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Eulerian series, zeta functions and the arithmetic of partitions

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Abstract

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In this dissertation we prove theorems at the intersection of the additive and multiplicative branches of number theory, bringing together ideas from partition theory, q-series, algebra, modular forms and analytic number theory. We present a natural multiplicative theory of integer partitions (which are usually considered in terms of addition), and explore new classes of partition-theoretic zeta functions and Dirichlet series — as well as "Eulerian" q-hypergeometric series — enjoying many interesting relations. We find a number of theorems of classical number theory and analysis arise as particular cases of extremely general combinatorial structure laws.

Among our applications, we prove explicit formulas for the coefficients of the q-bracket of Bloch-Okounkov, a partition-theoretic operator from statistical physics related to quasi-modular forms; we prove partition formulas for arithmetic densities of certain subsets of the integers, giving q-series formulas to evaluate the Riemann zeta function; we study q-hypergeometric series related to quantum modular forms and the "strange" function of Kontsevich; and we show how Ramanujan's odd-order mock theta functions (and, more generally, the universal mock theta function g_3 of Gordon-McIntosh) arise from the reciprocal of the Jacobi triple product via the q-bracket operator, connecting also to unimodal sequences in combinatorics and quantum modular-like phenomena.

"All of analysis will one day be subsumed by the theory of partitions."

– J. J. Sylvester[1]

[1] As paraphrased to the author by G. E. Andrews

For Marci and Max

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Robert Schneider

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Contents

Chapter 1

Setting the stage: Introduction, background and summary of results

1.1 Visions of Euler and Ramanujan

In antiquity, storytellers began their narratives by invoking the Muses, whose influence would guide the unfolding imagery. It is fitting, then, that we begin this work by praising Euler and Ramanujan, whose imaginations ranged playfully across much of the landscape of modern mathematical thought.

1.1.1 Zeta functions, partitions and q-series

One marvels at the degree to which our contemporary understanding of q-series, integer partitions, and what is now known as the Riemann zeta function all emerged nearly fully-formed from Euler's pioneering work [And98, Dun99].

Euler made spectacular use of product-sum relations, often arrived at by unexpected avenues, thereby inventing one of the principle archetypes of modern number theory. Among his many profound identities is the product formula for the Riemann zeta function, in which the sum and product converge for $\mathrm{Re}(s) > 1$:

$$\zeta(s) := \sum_{n=1}^{\infty} n^{-s} = \prod_{p \in \mathbb{P}} (1 - p^{-s})^{-1}. \tag{1.1}$$

With this relation, Euler connected the (at the time) cutting-edge theory of infinite series to the timeless set $\mathbb{P}$ of prime numbers — and founded the modern theory of L-functions. Moreover, in his famed 1744 solution of the "Basel problem" posed a century earlier by Pietro Mengoli, which was to find the value of $\sum_{n=1}^{\infty} 1/n^2$, Euler showed how to compute even powers of π — a constant of interest to mathematicians since ancient times — using the zeta function, giving explicit formulas of the shape

$$\zeta(2N) = \pi^{2N} \times \text{ rational}. \tag{1.2}$$

The evaluation of special functions such as $\zeta(s)$ is another rich thread of number theory. As we will show in this work, there are other classes of zeta functions (not to mention other formulas for π) arising from the theory of partitions.

Let $\mathbb{N}$ denote the natural numbers $1, 2, 3, 4, 5, \ldots$, i.e., the positive integers $\mathbb{Z}^+$ (we use both notations interchangeably)[1]. We shall now fix some standard notations.

Definition 1.1.1. *Let $\mathcal{P}$ denote the set of all integer partitions. For $\lambda_i \in \mathbb{N}$, let*

$$\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_r), \qquad \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_r \geq 1,$$

denote a generic partition, including the empty partition $\emptyset$. Alternatively, we sometimes write partitions in the form $\lambda = (1^{m_1}\ 2^{m_2}\ 3^{m_3} \ldots\ k^{m_k} \ldots)$ with $m_k = m_k(\lambda) \geq 0$ representing the multiplicity of k as a part of $\lambda \in \mathcal{P}$ (we adopt the convention $m_k(\emptyset) := 0$ for all $k \geq 1$). We note that λ has only finitely many parts with nonzero multiplicity.

[1] Prof. Paul Eakin at University of Kentucky once said, "Whenever integers appear, magic happens."

Definition 1.1.2. *Let*

$$\ell(\lambda) \ := \ r = \ m_1 + m_2 + m_3 + ... + m_k + ...$$

denote the length of λ (the number of parts), and

$$|\lambda| \ := \ \lambda_1 + \lambda_2 + \lambda_3 + \cdots + \lambda_r \ = \ m_1 + 2m_2 + 3m_3 + ... + km_k + ...$$

denote its size (the number being partitioned), with the conventions $\ell(\emptyset) := 0$, $|\emptyset| := 0$. We write "$\lambda \vdash n$" to mean λ is a partition of n, and "$\lambda_i \in \lambda$" to indicate $\lambda_i \in \mathbb{N}$ is one of the parts of λ.

For example, we might take $\lambda = (4, 3, 2, 2, 1) = (1^1\ 2^2\ 3^1\ 4^1)$, using both notational variants. Then $\ell(\lambda) = 1 + 2 + 1 + 1 = 5$ and $|\lambda| = 4 + 3 + 2 + 2 + 1 = 12$. It is often useful — and enlightening — to write a partition as a Ferrers-Young diagram[2], such as this visual representation of $(4, 3, 2, 2, 1)$, where the first row associates to the largest part $\lambda_1 = 4$, the second row represents $\lambda_2 = 3$, and so on:

$$\begin{array}{llll}
\bullet & \bullet & \bullet & \bullet \\
\bullet & \bullet & \bullet & \\
\bullet & \bullet & & \\
\bullet & \bullet & & \\
\bullet & & &
\end{array}$$

We also define the *conjugate* λ^* of partition λ to be the partition given by the transpose of the Ferrers-Young diagram, i.e., the columns of λ form the rows of λ^*. Thus the conjugate of $(4, 3, 2, 2, 1)$ is $(5, 4, 2, 1)$ by the diagram above.

[2] Strictly speaking, the one pictured is a Ferrers diagram; a Young diagram uses unit squares instead of dots.

Much like the set of positive integers, but perhaps even more richly, the set of integer partitions ripples with striking patterns and beautiful number-theoretic phenomena. In fact, the positive integers $\mathbb{N}$ are embedded in $\mathcal{P}$ in a number of ways: obviously, positive integers themselves represent the set of partitions into one part; less trivially, the prime decompositions of integers are in bijective correspondence with the set of prime partitions, i.e., the partitions into prime parts (if we map the number 1, the "empty prime" so to speak, to the empty partition $\emptyset$), as Alladi and Erdős note [AE77]. We might also identify the divisors of n with the partitions of n into identical parts, and there are many other interesting ways to associate integers to the set of partitions.

Partitions of n are notoriously challenging to enumerate[3] — there are just so many of them. Euler found another profound product-sum identity, the generating function for the so-called *partition function* $p(n)$ equal to the number of partitions of $n \geq 0$, with the convention $p(0) := 1$, viz.

$$\sum_{n=0}^{\infty} p(n)q^n = (q;q)_\infty^{-1}, \tag{1.3}$$

where on the right-hand side we use the usual q-Pochhammer symbol notation.

Definition 1.1.3. *For $z, q \in \mathbb{C}$, $|q| < 1$, the q-Pochhammer symbol is defined by $(z;q)_0 :=$ 1 and, for $n \geq 1$,*

$$(z;q)_n := \prod_{i=0}^{n-1}(1 - zq^i).$$

In the limit as $n \to \infty$, we write

$$(z;q)_\infty := \lim_{n\to\infty}(z;q)_n.$$

With the relation (1.3) and others like it, such as his pentagonal number theorem and q-binomial theorem [Ber06], Euler single-handedly established the theory of integer partitions[4]. In particular, much as with the zeta function above, he innovated the use

[3] See Appendix A for some elementary approaches to counting partitions.

[4] We note that Leibniz appears to have been the first to ask questions about partitions [And00].

of product-sum generating functions to study partitions, discovering subtle bijections between certain subsets of $\mathcal{P}$ and other interesting properties of partitions, often with connections to diverse forms of q-hypergeometric series (see [Fin88]).

1.1.2 Mock theta functions and quantum modular forms

Flashing forward almost two centuries from Euler's time, another highly creative explorer ventured into the waters of partitions and q-series. When Ramanujan put to sea from India for Cambridge University in 1914, destined to revolutionize number theory, a revolution in physics was already full-sail in Europe. Just one year earlier, the Rutherford–Bohr model of atomic shells heralded the emergence of a paradoxical new *quantum theory* of nature that contradicted common sense. In 1915, Einstein would describe how space, light, matter, geometry itself, warp and bend in harmonious interplay. The following year, Schwarzschild found Einstein's equations to predict the existence of monstrously inharmonious *black holes*, that we can now study directly (just very recently) using interstellar gravitational waves [Aea16].

During Ramanujan's five years working with G. H. Hardy, news of the paradigm shift in physics must have created a thrill among the mathematicians at Trinity College, Isaac Newton's *alma mater*. Certainly Hardy would have been aware of the sea change. After all, J. J. Thomson's discovery of the electron, as well as his subsequent "plum-pudding" atomic model, had been made at Cambridge's Cavendish Laboratory; Rutherford had done his post-doctoral work with Thomson there; and Niels Bohr came to Cambridge to work under Thomson in 1911 [Gam85]. Moreover, Hardy's intellectual colleague David Hilbert was in a public race with Einstein to write down the equations of General Relativity [Isa15].

We don't know how aware Ramanujan was of these happenings in physics, yet his flights of imagination and break with academic tradition were expressions of the scientific *Zeitgeist* of the age. In Cambridge, he made innovative discoveries in an array of classical topics, from prime numbers to the evaluation of series, products and integrals, to the theory of partitions — in particular, he discovered startling "Ramanujan congruences" relating the partition function $p(n)$ to primes, bridging additive and multiplicative number theory — all of which would have been accessible to Euler. After returning to India in 1919, as he approached his own tragic event horizon, Ramanujan's thoughts ventured into realms that — like the domains of subatomic particles and gravitational waves — would require the technology of a future era to navigate [PS15].

In the final letter he sent to Hardy, dated 12 January, 1920 (only a few months before he tragically passed away at age 32), Ramanujan described a new class of mathematical objects he called *mock theta functions* [Ram00], that mimic certain behaviors of classical modular forms (see [Apo13, Ono04] for details about modular forms). These interesting q-hypergeometric series — or "Eulerian" series, as Ramanujan referred to q-series — turn out to have profoundly curious analytic, combinatorial and algebraic properties. Ramanujan gave a prototypical example $f(q)$ of a mock theta function, defined by the series

$$f(q) := \sum_{n=0}^{\infty} \frac{q^{n^2}}{(-q;q)_n^2}, \tag{1.4}$$

where $|q| < 1$. Ramanujan claimed that $f(q)$ is "almost" modular in a number of ways. For instance, he provided a pair of functions $\pm b(q)$ with

$$b(q) := (q;q)_\infty(-q;q)_\infty^{-2}$$

that are modular up to multiplication by $q^{-1/24}$ when $q := e^{2\pi i\tau}$, $\tau \in \mathbb{H}$ (the upper half-plane), to compensate for the singularities arising in the denominator of (1.4) as q approaches an even-order root of unity ζ_{2k} (where we define $\zeta_m := e^{2\pi i/m}$) radially from within the unit circle[5]:

$$\lim_{q\to\zeta_{2k}} \left(f(q) - (-1)^k b(q)\right) = \mathcal{O}(1). \tag{1.5}$$

[5]At even-order roots of unity this limiting procedure isn't necessary as there is no pole to reckon with in the denominator and $f(q)$ converges (see Chapter 8).

This type of behavior was first rigorously investigated by Watson in 1936 [Wat36], and quantifies to some degree just how "almost" modular $f(q)$ is: at least at even-order roots of unity, $f(q)$ looks like a modular form plus a constant.

Only in the twenty-first century have we begun to grasp the larger meaning of functions such as this, beginning with Zwegers's innovative Ph.D. thesis [Zwe08] of 2002, and developed in work of other researchers. We now know Ramanujan's mock theta functions are examples of *mock modular forms*, which are the holomorphic parts of even deeper objects called *harmonic Maass forms* (see [BFOR17] for background).

In 2012, Folsom–Ono–Rhoades [FOR13] made explicit the limit in (1.5), showing that

$$\lim_{q \to \zeta_{2k}} \left(f(q) - (-1)^k b(q) \right) = -4U(-1, \zeta_{2k}), \tag{1.6}$$

where $U(z,q)$ is the *rank generating function for strongly unimodal sequences* in combinatorics (see [BFR15]), and is closely related to *partial theta functions* and mock modular forms. By this connection to U, the work of Folsom–Ono–Rhoades along with Bryson–Ono–Pitman–Rhoades [BOPR12] reveals that the mock theta function $f(q)$ is strongly connected to the newly-discovered species of *quantum modular forms* in the sense of Zagier: functions that are modular on the rational or real numbers (see the definition below) up to the addition of some "suitably nice" function, and (in Zagier's words) have "the 'feel' of the objects occurring in perturbative quantum field theory" [Zag10].[6]

Definition 1.1.4. *Following Zagier [Zag10], we say a function $f\colon \mathbb{P}^1(\mathbb{Q})\backslash S \to \mathbb{C}$, for a discrete subset S, is a quantum modular form if $f(x) - f|_k\gamma(x) = h_\gamma(x)$ for a "suitably nice" function $h_\gamma(x)$, with $\gamma \in \Gamma$ a congruence subgroup of $\mathrm{SL}_2(\mathbb{Z})$.*

Remark. In this definition, $|_k$ is the usual Petersson slash operator (see [Ono04]), and "suitably nice" implies some pertinent analyticity condition, e.g. $\mathcal{C}_k, \mathcal{C}_\infty$, etc.

As a prototype of this new "quantum" class of objects, Zagier pointed to a class

[6] See, for instance, [Rea16] about perturbative QFT.

of "strange" functions of $q \in \mathbb{C}$ that diverge almost everywhere in the complex plane — except at certain roots of unity, where they are perfectly well-behaved and turn out to obey modular transformation laws. One prototypical example of such an object is known in the literature as *Kontsevich's "strange" function*, an almost nonsensical q-hypergeometric series introduced in a 1997 lecture at the Max Planck Institute for Mathematics by Maxim Kontsevich [Zag01].

Definition 1.1.5. *The "strange" function $F(q)$ is defined by the series*

$$F(q) := \sum_{n=0}^{\infty} (q;q)_n. \tag{1.7}$$

Observing that $(q;q)_\infty$ converges inside the unit circle and diverges when $|q| \geq 1$ except at roots of unity, where it vanishes, gives an indication of what we think of as "strange" behavior in a function on $\mathbb{C}$: if we let q scan around the complex plane, $F(q)$ is only non-infinite at isolated points, flickering in and out of comprehensibility along the unit circle.[7]

Modular forms are well known to be connected to partition theory — the partition generating function $(q;q)_\infty^{-1}$ is essentially modular — as well as to zeta functions and other classical Dirichlet series by the theory of Hecke (see [Apo13], Ch. 6). But these new-found objects such as mock theta functions and almost-everywhere-divergent "strange" functions seem to dwell in a different dimension from classical number theory.

1.1.3 Glimpses of an arithmetic of partitions

In a series of papers in the early 1970s (e.g. [And72, And75]), Andrews introduced the theory of *partition ideals*, a deep explanation of generating functions and bijection identities. Using ideas from lattice theory, Andrews provides examples of beautiful algebraic

[7] Define $\chi_A(z) = 1$ if $z \in A \subseteq \mathbb{C}$ and $= 0$ otherwise. Then for any $f(z)$ defined on $B \subseteq \mathbb{C}$, and A a discrete subset of B (with $f(z) \neq 0$ except possibly if $z \in A$), one might think of $f(z)/\chi_A(z)$ as a toy model "strange" function — it is only finite on the points comprising A.

structures within the set $\mathcal{P}$ of integer partitions, and unifies classical partition identities of Euler, Rogers-Ramanujan (see [Sil17b]), and other authors. Andrews summarized his ideas on partition ideals in his seminal 1976 book [And98]. The following year, Alladi and Erdős published another innovative study [AE77] fusing partition theory with analytic number theory to investigate arithmetic functions, and drew a bijection between the set of positive integers $\mathbb{Z}^+$ and the set of partitions into prime parts (the so-called "prime partitions"), pointing to deeper arithmetic connections between $\mathbb{Z}^+$ and $\mathcal{P}$.

In light of these modern, far-reaching ideas, one wonders: to what degree might classical theorems from arithmetic arise as images in $\mathbb{N}$ (i.e., in prime partitions) of larger algebraic and set-theoretic structures in $\mathcal{P}$ such as those discovered by Andrews?

1.2 The present work

The partition generating formula (1.3) doesn't look very much like the zeta function identity (1.1), beyond the "sum = product" form of both identities. However, generalizing Euler's proofs of these theorems leads to a new class of "partition zeta functions", which we define and examine in this work, containing $\zeta(s)$ and classical Dirichlet series as special cases, and intersecting q-series generating functions in diverse ways. Further Eulerian methods, together with work of Alladi, Andrews, Fine, Ono, Ramanujan, Zagier and other researchers, give hints of combinatorial structures unifying aspects of multiplicative and additive number theory[8].

The pursuit of such structures is the central motivation for this work. Through a number of theorems, examples and applications, we propose a philosophical heuristic:

1. Classical multiplicative number theory is a special case (the restriction to prime partitions) of much more general theorems in the universe of partition theory.

2. One expects multiplicative functions and phenomena to have partition counterparts.

[8] See [MS18, Wak16] for recent work at the intersection of additive and multiplicative number theory.

1.2.1 Intersections of additive and multiplicative number theory

Chapter 2 preview

In Chapter 2, we set the stage for this dissertation by proving classical-type connections between the Möbius function $\mu(n)$ (for $n \in \mathbb{N}$) and integer partitions. One such result is the following. Let $p_a(n)$ denote the number of partitions of n having length equal to a, and define $\widehat{p}_a(n)$ to be the number of partitions of n with length some positive multiple of a, i.e., $\widehat{p}_a(n) = \sum_{j=1}^{\infty} p_{aj}(n)$. Let $P_a(q) := \sum_{k=0}^{\infty} p_a(k)q^k$ and $\widehat{P}_a(q) := \sum_{k=0}^{\infty} \widehat{p}_a(k)q^k$.

Proposition 1.2.1 (Theorem 2.1.2 in Chapter 2)**.** *We have the following pair of identities:*

$$P_a(q) = \sum_{j=1}^{\infty} \mu(j)\widehat{P}_{aj}(q),$$
$$p_a(n) = \sum_{j=1}^{\infty} \mu(j)\widehat{p}_{aj}(n).$$

In proving these partition identities, μ plays a key role, but with respect to the partition lengths aj, not the size n as one might anticipate. It is interesting in this theorem and others proved in Chapter 2, to see the interaction of this classical multiplicative function with additive partitions.

Chapter 3 preview

Following up on this multiplicative lead, Chapter 3 is one of the central chapters of this work. We define a partition version of the Möbius function, also studied privately by Alladi[9], and use it in various settings in subsequent chapters.

Furthermore, we present a natural multiplicative theory of integer partitions, and find many theorems of classical number theory and analysis arise as particular cases of

[9] K. Alladi, private communication, December 21, 2015

extremely general combinatorial structure laws. Let us define a new partition statistic, the *norm* of the partition, to complement the length $\ell(\lambda)$ and size $|\lambda|$.

Definition 1.2.1. *We define the norm of λ, notated n_λ, by $n_\emptyset := 1$ and, for λ nonempty, by the product of the parts:*

$$n_\lambda := \lambda_1\lambda_2\cdots\lambda_r.$$

Pushing further in the multiplicative direction, we can define a multiplication operation on the elements of $\mathcal{P}$, as well as division of partitions.

Definition 1.2.2. *We define the product $\lambda\lambda'$ of two partitions $\lambda, \lambda' \in \mathcal{P}$ as the multi-set union of their parts listed in weakly decreasing order, e.g., $(5,2,2)(6,5,1) = (6,5,5,2,2,1)$. The empty partition $\emptyset$ serves as the multiplicative identity.*

Definition 1.2.3. *We say a partition δ divides (or is a "subpartition" of) λ and write $\delta|\lambda$, if all the parts of δ are also parts of λ, including multiplicity, e.g., $(6,5,1)|(6,5,5,2,2,1)$. When $\delta|\lambda$ we define the quotient $\lambda/\delta \in \mathcal{P}$ formed by deleting the parts of δ from λ. We note that $\emptyset$ divides every partition.*

Note that in this setting, the partitions $(1), (2), (3), (4), \ldots$, of length one play the role of primes. We can now discuss the partition-theoretic analog of $\mu(n)$ mentioned above.

Definition 1.2.4. *For $\lambda \in \mathcal{P}$ we define a partition-theoretic Möbius function $\mu_{\mathcal{P}}(\lambda)$ as follows:*

$$\mu_{\mathcal{P}}(\lambda) := \begin{cases} 1 & \text{if } \lambda = \emptyset, \\ 0 & \text{if } \lambda \text{ has any part repeated,} \\ (-1)^{\ell(\lambda)} & \text{otherwise.} \end{cases}$$

Note that if λ is a prime partition, $\mu_{\mathcal{P}}(\lambda)$ reduces to $\mu(n_\lambda)$. Just as in the classical case, we have the following, familiar relations.

Proposition 1.2.2 (Proposition 3.3.1 in Chapter 3)**.** *Summing $\mu_{\mathcal{P}}(\delta)$ over the subparti-*

tions δ of $\lambda \in \mathcal{P}$ gives

$$\sum_{\delta|\lambda} \mu_{\mathcal{P}}(\delta) = \begin{cases} 1 & \text{if } \lambda = \emptyset, \\ 0 & \text{otherwise.} \end{cases}$$

We also have a partition-theoretic version of Möbius inversion.

Proposition 1.2.3 (Proposition 3.3.2 in Chapter 3)**.** *For $f\colon \mathcal{P} \to \mathbb{C}$ define*

$$F(\lambda) := \sum_{\delta|\lambda} f(\delta).$$

Then we also have

$$f(\lambda) = \sum_{\delta|\lambda} F(\delta)\mu_{\mathcal{P}}(\lambda/\delta).$$

Now, the classical Möbius function has a close companion in the Euler phi function $\varphi(n)$, and $\mu_{\mathcal{P}}$ has a companion as well.

Definition 1.2.5. *For $\lambda \in \mathcal{P}$ we define a partition-theoretic phi function*

$$\varphi_{\mathcal{P}}(\lambda) := \ n_\lambda \prod_{\substack{\lambda_i \in \lambda \\ \textit{without} \\ \textit{repetition}}} (1 - \lambda_i^{-1}).$$

Clearly $\varphi_{\mathcal{P}}(\lambda)$ reduces to $\varphi(n_\lambda)$ if λ is a prime partition, and, as with $\mu_{\mathcal{P}}$, generalizes classical results.

Proposition 1.2.4 (Propositions 3.3.4 and 3.3.5 in Chapter 3)**.** *We have that*

$$\sum_{\delta|\lambda} \varphi_{\mathcal{P}}(\delta) = n_\lambda, \qquad \varphi_{\mathcal{P}}(\lambda) = n_\lambda \sum_{\delta|\lambda} \frac{\mu_{\mathcal{P}}(\delta)}{n_\delta}.$$

There are generalizations in partition theory of many other arithmetic objects and theorems, for example, a partition version $\sigma_{\mathcal{P}}$ of the sum of divisors function $\sigma(n)$, and a partition version of the Cauchy product formula for the product of two infinite series.

Proposition 1.2.5 (Proposition 3.3.7 in Chapter 3). *For $f, g\colon \mathcal{P} \to \mathbb{C}$, we have that*

$$\left(\sum_{\lambda\in\mathcal{P}} f(\lambda)\right)\left(\sum_{\lambda\in\mathcal{P}} g(\lambda)\right) = \sum_{\lambda\in\mathcal{P}}\sum_{\delta|\lambda} f(\delta)g(\lambda/\delta),$$

so long as the sums on the left both converge absolutely.

As our first application of these ideas, we investigate the relatively recently-defined q-bracket operator $\langle f\rangle_q$ which represents certain expected values in statistical physics, studied by Bloch–Okounkov, Zagier, and others for its quasimodular[10] properties.

Definition 1.2.6. *We define the q-bracket $\langle f\rangle_q$ of a function $f\colon \mathcal{P} \to \mathbb{C}$ by the expected value*

$$\langle f\rangle_q := \frac{\sum_{\lambda\in\mathcal{P}} f(\lambda)q^{|\lambda|}}{\sum_{\lambda\in\mathcal{P}} q^{|\lambda|}} \in \mathbb{C}[[q]].$$

Here, we take the resulting power series to be indexed by partitions, unless otherwise specified.

This q-series operator turns out to play a nice role in the theory of partitions, quite apart from questions of modularity. Conversely, in analogy to antiderivatives, we define here an inverse "q-antibracket" of f.[11]

Definition 1.2.7. *We call $F\colon \mathcal{P} \to \mathbb{C}$ a q-antibracket of f if $\langle F\rangle_q = \sum_{\lambda\in\mathcal{P}} f(\lambda)q^{|\lambda|}$.*

As in antidifferentiation, the function F is not unique. Using the partition-theoretic ideas we develop, we can give an explicit formula for coefficients of the q-bracket and q-antibracket of any function f defined on partitions.

Proposition 1.2.6 (Theorems 3.4.1 and 3.4.2 in Chapter 3). *The q-bracket of $f\colon \mathcal{P} \to \mathbb{C}$ is given by*

$$\langle f\rangle_q = \sum_{\lambda\in\mathcal{P}} \widetilde{f}(\lambda)q^{|\lambda|},$$

[10]Quasimodular forms are a class containing integer-weight holomorphic modular forms generated by the Eisenstein series E_2, E_4, E_6, as opposed to just by E_4, E_6 as in the modular case.

[11]We will refer to the act of obtaining $\langle f\rangle_q$ and F as "applying the q-bracket/antibracket".

where $\widetilde{f}(\lambda) = \sum_{\delta|\lambda} f(\delta)\mu_{\mathcal{P}}(\lambda/\delta)$. *Moreover, let* $F(\lambda) := \sum_{\delta|\lambda} f(\delta)$*; then a* q*-antibracket of* f *is given by the coefficients* F *of*

$$\langle f \rangle_q^{-1} = \sum_{\lambda \in \mathcal{P}} F(\lambda) q^{|\lambda|}.$$

We apply this q-bracket formula to compute coefficients of the reciprocal of the Jacobi triple product (see [Ber06])

$$j(z;q) := (z;q)_\infty (z^{-1}q;q)_\infty (q;q)_\infty.$$

Proposition 1.2.7 (Corollary 3.6.2 of Chapter 3)**.** *For* $z \neq 1$ *the reciprocal of the triple product is given by*

$$\frac{1}{j(z;q)} = \sum_{n \geq 0} c_n q^n \quad \textit{with} \quad c_n = c_n(z) = (1-z)^{-1} \sum_{\lambda \vdash n} \sum_{\delta|\lambda} \sum_{\varepsilon|\delta} z^{\mathrm{crk}(\varepsilon)},$$

where $\mathrm{crk}(*)$ *denotes the crank of a partition as defined by Andrews-Garvan [AG88].*[12]

We see in Chapter 8 this identity is connected to Ramanujan's mock theta functions.

1.2.2 Partition zeta functions

Chapter 4 preview

Arithmetic functions and divisor sums are not the only multiplicative phenomena with connections in partition theory. In Chapter 4 we introduce a broad class of *partition zeta functions* (and in Chapter 5, partition Dirichlet series) arising from a fusion of Euler's product formulas for both the partition generating function and the Riemann zeta function, which admit interesting structure laws and evaluations as well as classical specializations.

[12] See Definition 3.6.1

Definition 1.2.8. *In analogy to the Riemann zeta function $\zeta(s)$, for a subset $\mathcal{P}'$ of $\mathcal{P}$ and value $s \in \mathbb{C}$ for which the series converges, we define a partition zeta function $\zeta_{\mathcal{P}'}(s)$ by*

$$\zeta_{\mathcal{P}'}(s) := \sum_{\lambda \in \mathcal{P}'} n_\lambda^{-s}.$$

If we let $\mathcal{P}'$ equal the partitions $\mathcal{P}_{\mathbb{X}}$ whose parts all lie in some subset $\mathbb{X} \subset \mathbb{N}$, there is also an Euler product

$$\zeta_{\mathcal{P}_{\mathbb{X}}}(s) = \prod_{n \in \mathbb{X}} (1 - n^{-s})^{-1}.$$

Of course, $\zeta(s)$ is the case $\mathbb{X} = \mathbb{P}$; and many classical zeta function identities generalize to partition identities. Furthermore, we show how partition zeta sums over other proper subsets of $\mathcal{P}$ can yield nice closed-form results. To see how subsets influence the evaluations, fix $s = 2$ and sum over three unrelated subsets of $\mathcal{P}$: partitions $\mathcal{P}_{even}$ into even parts, partitions $\mathcal{P}_{prime}$ into prime parts, and partitions $\mathcal{P}_{dist}$ into distinct parts.

Proposition 1.2.8 (Corollaries 4.2.1 and 4.2.10 in Chapter 4)**.** *We have the identities*

$$\zeta_{\mathcal{P}_{even}}(2) = \frac{\pi}{2}, \qquad \zeta_{\mathcal{P}_{prime}}(2) = \frac{\pi^2}{6}, \qquad \zeta_{\mathcal{P}_{dist}}(2) = \frac{\sinh \pi}{\pi}.$$

Notice how different choices of partition subsets induce very different partition zeta values for fixed s. Interestingly, differing powers of π appear in all three examples. Another curious formula involving π arises if we take $s = 3$ and sum on partitions $\mathcal{P}_{\geq 2}$ with all parts ≥ 2 (that is, no parts equal to 1)[13].

Proposition 1.2.9 (Proposition 4.2.3 in Chapter 4)**.** *We have that*

$$\zeta_{\mathcal{P}_{\geq 2}}(3) = \frac{3\pi}{\cosh\left(\frac{1}{2}\pi\sqrt{3}\right)}.$$

These formulas are appealing, but they look a little too motley to comprise a *family*

[13] We call these "nuclear" partitions in Appendix A, and see that they encode, in a sense, all of $\mathcal{P}$.

like Euler's values

$$\zeta(2k) \in \mathbb{Q}\pi^{2k}.$$

We produce a class of partition zeta functions that does yield nice evaluations like this.

Definition 1.2.9. *We define*

$$\zeta_{\mathcal{P}}(\{s\}^k) := \sum_{\ell(\lambda)=k} n_\lambda^{-s},$$

where the sum is taken over all partitions of fixed length $k \geq 1$ (the $k = 1$ case is just $\zeta(s)$).

Proposition 1.2.10 (Corollary 4.2.4 in Chapter 4)**.** *For $s = 2, k \geq 1$, we have the identity*

$$\zeta_{\mathcal{P}}(\{2\}^k) = \frac{2^{2k-1}-1}{2^{2k-2}}\zeta(2k).$$

For example, we give the following values:

$$\zeta_{\mathcal{P}}(\{2\}^2) = \frac{7\pi^4}{360}, \quad \zeta_{\mathcal{P}}(\{2\}^3) = \frac{31\pi^6}{15120}, \ \ldots \ , \ \zeta_{\mathcal{P}}(\{2\}^{13}) = \frac{22076500342261\pi^{26}}{93067260259985915904000000}$$

We prove increasingly complicated identities for $\zeta_{\mathcal{P}}(\{2^t\}^k), t \geq 1$, as well.

Chapter 5 preview

In Chapter 5, we more deeply probe certain aspects of partition zeta functions. For example, we are able to prove more than in Proposition 1.2.10 above with respect to rational multiples of powers of π.

Proposition 1.2.11 (Corollary 5.2.5 in Chapter 5)**.** *For $m > 0$ even, we have*

$$\zeta_{\mathcal{P}}(\{m\}^k) \in \mathbb{Q}\pi^{mk}.$$

So these zeta sums over partitions of fixed length really do form a family like Euler's zeta values. Inspired by work of Chamberland-Straub [CS13], in Chapter 5 we also evaluate partition zeta functions over partitions $\mathcal{P}_{a+m\mathbb{N}}$ whose parts are all $\equiv a$ modulo m. Let Γ denote the usual gamma function, and let $e(x) := e^{2\pi i x}$.

Proposition 1.2.12 (Proposition 5.2.2 in Chapter 5)**.** *For $n \geq 2$, we have*

$$\zeta_{\mathcal{P}_{a+m\mathbb{N}}}(n) = \Gamma(1+a/m)^{-n} \prod_{r=0}^{n-1} \Gamma\left(1 + \frac{a - e(r/n)}{m}\right).$$

We also address analytic continuation of certain partition zeta functions, which is somewhat rare. In Chapter 4, Corollary 4.2.7, the analytic continuation of $\zeta_{\mathcal{P}}(\{s\}^k)$ is given for fixed length $k = 2$; for $\mathrm{Re}(s) > 1$, we can write

$$\zeta_{\mathcal{P}}(\{s\}^2) = \frac{\zeta(2s) + \zeta(s)^2}{2}, \tag{1.8}$$

thus $\zeta_{\mathcal{P}}(\{s\}^2)$ inherits analytic continuation from the Riemann zeta functions on the right. We study analytic continuation more broadly in Chapter 5, and in Corollary 5.3.1 prove the meromorphic extension of $\zeta_{\mathcal{P}_{m\mathbb{N}}}(s)$ to the right half-plane of $\mathbb{C}$. Moreover, following ideas of Kubota and Leopoldt [KL64], in Theorem 5.3.1 we give p-adic interpolations for modified versions of $\zeta_{\mathcal{P}}(\{m\}^k)$ in the m-aspect.

Finally, we give applications in the theory of multiple zeta values, and note examples of *partition Dirichlet series* which generalize classical results. For instance, for appropriate $s \in \mathbb{C}$, $\mathbb{X} \subset \mathbb{N}$, we get familiar-looking relations like these.

Proposition 1.2.13 (Proposition 5.6.1 in Chapter 5)**.** *Just as in the classical cases, we have the following identities:*

$$\sum_{\lambda \in \mathcal{P}_{\mathbb{X}}} \mu_{\mathcal{P}}(\lambda) n_\lambda^{-s} = \frac{1}{\zeta_{\mathcal{P}_{\mathbb{X}}}(s)}, \qquad \sum_{\lambda \in \mathcal{P}_{\mathbb{X}}} \varphi_{\mathcal{P}}(\lambda) n_\lambda^{-s} = \frac{\zeta_{\mathcal{P}_{\mathbb{X}}}(s-1)}{\zeta_{\mathcal{P}_{\mathbb{X}}}(s)}.$$

1.2.3 Partition formulas for arithmetic densities

Chapter 6 preview

In Chapter 6 we explore a different connection between partitions and zeta functions. Alladi proves in [All77] a surprising *duality principle* connecting arithmetic functions to sums over smallest or largest prime factors of divisors, and applies this principle to prove for $\gcd(r,t)=1$ that

$$-\sum_{\substack{n\geq 2\\ p_{\min}(n)\equiv r \pmod{t}}} \mu(n)n^{-1} = \frac{1}{\varphi(t)}, \tag{1.9}$$

where $p_{\min}(n)$ denotes the smallest prime factor of n, and $1/\varphi(t)$ represents the proportion of primes in a fixed arithmetic progression modulo t. Using analogous dualities from partition generating functions (smallest/largest parts instead), and replacing μ with $\mu_{\mathcal{P}}$, in Chapter 6 we extend Alladi's ideas to compute arithmetic densities of other subsets of $\mathbb{N}$ using partition-theoretic q-series.

Proposition 1.2.14 (Theorems 6.1.3–6.1.4 of Chapter 6)**.** *For suitable subsets $\mathbb{S}$ of $\mathbb{N}$ with arithmetic density $d_{\mathbb{S}}$,*

$$-\lim_{q\to 1}\sum_{\substack{\lambda\in\mathcal{P}\\ \mathrm{sm}(\lambda)\in\mathbb{S}}} \mu_{\mathcal{P}}(\lambda)q^{|\lambda|} = d_{\mathbb{S}},$$

where $\mathrm{sm}(\lambda)$ *denotes the smallest part of λ, and $q\to 1$ from within the unit circle.*

In particular, if we denote kth-power-free integers by $\mathbb{S}_{\mathrm{fr}}^{(k)}$, we prove a partition formula to compute $1/\zeta(k)$ as the limiting value of a partition-theoretic q-series as $q\to 1$.

Proposition 1.2.15 (Corollary 6.1.2 of Chapter 6)**.** *If $k\geq 2$, then*

$$-\lim_{q\to 1}\sum_{\substack{\lambda\in\mathcal{P}\\ \mathrm{sm}(\lambda)\in\mathbb{S}_{\mathrm{fr}}^{(\mathrm{k})}}} \mu_{\mathcal{P}}(\lambda)q^{|\lambda|} = \frac{1}{\zeta(k)}.$$

We discuss further consequences, such as an interesting bijection between subsets of partitions.

1.2.4 "Strange" functions, quantum modularity, mock theta functions and unimodal sequences

Chapter 7 summary

In Chapter 7 we turn our attention to quantum modular forms, which figure into Chapter 8 as well. These are q-series that, in addition to being "almost" modular, generically "blow up" as q approaches the unit circle from within, but are finite when q radially approaches certain roots of unity or other isolated points — in which case the limiting values have been related to special values of L-functions [BFOR17] — and might even extend to the complex plane beyond the unit circle in the variable q^{-1}, a phenomenon called *renormalization* (see [LNR13]).

Inspired by Zagier's work [Zag10] with Kontsevich's "strange" function $F(q)$ defined above, as well as work by Andrews, Jiménez-Urroz and Ono [AJUO01], we construct a vector-valued quantum modular form $\phi(x) := \begin{pmatrix} \theta_1^S(e^{2\pi i x}) & \theta_2^S(e^{2\pi i x}) & \theta_3^S(e^{2\pi i x}) \end{pmatrix}^T$ whose components $\theta_i^S \colon \mathbb{Q} \to \mathbb{C}$ are similarly "strange".

Proposition 1.2.16 (Theorem 7.1.1 of Chapter 7)**.** *We have that $\phi(x)$ is a weight $3/2$ vector-valued quantum modular form. In particular, we have that*

$$\phi(z+1) - \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & \zeta_{12} \\ 0 & \zeta_{24} & 0 \end{pmatrix} \phi(z) = 0,$$

and we also have

$$\left(\frac{z}{-i}\right)^{-3/2}\phi(-1/z)+\begin{pmatrix}0 & \sqrt{2} & 0\\ 1/\sqrt{2} & 0 & 0\\ 0 & 0 & 1\end{pmatrix}\phi(z)=\begin{pmatrix}0 & \sqrt{2} & 0\\ 1/\sqrt{2} & 0 & 0\\ 0 & 0 & 1\end{pmatrix}g(z),$$

where $g(z)$ is a 3-dimensional vector of smooth functions defined as period integrals.

Moreover, finite evaluations of θ_i^S at odd-order roots of unity lead to closed-form evaluations of complicated-looking period integrals, via relations between certain L-functions and "strange" series.

