, most research has focused on students' heuristic reasoning and problem-solving behavior, while less attention has been given to the mathematical tasks in which students were engaged during problem-solving episodes. This is unfortunate because some researchers have found variation in students' thinking processes, their development of mathematical knowledge, and their conceptualization of mathematics depending on the mathematical task posed (Hiebert & Wearne, 1993; Stein & Lane, 1996). In particular, mathematical tasks that focus on mathematical ideas, as opposed to mechanical procedures, are associated with greater student understanding (Hiebert & Wearne, 1993). Mathematical tasks that elicit more reflective and analytic thought and engage students in non-algorithmic forms of thinking tend to have the greatest impact on students' mathematics gains (Hiebert & Wearne, 1993; Stein & Lane, 1996).

As a contemporary mathematical task, robotics-based tasks are attracting interest in education for their potential to engage students in nonroutine problem solving. In his seminal work, *Mindstorms*, Papert (1980) suggested that Turtle Geometry promoted nonroutine problem solving. He claimed that microcomputers and Turtle Geometry, an animal-like physical robot on four wheels, which was also represented digitally by a cursor on the computer screen, had the potential to enrich learning environments by making complex mathematical concepts accessible to the learner. He proposed that students' lack of success and engagement with mathematics could be attributed to the "poverty in materials from which the apparently 'more advanced' intellectual structures can be built" (p. 19).

Further, Papert argued Turtle Geometry could serve as an entry point to Polya's heuristics. For example, the robotics-based tasks in Turtle Geometry constantly elicit the heuristic of "Problem Decomposition" because tasks in Turtle Geometry are most efficiently solved when divided into sub-problems. Also, most robotics-based tasks lend themselves naturally to using the heuristic of considering "Simpler Auxiliary Problems," and so on. Today, researchers are starting to find evidence that robotics-based tasks do indeed stimulate problem solvers to use heuristic processes (Barack & Zadok, 2009; Castledine & Chalmers, 2012; Fernandes, Fermé, & Oliveira, 2009; Petre & Price, 2004; Silk, 2011).

Statement of the Problem

Despite efforts to improve students' mathematical problem solving, the majority of American students, including college students who are majoring in fields associated with mathematics, still face great difficulty with problems that require analysis or creative mathematical thinking – an activity known as nonroutine problem solving (Carpenter, Lindquist, Matthews, & Silver, 1984; Clement, 1982; Lindquist, Carpenter, Silver, Matthews, 1983; Schoenfeld, 1985; NAEP, 2013). This is concerning because U.S. economic growth is tightly connected with technological innovation (Bonvillian, 2002), and mathematics is part of the foundational knowledge of technology-related fields. Hence, for this growth to continue or at least to be sustained, students' engagement and proficiency in mathematics need to be given more attention.

While research findings suggest teachers are the variable with the greatest impact on students' learning (Darling-Hammond, 2000), Boaler (2016) proposes that teachers are not working alone. Teachers' success depends on the rich mathematical tasks and the questions that they bring into the classroom (Stein & Lane, 1996; Boaler, 2016). These resources can help educators engage students with mathematics or vice versa (Boaler, 2016).

Assuming that students are presented with a mathematical task that is of interest to them, Polya (1945) identified a list of heuristics, problem-solving strategies, in the form of questions that educators could introduced to students to help students remain engage with the problem and to progress towards a solution. However, empirical studies have yielded mix results (Schoenfeld, 1985). Notwithstanding, mathematicians and mathematics educators still uphold the heuristics identified by Polya (1945) as fruitful when solving non-routine problems (Schoenfeld, 1985, 2007). Schoenfeld (1985) suggested that the mathematics education community needs to change their simplistic views of heuristics and distill each of Polya's heuristics to its basic elements to help students adopt them. Furthermore, it will be necessary to understand how and why problem solvers make the decisions they make (Schoenfeld, 2007).

As an example of a mathematical task that could introduce students to Polya's heuristics, Papert (1980) suggested using Turtle Geometry, an animal-liked robot. Papert (1980) observed children employing Polya's heuristics when engaged with Turtle Geometry even though they were not taught or asked to employ the heuristics. McClintock (1980) suggested that we learn from specific mathematical tasks and capitalize on their particularities to focus on certain heuristics. Given that today's youth appear to find robotics activities interesting, study of prototypic robotics-based tasks is called for at this time.

Purpose of the Study

The purpose of the study was to examine the heuristics a nonroutine robotics-based task stimulated problem solvers to use. I examined variations in the heuristics employed depending upon the mathematical sophistication of the problem solver engaged in the task. I recruited 12 participants who enjoy solving mathematics problems – six high school students, and six undergraduate students.

The findings can inform the development of curricular units that use robotics-based tasks to introduce and support the teaching and learning of nonroutine mathematical problem solving at the high school level.

CHAPTER 2

REVIEW OF THE LITERATURE

Researchers have studied the topic of problem solving for many years, with different researchers using the term "problem solving" in different ways. Thus, before presenting a review of the literature, I provide definitions pertaining to how problem solving is used in this study. The definitions are combined with a brief overview of Polya's writings on problem solving (i.e. his identified heuristics and problem-solving phases) which I used as my theoretical lens for analyzing the participants' problem-solving episodes.

Definitions and Theoretical Framework

Nonroutine Problem Solving

Because the term 'problem solving' has been applied to a wide range of situations from completing rote exercises to "doing mathematics as a professional" (Schoenfeld, 1992), in the recent mathematics education literature distinctions are made between routine and nonroutine problem solving (Stanic & Kilpatrick, 1988, p. 15). Mathematical tasks that involve nonroutine problem solving engage students in the process of discovering solutions. When Polya (1947/1957, 1954, 1962-1981) refers to problem solving, he is referring to what is now classified as nonroutine problem solving. According to Polya (1957), "a great discovery solves a great problem" (p. v), but even when a problem does not lead to a great discovery, individuals may experience the joy of discovery by solving problems that challenge their curiosity and elicit their inventive talents. Polya (1957) goes on to say that:

> Solving a problem means finding a way out of a difficulty, a way around an obstacle, attaining an aim that was not immediately attainable. Solving problems is the specific achievement of intelligence, and intelligence is the specific gift of mankind: solving problems can be regarded as the most characteristically human activity (p. vii).

In accordance with Polya, Schoenfeld (1982) adds that during the act of solving a nonroutine task, there is a need to think. To define thinking, he gives examples of what thinking is not. For Schoenfeld (1982), a mathematician is not "thinking when he uses the quadratic formula" (p. 32). A problem that can be done routinely with ease by a person does not require thinking. Problems that require thinking force mathematicians to embark on an exploration until they feel comfortable with the problem and have gained an understanding of it. Besides becoming acquainted with the problem, problem-solving also requires an active search for possible alternative solutions along with concurrent mental evaluation of the viability of the various alternatives (Shuell, 1990). In short, nonroutine problem solving is both an active and a creative endeavor.

For the purpose of this dissertation, a *nonroutine problem task* refers to a mathematical task that makes the learner think. It elicits the inventive talents of the problem solver to achieve a goal that is not immediately attainable. Likewise, *routine problem solving* or a routine task will be used to refer to a mathematical task that does not make the learner think. Hence, even when a problem is supposed to be challenging, if the learner is familiar with the problem, and solves it with ease without thinking, the problem will be considered a routine problem for that person. That is, a problem task may be either routine or nonroutine depending on the person doing the solving.

Heuristics and Their Role in Nonroutine Problem Solving

Because nonroutine problem solving is a complex mental activity, some researchers and educators including Dewey (1910), Polya (1945/1957, 1954, 1962-1981) and Newell and Simon (1972) have suggested problem-solving models. In particular, George Polya's series of books were written in hopes of facilitating the teaching of problem solving. In his book, *How to Solve it* (1945/1957), the distinguished mathematician shared, 67, questions and suggestions that he used to stimulate the thinking of his students when solving nonroutine problems.

Although some scholars believe that the complexity of problem solving cannot be captured in separate steps or phases (Hayes, 1978; McCormick, 1997), Polya organized his list of questions into four problem-solving phases: Understanding the Problem, Devising a Plan, Carrying Out the Plan, and Looking Back. However, his accomplishment came with some ambiguity. The four phases are not necessarily entered and exited in a linear manner. Depending on the mental processes of the problem solver and the nonroutine task itself, the sequence and the number of times a problem solver enters, exits, re-enters, and re-exits the phases could vary.

In addition, Polya (1957) introduced the notion of *heuristics,* which he defined as the means and methods a person uses to solve nonroutine problems (p. vi). Among the 67 heuristics he considered helpful are Exploiting Analogies, Varying Specific Aspects of the Problem, Using Auxiliary Problems, Utilizing Related Problems for which a Solution is Known, Decomposing and Recombining the Elements of the Problem in New Ways, Working Backwards, and Drawing a Picture or Diagram. He also specified that beyond understanding the means and methods of nonroutine problem solving, a “heuristic endeavors to understand…the mental operations typically useful in this process” (Polya, 1957, p. 129-130). In this sense, Polya’s list of questions and suggestions also serve as heuristics because their purpose is to stimulate the thinking of the problem solver. Because experienced problem solvers often use the above heuristics, Polya’s work on problem solving is held in high regard by mathematicians and mathematics educators (Schoenfeld, 1985).

Despite agreement among expert problem solvers on the powerful impact of heuristics for nonroutine problem solving, it is not obvious to educators how to empower novice problem solvers with heuristics in school mathematics. Polya (1957) envisioned that a novice problem solver could

> absorb a few questions… so well that he is finally able to put to himself the right question in the right moment and perform the corresponding mental operation naturally and vigorously. (p. 4)

However, he did not anticipate that it would be easy for novice problem solvers to absorb heuristics. On the contrary, he intended for his work to be useful to teachers “who wish to develop their students’ ability to solve [nonroutine] problems, and to students who are keen on developing their own abilities” (Polya, 1957, p. vi). He wrote for an audience of proficient problem solvers who could use their understanding and experiences to help others absorb heuristics and for students who know how to teach themselves. Perhaps he assumed his audience was keen enough to know how to pose the right question at the right time to stimulate thinking processes.

Heuristics in Mathematics Education

Over time, mathematics education researchers have explored the purported potential of Polya's heuristics to improve students' performance on problem-solving tasks, but results have been mixed. Ashton (1962) conducted a ten-week study with ten 9th grade algebra classes selected from five schools. From each school, two algebra classes with students of above average ability participated. One of the classes was taught with their normal textbook method while the other was guided to discover the solutions of the algebra problems through questions from Polya's list. All students took a pre-test consisting of word problems. The same set of problems served as the post-test at the end of the study. The five classes that used Polya's heuristics showed greater gains than the comparison group in the pre-posttest. The gains of the experimental group were statistically significant, suggesting an improvement in their problem-solving skills at the conclusion of the ten-week treatment.

Although Ashton's results are encouraging, other researchers have not experienced the same success with heuristics (Goldberg, 1974; Loomer, 1976). On the contrary, Begle (1979) suggested that empirical studies on problem-solving strategies had not yielded information that could guide the mathematics education community regarding which heuristics to concentrate on for all students. Instead, these studies suggested that heuristics are participant and problem specific – challenging hope that a few heuristics can benefit most students.

In spite of these inconclusive findings, Schoenfeld (1985) suggested that the lack of consistent gains lay in a simplistic view of heuristic instruction, not necessarily in the lack of potential of heuristics on students' nonroutine problem-solving achievement. He suggested the impact of heuristics and their benefit could become negligible if the problem solver is not trained properly to distinguish when and how certain heuristics can be used – as Polya (1945/1957) anticipated. He also added that each of Polya's heuristics was dense with sub processes. For instance, to simply tell a student to consider using "Special Cases" is too vague. He advised that there was a need for each of Polya's heuristics to be "fully explicated before [they could] be use[d] reliably by students" (Schoenfeld, 1985, p. 73). In a nutshell, Schoenfeld suggested that the teaching of heuristics involved more than sharing Polya's list with students. Unfortunately, "in most studies, the characterization of heuristic strategies was not sufficiently prescriptive. Not nearly enough detail was provided for the characterizations to serve as guides to the problem solving process" (p. 73).

In addition, even though many of the empirical studies of heuristics are participant and problem specific, some researchers' findings parallel Polya's list of heuristics. Kilpatrick (1967) conducted a study with 56 above-average ability students who had just completed 8th grade to identify which of Polya's heuristics they employed when attempting to solve nonroutine problems. The participants had to think aloud as they attempted to solve the problems. Kilpatrick (1967) observed the participants in his study employing the following heuristics from Polya's list: Drawing a Figure, Using Successive Approximations, Questioning the Existence/Uniqueness of the Solution, Using a Deduction Process, Using an Equation, Using Trial and Error, and Checking the Solution. However, Kilpatrick (1967) observed that although participants employed some of the heuristics identified by Polya, they did not take full advantage of the heuristics. For instance, he noted that although the participants in his study drew figures, they did not use the figure to improve their understanding of the problem. They merely drew figures to keep a record of their thinking. Therefore, good problem solvers drew good figures and bad problem solvers drew bad figures. Kilpatrick (1967) concluded that instruction on how to draw a figure was needed for problem solvers to take advantage of the heuristic.

Other researchers have found that instruction in Polya's heuristics does seem to influence students' approach to problem solving. Lucas (1972) found that targeted instruction changed calculus students use of five heuristics in his study. These included Using New Notation, Applying the Method of a Related Problem, Applying the Result of a Related Problem, Reasoning by Analysis and Organizing Data. Kantowski (1974) added to the list Patterns of Analysis (Decompose heuristic) and Synthesis (Recombine heuristic).

However, there is disagreement over which heuristics can be influenced through instruction. For example, Lucas (1972) did not find evidence that he could influence heuristic behaviors such as Drawing a Diagram or Modifying a Diagram, Reasoning by Synthesis, Trial and Error, and heuristics related to the Looking Back problem-solving phase. Unlike Lucas, Kantowski (1974) found that it was possible to influence the Reasoning by Synthesis heuristic through instruction, but similarly to Lucas, she found little evidence in her study of students' use of the heuristics related to the Looking Back problem-solving phase.

The used of heuristics related to the Looking Back problem-solving phase have been identified as difficult to influence. Besides Lucas and Kantowski, other researchers have noted that students do not tend to use heuristics to mentally evaluate their work such as Checking the Argument, Looking for Alternative Solutions, or simply Checking their

Solution (Lee, 2015). Even when students are asked explicitly to Check their Solution, students tend to not repeat the heuristic strategy in the next problem (Oehmke, 1979, p. 53). The lack of use of heuristics related to the Looking Back problem-solving phase was also identified by Polya (1957), who noticed that "even fairly good students, when they have obtained the solution of the problem…write down neatly the argument, shut their books and look for something else" (p.14). Polya (1957) warned that "some of the best effects may be lost if the student fails to reexamine and to reconsider the completed solution" (p. 6). In fact, students who engaged in the "Looking Back" problem-solving phase more frequently tend to perform better in multiple solution problem tasks (Lee, 2015). Thus, the heuristics related to the "Looking Back" problem-solving phase might be good candidates to teach to all students.

Whereas the research in mathematics education suggests that students rarely engaged in the Looking Back problem-solving phase, preliminary findings from robotics-based instruction reveal students routinely engaged in the Looking Back problem-solving phase – even though they had not been instructed to do so. If different mathematical tasks indeed foster the use of particular heuristics in a natural way, I propose exploiting this finding to engineer problem-solving tasks that stimulate particular desired student thinking processes.

In fact, McClintock (1980) suggested heuristics be viewed as inherent to the stated problem task. He believed the level of this inherency will vary from problem to problem. For example, the famous Four-Ring Tower of Hanoi problem provides a rich opportunity for implementing the heuristic strategy of Subproblem Decomposition, because the Three-Ring Tower of Hanoi problem must be accomplished several times in the process of solving the Four-Ring Tower of Hanoi problem. That is, the Three-Ring Tower of Hanoi problem is a subproblem of the Four-Ring Tower of Hanoi problem. Even though problem solvers might not be aware of the specific heuristic that they are using to accomplish this task, without using some form of the Subproblem Decomposition heuristic, they will most likely not accomplish their goal. Even when the heuristic is not task specific, the task still provides the learning environment that stimulates the thinking of the student and initiates the reasoning process of the problem solver. It is the interaction between the task and the problem solver's mental processes that evoke the heuristics.

McClintock (1980) was not the only one who believed tasks subsume inherent mathematical characteristics. According to Krutetskii (1976), problems are apt to evoke certain mathematics "abilities" (e.g. flexibility of thinking, reversibility of mental processes, etc.) from students when the ability exists. He used think-aloud protocols and a set of carefully designed problems to demonstrate that by tinkering with the content and presentation of the task different mathematical behaviors can be evoked. Furthermore, Krutetskii was able to use the think-aloud data to gain a sense of the quality of interviewed students' thought processes and their ability level.

Robotics-Based Tasks and Heuristics

Similarly, in robotics learning environments, researchers have attributed the learning outcomes and learning behaviors of students to aspects of the robotics learning environment, starting with Papert's (1980) pioneer work on educational technology as a focal point. Papert introduced the education community to a new learning environment, Turtle Geometry. Turtle Geometry was a physical animal-like robot that was controlled via the computer. It was also represented digitally by the computer cursor. Papert suggested that learners could "identify" with Turtle because they could apply their knowledge about their own body movements in Turtle Geometry's learning environment. Additionally, in this learning environment, learners transitioned from viewing their work as right or wrong to wondering whether their work was fixable. Learners began debugging – finding the errors in their work and searching for ways to fix their work. Debugging was facilitated by the technology because it enabled learners to see their work progress towards a solution throughout the debugging process. Similarly, Sullivan (2008) identified behaviors that robotics learning environments support – computation, estimation, manipulation, and observation. Lastly, by surveying 127 educational robotics teachers, a group of researchers found that there were ten attributes that teachers perceived were present in robotics-based tasks; namely, technological, active, manipulative, collaborative, constructive, intentional, reflexive, contextual, conversation, and complex (Patiño, Diego, Rodilla, Conde, & Rodríguez-Aragón, 2014).

Papert (1980) argued that the unique attributes of Turtle Geometry's learning environment could be used to bridge to Polya's heuristics. To illustrate, he described the process that a Turtle Geometry learner experienced to build a house. To start the learner is given a set of existing commands, using the programming language LOGO. Normally, the learner used pre-existing commands to create new commands. To create a command to build a house, the learner could use preexisting commands for building squares and triangles. That is, to create the new command, the learner needs to divide the main goal into sub-problems where some of the sub-problems are existing commands, one of Polya's heuristics. Hence, "Turtle Geometry serves as a carrier for the general ideas of a heuristic strategy" (Papert, 1980, p. 64). Papert (1980) suggested that in learning Turtle Geometry, students were therefore introduced to Polya's principles for problem solving.

Barack and Zadok (2009) found years later that robotics-based tasks could similarly engage learners with problem solving strategies. They conducted a three-year study in which, each year, they met with 7th and 8th graders for two hours a week for 15 weeks during the participants' robotics class. During the first year, 80 students participated in the study. During the second year, 76 students took the class and continued participating during the third year of the study. Additionally, during the third year, another 116 students took the initial robotics class. The authors used qualitative methods to collect the data, preparing a detailed journal of each class meeting, documenting spontaneous conversations with students of unique class events, keeping records of students' computer files, photographing students' work, videotaping selected lessons, and conducting discussions with parents, teachers, and the school principal. They found that students initially relied heavily on trial and error to complete robotics-based tasks. But, as they gained more experience, students started to consider different approaches to robotics-based tasks, including the use of additional heuristics. For example, the authors identified three heuristics the students used: Planning by Using Modeling, Analogies, and Abstraction. To illustrate, a team reported that to prevent their car "from slipping on a tangential surface, they thought about skis, on the one hand, and tractor wheels, on the other" (p. 303).

Castledine and Chalmers (2012) conducted a qualitative study involving 23 six-graders. The collected data consisted of observations of students' discussions while solving robotics-based tasks, students' software programs, and students' questionnaires.

They found that robotics-based tasks helped students reflect on their problem-solving decisions – Polya's Looking Back problem-solving phase. The researchers remarked that if instead of students working on robotics-based tasks alone, a teacher gives them careful guidance, they believed LEGO robotics-based tasks could be a problem-solving tool.

Petre and Price (2004) also found that robotics-based tasks naturally encouraged self-reflection. They observed and interviewed 13 teams at the RoboFesta MK 2002 competition and 17 teams at the RoboCup Junior 2001 competition. They also followed one team of two members, aged 8 and 6, at the start of the study for two years. Many of the teams talked about their improved problem-solving skills and how these skills were helping them in mathematics and science classes. Although the children did not specifically mention what problem-solving strategies or skills they developed from their robotics-based tasks, the authors provide a table with advice from the children to new robotics competitors that reveals some of their acquired heuristics. For example, one of the students' advice is to "know what you are building" – to "plan it out first" in order to have a "better clear concept" (p. 153). This comment from the student contains two of Polya's problem-solving phases; "Understanding the Problem" and "Devising a Plan." There is also evidence of thinking from their quotes, "I never knew I could think that much. I thought my brain was the size of an acorn" (p. 152). Because the focus of the study was learning and motivation, the authors did not present an analysis of the students' problem-solving processes, but the quotes from the students contain evidence of problem-solving phases and heuristics used.

Robotics-Based Tasks and Mathematical Nonroutine Problem Solving

The potential of educational robotics-based tasks has also reached the mathematics classroom. Silk (2011) analyzed the heuristics related to proportional reasoning used by students working on teams to make their four wheel NXT 2.0 robots synchronize dance. He noticed that most students identified their heuristic as Guess and Check. By discussing the problem-solving strategies with each team, he realized that there was great variation under what students call Guess and Check – some of the students' Guess and Check heuristic had mathematical thinking. For instance, one group noticed that their second robot had wheels roughly twice as big as their first robot. Therefore, to calculate the second robot's number of rotations, they divided the number of rotations of their first robot by two – then they subtracted an adjustment to get closer to the desired answer.

Although Silk (2011) was not using Polya's heuristics to categorize the identified problem-solving strategies, it is obvious that students unconsciously, initially imagined a more accessible Related Problem, one of Polya's heuristics. As a result, they Varied the Original Problem, another of Polya's heuristics. Instead of solving the problem of a robot with wheels that were roughly twice as big as their first robot, they solved the problem of a robot with wheels exactly twice as big. By using this heuristic, they got very close to the desired solution. In the second phase of their problem-solving scenario, the students adjusted their result, by subtracting from it, to get even closer to the desired solution.

Though robotics-based tasks have the potential to help students cultivate heuristics that might help them during nonroutine problem solving, contemporary researchers have mostly focused on students' motivation, on how to teach mathematics concepts, on how to reinforce mathematics concepts, or on whether or not robotics-based tasks improve students' problem solving as measured by correct answers in a written test (Benitti, 2011; Fernandes, Fermé, & Oliveira, 2006, 2009; Hussain, Lindh, & Shukur, 2006; Iturrizaga, 2000; Norton, 2004, 2006). Less attention has been given to analyzing the heuristics that students are using to solve nonroutine problems in robotics-based tasks. However, because the nature of robotics-based tasks and, consequently, robotics-based instruction is to expose students to nonroutine problems, even if the focus of the studies was not to document the heuristics used by students when solving robotics-based tasks, the documented students' discussions exhibit the use of heuristics and students engaging in executing Polya's problem-solving phases.

For instance, the problem-solving phase Looking Back is constantly present in students' mathematical behavior in robotics-based tasks. In Fernandes, Fermé , and Oliveira's (2009) study, students recorded the distance travelled, in centimeters, by a four wheel Lego NXT 2.0 robot for one, three, and six seconds. Then, they calculated the quotient of the distance travelled and time. For one and three seconds, the quotient was 33. However, for six seconds, the quotient was 29.6666. Because six seconds had an unexpected result, one of the students said, "it can't be. It has to be 33." Another student followed by saying, " Let's program and measure all again. Something is wrong." Students' perceived inconsistency in their findings, which prompted them to engage in Polya's Looking Back problem-solving phase. The robotics-based task made them become more aware of their results.

Summary

It is clear many students are in need of improving their nonroutine problem-solving skills. In theory, effective use of heuristics could help novice problem solvers become more proficient. Previous researchers have found that proficient problem solvers use heuristics while in the act of solving problems. In fact, advanced problem solvers such as professional mathematicians agree to the effectiveness of Polya's identified heuristics during their own problem-solving episodes.

On the other hand, the mathematics education community has not reached consensus regarding which heuristics are most important to teach to students or how to teach them. These and other issues related to how to apply the theoretical knowledge of heuristics are still unresolved (Schoenfeld, 2007). This is not because these issues have proven to be unsolvable, instead, the lack of progress may be attributed to the limited attention given to heuristics in problem-solving research and instruction since the early 1990s (Schoenfeld, 2007), which is reflected in my review of the literature.

Schoenfeld (1985) argued that although the status quo of heuristics research might appear to present a pessimistic picture, it also serves as an entry point to change simplistic views of heuristics. I believe this is a good time to pick up again the research on the role of heuristics in problem solving, particularly because the research done from the 1990s to the present has both expanded and nuanced scholars' view of problem solving. It highlights the importance of other factors in the problem-solving process such as people's belief systems, people's affect towards problem solving, social influences, and situated problem solving (Schoenfeld, 2007). For me, this enriched understanding of problem solving provides an opportunity to revisit the use of heuristics in order to explore the possibility of fine-tuning understanding of the relationship between certain tasks – robotics-based tasks, in the case of my proposed research—and the heuristics they tend to stimulate problem solvers to use.

Hence, I propose to analyze which heuristics robotics-based tasks stimulate problem-solvers to implement. I believe that a greater understanding is needed about the complexity of each of Polya's heuristics before designing robotics-based tasks. Of particular interest are heuristics associated with Polya's Looking Back problem-solving phase because researchers have reported participants engaging in this phase during robotics-based tasks. Given the importance of self-reflection during nonroutine problem solving, it is worth investigating if in fact robotics-based tasks can promote the natural development of self-evaluation behavior – that is, engagement in Polya's Looking Back problem-solving phase – and how the heuristics were used.

Research Questions

The mathematics education community appears to be a good arena for initiating research in this area. Polya was more than a famous mathematician. He was a highly skilled mathematics educator, and his list of heuristics is the product of his many years assisting people to become mathematicians and mathematics educators. He not only believed that humans could become proficient problem solvers, he used his identified heuristics to produce many proficient problem solvers. In this study, I sought to understand the potential of a robotics-based task to stimulate problem solvers to employ Polya's heuristics by identifying:

1. What heuristics are employed by high school and undergraduate students during their attempt to solve a robotic arm task designed to require applications of trigonometry in its solution?
2. In what ways did the participants use of a specific heuristic differ? In what

ways were the use of specific heuristics similar?

3. What differences do participants note, retrospectively, in the approach they employed in attempting to solve the robotics-based task as compared to approaches they typically use to solve nonroutine problems in trigonometry?