Chapter 8 summary

Some of the interesting properties of quantum modular forms, such as finiteness at roots of unity and renormalization phenomena, extend to other q-hypergeometric series such as the "universal" mock theta function g_3. In Chapter 8, we apply partition-theoretic results from Chapter 3 as well as ideas from statistical physics, to show that g_3 arises naturally from the reciprocal of the classical Jacobi triple product $j(z;q)$ — and is intimately tied to rank generating functions for unimodal sequences — under the action of the q-bracket.

Let $j_z\colon \mathcal{P}\to\mathbb{C}$ denote the partition-indexed coefficients of $j(z,q)^{-1}=\sum_{\lambda\in\mathcal{P}} j_z(\lambda)q^{|\lambda|}$. It turns out the odd-order universal mock theta function g_3 (in an "inverted" form) and the rank generating function $\widetilde{U}(z,q)$ for unimodal sequences arise together as components of $\left\langle j_z\right\rangle_q$.

Proposition 1.2.17 (Theorem 8.2.1 in Chapter 8)**.** *For $0<|q|<1, z\neq 0, z\neq 1,$ the following statements are true:*

(i) We have the q-bracket formula

$$\left\langle j_z\right\rangle_q = 1 \ + \ \left[z(1-q)+z^{-1}q\right] g_3(z^{-1},q^{-1}) \ + \ \frac{zq^2}{1-z}\widetilde{U}(z,q).$$

(ii) The "inverted" mock theta function component in part (i) converges, and can be written in the form

$$g_3(z^{-1}, q^{-1}) = \sum_{n=1}^{\infty} \frac{q^n}{(z;q)_n (z^{-1}q;q)_n}.$$

Let $\zeta_m := e^{2\pi \mathrm{i}/m}$ be a primitive mth root of unity. Define the rank generating function $\widetilde{U_k}(z,q)$ (resp. $U_k(z,q)$) for unimodal (resp. strongly unimodal) sequences with k-fold peak. Then we have interesting relations between g_3 and $\widetilde{U}_k, U_k$.

Proposition 1.2.18 (Corollaries 8.2.2 and 8.3.1 of Chapter 8)**.** *For $|q| < 1 < |z|$, we have*

$$g_3(z^{-1}, q^{-1}) = \frac{z}{1-z} \sum_{k=1}^{\infty} \widetilde{U}_k(z,q) z^{-k} q^k.$$

For $|z| < 1$, the radial limit as $q \to \zeta_m$ an mth order root of unity is given by

$$g_3(z, \zeta_m) = \frac{z-1}{z} \sum_{k=1}^{\infty} U_k(-z, \zeta_m) z^k \zeta_m^{-k}.$$

We then find $g_3(z,q)$ to extend in q to the entire complex plane minus the unit circle, and give a finite formula for g_3 (as well as other q-series) at roots of unity, that is simple by comparison to other such formulas in the literature. For instance, we prove the following, simple formula for the mock theta function $f(q)$.

Proposition 1.2.19 (Example 8.3.3 in Chapter 8)**.** *For ζ_m an odd-order root of unity we have*

$$f(\zeta_m) = \frac{4}{3} \sum_{n=1}^{m} (-1)^n (-\zeta_m^{-1}; \zeta_m^{-1})_n.$$

We indicate similar formulas for other q-hypergeometric series and q-continued fractions, and look at interesting "quantum"-type behaviors of mock theta functions and other q-series inside, outside, and on the unit circle. Finally, we speculate about the nature of connections between partition theory, q-series and physical reality.

Remark. In the Appendices, we give follow-up points and observations related to work in various chapters.

Chapter 2

Combinatorial applications of Möbius inversion

Adapted from [JS14], a joint work with Marie Jameson

2.1 Introduction and Statement of Results

In this chapter we glimpse connections between additive number theory and the multiplicative branch of the theory, which we will follow up on in subsequent chapters. As we noted in the previous section, product-sum identities are ubiquitous in number theory and the theory of q-series. For example, recall Euler's identity

$$\prod_{n=1}^{\infty}(1-q^n)=\sum_{k=-\infty}^{\infty}(-1)^k q^{k(3k-1)/2},$$

and Jacobi's identity

$$\prod_{n=1}^{\infty}(1-q^n)^3=\sum_{n=0}^{\infty}(-1)^n(2n+1)q^{n(n+1)/2}.$$

More recently, Borcherds defined "infinite product modular forms"

$$F(z) = q^h \prod_{n=1}^{\infty} (1-q^n)^{a(n)},$$

where $q := e^{2\pi i z}$ and the $a(n)$'s are coefficients of certain weight $1/2$ modular forms (see Chapter 4 of [Ono10]). This was generalized by Bruinier and Ono [BO03] to the setting where the exponents $a(n)$ are coefficients of harmonic Maass forms.

At first glance, this does not look like the stuff of combinatorics. However, one might consider the partition function $p(n)$ and ask whether the product

$$\prod_{n=1}^{\infty} (1-q^n)^{p(n)} \tag{2.1}$$

has any special properties. In this direction, recent work of Ono [Ono10] studies the parity of $p(n)$. For $1 < D \equiv 23 \pmod{24}$, Ono defined

$$\Psi_D(q) := \prod_{m=1}^{\infty} \prod_{0 \le b \le D-1} \left(1 - \zeta_D^{-b} q^m\right)^{\left(\frac{-D}{b}\right) C(\overline{m}; Dm^2)},$$

where $\overline{m}$ is the reduction of $m \pmod{12}$, $\zeta_D := e^{2\pi i/D}$, and $C(\overline{m}; Dm^2)$ is the coefficient of a mock theta function. It turns out that

$$C(\overline{m}; n) \equiv \begin{cases} p\left(\frac{n+1}{24}\right) \pmod{2} & \text{if } \overline{m} \equiv 1, 5, 7, 11 \pmod{12}, \\ 0 & \text{otherwise.} \end{cases}$$

Ono considers the logarithmic derivative

$$\sum_{n=1}^{\infty} B_D(n) q^n := \frac{1}{\sqrt{-D}} \cdot \frac{q \frac{d}{dq} \Psi_D(q)}{\Psi_D(q)} = \sum_{m=1}^{\infty} m C(\overline{m}; Dm^2) \sum_{n=1}^{\infty} \left(\frac{-D}{n}\right) q^{mn} \tag{2.2}$$

and notes that reducing mod 2 gives

$$\frac{1}{\sqrt{-D}} \cdot \frac{q\frac{d}{dq}\Psi_D(q)}{\Psi_D(q)} \equiv \sum_{\substack{m\geq 1\\ \gcd(m,6)=1}} p\left(\frac{Dm^2+1}{24}\right) \sum_{\substack{n\geq 1\\ \gcd(n,D)=1}} q^{mn} \pmod 2. \tag{2.3}$$

This observation was instrumental in proving results regarding the parity of the partition function [Ono10]. However, if one desires to establish identities rather than congruences, it seems pertinent to again consider products of the form (2.1), but now at the level of q-series identities.

From this perspective, we wish to explore the logarithmic derivative of

$$\prod_{n=1}^{\infty}(1-q^n)^{a(n)} \tag{2.4}$$

for other, more general combinatorial functions $a(n)$. Then for a nonnegative integer n, define

$Q(n) :=$ number of partitions of n into distinct parts

$\widehat{Q}(n) :=$ number of partitions of n whose parts occur with the same multiplicity

and

$$F_Q(q) := \sum_{n=1}^{\infty} Q(n)q^n$$

$$F_{\widehat{Q}}(q) := \sum_{n=1}^{\infty} \widehat{Q}(n)q^n$$

$$\Psi(Q;q) := \prod_{n=1}^{\infty}(1-q^n)^{Q(n)/n}.$$

Theorem 2.1.1. *We have that*

$$\frac{q\frac{d}{dq}\Psi(Q;q)}{\Psi(Q;q)} = -F_{\widehat{Q}}(q).$$

Moreover, for all $n \geq 1$ we have

$$Q(n) = \sum_{d|n} \mu(d)\widehat{Q}(n/d),$$

where μ denotes the Möbius function.

For example, one can compute that

$$\begin{aligned}
\Psi(Q;q) &= 1 - q - \frac{1}{2}q^2 - \frac{1}{6}q^3 + \frac{1}{24}q^4 + \frac{43}{120}q^5 - \frac{233}{720}q^6 + \cdots \\
\frac{q\frac{d}{dq}\Psi(Q;q)}{\Psi(Q;q)} &= -q - 2q^2 - 3q^3 - 4q^4 - 4q^5 - 8q^6 - \cdots \\
F_{\widehat{Q}}(q) &= q + 2q^2 + 3q^3 + 4q^4 + 4q^5 + 8q^6 + \cdots = -\frac{q\frac{d}{dq}\Psi(Q;q)}{\Psi(Q;q)}.
\end{aligned}$$

In fact, while it is not obvious from a combinatorial perspective, this theorem is simple; it follows from the straightforward observation that

$$\widehat{Q}(n) = \sum_{d|n} Q(d).$$

Now we present two results in a slightly different direction that are perhaps more surprising. Looking again to the work of Ono [Ono10], we can apply Möbius inversion to (2.2) to find

$$C(\overline{n}; Dn^2) = \frac{1}{n}\sum_{d|n} \mu(d)\left(\frac{-D}{d}\right)B_D(n/d). \tag{2.5}$$

It is natural to ask whether there are analogs of this statement for related q-series, even if the series do not arise as logarithmic derivatives of Borcherds products.

We begin our search of interesting combinatorial functions by noting that the generating function for the partition function $p(n)$ obeys the identity of Euler

$$P(q) := \sum_{n=0}^{\infty} p(n)q^n = \sum_{n=0}^{\infty} \frac{q^{n^2}}{(q)_n^2}$$

where $(q)_n$ is the q-Pochhammer symbol, defined by $(q)_0 = 1$ and $(q)_n = \prod_{k=1}^{n}(1-q^k)$ for $n \geq 1$. We wish to investigate other functions of a similar form, such as those presented in the following theorems, which are formally analogous to (2.5) but involving other combinatorial functions.

Let $p_a(n)$ denote the number of partitions of n into a parts, and define $\widehat{p}_a(n)$ to be the number of partitions of n into ak parts for some integer $k \geq 1$, i.e.,

$$\widehat{p}_a(n) := \sum_{j=1}^{\infty} p_{aj}(n).$$

On analogy to the identities for $P(q)$ above, we let $P_a(q)$ and $\widehat{P}_a(q)$ denote the generating functions of $p_a(n)$ and $\widehat{p}_a(n)$, respectively. Then we have the following identities for $P_a(q)$ and $\widehat{P}_a(q)$.

Theorem 2.1.2. *We have that*

$$P_a(q) = \sum_{n=1}^{\infty} \mu(n)\widehat{P}_{an}(q)$$
$$p_a(n) = \sum_{j=1}^{\infty} \mu(j)\widehat{p}_{an}(n).$$

Observe that for $a = 1$, we have that $p_1(n) = 1$ for all integers n, and also that

$$\widehat{p}_1(n) = \sum_{j=1}^{\infty} p_j(n) = p(n).$$

In this case, the generating functions are given by

$$P_1(q) = \sum_{n=1}^{\infty} p_1(n)q^n = \sum_{n=1}^{\infty} q^n = \frac{q}{1-q}$$

and

$$\widehat{P}_1(q) = \sum_{n=1}^{\infty} \widehat{p}_1(n)q^n = \sum_{n=1}^{\infty} p(n)q^n.$$

Thus by Theorem 2.1.2, we have the explicit identities

$$P_1(q) = \sum_{n=1}^{\infty} \mu(n)\widehat{P}_n(q) = \frac{q}{1-q}$$

and, perhaps more interestingly,

$$p_1(n) = \sum_{j=1}^{\infty} \mu(j)\widehat{p}_j(n) = 1.$$

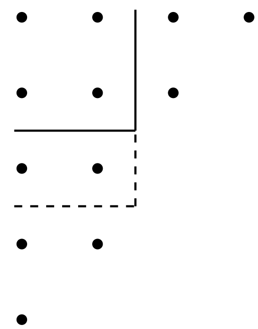
Looking again for identities similar to those given above for $P(q)$, for a positive integer a set

$$\begin{aligned} B_a(q) &:= \sum_{n=1}^{\infty} \frac{q^{n^2+an}}{(q)_n^2} =: \sum_{N=1}^{\infty} b_a(N)q^N \\ \widehat{B}_a(q) &:= \sum_{n=1}^{\infty} \frac{q^{n^2+an}}{(q)_n^2\,(1-q^{an})} =: \sum_{N=1}^{\infty} \widehat{b}_a(N)q^N. \end{aligned}$$

Generalizations of q-series such as $B_a(q)$ and $\widehat{B}_a(q)$ have been studied by Andrews [And98]. One can give a combinatorial interpretation for the coefficients $b_a(N)$ and $\widehat{b}_a(N)$ as follows.

Consider the Ferrers diagram of a given partition of an integer N with an $n \times n$ Durfee square, and having a rectangle of base n and height m adjoined immediately below the $n \times n$ Durfee square. For example, the partition of $N = 12$ shown below has a 2×2 Durfee square (marked by a solid line), and either a 2×2 or 2×1 rectangle below it (the

2×1 rectangle is marked by a dashed line).



We refer to this rectangular region of the diagram as an $n \times m$ "Durfee rectangle," and note that a given Ferrers diagram may have nested Durfee rectangles of sizes $n \times 1, n \times 2, \ldots, n \times M$, where M is the height of the largest such rectangle (assuming that at least one Durfee rectangle is present in the diagram).

We then have that

$b_a(N) =$# of partitions of N having an $n \times n$ Durfee square and at least an $n \times a$ Durfee rectangle

$\widehat{b}_a(N) =$# of partitions of N having an $n \times n$ Durfee square and at least an $n \times a$ Durfee rectangle (counted with multiplicity as an $n \times a$ rectangle may be nested within taller Durfee rectangles of size $n \times ak,\ $ for $k \geq 1$).

Assuming these notations, we have the following result.

Theorem 2.1.3. *We have that*

$$\widehat{b}_a(n) = \sum_{j=1}^{\infty} b_{aj}(n).$$

Moreover, we have

$$B_a(q) = \sum_{n=1}^{\infty} \mu(n) \widehat{B}_{an}(q)$$
$$b_a(n) = \sum_{j=1}^{\infty} \mu(j) \widehat{b}_{aj}(n).$$

2.2 Proof of Theorem 2.1.1

First we prove a lemma regarding logarithmic derivatives.

Lemma 2.2.1. *For any sequence $\{a(n)\}$, we have that*

$$\frac{q\frac{d}{dq}\left(\prod_{n=1}^{\infty}(1-q^n)^{a(n)}\right)}{\prod_{n=1}^{\infty}(1-q^n)^{a(n)}} = -\sum_{n=1}^{\infty}\sum_{d|n} a(d) d q^n.$$

Proof. Since $\log(1-x) = -\sum_{m=1}^{\infty} \frac{x^m}{m}$, we have that

$$\begin{aligned}
\frac{q\frac{d}{dq}\left(\prod_{n=1}^{\infty}(1-q^n)^{a(n)}\right)}{\prod_{n=1}^{\infty}(1-q^n)^{a(n)}} &= q\frac{d}{dq}\left(\log\left(\prod_{n=1}^{\infty}(1-q^n)^{a(n)}\right)\right) = q\frac{d}{dq}\left(\sum_{n=1}^{\infty} a(n)\log\left(1-q^n\right)\right) \\
&= -q\frac{d}{dq}\left(\sum_{n=1}^{\infty} a(n)\sum_{m=1}^{\infty}\frac{q^{mn}}{m}\right) = -\left(\sum_{n=1}^{\infty} a(n)\sum_{m=1}^{\infty} n q^{mn}\right) \\
&= -\sum_{n=1}^{\infty}\sum_{d|n} a(d) d q^n
\end{aligned}$$

as desired. □

Proof of Theorem 2.1.1. First note that for all $n \geq 1$ we have

$$\widehat{Q}(n) = \sum_{d|n} Q(d),$$

so $Q(n) = \sum_{d|n} \mu(d)\widehat{Q}(n/d)$ by Möbius inversion. By Lemma 2.2.1, we have that

$$\frac{q\frac{d}{dq}\Psi(Q;q)}{\Psi(Q;q)} = -\sum_{n=1}^{\infty}\sum_{d|n} Q(d)q^n = -\sum_{n=1}^{\infty}\widehat{Q}(n)q^n$$

as desired. □

2.3 Proof of Theorems 2.1.2 and 2.1.3

Suppose that for each positive integer a, we have two arithmetic functions $f(a;n)$ and $\widehat{f}(a;n)$ such that

$$\widehat{f}(a;n) = \sum_{j=1}^{\infty} f(aj;n),$$

where the above sum converges absolutely. We will define their generating functions as follows:

$$F(a;q) := \sum_{n=1}^{\infty} f(a;n)q^n$$
$$\widehat{F}(a;q) := \sum_{n=1}^{\infty} \widehat{f}(a;n)q^n.$$

We then have the following result.

Lemma 2.3.1. *We have that*

$$F(a;q) = \sum_{n=1}^{\infty} \mu(n)\widehat{F}(an;q)$$

and

$$f(a;n) = \sum_{j=1}^{\infty} \mu(j)\widehat{f}(aj;n).$$

Proof. Recall that

$$\sum_{d|n} \mu(n) = \begin{cases} 1 & \text{if } n = 1 \\ 0 & \text{otherwise.} \end{cases}$$

It follows that

$$\begin{aligned} F(a;q) &= \sum_{n=1}^{\infty} \left(\sum_{k \geq 1} f(an;k) q^k \right) \sum_{d|n} \mu(d) \\ &= \sum_{n=1}^{\infty} \mu(n) \sum_{k \geq 1} \left(\sum_{j=1}^{\infty} f(anj;k) \right) q^k \\ &= \sum_{n=1}^{\infty} \mu(n) \sum_{k \geq 1} \widehat{f}(an;n) q^k \\ &= \sum_{n=1}^{\infty} \mu(n) \widehat{F}(an;q). \end{aligned}$$

Then by comparing coefficients, one finds that $f(a;n) = \sum_{j=1}^{\infty} \widehat{f}(aj;n)$, as desired. $\square$

This lemma can be used to prove both Theorem 2.1.2 and Theorem 2.1.3. We note that Lemma 2.3.1 can be applied in extremely general settings, and one has great freedom in creatively choosing the constant a to be varied. For instance, taking $a = 1$ gives rise to any number of identities, as 1 can be inserted as a factor practically anywhere in a given expression.

Proof of Theorem 2.1.2. The theorem follows by a direct application of Lemma 2.3.1. $\square$

Proof of Theorem 2.1.3. First note that

$$\widehat{b}_a(N) = \sum_{j=1}^{\infty} b_{aj}(N),$$

since

$$\begin{aligned}\widehat{B}_a(q) &= \sum_{n=1}^{\infty} \frac{q^{n^2+an}}{(q)_n^2\,(1-q^{an})} = \sum_{n=1}^{\infty} \frac{q^{n^2+an}}{(q)_n^2} \sum_{j=0}^{\infty} q^{ajn} \\ &= \sum_{j=1}^{\infty} \sum_{n=1}^{\infty} \frac{q^{n^2+ajn}}{(q)_n^2} = \sum_{j=1}^{\infty} B_{aj}(q).\end{aligned}$$

The rest follows by applying Lemma 2.3.1. $\square$

Chapter 3

Multiplicative arithmetic of partitions and the q-bracket

Adapted from [Sch17]

3.1 Introduction: the q-bracket operator

In the previous chapter, we fused techniques from the theory of partitions and q-series with applications of the Möbius function, which is central to multiplicative number theory. Here we develop further ideas at the intersection of the additive and multiplicative branches of number theory, with applications to a q-series operator from statistical physics.

In a groundbreaking paper of 2000 [BO00], Bloch and Okounkov introduced the q-bracket operator $\langle f \rangle_q$ of a function f defined on the set of integer partitions, and showed that the q-bracket can be used to produce the complete graded ring of quasimodular forms. We note that Definition 1.2.6 in Section 1 extends the range of the q-bracket somewhat; the operator is defined in [BO00] and [Zag16] to be a power series in $\mathbb{Q}[[q]]$ instead of $\mathbb{C}[[q]]$, as those works take $f\colon \mathcal{P} \to \mathbb{Q}$. We may write the q-bracket in equivalent forms that will prove useful here:

$$\langle f \rangle_q = (q;q)_\infty \sum_{\lambda \in \mathcal{P}} f(\lambda) q^{|\lambda|} = (q;q)_\infty \sum_{n=0}^{\infty} q^n \sum_{\lambda \vdash n} f(\lambda). \tag{3.1}$$

A recent paper [Zag16] by Zagier examines the q-bracket operator from a number of enlightening perspectives, and finds broader classes of quasimodular forms arising from its application. This chapter is inspired by Zagier's treatment, as well as by ideas of Andrews [And98] and Alladi–Erdős [AE77].

While computationally, the q-bracket operator boils down to multiplying a power series by $(q;q)_\infty$ as in (3.1), conceptually the q-bracket represents a sort of weighted average of the function f over all partitions. Zagier gives an interpretation of the q-bracket as the "expectation value of an observable f in a statistical system whose states are labelled by partitions" [Zag16]. Such sums over partitions are ubiquitous in statistical mechanics and quantum physics. We will keep in the backs of our minds the poetic feeling that the partition-theoretic structures we encounter are, somehow, part of the fabric of physical reality.

We begin this chapter's study by considering the q-bracket of a prominent statistic in partition theory, the *rank* function $\mathrm{rk}(\lambda)$ introduced by Freeman Dyson [Dys44] to give combinatorial explanations for the Ramanujan congruences[1] $p(5n+4) \equiv 0 \pmod 5$ and $p(7n+5) \equiv 0 \pmod 7$, which we will define by $\mathrm{rk}(\emptyset) := 1$[2] and, for nonempty λ, by

$$\mathrm{rk}(\lambda) := \mathrm{lg}(\lambda) - \ell(\lambda)$$

where we let $\mathrm{lg}(\lambda)$ denote the *largest part* of the partition (similarly, we write $\mathrm{sm}(\lambda)$ for the *smallest part*). Noting that $\sum_{\lambda \vdash n} \mathrm{rk}(\lambda) = 1$ if $n = 0$ (i.e., if $\lambda = \emptyset$) and is equal to 0 otherwise, as conjugate partitions cancel in the sum and self-conjugate partitions have

[1] See [And98]

[2] This definition of $\mathrm{rk}(\emptyset)$ is nonstandardized, but fits conveniently with the ideas of this chapter.

rank zero, then

$$\sum_{\lambda\in\mathcal{P}} \mathrm{rk}(\lambda)q^{|\lambda|} = \sum_{n=0}^{\infty} q^n \sum_{\lambda\vdash n} \mathrm{rk}(\lambda) = 1.$$

Therefore we have that

$$\langle \mathrm{rk} \rangle_q = (q;q)_\infty. \tag{3.2}$$

We see by comparison with the Dedekind eta function $\eta(\tau) := q^{\frac{1}{24}}(q;q)_\infty$ that $\langle \mathrm{rk} \rangle_q$ is very nearly a weight-1/2 modular form.

Now, recall the weight-$2k$ Eisenstein series central to the theory of modular forms [Ono04]

$$E_{2k}(\tau) = 1 - \frac{4k}{B_{2k}} \sum_{n=1}^{\infty} \sigma_{2k-1}(n)q^n, \tag{3.3}$$

where $k \geq 1$, B_j denotes the jth Bernoulli number, σ_* is the classical sum-of-divisors function, and $q = e^{2\pi i\tau}$ as above. It is not hard to see the q-bracket of the "size" function

$$\langle | \cdot | \rangle_q = -q\frac{\frac{d}{dq}(q;q)_\infty}{(q;q)_\infty} = \frac{1 - E_2(\tau)}{24}$$

is essentially quasimodular; the series $E_2(\tau)$ is the prototype of a quasimodular form.

The near-modularity of the q-bracket of basic partition-theoretic functions is among the operator's most fascinating features. Bloch–Okounkov give a recipe for constructing quasimodular forms using q-brackets of shifted symmetric polynomials [BO00]. Zagier expands on their work to find infinite families of quasimodular q-brackets, including families that lie outside Bloch and Okounkov's methods [Zag16]. Griffin–Jameson–Trebat-Leder build on these methods to find p-adic modular and quasimodular forms as well [GJTL16]. While it appears at first glance to be little more than convenient shorthand, the q-bracket notation identifies — induces, even — intriguing classes of partition-theoretic phenomena.

In this study, we give an exact formula for the coefficients of $\langle f \rangle_q$ for any function $f\colon \mathcal{P} \to \mathbb{C}$. We also answer the converse problem, viz. for an arbitrary power series $\widehat{f}(q)$ we give a function F defined on $\mathcal{P}$ such that $\langle F \rangle_q = \widehat{f}(q)$ exactly. The main theorems appear in Section 3.4.

Along the way, we outline a simple, general multiplicative theory of integer partitions, which specializes to many fundamental results in classical number theory. In hopes of presenting a continuous story arc and preserving the flow of ideas, and because most of the proofs are closely analogous to classical cases, we suppress explicit proofs in this chapter, giving gestures and pertinent steps within the exposition, as needed.

We present an idealistic perspective: Multiplicative number theory in $\mathbb{Z}$ is a special case of vastly general combinatorial laws, one out of an infinity of parallel number theories in a partition-theoretic multiverse. It turns out the q-bracket operator plays a surprisingly natural role in this multiverse.

3.2 Multiplicative arithmetic of partitions

In Definition 1.2.1 we introduced a complementary statistic to the length $\ell(\lambda)$ and size $|\lambda|$ of $\lambda = (\lambda_1, \lambda_2, ..., \lambda_r)$, that we call the *norm* of the partition, viz.

$$n_\lambda := \lambda_1 \lambda_2 \cdots \lambda_r$$

with the convention $n_\emptyset := 1$ (it is an empty product). The norm may not seem to be a very natural statistic — after all, partitions are defined additively with no straightforward connection to multiplication — but this product of the parts shows up in partition-theoretic formulas scattered throughout the literature [And98, Fin88], and will prove to be important to the theory indicated here as well[3].

Recall from Definition 1.2.2 the product $\lambda\gamma$ of two partitions λ, γ (combine the parts and reorder into canonical weakly decreasing form). Then it makes sense to write $\lambda^2 :=$

[3]This statistic was first introduced in [Sch16, Sch17] as the "integer" of λ.

$\lambda\lambda, \lambda^3 := \lambda\lambda\lambda$, so on. It is easy to see that we have the following relations:

$$n_{\lambda\lambda'} = n_\lambda n_{\lambda'}, \qquad n_{\lambda^a} = n_\lambda^a,$$
$$\ell(\lambda\lambda') = \ell(\lambda) + \ell(\lambda'), \qquad \ell(\lambda^a) = a \cdot \ell(\lambda),$$
$$|\lambda\lambda'| = |\lambda| + |\lambda'|, \qquad |\lambda^a| = a|\lambda|.$$

Note that length and size both resemble logarithms.

In Definition 1.2.3 we also define division λ/δ of partitions λ, δ if δ is a subpartition of γ (delete the parts of δ from λ). Note that both the empty partition $\emptyset$ and λ itself are divisors of every partition λ. Then we also have the following relations:

$$n_{\lambda/\lambda'} = \frac{n_\lambda}{n_{\lambda'}}, \quad \ell(\lambda/\lambda') = \ell(\lambda) - \ell(\lambda'), \quad |\lambda/\lambda'| = |\lambda| - |\lambda'|.$$

On analogy to the prime numbers in classical arithmetic, the partitions into one part (e.g. $(1), (3), (4)$) are both prime and irreducible under this simple multiplication. The analog of the Fundamental Theorem of Arithmetic is trivial: of course, every partition may be uniquely decomposed into its parts. Thus we might rewrite a partition λ in terms of its "prime" factorization $\lambda = (a_1)^{m_1}(a_2)^{m_2}...(a_t)^{m_t}$, where $a_1 > a_2 > ... > a_t \geq 1$ are the distinct numbers appearing in λ such that $a_1 = \mathrm{lg}(\lambda)$ (the largest part of λ), $a_t = \mathrm{sm}(\lambda)$ (the smallest part), and m_i denotes the multiplicity of a_i as a part of λ. Clearly, then, we have

$$n_\lambda = a_1^{m_1} a_2^{m_2} \cdots a_t^{m_t}. \tag{3.4}$$

Remark. We note in passing that we also have a dual formula for the norm n_{λ^*} of the conjugate λ^* of λ, written in terms of λ, viz.

$$n_{\lambda^*} = M_1^{a_1-a_2} M_2^{a_2-a_3} \cdots M_{t-1}^{a_{t-1}-a_t} M_t^{a_t}, \tag{3.5}$$

where $M_k := \sum_{i=1}^{k} m_i$ (thus $M_t = \ell(\lambda)$), which is clear from the Ferrers-Young diagrams.

Fundamental classical concepts such as coprimality, greatest common divisor, least common multiple, etc., apply with exactly the same meanings in the partition-theoretic setting, if one replaces "prime factors of a number" with "parts of a partition" in the classical definitions.

Remark. If $\mathcal{P}' \subseteq \mathcal{P}$ is an infinite subset of $\mathcal{P}$ closed under partition multiplication and division, then the multiplicative theory presented in this study still holds when the relations are restricted to $\mathcal{P}'$.

3.3 Partition-theoretic analogs of classical functions

A number of important functions from classical number theory have partition-theoretic analogs, giving rise to nice summation identities that generalize their classical counterparts. One of the most fundamental classical arithmetic functions, related to factorization of integers, is the Möbius function. As in Definition 1.2.4, we can define a natural partition-theoretic analog of μ as well:

$$\mu_{\mathcal{P}}(\lambda) := \begin{cases} 1 & \text{if } \lambda = \emptyset, \\ 0 & \text{if } \lambda \text{ has any part repeated,} \\ (-1)^{\ell(\lambda)} & \text{otherwise.} \end{cases}$$

Just as in the classical case, we have by inclusion-exclusion the following, familiar relation.

Proposition 3.3.1. *Summing $\mu_{\mathcal{P}}(\delta)$ over the divisors δ of $\lambda \in \mathcal{P}$, we have*

$$\sum_{\delta|\lambda} \mu_{\mathcal{P}}(\delta) = \begin{cases} 1 & \textit{if } \lambda = \emptyset, \\ 0 & \textit{otherwise.} \end{cases}$$

Furthermore, we have a partition-theoretic generalization of the Möbius inversion formula, which is proved along the lines of proofs of the classical formula.

Proposition 3.3.2. *For a function $f\colon \mathcal{P} \to \mathbb{C}$ we have the equivalence*

$$F(\lambda) = \sum_{\delta|\lambda} f(\delta) \iff f(\lambda) = \sum_{\delta|\lambda} F(\delta)\mu_{\mathcal{P}}(\lambda/\delta).$$

Remark. Alladi has also considered the above partition Möbius function identities, in unpublished work[4].

In classical number theory, Möbius inversion is often used in conjunction with order-of-summation swapping principles for double summations. These have an obvious partition-theoretic generalization as well, reflected in the following identity.

Proposition 3.3.3. *Consider a double sum involving functions $f, g\colon \mathcal{P} \to \mathbb{C}$. Then we have the formula*

$$\sum_{\lambda\in\mathcal{P}} f(\lambda) \sum_{\delta|\lambda} g(\delta) = \sum_{\lambda\in\mathcal{P}} g(\lambda) \sum_{\gamma\in\mathcal{P}} f(\lambda\gamma).$$

The preceding propositions will prove useful in the next section, to evaluate the coefficients of the q-bracket operator.

Now, the classical Möbius function has a close companion in the Euler phi function $\varphi(n)$, also known as the totient function, which counts the number of natural numbers less than n that are coprime to n. This sort of statistic does not seem meaningful in the partition-theoretic frame of reference, as there is not generally a well-defined greater- or less-than ordering of partitions. However, if we sidestep this business of ordering and counting for the time being, we find it is possible to define a partition analog of φ which is naturally compatible with the identities above, as well as with classical identities involving the Euler phi function.

Recall that n_λ denotes the norm of λ, i.e., the product of its parts.

[4]K. Alladi, private communication, Dec. 21, 2015

Definition 3.3.1. *For $\lambda \in \mathcal{P}$ we define a partition-theoretic phi function $\varphi_{\mathcal{P}}(\lambda)$ by*

$$\varphi_{\mathcal{P}}(\lambda) := \ n_\lambda \prod_{\substack{\lambda_i \in \lambda \\ \textit{without} \\ \textit{repetition}}} (1 - \lambda_i^{-1}),$$

where the product is taken over only the distinct numbers composing λ, that is, the parts of λ without repetition.

Clearly if $1 \in \lambda$ then $\varphi_{\mathcal{P}}(\lambda) = 0$, which is a bit startling by comparison with the classical phi function that never vanishes. This phi function filters out partitions containing 1's.

As with the Möbius function above, the partition-theoretic $\varphi_{\mathcal{P}}(\lambda)$ yields generalizations of many classical expressions. For instance, there is a familiar-looking divisor sum, which is proved along classical lines.

Proposition 3.3.4. *We have that*

$$\sum_{\delta|\lambda} \varphi_{\mathcal{P}}(\delta) = n_\lambda.$$

We also find a partition analog of the well-known relation connecting the μ and φ.

Proposition 3.3.5. *We have the identity*

$$\varphi_{\mathcal{P}}(\lambda) = n_\lambda \sum_{\delta|\lambda} \frac{\mu_{\mathcal{P}}(\delta)}{n_\delta}.$$

Combining the above relations, we arrive at a nicely balanced identity.

Proposition 3.3.6. *For $f \colon \mathcal{P} \to \mathbb{C}$ let $F(\lambda) := \sum_{\delta|\lambda} f(\delta)$. Then we have*

$$\sum_{\lambda \in \mathcal{P}} \frac{\mu_{\mathcal{P}}(\lambda) F(\lambda)}{n_\lambda} = \sum_{\lambda \in \mathcal{P}} \frac{\varphi_{\mathcal{P}}(\lambda) f(\lambda)}{n_\lambda}.$$

Remark. Replacing $\mathcal{P}$ with the set $\mathcal{P}_{\mathbb{P}}$ of partitions into prime parts (the so-called "prime partitions"), then the divisor sum above takes the form $F(n) = \sum_{d|n} f(d)$ (with $n = n_\lambda$) and Proposition 3.3.6 specializes to the following identity, which is surely known classically:

$$\sum_{n=1}^{\infty} \frac{\mu(n)F(n)}{n} = \sum_{n=1}^{\infty} \frac{\varphi(n)f(n)}{n}.$$

A number of other important arithmetic functions have partition-theoretic analogs, too, such as the sum-of-divisors function σ_a.

Definition 3.3.2. *For $\lambda \in \mathcal{P}, a \in \mathbb{Z}_{\geq 0}$, we define the function*

$$\sigma_{\mathcal{P},a}(\lambda) := \sum_{\delta|\lambda} n_\delta^a,$$

with the convention $\sigma_{\mathcal{P}}(\lambda) := \sigma_{\mathcal{P},1}(\lambda)$.

One might wonder about "perfect partitions" or other analogous phenomena related to σ_a classically. This partition sum-of-divisors function will come into play in the next section. We note that the functions $\mu_{\mathcal{P}}, \varphi_{\mathcal{P}}$ and $\sigma_{\mathcal{P},a}$ are, just as in the classical cases, multiplicative in a partition sense.

Definition 3.3.3. *We say a function $f\colon \mathcal{P} \to \mathbb{C}$ is multiplicative (resp. completely multiplicative) if for $\lambda, \gamma \in \mathcal{P}$ with $\gcd(\lambda, \gamma) = \emptyset$ (resp. with no condition on the $\gcd$),*

$$f(\lambda\gamma) = f(\lambda)f(\gamma)$$

Another classical principle central to analysis is the Cauchy product formula, which gives the product of two infinite series in terms of the convolution of their summands. In the partition-theoretic setting, the Cauchy product takes the following form, in which the summands effectively give a partition version of Dirichlet convolution from multiplicative number theory (see [Apo13]).

Proposition 3.3.7. *Consider the product of two absolutely convergent sums over partitions, whose summands involve the functions $f, g\colon \mathcal{P} \to \mathbb{C}$. Then we have the formula*

$$\left(\sum_{\lambda\in\mathcal{P}} f(\lambda)\right)\left(\sum_{\lambda\in\mathcal{P}} g(\lambda)\right) = \sum_{\lambda\in\mathcal{P}}\sum_{\delta|\lambda} f(\delta)g(\lambda/\delta).$$

The proof of this partition Cauchy product proceeds exactly as in the classical case: we expand the left-hand side and compare the resulting terms[5]. Then it is immediate that the product of two partition-indexed power series for $|q| < 1$ is

$$\left(\sum_{\lambda\in\mathcal{P}} f(\lambda)q^{|\lambda|}\right)\left(\sum_{\lambda\in\mathcal{P}} g(\lambda)q^{|\lambda|}\right) = \sum_{\lambda\in\mathcal{P}} q^{|\lambda|}\sum_{\delta|\lambda} f(\delta)g(\lambda/\delta). \tag{3.6}$$

We reiterate, these familiar-looking identities not only mimic classical theorems, they fully generalize the classical cases. The definitions and propositions above all specialize to their classical counterparts when we restrict our attention to the set $\mathcal{P}_{\mathbb{P}}$ of prime partitions; then, as a rule-of-thumb, we just replace partitions with their "norms" in the formulas (other parameters may need to be adjusted too). This is due to the bijective correspondence between natural numbers and $\mathcal{P}_{\mathbb{P}}$ noted by Alladi and Erdős [AE77]: the set of "norms" of prime partitions (including $n_{\emptyset}$) is precisely the set of positive integers $\mathbb{Z}^+$, by the Fundamental Theorem of Arithmetic. Yet prime partitions form a narrow slice, so to speak, of the set $\mathcal{P}$ over which these general relations hold sway.

Many well-known laws of classical number theory arise as special cases of underlying partition-theoretic structures.

[5]In Appendix B we apply this partition Cauchy product formula to give coefficients of Ramanujan's tau function and the counting function for k-color partitions.

3.4 Role of the q-bracket

We return now to the q-bracket operator of Bloch–Okounkov, which we recall from Definition 1.2.6. The q-bracket arises naturally in the multiplicative theory outlined above. To see this, take $F(\lambda) = \sum_{\delta|\lambda} f(\delta)$ for $f\colon \mathcal{P} \to \mathbb{C}$. It follows from Proposition 3.3.3 that

$$\begin{aligned}\sum_{\lambda\in\mathcal{P}} F(\lambda)q^{|\lambda|} &= \sum_{\lambda\in\mathcal{P}} q^{|\lambda|} \sum_{\delta|\lambda} f(\delta) = \sum_{\lambda\in\mathcal{P}} f(\lambda) \sum_{\gamma\in\mathcal{P}} q^{|\lambda\gamma|} \\ &= \sum_{\lambda\in\mathcal{P}} f(\lambda) \sum_{\gamma\in\mathcal{P}} q^{|\lambda|+|\gamma|} = \left(\sum_{\lambda\in\mathcal{P}} f(\lambda)q^{|\lambda|}\right)\left(\sum_{\gamma\in\mathcal{P}} q^{|\gamma|}\right).\end{aligned}$$

Observing that the rightmost sum above is equal to $(q;q)_\infty^{-1}$, then by comparison with Definition 1.2.6 of the q-bracket operator, we arrive at the two central theorems of this study. Together they give a type of q-bracket inversion, converting divisor sums into power series, and vice versa.

Theorem 3.4.1. *For an arbitrary function $f\colon \mathcal{P} \to \mathbb{C}$, if*

$$F(\lambda) = \sum_{\delta|\lambda} f(\delta)$$

then

$$\langle F\rangle_q = \sum_{\lambda\in\mathcal{P}} f(\lambda)q^{|\lambda|}.$$

In the converse direction, we can also write down a simple function whose q-bracket is a given partition-indexed power series.

Theorem 3.4.2. *Consider an arbitrary power series of the form*

$$\sum_{\lambda\in\mathcal{P}} f(\lambda)q^{|\lambda|}.$$

Then we have the function $F\colon \mathcal{P} \to \mathbb{C}$ *given by*

$$F(\lambda) = \sum_{\delta|\lambda} f(\delta),$$

such that $\langle F \rangle_q = \sum f(\lambda) q^{|\lambda|}$.

These theorems are consequences of Theorem 3.3.7. We wish to apply Theorems 3.4.1 and 3.4.2 to examine the q-brackets of partition-theoretic analogs of classical functions introduced in Section 3.3.

Recall Definition 3.3.2 of the sum of divisors function $\sigma_{\mathcal{P},a}(\lambda)$. Then $\sigma_{\mathcal{P},0}(\lambda) = \sum_{\delta|\lambda} 1$ counts the number of partition divisors (i.e., sub-partitions) of $\lambda \in \mathcal{P}$, much as in the classical case. It is immediate from Theorem 3.4.1 that

$$\langle \sigma_{\mathcal{P},0} \rangle_q = (q;q)_\infty^{-1}. \tag{3.7}$$

If we note that $(q;q)_\infty$ is also a factor of the q-bracket on the left-hand side, we can see as well

$$\sum_{\lambda \in \mathcal{P}} \sigma_{\mathcal{P},0}(\lambda) q^{|\lambda|} = (q;q)_\infty^{-2}. \tag{3.8}$$

Remembering also from Equation 3.2 the identity $\langle \mathrm{rk} \rangle_q = (q;q)_\infty$, we have seen a few instances of interesting power series connected to powers of $(q;q)_\infty$ via the q-bracket operator.

Now let us recall the handful of partition divisor sum identities from Section 3.3 involving the partition-theoretic functions $\varphi_{\mathcal{P}}$, $\sigma_{\mathcal{P},a}$, and the "norm of a partition" function n_*. Theorem 3.4.1 reveals that these three functions form a close-knit family, related through (double) application of the q-bracket.

Corollary 3.4.1. *We have the pair of identities*

$$\langle \sigma_{\mathcal{P}} \rangle_q = \sum_{\lambda \in \mathcal{P}} n_\lambda q^{|\lambda|}$$
$$\langle n_* \rangle_q = \sum_{\lambda \in \mathcal{P}} \varphi_{\mathcal{P}}(\lambda) q^{|\lambda|}.$$

The coefficients of $\langle \sigma_{\mathcal{P}} \rangle_q$ are of the form n_*; applying the q-bracket a second time to the function n_* gives us the rightmost summation, whose coefficients are the values of $\varphi_{\mathcal{P}}$.

In fact, it is evident that this operation of applying the q-bracket more than once can be continued indefinitely; thus we feel the need to introduce a new notation, on analogy to differentiation.

Definition 3.4.1. *If we apply the q-bracket repeatedly, say $n \geq 0$ times, to the function f, we denote this operator by $\langle f \rangle_q^{(n)}$. We define $\langle f \rangle_q^{(n)}$ by the equation*

$$\langle f \rangle_q^{(n)} := (q;q)_\infty^n \sum_{\lambda \in \mathcal{P}} f(\lambda) q^{|\lambda|} \in \mathbb{C}[[q]].$$

Remark. It follows from the definition above that $\langle f \rangle_q^{(0)} = \sum_{\lambda \in \mathcal{P}} f(\lambda) q^{|\lambda|}$, $\langle f \rangle_q^{(1)} = \langle f \rangle_q$.

Theorem 3.4.2 gives us a converse construction as well, allowing us to write down a function $F(\lambda)$ whose q-bracket is a given power series, i.e., a *q-antibracket* from Definition 1.2.7. The act of taking the antibracket might be carried out repeatedly as well. We define a canonical class of q-antibrackets related to f by extending Definition **??** to allow for negative values of n.

Definition 3.4.2. *If we repeatedly divide the power series $\sum_{\lambda \in \mathcal{P}} f(\lambda) q^{|\lambda|}$ by $(q;q)_\infty$, say $n > 0$ times, we notate this operator as*

$$\langle f \rangle_q^{(-n)} := (q;q)_\infty^{-n} \sum_{\lambda \in \mathcal{P}} f(\lambda) q^{|\lambda|} \in \mathbb{C}[[q]].$$

We take the resulting power series to be indexed by partitions, unless otherwise specified.

We call the function on $\mathcal{P}$ defined by the coefficients of $\langle f\rangle_q^{(-1)}$ the "canonical q-antibracket" of f (or sometimes just "the antibracket").

Taken together, Definitions **??** and 3.4.2 describe an infinite family of q-brackets and antibrackets. The following identities give an example of such a family (and of the use of these new bracket notations).

Corollary 3.4.2. *Corollary 3.4.1 can be written more compactly as*

$$\langle\sigma_{\mathcal{P}}\rangle_q^{(2)} = \langle n_*\rangle_q^{(1)} = \langle\varphi_{\mathcal{P}}\rangle_q^{(0)}.$$

We can also condense Corollary 3.4.1 by writing

$$\langle\sigma_{\mathcal{P}}\rangle_q^{(0)} = \langle n_*\rangle_q^{(-1)} = \langle\varphi_{\mathcal{P}}\rangle_q^{(-2)}.$$

Both of the compact forms above preserve the essential message of Corollary 3.4.1, that these three partition-theoretic functions are directly connected through the q-bracket operator, or more concretely (and perhaps more astonishingly), simply through multiplication or division by powers of $(q;q)_\infty$.

Along similar lines, we can encode Equations (3.2), (3.7), and (3.8) in a single statement, noting an infinite family of power series that contains $\langle\mathrm{rk}\rangle_q$ and $\langle\sigma_{\mathcal{P},0}\rangle_q$.

Corollary 3.4.3. *For $n \in \mathbb{Z}$, we have the family of q-brackets*

$$\langle\mathrm{rk}\rangle_q^{(n)} = \langle\sigma_{\mathcal{P},0}\rangle_q^{(n+2)} = (q;q)_\infty^n.$$

Remark. Here we see the q-bracket connecting with modularity properties. For instance, another member of this family is $\langle\mathrm{rk}\rangle_q^{(24)} = q^{-1}\Delta(\tau)$, where Δ is the important modular discriminant function having Ramanujan's tau function as its coefficients [Ono04].

The identities above worked out easily because we knew in advance what the coefficients of the q-brackets should be, due to the divisor sum identities from Section 3.3. Theorems 3.4.1 and 3.4.2 provide a recipe for turning partition divisor sums F into coefficients f of power series, and vice versa.

However, generally a function $F\colon \mathcal{P} \to \mathbb{C}$ is not given as a sum over partition divisors. If we wished to write it in this form, what function $f\colon \mathcal{P} \to \mathbb{C}$ would make up the summands? In classical number theory this question is answered by the Möbius inversion formula; indeed, we have the partition-theoretic analog of this formula in Equation (3.3.2).

Recall the "divided by" notation λ/δ from Definition 1.2.3. Then we may write the function f (and thus the coefficients of $\langle F\rangle_q$) explicitly using partition Möbius inversion.

Theorem 3.4.3. *The q-bracket of the function $F\colon \mathcal{P} \to \mathbb{C}$ is given explicitly by*

$$\langle F\rangle_q = \sum_{\lambda\in\mathcal{P}} f(\lambda) q^{|\lambda|},$$

where the coefficients can be written in terms of F itself:

$$f(\lambda) = \sum_{\delta|\lambda} F(\delta)\mu_{\mathcal{P}}(\lambda/\delta).$$

We already know from Theorem 3.4.2 that the coefficients of the canonical antibracket of f are written as divisor sums over values of f. Thus, much like $\mathrm{rk}(\lambda)$ in Corollary 3.4.3, every function f defined on partitions can be viewed as the generator, so to speak, of the (possibly infinite) family of power series $\langle f\rangle_q^{(n)}$ for $n \in \mathbb{Z}$, whose coefficients can be written in terms of f as n-tuple sums of the shape $\sum_{\delta_1|\lambda}\sum_{\delta_2|\delta_1}\dots\sum_{\delta_n|\delta_{n-1}}$ constructed by repeated application of the above theorems.

This suggests the following useful fact.

Corollary 3.4.4. *If two power series are members of the family $\langle f\rangle_q^{(n)}$ $(n\in\mathbb{Z})$, then the coefficients of each series can be written explicitly in terms of the coefficients of the other.*

3.5 The q-antibracket and coefficients of power series over $\mathbb{Z}_{\geq 0}$

Theorems 3.4.1, 3.4.2, and 3.4.3 together provide a two-way map between the coefficients of families of power series indexed by partitions. In this section, we address the question of computing the antibracket (loosely speaking) of coefficients indexed not by partitions, but by natural numbers as usual. We remark immediately that a function defined on $Z_{\geq 0}$ may be expressed in terms of partitions in a number of ways, which are generally not equivalent. Thus there is more than one function $F\colon \mathcal{P} \to \mathbb{C}$ such that $\langle F \rangle_q = \sum_{n=0}^{\infty} c_n q^n$ for a given sequence c_n of coefficients. Here we treat only the canonical antibracket found using Theorem 3.4.2.

There are three classes of power series of the form $\sum_{n=0}^{\infty} c_n q^n$ that we examine: (1) the coefficients c_n are sums $\sum_{\lambda \vdash n}$ over partitions of n; (2) the coefficients c_n are sums $\sum_{d|n}$ over divisors of n; and (3) the coefficients c_n are an arbitrary sequence of complex numbers.

The class (1) above is already given by Theorem 3.4.1; to keep this section relatively self-contained, we rephrase the result here.

Corollary 3.5.1. *For $c_n = \sum_{\lambda \vdash n} f(\lambda)$ we can write*

$$\sum_{n=0}^{\infty} c_n q^n = \sum_{\lambda \in \mathcal{P}} f(\lambda) q^{|\lambda|}.$$

Then we have a function $F(\lambda) = \sum_{\delta | \lambda} f(\delta)$ such that $\langle F \rangle_q = \sum_{n=0}^{\infty} c_n q^n$.

Thus the power series of class (1) are already in a form subject to the q-bracket machinery detailed in the previous section. The class (2) with coefficients of the form $\sum_{d|n}$ is a little more subtle. We introduce a special subset $\mathcal{P}_{=}$ which bridges sums over partitions and sums over the divisors of natural numbers.

Definition 3.5.1. *We define the subset $\mathcal{P}_= \subseteq \mathcal{P}$ to be the set of partitions into equal parts, that is, whose parts are all the same positive number, e.g.* $(1), (1,1), (4,4,4)$. *We make the assumption $\emptyset \notin \mathcal{P}_=$, as the empty partition has no positive parts.*

The divisors of n correspond exactly (in two different ways) to the set of partitions of n into equal parts, i.e., partitions of n in $\mathcal{P}_=$. For example, compare the divisors of 6

$$1, 2, 3, 6$$

with the partitions of 6 into equal parts

$$(6), (3,3), (2,2,2), (1,1,1,1,1,1).$$

Note that for each of the above partitions $(a, a, ..., a) \vdash 6$, we have that $a \cdot \ell((a, a, ..., a)) = 6$. We see from this example that for any $n \in \mathbb{Z}^+$ we can uniquely associate each divisor $d|n$ to a partition $\lambda \vdash n, \lambda \in \mathcal{P}_=$, by taking d to be the length of λ. (Alternatively, we could identify the divisor d with $\mathrm{lg}(\lambda)$ or $\mathrm{sm}(\lambda)$, as defined above, which of course are the same in this case. We choose here to associate divisors to $\ell(\lambda)$ as length is a universal characteristic of partitions, regardless of the structure of the parts.)

By the above considerations, it is clear that

$$\sum_{d|n} f(d) = \sum_{\substack{\lambda \vdash n \\ \lambda \in \mathcal{P}_=}} f(\ell(\lambda)). \tag{3.9}$$

This leads us to a formula for the coefficients of a power series of the class (2) discussed above.

Corollary 3.5.2. *For $c_n = \sum_{d|n} f(d)$ we can write*

$$\sum_{n=0}^{\infty} c_n q^n = \sum_{\lambda \in \mathcal{P}_=} f(\ell(\lambda))\, q^{|\lambda|}.$$

Then we have a function

$$F(\lambda) = \sum_{\substack{\delta|\lambda \\ \delta\in\mathcal{P}_=}} f\left(\ell(\delta)\right)$$

such that $\langle F\rangle_q = \sum_{n=0}^{\infty} c_n q^n$.

The completely general class (3) of power series with arbitrary coefficients $c_n \in \mathbb{C}$ follows right away from Corollary 3.5.2 by classical Möbius inversion, as $f(n) := \sum_{d|n} c_d\ \mu_{\mathcal{P}}(n/d) \Rightarrow c_n = \sum_{d|n} f(d)$.

Corollary 3.5.3. *For* $c_n \in \mathbb{C}$ *we can write*

$$\sum_{n=0}^{\infty} c_n q^n = \sum_{\lambda\in\mathcal{P}_=} q^{|\lambda|} \sum_{d|\ell(\lambda)} c_d\ \mu\left(\frac{\ell(\lambda)}{d}\right).$$

Then we have a function

$$F(\lambda) = \sum_{\substack{\delta|\lambda \\ \delta\in\mathcal{P}_=}} \sum_{d|\ell(\delta)} c_d\ \mu\left(\frac{\ell(\delta)}{d}\right)$$

such that $\langle F\rangle_q = \sum_{n=0}^{\infty} c_n q^n$.

Remark. We point out an alternative expression for sums of the shape of $F(\lambda)$ here, that can be useful for computation. If we write out the factorization of a partition $\lambda = (a_1)^{m_1}(a_2)^{m_2}\ldots\ (a_t)^{m_t}$ as in Section 3.2, a divisor of λ lying in $\mathcal{P}_=$ must be of the form $(a_i)^m$ for some $1 \leq i \leq t$ and $1 \leq m \leq m_i$. Then for any function ϕ defined on $\mathbb{Z}^+$ we see

$$\sum_{\substack{\delta|\lambda \\ \delta\in\mathcal{P}_=}} \phi\left(\ell(\delta)\right) = \sum_{i=1}^{t}\sum_{j=1}^{m_i} \phi\left(\ell((a_i)^j)\right) = \sum_{i=1}^{t}\sum_{j=1}^{m_i} \phi(j). \tag{3.10}$$

Given the ideas developed above, we can now pass between q-brackets and arbitrary power series, summed over either natural numbers or partitions.

3.6 Applications of the q-bracket and q-antibracket

We close this report by briefly illustrating some of the methods of the previous sections through two examples.

3.6.1 Sum of divisors function

In classical number theory, for $a \geq 0$ the divisor sum $\sigma_{\mathcal{P},a}(n) := \sum_{d|n} d^a$ is particularly important to the theory of modular forms; as seen in Equation 3.3, for odd values of a, power series of the form

$$\sum_{n=0}^{\infty} \sigma_{\mathcal{P},a}(n) q^n$$

comprise the Fourier expansions of Eisenstein series [Ono04], which are the building blocks of modular and quasimodular forms. As a straightforward application of Corollary 3.5.2 following directly from the definition of $\sigma_{\mathcal{P},a}(n)$, we give a function $\mathcal{S}_a$ defined on partitions whose q-bracket is the power series above.

Corollary 3.6.1. *We have the partition-theoretic function*

$$\mathcal{S}_a(\lambda) := \sum_{\substack{\delta|\lambda \\ \delta \in \mathcal{P}_=}} \ell(\delta)^a$$

such that

$$\langle \mathcal{S}_a \rangle_q = \sum_{n=0}^{\infty} \sigma_{\mathcal{P},a}(n) q^n.$$

Remark. We note that Zagier gives a different function $S_{2k-1}(\lambda) = \sum_{\lambda_i \in \lambda} \lambda_i^{2k-1}$ (the moment function) that also has the q-bracket $\sum \sigma_{2k-1}(n) q^n$ [Zag16]. This is an example of the non-uniqueness of antibrackets of functions defined on natural numbers noted previously.

Thus we see the q-bracket operator brushing up against modularity, once again.

3.6.2 Reciprocal of the Jacobi triple product

We turn our attention now to another fundamental object in the subject of modular forms. Let $j(z;q)$ denote the classical *Jacobi triple product* [Ber06]

$$j(z;q) := (z;q)_\infty (z^{-1}q;q)_\infty (q;q)_\infty. \tag{3.11}$$

The reciprocal of the triple product

$$j(z;q)^{-1} = \sum_{\lambda\in\mathcal{P}} j_z(\lambda) q^{|\lambda|}$$

is interesting in its own right. For instance, $j(z;q)^{-1}$ plays a role not unlike the role played by $(q;q)_\infty$ in the q-bracket operator, for the Appell–Lerch sum $m(x,q,z)$ important to the study of mock modular forms (see [BFOR17, HM14]).

Our goal will be to derive a formula for the coefficients $j_z(\lambda)$ above. If we multiply $j(z;q)^{-1}$ by $(1-z)$ to cancel the pole at $z=1$, it behaves nicely under the action of the q-bracket. Let us write

$$(1-z)j(z;q)^{-1} = \frac{1}{(zq;q)_\infty (z^{-1}q;q)_\infty (q;q)_\infty} = \sum_{\lambda\in\mathcal{P}} J_z(\lambda) q^{|\lambda|}. \tag{3.12}$$

Let $\operatorname{crk}(\lambda)$ denote the *crank* of a partition, an important partition-theoretic statistic whose existence was conjectured by Dyson [Dys44] to explain the Ramanujan congruence $p(11n+7) \equiv 0 \pmod{11}$, and which was written down almost half a century later by Andrews and Garvan [AG88]. Crank is not unlike Dyson's rank, but is a bit more complicated.

Definition 3.6.1. *The crank* $\operatorname{crk}(\lambda)$ *of a partition* λ *is equal to its largest part if the multiplicity* $m_1(\lambda)$ *of 1 as a part of* λ *is* $=0$ *(that is, there are no 1's), and if* $m_1(\lambda) > 0$ *then* $\operatorname{crk}(\lambda) = \#\{$*parts of* λ *that are larger than* $m_1(\lambda)\} - m_1(\lambda)$.

We define $M(n,m)$ to be the number of partitions of n having crank equal to $m \in \mathbb{Z}$; then the Andrews–Garvan *crank generating function* $C(z;q)$ is given by

$$C(z;q) := \frac{(q;q)_\infty}{(zq;q)_\infty(z^{-1}q;q)_\infty} = \sum_{n=0}^{\infty} M_z(n)q^n, \tag{3.13}$$

where we set

$$M_z(n) := \sum_{\lambda \vdash n} z^{\mathrm{crk}(\lambda)} = \sum_{m=-\infty}^{\infty} M(n,m)z^m. \tag{3.14}$$

The function $C(z;q)$ has deep connections. When $z = 1$, Equation 3.13 reduces to Euler's partition generating function formula [Ber06]. For $\zeta \neq 1$ a root of unity, $C(\zeta;q)$ is a modular form, and Folsom–Ono–Rhoades show the crank generating function to be related to the theory of quantum modular forms [FOR13].

Comparing Equations 3.12 and 3.13, we have the following relation:

$$\langle J_z \rangle_q^{(2)} = C(z;q). \tag{3.15}$$

We see $J_z(\lambda)$ and $z^{\mathrm{crk}(\lambda)}$ are related through a family of q-brackets; then using Corollaries 3.5.1, 3.5.2, and 3.5.3, we can write $J_z(\lambda)$ explicitly. Noting that $J_z(\lambda) = (1-z)j_z(\lambda)$, we arrive at the formula we seek.

Corollary 3.6.2. *The partition-indexed coefficients of $j(z;q)^{-1}$ are*

$$j_z(\lambda) = (1-z)^{-1} \sum_{\delta|\lambda} \sum_{\varepsilon|\delta} z^{\mathrm{crk}(\varepsilon)}$$

for $z \neq 1$. In terms of the coefficients $M_z(*)$ given by Equation 3.14:

$$j_z(\lambda) = (1-z)^{-1} \sum_{\delta|\lambda} \sum_{\substack{\varepsilon|\delta \\ \varepsilon \in \mathcal{P}_=}} \sum_{d|\ell(\varepsilon)} M_z(d)\mu\left(\frac{\ell(\varepsilon)}{d}\right).$$

Remark. By Corollary 3.4.4, we can also write $M_z(n)$ in terms of the coefficients of

$j(z;q)^{-1}$.

We apply the q-bracket operator to the function $j(z;q)^{-1}$ from a somewhat different perspective in Chapter 8.

Remark. See Appendix B for further notes on Chapter 3.

Chapter 4

Partition-theoretic zeta functions

Adapted from [Sch16]

4.1 Introduction, notations and central theorem

The additive-multiplicative connections highlighted in the previous chapter extend to other realms of multiplicative number theory as well, viz., the study of zeta functions and Dirichlet series in analytic number theory[1].

We need to introduce one more notation, in order to state the central theorem of this chapter. Define $\varphi_n(f;q)$ by $\varphi_0(f;q) := 1$ and

$$\varphi_n(f;q) := \prod_{k=1}^{n}(1 - f(k)q^k)$$

where $n \geq 1$, for an arbitrary function $f\colon \mathbb{N} \to \mathbb{C}$. When the infinite product converges, let $\varphi_\infty(f;q) := \lim_{n\to\infty} \varphi_n(f;q)$. We think of φ as a generalization of the q-Pochhammer symbol. Note that if we set f equal to a constant z, then φ does specialize to the q-Pochhammer symbol, as $\varphi_n(z;q) = (zq;q)_n$ and $\varphi_\infty(z;q) = (zq;q)_\infty$.

[1] See [Apo90], Ch. 6, for connections between Dirichlet series and q-series via the theory of modular forms.

As with Euler product formula and partition generating function formula (1.1) and (1.3), respectively, it is the reciprocal $1/\varphi_\infty(f;q)$ that interests us. With the above notations, we have the following system of identities.

Theorem 4.1.1. *If the product converges, then* $1/\varphi_\infty(f;q) = \prod_{n=1}^{\infty}(1-f(n)q^n)^{-1}$ *may be expressed in a number of equivalent forms, viz.*

$$\frac{1}{\varphi_\infty(f;q)} = \sum_{\lambda\in\mathcal{P}} q^{|\lambda|} \prod_{\lambda_i\in\lambda} f(\lambda_i) \tag{4.1}$$

$$= 1 + \sum_{n=1}^{\infty} q^n \frac{f(n)}{\varphi_n(f;q)} \tag{4.2}$$

$$= 1 + \frac{1}{\varphi_\infty(f;q)} \sum_{n=1}^{\infty} q^n f(n) \varphi_{n-1}(f;q) \tag{4.3}$$

$$= 1 + \sum_{n=1}^{\infty} \frac{(-1)^n (q^{-1})^{\frac{n(n-1)}{2}}}{\varphi_n\left(\frac{1}{f};q^{-1}\right)\prod_{k=1}^{n-1} f(k)} \tag{4.4}$$

$$= 1 + \cfrac{\sum_{(6)}}{1 - \cfrac{\sum_{(5)}}{1 + \cfrac{\sum_{(6)}}{1 - \cfrac{\sum_{(5)}}{1+\cdots}}}} \tag{4.5}$$

where $\sum_{(5)}, \sum_{(6)}$ *in* (4.5) *denote the summations appearing in* (4.2) *and* (4.3)*, respectively.*

The product on the right-hand side of identity (4.1) above is taken over the parts λ_i of λ. Note that the summation in (4.4) converges for q^{-1} outside the unit circle (it may converge inside the circle as well). Note also that, by L'Hospital's rule, any power series $\sum_{n=1}^{\infty} f(n)q^n$ with constant term zero can be written as the limit

$$\sum_{n=1}^{\infty} f(n)q^n = \lim_{z\to 0} z^{-1}\left(\frac{1}{\varphi_\infty(zf;q)} - 1\right).$$

It is obvious that if f is completely multiplicative, then $\prod_{\lambda_i \in \lambda} f(\lambda_i) = f(n_\lambda)$, where n_λ is the *norm* of λ defined above. We record one more, obvious consequence of Theorem 4.1.1, as we assume it throughout this paper. As before, let $\mathbb{X} \subseteq \mathbb{Z}^+$, and take $\mathcal{P}_\mathbb{X} \subseteq \mathcal{P}$ to be the set of partitions into elements of $\mathbb{X}$. Then clearly by setting $f(n) = 0$ if $n \notin \mathbb{X}$ in Theorem 4.1.1, we see

$$\frac{1}{\prod_{n\in\mathbb{X}}(1-f(n)q^n)} = \sum_{\lambda\in\mathcal{P}_\mathbb{X}} q^{|\lambda|} \prod_{\lambda_i\in\lambda} f(\lambda_i).$$

The remaining summations in the theorem (aside from (4.4), which may not converge) are taken over $n \in \mathbb{X}$.

We see from Theorem 4.1.1 that we may pass freely between the shapes (4.1) – (4.5), which specialize to a number of classical expressions. For example, taking $f \equiv 1$ in the theorem gives the following fact.

Corollary 4.1.2. *The partition generating function formula* (1.3) *is true.*

Assuming $\mathrm{Re}(s) > 1$, if we take $q = 1$, $f(n) = 1/n^s$ if n is prime and $= 0$ otherwise, then Theorem 4.1.1 yields another classical fact, plus a formula giving the zeta function as a sum over primes.

Corollary 4.1.3. *The Euler product formula* (1.1) *for the zeta function is true. We also have the identity*

$$\zeta(s) = 1 + \sum_{p\in\mathbb{P}} \frac{1}{p^s \prod_{r\in\mathbb{P},\, r\leq p}\left(1-\frac{1}{r^s}\right)}.$$

Finally, tying this section in with the previous chapter, we give generating functions for the partition-theoretic phi function $\varphi_\mathcal{P}$ and sum of divisors function $\sigma_\mathcal{P}$. Setting $f(n) = n$ in Theorem 4.1.1, it is clear we have

$$\prod_{n=1}^{\infty}(1-nq^n)^{-1} = \sum_{\lambda\in\mathcal{P}} n_\lambda q^{|\lambda|},$$

the generating function for the norm n_λ. Then Proposition 3.4.1 yields the following.

Corollary 4.1.4. *We have the identities*

$$\prod_{n=1}^{\infty} \frac{1-q^n}{1-nq^n} = \sum_{\lambda\in\mathcal{P}} \varphi_{\mathcal{P}}(\lambda) q^{|\lambda|}, \qquad \prod_{n=1}^{\infty} \frac{1}{(1-q^n)(1-nq^n)} = \sum_{\lambda\in\mathcal{P}} \sigma_{\mathcal{P}}(\lambda) q^{|\lambda|}.$$

4.2 Partition-theoretic zeta functions

A multitude of nice specializations of Theorem 4.1.1 may be obtained. We would like to focus on an interesting class of partition sums arising from Euler's product formula for the sine function

$$x\prod_{n=1}^{\infty}\left(1-\frac{x^2}{\pi^2 n^2}\right) = \sin x, \tag{4.6}$$

combined with Theorem 4.1.1. Taking $q = 1$ (as we have done in Corollary 4.1.3), we begin by noting an easy partition-theoretic formula that may be used to compute the value of π.

Let $\mathcal{P}_{m\mathbb{Z}} \subseteq \mathcal{P}$ denote the set of partitions into multiples of m. Recall from above that the *norm* n_λ of a partition λ is the product $\lambda_1\lambda_2\cdots\lambda_r$ of its parts.

Corollary 4.2.1. *Summing over partitions into even parts, we have the formula*

$$\frac{\pi}{2} = \sum_{\lambda\in\mathcal{P}_{2\mathbb{Z}}} \frac{1}{n_\lambda^2}.$$

We notice that the form of the sum of the right-hand side resembles $\zeta(2)$. Based on this similarity, we wonder if there exists a nice partition-theoretic analog of $\zeta(s)$ possessing something of a familiar zeta function structure — perhaps Corollary 4.2.1 gives an example of such a function? However, in this case it is not so: the above identity arises from different types of phenomena from those associated with $\zeta(s)$. We have an infinite family of formulas of the following shapes.

Corollary 4.2.2. *Summing over partitions into multiples of any whole number $m > 1$, we have*

$$\sum_{\lambda \in \mathcal{P}_{m\mathbb{Z}}} \frac{1}{n_\lambda^2} = \frac{\pi}{m \sin\left(\frac{\pi}{m}\right)} \tag{4.7}$$

$$\sum_{\lambda \in \mathcal{P}_{m\mathbb{Z}}} \frac{1}{n_\lambda^4} = \frac{\pi^2}{m^2 \sin\left(\frac{\pi}{m}\right) \sinh\left(\frac{\pi}{m}\right)}, \tag{4.8}$$

and increasingly complicated formulas can be computed for $\sum_{\lambda \in \mathcal{P}_{m\mathbb{Z}}} 1/n_\lambda^{2^t}$, $t \in \mathbb{Z}^+$.

Examples like these are appealing, but their right-hand sides are not entirely reminiscent of the Riemann zeta function, aside from the presence of π. Certainly they are not as tidy as expressions of the form $\zeta(2k) =$ "$\pi^{2k} \times$ rational". Based on the previous corollaries, a reasonable first guess at a partition-theoretic analog of $\zeta(s)$ might be to define

$$\zeta_{\mathcal{P}}(s) := \sum_{\lambda \in \mathcal{P}} \frac{1}{n_\lambda^s} = \frac{1}{\prod_{n=1}^{\infty}\left(1 - \frac{1}{n^s}\right)}, \mathrm{Re}(s) > 1.$$

Of course, neither side of this identity converges, but using Definition 1.2.8 of a partition zeta function $\zeta_{\mathcal{P}'}(s)$ from Chapter 1, viz.

$$\zeta_{\mathcal{P}'}(s) := \sum_{\lambda \in \mathcal{P}'} n_\lambda^{-s}$$

for a subset $\mathcal{P}'$ of $\mathcal{P}$ and $s \in \mathbb{C}$ for which the series converges, we obtain convergent expressions if we omit the first term and perhaps subsequent terms of the product to yield $\zeta_{\mathcal{P}_{\geq a}}(s) := \sum_{\lambda \in \mathcal{P}_{\geq a}} 1/n_\lambda^s = \prod_{n=a}^{\infty}(1 - 1/n^s)^{-1}$ $(a \geq 2)$, where $\mathcal{P}_{\geq a} \subset \mathcal{P}$ denotes the set of partitions into parts greater than or equal to a. For instance, we have the following formula.

Corollary 4.2.3. *Summing over partitions into parts greater than or equal to* 2*, we have*

$$\zeta_{\mathcal{P}_{\geq 2}}(3) = \sum_{\lambda \in \mathcal{P}_{\geq 2}} \frac{1}{n_\lambda^3} = \frac{3\pi}{\cosh\left(\frac{1}{2}\pi\sqrt{3}\right)}.$$

While it is an interesting expression, stemming from an identity of Ramanujan [Ram00], once again this formula does not seem to evoke the sort of structure we anticipate from a zeta function — of course, the value of $\zeta(3)$ is not even known. We need to find the "right" subset of $\mathcal{P}$ to sum over, if we hope to find a nice partition-theoretic zeta function. As it turns out, there are subsets of $\mathcal{P}$ that naturally produce analogs of $\zeta(s)$ for certain arguments s.

Definition 4.2.1. *We define a partition-theoretic generalization $\zeta_{\mathcal{P}}(\{s\}^k)$ of the Riemann zeta function by the following sum over all partitions λ of fixed length $\ell(\lambda) = k \in \mathbb{Z}_{\geq 0}$ at argument $s \in \mathbb{C}$, $\mathrm{Re}(s) > 1$:*

$$\zeta_{\mathcal{P}}(\{s\}^k) := \sum_{\ell(\lambda)=k} \frac{1}{n_\lambda^s}. \tag{4.9}$$

Remark. This is a fairly natural formation, being similar in shape (and notation) to the weight k multiple zeta function $\zeta(\{s\}^k)$, which is instead summed over length-k partitions into distinct parts; Hoffman gives interesting formulas relating $\zeta_{\mathcal{P}}(\{s\}^k)$ (in different notation) to combinations of multiple zeta functions [Hof92], which exhibit rich structure.

We have immediately that $\zeta_{\mathcal{P}}(\{s\}^0) = 1/n_\emptyset^s = 1$ and $\zeta_{\mathcal{P}}(\{s\}^1) = \zeta_{\mathcal{P}}(\{s\}) = \zeta(s)$. Using Theorem 4.1.1 and proceeding (see Section 4.3) much as Euler did to find the value of $\zeta(2k)$ [Dun99], we are able to find explicit values for $\zeta_{\mathcal{P}}(\{2\}^k)$ at every positive integer $k > 0$. Somewhat surprisingly, we find that in these cases $\zeta_{\mathcal{P}}(\{2\}^k)$ is a rational multiple of $\zeta(2k)$.

Corollary 4.2.4. *For $k > 0$, we have the identity*

$$\zeta_{\mathcal{P}}(\{2\}^k) = \sum_{\ell(\lambda)=k} \frac{1}{n_\lambda^2} = \frac{2^{2k-1}-1}{2^{2k-2}}\zeta(2k).$$

For example, we have the following values:

$$\begin{aligned}
\zeta_{\mathcal{P}}(\{2\}) &= \zeta(2) = \frac{\pi^2}{6},\\
\zeta_{\mathcal{P}}(\{2\}^2) &= \frac{7}{4}\zeta(4) = \frac{7\pi^4}{360},\\
\zeta_{\mathcal{P}}(\{2\}^3) &= \frac{31}{16}\zeta(6) = \frac{31\pi^6}{15120},\ldots,\\
\zeta_{\mathcal{P}}(\{2\}^{13}) &= \frac{33554431}{16777216}\zeta(26) = \frac{22076500342261\pi^{26}}{93067260259985915904000000},\ldots
\end{aligned}$$

Corollary 4.2.4 reveals that $\zeta_{\mathcal{P}}(\{2\}^k)$ is indeed of the form "π^{2k} × rational" for all positive k, like the zeta values $\zeta(2k)$ given by Euler (we note that $\zeta(26)$ is the highest zeta value Euler published) [Dun99]. We have more: we can find $\zeta_{\mathcal{P}}(\{2^t\}^k)$ explicitly for all $t \in \mathbb{Z}^+$. These values are finite combinations of well-known zeta values, and are also of the form "$\pi^{2^t k}$ × rational".

Corollary 4.2.5. *For $k > 0$ we have the identity*

$$\begin{aligned}
\zeta_{\mathcal{P}}(\{4\}^k) &= \sum_{\ell(\lambda)=k} \frac{1}{n_\lambda^4}\\
&= \frac{1}{16^{k-1}}\left(\sum_{n=0}^{2k}(-1)^n(2^{2n-1}-1)(2^{4k-2n-1}-1)\zeta(2n)\zeta(4k-2n)\right),
\end{aligned}$$

and increasingly complicated formulas can be computed for $\zeta_{\mathcal{P}}(\{2^t\}^k), t \in \mathbb{Z}^+$.

Remark. The summation on the far right above may be shortened by noting the symmetry of the summands around the $n = k$ term.

It would be desirable to understand the value of $\zeta_{\mathcal{P}}(\{s\}^k)$ at other arguments s; the proof we give below (see Section 4.3) does not shed much light on this question, being based very closely on Euler's formula (4.1.3), which forces s be a power of 2. Also, if we solve Corollary 4.2.3 for $\zeta(0)$, we conclude that $\zeta(0) = \frac{2^{-2}}{2^{-1}-1}\zeta_{\mathcal{P}}(\{2\}^0) = -1/2$, which is the value of $\zeta(0)$ under analytic continuation. Can $\zeta_{\mathcal{P}}(\{s\}^k)$ be extended via analytic continuation for values of $k > 1$? In a larger sense we wonder: do nice zeta function analogs exist if we sum over other interesting subsets of $\mathcal{P}$? In Chapter 5 we will follow up on these questions.

We do have a few general properties shared by convergent series $\sum 1/n_\lambda^s$ summed over large subclasses of $\mathcal{P}$. First we need to refine some of our previous notations.

Definition 4.2.2. *Take any subset of partitions $\mathcal{P}' \subseteq \mathcal{P}$. Then for $\mathrm{Re}(s) > 1$, on analogy to classical zeta function theory, when these expressions converge we define*

$$\zeta_{\mathcal{P}'} := \sum_{\lambda \in \mathcal{P}'} \frac{1}{n_\lambda^s}, \qquad \eta_{\mathcal{P}'}(s) := \sum_{\lambda \in \mathcal{P}'} \frac{(-1)^{\ell(\lambda)}}{n_\lambda^s}, \qquad \zeta_{\mathcal{P}'}(\{s\}^k) := \sum_{\substack{\lambda \in \mathcal{P}' \\ \ell(\lambda) = k}} \frac{1}{n_\lambda^s}. \tag{4.10}$$

Remark. As important special cases, we have $\zeta_{\mathcal{P}_{\mathbb{P}}}(s) = \zeta(s)$ and $\zeta_{\mathcal{P}_{\mathbb{Z}^+}}(\{s\}^k) = \zeta_{\mathcal{P}}(\{s\}^k)$. It is also easy to see that $\zeta_{\mathcal{P}'}(s) = \sum_{k=0}^{\infty} \zeta_{\mathcal{P}'}(\{s\}^k)$ and $\eta_{\mathcal{P}'}(s) = \sum_{k=0}^{\infty} (-1)^k \zeta_{\mathcal{P}'}(\{s\}^k)$ if we assume absolute convergence. Moreover, given absolute convergence, we may write $\zeta_{\mathcal{P}'}(s)$, $\zeta_{\mathcal{P}'}(\{s\}^k)$ as classical Dirichlet series related to multiplicative partitions: we have $\zeta_{\mathcal{P}'}(s) = \sum_{j=1}^{\infty} \#\{\lambda \in \mathcal{P}' \mid n_\lambda = j\}\, j^{-s}$ and $\zeta_{\mathcal{P}'}(\{s\}^k)(s) = \sum_{j=1}^{\infty} \#\{\lambda \in \mathcal{P}' \mid \ell(\lambda) = k, n_\lambda = j\}\, j^{-s}$ (see [CS13] for more about multiplicative partitions).