CHAPTER 3

METHODOLOGY

In this study, I used structured, task-based interviews, which is a qualitative method used in mathematics education to observe and interpret mathematical behavior (Goldin, 2000, p. 517). This technique was pioneered by Piaget to study the form of knowledge structures and the reasoning processes of children. The interviews minimally involve a clinician (the researcher/interviewer), the participant, and the questions/problems/activities. Structured, task-based interviews are designed with the purpose of the study in mind. The interviews usually consist of four stages: (1) posing a question (design for free problem solving), (2) giving students minimal heuristic suggestions (when needed), (3) guided use of heuristics (when needed), and (4) exploratory, metacognitive questions (which was the final interview in this study). The interview guide for structured, task-based interviews uses a highly detailed script where the researcher anticipates the milestones participants will reach during their path to the solution. In the script, the researcher notes points where participants may need guidance and provides prompts to help the participants. The goal is to provide as much detail as possible to allow another researcher to conduct the "same" or a similar study elsewhere. It also creates a baseline that can be used as an assessment tool. For the purpose of this study, the structured, task-based interview did not include the second and third stages.

The guided use of heuristics stage was not included because it did not apply to this study.

For this study, the purpose of the structured, task-based interview was an exploratory investigation to identify the heuristics a robotics-based task stimulated problem solvers to use. Structured, task-based interviews were a good fit for this study because they provided a medium to explore the observable heuristics used by participants. They also allowed for interaction between the researcher and the participant. This was critical for my study because I needed to be able to prompt participants to continue thinking when they were about to give up. Likewise, I wanted participants to feel comfortable asking questions when they felt that they wanted to give up.

A strength and weakness of this methodology was the use of the think-aloud method. Kilpatrick (1967) gives a detailed analysis of what is gained and lost by using the think-aloud method to analyze problem solving processes. He listed the following as strengths of the think-aloud method: it does not require any special training, it permits lengthy observations, and the participant is only asked to report their thinking (they do not need to analyze their thought processes). Regarding its limitations, researchers have voiced the following concerns: the participant may not be able to verbalize their thinking, the thinking can be so fast that the participant cannot verbalize fast enough to keep up with their thinking, participants may remain silent when their thoughts are most active, and verbalizing information during problem solving can influence the thinking of the participants, to name a few (Kilpatrick, 1967; Lester, 1982).

Attempts have been made to address some of these concerns. Roth (1965) found no difference in the number of correct answers between participants in his study who solved problems thinking-aloud versus those who did not. Hafner (1957) found that the vocalization group in his study took slightly less time to complete problems, and they made fewer moves than the non-vocalization group. Gagne and Smith (1962) found that when asking participants to state aloud a reason for each of their decisions, their performance improved (e.g. they had shorter solution times and experienced more success at solving the problem). Although Gagne and Smith's (1962) findings do not imply that think-aloud enhances problem-solving performance, their findings support Ericsson and Simon's (1980) conjecture that thinking-aloud affects cognitive processes when participants are asked to verbalize information that they would otherwise not attend to. In short, the think-aloud method is not flawless and much still needs to be learned about the method. However, in the mathematics education community, it is the most valid means known for collecting data on mathematical problem solving (Hackenberg & Lee, 2015; Kantowski, 1974; Kilpatrick, 1964).

For the present study, structured, task-based interviews, along with the think-aloud method, was used to identify the heuristics a robotics-based task elicited from problem solvers. The researcher remained cognizant of the limitations of the think-aloud method while trying to answer the following research questions:

1. What heuristics are employed by high school and undergraduate students during their attempt to solve a robotic arm task designed to require applications of trigonometry in its solution?
2. In what ways did the participants use of a specific heuristic differ? In what ways were the use of specific heuristics similar?
3. What differences do participants note, retrospectively, in the approach they employed in attempting to solve the robotics-based task as compared to

approaches they typically use to solve nonroutine problems in trigonometry?

Pilot Studies

First Pilot Study

During February, 2015, I conducted an informal pilot study to gather feedback regarding the robotics-based task presentation. Three participants participated and gave feedback: One professor, one graduate student, and one high school student. The professor enjoyed solving mathematics problems, the graduate student reported feeling neutral towards mathematics, and the high school student disliked mathematics. They all had taken high school geometry.

Using the feedback from the participants in the pilot of the robotics-based task, I made the following changes to the robotics-based task:

1. Participants were told upfront that the robotics-based task was designed to take a long time to avoid making participants feel anxious or stressed about time.
2. Initially, the robotics-based task consisted of three parts. During the pilot, I reduced the robotics-based task to two parts. Participants of the study were not required to pick up the apple with the robot arm when they were using mathematics. Based on the feedback, using mathematics to position the gripper of the robot arm above the apple was challenging enough and it was rewarding to reach that point.
3. During the pilot, the initial and final interview took about 15 minutes each. The robotics-based tasks took from 1 hour to 3 hours. Hence, I told participants in the study that they could work on the robotics-based task for as long as they wanted but could also stop at any time and for any reason.

4. The informal pilot also informed my selection of participants for the study. In order to minimize my involvement with the participants during problem-solving episodes, I decided to limit the study to participants who enjoy mathematics and solving problems.

Second Pilot Study

From May to September, 2015, I conducted a second pilot study to test the feasibility of using Kilpatrick's (1967) coding system to code the problem-solving episodes, and to test the feasibility of the robotics-based task to stimulate problem solvers to use Polya's heuristics. I used my social network to find four participants who were known for loving mathematics; one engineering professor, one electrical engineering graduate student, one psychology undergraduate student with a minor in physics, and one high school student who was conducting research in mechanical engineering at a local university. All participants had taken a pre-calculus class at some point in their academic career. The study consisted of initial interviews (to gain an insight of the mathematical background of the participants), and structured, task-based interviews (which included a final interview to gather participants feedback and retrospective accounts of the problem-solving episode). I used a matrix to analyze data from the initial and final interviews. I used a modified version of Kilpatrick's (1967) coding system to code data from the problem-solving episodes.

To enhance the reliability and trustworthiness of the data, I recoded the data of one case, the problem-solving episode of the undergraduate student, multiple times until I achieved 100% internal reliability. After each iteration, I updated descriptions of the heuristics that needed more clarification. I calculated internal reliability of the coding using the formula given by Miles and Huberman (1994)

$$reliability = \frac{number\_of\_agreements}{total\_number\_of\_agreements + disagreements}$$ (p. 64). Then, I coded the other three cases.

To verify inter-coder reliability, a high school mathematics teacher who had nine years of teaching experience and a PhD in mathematics education coded portions of the data. To start, he coded 10 minutes of the undergraduate problem-solving episode to compare with my coding to check for inter-coder reliability. Individually, we coded five minutes of the recordings. Then, we went over our agreements and disagreements. I revised the descriptions of heuristic codes, and removed heuristics to have a more manageable list of heuristics. Initially, we had 33 percent inter-coder reliability based on Miles and Huberman's (1994) formula. We re-coded individually, checked our coding, updated descriptions of the heuristic codes, and removed codes until we agreed on the first 10 minutes of the undergraduate problem-solving episode. In total, I removed four heuristics and combined two heuristics into one. It took three meetings of about three hours each to reach agreement. Then, we coded the first 25 minutes of the high school student's problem-solving episode, individually. We achieved 91 percent inter-coder reliability and decided that we had reached strong inter-coder reliability. Lastly, I recoded the high school student's problem-solving episode with the updated modified version of Kilpatrick's (1967) heuristic coding system at two different times, leaving one week in between coding times. I achieved 100 percent internal reliability and I proceeded to recode the other three cases using the updated heuristic codes.

The following is a summary of the results of the second pilot study.

1. All participants in the study used Polya's heuristics when engaged in the robotics-based task.
2. The presence of the robotic arm seemed to have influenced the problem-solving behavior of the high school and undergraduate student. They both told me that they do not tend to give mathematical justifications for their mathematical decisions, but they said that the robotic arm influenced them to find a mathematical justification for their decisions.
3. Both the high school student and the undergraduate student used Polya's "Varying the Problem" heuristic, which occurs rarely with students at these levels.
4. In order for the robotics-based task to be effective at engaging the participants and to stimulate the use of heuristics, I found the investigator should keep interactions with the problem solvers at minimum during the problem solving episode.

Sampling and Participants

For the main study, I used purposeful sampling in this study. Purposeful sampling allows the researcher to select participants who are "information-rich cases for study in depth… those [cases] from which one can learn a great deal about issues of central importance to the purpose of the inquiry" (Patton, 2002, p. 230). For the purpose of this study, information-rich cases consisted of individuals who professed to enjoy learning about mathematics and had taken at least high school algebra2/trigonometry. To enjoy solving mathematical tasks is important because participants who do not enjoy learning about mathematics concepts might feel threatened when asked to solve a challenging mathematics problem in an interview setting and might not yield observable heuristic behavior (Kilpatrick, 1967). I verified the participants' attitudes towards solving mathematical problems by asking the participants to self-report whether they like or dislike mathematics.

Data Collection

I used my social network to recruit a total of 12 participants – six undergraduate students and six high school students. I assigned pseudonyms to each participant. I chose participants from different academic levels to participate because I wanted to gather baseline data as to whether the problem-solving processes implemented differ across age and mathematical sophistication.

I obtained written permission to conduct this study from the Georgia Institute of Technology's Institutional Review Board (IRB) which was acknowledged by Emory's IRB. I obtained written permission from the parents of the participating high school students, from the high school students, and from all the adult participants to ensure informed consent (consent forms available from the author).

In order to maximize the comfort level and the natural problem-solving processes of the participants, I conducted the research wherever the participant normally studies or feels comfortable thinking. They all told me that they could think anywhere as long as it was a quiet place without people around. I met with all the undergraduate students at their university campus in a quiet room (e.g. classroom and/or conference room). I met four high school students in a quiet room at a local university near them, and two high school students at a community center near their house, also in a quiet room. Lastly, participants each received a $50 gift card to compensate them for their time and as a concrete way for me to thank them each for their contribution to my study.

I employed three sources for data collection in this study: Initial interviews to document the participants' mathematics background and interest in the field of robotics, a robotics activity along with a scripted interview guide to generate data on problem-solving processes, and a final interview to gather feedback from the participant regarding the robotics-based task as well as the participant's reflection on their problem-solving processes. All the instruments (Appendices A, B, C, and D) were largely my own design with some of the interview questions borrowed, modified or inspired by items from the Intrinsic Motivation Inventory (IMI) (Deci & Ryan, n.d.). I also developed the scripted interview guide, which was inspired by Goldin's previous work (Goldin, 1985; Goldin & Landis, 1985). Descriptions of the instruments follow:

Initial Background Interview

The purpose of the initial interview was to capture participants' mathematics background. Although I recruited participants who expressed enjoyment in solving mathematical problems, it was still important to ask participants to verify this assumption. The initial interview included questions related to feelings towards mathematics and robotics. The emotions and beliefs learners have towards a topic have been documented as influencing students' success or failure with problem solving (Goldin, 1988, Goldin & Landis, 1985; Krutetskii, 1976; McClintock, 1980; Papert, 1957; Polya,1945/1957, 1962, 1965; Schoenfeld, 1985). Even Polya (1957) included emotions in his list of heuristics. Under the heuristic "Determination, hope, success," Polya (1957) suggested that "it would be a mistake to think that solving problems is a purely 'intellectual affair;' determination and emotions play an important role" (p. 93).

Hence, for this study, it was critical to start with a clear picture of the participants' relationship with mathematics; it was important to assure that the factor of emotions towards mathematics was an existing condition and that it was a positive one.

I used a semi-structured interview to allow participants to talk about a wide range of experiences. The questions were open-ended, and were carried out in an informal, conversational style (see Appendix A). I asked questions such as "What has it been like to be a mathematics student (in elementary school through high school)"? Because participants were not required to have experiences with robotics, I only asked one question involving robotics to ensure that they did not dislike robotics. I asked for more details from the participants when clarification of information was needed or more in-depth information was needed about a topic with potential to influence the results of the study or the participants' experience in the study. I audio recorded the interviews.

Robotics-Based Task

The robotics-based task was of my own creation (see Appendix B). For this study, it was important to find a mathematical task that was interesting and realistic for high school students. The robotics-based task was motivated by the problems given to college students in robotics classes, pick and place problems. I embedded the robotics problem in a real world application. I explored many different applications with high school students (e.g. picking up explosives, picking up items in a factory, etc.) until I found an application that high school students considered interesting and that they perceived mathematics was needed to solve the problem. Students responded favorably to using mathematics to write an algorithm for controlling a robotic arm to assist people who are without use of their own arms.

The task was to use concepts from high school trigonometry (e.g. law of cosines, trigonometric ratios, inverse trigonometric ratios, Pythagorean Theorem, and the distance formula) to control a robotic arm. The goal was to move the robotic arm such that the gripper was right above an apple. The robotic arm task was intended to stimulate participants to use problem-solving heuristics such as Decomposing and Recombining, Using Auxiliary Elements, Varying the Problem, Drawing a Figure, Working Backwards, and others. Along with the robotic arm activity, I created a detailed script of milestones that I anticipated the participants would reach while solving the robotic arm challenge (see Appendix C). The script also includes some suggestions regarding how to prompt participants if they needed guidance during the different milestones. The robotics-based task was audio-and-video recorded. Participants were given as much time as they wanted to work on the robotics-based task within reason but were also given the option to end the session for whatever reason if they chose to do so before completing the task.

Final Interview

I followed up the robotics-based task with an interview to capture heuristics from the participants' point of view and to gather feedback regarding the robotics-based task. However, I recognized that the self-reporting problem-solving behavior after the activity might be incomplete. Nisbett and Wilson (1977) discussed the conditions under which retrospection reports are likely to produce accurate and inaccurate reports. They suggested keeping the gap of time between the activity and the interview minimal because time can distort the participants' memories. Also, they warned researchers of the possibility that some individuals are not able to access their higher-order mental processes. For this reason, I conducted the final interview immediately after the robotics-based task and kept in mind Nisbett and Wilson's concerns during data analysis. Additionally, the recorded problem-solving episodes were my main source of data to identify heuristics used by the problem solver. Hence, the purpose of the data obtained from the final interview was to enrich my analysis of the problem-solving episodes.

Lastly, as in the initial interview, I collected data about the participant's emotions/feelings towards the robotic-based task to assure that the robotic-based task stimulated emotions in the participant that continue nurturing a positive engagement with mathematics and made the participant desire the solution. Papert (1980) recalled that his learning mathematical concepts as a child was facilitated by the physical representation of gears; he was cognizant that this relationship was more than a cognitive transaction. He remembered "feeling love, as well as understanding" in his relationship with gears (p. vii). Mathematicians such as Descartes and Polya concurred with Papert. They, too, believed that emotions are a key factor in the learning process (Polya, 1957, p. 93). Polya (1957) even urged educators to give students problems that make students desire the solution (p. 6). He thought it was "sad to work for an end that you do not desire" (Polya, 1957, p. 6). Other researchers have also noted the presence of an affective component in learning experiences, and have suggested that affective knowledge can be taught (DeBellis, 1996; Goldin, 1988; McLeod & Adams, 1989). DeBellis (1996) gives an example where the observer in a research setting laughs with the participant when the participant faces difficulties during problem solving. DeBellis explained that the external positive affect – the observer laughs – helped the participant continue tackling the problem while experiencing negative emotions. If external sources, such as laughs of an observer, nurture affective areas while problem solving, DeBellis conjectured that other external sources, such as teachers, may do the same (p. 255). Hence, it was crucial, for this study, to verify that the robotics-based task did not create any animosity towards mathematics or robotics. And if it did, it was important to identify such moments.

The questions were open-ended and were asked in an informal, conversational style (see Appendix D). Questions included: “Were you frustrated at any point during the activity? What did you do to help you cope with the frustration? Do you always do this when you are frustrated”? I asked questions about moments when I observed a behavior that implied a change of affect or demonstrated affect towards the activity to document more in-depth information regarding the internal processing of the participant. The interviews were audio recorded.

Materials

The robotic arm used in this study was designed by a team of researchers from The Georgia Institute of Technology (see Appendix E). Robotic arms have a motion type that is either spatial (3D) or planar (2D). For high school mathematics a robotic arm with planar motion type is best. The planar nature of actuation simplifies the math to a level appropriate for high school mathematics. The two-link geometry enables the exploration of mathematics related to the circle, the triangle, and planar vectors. The design also provides sufficient freedom to explore simple spatial (3D) mathematics in the future.

Prices of existing low-cost robotic arms with planar motion type start at $300 (LEGO) with additional motors and parts needed. Additionally, they need to be programmed. The robotic arm designed by Georgia Tech’s researchers cost around $100 to build. To reduce costs, the robotic arm fabrication involved laser cutting sheets of 1/8 and 1/4” plastic (acrylic). Further price reductions were achieved by using five low-cost, low-torque servo motors. Additionally, the researchers designed a Graphical User Interface (GUI) to allow users to control the robotic arm by entering numerical values obtained using mathematics procedures typically taught in high school and early college mathematics courses. Geometry and trigonometry were all that was needed for my study, no computer programming was needed. Although still in the prototype phase the robotic arm was, nevertheless, fully functioning for the purpose of this study.

A smart pen was used to record the writing and voices of the participants (see Appendix F). The smart pen recorded simultaneously what participants wrote and said. In a way, it created a movie of their writing with the audio synchronized. I created a set of flashcards (see Appendix G) with commands using the programming language Matlab for potential mathematics concepts that participants might need. The purpose of the Matlab commands was to free participants from tedious calculations. My hope was that this would allow participants to spend more time thinking and less time performing calculations. I only offered the flashcard to one of the high school students because he was the only one that needed assistance to do the calculations for the law of cosines.

Procedure

The interviews were conducted from June through July, 2016. The location of the interviews was different for participants because participants were given the freedom to pick a place and time of the day that was optimal for their thinking. All interviews were one-on-one and were conducted by the researcher.

Participants were told that they could work on the robotics-based task for as long as they wanted, but the overall study (two interviews and task session) would not exceed six hours. As interviewer, I explained that the robotics-based task could take a considerable length of time because I would be trying to capture their creative thinking processes as much as possible. I emphasized to the participants that it was critical for the study that they said aloud everything that they were thinking as they solved the robotics-based task, and to not feel pressured by time because the task was designed to take time. If anything, I explained to them that the more time they took to think and to verbalize what they were thinking, the better. Also, participants were advised to take any breaks that they needed to go to the bathroom, drink water, eat a snack or to have a meal. Participants were also advised to stop working on the robotics activity if they reached a point of fatigue. Lastly, I clarified to the participants that the goal of the study was not to evaluate their mathematics knowledge. The goal was to document their thinking processes as they solved the activity. After explaining the study to the participants and explaining to them that they could stop participating at any point of the study, I picked a pseudonym with the participants to start the study.

The study started with an initial interview (see Appendix A). I reminded the participants that the interview would be audio recorded and that the goal of the interview was to capture their history with mathematics and robotics. I sat next to the participants and conducted the interview in a conversational style.

Next, the robotics-based task began. I gave participants a copy of the robotics-based task (see Appendix B). I explained verbally the robotics-based task to the participants, and reminded the participants that this part of the study would be audio- and video-recorded. I reminded participants that they could ask for hints if they ran out of ideas and were about to give up, but this should be their last resource. I told them that they should only ask for a hint if they were about to give up.

The robotics-based task consisted of two parts. I told participants that the goal of the first part was to give them some practice using the think-aloud method and to get them comfortable with the robot arm. For the first part, I showed participants how to control the robot arm manually via a computer screen and asked them to verbalize what they were thinking as they controlled the robot arm to move the gripper to be right above the apple. For the second part, I asked participants to control the robotic arm using any mathematics that they knew. I reminded them to do all their thinking aloud. That is, to say what they were looking at, what they were writing, what they were thinking as they worked, what they were feeling, any ideas that came to their mind (even if the ideas seemed unimportant). In short, they were asked to say aloud everything that they were thinking.

In addition to placing a video camera to capture the interaction between the robot and the participants, I asked participants to use a smart pen to write. I sat next to the participants, but slightly behind. This allowed me to observe and write some notes during the robotics-based task while giving participants some privacy. If participants remained silent for about ten seconds, I prompted them to share what they were thinking.

I strived to have minimal interaction with participants during the second part of the robotics-based task, but I prompted some participants when they sought my help. My prompts were initially intended to motivate them to continue thinking but not to influence their thinking. I prompted them with questions such as, what would you do if you were alone at home? Lastly, the participants were audio recorded during the final interview after completing the robotics activity (see Appendix D).

Qualitative Analysis

Data obtained from the problem-solving episode was coded and analyzed using a modified coding system for the analysis of problem solving developed by Kilpatrick (1967) (see Appendix H). Polya's (1945) work on problem solving served as the theoretical foundation for Kilpatrick's coding scheme. Initially, Kilpatrick included all the heuristics identified by Polya, but he decided to reduce the list as some heuristics were never used or hardly used by his participants. The reduction of heuristics resulted in a more manageable coding system. I added back to Kilpatrick's coding system some heuristics that I anticipated my participants would use. Miles and Huberman (1994) argue that matrices allow the researcher to reduce, focus, and organize data to facilitate analysis (p. 93). Therefore, because Kilpatrick's coding sheets are only checklists that keep track of the specific heuristics used by a participant, I created and maintained a large matrix containing the details of each participant's use of specific heuristics to help in my analysis for Research Question 2. Appendix I shows a visual of the entire matrix and enlargements of two sample cells to give a sense of the type of data this matrix captured. This matrix enabled me to identify similarities and differences in how specific heuristics were used across participants.

The steps in analyzing the problem-solving episode were: (1) I listened to the recordings of the entire problem-solving episode from the video camera and the smartpen simultaneously and wrote notes (see Appendix J); (2) I coded the data using Kilpatrick's modified coding system (see Appendix H). To code the problem-solving episodes, I analyzed the recordings from the smartpen and the video camera in 5 minute intervals, and replayed them multiple times until the data were coded; (3) I included a description of how each participant employed each of the heuristics on the matrix described above and shown in Appendix I; (4) I summarized the findings and wrote a narrative.

I displayed data collected from the initial and final interviews in a matrix (see Appendix K). I used a matrix to gain an understanding of the data in three general domains: (1) the math/robotics background of the participants to gain insight to the context from which their problem-solving behavior derived (e.g. verifying participants did not dislike math or robotics, learned about participants and their history with math, etc.); (2) the feedback from the participants about the robotics-based task (e.g. what they liked/disliked about the robotics-based task to ensure no affective domains were harmed by the robotics-based task); and (3) the participants' reflections regarding the heuristics they used and what triggered/influenced their implementation. The data were extracted from the audio recordings as follows: (1) I listened to five minute intervals of recordings and wrote down direct quotes from participants in the corresponding cells of the matrix; (2) I listened to the audio recordings again to check the data extracted from the audio recordings; and (3) I condensed the data from the direct quotes in another table. In short, the matrix served as an initial display of the data to help me verify assumptions I made, and to give me a brief summary of how the interview data could enrich the data from the problem-solving episodes (gave context to the problem-solving episodes).

Reliability and Validity

Yin (2003) describes reliability as a way to "minimize the errors and biases in a study." That is, if another researcher had done the study and had followed the same procedures, he/she will have arrived at the same findings if he/she followed the same procedures with the same case study. Hence, the researcher must document all the work with as much detail as possible. This will ensure that someone else can follow or could replicate the study if he or she had a similar case.

In this study, I am concerned with both internal and external validity. Internal validity concerns making inferences from the researcher's observations (Yin, 2003). What was observed and what was inferred should always be made clear to the audience. For external validity, the concern is how generalizable the results of the study are. In qualitative studies, Yin (2003) advises that researchers focus on generalizing results to "theory," not from one case to another (p. 38).

In this empirical study, I attempted to reduce the biases in the study and protected its validity in the following ways: (a) by providing a researcher's reflection section to accompany my findings; (b) by establishing a clear chain of evidence of how the data were collected; (c) and by coding the data a second time to check for internal reliability using the coding scheme that was a check for inter-rater reliability during the second pilot of this study (Miles & Huberman, 1994).

Role of the researcher. Because in most qualitative studies researchers have an active role in the collection and interpretation of data, it is crucial for me, as researcher, to reflect and identify my personal perspectives and potential biases prior to the study and during the study. To begin with, I was a high school mathematics educator for nine years in a school that embraces pedagogical innovation and principles from social justice. I had the opportunity to meet with my students one-on-one when they were struggling because the school had scheduled office hours for teachers. Students could either work on homework or meet with teachers during this time. Hence, I had plenty of opportunities to explore the thinking processes of my students, and develop an intuition to find misconceptions.

Additionally, I am fond of the field of robotics. I fell in love with the field of robotics during one of my summer research experiences as an undergraduate student. I was excited to find out that mathematics was more powerful than I thought. I was excited to use mathematics to control a physical robot via a computer. Even more amazing, I could control a second robot, which was located on the other side of the country, by using mathematics as well. The field of robotics seemed to me like a dream land where mathematics and technology merged and became one entity. Years later, I observed the same excitement for learning robotics among some of my high school mathematics students. Needless to mention, I was delighted to explore robotics activities with my students, and as a result, became interested in exploring the potential of robotics-based tasks.

These experiences continue to influence my views and what I value in mathematics education. I believe that my perspectives will not hinder this study. However, they will influence my work. First, this study involves two fields that are interesting to me. Therefore, my excitement for these fields may come through during the study. My background in the field of social justice may influence my behavior towards my participants. Because I believe that most people, when nurtured properly, can learn to think mathematically, I tend to treat everyone as mathematically capable individuals. Hence, I perceive the role of a mathematics educator as the person that needs to figure out at what rate and in what ways different learners need to be supported.

I kept in mind my potential biases and sought the opinion of my peers as much as possible. I took the following steps to control the effects of such potential biases.

Chain of evidence. I documented the development of the data sources, the collection process, and the data analysis in detail. I kept records of when I communicated with participants and how I recruited them for the study. I documented how I gained access to the participants. I described why each participant was selected for the study, how the instruments were developed, how the data were collected, how the interviews were recorded, how the interviews were analyzed, and why some heuristics were added or removed.

Internal and inter-coder reliability. To enhance the reliability and trustworthiness of the data, I coded the data using the coding scheme that was a check for inter-rater reliability during the second pilot of this study. Additionally, I coded two problem-solving episodes a second time to check for internal reliability, using the data from participant 5 and participant 10. I left two weeks between coding times. This assured that after coding for a long time, I did not lose accuracy while coding due to coder fatigue. I calculated the internal reliability of my coding using the formula given by Miles and Huberman (1994)

Data Storage

I digitized and stored the data in one location in a secured, password protected network. I also stored detailed descriptions of how and when the data were collected.

CHAPTER 4

RESULTS

The 12 participants selected for this study are known by their families and peers as people who enjoy solving problems and tend to excel in mathematics. Their academic levels vary from a rising high school student to a rising fifth year undergraduate student. All participants were given pseudonyms and were given the same robotics-based-task to be solved. Because Polya (1957) emphasizes that students' emotions and students' interest in the problem are critical in the problem-solving process, a synopsis of the participants' mathematics background along with their thoughts about the robotics-based task is presented first. Additionally, a brief overview of their problem-solving episodes is included. Then, I present the findings of the study for each research question.