As previously, take $\mathbb{X} \subseteq \mathbb{Z}^+$ and take $\mathcal{P}_{\mathbb{X}} \subseteq \mathcal{P}$ to denote partitions into elements of $\mathbb{X}$ (thus $\mathcal{P}_{\mathbb{Z}^+} = \mathcal{P}$). Note that $\zeta_{\mathcal{P}_{\mathbb{X}}}(s) = \prod_{n \in \mathbb{X}} \left(1 - \frac{1}{n^s}\right)^{-1}$ is divergent if $1 \in \mathbb{X}$ and, when $\mathbb{X}$ is finite (thus there is no restriction on the value of $\mathrm{Re}(s)$), if $s = i\pi j/\log n$ for any $n \in \mathbb{X}$ and even integer j. Similarly, when $\mathbb{X}$ is finite, $\eta_{\mathcal{P}_{\mathbb{X}}}(s) = \prod_{n \in \mathbb{X}} \left(1 + \frac{1}{n^s}\right)^{-1}$ is divergent if $s = i\pi k/\log n$ for any $n \in \mathbb{X}$ and odd integer k. Clearly if $\mathbb{Y} \subseteq \mathbb{Z}^+$, then from the product representations we also have $\zeta_{\mathcal{P}_{\mathbb{X}}}(s)\zeta_{\mathcal{P}_{\mathbb{Y}}}(s) = \zeta_{\mathcal{P}_{\mathbb{X} \cup Y}}(s)\zeta_{\mathcal{P}_{\mathbb{X} \cap \mathbb{Y}}}(s)$ and $\eta_{\mathcal{P}_{\mathbb{X}}}(s)\eta_{\mathcal{P}_{\mathbb{Y}}}(s) = \eta_{\mathcal{P}_{\mathbb{X} \cup \mathbb{Y}}}(s)\eta_{\mathcal{P}_{\mathbb{X} \cap \mathbb{Y}}}(s)$.

Many interesting subsets of partitions have the form $\mathcal{P}_{\mathbb{X}}$, in particular those to which Theorem 4.1.1 most readily applies. Note that such subsets $\mathcal{P}_{\mathbb{X}}$ are partition ideals of order 1, in the sense of Andrews [And98]. With the above notations, we have the following useful "doubling" formulas.

Corollary 4.2.6. *If $\zeta_{\mathcal{P}_{\mathbb{X}}}(s)$ converges over $\mathcal{P}_{\mathbb{X}} \subseteq \mathcal{P}$, then*

$$\zeta_{\mathcal{P}_{\mathbb{X}}}(2s) = \zeta_{\mathcal{P}_{\mathbb{X}}}(s)\eta_{\mathcal{P}_{\mathbb{X}}}(s). \tag{4.11}$$

Furthermore, for $n \in \mathbb{Z}_{\geq 0}$ we have the identity

$$\zeta_{\mathcal{P}_{\mathbb{X}}}\left(\{2^{n+1}s\}^k\right) = \sum_{j=0}^{2^n k}(-1)^j \zeta_{\mathcal{P}_{\mathbb{X}}}\left(\{2^n s\}^j\right)\zeta_{\mathcal{P}_{\mathbb{X}}}\left(\{2^n s\}^{2^n k - j}\right). \tag{4.12}$$

Remark. As in Corollary 4.2.5, the summation on the right-hand side of (4.12) may be shortened by symmetry.

If we take $\mathbb{X} = \mathbb{P}$, then (4.11) reduces to the classical identity $\zeta(2s) = \zeta(s)\sum_{n=1}^{\infty}\lambda(n)/n^s$, where $\lambda(n)$ is Liouville's function. Another specialization of Corollary 4.2.6 leads to new information about $\zeta_{\mathcal{P}}(\{s\}^k)$: we may extend the domain of $\zeta_{\mathcal{P}}(\{s\}^k)$ to $\mathrm{Re}(s) > 1$ if we take $\mathbb{X} = \mathbb{Z}^+$, $n = 0$, $k = 2$. We find $\zeta_{\mathcal{P}}(\{s\}^2)$ inherits analytic continuation from the sum on the right-hand side below.

Corollary 4.2.7. *For* $\mathrm{Re}(s) > 1$, *we have*

$$\zeta_{\mathcal{P}}(\{s\}^2) = \frac{\zeta(2s) + \zeta(s)^2}{2}.$$

Remark. This resembles the series shuffle product formula for multiple zeta values [BF06].

Another interesting subset of $\mathcal{P}$ is the set of partitions $\mathcal{P}^*$ into *distinct* parts; also of interest is the set of partitions $\mathcal{P}^*_{\mathbb{X}}$ into distinct elements of $\mathbb{X} \subseteq \mathbb{Z}^+$ (thus $\mathcal{P}^*_{\mathbb{Z}^+} = \mathcal{P}^*$). However, partitions into distinct parts are not immediately compatible with the identities in Theorem 4.1.1. Happily, we have a dual theorem that leads us to zeta functions summed over $\mathcal{P}^*_{\mathbb{X}}$ for any $\mathbb{X} \subseteq \mathbb{Z}^+$. Let us recall the infinite product $\varphi_\infty(f;q)$ from Theorem 4.1.1.

Theorem 4.2.8. *If the product converges, then* $\varphi_\infty(f;q) = \prod_{n=1}^{\infty}(1 - f(n)q^n)$ *may be expressed in a number of equivalent forms, viz.*

$$\varphi_\infty(f;q) = \sum_{\lambda\in\mathcal{P}^*}(-1)^{\ell(\lambda)}q^{|\lambda|}\prod_{\lambda_i\in\lambda}f(\lambda_i) \tag{4.13}$$

$$= \ 1 - \sum\nolimits_{(6)} \tag{4.14}$$

$$= \ 1 - \varphi_\infty(f;q)\sum\nolimits_{(5)} \tag{4.15}$$

$$= 1 - \cfrac{\sum_{(5)}}{1 + \cfrac{\sum_{(6)}}{1 - \cfrac{\sum_{(5)}}{1 + \cfrac{\sum_{(6)}}{1 - \cdots}}}} \tag{4.16}$$

where $\sum_{(5)}, \sum_{(6)}$ *are exactly as in Theorem 4.1.1, and the sum in (4.13) is taken over the partitions into distinct parts.*

Remark. Note that there is not a nice "inverted" sum of the form (4.4) here.

Just as with Theorem 4.1.1, we may write arbitrary power series as limiting cases, and we have the obvious identity

$$\prod_{n\in\mathbb{X}}(1 - f(n)q^n) = \sum_{\lambda\in\mathcal{P}^*_{\mathbb{X}}}(-1)^{\ell(\lambda)}q^{|\lambda|}\prod_{\lambda_i\in\lambda}f(\lambda_i), \tag{4.17}$$

with the remaining summations in Theorem 4.2.8 being taken over elements of $\mathbb{X}$.

Remark. Clearly, the summation on the right-hand side of (4.17), as well as the $\mathbb{X} = \mathbb{Z}^+$ case (4.13), can be rewritten in the form

$$\sum_{\lambda\in\mathcal{P}_{\mathbb{X}}}\mu_{\mathcal{P}}(\lambda)q^{|\lambda|}\prod_{\lambda_i\in\lambda}f(\lambda_i).$$

However, to keep our notations absolutely general in this chapter from a set-theoretic perspective, we will for the most part label subsets of partitions into distinct parts with the "$*$" superscript, as opposed to filtering out terms with repeated parts using $\mu_{\mathcal{P}}$.

For completeness, we record another obvious but useful consequence of Theorems 4.1.1 and 4.2.8. The following statement might be viewed as a generalized eta quotient formula, with coefficients given explicitly by finite combinatorial sums[2].

Corollary 4.2.9. *For f_j defined on $\mathbb{X}_j \subseteq \mathbb{Z}^+$, consider the double product*

$$\prod_{j=1}^{n} \prod_{k_j \in \mathbb{X}_j} \left(1 \pm f_j(k_j) q^{k_j}\right)^{\pm 1} = \sum_{k=0}^{\infty} c_k q^k,$$

where the $\pm$ sign is fixed for fixed j, but may vary as j varies. Then the coefficients c_k are given by the $(n-1)$-tuple sum

$$c_k = \sum_{k_2=0}^{k} \sum_{k_3=0}^{k_2} \cdots \sum_{k_n=0}^{k_{n-1}} \left(\sum_{\substack{\lambda \vdash k_n \\ \lambda \in \mathcal{P}^{\pm}_{\mathbb{X}_n}}} \prod_{\lambda_i \in \lambda} f_n(\lambda_i) \right) \left(\sum_{\substack{\lambda \vdash (k_{n-1} - k_n) \\ \lambda \in \mathcal{P}^{\pm}_{\mathbb{X}_{n-1}}}} \prod_{\lambda_i \in \lambda} f_{n-1}(\lambda_i) \right)$$
$$\times \left(\sum_{\substack{\lambda \vdash (k_{n-2} - k_{n-1}) \\ \lambda \in \mathcal{P}^{\pm}_{\mathbb{X}_{n-2}}}} \prod_{\lambda_i \in \lambda} f_{n-2}(\lambda_i) \right) \cdots \left(\sum_{\substack{\lambda \vdash (k - k_2) \\ \lambda \in \mathcal{P}^{\pm}_{\mathbb{X}_1}}} \prod_{\lambda_i \in \lambda} f_1(\lambda_i) \right)$$

in which we have set $\mathcal{P}^{-}_{\mathbb{X}_j} := \mathcal{P}_{\mathbb{X}_j}$ and $\mathcal{P}^{+}_{\mathbb{X}_j} := \mathcal{P}^{*}_{\mathbb{X}_j}$ with the $\pm$ sign as associated to each j above.

Remark. The $+$ or $-$ signs in the formula for c_k indicate partitions arising from the numerator or denominator, respectively, of the double product. One may replace f_j with $-f_j$ to effect further sign changes.

Analogous corollaries to those following Theorem 4.1.1 are available, but we wish right away to apply this theorem to the problem at hand, the investigation of partition zeta

[2] In Appendix C we discuss an interesting class of "sequentially congruent" partitions suggested by this formula.

functions. We have

$$\zeta_{\mathcal{P}^*_{\mathbb{X}}}(s) = \prod_{n\in\mathbb{X}}(1+\frac{1}{n^s}), \qquad \eta_{\mathcal{P}^*_{\mathbb{X}}}(s) = \prod_{n\in\mathbb{X}}(1-\frac{1}{n^s}), \tag{4.18}$$

again noting that in fact

$$\eta_{\mathcal{P}^*_{\mathbb{X}}}(s) = \sum_{\lambda\in\mathcal{P}}\mu_{\mathcal{P}}(\lambda)n_\lambda^{-s}.$$

It is immediate then from (4.13) that for $\mathrm{Re}(s) > 1$ we also have the following relations, where the sum on the left-hand side of each equation is taken over the partitions into distinct elements of $\mathbb{X}$:

$$\zeta_{\mathcal{P}^*_{\mathbb{X}}}(s) = \frac{1}{\eta_{\mathcal{P}_{\mathbb{X}}}(s)}, \qquad \eta_{\mathcal{P}^*_{\mathbb{X}}}(s) = \frac{1}{\zeta_{\mathcal{P}_{\mathbb{X}}}(s)}. \tag{4.19}$$

Note that $\zeta_{\mathcal{P}^*_{\mathbb{X}}}(s)$ and $\eta_{\mathcal{P}^*_{\mathbb{X}}}(s)$ are finite sums (and entire functions of s) if $\mathbb{X}$ is a finite set, unlike $\zeta_{\mathcal{P}_{\mathbb{X}}}(s)$ and $\eta_{\mathcal{P}_{\mathbb{X}}}(s)$. Note also that $\eta_{\mathcal{P}^*_{\mathbb{X}}}(s) = 0$ identically if $1 \in \mathbb{X}$, with zeros when $\mathbb{X}$ is finite at the values $s = i\pi j/\log n$ for any $n \in \mathbb{X}$ and j even. Unlike $\zeta_{\mathcal{P}}(s)$, we can see from (4.19) that $\zeta_{\mathcal{P}^*}(s)$ is well-defined on $\mathrm{Re}(s) > 1$ (thus both $\zeta_{\mathcal{P}^*_{\mathbb{X}}}$ and $\eta_{\mathcal{P}^*_{\mathbb{X}}}$ are well-defined over all subsets $\mathcal{P}^*_{\mathbb{X}}$ of $\mathcal{P}^*$); when $\mathbb{X}$ is finite, $\zeta_{\mathcal{P}^*}(s)$ has zeros at $s = i\pi k/\log n$ for $n \in \mathbb{X}$ and k odd. Morever, we have $\zeta_{\mathcal{P}^*_{\mathbb{X}}}(s)\zeta_{\mathcal{P}^*_{\mathbb{Y}}}(s) = \zeta_{\mathcal{P}^*_{\mathbb{X}\cup\mathbb{Y}}}(s)\zeta_{\mathcal{P}^*_{\mathbb{X}\cap\mathbb{Y}}}(s)$ and $\eta_{\mathcal{P}^*_{\mathbb{X}}}(s)\eta_{\mathcal{P}^*_{\mathbb{Y}}}(s) = \eta_{\mathcal{P}^*_{\mathbb{X}\cup\mathbb{Y}}}(s)\eta_{\mathcal{P}^*_{\mathbb{X}\cap\mathbb{Y}}}(s)$. Here is an example of a zeta sum of this form.

Corollary 4.2.10. *Summing over partitions into distinct parts, we have that*

$$\zeta_{\mathcal{P}^*}(2) = \sum_{\lambda\in\mathcal{P}^*}\frac{1}{n_\lambda^2} = \frac{\sinh\pi}{\pi}.$$

Zeta sums over partitions into distinct parts admit an important special case: as we remarked beneath definition (4.9), the multiple zeta function $\zeta(\{s\}^k)$ can be written

$$\zeta(\{s\}^k) := \sum_{\lambda_1>\lambda_2>\cdots>\lambda_k\geq 1} \frac{1}{\lambda_1^s\lambda_2^s\cdots\lambda_k^s} = \sum_{\substack{\lambda\,\in\,\mathcal{P}^*\\ \ell(\lambda)\,=\,k}} \frac{1}{n_\lambda^s} = \zeta_{\mathcal{P}^*}(\{s\}^k). \tag{4.20}$$

Using this notation, we can derive even simpler formulas for the multiple zeta values $\zeta(\{2^t\}^k)$ than those found for $\zeta_{\mathcal{P}}(\{2^t\}^k)$ in Corollaries 4.2.4 and 4.2.5, such as these.

Corollary 4.2.11. *For $k > 0$ we have the identities*

$$\begin{aligned}
\zeta(\{2\}^k) &= \frac{\pi^{2k}}{(2k+1)!},\\
\zeta(\{4\}^k) &= \pi^{4k}\sum_{n=0}^{2k}\frac{(-1)^n}{(2n+1)!(4k-2n+1)!},\\
\zeta(\{8\}^k) &= \pi^{8k}\sum_{n=0}^{4k}(-1)^n\left(\sum_{i=0}^{n}\frac{(-1)^i}{(2i+1)!(2n-2i+1)!}\right)\\
&\qquad\qquad\times\left(\sum_{i=0}^{4k-n}\frac{(-1)^i}{(2i+1)!(8k-2n-2i+1)!}\right),
\end{aligned}$$

and increasingly complicated formulas of the shape "$\pi^{2^t k}$ × finite sum of fractions" can be computed for multiple zeta values of the form $\zeta(\{2^t\}^k)$, $t \in \mathbb{Z}^+$.

Remark. The first identity above is proved in [Hof92] by a different approach from that taken here (see Section 4.3); it is possible the other identities in the corollary are also known.

The summations in Corollary 4.2.11 arise from quite general properties: we have these "doubling" formulas comparable to Corollary 4.2.6.

Corollary 4.2.12. *If $\zeta_{\mathcal{P}^*_{\mathbb{X}}}(s)$ converges over $\mathcal{P}^*_{\mathbb{X}} \subseteq \mathcal{P}$, then*

$$\eta_{\mathcal{P}^*_{\mathbb{X}}}(2s) = \eta_{\mathcal{P}^*_{\mathbb{X}}}(s)\zeta_{\mathcal{P}^*_{\mathbb{X}}}(s). \tag{4.21}$$

Furthermore, for $n \in \mathbb{Z}_{\geq 0}$ we have

$$\zeta_{\mathcal{P}^*_{\mathbb{X}}}\left(\{2^{n+1}s\}^k\right) = \sum_{j=0}^{2^n k} (-1)^j \zeta_{\mathcal{P}^*_{\mathbb{X}}}\left(\{2^n s\}^j\right) \zeta_{\mathcal{P}^*_{\mathbb{X}}}\left(\{2^n s\}^{2^n k - j}\right). \tag{4.22}$$

Remark. Once again, the summation on the right-hand side of (4.22) may be be shortened by symmetry. Equation (4.22) yields a family of multiple zeta function identities when we let $\mathbb{X} = \mathbb{Z}^+$.

We note that by recursive arguments, from (4.11) and (4.21) together with (4.5), we have these curious product formulas connecting sums over partitions into distinct parts to their counterparts involving unrestricted partitions:

$$\zeta_{\mathcal{P}^*_{\mathbb{X}}}(s)\zeta_{\mathcal{P}^*_{\mathbb{X}}}(2s)\zeta_{\mathcal{P}^*_{\mathbb{X}}}(4s)\zeta_{\mathcal{P}^*_{\mathbb{X}}}(8s)\cdots = \zeta_{\mathcal{P}_{\mathbb{X}}}(s),$$

$$\eta_{\mathcal{P}_{\mathbb{X}}}(s)\eta_{\mathcal{P}_{\mathbb{X}}}(2s)\eta_{\mathcal{P}_{\mathbb{X}}}(4s)\eta_{\mathcal{P}_{\mathbb{X}}}(8s)\cdots = \eta_{\mathcal{P}^*_{\mathbb{X}}}(s).$$

Now, if we take $\mathbb{X} = \mathbb{P}$ then (4.21) becomes the well-known classical identity $\zeta(2s)^{-1} = \zeta(s)^{-1}\sum_{n=1}^{\infty} |\mu(n)|/n^s$, where $\mu(n)$ is the Möbius function. As we have noted, the quantity $(-1)^{\ell(\lambda)}$ is exactly $\mu_{\mathcal{P}}(\lambda)$ when λ is in $\mathcal{P}^*$, and otherwise is a partition version of Liouville's function which specializes to the classical Liouville's function when we consider unrestricted prime partitions.

The literature abounds with product formulas which, when fed through the machinery of the identities noted here, produce nice partition zeta sum variants; the interested reader is referred to [CS13] as a starting point for further study.

4.3 Proofs of theorems and corollaries

Proof of Theorem 4.1.1. Identity (4.1) appears in a different form as [Fin88, Eq. 22.16]. The proof proceeds formally, much like the standard proof of (4.1.1); we expand $1/\varphi_\infty(f;q)$

as a product of geometric series

$$\frac{1}{\varphi_\infty(f;q)} = (1 + f(1)q + f(1)^2q^2 + f(1)^3q^3 + \dots) \\ \times (1 + f(2)q^2 + f(2)^2q^4 + f(2)^3q^6 + \dots) \times \cdots$$

and multiply out all the terms (without collecting coefficients in the usual way). The result is the partition sum in (4.1).

Identities (4.2) and (4.3) are proved using telescoping sums. Consider that

$$\begin{aligned}\frac{1}{\varphi_\infty(f;q)} &= \frac{1}{\varphi_0(f;q)} + \sum_{n=1}^{\infty}\left(\frac{1}{\varphi_n(f;q)} - \frac{1}{\varphi_{n-1}(f;q)}\right) \\ &= 1 + \sum_{n=1}^{\infty}\frac{1}{\varphi_{n-1}(f;q)}\left(\frac{1}{1-f(n)q^n} - 1\right) \\ &= 1 + \sum_{n=1}^{\infty} q^n \frac{f(n)}{\varphi_n(f;q)} = \ 1 + \sum\nolimits_{(5)}\ ,\end{aligned}$$

recalling the notation $\sum_{(5)}$ (as well as $\sum_{(6)}$) from the theorem, which is (4.2). Similarly, we can show

$$\begin{aligned}\varphi_\infty(f;q) &= \varphi_0(f;q) + \sum_{n=1}^{\infty}(\varphi_n(f;q) - \varphi_{n-1}(f;q)) \\ &= 1 - \sum_{n=1}^{\infty} q^n f(n)\varphi_{n-1}(f;q) = \ 1 - \sum\nolimits_{(6)}\ .\end{aligned}$$

Thus we have

$$\sum\nolimits_{(5)} \ = \frac{1}{\varphi_\infty(f;q)} - 1 = \frac{1-\varphi_\infty(f;q)}{\varphi_\infty(f;q)} = \frac{\sum_{(6)}}{\varphi_\infty(f;q)},$$

which leads to (4.3).

To prove (4.4), substitute the identity

$$\varphi_n(f;q) = \prod_{k=1}^{n}(1-f(k)q^k) = (-1)^n q^{n(n+1)/2}\varphi_n(1/f;q^{-1})\prod_{k=1}^{n} f(k)$$

term-by-term into the sum (4.2) and simplify to find the desired expression.

The proof of (4.5) is inspired by the standard proof of the continued fraction representation of the golden ratio. It follows from the proof above of (4.2) and (4.3) that

$$\begin{aligned}\frac{1}{\varphi_\infty(f;q)} &= 1 + \frac{\sum_{(6)}}{\varphi_\infty(f;q)} \\ &= 1 + \frac{\sum_{(6)}}{1 - \varphi_\infty(f;q)\sum_{(5)}} \\ &= 1 + \cfrac{\sum_{(6)}}{1 - \cfrac{\sum_{(5)}}{1/\varphi_\infty(f;q)}}.\end{aligned}$$

We notice that the expression on the left-hand side is also present on the far right in the denominator. We replace this term $1/\varphi_\infty(f;q)$ in the denominator with the entire right-hand side of the equation; reiterating this process indefinitely gives (4.5). □

Remark. The series $\sum_{(5)}, \sum_{(6)}$ enjoy other nice, golden ratio-like relationships. For instance, because

$$(1 + \textstyle\sum_{(5)})(1 - \sum_{(6)}) = 1 \ ,$$

it is easy to see that

$$\textstyle\sum_{(5)} - \sum_{(6)} = \sum_{(5)} \sum_{(6)},$$

which resembles the formula $\varphi - 1/\varphi = \varphi \cdot 1/\varphi$ involving the golden ratio φ and its reciprocal.

Proof of Corollary 4.1.2. This is immediate upon letting $f \equiv 1$ in (4.1), as

$$\sum_{\lambda \in \mathcal{P}} q^{|\lambda|} \prod_{\lambda_i \in \lambda} f(\lambda_i) = 1 + \sum_{n=1}^{\infty} q^n \sum_{\lambda \vdash n} \prod_{\lambda_i \in \lambda} f(\lambda_i).$$

□

Proof of Corollary 4.1.3. As noted above, we assume $\mathrm{Re}(s) > 1$. Let $q = 1$, $f(n) = 1/n^s$ if n is prime and $= 0$ otherwise; then by (4.1)

$$\frac{1}{\prod_{p\in\mathbb{P}}\left(1-\frac{1}{p^s}\right)} = \sum_{\lambda\in\mathcal{P}_{\mathbb{P}}} \frac{1}{n_\lambda^s}.$$

Consider the prime decomposition of a positive integer $n = p_1^{a_1}p_2^{a_2}\cdots p_r^{a_r}$, $p_1 > p_2 > \cdots > p_r$. We will associate this decomposition to the unique partition into prime parts $\lambda = (p_1, \ldots, p_1, p_2, \ldots, p_2, \ldots, p_r, \ldots, p_r) \in \mathcal{P}$, where $p_k \in \mathbb{P}$ is repeated a_k times (thus n is equal to n_λ). As he have discussed previously, every positive integer $n \geq 1$ is associated to exactly one partition into prime parts (with $n = 1$ associated to $\emptyset \in \mathcal{P}_{\mathbb{P}}$), and conversely: there is a bijective correspondence between $\mathbb{Z}^+$ and $\mathcal{P}_{\mathbb{P}}$. Therefore we see by absolute convergence that

$$\sum_{n\geq 1}\frac{1}{n^s} = \sum_{\lambda\in\mathcal{P}_{\mathbb{P}}}\frac{1}{n_\lambda^s}.$$

Equating the left-hand sides of the above two identities gives Euler's product formula (1.1). The series given for $\zeta(s)$ follows immediately from Theorem (4.2) with the above definition of f. $\square$

Proof of Corollary 4.2.1. This is actually a special case of the subsequent Corollary 4.2.2, setting $m = 2$ in the first equation (see below). $\square$

Proof of Corollary 4.2.2. We begin with an identity equivalent to (4.6) and its "+" companion:

$$\frac{\pi z}{\sin(\pi z)} = \frac{1}{\prod_{n=1}^{\infty}\left(1-\frac{z^2}{n^2}\right)}, \qquad \frac{\pi z}{\sinh(\pi z)} = \frac{1}{\prod_{n=1}^{\infty}\left(1+\frac{z^2}{n^2}\right)}.$$

If $\omega_k := e^{2\pi i/k}$, then $\omega_{2k}^2 = \omega_k$ and we have, by multiplying the above two identities, the pair

$$\frac{\pi^2 z^2}{\sin(\pi z)\sinh(\pi z)} = \frac{1}{\prod_{n=1}^{\infty}\left(1-\frac{z^4}{n^4}\right)}, \quad \frac{\omega_4 \pi^2 z^2}{\sin(\omega_8 \pi z)\sinh(\omega_8 \pi z)} = \frac{1}{\prod_{n=1}^{\infty}\left(1+\frac{z^4}{n^4}\right)}.$$

Multiplying these two equations, and repeating this procedure indefinitely, we find identities like

$$\frac{\omega_4 \pi^4 z^4}{\sin(\pi z)\sinh(\pi z)\sin(\omega_8 \pi z)\sinh(\omega_8 \pi z)} = \frac{1}{\prod_{n=1}^{\infty}\left(1-\frac{z^8}{n^8}\right)},$$

$$\begin{aligned}\frac{\omega_4^2 \pi^8 z^8}{\sin(\pi z)\sinh(\pi z)\sin(\omega_8 \pi z)\sinh(\omega_8 \pi z)}& \\ \times \frac{1}{\sin(\omega_{16}\pi z)\sinh(\omega_{16}\pi z)\sin(\omega_8\omega_{16}\pi z)\sinh(\omega_8\omega_{16}\pi z)}& \\ = \frac{1}{\prod_{n=1}^{\infty}\left(1-\frac{z^{16}}{n^{16}}\right)},&\end{aligned}$$

as well as their “+" companions, and so on. On the other hand, it follows from (4.1) that

$$\frac{1}{\prod_{n=1}^{\infty}\left(1-\frac{zq^n}{n^s}\right)} = \sum_{\lambda\in\mathcal{P}} q^{|\lambda|}\prod_{\lambda_i\in\lambda}\frac{z}{\lambda_i^s} = \sum_{\lambda\in\mathcal{P}}\frac{z^{\ell(\lambda)}q^{|\lambda|}}{n_\lambda^s}.$$

Replacing z with $\pm z^{2^t}$ and taking $q=1$ in the above expression, it is easy to see that we have

$$\frac{1}{\prod_{n=1}^{\infty}\left(1-\frac{z^{2^t}}{n^{2^t}}\right)} = \sum_{\lambda\in\mathcal{P}}\frac{z^{2^t\ell(\lambda)}}{n_\lambda^{2^t}}, \qquad \frac{1}{\prod_{n=1}^{\infty}\left(1+\frac{z^{2^t}}{n^{2^t}}\right)} = \sum_{\lambda\in\mathcal{P}}\frac{(-1)^{\ell(\lambda)}z^{2^t\ell(\lambda)}}{n_\lambda^{2^t}}.$$

These series have closed forms given by complicated trigonometric and hyperbolic expressions such as the ones above. Setting $z = 1/m$ in such expressions yields the explicit values advertised in the corollary for

$$\frac{1}{\prod_{n=1}^{\infty}\left(1-\frac{1}{m^{2^t}n^{2^t}}\right)} = \frac{1}{\prod_{n=1}^{\infty}\left(1-\frac{1}{(mn)^{2^t}}\right)}$$
$$= \frac{1}{\prod_{n\equiv 0 \,(\mathrm{mod}\ m)}\left(1-\frac{1}{n^{2^t}}\right)} = \sum_{\lambda\in\mathcal{P}_{m\mathbb{Z}}} \frac{1}{n_\lambda^{2^t}}.$$

□

Remark. More generally, let $\mathcal{P}_{a(m)}$ denote the set of partitions into parts $\equiv a \pmod m$ (so $\mathcal{P}_{m\mathbb{Z}}$ is $\mathcal{P}_{0(m)}$ in this notation). It is clear that if $\lambda \in \mathcal{P}_{a(m)}$ then $n_\lambda^s \equiv a^s \pmod m$, thus we find

$$\frac{1}{\prod_{n\equiv a\,(\mathrm{mod}\ m)}(1-n^s q^n)} = \sum_{\lambda\in\mathcal{P}_{a(m)}} n_\lambda^s q^{|\lambda|} \equiv \frac{1}{(a^s q^a; q^m)_\infty} \pmod m.$$

Of course, these expressions diverge as $q \to 1$ so $\zeta_{\mathcal{P}_{a(m)}}(-s)$ does not make sense, but we wonder: do there exist similarly nice relations that involve $\zeta_{\mathcal{P}_{a(m)}}(s)$ or a related form?

Proof of Corollary 4.2.3. We apply (4.1) to the following formula submitted by Ramanujan as a problem to the *Journal of the Indian Mathematical Society*, reprinted as [Ram00, Question 261]:

$$\prod_{n=2}^{\infty}\left(1-\frac{1}{n^3}\right) = \frac{\cosh\left(\frac{1}{2}\pi\sqrt{3}\right)}{3\pi}.$$

Take $q = 1$, $f(n) = 1/n^3$ if $n > 1$ and $= 0$ otherwise in (4.1). Comparing the result with the above formula gives the corollary. □

Remark. Ramanujan gives a companion formula $\prod_{n=1}^{\infty}\left(1+\frac{1}{n^3}\right) = \cosh\left(\frac{1}{2}\pi\sqrt{3}\right)/\pi$ in the same problem [Ram00]. Multiplying this infinite product by the one above and using (4.1) yields a closed form for $\sum_{\lambda\in\mathcal{P}_{\geq 2}} 1/n_\lambda^6$ as well.

Proof of Corollary 4.2.4. Consider the sequence β_{2k} of coefficients of the expansion

$$\frac{z}{\sinh z} = \frac{1}{\prod_{n=1}^{\infty}\left(1+\frac{z^2}{\pi^2 n^2}\right)} = \sum_{k=0}^{\infty} \beta_{2k} z^{2k}. \tag{4.23}$$

From the Maclaurin series for the hyperbolic cosecant and Euler's work relating the zeta function to the Bernoulli numbers, it follows that

$$\beta_{2k} = \frac{4(-1)^k(2^{2k-1}-1)\zeta(2k)}{(2\pi)^{2k}}. \tag{4.24}$$

On the other hand, from (4.1) we have

$$\frac{1}{\prod_{n=1}^{\infty}\left(1+\frac{z^2}{\pi^2 n^2}\right)} = \sum_{\lambda\in\mathcal{P}} \frac{(-1)^{\ell(\lambda)} z^{2\ell(\lambda)}}{\pi^{2\ell(\lambda)} n_\lambda^2} = \sum_{k=0}^{\infty} \frac{(-1)^k z^{2k}}{\pi^{2k}} \sum_{\ell(\lambda)=k} \frac{1}{n_\lambda^2},$$

thus

$$\beta_{2k} = \frac{(-1)^k}{\pi^{2k}} \zeta_{\mathcal{P}}(\{2\}^k).$$

The corollary is immediate by comparing the two expressions for β_{2k} above. □

Proof of Corollary 4.2.5. Much as in the proof of Corollary 4.2.4 above, we have from (4.6) that

$$\frac{z}{\sin z} = \sum_{k=0}^{\infty} \frac{z^{2k}}{\pi^{2k}} \sum_{\ell(\lambda)=k} \frac{1}{n_\lambda^2} = \sum_{k=0}^{\infty} \alpha_{2k} z^{2k}$$

with

$$\alpha_{2k} = \frac{4(2^{2k-1}-1)\zeta(2k)}{(2\pi)^{2k}} = (-1)^k \beta_{2k}. \tag{4.25}$$

Using the Cauchy product

$$\left(\sum_{k=0}^{\infty} a_k z^k\right)\left(\sum_{k=0}^{\infty} b_k z^k\right) = \sum_{k=0}^{\infty} z^k \sum_{n=0}^{k} a_n b_{k-n}, \tag{4.26}$$

we see after some arithmetic

$$\frac{z^2}{\sin z \sinh z} = \left(\sum_{k=0}^{\infty} \alpha_{2k} z^{2k}\right)\left(\sum_{k=0}^{\infty} \beta_{2k} z^{2k}\right) = \sum_{k=0}^{\infty} \gamma_{4k} z^{4k},$$

where

$$\gamma_{4k} = \sum_{n=0}^{2k} \alpha_{2n}\beta_{4k-2n},$$

with α_*, β_* as in (4.25),(4.26) respectively. On the other hand, the proof of Corollary 4.2.2 implies

$$\frac{z^2}{\sin z \sinh z} = \frac{1}{\prod_{n=1}^{\infty}\left(1 - \frac{z^4}{\pi^4 n^4}\right)} = \sum_{k=0}^{\infty} \frac{z^{4k}}{\pi^{4k}} \sum_{\ell(\lambda)=k} \frac{1}{n_\lambda^4},$$

thus

$$\gamma_{4k} = \frac{1}{\pi^{4k}} \zeta_{\mathcal{P}}(\{4\}^k).$$

Comparing the two expressions for γ_{4k} above, the theorem follows, just as in the previous proof.

We can carry this approach further to find $\zeta_{\mathcal{P}}(\{2^t\}^k)$ for $t > 2$, much as in the proof of Corollary 4.2.2. For instance, to find $\zeta_{\mathcal{P}}(\{8\}^k)$ we begin by noting

$$\begin{aligned}\left(\sum_{k=0}^{\infty} \frac{z^{4k}}{\pi^{4k}} \zeta_{\mathcal{P}}(\{4\}^k)\right)\left(\sum_{k=0}^{\infty} \frac{(-1)^k z^{4k}}{\pi^{4k}} \zeta_{\mathcal{P}}(\{4\}^k)\right) &= \frac{1}{\prod_{n=1}^{\infty}\left(1 - \frac{z^4}{\pi^4 n^4}\right)\left(1 + \frac{z^4}{\pi^4 n^4}\right)} \\ &= \sum_{k=0}^{\infty} \frac{z^{8k}}{\pi^{8k}} \zeta_{\mathcal{P}}(\{8\}^k).\end{aligned}$$

We compare the coefficients on the left-and right-hand sides, using (4.26) to compute the coefficients on the left. Likewise, for $\zeta_{\mathcal{P}}(\{16\}^k)$ we compare the coefficients on both sides of the equation

$$\left(\sum_{k=0}^{\infty} \frac{z^{8k}}{\pi^{8k}} \zeta_{\mathcal{P}}(\{8\}^k)\right)\left(\sum_{k=0}^{\infty} \frac{(-1)^k z^{8k}}{\pi^{8k}} \zeta_{\mathcal{P}}(\{8\}^k)\right) = \sum_{k=0}^{\infty} \frac{z^{16k}}{\pi^{16k}} \zeta_{\mathcal{P}}(\{16\}^k),$$

and so on, recursively, to find $\zeta_{\mathcal{P}}(\{2^t\}^k)$ as t increases. It is clear from induction that

$\zeta_{\mathcal{P}}(\{2^t\}^k)$ is of the form "π^{2^t} $\times$ rational" for all $t \in \mathbb{Z}^+$. $\square$

Proof of Corollary 4.2.6. We have already seen these principles at work in the proofs of Corollaries 4.2.2 and 4.2.5. We have

$$\left(\sum_{\lambda\in\mathcal{P}_{\mathbb{X}}} \frac{z^{\ell(\lambda)}}{n_\lambda^s}\right)\left(\sum_{\lambda\in\mathcal{P}_{\mathbb{X}}} \frac{(-1)^{\ell(\lambda)} z^{\ell(\lambda)}}{n_\lambda^s}\right) = \frac{1}{\prod_{n\in\mathbb{X}}\left(1-\frac{z}{n^s}\right)\left(1+\frac{z}{n^s}\right)} = \sum_{\lambda\in\mathcal{P}_{\mathbb{X}}} \frac{z^{2\ell(\lambda)}}{n_\lambda^{2s}}.$$

Letting $z = 1$ gives (4.11). If we replace z with z^s we may rewrite the above equation in the form

$$\left(\sum_{k=0}^{\infty} z^{sk}\zeta_{\mathcal{P}_{\mathbb{X}}}(\{s\}^k)\right)\left(\sum_{k=0}^{\infty}(-1)^k z^{sk}\zeta_{\mathcal{P}_{\mathbb{X}}}(\{s\}^k)\right) = \sum_{k=0}^{\infty} z^{2sk}\zeta_{\mathcal{P}_{\mathbb{X}}}(\{2s\}^k).$$

Using (4.26) on the left and comparing coefficients on both sides gives the $n = 0$ case of (4.12); the general formula follows from the $n = 0$ case by induction. $\square$

Proof of comments following Corollary 4.2.6. Taking $\mathbb{X} = \mathbb{P}$ we see $(-1)^{\ell(\lambda)}$ specializes to Liouville's function $\lambda(n_\lambda) = (-1)^{\Omega(n_\lambda)}$ (here we are using "λ" in two different ways), where $\Omega(N)$ is the number of prime factors of N with multiplicity. That (4.11) therefore becomes $\zeta(s)\sum_{n=1}^{\infty}\lambda(n)/n^s = \zeta(2s)$ follows from arguments similar to the proof of Corollary 4.1.3. $\square$

Proof of Corollary 4.2.7. This identity follows immediately by taking $\mathcal{P}_{\mathbb{X}} = \mathcal{P}$, $n = 0$, $k = 2$ in (4.12) and simplifying. $\square$

Proof of Theorem 4.2.8. The proof of (4.13) is similar to Euler's proof that the number of partitions of n into distinct parts is equal to the number of partitions into odd parts [Ber06]. We expand the product

$$\varphi_\infty(f;q) = (1-f(1)q)(1-f(2)q^2)(1-f(3)q^3)\cdots,$$

which results in (4.13).

Identities (4.14) and (4.15) follow directly from the proof of (4.2),(4.3) above. Moreover, the proof of (4.16) is much like the proof of (4.5). We note that

$$\begin{aligned}\frac{1}{\varphi_\infty(f;q)} &= 1 - \varphi_\infty(f;q)\ \textstyle\sum_{(5)} \\ &= 1 - \frac{\sum_{(5)}}{1/\varphi_\infty(f;q)},\end{aligned}$$

and replace the term $1/\varphi_\infty(f;q)$ in the denominator on the right with the continued fraction in (4.5). □

Proof of Corollary 4.2.9. The formula follows easily from the leading identities in Theorems 4.1.1 and 4.2.8. We note that

$$\begin{aligned}\prod_{j=1}^{n} \prod_{k_j \in \mathbb{X}_j} \left(1 \pm f_j(k_j) q^{k_j}\right)^{\pm 1} &= \prod_{j=1}^{n} \left(\sum_{\lambda \in \mathcal{P}^{\pm}_{\mathbb{X}_j}} q^{|\lambda|} \prod_{\lambda_i \in \lambda} f_j(\lambda_i) \right) \\ &= \prod_{j=1}^{n} \left(\sum_{k_j=0}^{\infty} q^{k_j} \sum_{\substack{\lambda \vdash k_j \\ \lambda \in \mathcal{P}^{\pm}_{\mathbb{X}_j}}} \prod_{\lambda_i \in \lambda} f_j(\lambda_i) \right)\end{aligned}$$

and repeatedly apply Equation (4.26) on the right. □

Proof of Corollary 4.2.10. The identity is immediate from Theorem 4.2.8 by letting $z = 1$ in

$$\frac{\sinh(\pi z)}{\pi z} = \prod_{n=1}^{\infty} \left(1 + \frac{z^2}{n^2}\right) = \sum_{\lambda \in \mathcal{P}^*} \frac{z^{\ell(\lambda)}}{n_\lambda^2}.$$

□

Proof of Corollary 4.2.11. This proof proceeds much like the proofs of Corollaries 4.2.2, 4.2.4, 4.2.5 above, only more easily. We have from (4.6) and Theorem 4.2.8, together with the Maclaurin expansion of the sine function, that

$$\frac{\sin z}{z} = \sum_{k=0}^{\infty} \frac{(-1)^k z^{2k}}{\pi^{2k}} \zeta(\{2\}^k) = \sum_{k=0}^{\infty} \frac{(-1)^k z^{2k}}{(2k+1)!}.$$

Comparing the coefficients of the two summations above gives $\zeta(\{2\}^k)$. We carry this approach further to find $\zeta(\{2^t\}^k)$ for $t > 1$. We proceed inductively from the case above. Take the identity

$$\left(\sum_{k=0}^{\infty} \frac{z^{2^{t-1}k}}{\pi^{2^{t-1}k}} \zeta\left(\left\{2^{t-1}\right\}^k\right)\right) \left(\sum_{k=0}^{\infty} \frac{(-1)^k z^{2^{t-1}k}}{\pi^{2^{t-1}k}} \zeta\left(\left\{2^{t-1}\right\}^k\right)\right) = \sum_{k=0}^{\infty} \frac{z^{2^t k}}{\pi^{2^t k}} \zeta(\{2^t\}^k)$$

and compare coefficients on the left- and right-hand sides, using (4.26) to compute the coefficients on the left; expressions such as the remaining ones in the statement of the corollary result. It is clear from induction that $\zeta(\{2^t\}^k)$ always has the form "$\pi^{2^t k} \times$ finite sum of fractions". $\square$

Proof of Corollary 4.2.12. This proof is nearly identical to the proof of Corollary 4.2.6. From the associated product representations it is clear that

$$\left(\sum_{\lambda \in \mathcal{P}_{\mathbb{X}}^*} \frac{z^{\ell(\lambda)}}{n_\lambda^s}\right) \left(\sum_{\lambda \in \mathcal{P}_{\mathbb{X}}^*} \frac{(-1)^{\ell(\lambda)} z^{\ell(\lambda)}}{n_\lambda^s}\right) = \sum_{\lambda \in \mathcal{P}_{\mathbb{X}}^*} \frac{(-1)^{\ell(\lambda)} z^{2\ell(\lambda)}}{n_\lambda^{2s}}.$$

Letting $z = 1$ gives (4.24). If we replace z with z^s we may rewrite the above equation as

$$\left(\sum_{k=0}^{\infty} z^{sk} \zeta_{\mathcal{P}_{\mathbb{X}}^*}(\{s\}^k)\right) \left(\sum_{k=0}^{\infty} (-1)^k z^{sk} \zeta_{\mathcal{P}_{\mathbb{X}}^*}(\{s\}^k)\right) = \sum_{k=0}^{\infty} (-1)^k z^{2sk} \zeta_{\mathcal{P}_{\mathbb{X}}^*}(\{2s\}^k).$$

Again using (4.26) on the left and comparing coefficients on both sides gives the $n = 0$ case of (4.22); the general formula follows by induction. $\square$

Proof of comments following Corollary 4.2.12. Taking $\mathbb{X} = \mathbb{P}$ in Theorem 4.2.8 and noting that $\lambda \in \mathcal{P}_{\mathbb{P}}^*$ implies n_λ is squarefree, we see $(-1)^{(\lambda)} = \mu(n_\lambda)$, where μ denotes the classical Möbius function; therefore, we have the identity

$$\sum_{n=1}^{\infty} \frac{\mu(n)}{n^s} = \sum_{\lambda \in \mathcal{P}_{\mathbb{P}}^*} \frac{\mu(n_\lambda)}{n_\lambda^s} = \eta_{\mathcal{P}_{\mathbb{P}}^*}(s) = \frac{1}{\zeta_{\mathcal{P}_{\mathbb{P}}(s)}} = \frac{1}{\zeta(s)}.$$

On the other hand, we have $\zeta_{\mathcal{P}_{\mathbb{P}}^*}(s) = \sum_{n \text{ squarefree}} 1/n^s = \sum_{n=1}^{\infty} |\mu(n)|/n^s$. $\square$

Remark. See Appendix C for further notes on Chapter 4.

Chapter 5

Partition zeta functions: further explorations

Adapted from [ORS17], a joint work with Ken Ono and Larry Rolen

5.1 Following up on the previous chapter

In Chapter 4, we see the Riemann zeta function as the prototype for a new class of combinatorial objects arising from Eulerian methods. In this chapter we record a number of further identities relating certain zeta functions arising from the theory of partitions to various objects in number theory such as Riemann zeta values, multiple zeta values, and infinite product formulas. Some of these formulas are related to results in the literature; they are presented here as examples of this new class of partition-theoretic zeta functions. We also give several formulas for the Riemann zeta function, and results on the analytic continuation (or non-existence thereof) of zeta-type series formed in this way. Furthermore, we discuss the p-adic interpolation of these zeta functions in analogy with the classical work of Kubota and Leopoldt on p-adic continuation of the Riemann zeta function [KL64].

5.2 Evaluations

We saw in the previous chapter a variety of simple closed forms for partition zeta functions, depending on the natures of the subsets of partitions being summed over. Different subsets induce different zeta phenomena. In what follows, we consider the evaluations of a small sampling of possible partition zeta functions having particularly pleasing formulas.

5.2.1 Zeta functions for partitions with parts restricted by congruence conditions

Our first line of study will concern sets $\mathbb{M} \subset \mathbb{N}$ that are defined by congruence conditions. Note by considering Euler products as in Definition 1.2.8 that for disjoint $\mathbb{M}_1, \mathbb{M}_2 \subset \mathbb{Z}^+$,

$$\zeta_{\mathcal{P}_{\mathbb{M}_1 \cup \mathbb{M}_2}}(s) = \zeta_{\mathcal{P}_{\mathbb{M}_1}}(s)\zeta_{\mathcal{P}_{\mathbb{M}_2}}(s).$$

Hence, to study any set of partitions determined by congruence conditions on the parts, it suffices to consider series of the form

$$\zeta_{\mathcal{P}_{a+m\mathbb{N}}}(s),$$

where $a \in \mathbb{Z}_{\geq 0}$, $m \in \mathbb{N}$ (excluding the case $a = 0$, $m = 1$, where the zeta function clearly diverges), and $\mathcal{P}_{a+m\mathbb{N}}$ is partitions into parts congruent to a modulo m. We see examples of the case $\zeta_{\mathcal{P}_{m\mathbb{N}}}(2^N) = \zeta_{\mathcal{P}_{0+m\mathbb{N}}}(2^N)$ in Corollaries 4.2.1 and 4.2.1; we are interested in the most general case, with $s = n \in \mathbb{N}$.

Our first main result is then the following, where Γ is the usual gamma function of Euler and $e(x) := e^{2\pi i x}$. The proof will use an elegant and useful formula highlighted by Chamberland and Straub in [CS13], which we note was also inspired by previous work on multiplicative partitions in [CJNW13]. In fact, the following result is a generalization of Theorem 8 of [CJNW13] which in our notation corresponds to $a = m = 1$.

Theorem 5.2.2. *For $n \geq 2$, we have*

$$\zeta_{\mathcal{P}_{a+m\mathbb{N}}}(n) = \Gamma(1+a/m)^{-n} \prod_{r=0}^{n-1} \Gamma\left(1 + \frac{a - e(r/n)}{m}\right).$$

Theorem 5.2.2 has several applications. By considering the expansion of the logarithm of the gamma function, we easily obtain the following result, in which γ is the Euler-Mascheroni constant and the principal branch of the logarithm is taken.

Corollary 5.2.1. *For any $m, n \geq 2$, we have that*

$$\begin{aligned}\log\left(\zeta_{\mathcal{P}_{a+m\mathbb{N}}}(n)\right) &= n\log(1+a/m) + \frac{a(n+1)}{m}(1-\gamma) - \sum_{r=0}^{n-1} \log\left(1 + \frac{a-e(r/n)}{m}\right) \\ &\quad + \sum_{r=0}^{n-1} \sum_{k\geq 2} \frac{(-1)^k(\zeta(k)-1)\left(a^k + (a-e(r/n))^k\right)}{km^k}.\end{aligned}$$

When $a = 0$ and $m \geq 2$, we obtain the following strikingly simple formula, which is similar to Theorem 7 of [CJNW13] that in our notation corresponds to the case $a = m = 1$.

Corollary 5.2.2. *For any $m, n \geq 2$, we have that*

$$\log\left(\zeta_{\mathcal{P}_{m\mathbb{N}}}(n)\right) = n \sum_{\substack{k\geq 2 \\ n|k}} \frac{\zeta(k)}{km^k}.$$

5.2.3 Connections to ordinary Riemann zeta values

In addition to providing interesting formulas for values of more exotic partition-theoretic zeta functions, the above results also lead to curious formulas for the classical Riemann zeta function. In fact, $\zeta(s)$ is itself a partition zeta function, summed over prime partitions, so it is perhaps not too surprising to find that we can learn something about it from a partition-theoretic perspective. Then we continue the theme of evaluations by recording a few results expressing the value of ζ at integer argument $n > 1$ in terms of gamma factors.

In the first, curious identity, let μ denote the classical Möbius function. We point out that this is essentially a generalization of a formula for the case $a = m = 1$ given in Equation 11 of [CJNW13].

Corollary 5.2.3. *For all $m, n \geq 2$, we have*

$$\zeta(n) = m^n \sum_{k\geq 1} \frac{\mu(k)}{k} \sum_{r=0}^{nk-1} \log\left(\Gamma\left(1 - \frac{e\left(\frac{r}{nk}\right)}{m}\right)\right).$$

The next identity gives $\zeta(n)$ in terms of the nth derivative of a product of gamma functions. The authors were not able to find this formula in the literature; however, given the well-known connections between Γ and ζ, as well as the known example below the following theorem, it is possible that the identity is known.

Theorem 5.2.4. *For integers $n > 1$, we have*

$$\zeta(n) = \frac{1}{n!} \lim_{z\to 0^+} \frac{\mathrm{d}^n}{\mathrm{d}z^n} \prod_{j=0}^{n-1} \Gamma\left(1 - ze(j/n)\right).$$

Example 5.2.5. *As an example of implementing the above identity, take $n = 2$; then using Euler's well-known product formula for the sine function, it is easy to check that*

$$\zeta(2) = \frac{1}{2!} \lim_{z\to 0^+} \frac{\mathrm{d}^2}{\mathrm{d}z^2} \Gamma\left(1+z\right)\Gamma\left(1-z\right) = \frac{1}{2!} \lim_{z\to 0^+} \frac{\mathrm{d}^2}{\mathrm{d}z^2} \frac{\pi z}{\sin(\pi z)} = \frac{\pi^2}{6}.$$

This last formula for $\zeta(n)$, following from a formula in Chapter 4 together with the preceding theorem, is analogous to some extent to the classical identity $\sin(n) = \frac{e^{in} - e^{-in}}{2i}$.

Corollary 5.2.4. *For integers $n > 1$, we have*

$$\zeta(n) = \lim_{z\to 0^+} \frac{\prod_{j=0}^{n-1} \Gamma\left(1 - ze(j/n)\right) - \prod_{j=0}^{n-1} \Gamma\left(1 - ze(j/n)\right)^{-1}}{2z^n}.$$

5.2.6 Zeta functions for partitions of fixed length

We now consider zeta sums of the shape $\zeta_{\mathcal{P}}(\{s\}^k)$ as in Definition 4.2.1. Our first aim will be to extend Corollary 4.2.4 from the previous chapter.

Let $[z^n]f$ represent the coefficient of z^n in a power series f. Using this notation, we show the following, which in particular gives an algorithmic way to compute each $\zeta_{\mathcal{P}}(\{m\}^k)$ in terms of Riemann zeta values for $m \in \mathbb{N}_{\geq 2}$.

Theorem 5.2.7. *For all $m \geq 2$, $k \in \mathbb{N}$, we have*

$$\begin{aligned}\zeta_{\mathcal{P}}(\{m\}^k) &= \pi^{mk}[z^{mk}] \prod_{r=0}^{m-1} \Gamma\left(1 - \frac{z}{\pi} e(r/m)\right) \\ &= \pi^{mk}[z^{mk}] \exp\left(\sum_{j\geq 1} \frac{\zeta(mj)}{j} \left(\frac{z}{\pi}\right)^{mj}\right).\end{aligned}$$

Generalizing the comments just below Corollary 4.2.5, the next corollary follows directly from Theorem 5.2.7 (using the fact that $\zeta(k) \in \mathbb{Q}\pi^k$ for even integers k).

Corollary 5.2.5. *For $m \in 2\mathbb{N}$ even, we have that*

$$\zeta_{\mathcal{P}}(\{m\}^k) \in \mathbb{Q}\pi^{mk}.$$

Remark. This can also be deduced from Theorem 2.1 of [Hof92].

To conclude this section, we note one explicit method for computing the values $\zeta_{\mathcal{P}}(\{m\}^k)$ at integral k, m (especially if m is even, in which case the zeta values below are completely elementary).

Corollary 5.2.6. *For $m \geq 2, k \in \mathbb{N}$, and $j \geq i$, set*

$$\alpha_{i,j} := \zeta(m(j-i+1)) \frac{(k-i)!}{\pi^{m(j-i+1)}(k-j)!}.$$

Then we have

$$\zeta_{\mathcal{P}}(\{m\}^k) = \frac{\pi^{mk}}{k!} \det \begin{pmatrix} \alpha_{1,1} & \alpha_{1,2} & \alpha_{1,3} & \dots & \alpha_{1,k} \\ -1 & \alpha_{2,2} & \alpha_{2,3} & \dots & \alpha_{2,k} \\ 0 & -1 & \alpha_{3,3} & \dots & \alpha_{3,k} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & -1 & \alpha_{k,k} \end{pmatrix}.$$

Remark. There are results resembling these in Knopfmacher and Mays [KM99].

5.3 Analytic continuation and p-adic continuity

If we jump forward about 100 years from the pathbreaking work of Euler concerning special values of the Riemann zeta function at even integers, we arrive at the famous work of Riemann in connection with prime number theory (see [Edw01]). Namely, in 1859, Riemann brilliantly described the most significant properties of $\zeta(s)$ following that of an Euler product: the analytic continuation and functional equation for $\zeta(s)$. It is for this reason, of course, that the zeta function is named after Riemann, and not Euler, who had studied this function in some detail, and even conjectured a related functional equation. In particular, this analytic continuation allowed Riemann to bring the zeta function, and indeed the relatively new field of complex analysis, to the forefront of number theory by connecting its roots to the distribution of prime numbers.

It is natural therefore, whenever one is faced with new zeta functions, to ask about their prospect for analytic continuation. Here, we offer a brief study of some of these properties, in particular showing that the situation for our zeta functions is much more singular. Partition-theoretic zeta functions in fact naturally give rise to functions with essential singularities. Here, we use Corollary 5.2.2 to study the continuation properties of partition zeta functions over partitions $\mathcal{P}_{m\mathbb{N}}$ into multiples of $m > 1$. In order to state the result we first define, for any $\varepsilon > 0$, the right half-plane $\mathbb{H}_\varepsilon := \{z \in \mathbb{C} : \mathrm{Re}(z) > \varepsilon\}$, and we denote by $\frac{1}{\mathbb{N}}$ the set $\{1/n : n \in \mathbb{N}\}$.

Corollary 5.3.1. *For any $\varepsilon > 0$ and $m > 1$, $\zeta_{\mathcal{P}_{m\mathbb{N}}}(s)$ has a meromorphic extension to $\mathbb{H}_\varepsilon$ with poles exactly at $\mathbb{H}_\varepsilon \cap \frac{1}{\mathbb{N}}$. In particular, there is no analytic continuation beyond the right half-plane $\mathrm{Re}(s) > 0$, as there would be an essential singularity at $s = 0$.*

Remark. For the function $\zeta_{\mathcal{P}_{\mathbb{N}}}(s)$, a related discussion of poles and analytic continuation was made by the user mohammad-83 in a MathOverflow.net question.

Finally, we follow Kubota and Leopoldt [KL64], who showed ζ could be modified slightly to obtain modified zeta functions for any prime p which extend ζ to the space of p-adic integers $\mathbb{Z}_p$, to yield further examples of p-adic zeta functions of this sort. These continuations are based on the original observations of Kubota and Leopoldt, and, in a rather pleasant manner, on the evaluation formulas discussed above.

In particular, we will use Corollary 5.2.6 to p-adically interpolate modified versions of $\zeta_{\mathcal{P}}(\{m\}^k)$ in the m-aspect. Given the connection discussed in Section 5.4 to multiple zeta values, these results should be compared with the literature on p-adic multiple zeta values (e.g., see [Fur04]), although we note that our p-adic interpolation procedure seems to be more direct in the special case we consider.

The continuation in the m-aspect of this function is also quite natural, as the case $k = 1$ is just that of the Riemann zeta function. Thus, it is natural to search for a suitable p-adic zeta function that specializes to the function of Kubota and Leopoldt when $k = 1$. It is also desirable to find a p-adic interpolation result which makes the partition-theoretic perspective clear.

Here, we provide such an interpretation. Let us first denote the set of partitions with parts not divisible by p as $\mathcal{P}_p$; then we consider the length-k partition zeta values $\zeta_{\mathcal{P}_p}(\{s\}^k)$. Note that for $k = 1$, $\zeta_{\mathcal{P}_p}(\{s\}^1)$ is just the Riemann zeta function with the Euler factor at p removed (as considered by Kubota and Leopoldt). We then offer the following p-adic interpolation result.

Theorem 5.3.1. *Let $k \geq 1$ be fixed, and let $p \geq k+3$ be a prime. Then $\zeta_{\mathcal{P}_p}(\{s\}^k)$ can be extended to a continuous function for $s \in \mathbb{Z}_p$ which agrees with $\zeta_{\mathcal{P}_p}(\{s\}^k)$ on a positive proportion of integers.*

5.4 Connections to multiple zeta values

Our final application of the circle of ideas related to partition zeta functions and infinite products will be in the theory of multiple zeta values.

Definition 5.4.1. *We define for natural numbers $m_1, m_2, \ldots, m_k$ with $m_k > 2$ the multiple zeta value (commonly written "MZV")*

$$\zeta(m_1, m_2, \ldots, m_k) := \sum_{n_1 > n_2 > \ldots > n_k \geq 1} \frac{1}{n_1^{m_1} \ldots n_k^{m_k}}.$$

We call k the length of the MZV. Furthermore, if $m_1 = m_2 = \ldots = m_k$ are all equal to some $m \in \mathbb{N}$, we use the common notation

$$\zeta(\{m\}^k) := \sum_{n_1 > n_2 > \ldots > n_k \geq 1} \frac{1}{(n_1 n_2 \ldots n_k)^m}. \tag{5.1}$$

Multiple zeta values have a rich history and enjoy widespread connections; interested readers are referred to Zagier's short note [Zag95], and for a more detailed treatment, the excellent lecture notes of Borwein and Zudilin [BZ11]. There are many nice closed-form identities in the literature; for example, one can show (see [Hof92]) on analogy to Corollary 4.2.4 that

$$\zeta(\{2\}^k) = \frac{\pi^{2k}}{(2k+1)!}, \tag{5.2}$$

which we prove, along with similar (but more complicated) expressions for $\zeta(\{2^t\}^k)$, in the previous chapter.

Observe from its definition that the partition zeta function $\zeta_{\mathcal{P}}(\{m\}^k)$ can be rewritten in a similar-looking form to (5.1) above:

$$\zeta_{\mathcal{P}}(\{m\}^k) = \sum_{n_1 \geq n_2 \geq \ldots \geq n_k \geq 1} \frac{1}{(n_1 n_2 \ldots n_k)^m}. \tag{5.3}$$

In fact, if we take $\mathcal{P}^*$ to denote partitions into distinct parts, then (5.1) reveals $\zeta(\{m\}^k)$ is equal to the partition zeta function $\zeta_{\mathcal{P}^*}(\{m\}^k)$ summed over length-k partitions into distinct parts, as pointed out in the preceding chapter. Series such as those in (5.3) have been considered and studied extensively by Hoffman (for instance, see [Hof92]).

By reorganizing sums of the shape (5.3), we arrive at interesting relations between $\zeta_{\mathcal{P}}(\{m\}^k)$ and families of MZVs. In order to describe these relations, we first recall that a *composition* is simply a finite tuple of natural numbers, and we call the sum of these integers the *size* of the composition. Denote the set of all compositions by $\mathcal{C}$ and write $|\lambda| = k$ for $\lambda = (a_1, a_2, \ldots, a_j) \in \mathcal{C}$ if $k = a_1 + a_2 + \ldots + a_j$. Then we obtain the following.

Proposition 5.4.1. *Assuming the notation above, we have that*

$$\zeta_{\mathcal{P}}(\{m\}^k) = \sum_{\substack{\lambda=(a_1,\ldots,a_j)\in\mathcal{C} \\ |\lambda|=k}} \zeta(a_1 m, a_2 m, \ldots, a_j m).$$

Remark. Proposition 5.4.1 is analogous to results of Hoffman; the reader is referred to Theorem 2.1 of [Hof92].

In particular, for any $n > 1$ we can find the following reduction of $\zeta(\{n\}^k)$ to MZVs of smaller length. We note that in Theorem 2.1 of [Hof92], Hoffman also shows directly how to write these values in terms of products (as opposed to simply linear combinations) of ordinary Riemann zeta values: hints, perhaps, of further connections. We remark in passing that this can be thought of as a sort of "parity result" (cf. [IKZ06, Tsu04]).

Corollary 5.4.1. *For any $n, k > 1$, the MZV $\zeta(\{n\}^k)$ of length k can be written as an explicit linear combination of MZVs of lengths less than k.*

As our final result, we give a simple formula for $\zeta(\{n\}^k)$. This formula is probably already known; if $k=2$ it follows from a well-known result of Euler (see the discussion of $H(n)$ on page 3 of [Zag16]), The idea of the proof is also similar to what has appeared in, for example, [Zag16]. However, the authors have decided to include it due to connections with the ideas used throughout this paper, and the simple deduction of the formula from expressions necessary for the proofs of the results described above.

Proposition 5.4.2. *The MZV $\zeta(\{n\}^k)$ of length k can be expressed as a linear combination of products of ordinary ζ values. In particular, we have*

$$\zeta(\{n\}^k) = (-1)^k \left[z^{nk}\right] \exp\left(-\sum_{j\geq 1} \frac{\zeta(nj)}{j} z^{nj}\right).$$

Remark. This formula is equivalent to a special case of Theorem 2.1 of [Hof92]. However, since the proof is very simple and ties in with the other ideas in this paper, we give a proof for the reader's convenience.

The proof of Corollary 5.2.6 yields a similar determinant formula here.

Corollary 5.4.2. *For $n \geq 2, k \in \mathbb{N}$, and $j \geq i$, set*

$$\beta_{i,j} := -\zeta(n(j-i+1))\frac{(k-i)!}{(k-j)!}.$$

Then we have

$$\zeta(\{n\}^k) = \frac{(-1)^k}{k!} \det \begin{pmatrix} \beta_{1,1} & \beta_{1,2} & \beta_{1,3} & \dots & \beta_{1,k} \\ -1 & \beta_{2,2} & \beta_{2,3} & \dots & \beta_{2,k} \\ 0 & -1 & \beta_{3,3} & \dots & \beta_{3,k} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & -1 & \beta_{k,k} \end{pmatrix}.$$

Remark. We can see from the above corollary that $\zeta(\{n\}^k)$ is a linear combination of products of zeta values, which is closely related to formulas of Hoffman [Hof92].

5.5 Proofs

5.5.1 Machinery

Useful formulas

In this section, we collect several formulas that will be key to the proofs of the theorems above. We begin with the following beautiful formula given by Chamberland and Straub in Theorem 1.1 of [CS13]). In fact, this formula has a long history, going back at least to Section 12.13 of [WW27], and we note that Ding, Feng, and Liu independently discovered this same result in Lemma 7 of [DFL14].

Theorem 5.5.2. *If $n \in \mathbb{N}$ and $\alpha_1, \ldots, \alpha_n$ and $\beta_1, \ldots, \beta_n$ are complex numbers, none of which are non-positive integers, with $\sum_{j=1}^n \alpha_j = \sum_{j=1}^n \beta_j$, then we have*

$$\prod_{k\geq 0}\prod_{j=1}^{n}\frac{(k+\alpha_j)}{(k+\beta_j)} = \prod_{j=1}^{n}\frac{\Gamma(\beta_j)}{\Gamma(\alpha_j)}.$$

We will also require two Taylor series expansions for $\log\Gamma$, both of which follow easily from Euler's product definition of the gamma function [Edw01]. The first expansion, known as Legendre's series, is valid for $|z| < 1$ (see (17) of [Wre68]):

$$\log\Gamma(1+z) = -\gamma z + \sum_{k\geq 2}\frac{\zeta(k)}{k}(-z)^k. \tag{5.4}$$

We also have the following expansion valid for $|z| < 2$[1]:

$$\log\Gamma(1+z) = -\log(1+z) + z(1-\gamma) + \sum_{k\geq 2}(-1)^k(\zeta(k)-1)\frac{z^k}{k}. \tag{5.5}$$

Furthermore, we need a couple of facts about Bell polynomials (see Chapter 12.3

[1] See for instance (5.7.3) of NIST Digital Library of Mathematical Functions, `http://dlmf.nist.gov/`, Release 1.0.6 of 2013-05-06.

of [And98]). The nth *complete Bell polynomial* is the sum

$$B_n(x_1, \ldots, x_n) := \sum_{i=1}^{n} B_{n,i}(x_1, x_2, \ldots, x_{n-i+1}).$$

The ith term here is the polynomial

$$\begin{aligned} &B_{n,i}(x_1, x_2, \ldots, x_{n-i+1}) \\ &\quad := \sum \frac{n!}{j_1! j_2! \cdots j_{n-i+1}!} \left(\frac{x_1}{1!}\right)^{j_1} \left(\frac{x_2}{2!}\right)^{j_2} \cdots \left(\frac{x_{n-i+1}}{(n-i+1)!}\right)^{j_{n-i+1}}, \end{aligned}$$

where we sum over all sequences $j_1, j_2, ..., j_{n-i+1}$ of nonnegative integers such that $j_1 + j_2 + \cdots + j_{n-i+1} = i$ and $j_1 + 2j_2 + 3j_3 + \cdots + (n-i+1)j_{n-i+1} = n$.

With these notations, we use a specialization of the classical Faà di Bruno formula [FdB55], which allows us to write the exponential of a formal power series as a power series with coefficients related to complete Bell polynomials[2]:

$$\exp\left(\sum_{j=1}^{\infty} \frac{a_j}{j!} x^j\right) = \sum_{k=0}^{\infty} \frac{B_k(a_1, \ldots, a_k)}{k!} x^k. \tag{5.6}$$

Faà di Bruno also gives an identity [FdB55] that specializes to the following formula for the kth complete Bell polynomial in the series above as the determinant of a certain $k \times k$ matrix:

$$B_k(a_1, \ldots, a_k) = \det \begin{pmatrix} a_1 & \binom{k-1}{1}a_2 & \binom{k-1}{2}a_3 & \binom{k-1}{3}a_4 & \cdots & \cdots & a_k \\ -1 & a_1 & \binom{k-2}{1}a_2 & \binom{k-2}{2}a_3 & \cdots & \cdots & a_{k-1} \\ 0 & -1 & a_1 & \binom{k-3}{1}a_2 & \cdots & \cdots & a_{k-2} \\ 0 & 0 & -1 & a_1 & \cdots & \cdots & a_{k-3} \\ 0 & 0 & 0 & -1 & \cdots & \cdots & a_{k-4} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & -1 & a_1 \end{pmatrix}. \tag{5.7}$$

[2]We prove Faà di Bruno's formula and give other partition-theoretic applications in Appendix D.

5.5.3 Proofs of Theorems 5.2.2 and 5.2.4, and their corollaries

We begin with the proof of our first main formula.

Proof of Theorem 5.2.2. By Euler products, as in the previous chapter, we find that

$$\zeta_{\mathcal{P}_{a+m\mathbb{N}}}(n) = \prod_{k\in a+m\mathbb{N}} \frac{k^n}{k^n-1} = \prod_{j\geq 1} \frac{(a+mj)^n}{(a+mj)^n-1} = \prod_{j\geq 0}\prod_{r=0}^{n-1} \frac{(j+1+a/m)^n}{\left(j+1+\frac{a-e(r/n)}{m}\right)}.$$

Using Theorem 5.5.2 and the well-known fact that

$$\sum_{j=0}^{n-1} e(j/n) = 0 \tag{5.8}$$

directly gives the desired result. □

Proof of Corollary 5.2.1. For this, we apply (5.5) and use (5.8), the obvious fact that

$$|(a-e(j/n))/m| < 2,$$

and the easily-checked fact that

$$1+(a-e(j/n))$$

is never a negative real number for $j = 0, \ldots, n-1$. □

Proof of Corollary 5.2.2. Here, we simply use (5.4). Again, the corollary is proved following a short, elementary computation, using the classical fact that

$$\sum_{r=0}^{n-1} e(rk/n) = \begin{cases} n & \text{if } n|k, \\ 0 & \text{else.} \end{cases}$$

□

Proof of Corollary 5.3.1. By Corollary 5.2.2, we find for $n \geq 2$ that

$$\log\left(\zeta_{\mathcal{P}_{m\mathbb{N}}}(n)\right) = \sum_{k\geq 1} \frac{\zeta(nk)}{km^{kn}}.$$

Suppose that $\mathrm{Re}(s) > 0$ and $s \notin \frac{1}{\mathbb{N}}$. Then letting

$$K := \max\{\lceil 1/\mathrm{Re}(s)\rceil + 1, \mathrm{Re}(s)\},$$

it clearly suffices to show that

$$\sum_{k\geq K} \frac{\zeta(sk)}{km^{ks}}$$

converges. But in this range on k, by choice we have $\mathrm{Re}(sk) > 1$, so that using the assumption $m \geq 2$, we find for $\mathrm{Re}(s) > 0$ the upper bound

$$\sum_{k\geq K} \frac{\zeta(sk)}{km^{ks}} \leq \zeta(Ks) \sum_{k\geq K} \frac{1}{k2^{k\,\mathrm{Re}(s)}} \leq \zeta(Ks) \sum_{k\geq 1} \frac{1}{k2^{k\,\mathrm{Re}(s)}}$$
$$= -\zeta(Ks)\log\left(2^{-\mathrm{Re}(s)}\left(2^{\mathrm{Re}(s)}-1\right)\right),$$

and note that in the argument of the logarithm in the last step, by assumption we have $2^{\mathrm{Re}(s)} - 1 > 0$.

Conversely, if $s \in \frac{1}{\mathbb{N}}$, then it is clear that this representation shows there is a pole of the extended partition zeta function, as one of the terms gives a multiple of $\zeta(1)$. $\square$

Proof of Corollary 5.2.3. We utilize a variant of Möbius inversion, reversing the order of summation in the double sum $\sum_{k\geq 1}\sum_{d|k} \mu(d) f(nk) k^{-s}$; if

$$g(n) = \sum_{k\geq 1} \frac{f(kn)}{k^s},$$

then

$$f(n) = \sum_{k\geq 1} \frac{\mu(k)g(kn)}{k^s}.$$

Applying this inversion procedure to Corollary 5.2.2, so that $g(n) = \log \zeta_{\mathcal{P}_{m\mathbb{N}}}(n)$ (taking $s = 1$), and $f(n) = \zeta(n)/m^n$, we directly find that

$$\zeta(n) = m^n \sum_{k\geq 1} \frac{\mu(k)}{k} \log\left(\zeta_{\mathcal{P}_{m\mathbb{N}}}(nk)\right).$$

Applying Theorem 5.2.2 then gives the result. □

Proof of Theorem 5.2.4. By the comments following Theorem 4.1.1, for $\mathbb{M} \in \mathbb{N}$ we have

$$\prod_{k\in\mathbb{M}} \left(1 - \frac{z^s}{k^s}\right)^{-1} = 1 + z^s \sum_{k\in\mathbb{M}} \frac{1}{k^s \prod_{\substack{j\in\mathbb{M} \\ j\leq k}} \left(1 - \frac{z^s}{j^s}\right)}; \tag{5.9}$$

thus

$$\sum_{k\in\mathbb{M}} k^{-s} = \lim_{z\to 0^+} \frac{\prod_{k\in\mathbb{M}} \left(1 - \frac{z^s}{k^s}\right)^{-1} - 1}{z^s}.$$

Taking $\mathbb{M} = \mathbb{N}, s = n \in \mathbb{Z}_{\geq 2}$, we apply L'Hospital's rule n times to evaluate the limit on the right-hand side. The theorem then follows by noting, from Theorem 5.5.2, that in fact

$$\prod_{k\in\mathbb{N}} \left(1 - \frac{z^n}{k^n}\right)^{-1} = \prod_{j=0}^{n-1} \Gamma\left(1 - ze(j/n)\right).$$

□

Proof of Corollary 5.2.4. Picking up from the proof of Theorem 5.2.4 above, it follows also from Theorem 4.1.1 that

$$\prod_{k\in\mathbb{M}} \left(1 - \frac{z^s}{k^s}\right) = 1 - z^s \sum_{k\in\mathbb{M}} \frac{\prod_{\substack{j\in\mathbb{M} \\ j<k}} \left(1 - \frac{z^s}{j^s}\right)}{k^s}.$$

Subtracting this equation from (5.9), making the substitutions $\mathbb{M} = \mathbb{N}$, $s = n \geq 2$ as in the proof above, and using Theorem 5.5.2, gives the corollary. □

5.5.4 Proof of Theorem 5.2.7 and its corollaries

Proof of Theorem 5.2.7. Using a similar method as in Chapter 4 and a similar rewriting to that used in the proof of Theorem 5.2.2, we note that a short elementary computation shows

$$\sum_{k\geq 0} \frac{z^{mk}}{\pi^{mk}} \zeta_{\mathcal{P}}(\{m\}^k) = \prod_{k\geq 1} \frac{1}{1-\frac{z^m}{\pi^m k^m}} = \prod_{k\geq 0} \prod_{r=0}^{m-1} \frac{(k+1)^m}{\left(k+1-\frac{z}{\pi}e(r/m)\right)}.$$

Much as in the proof of Theorem 5.2.4, using Theorem 5.5.2, we directly find that this is equal to

$$\prod_{r=0}^{m-1} \Gamma\left(1-\frac{z}{\pi}e(r/m)\right),$$

which gives the first equality in the theorem. Applying Equation (5.4) (formally we require $|z| < \pi$, but we are only interested in formal power series here anyway), we find immediately, using a very similar calculation to that in the proof of Corollary 5.2.2, that

$$\begin{aligned} \sum_{k\geq 0} \left(\frac{z}{\pi}\right)^{mk} \zeta_{\mathcal{P}}(\{m\}^k) &= \exp\left(\sum_{r=0}^{m-1} \sum_{j\geq 2} \frac{\zeta(j)}{j} \left(\frac{z}{\pi}\right)^{mj} e(rj/m)\right) \\ &= \exp\left(m \sum_{\substack{j\geq 2 \\ m\mid j}} \frac{\zeta(j)}{j} \left(\frac{z}{\pi}\right)^{j}\right), \end{aligned} \tag{5.10}$$

which is equivalent to the second equality in the theorem. □

Proof of Corollary 5.2.6. Replace x with z^m in Equation 5.6, and set

$$a_j = \frac{(j-1)!\zeta(mj)}{\pi^{mj}}$$

on the left-hand side (which becomes the right-hand side of (5.10)). Then comparing the right side of 5.6 to the left side of 5.10, we deduce that

$$\zeta_{\mathcal{P}}(\{m\}^k) = \frac{\pi^{mk}}{k!} B_k(a_1, \ldots, a_k).$$

To complete the proof, we substitute the determinant in 5.7 for $B_k(a_1, \ldots, a_k)$ and rewrite the terms in the upper half of the resulting matrix as $\alpha_{i,j}$, as defined in the statement of the corollary. $\square$

Proof of Theorem 5.3.1. In analogy with the calculation of Theorem 5.2.7, we find that

$$\begin{aligned}\sum_{k\geq 0} z^{mk} \zeta_{\mathcal{P}_p}(\{m\}^k) &= \prod_{\substack{k\geq 1\\ p\nmid k}} \frac{1}{1-\frac{z^m}{k^m}} = \frac{\prod_{k\geq 0}\prod_{r=0}^{m-1} \frac{(k+1)^m}{(k+1-ze(r/m))}}{\prod_{k\geq 0}\prod_{r=0}^{m-1} \frac{(k+1)^m}{\left(k+1-\frac{z}{p}e(r/m)\right)}} \\ &= \prod_{r=0}^{m-1} \frac{\Gamma\left(1-ze(r/m)\right)}{\Gamma\left(1-\frac{z}{p}e(r/m)\right)}.\end{aligned}$$

As in the calculation of (5.10), this is equal to

$$\exp\left(\sum_{j\geq 1} \frac{\zeta(mj)}{j} (z)^{mj} \left(1 - 1/p^{mj}\right)\right),$$

so if we set

$$\alpha_{i,j}^{(p)}(m) := \zeta^*(m(j-i+1)) \frac{(k-i)!}{(k-j)!} \quad \text{where} \quad \zeta^*(s) := (1-p^{-s})\zeta(s),$$

then we have

$$\zeta_{\mathcal{P}_p}(\{m\}^k) = \frac{1}{k!} \det \begin{pmatrix} \alpha_{1,1}^{(p)} & \alpha_{1,2}^{(p)} & \alpha_{1,3}^{(p)} & \ldots & \alpha_{1,k}^{(p)} \\ -1 & \alpha_{2,2}^{(p)} & \alpha_{2,3}^{(p)} & \ldots & \alpha_{2,k}^{(p)} \\ 0 & -1 & \alpha_{3,3}^{(p)} & \ldots & \alpha_{3,k}^{(p)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & -1 & \alpha_{k,k}^{(p)} \end{pmatrix}.$$

We further define $\zeta_{\mathcal{P}_p}(\{m\}^k)$ for more general values in $\mathbb{C}$, such as $m \in -\mathbb{N}$ using the analytic continuation of ζ in each of the factors $\alpha_{i,j}^{(p)}(m)$. Next we recall the Kummer congruences, which state that if k_1, k_2 are positive even integers not divisible by $(p-1)$ and $k_1 \equiv k_2 \pmod{p^{a+1} - p^a}$ for $a \in \mathbb{N}$ where $p > 2$ is prime, then

$$\left(1 - p^{k_1 - 1}\right) \frac{B_{k_1}}{k_1} \equiv \left(1 - p^{k_2 - 1}\right) \frac{B_{k_2}}{k_2} \pmod{p^{a+1}}.$$

Let us take S_{s_0} to be the set of natural numbers congruent to s_0 modulo $p-1$. The Kummer congruences then imply that for any $s_0 \not\equiv 0 \pmod{p-1}$, and for any $k_1, k_2 \in S_{s_0}$ with $k_1 \equiv k_2 \pmod{p^a}$ and $k_1, k_2 > 1$, that

$$\zeta^*(1 - k_1) \equiv \zeta^*(1 - k_2) \pmod{p^{a+1}}.$$

If we choose $m_1, m_2 \in S_{s_0}$ with $m_1 \equiv m_2 \pmod{p^a}$, then the values $1-(1-m_1)(j-i+1)$, $1-(1-m_2)(j-i+1)$ are in $S_{1+(s_0-1)(j-i-1)}$ and are congruent modulo p^a, and as $p > k$ the additional factorial terms (inside and outside the determinant) are p-integral. Now in our determinant, $j - i + 1$ ranges through $\{1, 2, \ldots, k\}$, and we want to find an s_0 such that $1 + (s_0 - 1)r \not\equiv 0 \pmod{p-1}$ for $r \in \{1, 2, \ldots k\}$. If we take $s_0 = 2$, then the largest value of $1 + (s_0 - 1)r$ is $k + 1$, which is by assumption less than $p - 1$, and hence not divisible by it. Hence, in our case, $s_0 = 2$ suffices. Thus, if $m_1, m_2 \in S_2$ with $m_1 \equiv m_2 \pmod{p^a}$, then

$$\zeta_{\mathcal{P}_p}(\{1 - m_1\}^k) \equiv \zeta_{\mathcal{P}_p}(\{1 - m_2\}^k) \pmod{p^{a+1}}.$$

This shows that our zeta function is uniformly continuous on S_2 in the p-adic topology. As this set is dense in $\mathbb{Z}_p$, we have shown the function extends in the m-aspect to $\mathbb{Z}_p$.