The 12 Problem Solvers

John – Participant 1

John was a 14 year old who described learning mathematics as fun, interesting and "just really cool." For him, mathematics is the "art of thinking through and solving problems logically and precisely." He told me that mathematics is a mystery with limitless possibilities for exploration and it is very useful when exploring other topics he is interested in, such as physics and computer science. John explores mathematics by creating his own problems, which he documents on a notebook that he carries with him. At the time we met, he was trying to use differential equations to describe the path that would be traveled by an ant on a curved surface.

While John had a positive relationship with mathematics, his learning experiences at school were not always positive. He told me he really liked mathematics at his first elementary school. However, towards the end of elementary school, he switched schools and he did not like the assignments he was getting at his new school. He did not feel he could express himself or create work that he would be proud of. By the time he reached 7^{th} grade, he had become so sad that his mom decided to take him out of school.

For the past 1.5 years, John had been homeschooled and had become a happy student again. During this time, he learned from Algebra II to Linear Algebra. He competed at the state science and mathematics fairs. He took the SAT physics and Mathematics II exams, earning a score of 800 on both. He took the AP Calculus, AP Physics, and AP Computer Science exams, earning a score of 5 on his AP Computer Science exam. He anticipated earning a score of 5 on the other exams as well.

When I met him, John's mathematical sophistication and appreciation for the elegance of mathematics was evident. He used proper mathematical notation and constantly searched for relations that will yield simpler and more elegant mathematical representations. Periodically, he reflected on the correctness of his thinking and work, and kept on reminding himself of any assumptions he had made. He started the problem by documenting what he knew. Then, he considered what he needed to figure out. Before approaching the problem, he thought about the different ways in which he could solve the problem. He contemplated two possibilities and he picked the one that he anticipated to be simpler.

He had some awareness regarding the problem-solving strategies that he used during his problem-solving episode. He told me that he visualized things in his head and he used diagrams. He told me his diagrams evolved and became more detailed as he gained more insight regarding what he needed to figure out to solve the problem (see

Figure 1). Once he understood the problem, he proceeded to solve the problem successfully.

During the final interview, he told me that he engaged with the problem because he found the math interesting. He did not care much for the robotic arm. He added that although he found the problem interesting, he wasn't excited because he had thought about something similar in the past. Although he had never solved a similar problem before, for him, the excitement comes from the uniqueness of the problem.

Adam – Participant 2

Adam was a computer science major who had just completed his first year of college. For him, mathematics is a "great tool." It is the purest form of problem solving. It is a needed logic to abstract things. As an example, he told me that without math people would have to eye-ball how to build a building, but with math, people can calculate the different forces that would be affecting the building and determine how high it can be.

As a child, he was attracted to mathematics for its logic, but things changed in high school. Due to his outstanding performance in mathematics, he was fast-tracked through mathematics starting in 6^{th} grade. By the time he was in 10^{th} grade, he was taking pre-calculus and statistics. He told me that this was detrimental for his education because after Geometry, mathematics was becoming exponentially more challenging. It became daunting. At this junction, he met his favorite teacher of all time – his computer science teacher.

Adam became mesmerized by the logic and the discovery he was exposed to in computer science. Computer science was "cool" because he could break down every problem into “bite size” sub-problems. He felt he had to internalize the problem to come up with the sub-problems that he created. On the other hand, in mathematics, he “never felt that the questions that [he] was given facilitated needing to break it down to internalize it the way [he] did in computer science.” He felt that in mathematics he was given a hammer in class. He was told how the hammer works. Then, he had to nail down problems with the hammer for homework.

I met Adam in a room full of robots in the computer science building at his university. He felt comfortable in the room because it was the room he was using to introduce middle school students to robotics. When he saw the robotics challenge, he immediately said that it reminded him of inverse kinematics, and he thought about writing a computer program. As he thought more about the problem, he said the problem was about two vectors with “different headings.”

His problem-solving episode consisted of three phases. During phase I, he approximated the angles of rotation of the shoulder and the elbow. He started by thinking about what measures will make sense for the angles and imagining the mathematical model. He thought about the shoulder not rotating and then dismissed that option. He thought about using trigonometric ratios to approximate the angle of rotation of the elbow, but he could not remember them at that moment (later, he did). Yet, he still wrote down the ratio $\frac{5}{4}$ and said that this ratio would yield an angle close to $45°$ but a bit bigger. Later, he added that there were two ways for the robotic arm to reach the apple and together they formed a “trapezoidal looking thing” (see Figure 2a). Throughout his problem solving episode, he continued to think deeply about his work before performing calculations. Next, he entered his approximated angles in the computer to test them, and changed the angles as he observed how the robotic arm moved.

During phase II, Adam thought about how to solve the robotics-based task using mathematics. He used vector addition to write two equations (see Figure 2b). He became overwhelmed by the calculations that he anticipated were going to be required to solve the system of equations. He told me that he did not consider it his problem to solve tedious calculations – that is the problem of the computer. Normally, he uses www.wolframalpha.com as a symbolic calculator. He told me Wolfram-Alpha is very useful. For him, technology frees him from calculations, which allows him to focus on the problem conceptually. As a child, he was proud to be a calculator, but now, he told me he did not have time for that.

During Phase III, Adam decided to use Wolfram-Alpha to solve the system of equations although he wanted to do it without it because a part of him felt that using the internet was cheating. Initially, the solutions from the system of equations did not work. He tried to reach the apple from the two different ways to verify that there was something wrong with the answers he had. He used a calculator to verify that the angles satisfied the equations. He was certain that his equations were correct. Then, he thought that the problem must be his understanding of the problem. As he focused on thinking about the problem, he thought about the vectors being relative. He said in graphics that is something he has to think about. He checked his diagram and noted that he was not thinking correctly about the angle of rotation of the elbow (see Figure 2c). He noticed that the orientation of the axis for the angle of rotation of the elbow changed after the shoulder rotation. He thought that was a “cool discovery.” He recalculated the angle of rotation of the elbow and was excited to have finally solved the problem correctly.

Sarah – Participant 3

Sarah was a rising senior at a high school for the arts. She had earned As in all her mathematics classes. When we met, she had just completed A.P. Calculus. She told me that although she liked mathematics, acting and singing were her passion. Then, she told me that if she were to find a field in mathematics "that was right" for her, mathematics could become her main interest. Based on the mathematics that she had learned thus far, for her, mathematics "is the study of how to use and manipulate numbers…How to take numbers and apply them to the real world in useful situations." Nevertheless, she told me her 14 year old brother, John, had exposed her to challenging mathematical problems and had made her aware that there are areas of mathematics that are not completely based on numbers.

During the problem-solving episode, Sarah's comfort with challenging problems was evident. She did not give up easily. Instead, she searched for new ways to think about the problem. For example, early on the problem-solving episode, she told me she had a feeling she needed the cosine rule that she learned in pre-calculus. Yet, she could not remember it because she said pre-calculus was "a bunch of rules" that she had to memorize for the test. She did well on the test, but she told me she did not get a "very solid understanding." She only remembered that the "cosine rule" was useful when working with non-right triangles. Instead of giving up, she told me she was going to assume she had a right triangle because she had plenty of experience with right triangles. With this assumption, she figured out that the angle of rotation of the elbow was not 90 degrees but it was very close to 90 degrees (see Figure 3a and 3b).

Although Sarah got very close to the apple, she was not satisfied with her work because she told me it was not good enough to help someone that cannot use their arms. She told me she needed to figure out a way that will yield the angles of rotation given any location of the apple. I asked her what she would do if she was at home. She replied that she would search online for the Law of Cosines. Then, I gave her access to the internet and she performed two Google searches – one for the Law of Cosines and the other for the Law of Sines.

As Sarah approached the problem, she had two lines of thinking occurring concurrently. In addition to focusing on performing the calculations correctly, she introduced secondary ways to verify her work. She tried to use the Law of Sines to verify her results from the Law of Cosines. She solved the equation of the Law of Cosines for the angle symbolically, and entered the values in the calculator to re-calculate the angle she had already found (see Figure 3d). Then, she added the angles of the triangle to make sure they added up to 180 degrees. Via this process, she was able to find fallacies in her thinking and careless mistakes.

After one hour and three minutes of working, Sarah found a solution that she was happy with and she told me that she was done, but she changed her mind after she found a possible mistake. She looked at the computer screen and she noticed that the computer had 90 degrees for the angle of rotation of the elbow. She told me she had entered 99 degrees. When she re-entered 99 degrees, the computer changed it to 90 degrees again. At this point, she suddenly remembered that the motors of the robotic arm could only rotate positive or negative 90 degrees. She told me that she needed to work on the problem again. This time she noticed visually that the elbow needed to rotate less than 90 degrees. She told me that the angle she calculated did not exist for the robot, and she showed me with her hands the actual angle of rotation of the elbow (see Figure 4). She entered her new solution in the computer. She was thrilled to have found the solution! She said multiple times that the robotic arm challenge was "fun," more fun than she expected it to be.

Juana – Participant 4

Juana was a rising high school senior who liked but was "not crazy" about mathematics. Because Juana is bilingual, she told me mathematics is easier to understand than languages and arts. Additionally, she was intrigued by how people can interpret mathematical problems differently, and yet still arrived at the same conclusion. In mathematics class, she looked forward to finding out how her peers interpreted problems and the methods they used to approach the problem. This was in accordance with her view of mathematics. For her, mathematics is about thinking processes and secondarily, it is also the study of numbers. She told me mathematics is not a sequential process because the same method cannot be used all the time. In mathematics, there are always different ways to solve and to avoid problems.

While Juana had earned As in all her high school mathematics classes, she had not been challenged until last year, in her pre-calculus class. Juana's family experienced tremendous adversity when she was a 7th grader. She stopped doing her homework, and talking to peers. Due to this, she was demoted to the classes for the students struggling academically. As things improved at home, she become more interested in learning and started doing her work. Eventually, she was tutoring all her peers in mathematics. Her 10th grade mathematics teacher noticed her interest in mathematics and recommended her for pre-calculus for 11th grade, an advanced mathematics class.

When I met with Juana, she displayed an emphasis on verifying her work and the information she recalled. She calculated the hypotenuse of a triangle twice which helped her find a calculation mistake. She calculated the same angle by employing sine inverse and cosine inverse to assure she would get the same answer (see Figure 5a). When she struggled to recall whether trigonometric ratios were squared or not, she calculated the two missing angles of a right triangle using both options (see Figure 5b). Then, she looked at the triangle and tried to decide which angle measurements made sense for that triangle. She decided that the not squaring the argument of the inverse tangent made more sense.

Although Juana made progress on the mathematical problem she formulated, she did not succeed at understanding the robotic-based task. At the end of her problem-solving episode, I asked her how she saw the right triangle that she tackled (see Figure 5a). She told me she saw two sides of the triangle when she drew the apple. This made her think that she needed to find the third side. From this point on, she focused her attention on the right triangle. Yet, she did not make a clear connection between the right triangle and the angles for which she was searching. Verbally, Juana mentioned that she needed to figure out the angle of rotation of the elbow and the shoulder of the robotic arm. Despite this, she did not include labels of these angles on her diagrams. On the contrary, she entered the measurements of the missing angles of the right triangle to the robotic arm without logic. First, she entered 23 degrees and 67 degrees for the rotation of the shoulder and the elbow, respectively. When this did not work, she switched them. Next, she used the angles she obtained by squaring the argument of tangent inverse, even though she had decided earlier that they were incorrect.

Not getting feedback during her problem-solving episode frustrated her and disturbed her problem-solving behavior. To solve problems, she told me she follows three activities. Usually, she starts by doing her best, alone. She tries to figure out if she can do the problem "little by little" or if she has to do it all at once. If she does not excel, then she quits and seeks for her peers help. If she does not understand her peers thinking, then she asks her teacher for help. She does not like to put herself down for a mathematics problem. The robotics-based task frustrated her because I did not give her feedback. She was happy that the gripper was near the apple at the end, but she still did not know what stopped her from getting to the apple. At this point, she would have reached out to her peers or teacher to see other ways to think about the problem. She told me that although she struggled to figure out whether she was right, she found it amazing to be able to see how the mathematics that she did affected the robotic arm.

Mark – Participant 5

Mark was a rising fifth year undergraduate student in mechanical engineering. He had always enjoyed solving mathematics problems. He told me that, as a child, he did his mathematics homework first. He added that he liked mathematics because he liked things with definite answers. For him, mathematics is a "man-made system for quantifying observed phenomena."

While Mark did not like to feel inferior to others, he tolerated the feeling of inferiority to be around people who had demonstrated mastery in a domain. He told me he tended to hang out with high achievers, from different domains, because that motivated him to strive to become better. He told me that he found it demoralizing to be view as ordinary. Throughout his problem-solving episode, he kept on making remarks alluding to how vulnerable he felt attempting to solve the problem because he knew I knew the solution. He told me it was challenging, for him, to think aloud because he kept on thinking about "minimalistic things" he could say that will give the impression that he knew what he was doing, while simultaneously making progress towards a solution. When I asked him what I could have done to make him feel more comfortable, he laughed and told me nothing. He told me he could only feel comfortable if he knew I was more clueless than him, if he knew I did not understand mathematics and I was there only to record.

Despite the discomfort, Mark told me he was striving to find the "smartest, most clever way" to solve the problem, a behavior motivated by his engineering professors. Before engaging in computations, Mark invested close to 80% of his problem-solving episode to understanding the problem and contemplating different ways to approach it. He started by drawing a diagram indicating the known and the unknown information (see Figure 6a). Hoping to simplify the robotics-based task, he drew another diagram to check if the angle of rotation of the elbow was 90 degrees. After excluding this option, he was certain that he needed to figure out the internal angles of an obtuse triangle. However, even though he recalled there were trigonometric properties to work with non-right triangles, he could not remember them. He thought about dividing the obtuse triangle into two right triangles, then he could use a system of equations to find the internal angles (see Figure 6b). Although Mark thought this option would work, he did not want to pursue it because he anticipated it would be computationally intense. For this reason, he decided to continue searching for additional entry points into the problem. In total, he drew 14 diagrams. After 45 minutes, he decided to Google "trig properties for nonright triangles."

Overall, Mark was cognizant of his problem-solving decisions and the problem-solving strategies he employed. He told me that if he was at home, he would have used Google when he started struggling. For the robotics-based task, he decided to postpone consulting Google for two reasons. He was "doing this problem for himself" and he knew the problem would end quickly once he had access to Google. His prediction was correct. He solved the problem 12 minutes after consulting Google (see Figure 6c). When I asked him if he had employed any problem-solving strategies to help him solve the problem, he told me he mainly used three strategies – writing down everything he knew, drawing without confusing details, and keeping in mind information he knew and information he needed to figure out, which is in accord with what I observed.

Ebony – Participant 6

Ebony was a rising fifth year university student. When we met, she was one semester away from completing her Bachelors in applied mathematics. She was excited to participate in the study because she cared about the teaching of mathematics. She told me she believed in community. For her, it was not enough to excel in mathematics; she wanted her community to excel with her. For now, she tutored elementary school students in mathematics and planned to pursue a career in education. She knew could help others understand mathematics because she had overcome mathematical difficulties herself. Before taking upper division mathematics, she had no exposure to proofs which, initially, hindered her from being successful in those classes. She earned a C in her upper level abstract algebra class. She told me it was "traumatic to not understand the one topic she always exceled in." However, she did not give up because she learned from observing her mother that no matter what happens in life, she can get through it. She had mostly earned A's in her other mathematics classes after abstract algebra.

Because of her experiences, Ebony told me mathematics is "critical thinking and being able to be creative." She demonstrated her creativity right from the beginning. She started by thinking about the simplest problem she could imagine with the robotic arm: only rotating the shoulder. When she noticed that the robotic arm was 2 inches longer than she needed, she drew an obtuse triangle to depict how the robotic arm should look when reaching the apple (see Figure 7a). She immediately recalled that she could use the Law of Sines and the Law of Cosines to find the internal angles for any triangle. However, she could not recall the formulas. Eventually, she recalled the Law of Sines but she did not have enough information to use it. Then, she tried to divide the obtuse triangle into right triangles (see Figure 7b).

For her last approach, Ebony thought about using vectors; this approach brought her much joy. When she recalled that "there is a cosine in vectors," she made the connection that the Law of Cosines, the Law of Sines, and vectors involved angles. She told me that vectors have some degrees of separation. She attempted to employ the geometric definition of the dot product to achieve the same results as using the Law of Cosines (see Figure 7c). She used trial and error using different vectors that might yield the length of the arm and forearm of the robotic arm. However, she did not see a way to derive the Law of Cosines using the dot product of two vectors.

At the end, Ebony was happy because she said "it was kind of cool to see how" she jumped from different mathematics topics to tackle her problem. She told me she saw the rationale behind the Law of Sines and Cosines. She told me that this was important for her because when she fails to remember something, it means that she had only memorized a fact. She did not understand it intrinsically. She told me she felt defeated for not solving the problem, but she felt good that she made it so far because she was certain that the dot product of the two vectors leads to the Law of Cosines. She was so excited to figure out the problem on her own that she refused to use Google. She told me she usually uses Google at home but she knew that if she continued thinking about it, she would figure it out.

Mena – Participant 7

Mena was a rising high school sophomore. He attributed his liking mathematics to the learning environment in his fifth grade class. He told me that he and another student were separated from their mathematics class to learn algebra. The teacher gave them worksheets and tests, but was not able to work with them because he had to teach the normal class. The situation gave Mena and his peer the opportunity to explore mathematics beyond the normal curriculum. He said that sometimes they did not understand the content in the book, which motivated them to think deeper about problems. By doing this, they learned that there were many ways to do mathematics. This experience influenced him to like mathematics, which he told me was a "big part of learning math" because he tended to pay more attention to things he liked.

For him, mathematics is "anything to do with advanced thinking." He liked to attempt to do proofs, like the Pythagorean Theorem, and imagine what mathematicians were thinking when proving the same proofs. However, he told me that, with the exception of his pre-calculus teacher, his math teachers did not explain the rationale behind math concepts. Most of his teachers tended to give students a formula, asked them to plug values into the formula, and get the answer.

Mena enjoyed feeling the excitement that comes with making progress when solving a problem, but he rarely experienced that feeling in school. When he was solving the robotics-based task, he got stuck and he told me that he did not know what to do because he does not experience getting stuck with school work. His teachers tended to review previously learned concepts before using them. During his problem-solving episode, he was very cheerful every time he found a way to continue progressing towards a solution. At the end, he told me that he felt “pure joy” every time he made a small breakthrough.

When we met, Mena noticed two ways to approach the robotics-based task. He first thought about simplifying the problem by only rotating the shoulder of the robotic arm. However, by finding the distance between the shoulder joint and the apple, he realized that the robotic arm was too long. Hence, he decided it was necessary to rotate both the shoulder and the elbow. Next, he drew the arm and the forearm of the robotic arm such that the gripper would be above the apple (see Figure 8a). Then, he added right triangles to his diagram which had hypotenuses with lengths as close as possible to the lengths of the arm and the forearm of the robotic arm, 4 and 4.5 (see Figure 8b). He was disappointed that he could not remember the “angle formulas.” Eventually, he told me he was going to use Google. First, he searched for “Law of Sines Formula,” then for “Law of Cosines Formula.” It bothered him to resort to Google because he wanted to do it on his own.

Also, Mena told me it bothered him that he did not figure out the problem. It hurt him to walk away because he wanted “to figure it out badly.” He also knew that his method was right. He was convinced that the problem was in the right triangles he constructed. He just could not see where the mistake was.

Imani – Participant 8

Imani was a rising third year college student majoring in biomedical engineering. She told me she had always excelled in mathematics and mathematics related classes. However, she did not always enjoy solving mathematics problems. When she was younger, 8 or 9 years old, she was not interested in mathematics because it was all memorization. She told me she started liking mathematics when it started being based on reasoning. From pre-algebra and on, mathematics became fun for her. She described mathematics as “principles of numbers and certain properties that kind of governed everything.” For her, mathematics is the foundation for everything, even music.

Imani seemed very calm during her problem-solving episode. At the end I asked her if she felt any emotions during her problem-solving episode, she told me that she felt sadness, happiness, and frustration. However, the emotions did not control her actions. She told me she was methodological about her approach to solve the problem. She was not sure if she said it aloud, but she told me she constantly asked herself in what other ways she could manipulate what she had, and how she could approach the problem differently. Also, she told me she took breaths when she felt stuck. This is in accordance with what I observed.

What I observed was also in accordance with her description of the problem-solving process she follows to solve non-routine problems. Prior to the robotics-based task, she told me she starts the problem-solving process by writing down any equations that might be useful to the problem. Then, she writes down what she knows. On a separate list, she writes down what she does not know. Lastly, she endeavors to find which equations connect the two lists. She did not mention making drawings, which was interesting because she only drew diagrams when I asked her to show me on paper what she was seeing or thinking.

When solving the robotics-based tasks, Imani followed her described problem-solving process. She started by writing down everything she knew. Next, she wrote down a system of two equations using vector addition (see Figure 9a). Then, she embarked on solving the system of equations. She described this part as the "annoying part." She could not remember all the trigonometric identities. To help her recall, she consulted Google. She performed 6 searches, all related to trigonometric identities with some repeated searches to retrieve websites she needed to reference again. She struggled to solve the system of equations because the calculations became messy due to the presence of trigonometric functions. In addition to using trigonometric identities to help her simplify the equations, she also searched for new equations that might make her computations easier. I was surprised when she thought about adding the magnitudes of the two vectors to create a third equation because it was a new idea for me (see Figure 9b).

As she became fatigued by solving the system of equations, she thought about creating a triangle. She told me she imagined the two vectors and it occurred to her that by adding a line to her vectors, she could form a triangle (see Figure 9c). This was promising because she told me that triangles have "lots of nice rules; that is where you get sine and cosine in the first place." Then, she consulted Google for "the Law of Cosines" and she solved the problem.

Maisha – Participant 9

Maisha was a rising 3rd year electrical engineering undergraduate student. She told me she liked mathematics because it came naturally to her. She liked its consistency and knowing that there is only one answer. For this reason, she did not like Linear Algebra, a theoretical mathematics class, as much as she has liked her previous mathematics classes. She told me theoretical mathematics focuses on thought processes, not on getting one answer. Before taking linear algebra, she was used to simple math based on performing computations, not critical thinking. Nonetheless, because she had only taken one theoretical mathematics class, for her, mathematics still was “the use of numbers to critically solve problems.”

Although Maisha was uncomfortable with creative-mathematical thinking, Maisha enjoyed the feeling of making connections between mathematical ideas. She told me she had a distant memory from 7th grade when she felt excited about seeing a connection between fractions and another mathematics concept. She was unable to remember the details but she remembered the excitement she felt. For the same reason, she told me she liked the robotics-based task because it allowed her to think critically and to make connections. She told me her computer science class gave her the same excitement. She was given all the tools, but she “needed to figure out how to solve the problem.” She added that schools should be about critical thinking not just about learning concepts.

When we met, Maisha attempted to solve the robotics-based task via a method consisting of three steps. To find the angle of rotation of the elbow, she created two steps. First, she rotated the elbow of the robotic arm 90 degrees. Then, she formed a right triangle using the coordinates of the location of the gripper – which she obtained by "eyeballing it" – and the coordinates of the location of the apple as vertices of the right triangle (see Figure 10a). She used the right triangle to figure out how much more to rotate the elbow from 90 degrees. To figure out the angle of rotation of the shoulder, she tried to form another right triangle using the location of the gripper and the apple (see Figure 10b). However, she struggled to identify where the angle of rotation of the shoulder was located in the right triangle. Eventually, she treated the angle she labeled elbow rotation as the rotation of the shoulder. Then, she drew two legs of a triangle with lengths 4 and 4.5, and calculated the angle formed by these two legs using cosine and cosine inverse (see Figure 10c). She concluded that either triangle would work to find the angle of rotation of the shoulder. When she entered the values into the computer, the gripper did not reach the apple. Then, she adjusted the values in the computer to get closer to the apple. She gave up after she moved the apple to another location and she could not replicate her three step process to get to the apple. She told me that the problem was hard because there were two axes of rotation.

At the end, she thought about using circles. She performed a few Google searchers to find information regarding circles. However, she was perplexed with the two axes of rotation. She told me she did not work on problems for more than two hours in a day because she becomes unproductive after two hours. Hence, she decided to end her problem-solving episode.

Lola – Participant 10

Lola was a rising 3rd year electrical engineering undergraduate student. For her mathematics is the "study of relations among the world." She told me that she found it rewarding when she exceled at making connections and solving mathematical problems. As opposed to most of her college friends, she told me that she only had female mathematics educators during high school and during her first year of college. Sometimes, she wondered if that influenced her positive relationship with mathematics. During her second year of college, she told me, her experience with mathematics was different. She said her linear algebra and statistics university professors were boring and had monotonic voices.

Lola decided to break down the problem into steps. She told me that in her computer science class it was a common practice to break a task into steps and she was trying to bring that practice into mathematics. She thought the robotic-based task lent itself to be divided into steps. To document the steps that the robotic arm needed to accomplish to reach the apple, she entered values into the computer. She wrote down three steps. Step 1 consisted of finding the y-coordinate of the location of the elbow after rotating the shoulder. To find it, she subtracted the y-coordinate of the location of the apple from the length of the forearm, which was $2.5 – 4 = -1.5$. She was not confident about this step because she told me it was not accurate. To be accurate, the forearm needed to be perpendicular to the x-axis when reaching the apple (see Figure 11a). Step 2 consisted of setting the shoulder to rotate -10 degrees. She drew a diagram identifying the angle of rotation of the shoulder and included information that she knew could help her find the angle (see Figure 11a). Step 3 consisted of moving the elbow to the apple. She drew the movements of the robotic arm to identify the angle of rotation of the elbow (see Figure 11b). Lastly, she drew a diagram depicting the two unknowns along with all the information she thought she would need to figure out the unknowns (see Figure 11c).

Although Lola was certain that she had the correct diagram to solve the problem, she did not excel in finding the unknowns. She told me that her attention was fixated on a right triangle and an isosceles triangle, which were not allowing her to progress towards a solution (see Figure 11c). However, she told me that she could not get these triangles out of her head. When this happens to her, she takes a break and attempts the problem again when she has a fresh mind. The break helps her see new possibilities to approach the problem. During her problem-solving episode, she took a break to eat after two hours, but when we came back, she was still thinking about the two triangles that she thought were not fruitful for finding the unknowns. Hence, she wrote equations using these triangles to get the solutions that she wanted by "brute force." Next, she tested her equations using another point and decided to stop working on the problem when the equations did not work.

Aniya – Participant 11

Aniya was a rising high school senior. She told me mathematics had always come easy to her. However, she liked it when it did not come easy because she liked to be challenged. She told me that for her mathematics is a "system using numbers and sometimes symbols." She hesitated to add to her definition that mathematics "is a way to solve problems" because she did not see the connection between mathematics and real life problems anymore.