$\square$

5.5.5 Proofs of results concerning multiple zeta values

Proof of Proposition 5.4.1. Recall from (5.3) that we need to study the sum

$$\sum_{n_1 \geq n_2 \geq \ldots \geq n_k \geq 1} \frac{1}{(n_1 n_2 \ldots n_k)^m}.$$

The proof is essentially combinatorial accounting, keeping track of the number of ways to split up a sum

$$\sum_{n_1 \geq n_2 \geq \ldots \geq n_k \geq 1}$$

over all all k-tuples of natural numbers into a chain of equalities and strict inequalities. Suppose that we have

$$n_1 \geq n_2 \geq \ldots \geq n_k \geq 1.$$

Then if any of these inequalities is an equality, say $n_j = n_{j+1}$, in the contribution to the sum

$$\sum_{n_1 \geq n_2 \geq \ldots \geq n_k \geq 1} (n_1 \ldots n_k)^{-m},$$

the terms n_j and n_{j+1} "double up". That is, we can delete the n_{j+1} and replace the n_j^{-m} in the sum with a n_j^{-2m}. Thus, the reader will find that our goal is to keep track of different orderings of $>$ and $=$, taking symmetries into account. The possible chains of $=$ and $>$ are encoded by the set of compositions of size k, by associating to the composition $(a_1, \ldots, a_j)$ the chain of inequalities

$$n_1 = \ldots = n_{a_1} > n_{a_1+1} = \ldots = n_{a_1+a_2} > n_{a_2+1} > \ldots > n_k.$$

That is, the number a_1 determines the number of initial terms on the right which are equal before the first inequality, a_2 counts the number of equalities in the next block of inequalities, and so on. It is clear that the sum corresponding to the each composition then contributes the desired amount to the partition zeta value in the corollary. $\square$

Proof of Corollary 5.4.1. In Proposition 5.4.1, comparison with Corollary 5.2.6 shows that we have a linear relation among MZVs and products of zeta values. Observe that in $\zeta_{\mathcal{P}}(\{m\}^k)$, the only composition of length k is $(1, 1, \ldots, 1)$, which contributes $k!\zeta(\{m\}^k)$ to the right-hand side of Proposition 5.4.1, and that the rest of the compositions are of lower length, hence giving MZVs of smaller length; the corollary follows immediately. □

Proof of Proposition 5.4.2. Consider the multiple zeta value $\zeta(\{n\}^k)$ of length k. Then we directly compute

$$\sum_{k\geq 0}(-1)^k\zeta(\{n\}^k)z^{nk} = \prod_{m\geq 1}\left(1-\left(\frac{z}{m}\right)^n\right) = \prod_{m\geq 0}\prod_{r=0}^{n-1}\frac{(m+1-ze(r/n))}{(m+1)^n}.$$

By Theorem 5.5.2, this equals

$$\prod_{r=0}^{n-1}\Gamma(1-ze(r/n))^{-1}.$$

Using precisely the same computation as was made in the proof of Theorem 5.2.7, we find that this is equal to

$$\exp\left(-n\sum_{\substack{j\geq 2\\ n|j}}\frac{\zeta(j)}{j}z^j\right).$$

Hence, we have that

$$\zeta(\{n\}^k) = (-1)^k\left[z^{nk}\right]\exp\left(-\sum_{j\geq 1}\frac{\zeta(nj)}{j}z^{nj}\right).$$

□

Proof of Corollary 5.4.2. Here we proceed exactly as in the proof of Corollary 5.2.6, except we make the simpler substitution

$$a_k = (k-1)!\zeta(nk)$$

into Equation 5.6, and compare with Proposition 5.4.2. In the final step, we replace the terms in the upper half of the matrix with $\beta_{i,j}$ as defined in the statement of the corollary. □

5.6 Partition Dirichlet series

We have presented samples of a few varieties of flora one finds at the fertile intersection of combinatorics and analysis. What unifies all of these is the perspective that they represent instances of partition zeta functions, with proofs that fit naturally into the Eulerian theory we propound.

We close this article by noting a general class of partition-theoretic analogs of classical Dirichlet series having the form

$$\mathcal{D}_{\mathcal{P}'}(f,s) := \sum_{\lambda \in \mathcal{P}'} f(\lambda) n_\lambda^{-s},$$

where $\mathcal{P}'$ is a proper subset of $\mathcal{P}$ and $f \colon \mathcal{P}' \to \mathbb{C}$. Of course, partition zeta functions arise from the specialization $f \equiv 1$, just as in the classical case.

Taking $\mathcal{P}' = \mathcal{P}_{\mathbb{M}}$ as defined previously, then if $f := f(n_\lambda)$ is completely multiplicative with appropriate growth conditions, it follows from Theorem 4.1.1 that $\mathcal{D}_{P_{\mathbb{M}}}(f,s)$ has the Euler product

$$\mathcal{D}_{P_{\mathbb{M}}}(f,s) = \prod_{j \in \mathbb{M}} \left(1 - \frac{f(j)}{j^s}\right)^{-1} \ (\mathrm{Re}(s) > 1)\,, \tag{5.11}$$

and nearly the entire theory of partition zeta functions developed in the previous chapter extends to these series as well. Moreover, incorporating partition-arithmetic functions from Chapter 3, by very much the same steps as proofs of the classical cases, we have familiar-looking formulas such as these. We take $\mathrm{Re}(s)$ so the series converge absolutely.

Theorem 5.6.1. *Generalizing the classical cases, we have the following identities:*

$$\sum_{\lambda\in\mathcal{P}_{\mathbb{X}}} \mu_{\mathcal{P}}(\lambda) n_\lambda^{-s} = \frac{1}{\zeta_{\mathcal{P}_{\mathbb{X}}}(s)}, \qquad \sum_{\lambda\in\mathcal{P}_{\mathbb{X}}} \varphi_{\mathcal{P}}(\lambda) n_\lambda^{-s} = \frac{\zeta_{\mathcal{P}_{\mathbb{X}}}(s-1)}{\zeta_{\mathcal{P}_{\mathbb{X}}}(s)}.$$

For $f, g\colon \mathcal{P} \to \mathbb{C}$, let us define a partition analog of Dirichlet convolution[3], viz.

$$(f * g)(\lambda) := \sum_{\delta|\lambda} f(\delta) g(\lambda/\delta). \tag{5.12}$$

Then the partition Cauchy product in Proposition 3.3.7 yields another familiar relation.

Theorem 5.6.2. *We have*

$$\left(\sum_{\lambda\in\mathcal{P}} f(\lambda) n_\lambda^{-s}\right)\left(\sum_{\lambda\in\mathcal{P}} g(\lambda) n_\lambda^{-s}\right) = \sum_{\lambda\in\mathcal{P}} (f * g)(\lambda) n_\lambda^{-s}. \tag{5.13}$$

Remark. See Appendix D for further notes on Chapter 5.

[3] An analogy which was suggested to the author by Olivia Beckwith

Chapter 6

Partition-theoretic formulas for arithmetic densities

Adapted from [OSW18], a joint work with Ken Ono and Ian Wagner

6.1 Introduction and statement of results

Consider again the classical Möbius function $\mu(n)$, and let us rewrite the well-known fact $\sum_{n=1}^{\infty} \mu(n)/n = 0$ in the form

$$-\sum_{n=2}^{\infty} \frac{\mu(n)}{n} = 1. \tag{6.1}$$

For notational convenience define $\mu^*(n) := -\mu(n)$. Now, (6.1) above can be interpreted as the statement that one-hundred percent of integers $n \geq 2$ are divisible by at least one prime. This idea was used by Alladi [All77] to prove that if $\gcd(r, t) = 1$, then

$$\sum_{\substack{n \geq 2 \\ p_{\min}(n) \equiv r \pmod{t}}} \frac{\mu^*(n)}{n} = \frac{1}{\varphi(t)}. \tag{6.2}$$

Here $\varphi(t)$ is Euler's phi function, and $p_{\min}(n)$ is the smallest prime factor of n.

Alladi has asked[1] for a partition-theoretic generalization of this result. We answer his question by obtaining an analog of a generalization that was recently obtained by Locus [Loc17]. Locus began by interpreting Alladi's theorem as a device for computing densities of primes in arithmetic progressions. She generalized this idea, and proved analogous formulas for the Chebotarev densities of Frobenius elements in unions of conjugacy classes of Galois extensions.

We recall Locus's result. Let S be a subset of primes with Dirichlet density, and define

$$\mathfrak{F}_S(s) := \sum_{\substack{n\geq 2\\ p_{\min}(n)\in S}} \frac{\mu^*(n)}{n^s}. \tag{6.3}$$

Suppose K is a finite Galois extension of $\mathbb{Q}$ and p is an unramified prime in K. Define

$$\left[\frac{K/\mathbb{Q}}{p}\right] := \left\{ \left[\frac{K/\mathbb{Q}}{\mathfrak{p}}\right] : \mathfrak{p} \subseteq \mathcal{O}_K \text{ is a prime ideal above } p \right\},$$

where $\left[\frac{K/\mathbb{Q}}{\mathfrak{p}}\right]$ is the Artin symbol (for example, see Chapter 8 of [Mar77]), and $\mathcal{O}_K$ is the ring of integers of K. It is well known that $\left[\frac{K/\mathbb{Q}}{p}\right]$ is a conjugacy class C in $G = \mathrm{Gal}(K/\mathbb{Q})$. If we let

$$S_C := \left\{ p \text{ prime} : \left[\frac{K/\mathbb{Q}}{p}\right] = \mathrm{C} \right\}, \tag{6.4}$$

then Locus proved (see Theorem 1 of [Loc17]) that

$$\mathfrak{F}_{S_C}(1) = \frac{\#C}{\#G}.$$

Remark. Alladi's formula (6.2) is the cyclotomic case of Locus's Theorem.

We now turn to Alladi's question concerning a partition-theoretic analog. Let $\mathrm{sm}(\lambda) := \lambda_{\ell(\lambda)}$ denote the smallest part of λ (resp. $\mathrm{lg}(\lambda) := \lambda_1$ the largest part of λ). Also, recall the partition-theoretic Möbius function $\mu_{\mathcal{P}}$ from previous chapters. Notice that $\mu_{\mathcal{P}}(\lambda) = 0$ if

[1]K. Alladi, "A duality between the largest and smallest prime factors via the Moebius function and arithmetical consequences", Emory University Number Theory Seminar, February 28, 2017.

λ has any repeated parts, which is analogous to the vanishing of $\mu(n)$ for integers n which are not square-free. In particular, the parts in partition λ play the role of prime divisors of n in this analogy, as in Chapter 3. We define $\mu^*_{\mathcal{P}}(\lambda) := -\mu_{\mathcal{P}}(\lambda)$ as in Locus's theorem, for aesthetic reasons.

The table below offers a description of the objects which are related with respect to this analogy. However, it is worthwhile to first compare the generating functions for $\mu(n)$ and $\mu_{\mathcal{P}}(\lambda)$. Using the Euler product for the Riemann zeta function, it is well known that the Dirichlet generating function for $\mu(n)$ is

$$\frac{1}{\zeta(s)} = \prod_{p \text{ prime}} \left(1 - \frac{1}{p^s}\right) = \sum_{m=1}^{\infty} \mu(m) m^{-s}. \tag{6.5}$$

As we noted in Chapter 3, the generating function for $\mu_{\mathcal{P}}(\lambda)$ is

$$(q;q)_\infty = \prod_{n=1}^{\infty}(1-q^n) = \sum_{\lambda} \mu_{\mathcal{P}}(\lambda) q^{|\lambda|}.$$

By comparing the generating functions for $\mu(n)$ and $\mu_{\mathcal{P}}(\lambda)$, we see that prime factors and integer parts of partitions are natural analogs of each other. The following table offers the identifications that make up this analogy.

Natural number m	Partition λ
Prime factors of m	Parts of λ
Square-free integers	Partitions into distinct parts
$\mu(m)$	$\mu_{\mathcal{P}}(\lambda)$
$p_{\min}(m)$	$\mathrm{sm}(\lambda)$
$p_{\max}(m)$	$\lg(\lambda)$
m^{-s}	$q^{\lvert\lambda\rvert}$
$\zeta(s)^{-1}$	$(q;q)_\infty$
$s=1$	$q \to 1$

Suppose that $\mathbb{S}$ is a subset of the positive integers with arithmetic density

$$\lim_{X\to\infty}\frac{\#\{n\in\mathbb{S}:n\leq X\}}{X}=d_{\mathbb{S}}.$$

The partition-theoretic counterpart to (6.3) is

$$F_{\mathbb{S}}(q):=\sum_{\substack{\lambda\in\mathcal{P}\\ \mathrm{sm}(\lambda)\in\mathbb{S}}}\mu^*_{\mathcal{P}}(\lambda)q^{|\lambda|}. \tag{6.6}$$

To state our results, we define

$$\mathbb{S}_{r,t}:=\{n\in\mathbb{Z}^+:n\equiv r\pmod t\}. \tag{6.7}$$

These sets are simply the positive integers in an arithmetic progression r modulo t.

Our first result concerns the case where $t=2$. Obviously, the arithmetic densities of $\mathbb{S}_{1,2}$ and $\mathbb{S}_{2,2}$ are both $1/2$. The theorem below offers a formula illustrating these densities and also offers curious lacunary q-series identities.

Theorem 6.1.1. *Assume the notation above.*

(1) The following q-series identities are true:

$$F_{\mathbb{S}_{1,2}}(q)=\sum_{n=1}^{\infty}(-1)^{n+1}q^{n^2},$$

$$F_{\mathbb{S}_{2,2}}(q)=1+\sum_{n=1}^{\infty}(-1)^n q^{n^2}-\sum_{m=-\infty}^{\infty}(-1)^m q^{\frac{m(3m-1)}{2}}.$$

(2) We have that

$$\lim_{q\to1}F_{\mathbb{S}_{1,2}}(q)=\lim_{q\to1}F_{\mathbb{S}_{2,2}}(q)=\frac{1}{2}.$$

Remark. The limits in Theorem 6.1.1 are understood as q tends to 1 from within the unit disk.

Example 6.1.2. *For complex z in the upper-half of the complex plane, let $q(z) := \exp\left(-\frac{2\pi \mathrm{i}}{\mathrm{z}}\right)$. Therefore, if $z \to 1$ in the upper-half plane, then $q(z) \to 1$ in the unit disk. The table below displays a set of such z beginning to approach 1 and the corresponding values of $F_{\mathbb{S}_{1,2}}(q(z))$.*

z	$F_{\mathbb{S}_{1,2}}(q(z))$
$1 + .10i$	0.458233...
$1 + .09i$	0.471737...
$1 + .08i$	0.482784...
$1 + .07i$	0.491003...
$1 + .06i$	0.496296...
$1 + .05i$	0.498998...
$1 + .04i$	0.499919...
$1 + .03i$	0.500048...
$1 + .02i$	0.500024...
$1 + .01i$	0.500006...

Theorem 6.1.1 (1) offers an immediate combinatorial interpretation. Let $D^+_{\text{even}}(n)$ denote the number of partitions of n into an even number of distinct parts with smallest part even, and let $D^+_{\text{odd}}(n)$ denote the number of partitions of n into an even number of distinct parts with smallest part odd. Similarly, let $D^-_{\text{even}}(n)$ denote the number of partitions of n into an odd number of distinct parts with smallest part even, and let $D^-_{\text{odd}}(n)$ denote the number of partitions of n into an odd number of distinct parts with smallest part odd. To make this precise, for integers k let $\omega(k) := \frac{k(3k-1)}{2}$ be the index k *pentagonal number*.

Corollary 6.1.1. *Assume the notation above. We have the following bijections:*

(1) For partitions into distinct parts whose smallest part is odd, we have

$$D^+_{\text{odd}}(n) - D^-_{\text{odd}}(n) = \begin{cases} 0 & \text{if } n \text{ is not a square} \\ 1 & \text{if } n \text{ is an even square} \\ -1 & \text{if } n \text{ is an odd square.} \end{cases}$$

(2) For partitions into distinct parts whose smallest part is even, we have

$$D^+_{\text{even}}(n) - D^-_{\text{even}}(n) = \begin{cases} -1 & \text{if } n \text{ is an even square and not a pentagonal number} \\ 1 & \text{if } n \text{ is an odd square and not a pentagonal number} \\ 1 & \text{if } n \text{ is an even index pentagonal number and not a square} \\ -1 & \text{if } n \text{ is an odd index pentagonal number and not a square} \\ 0 & \text{otherwise.} \end{cases}$$

Question 1. *It would be interesting to obtain a combinatorial proof of Corollary 6.1.1 by refining Franklin's proof of Euler's Pentagonal Number Theorem (see pages 10-11 of [And98]).*

Our proof of Theorem 6.1.1 makes use of the q-Binomial Theorem and some well-known q-series identities. It is natural to ask whether such a relation holds for general sets $\mathbb{S}_{r,t}$. The following theorem shows that Theorem 6.1.1 is indeed a special case of a more general phenomenon.

Theorem 6.1.3. *If $0 \leq r < t$ are integers and $\gcd(m,t) = 1$, then we have that*

$$\lim_{q \to \zeta} F_{\mathbb{S}_{r,t}}(q) = \frac{1}{t},$$

where ζ is a primitive mth root of unity.

Remark. The limits in Theorem 6.1.3 are understood as q tends to ζ from within the unit disk.

Obviously, these results hold for any set $\mathbb{S}$ of positive integers that is a finite union of arithmetic progressions. It turns out that this theorem can also be used to compute arithmetic densities of more complicated sets arising systematically from a careful study of arithmetic progressions. We focus on the sets of positive integers $\mathbb{S}_{\mathrm{fr}}^{(k)}$ which are kth power-free. In particular, we have that

$$\mathbb{S}_{\mathrm{fr}}^{(2)} = \{1, 2, 3, 5, 6, 7, 10, 11, 13, \ldots\}.$$

It is well known that the arithmetic densities of these sets are given by

$$\lim_{X \to +\infty} \frac{\#\left\{1 \leq n \leq X \ : \ n \in \mathbb{S}_{\mathrm{fr}}^{(k)}\right\}}{X} = \prod_{p \text{ prime}} \left(1 - \frac{1}{p^k}\right) = \frac{1}{\zeta(k)}.$$

To obtain partition-theoretic formulas for these densities, we first compute a partition-theoretic formula for the density of

$$\mathbb{S}_{\mathrm{fr}}^{(k)}(N) := \{n \geq 1 \ : \ p^k \nmid n \text{ for every } p \leq N\}. \tag{6.8}$$

Theorem 6.1.4. *If $k, N \geq 2$ are integers, then we have that*

$$\lim_{q \to 1} F_{\mathbb{S}_{\mathrm{fr}}^{(k)}(N)}(q) = \prod_{p \leq N \text{ prime}} \left(1 - \frac{1}{p^k}\right).$$

The constants in Theorem 6.1.4 are the arithmetic densities of positive integers that are not divisible by the kth power of any prime $p \leq N$, namely $\mathbb{S}_{\mathrm{fr}}^{(k)}(N)$. Theorem 6.1.4 can be used to calculate the arithmetic density of $\mathbb{S}_{\mathrm{fr}}^{(k)}$ by letting $N \to +\infty$.

Corollary 6.1.2. *If $k \geq 2$, then*

$$\lim_{q \to 1} F_{\mathbb{S}_{\mathrm{fr}}^{(k)}}(q) = \frac{1}{\zeta(k)}.$$

Furthermore, if $k \geq 2$ is even, then

$$\lim_{q \to 1} F_{\mathbb{S}_{\mathrm{fr}}^{(k)}}(q) = (-1)^{\frac{k}{2}+1} \frac{k!}{B_k \cdot 2^{k-1}} \cdot \frac{1}{\pi^k},$$

where B_k is the kth Bernoulli number.

This chapter is organized as follows. In Section 6.2.1 we discuss the q-Binomial Theorem, which will be an essential tool for our proofs, as well as a duality principle for partitions related to ideas of Alladi. In Section 6.2.2 we will use the q-Binomial Theorem to prove results related to Theorem 6.1.3. Section 6.3 will contain the proofs of all of the theorems, and Section 6.4 will contain some nice examples.

6.2 The q-Binomial Theorem and its consequences

In this section we recall elementary q-series identities, and we offer convenient reformulations for the functions $F_{\mathbb{S}}(q)$.

6.2.1 Nuts and bolts

Let us recall the classical q-Binomial Theorem (see [And98] for proof).

Lemma 6.2.1. *For $a, z \in \mathbb{C}, |q| < 1$ we have the identity*

$$\frac{(az;q)_\infty}{(z;q)_\infty} = \sum_{n=0}^{\infty} \frac{(a;q)_n}{(q;q)_n} z^n.$$

We recall the following well-known q-product identity (for proof, see page 6 of [Fin88]).

Lemma 6.2.2. *Using the above notations, we have that*

$$\frac{(q;q)_\infty^2}{(q^2;q^2)_\infty} = 1 + 2\sum_{n=1}^{\infty}(-1)^n q^{n^2}.$$

The following elementary lemma plays a crucial role in this paper.

Lemma 6.2.3. *If $\mathbb{S}$ is a subset of the positive integers, then the following are true:*

$$F_{\mathbb{S}}(q) = \sum_{n\in\mathbb{S}} q^n \prod_{m=1}^{\infty}(1-q^{m+n}) = (q;q)_\infty \cdot \sum_{\substack{\lambda\in\mathcal{P}\\ \mathrm{lg}(\lambda)\in\mathbb{S}}} q^{|\lambda|}.$$

Remark. *Lemma 6.2.3 may be viewed as a partition-theoretic case of Alladi's duality principle, which was originally stated in [All77] as a relation between functions on smallest versus largest prime divisors of integers, and was given in full partition-theoretic generality by Alladi in a lecture at Emory University[2], although we don't use that formula here.*

Proof. By inspection, we see that

$$F_{\mathbb{S}}(q) = \sum_{\substack{\lambda\in\mathcal{P}\\ \mathrm{sm}(\lambda)\in\mathbb{S}}} \mu^*_{\mathcal{P}}(\lambda) q^{|\lambda|} = \sum_{n\in\mathbb{S}} q^n \prod_{m=1}^{\infty}(1-q^{m+n}).$$

By factoring out $(q;q)_\infty$ from each summand, we find that

$$\begin{aligned} F_{\mathbb{S}}(q) &= \sum_{n\in\mathbb{S}} q^n \prod_{m=1}^{\infty}(1-q^{m+n}) = (q;q)_\infty \cdot \sum_{n\in\mathbb{S}} \frac{q^n}{(q;q)_n} \\ &= (q;q)_\infty \cdot \sum_{\substack{\lambda\in\mathcal{P}\\ \mathrm{lg}(\lambda)\in\mathbb{S}}} q^{|\lambda|}. \end{aligned}$$

□

[2] See previous footnote in this chapter

6.2.2 Case of $F_{\mathbb{S}_{r,t}}(q)$

Here we specialize Lemma 6.2.3 to the sets $\mathbb{S}_{r,t}$. The next lemma describes the q-series $F_{\mathbb{S}_{r,t}}(q)$ in terms of a finite sum of quotients of infinite products. To prove this lemma we make use of the q-Binomial Theorem.

Lemma 6.2.4. *If t is a positive integer and $\zeta_t := e^{2\pi i/t}$, then*

$$F_{\mathbb{S}_{r,t}}(q) = (q;q)_\infty \cdot \frac{1}{t}\left[\sum_{m=1}^{t} \frac{\zeta_t^{-mr}}{(\zeta_t^m q;q)_\infty}\right].$$

Proof. From Lemma 6.2.3, we have that

$$F_{\mathbb{S}_{r,t}}(q) = (q;q)_\infty \cdot \sum_{n=0}^{\infty} \frac{q^{tn+r}}{(q;q)_{tn+r}}.$$

By applying the q-Binomial Theorem (see Lemma 6.2.1) with $a = 0$ and $z = \zeta_t^m q$, we find that

$$\frac{1}{t}\left[\sum_{m=1}^{t} \frac{\zeta_t^{-mr}}{(\zeta_t^m q;q)_\infty}\right] = \frac{1}{t}\left[\sum_{m=1}^{t}\sum_{n=0}^{\infty} \frac{\zeta_t^{m(n-r)} q^n}{(q;q)_n}\right].$$

Due to the orthogonality of roots of unity we have

$$\sum_{m=1}^{t} \zeta_t^{m(n-r)} = \begin{cases} t & \text{if } n \equiv r \pmod{t} \\ 0 & \text{otherwise.} \end{cases}$$

Hence, this sum allows us to sieve on the sum in n leaving only those summands with $n \equiv r \pmod{t}$, namely the series

$$\sum_{n=0}^{\infty} \frac{q^{tn+r}}{(q;q)_{tn+r}}.$$

Therefore, it follows that

$$F_{\mathbb{S}_{r,t}}(q) = (q;q)_\infty \cdot \frac{1}{t}\left[\sum_{m=1}^{t}\sum_{n=0}^{\infty}\frac{\zeta_t^{m(n-r)}q^n}{(q;q)_n}\right].$$

□

Lemma 6.2.5. *If a and m are positive integers and ζ is a primitive mth root of unity, then*

$$\lim_{q\to 1}\frac{(q;q)_\infty}{(\zeta^a q;q)_\infty} = \begin{cases} 1 & \text{if } m \mid a \\ 0 & \text{otherwise.} \end{cases}$$

Proof. Since $(aq;q)_\infty^{\pm 1}$ is an analytic function of q inside the unit disk (i.e., of $q := e^{2\pi i z}$ with z in the upper half-plane) when $|a| \leq 1$, the quotient on the left-hand side of Lemma 6.2.5 is well-defined as a function of q (resp. of z), and we can take limits from inside the unit disk. When $m \mid a$, the q-Pochhammer symbols cancel and the quotient is identically 1. When $m \nmid a$, then $(q;q)_\infty$ clearly vanishes as $q \to 1$ while $(\zeta^a q;q)_\infty$ is non-zero; thus the quotient is zero. □

6.3 Proofs of these results

6.3.1 Proof of Theorem 6.1.1

Here we prove Theorem 6.1.1 (1); we defer the proof of the second part until the next section because it is a special case of Theorem 6.1.3.

Proof of Theorem 6.1.1 (1). By Lemma 6.2.4 we have

$$\begin{aligned} F_{\mathbb{S}_{1,2}}(q) &= (q;q)_\infty \cdot \frac{1}{2}\left[\frac{1}{(q;q)_\infty} - \frac{1}{(-q;q)_\infty}\right] \\ &= \frac{1}{2}\left[1 - \frac{(q;q)_\infty}{(-q;q)_\infty}\right] \\ &= \frac{1}{2}\left[1 - \frac{(q;q)_\infty^2}{(q^2;q^2)_\infty}\right]. \end{aligned}$$

Lemma 6.2.2 now implies that

$$F_{\mathbb{S}_{1,2}}(q) = \sum_{n=1}^{\infty}(-1)^{n+1}q^{n^2}.$$

To prove the $F_{\mathbb{S}_{2,2}}(q)$ identity, first recall that $\sum_{\lambda\in\mathcal{P}}\mu^*_{\mathcal{P}}(\lambda)q^{|\lambda|} = -(q;q)_\infty$. From this we know $F_{\mathbb{S}_{1,2}}(q) + F_{\mathbb{S}_{2,2}}(q) = 1 - (q;q)_\infty$. Using the identity for $F_{\mathbb{S}_{1,2}}(q)$ and Euler's Pentagonal Number Theorem completes the proof. □

Proof of Corollary 6.1.1. Case (1). This corollary follows immediately from Theorem 6.1.1 (1). The reader should recall that $F_{\mathbb{S}_{1,2}}(q)$ is the generating function for $\mu^*_{\mathcal{P}}(\lambda) = -\mu_{\mathcal{P}}(\lambda)$.

Case (2). This corollary is not as immediate as case (1). Of course, we must classify the integer pairs m and n for which $n^2 = m(3m-1)/2$. After simple manipulation, we find that this holds if and only if

$$(6m-1)^2 - 6(2n)^2 = 1.$$

In other words, we require that $(x,y) = (6m-1, 2n)$ be a solution to the Pell equation

$$x^2 - 6y^2 = 1.$$

It is well known that all of the positive solutions to Pell's equation are of the form (x_k, y_k), where

$$x_k + \sqrt{6} \cdot y_k = (5 + 2\sqrt{6})^k.$$

Using this description and the elementary congruence properties of (x_k, y_k), one easily obtains Corollary 6.1.1 (2).

$\square$

6.3.2 Proof of Theorem 6.1.3

Here we prove the general limit formulas for the arithmetic densities of $\mathbb{S}_{r,t}$.

Proof of Theorem 6.1.3. From Lemma 6.2.4 we have

$$F_{\mathbb{S}_{r,t}}(q) = (q;q)_\infty \cdot \frac{1}{t} \left[\sum_{m=1}^{t} \frac{\zeta_t^{-mr}}{(\zeta_t^m q; q)_\infty} \right].$$

We stress that we can take a limit here because we have a finite sum of functions which are analytic inside the unit disk. Using Lemma 6.2.5 we see that

$$\lim_{q \to 1} \frac{(q;q)_\infty}{(\zeta_t^m q; q)_\infty} = \begin{cases} 1 & \text{if } m = t \\ 0 & \text{otherwise.} \end{cases}$$

From this we have

$$\lim_{q \to 1} F_{\mathbb{S}_{r,t}}(q) = \frac{1}{t}.$$

The proof for $q \to \zeta$ where ζ is a primitive mth root of unity with $\gcd(m, t) = 1$ follows the exact same steps.

$\square$

6.3.3 Proofs of Theorem 6.1.4 and Corollary 6.1.2

Here we will prove Theorem 6.1.4 and Corollary 6.1.2 by building up kth power-free sets using arithmetic progressions. We prove Theorem 6.1.4 first.

Proof of Theorem 6.1.4. The set of numbers not divisible by p^k for any prime $p \le N$ can be built as a union of sets of arithmetic progressions. Therefore, for a given fixed N we only need to understand divisibility by p^k for all primes $p \le N$. Because the divisibility condition for each prime is independent from the other primes, we have

$$F_{\mathbb{S}_{\mathrm{fr}}^{(k)}(N)}(q) = \sum_{\substack{0 \le r < M \\ p^k \nmid r}} F_{\mathbb{S}_{r,M}}(q),$$

where $M := \prod_{\substack{p \le N \\ \text{prime}}} p^k$. We have a finite number of summands, and the result now follows immediately from Theorem 6.1.3. □

Next, we will prove Corollary 6.1.2.

Proof of Corollary 6.1.2. For fixed N define $\zeta_N(k) := \prod_{p \le N \text{prime}} \left(\frac{1}{1-p^k}\right)$, so $\lim_{q\to 1} F_{\mathbb{S}_{\mathrm{fr}}^{(k)}(N)}(q)$ $= \frac{1}{\zeta_N(k)}$. It is clear $\lim_{N\to\infty} \zeta_N(k) = \zeta(k)$. It is in this sense that we say $\lim_{q\to 1} F_{\mathbb{S}_{\mathrm{fr}}^{(k)}}(q) =$ $\frac{1}{\zeta(k)}$. □

6.4 Examples

Example 6.4.1. *In the case of $\mathbb{S}_{1,3}$, which has arithmetic density 1/3, Theorem 6.1.3 holds for any mth root of unity where $3 \nmid m$. The two tables below illustrate this as q approaches $\zeta_1 = 1$ and $\zeta_4 = i$, respectively, from within the unit disk.*

q	$F_{\mathbb{S}_{1,3}}(q)$
0.70	0.340411885...
0.75	0.335336994...
0.80	0.333552814...
0.85	0.333331545...
0.90	0.333333329...
0.95	0.333333333...

q	$F_{\mathbb{S}_{1,3}}(q)$
$0.70i$	$\approx 0.034621 + 0.793781i$
$0.75i$	$\approx 0.057890 + 0.802405i$
$0.80i$	$\approx 0.097030 + 0.771774i$
$0.85i$	$\approx 0.167321 + 0.674712i$
$0.90i$	$\approx 0.294214 + 0.454400i$
$0.95i$	$\approx 0.424978 + 0.067775i$
$0.97i$	$\approx 0.376778 - 0.016187i$
$0.98i$	$\approx 0.340170 + 0.005772i$
$0.99i$	$\approx 0.332849 + 0.000477i$

Example 6.4.2. *The table below illustrates Theorem 6.1.4 for the set* $\mathbb{S}_{\mathrm{fr}}^{(2)}(5)$*, which has arithmetic density* $16/25 = 0.64$*. These are the positive integers which are not divisible by 4, 9 and 25. Here we give numerics for the case of* $F_{\mathbb{S}_{\mathrm{fr}}^{(2)}(5)}(q)$ *as* $q \to 1$ *along the real axis.*

q	$F_{\mathbb{S}_{\mathrm{fr}}^{(2)}(5)}(q)$
0.90	0.615367...
0.91	0.619346...
0.92	0.625991...
0.93	0.631607...
0.94	0.631748...
0.95	0.631029...
0.96	0.638291...
0.97	0.639893...

Example 6.4.3. *Here we approximate the density of* $\mathbb{S}_{fr}^{(4)}$*, the fourth power-free positive integers. Since* $\zeta(4) = \pi^4/90$*, it follows that the arithmetic density of* $\mathbb{S}_{fr}^{(4)}$ *is* $90/\pi^4 \approx 0.923938....$ *Here we choose* $N = 5$ *and compute the arithmetic density of* $\mathbb{S}_{fr}^{(4)}(5)$*, the positive integers which are not divisible by* $2^4, 3^4$*, and* 5^4*. The density of this set is* $208/225 \approx 0.924444....$ *This density is fairly close to the density of fourth power-free integers because the convergence in the* N *aspect is significantly faster for fourth power-free integers than for square-free integers, as discussed above.*

q	$F_{\mathbb{S}_{fr}^{(4)}(5)}(q)$
0.90	0.934926...
0.91	0.936419...
0.92	0.936718...
0.93	0.935027...
0.94	0.931517...
0.95	0.925619...
0.96	0.921062...
0.97	0.925998...
0.98	0.924967...

Remark. See Appendix E for further notes on Chapter 6.

Chapter 7

“Strange” functions and a vector-valued quantum modular form

Adapted from [RS13], a joint work with Larry Rolen

7.1 Introduction and Statement of Results

In this chapter and the next, we pivot away from partition theory (at least explicitly) to focus on certain interesting classes of q-series, which we will then tie back to the ideas of the previous sections in the final chapter.

In a seminal 2010 Clay lecture, Zagier defined a new class of function with certain automorphic properties called a “quantum modular form” [Zag10], as in Definition 1.1.4. Roughly speaking, this is a complex function on the rational numbers which has modular transformations modulo “nice” functions. Although the definition is intentionally vague, Zagier gave a handful of motivating examples to serve as prototypes of quantum behavior. For example, he defined quantum modular forms related to Dedekind sums, sums over quadratic polynomials, Eichler integrals and other interesting objects. One of the most striking examples of quantum modularity is described in Zagier's paper on Vassiliev invariants [Zag01], in which he studies the Kontsevich "strange" function introduced in Definition 1.1.5, viz.

$$F(q) := \sum_{n=0}^{\infty} (q;q)_n, \tag{7.1}$$

where we take $q := e^{2\pi i z}$ with $z \in \mathbb{C}$.

This function is strange indeed, as it does not converge on any open subset of $\mathbb{C}$, but converges (as a finite sum) for q any root of unity. In 2012, Bryson, Pitman, Ono, and Rhoades showed [BOPR12] that $F(q^{-1})$ agrees to infinite order at roots of unity with a function $U(-1,q)$ which is also well-defined on the upper-half plane $\mathbb{H}$, obtaining a quantum modular form that is a "reflection" of $F(q)$ and that naturally extends to $\mathbb{H}$. Moreover, $U(-1,q)$ counts unimodal sequences having a certain combinatorial statistic.

Zagier's study of $F(q)$ depends on the formal q-series identity

$$\sum_{n=0}^{\infty} \big(\eta(24z) - q(1-q^{24})(1-q^{48})\cdots(1-q^{24n})\big) = \eta(24z)D(q) + E(q), \tag{7.2}$$

where $\eta(z) := q^{1/24}(q;q)_\infty$, $D(q)$ is an Eisenstein-type series, and $E(q)$ is a "half-derivative" of $\eta(24z)$ (such formal half-derivatives will be discussed in Section 7.2). We refer to such an identity as a "sum of tails" identity. In this chapter we revisit Zagier's construction using work of Andrews, Jiménez-Urroz, and Ono on more general sums of tails formulas [AJUO01] (see also [And86b]). We construct a natural three-dimensional vector-valued quantum modular form associated to tails of infinite products. Moreover, the components are analogous "strange" functions; they do not converge on any open subset of $\mathbb{C}$ but make sense for an infinite subset of $\mathbb{Q}$. We define:

$$H(q) = \begin{pmatrix} \theta_1 \\ \theta_2 \\ \theta_3 \end{pmatrix} := \begin{pmatrix} \eta(z)^2/\eta(2z) \\ \eta(z)^2/\eta(z/2) \\ \eta(z)^2/\eta(\frac{z}{2}+\frac{1}{2}) \end{pmatrix}. \tag{7.3}$$

We also note that $\theta_3 = \zeta_{48}^{-1} \cdot \frac{\eta(z/2)\eta(2z)}{\eta(z)}$ by the following identity which is easily seen by expanding the product definition of $\eta(z)$:

$$\eta(z+1/2) = \zeta_{48} \cdot \frac{\eta(2z)^3}{\eta(z)\cdot\eta(4z)}, \tag{7.4}$$

where $\zeta_k := e^{2\pi i/k}$. From this it follows that if we let

$$F_9(z) := \eta(z)^2/\eta(2z), \qquad\qquad F_{10}(z) := \eta(16z)^2/\eta(8z)$$

then

$$H(q) = \begin{pmatrix} F_9(q) & F_{10}(q^{1/16}) & \zeta_{12}^{-1} F_{10}(\zeta_{16}\cdot q^{1/16}) \end{pmatrix}^T \tag{7.5}$$

(the notations F_9 and F_{10} come from [AJUO01]). For convenience, we recall the classical theta-series identities for F_9 and F_{10}:

$$F_9(q) = 1 + 2\sum_{n=1}^{\infty}(-1)^n q^{n^2}, \qquad\qquad F_{10}(q) = \sum_{n=0}^{\infty} q^{(2n+1)^2}. \tag{7.6}$$

It is simple to check that $H(z)$ is a 3-dimensional vector-valued modular form using basic properties of $\eta(z)$, as we describe in Section 7.4. To each component θ_i we associate for all $n \geq 0$ a finite product $\theta_{i,n}$:

$$\theta_{1,n} := \frac{(q;q)_n}{(-q;q)_n}, \qquad \theta_{2,n} := q^{\frac{1}{16}} \cdot \frac{(q;q)_n}{(q^{\frac{1}{2}};q)_{n+1}}, \qquad \theta_{3,n} := \frac{\zeta_{16}}{\zeta_{12}} \cdot q^{\frac{1}{16}} \cdot \frac{(q;q)_n}{(-q^{\frac{1}{2}};q)_{n+1}}, \tag{7.7}$$

such that $\theta_{i,n} \to \theta_i$ as $n \to \infty$. Next, we construct corresponding "strange" functions $\theta_i^S := \sum_{n=0}^{\infty} \theta_{i,n}$. Note that these functions do not make sense on any open subset of $\mathbb{C}$, but that each θ_i^S is defined for an infinite set of roots of unity and, in particular, θ_2^S is defined for all roots of unity. Our primary object of study will then be the vector of "strange" series $H_Q(z) := \begin{pmatrix} \theta_1^S(z) & \theta_2^S(z) & \theta_3^S(z) \end{pmatrix}^T$. In order to obtain a quantum modular form, we first define $\phi_i(x) := \theta_i^S(e^{2\pi i x})$ from a subset of $\mathbb{Q}$ to $\mathbb{C}$, and let $\phi(x) :=$

$\begin{pmatrix} \phi_1(x) & \phi_2(x) & \phi_3(x) \end{pmatrix}^T$. We then show the following result.

Theorem 7.1.1. *Assume the notation above. Then the following are true:*

(1) There exist q-series G_i (see Section 7.4) which are well-defined for $|q| < 1$, such that $\theta_i^S(q^{-1}) = G_i(q)$ for any root of unity for which θ_i^S converges.

(2) We have that $\phi(x)$ is a weight $3/2$ vector-valued quantum modular form. In particular, we have that

$$\phi(z+1) - \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & \zeta_{12} \\ 0 & \zeta_{24} & 0 \end{pmatrix} \phi(z) = 0,$$

and we also have that

$$\left(\frac{z}{-i}\right)^{-3/2} \phi(-1/z) + \begin{pmatrix} 0 & \sqrt{2} & 0 \\ 1/\sqrt{2} & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \phi(z) = \begin{pmatrix} 0 & \sqrt{2} & 0 \\ 1/\sqrt{2} & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} g(z),$$

where $g(z)$ is a 3-dimensional vector of smooth functions defined as period integrals in Section 7.3.

In addition, we deduce the following corollary regarding generating functions of special values of zeta functions from the sums of tails identities. Let

$$H_9(t,\zeta) := -\frac{1}{4}\sum_{n=0}^{\infty} \frac{(1-\zeta e^{-t})(1-\zeta^2 e^{-2t})\cdots(1-\zeta^n e^{-nt})}{(1+\zeta e^{-t})(1+\zeta^2 e^{-2t})\cdots(1+\zeta^n e^{-nt})}, \tag{7.8}$$

$$H_{10}(t,\zeta) := -2(\zeta e^{-t})^{1/8}\sum_{n=0}^{\infty} \frac{(1-\zeta e^{-2t})(1-\zeta^2 e^{-4t})\cdots(1-\zeta^n e^{-2nt})}{(1-\zeta e^{-t})(1-\zeta^2 e^{-3t})\cdots(1-\zeta^n e^{-(2n+1)t})}. \tag{7.9}$$

Remark. *Note that there are no rational numbers for which all three components of ϕ make sense simultaneously. To be specific, $\phi_1(z)$ makes sense for rational numbers which correspond to primitive odd order roots of unity, $\phi_2(z)$ makes sense for all rational* *numbers, and $\phi_3(z)$ converges at even order roots of unity. Hence, by (2) of Theorem 7.1.1, we understand that each of the six equations of the vector-valued transformation laws is true where the corresponding component in the equation is well-defined; as there are no equations in which ϕ_1 and ϕ_3 both appear, then for all the equations there is an infinite subset of rationals on which this is possible.*

For a root of unity ζ, we define the following two L-functions

$$L_1(s,\zeta) := \sum_{n=1}^{\infty} \frac{(-\zeta)^{n^2}}{n^s},$$

$$L_2(s,\zeta) := \sum_{n=1}^{\infty} \left(\frac{2}{n}\right)^2 \cdot \frac{\zeta^{\frac{n^2}{8}}}{n^s}.$$

Then we have the following.

Corollary 7.1.1. *Let $\zeta = e^{2\pi i\alpha}$ be a primitive k^{th} root of unity, k odd for H_9 and k even for H_{10}. Then as $t \searrow 0$, we have as power series in t*

$$H_9(t,\zeta) = \sum_{n=0}^{\infty} \frac{L_1(-2n-1,\zeta)(-t)^n}{n!}, \tag{7.10}$$

$$H_{10}(t,\zeta) = \sum_{n=0}^{\infty} \frac{L_2(-2n-1,\zeta)(-t)^n}{8^n n!}. \tag{7.11}$$

To illustrate our results by way of an application, we provide a numerical example which gives finite evaluations of seemingly complicated period integrals. First define

$$\Omega(x) := \int_x^{i\infty} \frac{\theta_1(z)}{(z-x)^{3/2}}\,dz$$

for $x \in \mathbb{Q}$, and consider $\theta_1^S(\zeta_k)$ for k odd, which is a finite sum of k^{th} roots of unity. Then the proof of Theorem 7.1.1 will imply that $\Omega(1/k) = \pi i(1+i)\theta_1^S(\zeta_k)$ by showing that the period integral $\Omega(x)$ is a "half-derivative" which is related to θ_1^S at roots of unity by a sum of tails formula. The following table gives finite evaluations of $\theta_1^S(\zeta_k)$ and numerical approximations to the integrals $\Omega(1/k)$.

k	$\pi i(i+1)\theta_1^S(\zeta_k)$	$\int_{1/k+10^{-9}}^{10^9 i} \frac{\theta_1(z)}{(z-1/k)^{\frac{3}{2}}}\,dz$
3	$\pi i(i+1)(-2\zeta_3+3) \sim -7.1250+18.0078i$	$-7.1249+18.0078i$
5	$\pi i(i+1)(-2\zeta_5^3-2\zeta_5^2-8\zeta_5+3) \sim 12.078+35.7274i$	$12.078+35.7273i$
7	$\pi i(i+1)(6\zeta_7^4-2\zeta_7^2-10\zeta_7+7) \sim 52.0472+25.685i$	$52.0474+25.685i$
9	$\pi i(i+1)(8\zeta_9^4-16\zeta_9+3) \sim 76.4120-28.9837i$	$76.4116-28.9836i$

The chapter is organized as follows. In Section 7.2 we recall the identities of [AJUO01], and in Section 7.3 we describe the modularity properties of Eichler integrals of half-integral weight modular forms. In Section 7.4 we complete the proof of Theorem 7.1.1. We finish with the proof of Corollary 7.1.1 in Section 7.5.

7.2 Preliminaries

In this section, we describe some of the machinery needed to prove Theorem 7.1.1.

7.2.1 Sums of Tails Identities

Here we recall the work of Andrews, Jiménez-Urroz, and Ono on sums of tails identities. To state their results for F_9 and F_{10} and connect θ_i^S to quantum modular objects, we formally define a "half-derivative operator" by

$$\sqrt{\theta}\left(\sum_{n=0}^{\infty} a(n)q^n\right) := \sum_{n=1}^{\infty} \sqrt{n}a(n)q^n. \tag{7.12}$$

If we have a generic q-series $f(q)$, we will also denote $\sqrt{\theta}f(q) := \widetilde{f}(q)$. Then Andrews, Jiménez-Urroz, and Ono show [AJUO01] that for finite versions $F_{9,i}, F_{10,i}$ associated to F_9, F_{10} the following holds true:

Theorem 7.2.2 (Andrews-Jiménez-Urroz-Ono)**.** *As formal power series, we have that*

$$\sum_{n=0}^{\infty}(F_9(z) - F_{9,n}(z)) = 2F_9(z)E_1(z) + 2\sqrt{\theta}(F_9(z)), \tag{7.13}$$

$$\sum_{n=0}^{\infty}(F_{10}(z) - F_{10,n}(z)) = F_{10}(z)E_2(z) + \frac{1}{2}\sqrt{\theta}(F_{10}(z)), \tag{7.14}$$

where the $E_i(z)$ are holomorphic Eisenstein-type series.

In particular, as F_9, F_{10} vanish to infinite order while E_1, E_2 are holomorphic at all cusps where the "strange" functions are well-defined, we have for q an appropriate root of unity that the "strange" function associated to F_i equals $\widetilde{F}_i$ to infinite order. As the series θ_2, θ_3 do not have integral coefficients, we make the definitions $\widetilde{\theta}_2(z) := \widetilde{F}_{10}(z/16)$ and $\widetilde{\theta}_3(z) := \widetilde{F}_{10}(z/16+1/16)$. By the definition of the strange series, we obtain the following.

Corollary 7.2.1. *At appropriate roots of unity where each "strange" series is defined, we have that*

$$\theta_1^S(q) = 2\widetilde{\theta}_1(q), \qquad \theta_2^S(q) = \frac{1}{2}\widetilde{\theta}_2(q), \qquad \theta_3^S(q) = \frac{1}{2}\widetilde{\theta}_3(q). \tag{7.15}$$

7.3 Properties of Eichler Integrals

In the previous section we have seen that at a rational point x, each component of $\phi(x)$ agrees up to a constant with a "half-derivative" of the corresponding theta function at $q = e^{2\pi i x}$. Thus, we can reduce part (2) of Theorem 7.1.1 to a study of modularity of such half-derivatives. We do so following the outline given in [Zag01], which is further explained in the weight 3/2 case in [LZ99]. Recall that in the classical setting of weight $2k$ cusp forms, $1 \leq k \in \mathbb{Z}$, we define the *Eichler integral* of $f(z) = \sum_{n=1}^{\infty} a(n)q^n$ as a formal $(k-1)^{\text{st}}$ antiderivative $\widetilde{f}(z) := \sum_{n=1}^{\infty} n^{1-k}a(n)q^n$. Then $\widetilde{f}$ is nearly modular of weight $2-k$, as the differentiation operator $\frac{d}{dq}$ does not preserve modularity but preserves near-modularity. More specifically, $\widetilde{f}(z+1) = \widetilde{f}(z)$ and $z^{k-2}\widetilde{f}(-1/z) - \widetilde{f}(z) = g(z)$

where $g(z)$ is the *period polynomial.* This polynomial encodes deep analytic information about f and can also be written as $g(x) = c_k \int_0^{i\infty} f(z)(z-x)^{k-2}\, dz$ for a constant c_k depending on k. Suppose we now begin with a weight 1/2 vector-valued modular form f with n components f_i such that and $f(-1/z) = M_S f(z)$, for M_S both $n \times n$ matrices (the transformation under translation is routine).

In this case, of course, it does not make sense to speak of a half-integral degree polynomial, and the integral above does not even converge. However, we may remedy the situation so that the analysis becomes similar to the classical case. We formally define $\widetilde{f}$ by taking a formal antiderivative (in the classical sense) on each component. As $1-k = 1/2$, we have in fact $\widetilde{f}_i = \sqrt{\theta} f_i$. We would like to determine an alternative way to write the Eichler integral as an actual integral, so that we may use substitution and derive modularity properties of $\widetilde{f}$ from f. However, the integral $g(z) = c_{1/2} \int_0^{i\infty} f(z)(z-x)^{-3/2}\, dz$ no longer makes sense. To remedy this in the weight 3/2 case, Lawrence and Zagier define another integral $f^*(x) := c_k \int_{\bar{x}}^{\infty} \frac{f(z)}{(z-x)^{\frac{1}{2}}}\, dz$, which is meaningful for x in the lower half plane $\mathbb{H}^-$.

Here we sketch their argument in the weight 1/2 case for completeness, and as the analysis involved in our own work differs slightly. Returning to our vector-valued form f, recall that the definition of the Eichler integral of f corresponds with $\sqrt{\theta} f$. For $x \in \mathbb{H}^-$, we define

$$f^*(x) = \left(\frac{-i}{\pi(1+i)}\right) \cdot \int_{\bar{x}}^{i\infty} \frac{f(z)}{(z-x)^{\frac{3}{2}}}\, dz. \tag{7.16}$$

To evaluate this integral, use absolute convergence to exchange the integral and the sum, and note that for $q_z = e^{2\pi i z}$,

$$\int_{\bar{x}}^{i\infty} \frac{q_z^n}{(z-x)^{\frac{3}{2}}}\, dz = \left((2+2i)\pi\sqrt{n} q_z^n \operatorname{erfi}\left((1+i)\sqrt{\pi n(z-x)}\right) - \frac{2q_x^n}{(z-x)^{\frac{1}{2}}} \right) \Bigg|_{z=\bar{x}}^{i\infty} \tag{7.17}$$

where $\operatorname{erfi}(x)$ is the *imaginary error function.* As in [LZ99], we have that $\widetilde{f}(x+iy) = f^*(x-iy)$ as full asymptotic expansions for $x \in \mathbb{Q}$, $0 < y \in \mathbb{R}$. To see this, note that at the lower limit, the antiderivative vanishes as $y \to 0$ as $\operatorname{erfi}(0) = 0$ and although the square root in the denominator goes to zero, for each rational at which we are evaluating our "strange" series, the corresponding theta functions vanish to infinite order, which makes this term converge. For the upper limit, the square root term immediately vanishes, and we use the fact that $\lim_{x\to\infty} \operatorname{erfi}(1+i)\sqrt{ix+y} = i$ for $x, y \in \mathbb{R}$.

Thus, as in [LZ99], we have that $\widetilde{f}(x) = f^*(x)$ to infinite order at rational points. In the case of θ_1, we have that $\widetilde{\theta}_1(x) = \theta^*(x)$, but for θ_2 and θ_3 we have to divide by $4 = \sqrt{16}$ due to the non-integrality of the powers of q in order to agree with the definition of $\widetilde{\theta}_i$. Using this together with Corollary 7.2.1, in all cases we find that $\theta_i^S(q) = \theta_i^*(q)$ at roots of unity where both sides are defined. Now, the modularity properties for the integral follow *mutatis mutandis* from [LZ99] using the modularity of f and a standard u-substitution. More precisely, suppose $f(-1/z)(z)^{-\frac{1}{2}} = M_S f(z)$. Then we have shown that the following modularity properties hold for $f^*(z)$ when $z \in \mathbb{H}^-$, and hence also hold for $\widetilde{f}(z)$ for each component at appropriate roots of unity where each "strange" function is defined. By this, we mean that the modularity conditions in the following proposition can be expressed as six equations, and each of these equations is true precisely where the corresponding "strange" series make sense.

Proposition 7.3.1. *If* $g(x) := \left(\frac{-i}{\pi(1+i)}\right) \cdot \int_0^{i\infty} \frac{f(z)}{(z-x)^{\frac{3}{2}}}\, dz$, *then*

$$\left(\frac{x}{-i}\right)^{-\frac{3}{2}} f(-1/x) + M_S f(x) = M_S g(x).$$

It is also explained in [LZ99] why $g_\alpha(z)$ is a smooth function for $\alpha \in \mathbb{R}$. Although $g(x)$ is *a priori* only defined in $\mathbb{H}^-$, we may take any path L connecting 0 to $i\infty$. Then we can holomorphically continue $g(x)$ to all of $\mathbb{C} - L$. Thus, we obtain a continuation of g which is smooth on $\mathbb{R}$ and analytic on $\mathbb{R} - \{0\}$.

7.4 Proof of Theorem 7.1.1

Here we complete the proofs of parts (1) and (2) of Theorem 7.1.1.

7.4.1 Proof of Theorem 7.1.1 (1)

We show that at appropriate roots of unity, our "strange" functions θ_i^S are reflections of q-series which are convergent on $\mathbb{H}$. Using (7.5), it suffices to show for θ_1^S that $\sum_{n=0}^{\infty} \frac{(q^{-1};q^{-1})_n}{(-q^{-1};q^{-1})_n}$ agrees at odd roots of unity with a q-series convergent when $|q| < 1$. To factor out inverse powers of q, we observe that

$$(a^{-1};q^{-\alpha})_n = (-1)^n a^n q^{\frac{\alpha n(n-1)}{2}} (a;q^\alpha)_n. \tag{7.18}$$

Applying this identity to the numerator and denominator term-by-term, we have at odd order roots of unity

$$\theta_1^S(q^{-1}) = \sum_{n=0}^{\infty} (-1)^n \frac{(q;q)_n}{(-q;q)_n} = 2\sum_{n=0}^{\infty} \frac{q^{2n+1}(q;q)_{2n}}{(1+q^{2n+1})(-q;q)_{2n}}. \tag{7.19}$$

The series on the right-hand side is clearly convergent for $|q| < 1$, and results from pairing consecutive terms of the left-hand series as follows:

$$\frac{(q;q)_{2n}}{(-q;q)_{2n}} - \frac{(q;q)_{2n+1}}{(-q;q)_{2n+1}} = \frac{(q;q)_{2n}}{(-q;q)_{2n}}\left(1 - \frac{1-q^{2n+1}}{1+q^{2n+1}}\right) = \frac{2q^{2n+1}(q;q)_{2n}}{(1+q^{2n+1})(-q;q)_{2n}}.$$

Remark. *Alternatively, one can show the convergence of $\theta_1^S(q^{-1})$ by letting $a = 1, b = -1, t = -1$ in Fine's identity [Fin88]*

$$\sum_{n=0}^{\infty} \frac{(aq;q)_n}{(bq;q)_n}(t)^n = \frac{1-b}{1-t} + \frac{b-atq}{1-t}\sum_{n=0}^{\infty} \frac{(aq;q)_n}{(bq;q)_n}(tq)^n, \tag{7.20}$$

giving

$$\theta_1^S(q^{-1}) = 1 + \frac{q-1}{2}\sum_{n=0}^{\infty}\frac{(q;q)_n}{(-q;q)_n}(-q)^n \tag{7.21}$$

which also converges for $|q| < 1$.

Similarly, we use (7.18) to study θ_2^S, θ_3^S. Note that it suffices by (7.5) to study $\sum_{n=0}^{\infty}\frac{(q^{-2};q^{-2})_n}{(q^{-3};q^{-2})_n}$. Factorizing as above, we find that

$$\sum_{n=0}^{\infty}\frac{(q^{-2};q^{-2})_n}{(q^{-3};q^{-2})_n} = \sum_{n=0}^{\infty}\frac{q^n(q^2;q^2)_n}{(q^3;q^2)_n}, \tag{7.22}$$

the right-hand side of which is clearly convergent on $\mathbb{H}$. We note that in general, similar inversion formulas result from applying (7.18) to diverse q-series and other expressions involving eta functions, q-Pochhammer symbols and the like.

7.4.2 Proof of Theorem 7.1.1 (2)

Proof. Here we complete the proof of Theorem 7.1.1 using the results of Sections 7.2 and 7.3. Note that by the Corollary (7.2.1) to the sums of tails formulas of Andrews, Jiménez-Urroz, and Ono in [AJUO01], each component of $H(q)$ agrees to infinite order at rational numbers with a multiple of the corresponding Eichler integral. By the discussion of Eichler integrals in Section 7.3, the value of each $\widetilde{\theta}_i$ agrees at rationals with the value of the corresponding θ_i^*. Therefore, by the discussion of the modularity properties of θ_i^*, we need only to describe the modularity of $H(q)$. This is simple to check using the usual transformation laws

$$\eta(z+1) = \zeta_{24}\eta(z), \tag{7.23}$$

$$\eta(-1/z) = \left(\frac{z}{i}\right)^{\frac{1}{2}}\eta(z), \tag{7.24}$$

and (7.4). Hence we see that

$$H(z+1)=\begin{pmatrix}1&0&0\\0&0&\zeta_{12}\\0&\zeta_{24}&0\end{pmatrix}H(z),\tag{7.25}$$

$$H(-1/z)=\left(\frac{z}{i}\right)^{\frac{1}{2}}\begin{pmatrix}0&\sqrt{2}&0\\1/\sqrt{2}&0&0\\0&0&1\end{pmatrix}H(z),\tag{7.26}$$

and the corresponding transformations of θ_i^* follow. □

7.5 Proof of Corollary 7.1.1

Proof. The proof of Corollary 1.1 is a generalization of and proceeds similarly to the proofs of Theorems 4 and 5 of [AJUO01]. As the sums of tails identities in Theorem 2.1 show that the "strange" functions F_9 and F_{10} agree *to infinite order* with the half derivatives of F_9 and F_{10} at the roots of unity under consideration, the coefficients in the asymptotic expansion of $H_i(t,\zeta)$ for $i=9,10$ agree up to a constant factor with the coefficients of the asymptotic expansion of $\sqrt{\theta}F_i(\zeta e^{-t})$. Recalling the classical theta series expansions for F_i in (1.6), the first part of Corollary 1.1 follows immediately from the following well-known fact:

Lemma 7.5.1 (Proposition 5 of [Kaz06])**.** *Let $\chi(n)$ be a periodic function with mean value zero and $L(s,\chi):=\sum_{n=0}^{\infty}\chi(n)n^{-s}$. As $t\searrow 0$, we have*

$$\sum_{n=0}^{\infty}n\chi(n)e^{-n^2t}\sim\sum_{n=0}^{\infty}L(-2n-1,\chi)\frac{(-t)^n}{n!}.$$

The proof follows from taking a Mellin transform, making a change of variables, and picking up residues at negative integers. The assumption on the coefficients $\chi(n)$ assures that $L(s,\chi)$ can be analytically continued to $\mathbb{C}$. The mean value zero condition is easily checked in our case; for example for F_9 one needs to verify that $\{(-\zeta)^{n^2}\}_{n\geq 0}$ is mean value zero for ζ a primitive order $2k+1$ root of unity, and for F_{10} one must check that $\{\zeta^{\frac{(2n+1)^2}{8}}\}_{n\geq 0}$ is mean value zero for an even order root of unity ζ. These may both be checked using well-known results for the generalized quadratic Gauss sum

$$G(a,b,c) := \sum_{n=0}^{c-1} e\left(\frac{an^2+bn}{c}\right). \tag{7.27}$$

In particular, for F_9, for an odd order root of unity ζ, $-\zeta$ is primitive of order k where $k \equiv 2 \pmod 4$, so we need that $G(a,0,k)=0$ when $k\equiv 2 \pmod 4$, which fact is well known. For F_{10}, we may use the standard fact that $G(a,b,c)=0$ whenever $4|c$, $(a,c)=1$, and $0<b\in 2\mathbb{Z}+1$ to obtain our result. This Gauss sum calculation follows, for instance, by using the multiplicative property of Gauss sums together with an application of Hensel's lemma.

In the case of F_{10}, note that the formula for $H_{10}(t,\alpha)$ is obtained by substituting $q=\zeta e^{-t}$ into the "strange" function for F_{10} after letting $q\to q^{\frac{1}{8}}$. A simple change of variables in the Mellin transform in the foregoing proof of the present Lemma adjusts for the $1/8$ powers by giving an extra factor of 8^s before taking residues. $\square$

Remark. See Appendix F for further notes on Chapter 7.

Chapter 8

Jacobi's triple product, mock theta functions, unimodal sequences and the q-bracket

Adapted from [Schar]

8.1 Introduction

We do not know what sparked Ramanujan to discover mock theta functions, but we see in this chapter that they are indeed natural functions to study from a classical perspective. It turns out in Section 8.2 all of the mock theta functions Ramanujan wrote to Hardy about — to be precise, the odd-order *universal mock theta function* of Gordon–McIntosh that essentially specializes to the odd-order mock theta functions Ramanujan wrote down [GM12] — arise from the *Jacobi triple product*, a fundamental object in number theory and combinatorics [Ber06], and are generally "entangled" with rank generating functions for unimodal sequences, under the action of the *q-bracket* operator from statistical physics and partition theory that we studied in Chapter 3, which boils down to multiplication by $(q;q)_\infty$. In Section 8.3 we find finite formulas for the odd-order universal mock theta function and indicate similar formulas for other q-hypergeometric series.

8.2 Connecting the triple product to mock theta functions via partitions and unimodal sequences

At the wildest boundaries of nature, we see tantalizing hints of q-series. In the previous chapter we investigated a class of almost-modular forms having the "feel" of quantum phenomena [Zag10]. In a different "quantum" connection, Borcherds proposed a proof of the Jacobi triple product identity

$$j(z;q) := (z;q)_\infty(z^{-1}q;q)_\infty(q;q)_\infty = \sum_{n=-\infty}^{\infty}(-1)^n z^n q^{n(n+1)/2} \tag{8.1}$$

where $q, z \in \mathbb{C}, |q| < 1, z \neq 0$[1], based on the *Dirac sea* model of the quantum vacuum, plus ideas from partition theory (see [Cam94]): the quantum states of fermions, which obey the Pauli exclusion principle, are conceptually analogous to partitions into distinct parts; quantum states of bosons, which are unrestricted in the number that can occupy any state, correspond to partitions with unrestricted multiplicities of parts[2]. The triple product is implicit in countless famous classical identities (see [Ber06]). Up to multiplication by rational powers of q, $j(z;q)$ specializes to the Jacobi theta function (that Ramanujan constructed "mock" versions of), a weight 1/2 modular form which is also important in physics as a solution to the heat equation. In Borcherds's proof, it is as if this beautiful, versatile identity (8.1) emerges from properties of empty space.

Also from the universe of q-hypergeometric series, mock theta functions and their generalization mock modular forms [BFOR17] are connected conjecturally to deep mysteries

[1] We note the simple zero at $z = 1$ from the $(1-z)$ factor in $(z;q)_\infty$.

[2] In Appendix B we draw further analogies to particle physics by introducing "antipartitions" that annihilate partitions, yielding a multiplicative group structure on the partitions.

in physics, like mind-bending phenomena at the edges of black holes [DMZ12, DGO15]. All the diversity of physical reality — and of our own mental experience — plays out quite organically between these enigmatic extremes. Perhaps not unrelatedly, in this chapter we see there is an organic connection between the Jacobi triple product and mock theta functions, under the action of the q-bracket of Bloch–Okounkov studied in Chapter 3.[3]

The odd-order *universal mock theta function* $g_3(z,q)$ of Gordon and McIntosh [GM12], which specializes to Ramanujan's original list of mock theta functions up to changes of variables and multiplication by rational powers of q and z (with z a rational power of q times a root of unity), is defined as

$$g_3(z,q) := \sum_{n=1}^{\infty} \frac{q^{n(n-1)}}{(z;q)_n(z^{-1}q;q)_n}, \tag{8.2}$$

and, like the triple product, is subject to all sorts of wonderful transformations.[4]

Let us recall that a *unimodal sequence* of integers is of the type

$$0 \leq a_1 \leq a_2 \leq \ldots \leq a_r \leq \overline{c} \geq b_1 \geq b_2 \geq \ldots \geq b_s \geq 0.$$

The term $\overline{c}$ is called the *peak* of the sequence; generalizing this concept, if $\overline{c}$ occurs with multiplicity $\geq k$, we might consider the unimodal sequence with a *k-fold peak*

$$0 \leq a_1 \leq a_2 \leq \ldots \leq a_r \leq \overline{c}\ \overline{c}\ \ldots\ \overline{c} \geq b_1 \geq b_2 \geq \ldots \geq b_s \geq 0,$$

where "$\overline{c}\ \overline{c}\ \ldots\ \overline{c}$" denotes k repetitions of $\overline{c}$. When all the inequalities above are strictly "$<$" or "$>$" the sequence is *strongly unimodal*.

If r is the number of a_i to the left and s is the number of b_j to the right of a unimodal sequence, the difference $s-r$ is called the *rank* of the sequence; and the sum of all the

[3]Indeed, there are many interesting connections between partition theory, q-series and statistical physics; for instance, see Ch. 8 of [And86a], Ch. 22 of [Zwi04], and work of the author and his collaborators through the Emory Working Group in Number Theory and Molecular Simulation [ZPBSea17].

[4]We note the simple pole at $z=1$.

terms including the peak is the *weight* of the sequence. Another series that plays a role here is the *rank generating function* $\widetilde{U}(z,q)$ *for unimodal sequences*, given by

$$\widetilde{U}(z,q) := \sum_{n=0}^{\infty} \frac{q^n}{(zq;q)_n(z^{-1}q;q)_n} = \sum_{n=0}^{\infty}\sum_{m=-\infty}^{\infty} \widetilde{u}(m,n)z^m q^n, \tag{8.3}$$

where $\widetilde{u}(m,n)$ is the number of unimodal sequences of rank m and weight n. Each summand of the first infinite series is the generating function for unimodal sequences with peak term n: the factor $(z^{-1}q;q)_n^{-1}$ generates $a_i \leq n$, $(zq;q)_n^{-1}$ generates $b_j \leq n$ and the q^n factor inserts n as the peak term $\overline{c}$ (following [BOPR12, KL14]). If we replace z with $-z$, the right-most series is actually the very first expression Andrews revealed from Ramanujan's "lost" notebook ([And79], Eq. 1.1) shortly after unearthing the papers at Trinity College [Sch12]. This form, which is related to partial theta functions [KL14], was swimming alongside mock theta functions in the Indian mathematician's imagination during his final year. Finally, following Bloch–Okounkov [BO00] as well as Zagier [Zag16], we define the *q-bracket* $\langle f \rangle_q$ of a function $f\colon \mathcal{P} \to \mathbb{C}$ to be given by

$$\langle f \rangle_q := \frac{\sum_{\lambda\in\mathcal{P}} f(\lambda)q^{|\lambda|}}{\sum_{\lambda\in\mathcal{P}} q^{|\lambda|}} = (q;q)_\infty \sum_{\lambda\in\mathcal{P}} f(\lambda)q^{|\lambda|}, \tag{8.4}$$

where the sums are taken over all partitions. This operator represents the expected value in statistical physics of a measurement over a grand ensemble whose states are indexed by partitions with weights f, for a canonical choice of q; this is the content of the quotient in the middle of (8.4).

However, we proceed formally here using the right-most expression, without drawing too much physical interpretation (while always keeping the mysterious feeling that our formulas resonate in physical reality). Simply multiplying by $(q;q)_\infty$ induces quite interesting q-series phenomena: Bloch–Okounkov [BO00], Zagier [Zag16], and Griffin–Jameson–Trebat-Leder [GJTL16] show that the q-bracket can produce families of modular, quasimodular and p-adic modular forms; and the present author finds the q-bracket to play a natural role in partition theory as well [Sch17, Wak16], modularity aside. (We highly recommend Zagier's paper [Zag16] for more about the q-bracket.)

We will see here that the reciprocal of the Jacobi triple product

$$j(z;q)^{-1} =: \sum_{\lambda \in \mathcal{P}} j_z(\lambda) q^{|\lambda|}$$

has a very rich and interesting interpretation in terms of the q-bracket operator, which (multiplying $j(z;q)^{-1}$ by $(q;q)_\infty$) has the shape

$$\langle j_z \rangle_q = \frac{1}{(z;q)_\infty (z^{-1}q;q)_\infty}.$$

Note that this q-bracket also has a simple pole at $z = 1$. We abuse notations somewhat in writing the coefficients j_z in this way, as if $z \in \mathbb{C}$ were a constant. In fact, j_z is a map from the partitions to $\mathbb{Z}[z]$, which we found in Chapter 3 to be given explicitly by

$$j_z(\lambda) = (1-z)^{-1} \sum_{\delta | \lambda} \sum_{\varepsilon | \delta} z^{\mathrm{crk}(\varepsilon)} \tag{8.5}$$

for $z \neq 1$, and "crk" is the *crank* statistic of Andrews–Garvan [AG88] from Definition 3.6.1.

Remark. The crank generating function (3.6.1) can be written

$$C(z;q) = \frac{(q;q)_\infty}{(zq;q)_\infty (z^{-1}q;q)_\infty} = (1-z)(q;q)_\infty \langle j_z \rangle_q .$$

In Chapter 3 we used the q-bracket operator to find the coefficients of $\langle j_z \rangle_q$ explicitly in terms of sums over subpartitions and the crank statistic, as well. Now we take a different approach, and look at $\langle j_z \rangle_q$ from the point-of-view of q-hypergeometric relations. It turns out the odd-order universal mock theta function g_3 (in an "inverted" form) and the unimodal rank generating function $\widetilde{U}$ naturally arise together as components of $\langle j_z \rangle_q$.

Theorem 8.2.1. *For $0<|q|<1, z\neq 0, z\neq 1$, the following statements are true:*

(i) We have the q-bracket formula

$$\langle j_z \rangle_q = 1 \ + \ \left[z(1-q)+z^{-1}q\right] g_3(z^{-1},q^{-1}) \ + \ \frac{zq^2}{1-z}\widetilde{U}(z,q).$$

(ii) The "inverted" mock theta function component in part (i) converges, and can be written in the form

$$g_3(z^{-1},q^{-1}) = \sum_{n=1}^{\infty} \frac{q^n}{(z;q)_n (z^{-1}q;q)_n}.$$

By considering the factor $z(1-q)+z^{-1}q$ as $|z|\to\infty$ and as $|z|\to 0$ in part (i) of Theorem 8.2.1, we get the following asymptotics.

Corollary 8.2.1. *We have the asymptotic estimates:*

(i) For $0<|q|<1\ll|z|$, we have

$$\langle j_z \rangle_q \sim z(1-q)g_3(z^{-1},q^{-1}) \ \text{as } |z|\to\infty.$$

(ii) For $0<|q|<1,\ 0<|z|\ll 1$, we have

$$\langle j_z \rangle_q \sim z^{-1}q \ g_3(z^{-1},q^{-1}) \ \text{as } |z|\to 0.$$

Thus the inverted mock theta function component dominates the behavior of the q-bracket for z not close to the unit circle (which is "most" of the complex plane).

Remark. So the universal mock theta function is the main influence on these expected values for large and small $|z|$, with appropriate choice of q.

Conversely, if we write

$$\langle j_z \rangle_q =: \sum_{n=0}^{\infty} c_n q^n, \qquad g_3(z^{-1}, q^{-1}) =: \sum_{n=0}^{\infty} \gamma_n q^n,$$

where the coefficients $c_n = c_n(z),\ \gamma_n = \gamma_n(z)$ also depend on z, then we proved an explicit combinatorial formula for the c_n in Chapter 3 using nested sums over subpartitions of n, viz.

$$c_n(z) = (1-z)^{-1} \sum_{\lambda \vdash n} \sum_{\delta | \lambda} \sum_{\epsilon | \delta} \sum_{\varphi | \epsilon} \mu(\lambda/\delta) z^{\mathrm{crk}(\varphi)}. \tag{8.6}$$

With (8.6) in hand, it follows from Corollary 8.2.1 that the coefficients of $g_3(z^{-1}, q^{-1})$ satisfy the asymptotic

$$\gamma_n(z) \sim \begin{cases} z^{-1}(c_1 + c_2 + ... + c_n) \text{ as } |z| \to \infty \\ \\ z c_{n-1} \text{ as } |z| \to 0,\ n \geq 1 \end{cases} \tag{8.7}$$

(which depends entirely on the growth of z, not n), as the coefficients enjoy the recursion

$$\gamma_n - \gamma_{n-1} \sim z^{-1} c_n \text{ for } |z| \gg 1.$$

It is a well-known fact (see, for instance, [Ono04]) that if $\zeta_* \neq 1$ is a root of unity, then

$$(\zeta_* q; q)_\infty (\zeta_*^{-1} q; q)_\infty$$

is, up to multiplication by a rational power of q, a modular function; but this product is the reciprocal of

$$(1 - \zeta_*) \cdot \langle j_z \rangle_q \big|_{z=\zeta_*}.$$

This is another example of the intersection of the q-bracket with modularity phenomena, and at the same time gives a feeling for the obstruction to the inverted mock theta function's sharing in this modularity at $z = \zeta_*$; for $g_3(z^{-1}, q^{-1})$ is not necessarily a dominating aspect of $\langle j_z \rangle_q$ for $z \neq 1$ near the unit circle, whereas the unimodal rank generating aspect $\widetilde{U}(z,q)$ makes a more noticeable contribution, and the two pieces work together to produce modular behavior.

Going a little farther in this direction, there is a close relation between g_3 and the more general class of k-fold unimodal rank generating functions. Let us define the *rank generating function* $\widetilde{U}_k(z,q)$ *for unimodal sequences with a* k*-fold peak* by the series

$$\widetilde{U}_k(z,q) := \sum_{n=0}^{\infty} \frac{q^{kn}}{(zq;q)_n (z^{-1}q;q)_n} = \sum_{n=0}^{\infty} \sum_{m=-\infty}^{\infty} \widetilde{u}_k(m,n) z^m q^n, \tag{8.8}$$

where $\widetilde{u}_k(m,n)$ is the number of k-fold peak unimodal sequences of rank m and weight n. This identity follows directly from the combinatorial definition of $\widetilde{U}_k$, as Lovejoy noted to the author[5]: the $(z^{-1}q;q)_n^{-1}$ and $(zq;q)_n^{-1}$ generate the a_i, b_j just as in (8.3), and q^{kn} inserts k copies of n as the k-fold peak.

Then it is not hard to find (see Theorem 4.1.1) relations like

$$\frac{1}{(zq;q)_\infty (z^{-1}q;q)_\infty} = 2 - z - z^{-1} + (z + z^{-1})\widetilde{U}_1(z,q) - \widetilde{U}_2(z,q), \tag{8.9}$$

which of course is equal to $(1-z)\langle j_z \rangle_q$ and is modular for $z = \zeta_*$, up to multiplication by a power of q. For example, noting that $z + z^{-1} = 0$ when $z = i$, then (8.9) yields

$$2 - \widetilde{U}_2(i,q) = (iq;q)_\infty^{-1} (-iq;q)_\infty^{-1} = (-q^2;q^2)_\infty^{-1}, \tag{8.10}$$

where $(-q^2;q^2)_\infty$ is essentially a modular function.

At this point we can compare (8.9) to Theorem 8.2.1(i) to solve for $g_3(z^{-1}, q^{-1})$ in terms of $\widetilde{U}_1, \widetilde{U}_2$, but it is a little messy. However, it follows from a convenient rewriting

[5] J. Lovejoy, Private communication, August 3, 2016.

of the right-hand side of Theorem 8.2.1(ii) using geometric series

$$\sum_{n=0}^{\infty} \frac{z}{(z;q)_{n+1}(z^{-1}q;q)_n} \left(\frac{z^{-1}q^{n+1}}{1-z^{-1}q^{n+1}} \right) = \frac{z}{1-z} \sum_{n=0}^{\infty} \sum_{k=1}^{\infty} \frac{z^{-k}q^{k(n+1)}}{(zq;q)_n(z^{-1}q;q)_n}$$

which converges absolutely for $|q| < |z|$, and then swapping order of summation, that in fact $g_3(z^{-1}, q^{-1})$ can be written nicely in terms of the $\widetilde{U}_k$.