Although Aniya still liked mathematics and had only earned As in her mathematics classes, she told me that after elementary school she did not see the connection between mathematics and real life problems. In elementary school, she was introduced to fun and challenging problems. After elementary school, instead of doing fun and challenging problems, she had to do more problems and covered topics faster. For pre-calculus, her last mathematics class, she watched Khan Academy videos for homework. The following day, her teacher went over the content of the Khan Academy video. She told me she slept through most of her pre-calculus class because she was bored. Her teacher did not mind because she was earning an A in class.

When we met, Aniya told me she only knew two strategies for solving problems, drawing figures and searching for terms, both of which she learned in her physics class. Her proficiency at using these strategies was evident during her problem-solving episode. As soon as I introduced her to the problem, she said she needed to draw it. She drew the robotic arm in its starting position. Then, she imagined rotating only the shoulder. Next, she imagined rotating only the elbow, but in both cases, she said she will miss the apple. At this point, she told me that she could not see how to use math to solve the robotics-based task. Suddenly, she said “it is a coordinate grid so it is about angles.” By thinking of the terms “coordinate grid” and “angles,” she thought about triangles.

She focused on two triangles, a right triangle and a scalene triangle. She found one of the angles of the right triangle but she did not verbalize what she was trying to accomplish (see Figure 12a). Next, she drew a scalene triangle. Although she did not have a right triangle, she incorrectly used trigonometric ratios to figure out the measurement of two of the internal angles of the scalene triangle. To find the angle of rotation of the shoulder, she divided an internal angle of the scalene triangle into two parts, and noted that the angle below the x-axis was the angle of rotation of the shoulder (see Figure 12b). She struggled to figure out the measurement of this angle. After a lunch break, she told me she had forgotten the right triangle she had drawn. She was upset about forgetting it because she told me she needed it to figure out the angle of rotation of the shoulder. For the angle of rotation of the elbow, initially, she thought it was one of the internal angles of the scalene triangle (see Figure 12b). However, after entering the angle into the computer, she realized that she was looking at the wrong angle. She told me that after seeing the robotic arm move, she imagined how her arm would move and she realized that the angle of rotation of the elbow was supplementary to the angle of the scalene triangle.

At the end of her problem-solving episode, she told me she felt happy. Although the gripper of the robotic arm did not reach the apple, she told me she felt confident about her mathematical thinking. She attributed the small discrepancy in the distance from the gripper to the apple to a mechanical error of the robot.

Pablo – Participant 12

Pablo was a rising high school senior. He was curious about this study because he was a member of the robotics club at his high school, and he liked mathematics. For him, “math is something you use with numbers to solve certain problems.” Mathematics had been easy for him. Although he was an A student, he did not have plans to attend college. He told me he already mastered the content from school. He was bored in school, even in his mathematics classes, because there was a lot of repetition, which he considered busy work. Instead of going to college, he was planning to work full time as a sound engineer. At the time of our meeting, he was working part time as a sound engineer. He told me he was using machines that only a few sound engineers in his company knew how to use, and he had already recorded songs for famous bands.

When we met, Pablo told me he only had one strategy for solving problems – start the problem. He told me that in order to start thinking, he had to start the problem with anything that came to his mind, even if it appeared to be a random idea. Later on, he could fix areas where his thinking was not on target or where he made mistakes. His problem-solving behavior during the robotics-based task, followed the strategy he described. When he got stuck, he immediately said, "I am going to write something to get started."

Pablo told me the robotics-based tasked was challenging because it gave him "too much freedom to think." He said that "usually, there is a straight forward way to do things." In his math class, the teacher tells him how to solve problems and in his high school robotics club, he tells the robot exactly which actions to perform (e.g. turned 90 degrees, etc.). For this reason, he started the robotics-based task by entering values to the computer to get to the apple. At his high school robotics club, he labels all the locations where the robot needs to go. Next, he programs the robot to perform the necessary actions to get from location A to location B. Due to this routine, he kept on rejecting the idea of writing equations. To him, the equations were not needed because the robot could be programmed to remember the rotations needed to arrive at the apple.

Once he accepted that equations would provide the flexibility to reach the apple regardless of its location, he approached the problem in two ways. First, he assumed he knew the angle of rotation of the elbow, then he noticed by observing the robotic arm that the elbow would move down about an inch after rotating the shoulder -10 degrees. This allowed him to imagine how to represent the problem in a diagram. From the diagram, he imagined two triangles – one for finding the angle of rotation of the elbow and the other for the angle of rotation of the shoulder (see Figure 13). He performed a Google search for the “cosine rule” to find the angle of rotation of the elbow, using a triangle with two approximated side lengths. To find the angle of rotation of the shoulder, he performed the Google searched to determine “how to find the third side of a triangle when it is not a right triangle.” He read the first two sentences of the top Google search results and told me that he needed to know two sides and an angle, to use the Law of Cosines. He said he did not know an angle because he was looking at Figure 13a. Yet, he had identified an angle earlier (see Figure 13b). Plus, he told me his brain was tired. Hence, he ended the problem-solving episode. At the end of the final interview, he said “I think high school mathematics is kind of easy now. Too easy! It is not challenging.”

Research Question 1: What heuristics are employed by participants during their attempt to solve a robotic arm task designed to require applications of trigonometry in its solution?

Of the 11 heuristics in this study, 10 were used by at least 7 participants (see Table 1). Five heuristics were employed by all 12 participants — Draws a Figure, Recalls a Related Problem, Uses a Related Problem, Uses Auxiliary Elements/Problems, and Checks Solution. Except for Juana, the other problem solvers employed the heuristic “Works Backwards.” The heuristic “Uses Successive Approximation” was employed by 10 of the participants. John and Mark did not employ this heuristic because they were determined to find the exact values for the unknowns symbolically. The heuristic “Decomposition” was employed by 10 participants. To my surprise, the heuristic “Changes Condition” was employed by 9 of the 12 participants. Kilpatrick (1967) mentioned that use of this heuristic rarely occurs. Despite the overlap in usage of heuristics among the participants, the way in which they used the heuristics sometimes differed – as I describe in the results for research question 2 below.

Although not in my initial coding scheme, a potential contemporary heuristic emerged during the problem-solving episodes – Googling. Ten participants reached a point at which they felt stuck. Some recalled a mathematical concept but not clearly enough to be able to apply the concept to the problem at hand. Others identified that they needed information that they did not have. When they explained this difficulty to me, I asked them what would they have done if they were at home, alone? They all replied that they would have consulted Google, a web search engine. One participant, Adam, told me that normally, he would use Google and wolframalpha.com as a symbolic calculator.

When I reminded them that they could do anything that they would normally do if I was not there, seven participants performed Google searches using their cell phones. Two told me that normally they would consult Google when they are alone and they get stuck, but they told me they did not want to use it during the study because they wanted to figure it out on their own. One participant, Adam, used the symbolic calculator and resources on wolframalpha.com instead of Googling.

Research Question 2: In what ways did the participants use of a specific heuristic differ? In what ways were the use of a specific heuristic similar?

As I mentioned in the methodology chapter, because Kilpatrick's (1967) coding sheets focused on identifying and keeping track of which heuristics were employed by the participant, I created a matrix to keep track of how each participant employed the heuristic(s) to aid me in identifying the similarities and differences in how a heuristic was used (see Appendix I). In the matrix, I included a list of the concepts each participant recalled and/or used during his or her problem-solving episode. To compare which participants made use of which heuristics, I consolidated the data from each participant's individual coding sheet into a summary table (see Table 1). Then, I made further use of Table 1 by counting how many of the 12 participants made use of each of the 11 heuristics under investigation in this study. As a measure of diagram use I examined each participant's work on paper, counting each separate figure they drew, without regard to quality of the figure. Appendix L illustrates an example of this diagram's display for one of the participants.

Draws a Figure Heuristic

All 12 participants produced a set of figures while attempting to solve the robotics task. The number of diagrams per participant range from 4 to 23, with a median of 11. Although I asked participants to draw everything that they imagined on the notebook, sometimes participants made references, while thinking aloud, to scenarios that were not recorded on their notebook. One participant, Imani, gave me an explanation for not drawing what she was seeing after I reminded her to draw for me what she was seeing in her head. She told me "sorry my hand doesn't work as fast as my head." Then, she proceeded to make a drawing to give me access to what she was seeing. Hence, the number of figures that a participant drew might not be indicative of all the scenarios a participant imagined. It is also possible that participants' diagrams are the last image they imagined from a quick thinking episode.

Participants did not always see all the relevant information that needed to be included in their diagrams or they did not see it initially. All participants verbalized that they needed to figure out the angle of rotation of the elbow and the shoulder. However, only Lola and Mark identified in their diagrams where the unknowns were at the beginning of their problem-solving episode. Seven participants identified the unknowns later in their problem-solving episode on their diagrams. John told me that he did not see the unknowns initially. He had to play with the problem to see them. For this reason, he started with a basic diagram and he added more details as he identified more information that should be added to his diagram. Of these nine participants who identified the unknowns in their diagrams, five solved the problem correctly. The other four were not successful because of misconceptions they had of mathematical concepts, because they were unable to see triangles that contain all the necessary information to solve the problem, or because they became overwhelmed by the freedom they were given (e.g. deciding what knowledge to use and when to use it). Lastly, the three participants who did not identify the unknowns in their diagrams did not solve the problem. They seemed to lack direction. They seemed to be driven to find any angle they could find.

Changes Condition Heuristic

According to Polya (1957), modifying the problem is crucial for sustaining the problem-solver's interest on the problem (p.209). It enables the problem solver to change a problem until the problem solver is empowered by seeing or finding a useful piece of information. In this study, nine of the problem-solvers Changed the Condition. Of these nine problem-solvers, eight imagined that there was only a rotation at the shoulder or only a rotation at the elbow at the beginning of their problem-solving episode. Also, four problem solvers who wanted to take advantage of the properties of right triangles fixed the angle of rotation of the elbow to be 90 degrees. Although eight of the problem-solvers that Changed the Condition were searching for an easier problem that they could explore, one problem solver changed the problem to be a specific triangle that she knew well, the 45-45-90 triangle, to evaluate the accuracy of her mathematical thinking.

Of the nine problem solvers who Changed the Condition, three made comments alluding to changing the problem in order to be able to start the problem solving process. For example, Lola said she just needed to "start with something." In a like manner, to justify to herself why it was acceptable to transform the problem, Sarah said "I have one triangle and I am trying to find another triangle. I know that the angle is not 90 degrees but it is close." The third participant, Pablo, mentioned that he was going to write something to get started. He told me he could fix his thinking later if he was wrong. For him, the important issue was to get started, not to be correct. In short, Lola, Sarah, and Pablo, verbalized awareness of taking a detour from the problem they needed to engage with. However, they acknowledged that, for them, it was only a starting point. Their ultimate goal was to learn and to go back to the initial problem. As Polya (1957) anticipated, these problem solvers were modifying the problem to remain engaged with the problem.

Recalls a Related Problem Heuristic

Polya (1957) suggested that if a problem existed for which a problem solver lacked related knowledge, the problem would probably be unsolvable. Participants recalled a total of 29 concepts. The number of related problems recalled per participant ranged from 6 to 13, with a median of 8. The related problems recalled with the most frequency were associated with the following mathematical concepts: trigonometric ratios (11 participants), inverse trigonometric functions (11 participants), the interior angles of a triangle sum to 180 degrees (11 participants), Pythagorean Theorem (10 participants), right triangles (10 participants), Law of Cosines (8 participants), Law of Sines (6 participants), vectors (5 participants), and supplementary angles (5 participants).

Right triangles were favored over other types of triangles. When the problem solvers reflected on how they found angles in the past, 11 of them thought about using triangles, with 10 of them thinking or searching for right triangles. Lola even verbalized not knowing why she did not have a right triangle.

Uses Related Problem in Solution Heuristic

Polya (1957) anticipated that problem solvers will be able to recall many related problems. In fact, he expressed concern that problem solvers could become overwhelmed with the multitude of options. In this study, participants used from 4 to 10 of the related problems they recalled while attempting to solve the robotics-based task, with a median of 6.

As predicted by Polya (1957), some problem solvers expressed confusion for having to choose what knowledge to use. Pablo told me that he had "too much freedom" during the robotics-based task. In his words, he "had too many options! Too much time! And the more [he] was thinking, the more [he] was getting confused." He told me that usually teachers tell him what knowledge he is going to need before giving him a problem set. Determining what knowledge to use to approach a problem was a new experience for him. Nine of the problem solvers told me something similar.

However, Mark told me that his struggle was remembering the details of mathematical concepts. Because his engineering professors tell him that his job is to know knowledge exists and to know how to find it, he does not need to remember everything anymore. Actually, he was told that there is so much information out there that it will be impossible to know it all. For this problem, he felt good because he knew what math he needed. He was certain that there were properties of nonright triangles and as soon as he gained access to technology he found the Law of Cosines, which he used to solve the problem.

Uses Successive Approximation Heuristic

This is a Polya-like heuristic that Kilpatrick (1967) created to identify a behavior that is used by problem solvers to generate a solution by consciously using information from previous trials. In this study, ten of the problem solvers employed the heuristic Successive Approximations. Seven of the ten problem solvers entered values for the angles of rotation of the elbow and the shoulder into the computer and observed the movement enacted by the robotic arm. They continued to adjust the values until they observed the robotic arm had reached the apple. The remaining three problem solvers first used a mathematical model to move the gripper close to the location of the apple. Then, they used the computer to adjust their solution until the gripper reached the location of the apple.

Sets up Equations Heuristic

This heuristic calls for the problem solver to use mathematical symbols and to express the problem using equations. To express the problem using equations, the problem solver needs to identify the unknown, the given data, and the condition (i.e. what is linking the unknown data to the known data). In this study, seven of the problem solvers set up equations as described by Polya (1957). Of these seven problem solvers, two of them set up equations based on mathematical misconceptions. Aniya employed trigonometry ratios when dealing with a non-right triangle. Mena built right triangles which contained a desired hypotenuse, and did not realize that these triangles where not unique. Lastly, in addition to the previously mentioned seven problem solvers, Lola set up equations for the problem as well, but she did not do so as described by Polya (1957). Although she had identified the given data and the unknown data, she did not see conditions linking the known with the unknown data. Instead, she engineered equations that would yield the answers she wanted. She thought about operations she could perform to the unknowns that would yield the elbow and shoulder rotations she found by entering values into the computer. That is, she had an answer in mind and she created equations that would yield her answer.

Uses Auxiliary Elements/Problems Heuristic

According to Polya (1957), this happens when problem solvers add an element (e.g. a line, a theorem, a variable, or a problem) to their work with the hope of making progress in the original problem. All 12 problem solvers employed auxiliary elements. Eleven of the problem solvers added lines to their work to form triangles with the hope of finding an angle or the distance between two points. Six participants added a line to an existing diagram to form an angle or to divide an angle. Five participants added a line to an existing diagram to form a right triangle inside a non-right triangle.

Works Backwards Heuristic

Polya (1957) described working backwards as starting the problem-solving process by visualizing the problem solved, and then figuring out what is required to achieve the imagined-solved problem. This heuristic was employed by 11 participants. They imagined the gripper over the apple and consequently imagined how the robotic arm needed to be positioned in order for the gripper to be over the apple. Then, they tried to figure out which angle of rotation at the shoulder and the elbow were required to achieve this solution.

Decomposition Heuristic

Polya (1957) suggested that sometimes problem solvers need to focus their attention on parts of the problem instead of tackling the whole problem. Ten of the problem solvers focused their attention on the angle of rotation of the shoulder first and then the angle of rotation of the elbow, or vice versa. Of these ten problem solvers, three found the angle of rotation of the elbow first, and then tried to find the angle of rotation of the shoulder based on the results from the elbow rotation. An additional participant, John, verbalized thinking about whether or not it mattered which angle was found first. However, he decided quickly that it did not matter and after that decision, he paid attention to both angles, concurrently, while formulating the problem.

Checks Solution Heuristic

For Polya (1957), checking the result of a problem improves the trust in the solution. In this study, all 12 participants checked their solution. Namely, they checked that the gripper of the robotic arm moved over the apple. They all wanted to make sure that the angles they had in fact moved the gripper of the robotic arm to be above the apple. Seven problem solvers entered values into the computer to find the angles by employing the heuristic Successive Approximation. Because they saw which angles of rotation of the elbow and the shoulder moved the gripper of the robotic arm to be over the apple, they used these values to gage whether they were on track to reach the apple, when they were working with equations.

In addition to checking their final solution, some problem solvers checked their intermediate work. Three problem solvers found a potential angle of rotation of the elbow first. After entering the value into the computer and seeing the robotic arm move, they decided that it looked promising and they proceeded to find the corresponding angle of rotation of the shoulder. John was prompted to check his work after obtaining a domain error while performing calculations with his calculator. Adam became concerned when he took the square root of an expression. The presence of the square root influenced him to stop and think about which angles will yield a nonnegative radicand, and wonder whether these angles made sense for his problem. Sarah and Imani tried to verify their work in progress by employing other mathematical procedures. For instance, Sarah tried to use the Law of Sines to check the results she obtained from using the Law of Cosines. Imani used a triangle that she knew, the 30-60-90 triangle, to verify that her thinking was correct.

Part of checking a solution includes checking that the solution makes sense. Only John verbalized that his solution made sense for the problem. Three participants realized that the elbow angle of rotation that they found did not make sense because the robotic arm only rotated 90 degrees when they entered a rotation over 90 degrees in the computer. Another problem solver, Adam, realized that the angle of rotation of the elbow did not make sense because he had an answer over 90 degrees and he remembered that the robotic arm could only rotate positive or negative 90 degrees.

Derives Solution by Another Method Heuristic

Polya (1957) suggested that problem solvers who engaged in solving problems via different methods strengthen their knowledge base and the reliability of their solution. In this study, none of the problem solvers attempted to solve the problem via another method after solving the problem successfully. John thought about alternative methods he could have employed to solve the problem at the beginning to determine the best way to approach the problem. However, when he solved the problem, he did not try to solve it again by employing the other methods he mentioned earlier.

Research Question 3: What differences do participants note, retrospectively, in the approach they employed in attempting to solve the robotics-based task as compared to approaches they typically use to solve nonroutine problems in trigonometry?

Of the 12 problem solvers in this study, 11 of them perceived differences in their problem-solving approaches during their problem-solving episode due to the presence of the robotic arm. One problem solver, John, did not perceive any differences in his problem-solving behavior due to the presence of the robotic arm. He enjoyed engaging with the problem because he found the problem itself interesting. He told me that what he enjoyed was the math because he likes "to do math for the sake of the math." He did not think it mattered whether he had a robot or not. Hence, the results presented below only pertain to the 11 problem solvers who perceived differences in their problem-solver behavior due to the presence of the robotic arm during their problem-solving episode. For these 11 problem solvers, the presence of the robotic arm influenced their motivation to engage with the problem and facilitated "scientific engagement," as coined by Sarah, that was connected with visualization and feedback.

The 11 problem solvers that were motivated to engage with the problem due to the presence of the robotic arm told me in different ways that they were interested due to the novelty of the problem-solving scenario. In Aniya's words, the robotic arm "was fun to use. It was hands on. It was real. It wasn't just paper." Likewise, Adam found the robotics-based task motivational because he found the robotic arm "cool." He said it was better than reviewing vectors on a board. He said that because he found it "cool," it caught his interest.

In addition to supporting a motivational component, the physical robot promoted a new approach in problem solving, scientific engagement. Sarah explained to me that in mathematics someone needs to tell her when she has the right answer. However, with the robotic arm, she was in control. She was able to use "the robot arm to test some theories." She could see what happened when an angle was set to 90 degrees. She told me that she was able to see the angles and to figure out what she was doing wrong. The robotic arm helped her decide "where to go next."

The other 10 problem solvers did not use the term scientific engagement. However, they described a similar problem-solving approach. Pablo told me that without the robotic arm, he would not have been able to guess because with worksheets, "you have nowhere to work from. You cannot test your ideas." Aniya told me that when she got stuck, she looked at the robot. Seeing it move influenced her to think about her own arm and to think about the different angles. It allowed her to "play with the robot to do the math." Similarly, Mark told me that when he saw the robotic arm moving, he "immediately started thinking where things were moving." He told me he could solve the problem if it was only on paper, "but it is just a different world to see it."

The importance of visualizing a problem was shared by the 11 participants. Mark told me that as he progressed through his undergraduate education and participated on research projects, he learned to appreciate concepts he learned in the past once he started to create visualizations for them. In his opinion, being able to visualize a concept strengthens his understanding and intuition of mathematical ideas. Adam also mentioned that ever since he was a child he was drawn to activities that enabled him to associate “math with something physical not just a piece of paper.”

Intertwined with visualizing, the 11 problem solvers mentioned the influence of immediate feedback on their thinking. Ebony told me that seeing the inconsistencies between what she was thinking and what the robot actually accomplished “changed what [she] was doing because physically, it did not make sense.” Three of the undergraduate students told me that getting feedback reminded them of computer science because, as in computer science, they were able to test their ideas. Lola and Adam had conflicting experiences in terms of receiving feedback from the robotic arm. While Lola mentioned that she broke down the problem into steps – a strategy that she learned in computer science – and attempted to test each step, Adam mentioned that the robotic arm failed to consistently reinforce him. He told me that he had to solve the entire problem to get feedback. He expressed wishing that the robotic arm could have told him whether he was going in the right direction or not throughout the problem, not just the end.

CHAPTER 5

DISCUSSION

How to improve students creative mathematical thinking – nonroutine problem solving – remains an unresolved challenge in the United States. Although there is evidence that the heuristics identified by Polya (1945) are fruitful when solving non-routine problems, it is still unclear how to teach Polya’s heuristics. To do so, it will be necessary to understand how and why problem solvers make the decisions they make (Schoenfeld, 2007). To address this point, Papert (1980) suggested that Turtle Geometry, an animal-like physical robot, could serve as an entry point to Polya's heuristics. His suggestion arose from observing children engaging with Polya's heuristics even though they were never taught Polya's heuristics nor were they asked to employ them. In this study, I sought to identify what heuristics problem solvers employ and how during their attempt to solve a robotics-based task designed to require the applications of trigonometry in its solution.

Overall, the 12 participants in this study resorted to heuristics identified by Polya to overcome obstacles and to understand the problem (see Table 1). Through this study, it became more evident to me that Polya's heuristics are not the fast-track to a solution. Problem solvers employed the heuristics in a manner that honored their cognitive processes. How much time they engaged with the problem varied depending on what each problem solver was able to see after employing each heuristic.

Beyond helping problem solvers to find a solution, heuristics can promote engagement with mathematics. As the 12 participants remained engaged with the robotics-based task, they were concurrently distilling the problem into its basic elements – discovering and seeing the structure of the problem. However, at the end of the problem-solving episode, they had more than an answer; they were excited. Motivation was something Polya (1945) held in high regard. In fact, Polya (1945) mentioned that it will be "sad" to have students work on problems that do not interest them. For this reason, it was refreshing to observe that the participants were interested and enjoyed the robotics-based tasks – irrespective of the problem-solving episode outcome.

Like Papert (1980), I also observed problem solvers' engagement with Polya's identified heuristics, including heuristics that are rarely observed such as Changing the Condition and Checking the Solution, when they were engaged in the robotics-based task (see Table 1). Thus, although the results of this study are not sufficient to support causal relations, the problem-solving episodes of the 12 participants in this study provide proof of existence supporting Papert's (1980) and Polya's (1945) theoretical contributions. The finding that heuristics were commonly used in the robotics task posed in this study should not be interpreted to imply that similar heuristic usage might not also be elicited by certain non-robotics-based tasks. Polya wrote about heuristics in 1945 before modern educational technologies were widely available. However, the results do suggest that when problem solvers are interested in the mathematical task, they become motivated to engage with the mathematical task for a long period of time – giving problem solvers time to employ Polya's heuristics. In this study, 11 of the participants told me that they engaged with the robotics-based task because they found the robotic arm cool and they wanted to see the robotic arm reach the apple. They told me the robotic arm served as a motivational component that sustained their perseverance.

When considered more broadly the study findings related to my specific research questions lead me to make three overarching observations that I will elaborate on in the discussion sections that follow. The first observation is based on findings from research questions 2 and 3. The second observation is based on findings from research question 2. Lastly, the third observation is an unintended new finding based on the results from research questions 2 and 3. Each of these three observations is followed by separate subsections on implications for theory and implications for educators.

Observation #1: Problem solvers reported that the robotics-based task prompted them to engage with the problem in a scientific manner during the solution process. They created conjectures by Changing the Condition(s) of the original problem or by employing Successive Approximations. They tested their conjectures using the robotic arm, and they updated their conjectures.

This observation is in support of Papert's (1980) assertion that in robotics-learning environments learners are inclined to wonder if their work is fixable – instead of viewing their work as entirely wrong. One of the participants described it as engaging "scientifically" with mathematics. Namely, the learning environment allows problem solvers to make conjectures, test their conjectures by observing the robotic arm, and update their conjectures. After noticing this inquiry behavior in robotics learning environments, Sullivan (2008) deduced that by engaging in robotics activities, students are experiencing the scientific method. In the context of this study, the participants engaged in scientific inquiry to attempt to solve a mathematical task.

Furthermore, participants were able to perceive that they had engaged in this exploratory behavior. For example, Pablo told me that with worksheets, he could not guess. He had to know the answer. In this sense, the presence of the robotic arm disturbed the way he experienced mathematics in the past. Schoenfeld (2008) mentioned that most students have experienced mathematics as a set of rules that need to be memorized. Few students have experienced mathematics as a discipline "in which one can explore, and make and verify discoveries" (Schoenfeld, 2008).

The exploratory behavior was supported by the feedback given by the robotic arm intertwined with some of Polya's identified heuristics. For example, eight problem solvers modified the original problem by assuming only a rotation at the shoulder or the elbow was needed. This is a phenomenon rarely observed (Kilpatrick 1967, 2016). However, in the context of the robotics-based task, it was common for the problem solvers to wonder what if instead of two rotations only one rotation was needed to move the gripper over the apple. Intuitively, they knew that the problem would be easier if it only involved one rotation, supporting McClintock's (1980) suggestion that mathematical tasks have particularities that call for the implementation of certain heuristics. In this case, it was overwhelming for the problem solvers to think about the two unknowns initially. This prompted them to think about considering the scenario with only one rotation. Because they had access to the robotic arm, after thinking about it, they could look at the robotic arm and imagine the outcome.

Better, they could enter their solution into the computer and test their idea. Problem solvers were cognizant about Checking their Solution because they wanted to see the gripper of the robotic arm reach the apple. Desiring to see something happen prompted the participants to Check their Solution. In other studies, participants did not tend to Check their Solutions (Lee, 2015). Even in a study where the researcher reminded participants to Check their Solution, participants did not Check the Solution of subsequent problems (Oehmke, 1979). In this study, however, the problem solvers checked the answer because that required using the robot, which they thought was "cool" to use. Nonetheless, problem solvers merely checked answers. They did not engage in the Looking Back phase as described by Polya (1945), thus confirming what seems to be a stubborn avoidance of this reflective stage in the problem-solving process.

The heuristic Successive Approximation also supported scientific engagement.

When participants were out of ideas, they approximated solutions and observed how the robotic arm moved after each approximation. Observing the robotic arm gave them a visual representation of the problem and gave them time to think. Some participants saw triangles or other elements of the problem while observing the robotic arm. Sullivan (2008) observed a similar behavior in other robotics activities but she described it as estimation – a skill needed in science.

Implications for Theory

Polya's (1945) intended audience was teachers keen enough to know how to pose the right heuristic at the right time to stimulate their students' thinking processes. However, the results of this study suggest that teachers do not have to do it alone. Mathematical tasks can help teachers by guiding students to start thinking about implementing a heuristic, as suggested by McClintock (1980). At this point, the observant teacher can take the students' lead; once the teacher sees that students are trying to implement a particular heuristic the teacher could use that as a motivating prompt to have a more formal class discussion focused on the heuristic she observed some of the students beginning to employ.

In this study, eight problem solvers thought about modifying the problem by only rotating the shoulder or the elbow of the robotic arm. Their motive to eliminate one rotation is in accordance with Polya's (1945) writings on problem solving. Polya (1945) described human problem solving as more successful than insect or animal problem solving because humans are able to modify the problem more intelligently. In this study, having two unknowns which depended on each other overwhelmed eight problem solvers and influenced them to consider the scenario where there is only one unknown. In this scenario, teachers could be relief from identifying when and which heuristic students should be introduced to. Instead, students thought about Varying the Problem by Changing the Conditions of the original problem. In doing this, they made visible for the observant teacher a good time to elaborate on the heuristic Varying the Problem by Changing the Condition.

Implications for Educators

For mathematics educators, the observation that participants Changed the Condition of the original problem without formal instruction is promising. The heuristic provides an entry point via which teachers can start portraying mathematics as a discipline of exploration, and positioning exploration as an essential component of the problem-solving process. For instance, imagine a classroom with the eight participants who Changed the Condition of the original problem by assuming only one rotation was required for the gripper to reach the location of the apple. Now, imagine a mathematics educator roaming around the classroom and engaging in conversations with the students. When the educator interacted with these eight students, she could have told them, great idea! What you just thought about is a strategy mathematicians use to stay engaged with a problem and to get to know a problem. In such a scenario, the conversation about the heuristic Varying the Problem by Changing the Condition will grow out of an idea students had. At this juncture, the educator has the door wide open to introduce students to Varying the Problem by Changing the Condition, to explain why generating new mathematics questions by reformulating the problem is powerful in mathematics, and to highlight the fact that they thought about using the strategy on their own.

By introducing students to the activity of generating mathematics questions by reformulating a given problem, students' gains are twofold. First, students will experience how mathematicians come to know a problem, or what it means to understand a problem (Brown & Walter, 1983). Second, they will be empowered to formulate their own questions. Teachers will no longer have to come to class with additional problems for those students who finish classwork early. Perhaps, teachers could motivate students to generate their own questions by modifying their solved problems. This will disturb the paradigm where teachers are the source of problems by positioning students as the generators of mathematics problems. Knowing how to pose questions was rated as the most valuable skill to have in today's workspace by Conrad Wolfram, the director of one of the most influential mathematical companies in the world (Boaler, 2016). Students posing questions is a rare phenomenon in the mathematics classroom (Boaler, 2016), but nonetheless, it is an important component of the mathematics discipline.

Observation # 2: When "Drawing a Figure," identifying the unknown might require arduous work from the problem solver.

Although Kilpatrick (1967) found in his dissertation study that Drawing a Figure "seemed to bear no relation to the mode or success of a solution," the participants in this study showed otherwise (p.104). Kilpatrick found that poor problem solvers drew bad figures and good problem solvers drew good figures. However, irrespective of being a good or bad problem solver, his participants did not use their drawings to their advantage. He concluded that instruction on Drawing a Figure was needed for the heuristic to be fruitful. Other researchers' findings supported Kilpatrick's conclusion, indicating instruction on Drawing a Figure was needed (Diezmann, 2000; Nunokawa, 1994).

In this study, Drawing a Figure was fruitful for participants who used the drawing as suggested by Polya (1945). Namely, they identified on their drawings the known data, the unknown data, and used the drawing to focus on different details of the drawing to figure out how the known data were connected to the unknown data. Participants who pursued and accomplished these three goals solved the robotics-based task during our meeting. The exceptions were participants who had misconceptions of the mathematical concepts they applied (e.g. using trigonometric ratios when having scalene triangles).

Although it was common for participants in this study to include in their drawings the known data right from the beginning, this was not the case for the unknown. It was surprising to me to observe that most participants did not see the unknown at the beginning. They verbalized that they had two unknowns, the angle of rotation of the elbow and the angle of rotation of the shoulder. Yet, they were unable to see them in their drawing. At the elementary school level, Ng (2003) also found that less than half of his participants were able to identify the unknowns in their diagrams. In this study, however, 7 out of 9 participants who did not see the unknowns, initially, were able to identify the unknowns in their diagrams later in their problem-solving episode. Even John, a participant that loves mathematics and engages in nonroutine problem solving during his free time, did not see the unknowns initially. He told me that sometimes he has to engage with the problem to see the unknowns.

Nunokawa (1994) observed a similar case when documenting a problem solving episode of a graduate student. The participant in his study drew 19 diagrams through the process of becoming acquainted with the problem. Nunokawa suggested that when problem solvers faced a nonroutine problem, they might not see the structure of the problem initially. Instead, problem solvers may create an initial drawing that reflects the structure of the problem they are able to perceive. As problem solvers remain engaged with the problem, their perspective of the problem may change as well as their drawings. In this manner, the problem solver goes through iterations of drawings until they see the structure of the problem. Like Nunokawa's participants' drawings, the drawings of the participants in this study evolved through their problem solving episode.

It is important to highlight that some of the participants that were able to see the unknown were not searching for it. They were able to visualize the unknown when addressing inconsistencies between where they thought the robotic arm would move and where it actually moved. This is concerning because without knowing where the unknown is located, problem solvers lack the aim of the mathematical problem and are unable to use their drawing to their advantage , to figure out how the known data link to the unknown data.

Implications for Theory

Although Polya (1945) wrote about which elements needed to be included in a figure such as the unknown data, Polya did not give insight regarding how to proceed when the problem solver could not see the unknown data. He suggested that problem solvers should "draw a hypothetical figure which supposes the condition of the problem satisfied in all its parts." However, this suggestion assumes that the problem solver excelled at visualizing the location of the unknown data in the figure. In this study, seven participants had to engage with the problem to identify the location of the unknown data in their figure. One participant, Lola, was able to see the two unknowns in her figure early in her problem-solving episode by writing methodologically, step by step, the movement of the robotic arm. Further research is needed to find out if this strategy can help others see the unknown data early in their problem-solving episodes. Other problem solvers realized that they were labeling the wrong angle for the angle of rotation of the shoulder. After finding discrepancies between the answer they were expecting and the movement the robotic arm performed, they saw the correct location of the angle of rotation of the shoulder in their diagram. Also, there might be other strategies available to aid the problem solver to see the unknown that I was not able to identify them at the time of this study.

Implications for Educators

Knowing that problem solvers might not see the unknown from the start has pedagogical implications. It is typical for mathematics educators to present a problem to the class and to translate the problem with the students on the board, at the beginning of class (Schoenfeld, 1985, 2008). However, this sequence for solving a problem is in conflict with the cognitive processes of the students in this study, and potentially for other students as well (Nunokawa, 1994, 2006). Arriving at a drawing that is in agreement with the structure of the problem is arduous work. Perhaps, it will be more compatible with students' cognitive processes if the aim of the class activity became to produce a fruitful drawing instead of starting class with a good drawing.

As a starting point, educators can focus on making problem solvers aware of Polya's (1945) recommendations for drawings. To be specific, the problem solver should identify the known data and the unknown data on their diagrams. Students should be advised that they might not be able to visualize the unknown data immediately. Lastly, participants should be reminded that the goal of a diagram is to help them explore the connections among the data to discover the connection between the known data and the unknown data. In this study, one problem solver saw the unknown data by writing down step by step the movements the robotic arm needed to accomplish, a strategy she learned in her computer science class.

Observation #3: Search engines such as Google add "Finding a Related Problem" to Polya's "Recall a Related Problem" heuristic.

As anticipated by Polya (1945), participants in this study were often overwhelmed by what a participant called "freedom to think." They felt they were wandering in a mathematical forest. For most of them, this was a unique experience. The typical sequence of events in their mathematics classrooms is such that they first cover a specific topic and only then do they get to apply it. Hence, to students, when they get a set of problems, it is implied that they should find a way to employ the concepts just covered in class. However, when solving nonroutine problems, it is unknown which mathematics concepts apply or how they should be used. Unfortunately, Polya (1945) did not provide a step by step process of how one could determine which related problem is most fruitful. The aim of his heuristic Recall a Related Problem was to mobilize the problem solvers knowledge. By doing so, the problem solver could create potential paths to engage with the problem. Yet, there is no way the problem solver can know in advance which path is most fruitful. Moreover, it is important to note that Polya (1945) wrote about how to mobilize ones prior knowledge, pre-internet era.

The participants in this study were not limited to the knowledge they could recall or they had previously acquired. Through the internet, they had access to other peoples' knowledge. In particular, participants resorted to Google, a web search engine, to gain clarity of the concepts they were recalling or to find the specific information they needed. For example, one of the participants, Mark, limited himself to figure out how to solve the robotics-based task using right triangles because this was the knowledge that was available to him. However, after resorting to Google and searching for "trig properties for nonright triangles," he expanded the knowledge to which he had access. That is, he combined Recalling a Related Problem with searching for what he needed. He recalled that the triangle that he was working with was called a nonright triangle and he recalled the angles are related to trigonometric properties. Then, he used technology to find the information he needed. In this sense, knowledge is still important (Polya, 1945; Schoenfeld, 1985), but lacking specific knowledge is not as detrimental as it was before the internet, as long as problem solvers can break the problem down to determine what new knowledge they need to acquire.

Implications for Theory

When Polya (1945) wrote about problem solving, he primarily focused on how to mobilize problem solvers existing knowledge by using the heuristic Recall a Related Problem. However, in addition, Polya (1945) suggested Consulting Definitions when problem solvers recalled mathematics terms but lack clarity. Today, in the year 2017, problem solvers have access to web search engines, like Google, to find information when they have insufficient information about a mathematical term. Additionally, web search engines enable problem solvers to search for any information that they need to solve the problem, not only Definitions. That is, in 2017, problem solvers can mobilize their existing knowledge by Recalling a Related Problem, and they can add to their existing knowledge by employing the heuristics Consulting Definitions, and Find Related Problems. Finding Related Problems is more easily achievable today due to web search engines.

Implications for Educators

The information age is changing the paradigm of knowledge used to creatively play in a problem-solving environment. In the past, students who could not recall or who lacked knowledge were doomed to fail when solving problems. However, this does not need to be the case anymore. Today, students and teachers have access to web search engines designed to help people find information, such as Google. Students are using these outside of the classroom and are learning to distill problems to figure out what information they need. As a mathematics educator, I am enthusiastic about the possibility to help my students find the information they need, which in some ways also mobilizes their existing knowledge.

As mentioned in the results chapter, engineering and computer science educators are communicating to their students that given all the information available today, it will be impossible for them to teach them everything. Instead, they are focusing on teaching them the basics and on motivating them to learn to identify what information they need and how to find it. Based on the Google searches performed by seven of the participants in this study, it is clear some students have become savvy at finding what they need while others have not. I propose that in this era of readily available information, learning to determine what information one needs to solve a problem ought to be embraced in the classroom.

Needed Future Research

The participants for this study were high school students and undergraduate college students who enjoyed solving mathematics problems, in one southern metropolitan area. Future studies could include participants who might not feel as comfortable with mathematics as those in this study. Using the results of this and other related studies, future researchers might be able to develop a set of minimal heuristic prompts that could provide the support these students might need to continue to make progress towards a solution rather than abandoning the problem altogether.

Also, as I stated in the methodology section, one concern when implementing the think-aloud method is that participants might not be able to verbalize their thinking and their thinking might be so fast that it impedes the participant to verbalize it. In this study, there were moments when participants told me that thinking-aloud impeded their mathematical thinking. For instance, I was able to infer when Imani was struggling to verbalize everything she was thinking because she remained quiet and had a blank stare. I had to prompt her often to continue verbalizing what she was thinking and to draw what she was imagining. At some point, she felt bad that she was not drawing what she was seeing even though I kept reminding her to do so. She told me "sorry my hand doesn't work as fast as my head." To draw what she was imagining for me meant that she had to stop thinking, at the risk of losing her train of thought. When she went into deep thinking, her brain focused on thinking, becoming unaware of whether she was thinking-aloud or not. Hence, if she was silent, she was not cognizant that this was happening until I prompted her.

Unlike Imani, other participants were cognizant that they were not talking but they told me they needed to remain silent to be able to process their thoughts. John stopped me from prompting him when he became silent by humming. At the end, he told me he was annoyed that I tried to interrupt his thinking. He was very verbal for the most part, except for a few moments when he went into deep thinking. On the other hand, Mark talked the whole time. However, he told me it was challenging, for him, to talk aloud because he kept on thinking about "minimalistic things" he could say that will give the impression that he knew what he was doing, while simultaneously making progress towards a solution. He told me the talking made his mind go in circles; he found it unnatural. He said that he had to justify the talking instead of conducting true thinking. In summary, thinking-aloud is challenging for some participants. There are moments when forcing participants to talk does impede their thinking by making them loose track of their thoughts. However, other participants, who usually talk when they are solving problems at home, did not find thinking-aloud interrupted their thinking.

In terms of methodology, future studies could explore whether there are differences between problem solvers who struggled to think-aloud when they have moments of deep thinking, and those who do not. I wonder if those who speak during their deep-thinking moments are saying "minimalistic things" while their brains are engaged on abstract thinking that they are unable to verbalize. In such a case, the information they are externalizing is not rich and it might be equivalent to allowing them to be silent during those moments. In other words, John stopped me with his humming from interrupting his deep-thinking moments. However, the data from his problem-solving episode was rich. How much richer could it have been if he had excelled at verbalizing this thinking during the deep-thinking moments? What did I lose?

Another limitation of the study was meeting with the participants only once for the problem-solving episode. Polya (1945) mentioned that there is such a thing as subconscious work. He gave the example of remembering a person, but been unable to recall the person's name. Then, all of a sudden the name came to him the following morning. He supposed his experience was not unique; he conjectured that many people might have experienced something similar. He proceeded to share that something similar could happen to problem solvers when solving mathematics problems. For instance, he mentioned that sometimes a problem solver might not succeed at solving the problem one day. However, after sleeping or after a few days, the problem solver might have a bright idea when attempting the problem again. During the second pilot of this study, one participant sent me an email the morning following his problem-solving episode, identifying all the different ways the robotics-based task could be solved and under what circumstances each approach should be the preferable one. In his email, he explained that he woke up and thought about the problem and decided to engage with it again with a fresh mind.

Other participants might similarly have continued to explore the problem after the problem-solving episode ended and might have subsequently solved the task successfully. Yet, they might not have thought about reaching out to me. To enable participants to see the problem with a fresh mind, I encouraged participants to take breaks during the problem-solving episode. For some, this was helpful. However, for others, it was not. For instance, Lola verbalized during her problem-solving episode that she needed to stop working with the triangles that she was using. She said she was looking at angles and sides she could find instead of looking for the angles and sides she needed.

Yet, she said she could not break from that thinking. She knew, as did I, that she needed to walk away from the problem for longer than a few minutes in order to be able to let go of her current line of thinking. Other problem solvers verbalized that they were going to work on the problem again later because they felt they were close to solving it.

Unfortunately, the research design of this study did not allow some participants to take the break that they needed to be able to see the problem with fresh eyes again, and to proceed to a solution. It is my hope that in the near future, I or other researchers will be able to design research studies that honor what Polya (1945) described as the subconscious work. These studies could provide insights regarding when a problem solver should take a break from a problem and the number of problem solvers able to see the problem with fresh eyes after a night's rest. This study and the studies included in its literature review are limited to reporting what problem solvers accomplish during one meeting. Future researchers should explore problem solving behavior over an extended time period.

Another limitation of the study is its sample size. By increasing the sample size from 4 to 12, from the second pilot to the dissertation study, my view was expanded. I was able to notice that visualizing the unknown is not an easy endeavor. And not seeing the unknown, makes it difficult for the problem solver to search for what is linking the unknown data to the unknown data. Given the data collected thus far, I was able to observe problem-solvers engaging with Polya's heuristics and I was able to document how these heuristics were used. However, I am not certain that I reached what Miles and Huberman (1994) characterize as "saturation." Future studies, replicating this study in a larger scale, are needed to assure that no further insights emerge when true saturation is reached.

To date, researchers have identified what it is important to focus on when people are engaged in problem solving, but little is still known about how and why people make the decisions they make (Schoenfeld, 2007), and how mathematical tasks stimulate their decisions. The participants in this study provide a glimpse into how heuristics aided them in their decision-making during problem-solving episodes or how the lack of mastery of a heuristic impeded their progress towards the solution. In the case of the heuristic “Drawing a Figure,” the participants who were cognizant of identifying the unknown in their diagrams and of searching for how the unknown data were linked to the known data solved the problem correctly. On the other hand, those participants who were not cognizant of this worked aimlessly – as anticipated by Polya (1945). Recognizing what unknown is critical to the solution process is not always an easy endeavor. Participants who excelled in the ability to connect the given problem situation with the appropriate unknown were those who thought about the problem systematically. One participant even wrote step by step the movements of the robotic arm, keeping careful track of where the movement started and where it ended, which was something she learned to do in her computer science class. Mathematics educators need to recognize the importance of the transfer of knowledge and skills learned in one subject to another, rather than considering mathematics independent from other disciplines.

Final Remarks

As I end this dissertation, I would like to leave the reader thinking about the importance of novelty in mathematics education. In addition to the visual and physical representation, the robotic arm added a motivational component to the problem-solving experience. With the exception of one participant, all the participants thought the robotic arm was "cool" and they wanted to see the gripper reach the apple. They searched intently for the solution because they were engaged with a task they found interesting – a novel experience. I started this dissertation hoping to incorporate many similar mathematical tasks into the secondary curriculum motivated by the robotic arm, but the results of this dissertation have led me to reflect on the importance of novelty. To incorporate similar activities in an academic year could make all the experiences more ordinary. Less might be more if the goal is to create some truly memorable moments with mathematics. Thus this is a line of inquiry that should be explored in future research. Polya (1957) warned it would be "sad to work for an end that you do not desire." The use of robotics-based tasks could be one tool in a teacher's repertoire, rather than a panacea for effective mathematics teaching.

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Table 1

List of Heuristics Used by the 12 Participants.

	1 John	2 Adam	3 Sarah	4 Juana	5 Mark	6 Ebony	7 Mena	8 Imani	9 Maisha	10 Lola	11 Aniya	12 Pablo
Draws Figure	X	X	X	X	X	X	X	X	X	X	X	X
Changes Condition		X	X		X	X	X	X		X	X	X
Recalls Same or Related Problem	X	X	X	X	X	X	X	X	X	X	X	X
Uses Related Problem in Solution	X	X	X	X	X	X	X	X	X	X	X	X
Uses Successive Approximation		X	X	X		X	X	X	X	X	X	X
Sets Up Equations	X	X	X		X		X	X			X	
Uses Auxiliary Elements/Problem	X	X	X	X	X	X	X	X	X	X	X	X
Works Backwards	X	X	X		X	X	X	X	X	X	X	X
Decomposition			X	X	X	X	X	X	X	X	X	X
Checks Solution	X	X	X	X	X	X	X	X	X	X	X	X
Derives Soln. by Another Method												

Figure 1: *John's Sample Work*

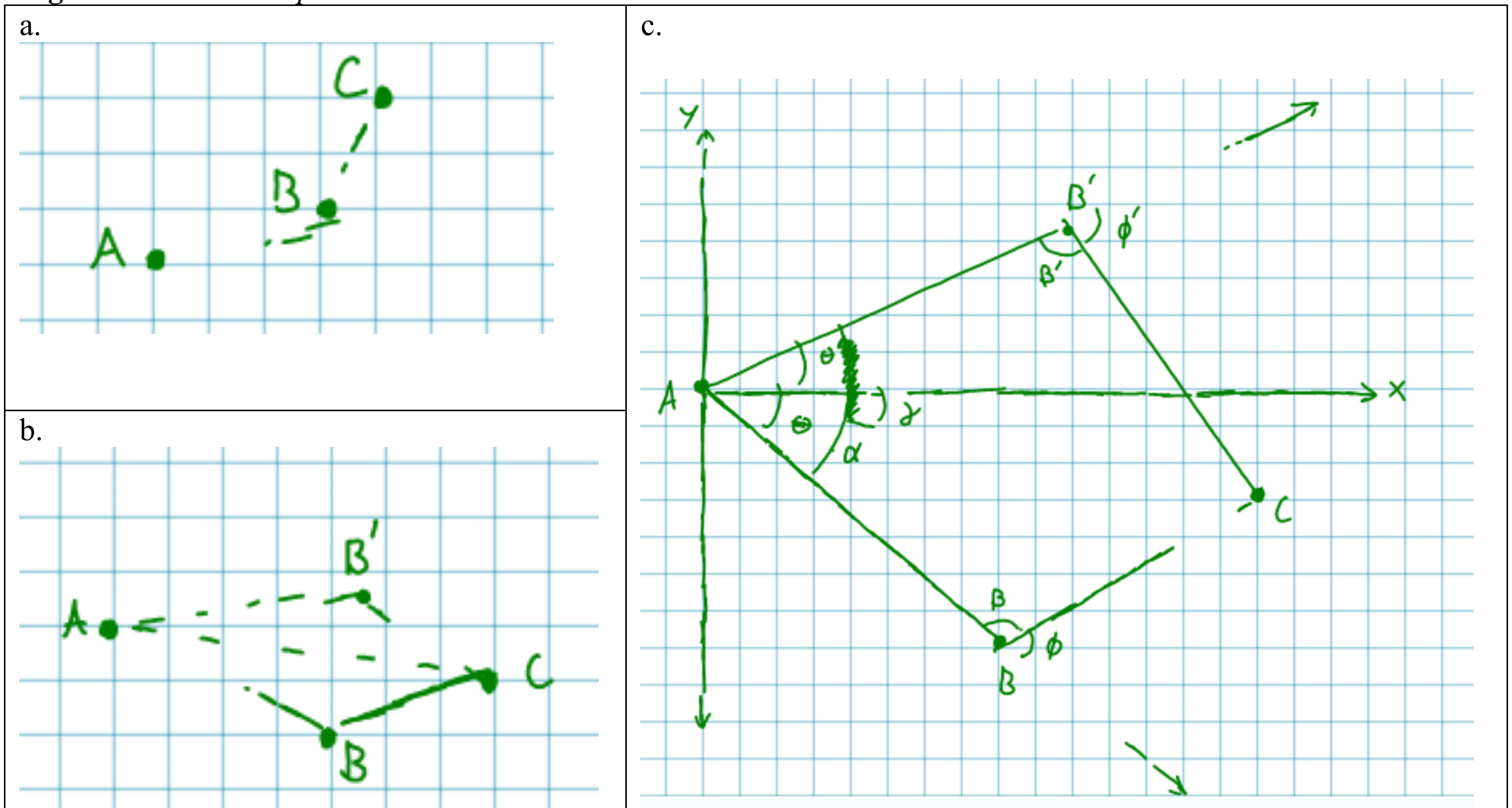


Figure 1: John's diagrams of the mathematical problems he was visualizing, with each diagram showing more details as he progressed through the problem-solving episode. He first visualized the three joints of the robotic arm with joint C over the apple. In his head, he said he was imagining a triangle (a). Then, he visualized two ways joint C to end over the apple. He said the two possible solutions formed a kite (b). John started visualizing the angles he needed to solve the problem (c).

Figure 2: *Adam's Sample Work*

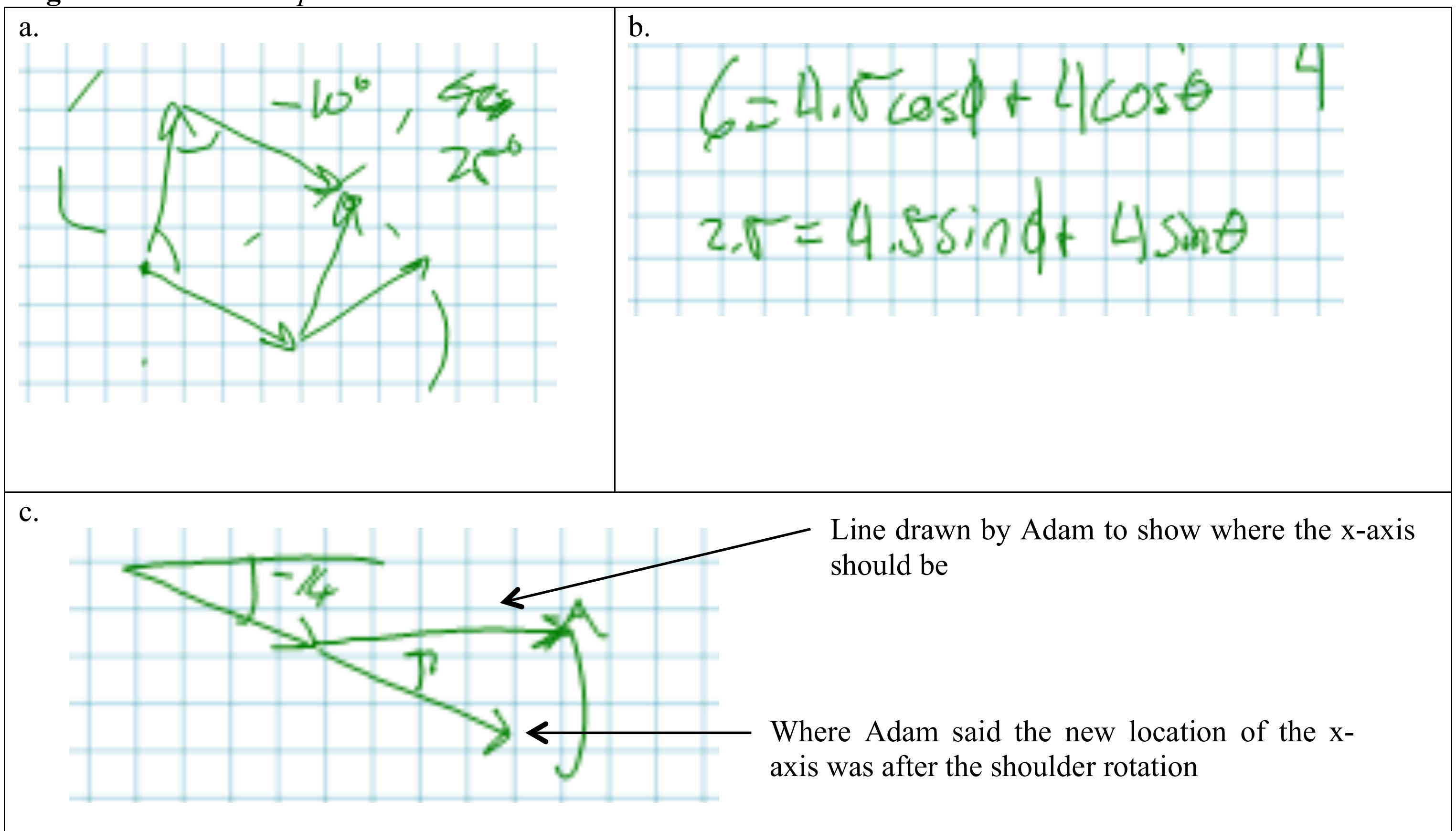


Figure 2: The two ways via which Adam imagined the robotic arm could reach the apple. Together, he described it as a "trapezoidal looking thing" (a). The system of equations Adam used to solve the robotic arm challenge (b). Adam's diagram depicting that the x-axis had a different location for the second vector.

Figure 3: *Sarah's Sample Work*

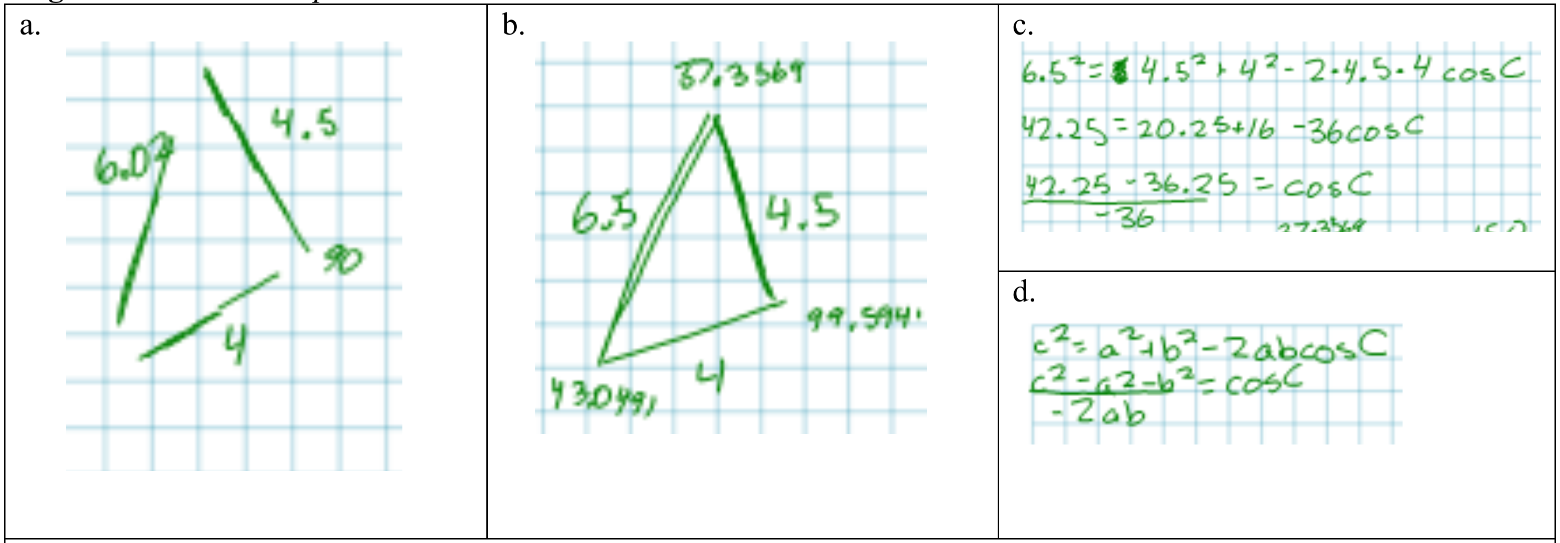


Figure 3: Sarah assumed one angle of the triangle was $90°$ and calculated the length of the side opposite to the angle to be 6.02. She noticed that it was close to 6.5 and then concluded that the angle opposite to 6.5 must be close to $90°$ (a). She used the Law of Cosines to figure out that the angle was in fact close to $90°$. It was $99.59°$ (b). She used the Law of Cosines to figure out the interior angles of the triangle (c). She double checked her computations from part b by solving the equation of the Law of Cosines symbolically for the angle. Then, she used a calculator to re-calculate the angles.

Figure 4: *Sarah Describing the Angle of Rotation of the Elbow*

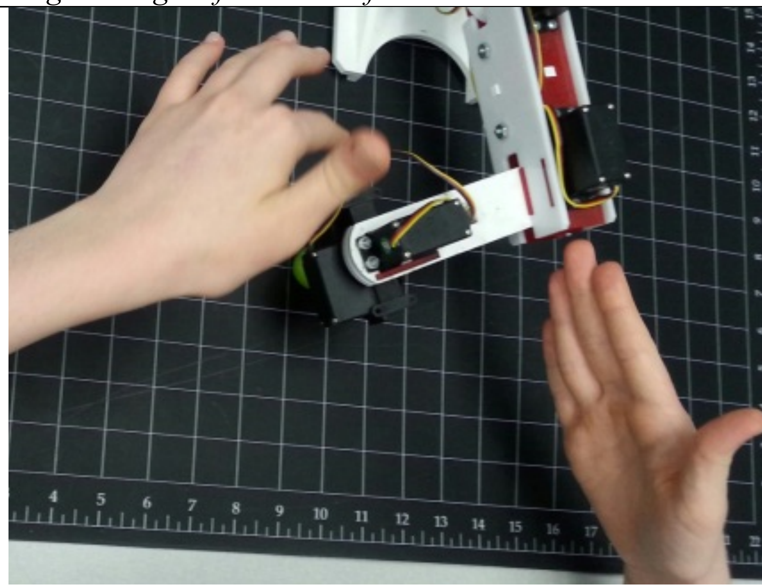
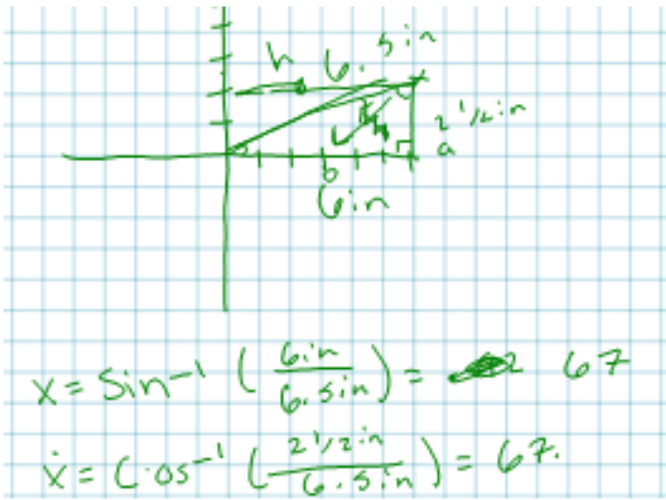


Figure 4: Sarah showed me with her hands the angle of rotation of the elbow.

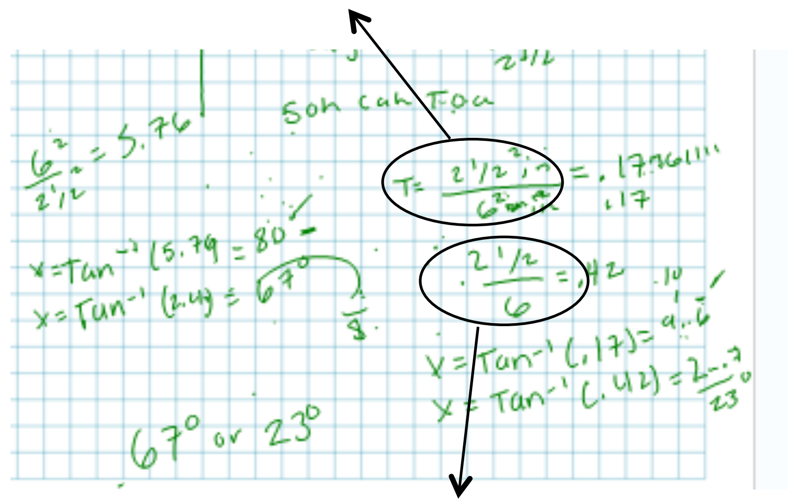
Figure 5: *Juana's Sample Work*

a.)



b.)

She squared the ratio (opposite/adjacent) resulting from tangent of an angle

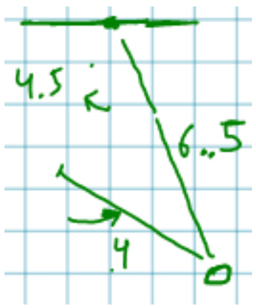


She did not squared the ratio (opposite/adjacent) resulting from taking the tangent of an angle

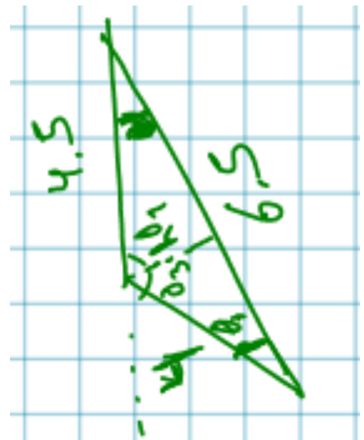
Figure 5: The right triangle Juana imagined followed by the two ways she used to find a missing angle (a). Juana's calculations when she was trying to recall whether the input of inverse trigonometry functions is squared or not (b).

Figure 6: *Mark's Sample Work*

a.



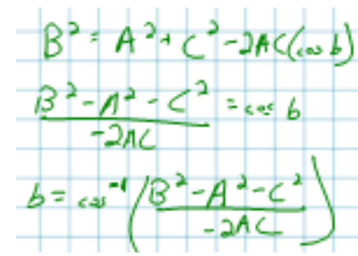
b.



c.

Angle of rotation of the elbow

Law of Cosines



Final answer for the shoulder rotation

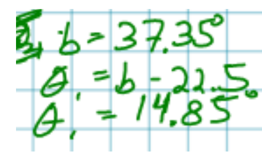
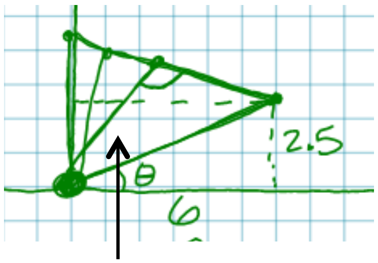


Figure 6: Mark's first diagram of the mathematical problem he imagined (a). Mark's diagram where Mark divided the obtuse triangle into two right triangles and thought about using a system of equations to figure out the internal angles (b). Mark's final diagram along with his calculations. He used the calculator for all his calculations (c).

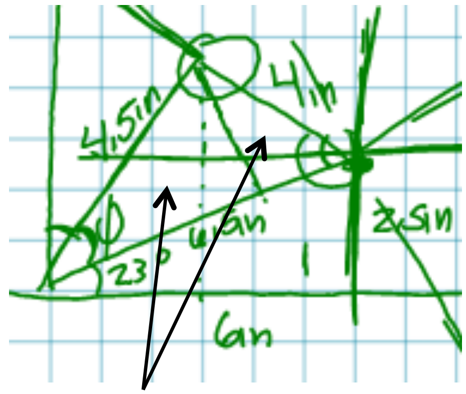
Figure 7: *Ebony's Sample Work*

a.



Obtuse triangle

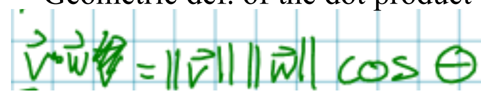
b.



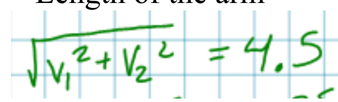
Obtuse triangle divided into two right triangles

c.

Geometric def. of the dot product



Length of the arm



Using trial and error to find a vector satistying the length of the arm

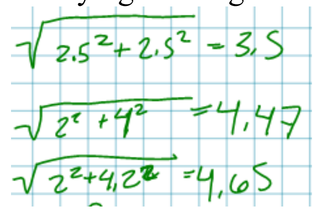
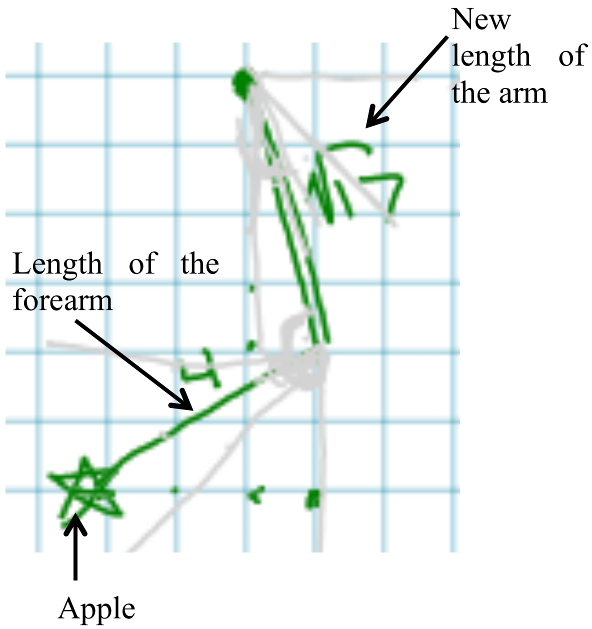


Figure 7. Ebony's diagram for how the robotic arm needed to be positioned to reach the apple – forming an obtuse triangle (a). Ebony divided the obtuse triangle into two right triangles (b). Ebony considered the geometric definition of the dot product of two vectors to find the interior angles of the obtuse triangle. She tried to find the vectors by using trial and error (c).

Figure 8: *Mena's Sample Work*

a.



b.

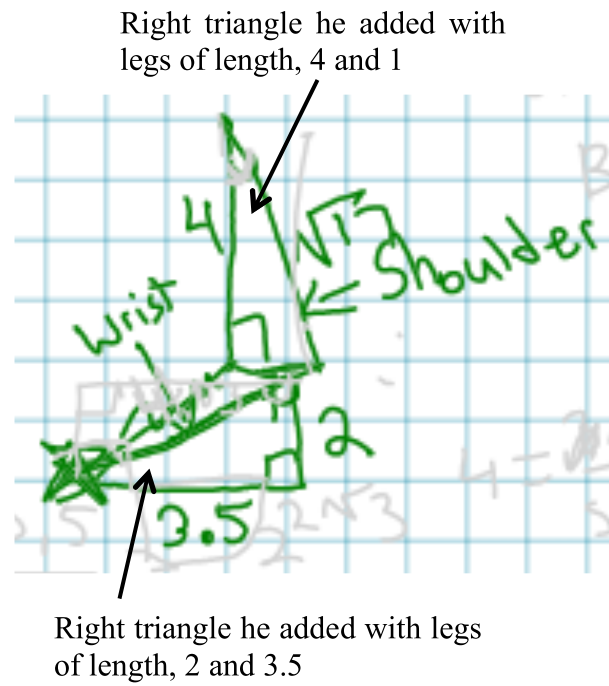


Figure 8: Mena's diagram of the robotic arm with the gripper over the apple. He changed the original length of the arm, 4.5, to $\sqrt{17}$ because this was the length of the hypotenuse of a right triangle he created which was close to 4.5 (a). Mena introduced right triangles to his diagram. He found the right triangles by trial and error. His goal was to find triangles with hypotenuse with lengths close to 4 and 4.5, the lengths of the arm and forearm of the robotic arm.

Figure 9: *Imani's Sample Work*

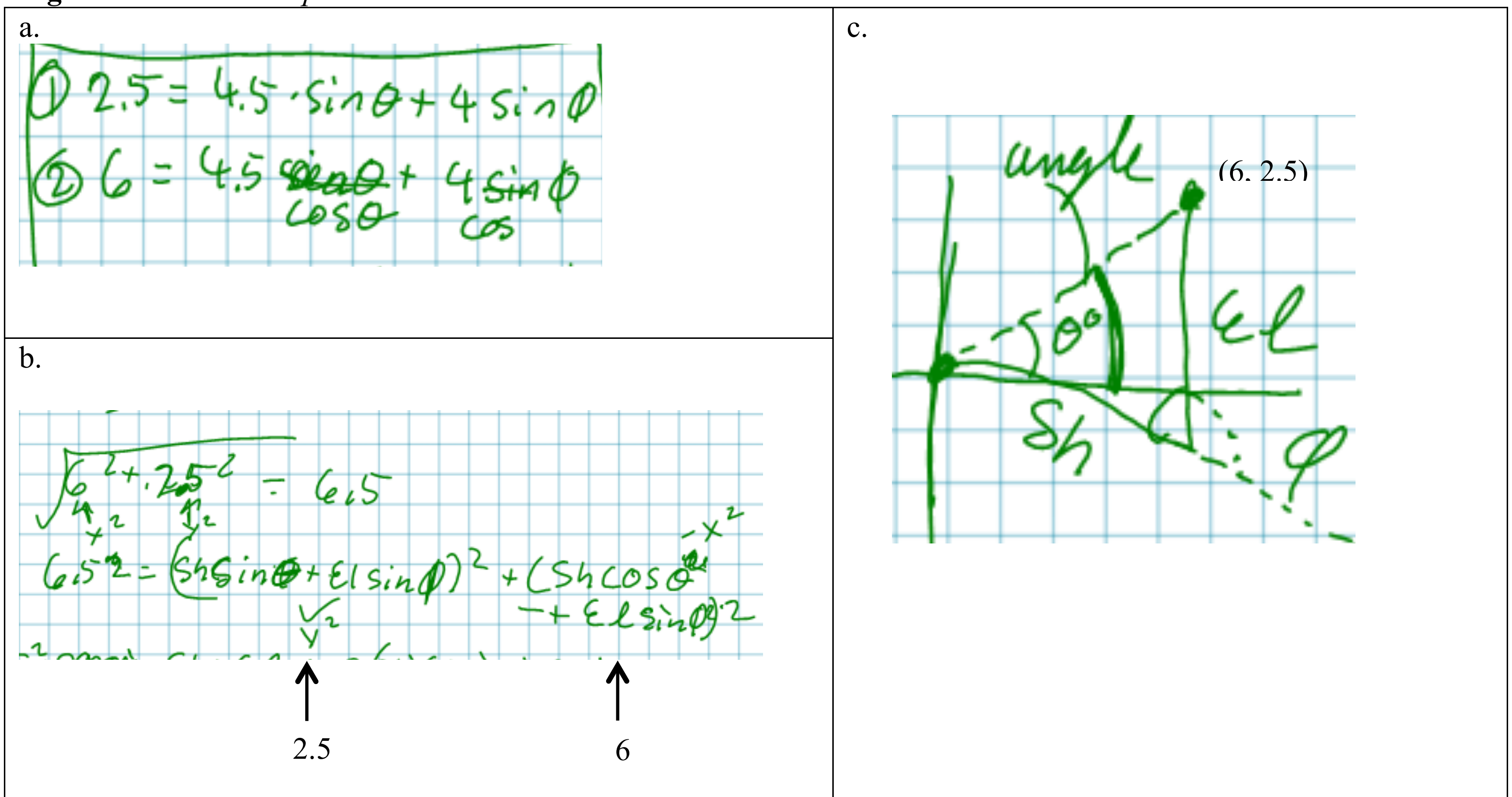


Figure 9. The system of equations Imani wrote by using vector addition (a). An additional equation Imani wrote by using the magnitude of the vectors and knowing that the distance from the apple to the shoulder joint was 6.5 (b). A diagram Imani drew after I prompted her to show me on paper what she was imagining. She drew the triangle she imagined and that influenced her to use the Law of Cosines. She told me the angle between the shoulder and θ was the angle of rotation of the shoulder and the angle φ was the angle of rotation of the elbow.

Figure 10: *Maisha's Sample Work*

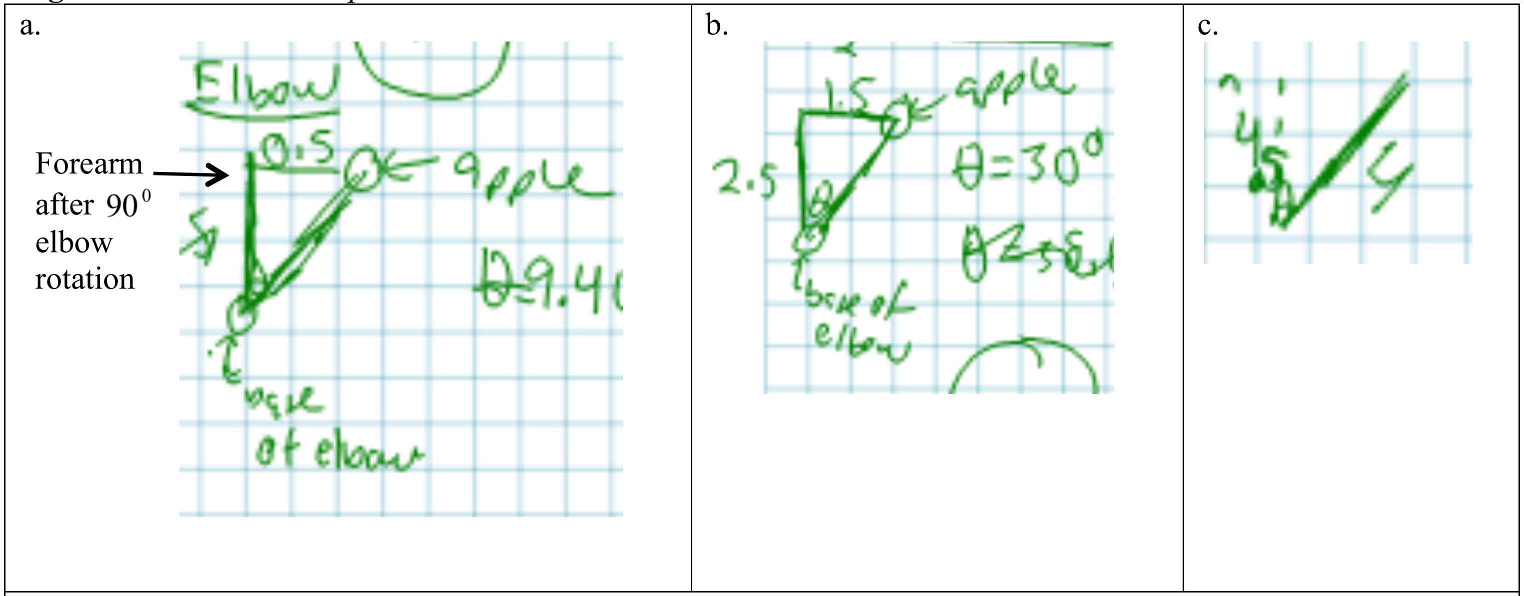


Figure 10. Maisha's diagram to find the additional elbow rotation after a 90^0 rotation at elbow (a). The right triangle she formed to find the rotation of the shoulder after the elbow rotation. She was confused at this point and she looked at the elbow even though she wanted to find the shoulder rotation (b). The second triangle she thought she could use to find the angle of rotation of the shoulder (c).

Figure 11: *Lola's Sample Work*

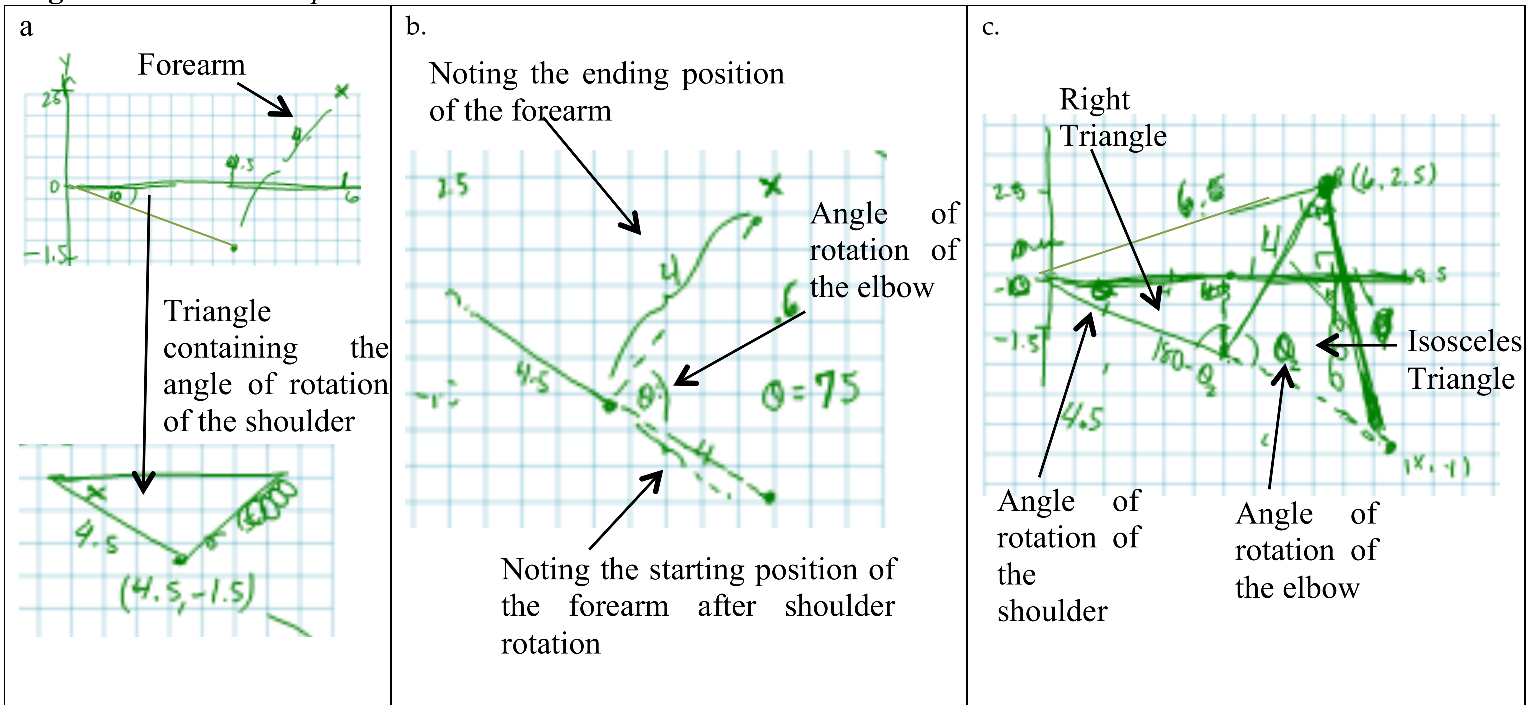


Figure 11. Although the forearm needed to be slanted to reach the apple, Lola decided to treat it as if it was perpendicular to the x-axis to find the y-coordinate of the location of the elbow after the shoulder rotation. The diagram also shows where Lola saw the angle of rotation of the shoulder (a). The diagram shows where Lola saw the angle of rotation of the elbow (b). Lola's diagram including all the information she knew about the problem and the unknowns she needed to find. The diagram also contains the right triangle and isosceles triangle that Lola focused on to attempt to solve the problem (c).

Figure 12: *Aniya's Sample Work*

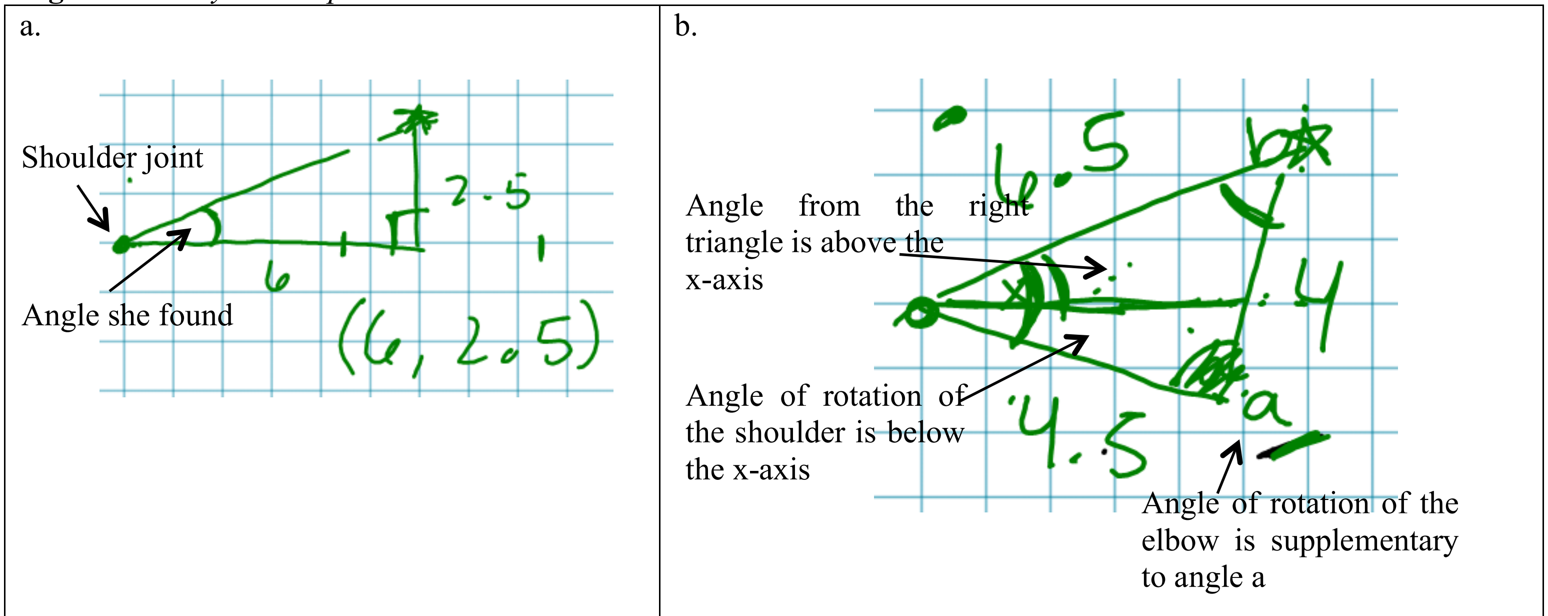


Figure 12. The right triangle Aniya used to figure out the angle of rotation of the shoulder (a). The scalene triangle that Aniya drew to find the angle of rotation of the shoulder and the elbow. She subtracted the angle she found of the right triangle from the angle, at the joint of the shoulder, of the scalene triangle to find the angle of rotation of the shoulder. Then, she noticed that the angle of rotation of the elbow was supplementary to angle a.

Figure 13: *Pablo's Sample Work*

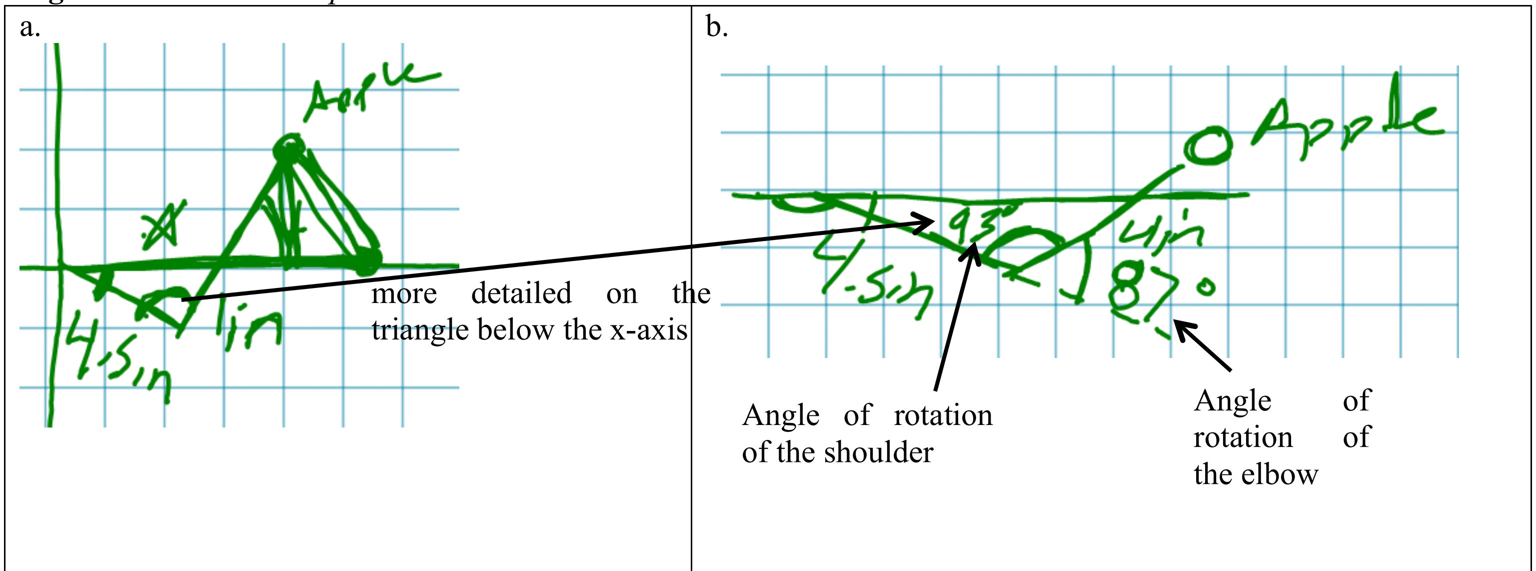


Figure 13. Pablos's diagram of the mathematical problem he imagined needed to be solved. The star denotes the length of the side of the triangle he did not know. He approximated one length of the second side of the triangle to be 1 inch, and he knew the arm was 4.5 inches (a). A more detailed diagram of the triangle Pablo was analyzing to figure out the angle of rotation of the shoulder. He found the 93° angle because he had calculated the angle of rotation of the shoulder to be 87°.

APPENDIX A:
Instrument for the Initial Interview

Participant's background

Hello, my name is Patty. What is your pseudonym? First, I want to thank you for making time to meet with me today. Before we start, I want to share with you more about my study. The main goal of my study is to understand how you think, and what you feel as you use mathematics to play with our robotic arm. That is, I am more interested in what you are thinking as you work to solve the robotics challenge, than I am in you getting the "right" answer. You can work on the robotics challenge for as long as you want to, but I am hoping to keep the entire study within a 6 hour interval. I will be asking you questions while you are tackling the robotics challenge to understand your thinking. Please don't get anxious or feel pressure if you think it is taking you too long to figure out the robotics challenge. I designed the robotics challenge to take a while to be solved - to give me time to understand what you are thinking. In short, I am slowing you down because it is hard for me to capture what you are thinking.

There are three parts to this study. First, there is an initial interview which will last at most half an hour. Second, there is a robotics challenge and you can work on it for as long as you want to. Lastly, the final interview will last at most half an hour. Please note that we can take any breaks in between that you may need. If you feel tired during the robotics activity, we can stop working and schedule another visit to continue working. I prefer to come another day or days to allow your mind to fully investigate the robotics challenge. Do you have any questions?

OK, we are going to get started with a few questions about your background in mathematics.

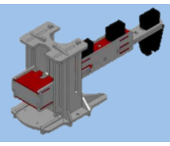
I. What has it been like to learn mathematics (in elementary school through the present)?"
   a. What makes you like or dislike learning about mathematics concepts?
   b. What motivates you to continue learning mathematics? Or, what discourages you from learning mathematics?
   c. Do your friends and family like mathematics?
   d. When you don't know how to solve a math problem, how do you approach the problem? What do you do to try to solve it?
   e. Do you think you are good at mathematics?
   f. What kind of grades do you earn in your math classes?
   g. What is the highest level of mathematics that you have taken? Years?
   h. How do you feel when you are solving math problems?
   i. Do you like robotics?
   j. On average how long does it take you to solve math problems?
   k. Have you ever lost track of time when you were solving a math problem?
   l. How much of your free time do you spend learning mathematics?
   m. Do you have a process/technique that you follow when you solve math problems?
   n. How was mathematics taught to you? Describe your learning experience (e.g. your teacher was working on the board most of the time, you had to

do constructions, you solve many problems that you had not seen before, etc.).

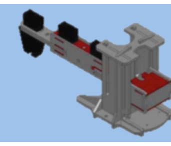
o. In your own words, what is mathematics?
p. Can everyone learn mathematics?
q. Is mathematics useful?

APPENDIX B:

Instrument for the Robotics-Based Task



Picking up an apple from a table



John suffers of quadriplegia – his arms and legs lost function after a car accident. As you may imagine, this has made it difficult for him to do many of his daily activities. He can't pick up his food, brush his teeth, change clothes, write, etc. John heard of software that can allow him to control a robot arm with his voice to help him accomplish some of his daily life activities again. This news was encouraging for John.

What John didn't know is that robotic arms, controlled by people with quadriplegia, are still under research. Today, you will have the opportunity to join the community of researchers who are working towards giving robotic arms the ability to function automatically – that is, a computer program controls its movements. Since eating on their own is of high importance, according to family members of people in need of a robotic arm, we are going to start by picking up apples with a robotic arm. The experiment will consist of two parts. The first part is designed to help you practice thinking aloud and to become comfortable with the robot arm.

Since I am interested in understanding your thinking processes, it will be most helpful if you would please say aloud what you are thinking from this point on. If you have not experienced saying out loud what you are thinking, it is going to feel weird at the beginning. It is also difficult to do it. You will be amazed at how much you think without being aware that you are thinking. For this reason, I will be asking you to tell me what you are thinking when you become silent. Also, if I need more details or don't understand something, I will ask you questions related to what you are thinking. Lastly, please share your feelings when you are talking out loud. For example, if you feel happy, motivated, excited, curious, frustrated, fearful, ashamed, proud, etc., make sure you say it out loud so that I know what you are feeling.

Please feel free to use any of the materials on the table (notebook, smart pen, and calculator). Do you have any questions? OK, let's get started.

Part 1: Use the computer keyboard to control the robotic arm to move where the two apples are, and pick up the apples.

Part 2: Use mathematics to control the robotic arm to move where the apple is. You don't have to pick up the apple, but the gripper of the robotic arm needs to be above the apple. If at any point you feel that you need someone to think with you, you can ask me questions or ask me for a hint. However, ask me for a hint as a last resource. It is important that you explore on your own as much as possible.

APPENDIX C:
Instrument for Robotics-Based Task Activity Script

Task-Based Interview Script-Picking up an apple from a table

Part I: Getting familiar with the robot arm. Controlling the robot arm with the computer keyboard

I. Pick up the apples or move the robotic arm to the location of the apples (the apple must be right below the gripper of the robotic arm). Inform the participant that the shoulder and elbow of the robotic arm only rotates $\pm 90^\circ$.
   a. Determine if the participant can approximate the angles of rotation of the shoulder and the elbow of the robot arm.
   b. Observe the heuristic process or processes the participant spontaneously use
   c. Observe other noteworthy occurrences, including expressions of affect;
   d. Note whether the participant states a coherent reason for what he or she did (e.g. circumference of circles).

Part II: Applying mathematics to picking up apples

I. Presentation of the main problem. Use mathematics to move the robotic arm so that the gripper is right above the apple. Avoid giving the participant heuristic suggestions.
   A. Understand the problem
      a. Determine if participant understands all the words used in the statement of the picking an apple robotic challenge.
      b. Determine if the participant understands what she was asked to find or show
      c. Observe if the participant can think of a picture or diagram that might help him/her understand the problem.
      d. Observe if the participant considered whether he/she has enough information to enable her to find a solution.
   B. Devise a plan
      a. Determine if the participant is willing to tackle the problem.
      b. Observe the heuristic process or processes the participant spontaneously use; Guess and Check, Look for a Pattern, Solve Simpler Problem, Draw a Figure, Work Backwards, Consider Special Cases, etc. Observe in detail how far the participant takes the process.
      c. Observe other noteworthy occurrences, including expressions of affect.
      d. Note whether the participant states a coherent reason for what she did.
      e. If participant seems to be lacking mathematics concepts such as knowledge of trigonometric ratios, inverses, Pythagorean Theorem, law of

cosines, etc., remind the participant that she can use any resources she wants.

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f. Angle of rotation of the elbow
    i. Observe whether the participant knows where the angle of rotation of the elbow starts and where it ends.
    ii. Observe if the participant uses Law of Cosine to find the angle adjacent to the angle of rotation of the elbow. Observe how the participant justifies her reasoning for using the Law of Cosines.
g. If participant asks for help, ask the participant what she/he will do if she/he was home alone.

II. Look back (see Appendix D for more detail)
    a. Note participant's feelings about the problem.
    b. Note participant's recognition of previously-encountered similar problems.
    c. Ask for a coherent retrospective account, asking for alternate methods or shortcuts;

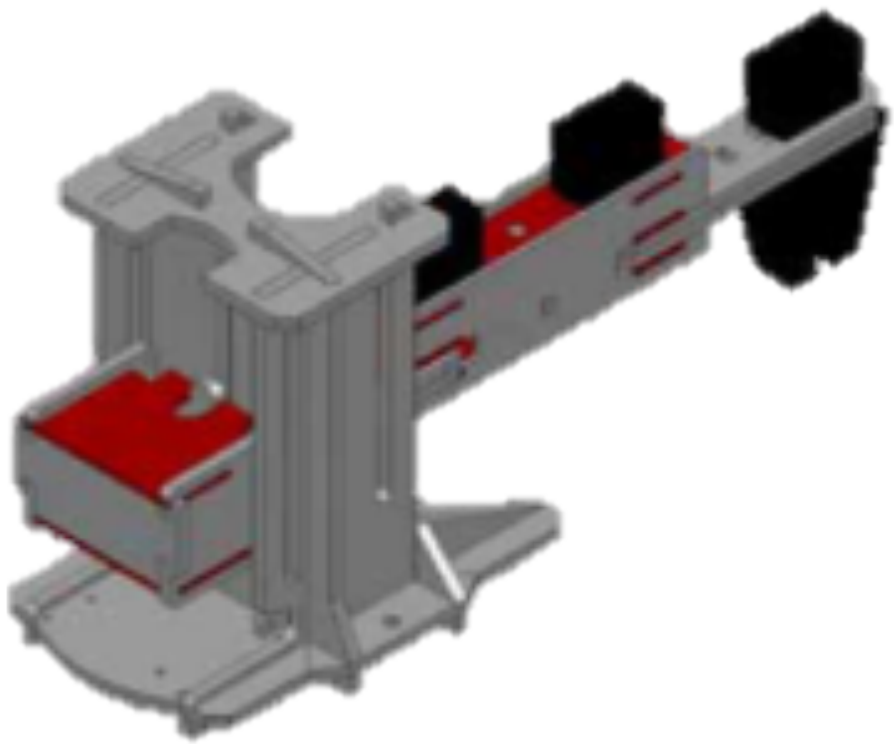
Correct conceptual misunderstandings which may have occurred during the problem-solving interview.

APPENDIX D:
Instrument for the Final Interview

1. How do you feel about the robotic arm challenge?
2. Have you solved other math problems similar to this one before?
3. Could you have solved the robotic arm challenge in a different way?
4. Tell me what did you think when you first started the robotic arm challenge? Did you create an initial plan about how to solve the problem?
5. Did you change your plans as you were solving the robotics arm activity? Why?
6. How did you decide to solve the problem the way that you did?
7. Tell me what strategies/ideas you used that helped you solve the robotic challenge? What made you think of that idea?
8. When you were doing the robotics challenge, it seemed that you were struggling at____________, then you figured it out. What did you think that helped you figure it out?
9. Did the robotics activity change the way you normally approach mathematics problems that you don't initially know how to solve? If yes, in what ways? How was it similar or different?
10. Tell me about a time when I asked you a question(s) that helped you think about what you could do to approach the robotic challenge?
11. Was there anything in the way the robotic challenge was presented to you that helped you think about possible ways to approach the problem? (e.g. wording, the way the robot arm was position, experiences you had in the past, etc.)
12. Do you think you will have approach this problem the same or differently without the robot arm? That is, if I had given you the problem in a worksheet, how do you think that will have changed how you approach the problem?
13. Do you feel that your understanding of a math concept improve during the robotics activity? If so, which ones and how?
14. Where you frustrated at any point? Did the frustration made you want to give up?
15. How will you say this robotic arm could be best used in a math class? Will you like to have this robotic arm in your math class?

APPENDIX E:
Robotic Arm Prototype Used in the Study

Robotic arm prototype



APPENDIX F:
Echo Pen Used in the Study

2GB Echo Smartpen

Saves notes and pencasts directly to the computer

- Records everything you hear, say and write, while linking your audio recordings to your notes.
- Replays audio directly from paper by tapping on your notes
- Echo Desktop software allows you to save, organize and play back interactive notes from your Mac or Windows computer.
- Share notes and pencasts as images.

APPENDIX G:
Flash Cards Used in the Study

Photograph of the set of flash cards

Front and back of each flash card

Front of Flash card

Law of Cosines--angles

$$c^2 = a^2 + b^2 - 2abCos(C)$$

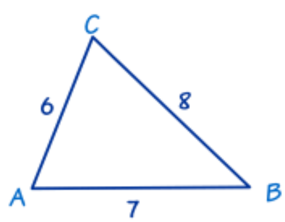
The law of cosines is a formula used to find a missing side or missing angle of any triangle – the triangle does not need to be a right triangle.

Missing: **an angle of the triangle**
Know: 3 sides of the triangle

Then you can use the command in matlab
Law_of_Cosines_missing_angle (side #1, side #2, side #3 is the side opposite to the missing angle)

Back of Flash card

Example: To find angle C in the triangle below, you use the command Law_of_Cosines_missing_angle (6, 8, 7) or
Law_of_Cosines_missing_angle (8,6,7) or



In short, it doesn't matter which sides of the triangle you label #1 and #2, but you must label side # 3, the side that is opposite to the angle that you are missing. In this case, the length of the side opposite to angle C is 7.

Law of Cosines—sides

$$c^2 = a^2 + b^2 - 2abCos(C)$$

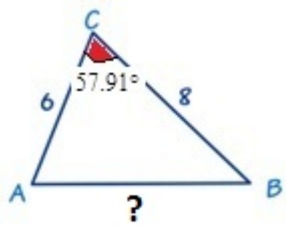
The law of cosines is a formula used to find a missing side or missing angle of any triangle – the triangle does not need to be a right triangle.

Missing: **a side of the triangle**
Know: 2 sides of the triangle and the angle between them

Then you can use the command in matlab
Law_of_Cosines_missing_side(known side #1, known side #2, known angle between side #1 and side #2)

Example: To find the length of the side opposite of angle C in the triangle below, you use the command

Law_of_Cosines_missing_side(6, 8, 57.91) or
Law_of_Cosines_missing_side (8,6,57.91) or



In short, it doesn't matter which sides of the triangle you label #1 and #2, but the missing angle must be between them.

Pythagorean Theorem

$$c^2 = a^2 + b^2$$

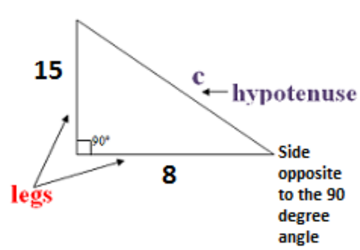
The Pythagorean Theorem is a formula used to find a missing side of a right triangle.

Missing: **the hypotenuse of a right triangle**
Know: 2 sides of the triangle (the two legs)

Then you can use the command in matlab
Pythagorean_Theorem(known leg #1, known leg #2)

Example: To find the hypotenuse, the longest side of a right triangle, in the triangle below, you use the command

Pythagorean_Theorem(15,8) or
Pythagorean_Theorem(8,15) or



In short, give the command the length of the two legs of the right triangle and the command will return the length of the hypotenuse, the side that is opposite to the 90 degrees angle.

Sine Inverse

Sine inverse is use to find angles in a right triangle.

Missing: **an angle of a right triangle**

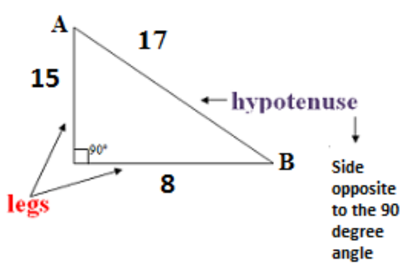
Known: 2 sides of the triangle- the side opposite to the angle and the hypotenuse

Then you can use the command in matlab

$$\text{asind}\left(\frac{\text{Length of the side opposite to the angle}}{\text{Lenght of the hypothenuse}}\right)$$

Example: To find angle A use the command

$$\text{asind}\left(\frac{8}{17}\right)$$



In short, give the command the 8 divided by 17 and the command will return the measurement of angle A in degrees. If you want the measurement of angle B, use the command asind($\frac{15}{17}$) because the length of the side opposite to angle B is 15 and the length of the hypotenuse is 17.

Cosine Inverse

Cosine inverse is use to find angles in a right triangle.

Missing: **an angle of a right triangle**

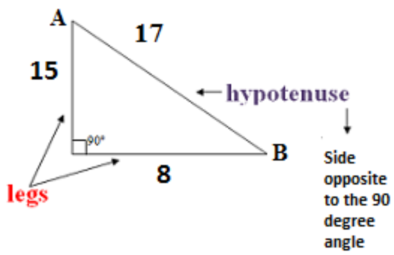
Known: 2 sides of the triangle- the side adjacent to the angle and the hypotenuse

Then you can use the command in matlab

$$\text{acosd}\left(\frac{\text{Length of the side adjacent to the angle}}{\text{Lenght of the hypothenuse}}\right)$$

Example: To find angle A use the command

$$\text{acosd}\left(\frac{15}{17}\right)$$



In short, give the command 15 divided by 17 and the command will return the measurement of angle A in degrees. If you want the measurement of angle B, use the command acosd($\frac{8}{17}$) because the length of the side adjacent to angle B is 8 and the hypotenuse is 17.

Tangent Inverse

Tangent inverse is use to find angles in a right triangle.

Missing: **an angle of a right triangle**

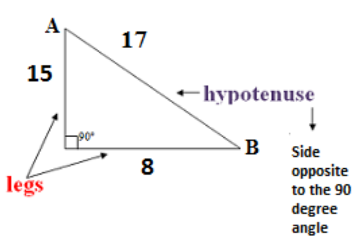
Known: 2 sides of the triangle- the side opposite to the angle and the side adjacent to the angle.

Then you can use the command in matlab

$$\text{atand}\left(\frac{\text{Length of the side opposite to the angle}}{\text{Length of the side adjacent to the angle}}\right)$$

Example: To find angle A use the command

$$\text{atand}\left(\frac{8}{15}\right)$$



In short, give the command 8 divided by 15 and the command will return the measurement of angle A in degrees. If you want the measurement of angle B, use the command atand($\frac{15}{8}$) because the length of the side opposite to angle B is 15 and the length of the side adjacent to angle B is 8.

APPENDIX H:
Instrument – Coding Sheet with Heuristic Definitions Used in the Study

Additional heuristics were added to Kilpatrick's (1967) coding system for this study.

MODIFIED CODING FORM FOR PROBLEM-SOLVING PROTOCOLS

Participant Pseudo name________________ Date_______________
Coding no.____________________________ Time_______________

PREPARATION
____ Draws Figure
____ Changes Condition (spec./gen./varying)
____ Performs Exploratory Manipulation

RECALL/ACCESS RESOURCES
____ Recalls Same or Related Problem
____ Uses Related Problem in Solution
____ Says he has forgotten procedure

PRODUCTION
____ Uses Successive Approximation
____ Sets Up Equation(s)
____ Misinterprets problem
____ Selects soln. on irrelevant basis
____Uses Auxiliary Elements/Problem
____ Works Backwards
____ Decomposition (use of Subproblems)

EVALUATION
____Checks Soln.
____Derives Soln. by Another Method

Other heuristic(s)

COMMENTS ABOUT SOLUTION
____ Questions existence of solution
____ Questions uniqueness of solution
____ Questions necessity/relevance of info.
____ Expresses uncertainty about final soln.
____ Says he doesn't know how to solve prob.

REQUESTS/SEARCHES
____ Requests assistance, more info.
____ Requests verification
____Searchers for______________
using_________

COMMENTS
____ Expresses enjoyment, liking for prob.
____ Expresses dislike, uninterested in prob.
____ Admits confusion
____ Shows concern for performance
____ Says procedure unorthodox
____ Says he can't explain result

EXECUTIVE ERRORS
Count / arith. Oper.
Alg. Manipulation
Other slips

PROCESS SEQUENCE:

Preparation Category

Draws Figure is checked if the participant makes any kind of figure or diagram, however primitive or incomplete, as long as it is apparently intended as an aid to understanding. A line or two drawn and then hastily erased does not count.

Changes Condition (Spec, /Gen./Vary) refers to any conscious attempt by the participant to modify the condition of the problem by imposing additional constraints (Specialization), by removing part of the condition (Generalization), or by changing the condition in some other way (Varying the Problem). This behavior occurs very rarely. It does not include inadvertent modifications of the problem by participants who have misread or forgotten parts of the condition.

Performs Exploratory Manipulation refers to situations in which the participant performs some operation on the data of the problem to see what the result would be rather than to test an idea he has about the solution. Such situations are usually accompanied by remarks that the participant is trying this operation just to see what he gets, but sometimes they can be identified when the participant simply manipulates the data and then goes on to something else. When the coder is in doubt, especially when the manipulation has led to a useful result, the participant is given credit for some (vague) plan and the behavior is not checked.

The Recall Category

This category includes behavior related to the participant 's memory of similar problems.

Recalls Same or Similar Problem is checked if the participant mentions that he has seen a particular problem or one like it before. This behavior is not coded if the participant merely mentions a related mathematical topic: "This is a trigonometry problem ," or "I'll need to set up a proportion."

Uses Related Problem in solution is checked if the participant uses either the method or result from a related problem in his solution. Sometimes this behavior must be inferred from the participant's statement that he recognizes the problem, followed by a straightforward solution. If the participant uses a related problem, it is assume that he recall a related problem as well and the coder should check the **recall same or similar problem** code.

If the participant says he has forgotten a particular procedure or technique, whether or not he recalls a related problem, the coder checks **Says he has forgotten procedure**. The participant may not directly say that he has forgotten the procedure, but he may mention that he is trying to remember how to solve the problem.

The Production Category

This category includes behavior related to the generation of a solution.

Uses Successive Approximation is checked if the participant's remarks indicate that he is consciously using information from previous trial values in making subsequent trials. It does not apply to haphazard trials, regardless of their success.

Sets Up Equations is checked if the participant expresses the math problem in mathematical symbols and writes equations using the mathematical symbols.

If the participant indicates through his activities and comments that he has somehow misread or misinterpreted a critical part of the problem, although he may understand very well various relationships among problem elements, the coder checks **Misinterprets problem**.

Selects soln. on irrelevant basis means that although the participant may have used correct reasoning for part of the problem, his final choice of a solution has an arbitrary, irrelevant, or superficial basis; e. g . , "I'll put that as my answer because it's about the only number I haven't tried, " or "The guilty one is Sam because he's the first one mentioned. "

Uses Auxiliary Element is coded when the participant adds an element (e.g. lines, unknowns, solves theorems to help find the solution of the problem at hand, etc.). For example, the participant might introduce a line to his image to form a triangle in order to help him solve the problem.

Uses Auxiliary Problem is coded when the participant introduces a new problem and solves it in the hope that the solution will help solve the original problem.

Works Backwards is coded when the participant visualizes the final solution to solve the problem. The participant starts solving the problem by assuming he has what he is looking for (the final solution). For example, the participant may need to move the robot arm to a certain location. The participant may attempt to solve the problem by assuming the robot arm is at the desire solution and inquiring what is needed to arrive at that result.

Decomposition is coded when the participant's attention is focus on parts of the problem instead of the entire problem. That is, the participant divides the problem into subproblems in order to solve the entire problem.

The Evaluation Category

This category applies only to the checking of values for unknowns or of the arguments used to derive such values. It does not apply to checks on an operation that are performed during the operation itself because such checks are too difficult to distinguish from other behavior.

Checks Soln. is coded if the participant explicitly repeats an operation after he has obtained a solution. Also, it is coded if the participant substitutes his solution value directly in the original problem to test whether the relationships among problem elements hold under such a substitution. Sometimes the participant merely reads parts of the problem over again, and the coder must infer whether such a check has been attempted. It is also check if the problem solver uses technology to check if his solution satisfies the condition. Checks soln. is also coded if the participant indicates that he has tested whether his solution is reasonable either in terms of the problem context alone or in terms of the "real world". Unfortunately such tests are seldom detected unless they fail; i.e, participants frequently report that a result is unrealistic or nonsense, but they rarely say that a result is reasonable even though they may think so.

Derives Soln. by Another Method is coded only if the participant, after using one procedure to derive a solution, attempts to derive the solution by a different procedure—a rare phenomenon. The phrase does not refer to a second attempt at a problem after the first solution has been discarded.

In coding **evaluation behavior**, one may find it difficult to distinguish between checking a provisional solution and trying to solve the problem again after an unsuccessful attempt. Unless the participant's other behavior and comments indicate that he is attempting to check his work, the behavior should not be counted as a check.

Comments About Solution

This category includes several kinds of comments that the participant may make about the solution.

Questions existence of solution refers to remarks indicating that the participant doubts there is a solution to the problem.

Questions uniqueness of solution refers to any remarks to the effect that there is insufficient information for a unique solution.

Remarks that there is no solution because of insufficient information are coded as "uniqueness" rather than "existence" comments.

Questions necessity/relevance of information refers to remarks indicating that the participant thinks some of the data are not necessary for a solution.

Expresses uncertainty about final soln. is checked only if the participant, as he leaves the problem, makes remarks indicating that he doubts the correctness of his solution. It is not checked if the participant expresses doubt and then attacks the problem again or if he has obtained no solution.

Says he doesn't know how to solve problem is almost self-explanatory. The participant must make the remark as he stops work on the problem. Statements that he does not understand the problem are not coded.

The Request Category

This category consists of two kinds of questions addressed to the interviewer.

Requests for clarification of the problem or for assistance in solving it are coded as **Requests assistance, more info**.

Requests for the correct solution or for some indication as to how well the participant is doing are coded as **Requests verification**. Rhetorical questions or procedural questions ("Where do I put my answer?") are not coded.

Searches for definitions is checked if the participant searches for the defition of a mathematics concept or for a theorem. The participant may look at a book, do a search on the web, look over old notes, etc. for the information.

The Comments Category

This category includes remarks tangential to the task of solving the problem. **Expresses enjoyment, liking for prob**. is coded if the participant shows pleasure in the task or expresses positive feelings about any aspect of the situation.

Expresses distaste, dislike for prob. is coded if the participant shows displeasure or expresses negative feelings.

Admits confusion is coded if the participant says he is confused at present or was confused earlier.

Shows concern for performance is coded if the participant makes remarks indicating that he is bothered by some aspect of his performance, such as slowness or ineptitude.

Says procedure unorthodox means that the participant believes there is some preferred method which he either does not know or would rather not use.

Says he can't explain result is coded if the participant says he does not know why he performed a particular operation or how he obtained a particular result.

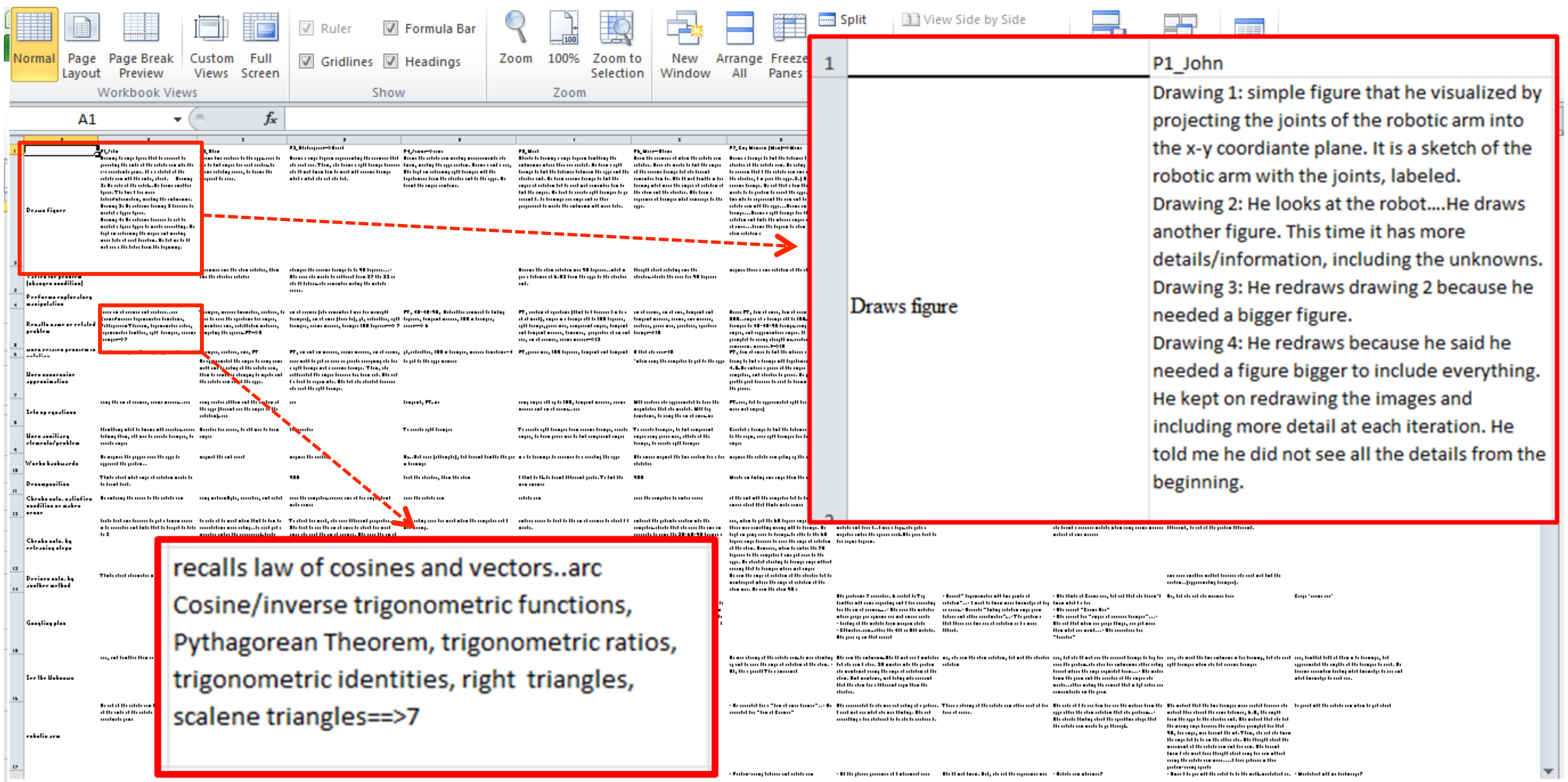
The Executive Errors category is used to code errors in carrying out manipulations. These errors result primarily from defects in concentration, attention, or immediate memory rather than from failures in understanding. Executive errors are tallied each time they are made; if an error is repeated it is tallied again. Errors are not coded if they are immediately retracted or corrected.

Errors in counting or in performing arithmetic operations are tallied as
Count, /arith. oper.

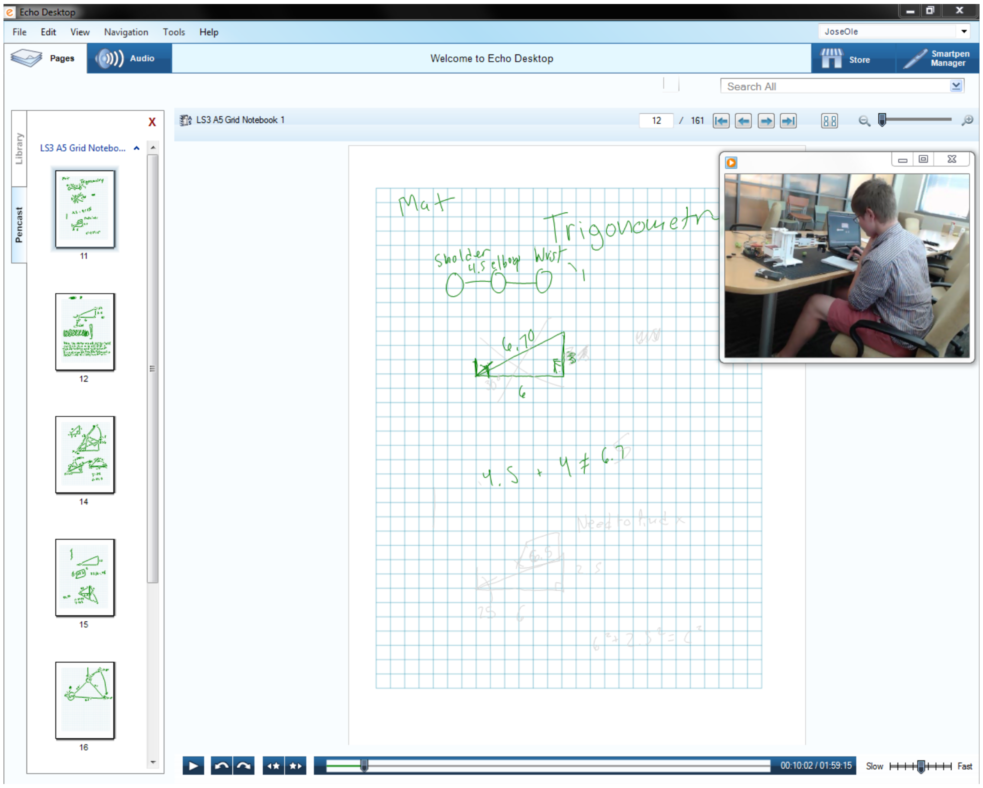
Non-arithmetic errors in the manipulation of equations are tallied as **Alg. Manipulation**.

All other executive errors, such as miscopying or misreading a number, are tallied as **Other slips**.

APPENDIX I:
Matrix Display Used to Keep Track of How Each Participant Employed Each Heuristic



APPENDIX J:
Technology Used for Recording and Analyzing the Problem-Solving Episode



To analyze the data recorded from the smartpen and the camera, I played both recordings simultaneously (as shown above). The above image of my computer screen illustrates how I view the recorded data. In the right-upper corner, I played the recording from the video camera. The data from the smartpen is in green and grey writing. The writing is initially in grey, but the livescribe software turns the writing into green as it plays the movie of what the participant said as he was writing.

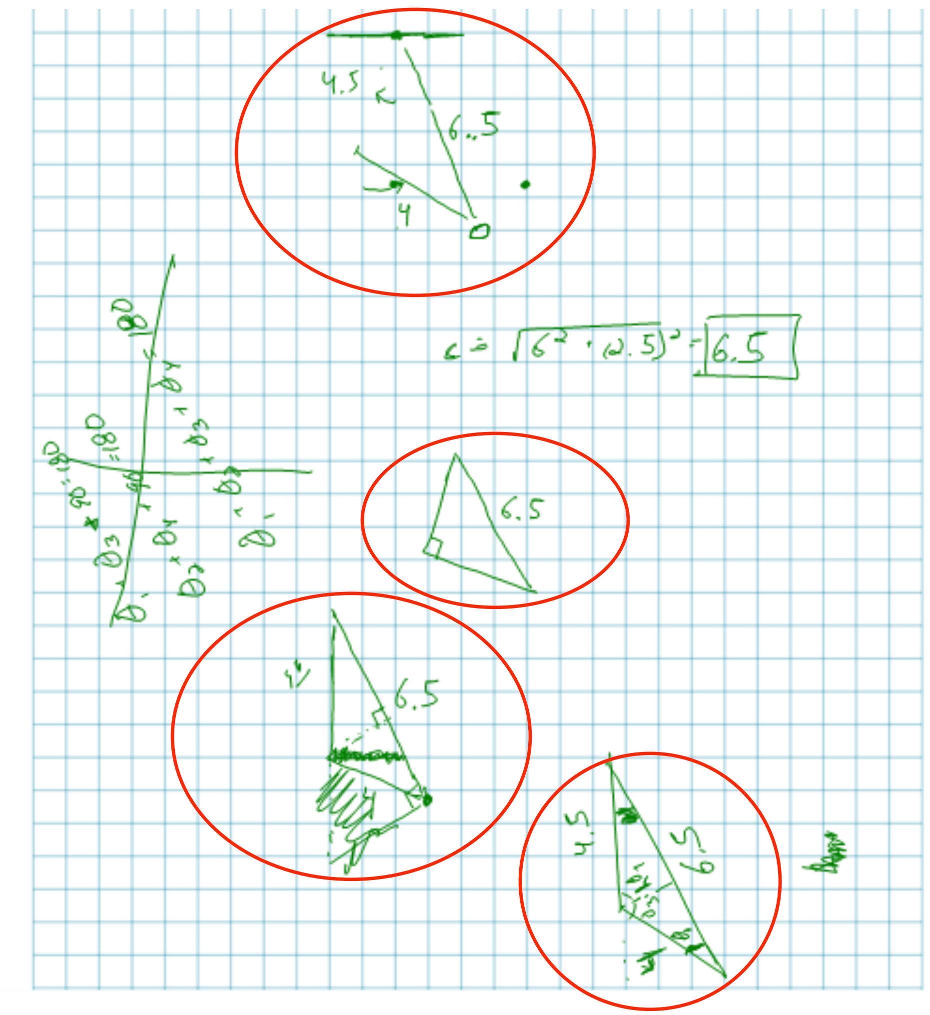
Although both recordings were played simultaneously, at times, I had to focus entirely on either the data from the smartpen or the video camera during the analysis of the data.

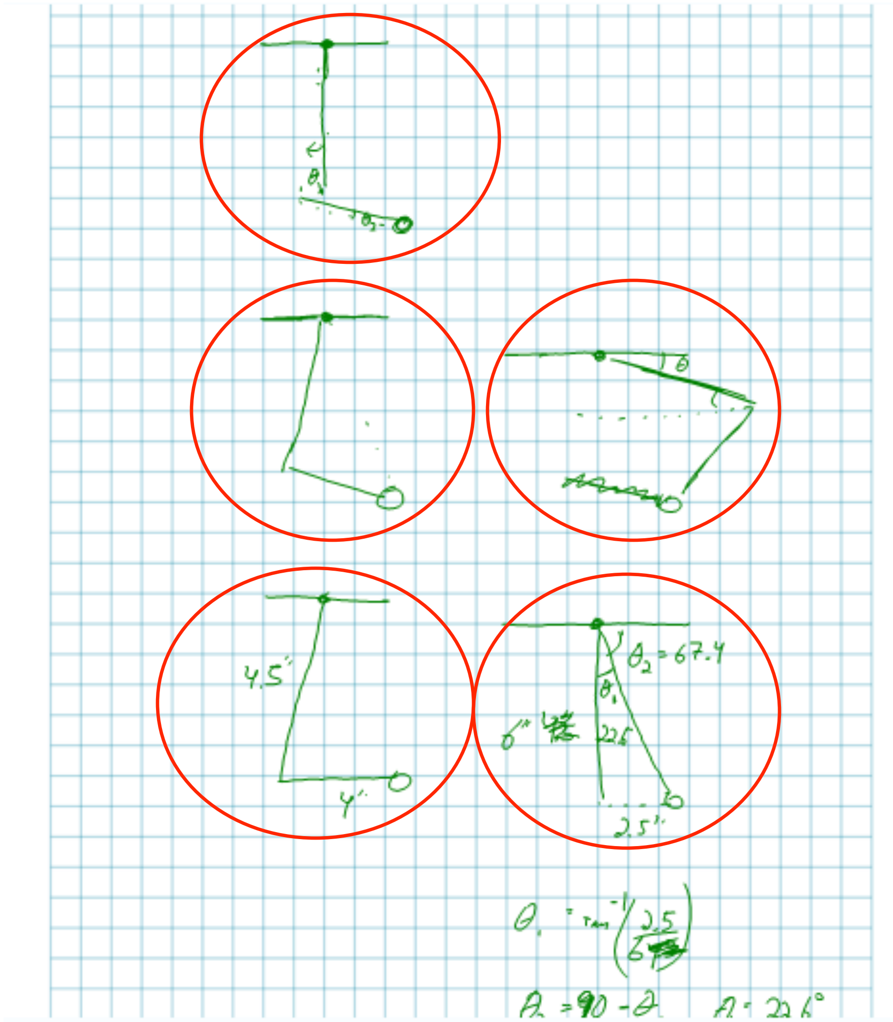
APPENDIX K:
Matrix Used to Keep Track of Information from the Initial and Final Interviews

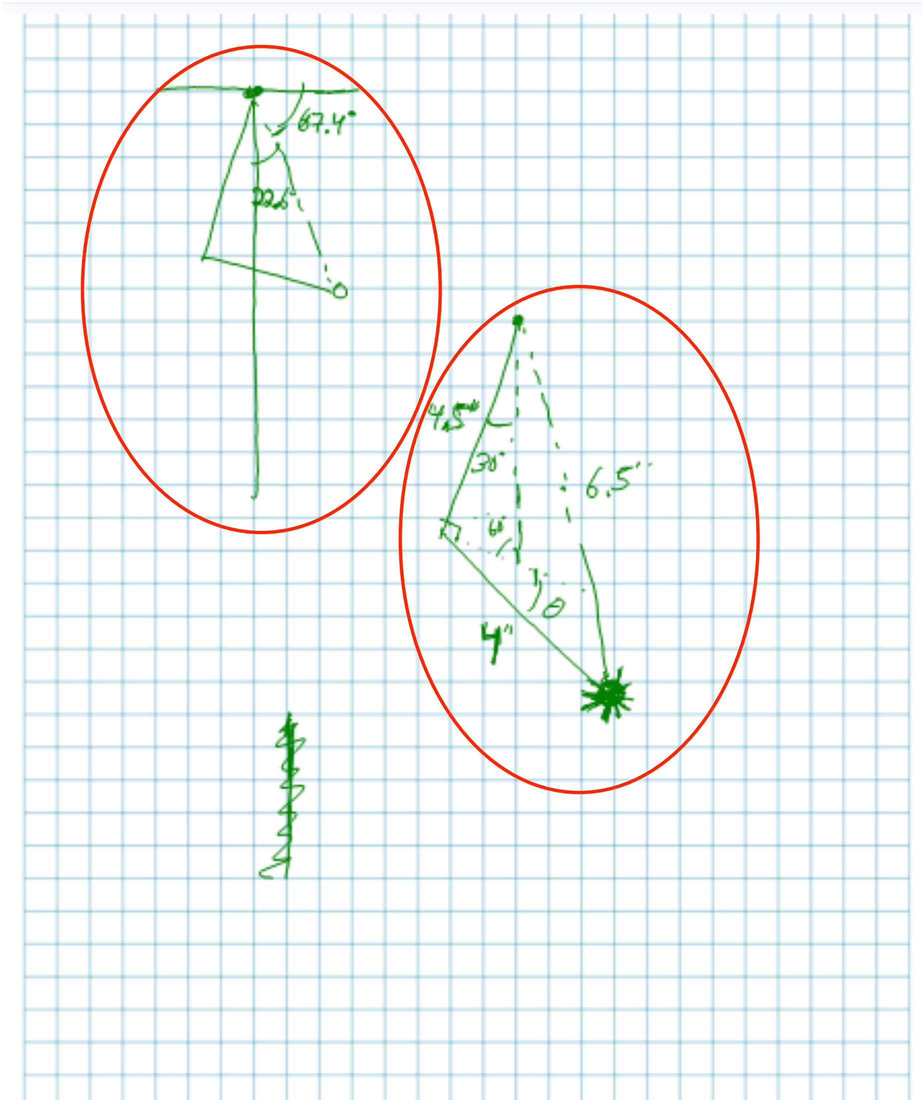
		PARTICIPANTS' MATH/ROBOTS BACKGROUND					PARTICIPANTS' FEEDBACK AND HEURISTICS USED REFLECTION				
		Likes/ Dislikes Math	Likes/ Dislikes Robots	Math Support System	Strategies Normally Used to Solve Nonroutine Problems	Def. of math	Likes/ Dislikes Robotics -Based Task	Cognizant Use of Heuristics and Triggers	Robotic-Arm Influenced on Solving Nonroutine Problem	Most Helpful Heuristic(s)	Notes
PARTICIPANTS	1										
	2										
	3										

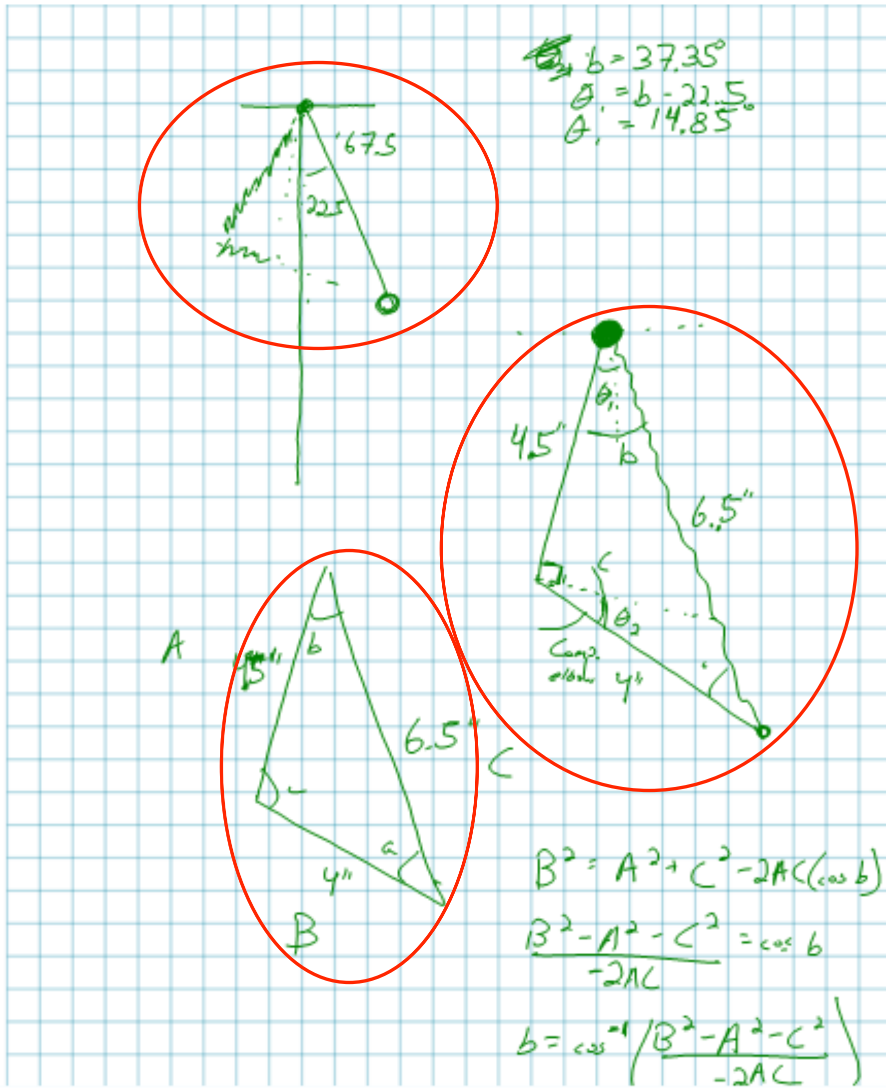
APPENDIX L:
Participant 5: Mark's Work and Sample of Counting of Figures Drawn

Mark drew 15 figures. I circled each figure that was counted to make it easier for the reader to identify the 15 figures. All the figures that he drew were counted, regardless of quality.




4.5''
4''
θ2 = 67.4
6''
22.6
2.5''


67.4°
22.6°
4.5"
30°
60°
6.5"
θ
4"


b = 37.35°
θ₁ = b - 22.5°
θ₁ = 14.85
67.5
22.5
4.5"
b
6.5"
θ₂
4"
A
b
6.5"
C
a
4"
B
B² = A² + C² - 2AC(cos b)
(B² - A² - C²) / -2AC = cos b
b = cos⁻¹((B² - A² - C²) / -2AC)

$$C^2 = A^2 + B^2 - 2AB(\cos C)$$

$$\cos^{-1}\left(\frac{C^2 - A^2 - B^2}{-2AB}\right) = C$$

$$C = 100^\circ$$

$$\theta_2 = 100 - 90 = 10^\circ$$

$$\theta_1 = 14.85$$

$$\theta_2' = 80^\circ$$