Corollary 8.2.2. *For $|q| < 1 < |z|$, we have*

$$g_3(z^{-1}, q^{-1}) = \frac{z}{1-z} \sum_{k=1}^{\infty} \widetilde{U}_k(z,q) z^{-k} q^k.$$

Thus the inverted universal mock theta function leads to a type of two-variable generating function for the sequence of rank generating functions for unimodal sequences with k-fold peaks, $k = 1, 2, 3, \ldots$.

Proof of Theorem 8.2.1. We begin by noting for $|q| < 1, z \neq 0$,

$$\langle j_z \rangle_q = (z;q)_\infty^{-1} (z^{-1}q;q)_\infty^{-1} = \prod_{n=0}^{\infty} \left(1 - q^n(z + z^{-1}q - q^{n+1})\right)^{-1},$$

where in the final step we multiplied together the nth terms from each q-Pochhammer symbol. Thus we have

$$\prod_{n=0}^{\infty} \left(1 - q^n(z + z^{-1}q - q^{n+1})\right)^{-1} = 1 + \sum_{n=1}^{\infty} \frac{q^n(z + z^{-1}q - q^{n+1})}{\prod_{j=0}^{n-1} (1 - q^j(z + z^{-1}q - q^{j+1}))}, \tag{8.11}$$

which is easily seen to be absolutely convergent, and can be shown by expanding the product on the left as the telescoping series

$$1 + \sum_{n=1}^{\infty} \left(\frac{1}{\prod_{i=0}^{n} (1 - q^i(z + z^{-1}q - q^i))} - \frac{1}{\prod_{i=0}^{n-1} (1 - q^{i-1}(z + z^{-1}q - q^{i-1}))} \right) \tag{8.12}$$

with a little arithmetic. Now, by the above considerations, (8.11) is equivalent to the following relation.

Lemma 8.2.1. *For* $|q| < 1, z \neq 0$, *we have*

$$\langle j_z \rangle_q = 1 + (z + z^{-1}q) \sum_{n=1}^{\infty} \frac{q^n}{(z;q)_n (z^{-1}q;q)_n} - q \sum_{n=1}^{\infty} \frac{q^{2n}}{(z;q)_n (z^{-1}q;q)_n}.$$

We cannot help but notice how both series on the right-hand side of Lemma 8.2.1 resemble the right-hand summation of identity (8.3) for $\widetilde{U}(z,q)$. This is not a coincidence; it follows right away from the simple observation

$$\widetilde{U}(z,q) = \sum_{n=0}^{\infty} \frac{q^n}{(zq;q)_n (z^{-1}q;q)_n} = q^{-1}(1-z) \sum_{n=0}^{\infty} \frac{q^{n+1}(1 - z^{-1}q^{n+1})}{(z;q)_{n+1} (z^{-1}q;q)_{n+1}},$$

that $\widetilde{U}$ splits off in a very similar fashion to $\langle j_z \rangle_q$ in Lemma 8.2.1, after taking into account $q \neq 0$:

$$\widetilde{U}(z,q) = q^{-1}(1-z) \sum_{n=1}^{\infty} \frac{q^n}{(z;q)_n (z^{-1}q;q)_n} - (zq)^{-1}(1-z) \sum_{n=1}^{\infty} \frac{q^{2n}}{(z;q)_n (z^{-1}q;q)_n}. \tag{8.13}$$

Comparing Lemma 8.2.1 and (8.13), plus a little bit of algebra, then gives

$$\langle j_z \rangle_q = 1 \; + \; \left[z(1-q) + z^{-1}q \right] \sum_{n=1}^{\infty} \frac{q^n}{(z;q)_n (z^{-1}q;q)_n} \; + \; \frac{zq^2}{1-z} \widetilde{U}(z,q). \tag{8.14}$$

Now, to connect the remaining summation in (8.14) to the universal mock theta function g_3, we apply a somewhat clever factorization strategy in the q-Pochhammer symbols to arrive at a useful identity (see [FG00], Appendix 1 (I.3)):

$$\begin{aligned} (z;q)_n (z^{-1}q;q)_n &= \prod_{j=0}^{n-1} \left[(-zq^j)(1 - z^{-1}(q^{-1})^j) \right] \left[(-z^{-1}q^{j+1})(1 - z(q^{-1})^{j+1}) \right] \\ &= q^{n^2} (z^{-1};q^{-1})_n (zq^{-1};q^{-1})_n. \end{aligned} \tag{8.15}$$

Thus

$$\sum_{n=1}^{\infty}\frac{q^n}{(z;q)_n(z^{-1}q;q)_n}=\sum_{n=1}^{\infty}\frac{q^n}{q^{n^2}(z^{-1};q^{-1})_n(zq^{-1};q^{-1})_n} \tag{8.16}$$

$$=\sum_{n=1}^{\infty}\frac{(q^{-1})^{n(n-1)}}{(z^{-1};q^{-1})_n(zq^{-1};q^{-1})_n}. \tag{8.17}$$

The right-hand side of (8.16) is $g_3(z^{-1},q^{-1})$, noting that it converges under the same conditions as the left side (being merely a term-wise rewriting), but with $q=0$ omitted from the domain.

Remark. Equivalently, identities like these result from the observation that

$$(1-zq^i)(1-z^{-1}q^{-i})^{-1}=-zq^i.$$

Taking the product over $0\le i\le n-1$ gives

$$(z;q)_n(z^{-1};q^{-1})_n^{-1}=(-1)^n z^n q^{n(n-1)/2}$$

and, proceeding in this manner, a variety of q-series summand forms can be produced (and inverted as above) by creative manipulation.

□

Remark. We note in passing that, using (7.2) and (8.2) of Fine [Fin88], Ch. 1, together with Theorem 8.2.1(ii), we can also write

$$g_3(z^{-1},q^{-1}) \;=\; (z^{-1}q;q)_\infty^{-1}(-z;q)_\infty^{-1}\sum_{n=0}^{\infty}(-1)^n z^{-2n}q^{\frac{n(n+1)}{2}}-\sum_{n=0}^{\infty}z^{-n+1}(z^{-1};q)_n. \tag{8.18}$$

Recall that many modular forms arise as specializations of $j(z;q)$ (because $j(z;q)$ is essentially a Jacobi form, see [BFOR17]), and that $g_3(z,q)$ is the prototype for the class of mock modular forms that (using Ramanujan's language) "enter into mathematics as beautifully" as the modular cases [Har59]. It is interesting that these important number-theoretic objects which are speculatively associated in the literature to opposite extremes of the universe — subatomic and supermassive — are themselves intertwined via the q-bracket from statistical physics, which applies to phenomena at every scale.

8.3 Approaching roots of unity radially from within (and without)

One point that arises in (8.15) and (8.16) above is that, evidently, one can construct pairs of q-series $\varphi_1(q), \varphi_2(q)$, convergent for $|q| < 1$, with the property

$$\varphi_1(q) = \varphi_2(q^{-1}) \tag{8.19}$$

(thus $\varphi_1(q) + \varphi_2(q)$, $\varphi_1(q)\varphi_2(q)$ are self-reciprocal. This type of phenomenon, relating functions inside and outside the unit disk, is studied in [BFR12, Fol16]. In particular, the universal mock theta function g_3 can be written as a piecewise function

$$g_3(z,q) = \begin{cases} \sum_{n=1}^{\infty} \frac{q^{n(n-1)}}{(z;q)_n (z^{-1}q;q)_n} & \text{if } |q| < 1, \\ \\ \sum_{n=1}^{\infty} \frac{(q^{-1})^n}{(z;q^{-1})_n (z^{-1}q^{-1};q^{-1})_n} & \text{if } |q| > 1, \end{cases} \tag{8.20}$$

for q inside or outside the unit circle, respectively, and $z \neq 0$ or 1.[6] What of $g_3(z,q)$ for q lying *on* the circle? Generically one expects this question to be somewhat dicey.

To be precise in what follows, for ζ on the unit circle we define $g_3(z,\zeta)$ to mean the limit of $g_3(z,q)$ as $q \to \zeta$ radially from within (or without if the context allows), when the limit exists. Recalling the notation $\zeta_m := e^{2\pi i/m}$, it turns out that for $\zeta = \zeta_*$ an appropriate root of unity, $g_3(z,\zeta_*)$ is finite, both in value and length of the sum.

[6] We note this does not yield analytic continuation as the unit circle presents a wall of singularities.

Theorem 8.3.1. *For $q = \zeta_m$ a primitive mth root of unity, $z \neq 0, 1$, or a rational power of ζ_m, and $z^m + z^{-m} \neq 1$, the odd-order universal mock theta function is given by the finite formula*

$$g_3(z, \zeta_m) \;=\; (1 - z^m - z^{-m})^{-1} \sum_{n=0}^{m-1} \zeta_m^n \, (z; \zeta_m)_n (z^{-1}\zeta_m; \zeta_m)_n.$$

Remark. Bringmann–Rolen [BR15] and Jang–Löbrich [JL17] have studied radial limits of universal mock theta functions from other perspectives.

Thus, under the right conditions, (8.20) together with Theorem 8.3.1 suggest $g_3(z, q)$ can, in a certain sense, "pass through" the unit circle at roots of unity (as a function of q following a radial path) into the complex plane beyond, and vice versa, while always remaining finite.

In the theory of quantum modular forms, one encounters functions that exhibit this renormalization behavior (see [BFOR17, RS13]). We see that g_3 exhibits this type of behavior.

Some mock theta functions are closely related to quantum modular forms. As we noted in Chapter 1, Ramanujan's mock theta function $f(q)$ (from (1.4)) is, at even-order roots of unity, essentially a quantum modular form plus a modular form[7], through its relation to another rank generating function, the *rank generating function $U(z, q)$ for strongly unimodal sequences* [BOPR12, FOR13], defined by

$$U(z, q) := \sum_{n=0}^{\infty} q^{n+1}(-zq; q)_n(-z^{-1}q; q)_n = \sum_{n=0}^{\infty} \sum_{m=-\infty}^{\infty} u(m, n) z^m q^n, \tag{8.21}$$

where $u(m, n)$ is the number of strongly unimodal sequences of rank m and weight n. As with $\widetilde{U}, \widetilde{U}_k$ previously, the identity follows directly from the combinatorial definition: here, the $(-z^{-1}q; q)_n^{-1}$ and $(-zq; q)_n^{-1}$ generate *distinct* $a_i \leq n, b_j \leq n$, respectively, and q^{n+1} inserts $n+1$ as the peak term.

[7] We give examples of similar cases in Appendix E.

This $U(z,q)$ is a function that strikes deep: up to multiplication by rational powers of q, $U(i,q)$ is mock modular, $U(1,q)$ is mixed mock modular, and $U(-1,q)$ is a quantum modular form that can be completed to yield a weight $3/2$ non-holomorphic modular form [BFOR17]; in fact, mock and quantum modular properties of $U(z,q)$ are proved in generality for z in an infinite set of roots of unity in [FKVY17].

Of course, U is the $k=1$ case of the *rank generating function* $U_k(z,q)$ *for strongly unimodal sequences with* k*-fold peak*, given by

$$U_k(z,q) := \sum_{n=0}^{\infty} q^{k(n+1)}(-zq;q)_n(-z^{-1}q;q)_n = \sum_{n=0}^{\infty}\sum_{m=-\infty}^{\infty} u_k(m,n)z^m q^n, \tag{8.22}$$

where $u_k(m,n)$ counts k-fold peak strongly unimodal sequences of rank m and weight n, as above. Once again, we note the symmetry $U_k(z^{-1},q) = U_k(z,q)$. As with $\widetilde{U}_k$ in (8.9), we can find (see Theorem 4.2.8) nice relations like

$$(zq;q)_\infty(z^{-1}q;q)_\infty = 1 - (z+z^{-1})U_1(z,q) + U_2(z,q), \tag{8.23}$$

which is modular for $z = \zeta_*$ a root of unity, up to multiplication by a power of q. For instance, at $z=i$, equation (8.23) gives

$$1 + U_2(i,q) = (iq;q)_\infty(-iq;q)_\infty = (-q^2;q^2)_\infty. \tag{8.24}$$

Remark. Multiplying (8.10) and (8.24) leads to a nice pair of identities relating U_2 and $\widetilde{U}_2$:

$$U_2(i,q) = \frac{1-\widetilde{U}_2(i,q)}{\widetilde{U}_2(i,q)-2}, \quad \widetilde{U}_2(i,q) = \frac{1+2U_2(i,q)}{1+U_2(i,q)}. \tag{8.25}$$

Now, taking a similar approach to that in Section 8.2 with regard to $\widetilde{U}_k$, we can find from Theorem 8.3.1, using an evaluation of $U_k(-z,q)$ at $q=\zeta_m$ much like the theorem[8]

$$U_k(-z,\zeta_m) = \frac{-1}{1-z^m-z^{-m}} \sum_{n=0}^{m-1} \zeta_m^{k(n+1)} (z\zeta_m;\zeta_m)_n (z^{-1}\zeta_m;\zeta_m)_n, \tag{8.26}$$

that the universal mock theta function g_3 also connects to these rank generating functions U_k at roots of unity, through a similar relation to Corollary 8.2.2.

Corollary 8.3.1. *For $|z|<1$, we have*

$$g_3(z,\zeta_m) = \frac{z-1}{z} \sum_{k=1}^{\infty} U_k(-z,\zeta_m) z^k \zeta_m^{-k}.$$

How suggestive it is, in light of the relationship between $f(q)$ and $U(-1,q)$ [FOR13], to see specializations of g_3 giving rise to both forms of k-fold unimodal rank generating functions in Corollaries 8.2.2 and 8.3.1.

Proof of Theorem 8.3.1 and Corollary 8.3.1. We start with an elementary observation. For an arbitrary q-series with coefficients d_n, then in the limit as q approaches an mth root of unity ζ_m radially from within the unit circle, we have

$$\lim_{q\to\zeta_m} \sum_{n=1}^{\infty} d_n q^n = \sum_{n=1}^{m} D_n \zeta_m^n \text{ where } D_n := \sum_{j=0}^{\infty} d_{n+mj}, \tag{8.27}$$

so long as $\sum_j d_{n+mj}$ converges. The moral of this example: q-series *want* to be finite at roots of unity.

In a similar direction, Theorem 8.3.1 arises from the following very general lemma, which the author has spoken on at Emory University since 2012 and used for heuristics, but has not published previously. It is really Lemma 8.3.1 below that is the pivotal result of Section 8.3; the applications to $g_3(z,\zeta_m)$ form an interesting exercise.

[8]We note for $k=1, z=1$, m even, that the summation in (8.26) appears in the right-hand side of (1.6).

Lemma 8.3.1. *Suppose $\phi\colon \mathbb{Z}^+ \to \mathbb{C}$ is a periodic function of period $m \in \mathbb{Z}^+$, i.e., $\phi(r + mk) = \phi(r)$ for all $k \in \mathbb{Z}$. Define $f\colon \mathbb{Z}^+ \to \mathbb{C}$ by the product*

$$f(j) := \prod_{i=1}^{j} \phi(i),$$

and its summatory function $F(n)$ by $F(0) := 0$ and, for $n \geq 1$,

$$F(n) := \sum_{j=1}^{n} f(j), \qquad F(\infty) := \lim_{n\to\infty} F(n) \text{ if the limit exists.}$$

Then the following statements are true:

(i) For $0 \leq r < m$ we have

$$F(mk + r) = \frac{1 - f(m)^k}{1 - f(m)} F(m) + f(m)^k F(r).$$

(ii) For $|f(m)| < 1$ we have the finite formula

$$F(\infty) = \frac{F(m)}{1 - f(m)}.$$

Proof of Lemma 8.3.1. First we observe that

$$f(mk) = \prod_{i=1}^{mk} \phi(i) = \left(\prod_{i=1}^{m} \phi(i) \right)^k = f(m)^k, \tag{8.28}$$

by the periodicity of ϕ. Then by the definition of $F(n)$ in Lemma 8.3.1 together with

(8.28) we can rewrite

$$\begin{aligned} &F(mk+r) \\ &= \sum_{j=1}^{m} f(j) + \sum_{j=m+1}^{2m} f(j) + \sum_{j=2m+1}^{3m} f(j) + \ldots + \sum_{j=m(k-1)+1}^{mk} f(j) + \sum_{j=mk+1}^{mk+r} f(j) \\ &= \left(1 + f(m) + f(m)^2 + \ldots + f(m)^{k-1}\right) \sum_{j=1}^{m} f(j) + f(m)^k \sum_{j=1}^{r} f(j). \end{aligned} \tag{8.29}$$

Recognizing the sum $1 + f(m) + f(m)^2 + \ldots$ as a finite geometric series completes the proof of (i). If $|f(m)| < 1$, the infinite case gives (ii). □

Remark. Euler's continued fraction formula [Eul85] allows one to rewrite any hypergeometric sum as a continued fraction, and vice versa. Then we get another finite formula for $F(\infty)$, which holds for any convergent continued fraction of the following shape with periodic coefficients, including q-hypergeometric series when q is replaced by appropriate ζ_m:

$$F(\infty) = \cfrac{\phi(1)}{1 - \cfrac{\phi(2)}{1+\phi(2) - \cfrac{\phi(3)}{1+\phi(3) - \cfrac{\phi(4)}{1+\phi(4)-\ldots}}}} = \frac{1}{1-f(m)} \left(\cfrac{\phi(1)}{1 - \cfrac{\phi(2)}{1+\phi(2) - \cfrac{\phi(3)}{1+\ldots - \cfrac{\phi(m)}{1+\phi(m)}}}} \right). \tag{8.30}$$

Therefore, the finiteness and renormalization considerations in this section also apply to q-continued fractions.

Clearly if we take ϕ to be sine, cosine, etc. in Lemma 8.3.1, we can produce a variety of trigonometric identities. More pertinently, if we replace $\phi(i)$ with $\widetilde{\phi}(t,i) := t\phi(i)$, this $\widetilde{\phi}$ also has period m; then we see $\widetilde{f}(j) := \prod_{i=1}^{j} \widetilde{\phi}(t,i) = t^j f(j)$. Thus the summatory functions $\widetilde{F}(n) = \widetilde{F}(t,n)$ and $\widetilde{F}(\infty) = \widetilde{F}(t,\infty)$ represent a polynomial and a power series in t, respectively — which are, respectively, subject to (i) and (ii) of Lemma 8.3.1. Then for ϕ with period k and the product f as defined above, we get identities like

$$\sum_{n=1}^{\infty} f(n)t^n = \frac{1}{1-f(k)t^k}\sum_{n=1}^{k} f(n)t^n. \tag{8.31}$$

(We could also take $\widetilde{\phi}(t,i)$ equal to $t^i\phi(i)$ or $t^{2i}\phi(i)$ or $t^{2i-1}\phi(i)$, to lead to power series of other familiar shapes; however, such $\widetilde{\phi}$ are not generally periodic.)

Thinking along these lines, if we set

$$\phi(i) = t\frac{(1-a_1q^{i-1})(1-a_2q^{i-1})...(1-a_rq^{i-1})}{(1-b_1q^{i-1})(1-b_2q^{i-1})...(1-b_sq^{i-1})}$$

for $a_*, b_* \in \mathbb{C}$, the product $f(j)$ becomes a quotient of q-Pochhammer symbols, producing the q-hypergeometric series

$$F(t,\infty) = \ _rF_s(a_1,...,a_r;b_1,...,b_s;t:q).$$

If $q \to \zeta_m$ an mth root of unity, then ϕ is also cyclic of period m, and in the radial limit ${}_rF_s(a_1,...,a_r;b_1,...,b_s;t:\zeta_m)$ is subject to Lemma 8.3.1(ii), so long as in the denominator $(1-b_*\zeta_m^i) \neq 0$ for any i.

Remembering the "moral" of equation (8.27), then similar considerations apply to almost all q-series and Eulerian series, for $q=\zeta_m$ a root of unity that does not produce singularities. In particular, so long as the choice of z also does not lead to singularities, it is immediate from Lemma 8.3.1 by the definition (8.2) of g_3 that

$$\begin{aligned} g_3(z,\zeta_m) &= \frac{1}{1-(z;\zeta_m)_m^{-1}(z^{-1}\zeta_m;\zeta_m)_m^{-1}}\sum_{n=1}^{m}\frac{\zeta_m^{n(n-1)}}{(z;\zeta_m)_n(z^{-1}\zeta_m;\zeta_m)_n} \\ &= \frac{2-z^m-z^{-m}}{1-z^m-z^{-m}}\sum_{n=1}^{m}\frac{\zeta_m^{n(n-1)}}{(z;\zeta_m)_n(z^{-1}\zeta_m;\zeta_m)_n}, \end{aligned} \tag{8.32}$$

where for the final equation we used the elementary fact that

$$(X;\zeta_m)_m = 1 - X^m$$

in the leading factor. For a slightly simpler formula, we apply Lemma 8.3.1 to the identity for $g_3(z^{-1};\zeta_m^{-1})$ in Theorem 8.2.1 instead, then take $z \mapsto z^{-1}$ and $\zeta_m \mapsto \zeta_m^{-1}$, to see

$$g_3(z,\zeta_m) = \frac{2 - z^m - z^{-m}}{1 - z^m - z^{-m}} \sum_{n=1}^{m} \frac{\zeta_m^{-n}}{(z^{-1};\zeta_m^{-1})_n (z\zeta_m^{-1};\zeta_m^{-1})_n}. \tag{8.33}$$

We note that the leading factor is symmetric under inversion of z.

Remark. Jang–Löbrich prove finite formulas similar to (8.32) and (8.33) for $g_3(z,\zeta_m)$ [JL17], by different methods.

A particularly lovely aspect of q-series such as these is that they transform into an infinite menagerie of shapes, limited only by the curiosity of the analyst. (For instance, see Fine [Fin88] for a stunning exploration of q-hypergeometric series.[9]) Then a form like g_3 might have a number of different finite formulas.

To derive Theorem 8.3.1, which is simpler than the preceding expressions for g_3, we use another factorization strategy in the q-Pochhammer symbols. Again we exploit that

$$(X;\zeta_m)_m = 1 - X^m = (X;\zeta_m^{-1})_m;$$

thus for $0 \leq n \leq m$, since $\zeta_m^{-j} = \zeta_m^{m-j}$ we have

$$\begin{aligned}(X;\zeta_m^{-1})_n &= (1-X)(1-X\zeta_m^{m-1})(1-X\zeta_m^{m-2})...(1-X\zeta_m^{m-(n-1)}) \\ &= \frac{(1-X)(X;\zeta_m)_m}{(X;\zeta_m)_{m-n+1}} = \frac{(1-X)(1-X^m)}{(X;\zeta_m)_{m-n+1}}.\end{aligned} \tag{8.34}$$

[9]Fine writes: "The beauty and surprising nearness to the surface of some of the results could easily lead one to embark on an almost uncharted investigation of [one's] own." ([Fin88], p. xi)

Making the change of indices $n \mapsto m-n+1$ in the summation in (8.33) then yields

$$\sum_{n=1}^{m} \frac{\zeta_m^{-(m-n+1)}}{(z^{-1};\zeta_m^{-1})_{m-n+1}(z\zeta_m^{-1};\zeta_m^{-1})_{m-n+1}} = \sum_{n=1}^{m} \frac{\zeta_m^{n-1}(z^{-1};\zeta_m)_n(z\zeta_m^{-1};\zeta_m)_n}{(1-z^{-1})(1-z\zeta_m^{-1})(2-z^m-z^{-m})}.$$

Substituting this final expression back into (8.33), with a little algebra and adjusting of indices, gives Theorem 8.3.1.

To prove Corollary 8.3.1, we use geometric series, along with an order-of-summation swap and index change, to rewrite Theorem 8.3.1 in the form

$$\begin{aligned} g_3(z,\zeta_m) &= \frac{1-z}{(1-z^m-z^{-m})} \sum_{n=0}^{m-1} \zeta_m^n \frac{(z\zeta_m;\zeta_m)_n(z^{-1}\zeta_m;\zeta_m)_n}{1-z\zeta_m^n} \\ &= \frac{z^{-1}(1-z)}{z(1-z^m-z^{-m})} \sum_{k=1}^{\infty} z^k \zeta_m^{-k} \sum_{n=0}^{m-1} \zeta_m^{k(n+1)}(\zeta_m;\zeta_m)_n(z^{-1}\zeta_m;\zeta_m)_n. \end{aligned} \tag{8.35}$$

Comparing this with the formula (8.26) for $U(-z,\zeta_m)$, which follows easily from Lemma 8.3.1, gives the corollary. (The sum on the right might be simplified further using (8.27).)

Remark. Convergence in these formulas depends on one's choice of substitutions; for a particular choice, careful analysis may be required to show boundedness as q approaches the natural boundary of a q-series (see Watson [Wat37] for examples).

□

We note that a slight variation on the proof above leads to finite formulas at applicable roots of unity for the even-order universal mock theta function $g_2(z,q)$ of Gordon–McIntosh [GM12] as well, by an alternative approach to that of Bringmann–Rolen [BR15]. Using transformations from Andrews [And98], Fine [Fin88], and other authors, still simpler formulas might be found for particular specializations of g_3 at roots of unity. We demonstrate this point below.

Example 8.3.2. *The limit of the mock theta function $f(q)$ at ζ_m an odd-order root of* *unity is given by*

$$f(\zeta_m) = 1 - \frac{2}{3}\sum_{n=1}^{m}(-1)^n \zeta_m^{-(n+1)}(-\zeta_m^{-1};\zeta_m^{-1})_n.$$

Proof of Example 8.3.2. The function $f(q)$ is convergent at odd roots of unity; however, for the reader's convenience, we will sketch a proof of convergence to the given value for just the case $q \to \zeta_m$ along a radial path. By (26.22) in [Fin88], Ch. 3, Ramanujan's mock theta function $f(q)$ defined in (1.4) can be rewritten

$$f(q) = 1 - \sum_{n=1}^{\infty} \frac{(-1)^n q^n}{(-q;q)_n}. \tag{8.36}$$

To show the summation on the right is bounded as q approaches an odd-order root of unity radially, we exactly follow the steps of Watson's analysis of the mock theta function $f_0(q)$ in [Wat37], Sec. 6. In Watson's nomenclature, take $M = 2$, N odd, to write $q = e^{2\pi i/N} = \zeta_N$. Then by replacing $q^{(nN+m)^2}$ with $(-1)^{nN+m}q^{nN+m}$ in the numerators of the $n \geq 1$ terms of the series $f_0(q)$ (we note that Watson's m is not the same as the subscript of ζ_m we use throughout this paper, which corresponds to N in this proof), one sees

$$\left|1 - \sum_{n=1}^{\infty} \frac{(-1)^n q^n}{(-q;q)_n}\right| \leq 2\sum_{n=0}^{N-1}\left|\frac{q^n}{(-q;q)_n}\right| < \infty$$

when $q = \rho\zeta_N$ with $0 \leq \rho \leq 1$. To see the value the series converges to, consider the $(Nk + r)$th partial sum, with $r < N$, of the right-hand side of (8.36) as $\rho \to 1^-$, in light of Lemma 8.3.1 (i). In fact, as $|(-1)^N\zeta_N^N/(-\zeta_N,\zeta_N)_N| = 1/2 < 1$, then part (ii) of Lemma 8.3.1 applies as $Nk + r \to \infty$ and (also taking into account that $f(\rho\zeta_N)$ converges uniformly for $\rho < 1$) we may write

$$\lim_{\rho\to 1^-}\left(1 - \sum_{n=1}^{\infty} \frac{(-1)^n(\rho\zeta_N)^n}{(-\rho\zeta_N;\rho\zeta_N)_n}\right) = 1 - \frac{2}{3}\sum_{n=1}^{N}\frac{(-1)^n\zeta_N^n}{(-\zeta_N;\zeta_N)_n}. \tag{8.37}$$

Now, observe that (8.34) gives

$$(-\zeta_N;\zeta_N)_n^{-1} = (-\zeta_N^{-1};\zeta_N^{-1})_{N-n-1}. \tag{8.38}$$

Applying (8.38) to the right-hand side of (8.37), then making the change $n \mapsto N-n-1$ in the indices, we arrive at the desired result. □

Example 8.3.3. *For ζ_m an odd-order root of unity we have*

$$f(\zeta_m) = \frac{4}{3}\sum_{n=1}^{m}(-1)^n(-\zeta_m^{-1};\zeta_m^{-1})_n.$$

Proof of Example 8.3.3. Here we use only finite sums, so we do not need to justify convergence. Let us define an auxiliary series

$$h(\zeta_m) = \frac{2}{3}\sum_{n=1}^{m}(-1)^n(-\zeta_m^{-1};\zeta_m^{-1})_n.$$

Then using Example 8.3.2, with a little arithmetic and adjusting of indices, gives

$$f(\zeta_m) - h(\zeta_m) \ = \ 1 - \frac{2}{3}\sum_{n=1}^{m}(-1)^n(-\zeta_m^{-1};\zeta_m^{-1})_{n+1} \ = \ h(\zeta_m).$$

Comparing the left- and right-hand sides above implies our claim. □

8.4 The "feel" of quantum theory

By the considerations here we can find both finite formulas at roots of unity, and inverted versions using factorizations such as in (8.15) leading to forms such as (8.19) and (8.20), for a great many q-hypergeometric series. Whether or not they enjoy modularity properties, these can display very interesting behaviors, emerging outside the unit circle radially from an entirely different point ζ_m^{-1} than the point on the circle ζ_m approached from within, and likewise when entering the circle radially at roots of unity from without. Moreover, the map inside the unit circle in the variable q looks like an "upside-down hyperbolic mirror-image" of the function's behavior on the outside. (Taking $q \mapsto \overline{q}$ in either the $|q| < 1$ or $|q| > 1$ piece of (8.20) turns the map "right-side up", but at the expense of holomorphicity[10].)

This imagery reminds the author of depictions of white holes and wormholes in science fiction[11]. Do there exist "points-of-exit" (and entry) analogous in some way to roots of unity, at the event horizon of a black hole? Is there a mirror-image universe contained within? We won't take these fantastical analogies too seriously, yet one is led to wonder: how deep is the connection of q-series and partition theory, to phenomena at nature's fringe?

Remark. See Appendix G for further notes on Chapter 8.

[10]As Tyler Smith, Emory University Department of Physics, noted to the author.
[11]Not to mention phenomena like quantum tunneling and wall crossing in physics

Appendix A

Notes on Chapter 1: Counting partitions

A.1 Elementary considerations

Here we point out some simple but useful relations. We adopt the conventions $p(0) := 1$, and $p(n) := 0$ for $n < 0$. Then we have the following elementary fact[1].

Proposition A.1.1. *The number of partitions of n with k appearing as a part at least once, is equal to $p(n-k)$.*

Proof. There is a bijective correspondence between the set of partitions of n having k as a part, and the partitions of $n-k$. Take any partition of n with k appearing at least once, and delete one part of size k to produce a unique partition of $n-k$. Conversely, take any partition of $n-k$ and adjoin a part k to produce a unique partition of n. □

For example, consider the seven partitions of 5:

$$(5), (4,1), (3,2), (3,1,1), (2,2,1), (2,1,1,1), (1,1,1,1,1).$$

[1] Wakhare exploits similar ideas in [Wak16].

The number $k = 2$ shows up in three partitions of 5, so $p(3) = p(5-2)$ must be equal to three, which is of course correct.

Proceeding further in this direction, a natural generalization of Proposition A.1.1 is the following statement.

Proposition A.1.2. *For any $\delta \in \mathcal{P}$ with $|\delta| < |\lambda|$, the number of partitions $\lambda \vdash n$ such that $\delta|\lambda$ is equal to $p(n - |\delta|)$.*

So for any $m < n$, we can recover $p(m)$ by a quick inspection of the partitions of n. For example, consider again the partitions of 5 listed above. The subpartition $(2, 1)$ shows up in two partitions of 5, giving the correct value of two for $p(2) = p(5 - |(2, 1)|)$.

Proof. To prove Proposition A.1.2 we show basically the same bijection as above. Take any partition $\lambda \vdash n$ such that $\delta|\lambda$, and delete the parts of δ to arrive at a partition of $n - |\delta|$. Conversely, setting $k = |\delta|$, then adjoin the parts of δ to any partition of $n-k$ to arrive at a partition of n. □

One immediate corollary of the propositions above is the following.

Proposition A.1.3. *For any $a, b, c \geq 1$, we have that $p(a)$ is equal to the number of partitions of $a + bc$ in which b occurs as a part with multiplicity $\geq c$.*

We point out the above statement is symmetric in b, c.

Remark. By the same token, the total number $M_k(n) := \sum_{\lambda \vdash n} m_k(\lambda)$ of k's appearing as parts over all partitions of n is given by $M_k(n-k) + p(n-k)$: adjoin k to partitions of $n-k$, including partitions already containing k, to yield partitions of n containing k.[2] Then by recursion,

$$M_k(n) = p(n-k) + p(n-2k) + p(n-3k) + ... + p\left(n - \left\lfloor \frac{n}{k} \right\rfloor k\right). \tag{A.1}$$

[2]One can prove the well-known generating function formula $\sum_{n=1}^{\infty} M_k(n)q^n = \frac{q^k}{(q;q)_\infty(1-q^k)}$ from ideas in Chapter 3. Write the right-hand side of the claimed identity as $(q;q)_\infty^{-1}\sum_{n=1}^{\infty} q^{nk} = (q;q)_\infty^{-1}\sum_{\lambda\in\mathcal{P}} \chi_k(\lambda)q^{|\lambda|} = \sum_{\lambda\in\mathcal{P}} q^{|\lambda|}\sum_{\delta|\lambda}\chi_k(\delta)$, where we set $\chi_k(\lambda) = 1$ if λ is a partition into all k's and $= 0$ otherwise. Then observe the number of subpartitions of λ into all k's is exactly $\sum_{\delta|\lambda}\chi_k(\delta) = m_k(\lambda)$.

We note that Ramanujan-like congruences yield congruences for M_k, too. For instance,

$$M_5(5n+4) \equiv 0 \pmod{5} \tag{A.2}$$

follows from $p(5n+4-5j) \equiv 0 \pmod{5}$ for $1 \le j \le n$. By the same argument,

$$M_7(7n+5) \equiv 0 \pmod{7}, \qquad M_{11}(11n+6) \equiv 0 \pmod{11}. \tag{A.3}$$

A.2 Easy formula for $p(n)$

Here we count partitions of n via a natural subclass of partitions we will refer to as *nuclear partitions*, which are partitions having no part equal to one. In Chapter 4 we call this set $\mathcal{P}_{\geq 2}$; here we will denote the "nuclear" partitions by $\mathcal{N} \subset \mathcal{P}$ and let $\mathcal{N}_n$ denote nuclear partitions of $n \geq 0$. Set $\nu(0) := 1$ and for $n \geq 1$, let $\nu(n)$ count the number of nuclear partitions of n (noting $\nu(1) = 0$). Clearly we have the recursive relation $p(n) = \nu(n) + p(n-1)$; thus $\nu(n)$ has the generating function $(q^2;q)_\infty^{-1}$. By recursion,

$$p(n) = \nu(0) + \nu(1) + \nu(2) + \nu(3) + ... + \nu(n). \tag{A.4}$$

So to count partitions of n, we need only keep track of nuclear partitions, a much sparser set. For instance, here is an algorithm to compute $p(n)$ from the nuclear partitions of n aside from the partition (n) itself, i.e., the set $\mathcal{N}_n \backslash (n)$, which is a considerably smaller set than $\mathcal{P}_n$. We let $\mu = (\mu_1, \mu_2, ..., \mu_r)$, $\mu_1 \geq \mu_2 \geq ... \geq \mu_r \geq 2$, denote a nuclear partition.

Theorem A.2.1. *We have that*

$$p(n) = n + \nu(n) - 1 + \sum_{\mu \in \mathcal{N}_n \backslash (n)} (\mu_1 - \mu_2),$$

with the right-hand sum taken over nuclear partitions of n excluding the partition (n).

Then to compute $p(n)$ one can follow these steps:

1. Write down the partitions of n containing no 1's aside from (n) itself, that is, the subset $\mathcal{N}_n \backslash (n)$. For example, to find $p(6)$ we use $\mathcal{N}_6 \backslash (6) = \{(4,2),(3,3),(2,2,2)\}$.

2. Write down the difference $\mu_1 - \mu_2 \geq 0$ between the first part and the second part of each partition from the preceding step. In the present example, we write down

$$4-2=2, \quad 3-3=0, \quad 2-2=0.$$

3. Add together the differences obtained in the previous step, then add the result to $n-1+\nu(n)$ to arrive at $p(n)$. In this example, we add $2+0+0=2$ from the previous step to $6+\nu(6)-1=6+4-1$, arriving at $p(6)=6+4-1+2=11$, which of course is correct.

Proof. Observe that every nuclear partition of n can be formed by adding $m_1(\lambda)$ to the largest part λ_1 of a "non-nuclear" partition $\lambda \vdash n$, and deleting all the 1's from λ, e.g., $(3,2,1,1) \to (5,2)$. Conversely, every nuclear partition $\mu \vdash n$ can be turned into a non-nuclear partition of n by decreasing the largest part μ_1 by some positive integer $j \leq \mu_1 - \mu_2$, and adjoining j 1's to form the non-nuclear partition. So the nuclear partitions of n "decay" (by giving up 1's from the largest part) into non-nuclear partitions of n, e.g., $(5,2) \to (4,2,1) \to (3,2,1,1) \to (2,2,1,1,1)$, of which the total number is $p(n)-\nu(n)$. Each nuclear partition μ decays into $\mu_1 - \mu_2$ different non-nuclear partitions except the partition (n), which decays into $n-1$ non-nuclear partitions, viz. $(n) \to (n-1,1) \to (n-2,1,1) \to ... \to (1,1,...,1)$, so the number of non-nuclear partitions of n is $p(n)-\nu(n) = (n-1) + \sum_{\mu \in \mathcal{N}_n \backslash (n)} (\mu_1 - \mu_2)$. □

It is interesting to see how the subset $\mathcal{N} \subset \mathcal{P}$ produces the entire set $\mathcal{P}$ by this simple "decay" process[3]. Now, let $\gamma(n)$ denote the number of nuclear partitions μ of n such that

[3] To prove this assume otherwise, that for some $n \geq 0$ there is a non-nuclear partition ϕ of n *not*

$\mu_1 = \mu_2$ (the first two parts are equal), setting $\gamma(0) := 0$ and noting $\gamma(1) = \gamma(2) = \gamma(3) = 0$, as well. Then for $n \geq 3$ the recursion $\nu(n) = \gamma(n) + \nu(n-1)$ holds (adding 1 to the largest part of every nuclear partition of $n-1$ gives the nuclear partitions μ of n with $\mu_1 > \mu_2$)[4], thus the generating function for $\gamma(n)$ is $\frac{1}{(1+q)(q^3;q)_\infty} - 1 + q - q^2$. Moreover, noting $\nu(2) = 1$, we have for $n \geq 3$ that

$$\nu(n) = 1 + \gamma(3) + \gamma(4) + ... + \gamma(n). \tag{A.5}$$

For $m \geq 1$, let $\nu(n, m)$ denote the number of nuclear partitions of n whose parts are all $\leq m$. Then it is easily verified that we can also compute $\nu(n)$ as follows[5].

Theorem A.2.2. *We have $n \geq 4$ that*

$$\nu(n) = \sum_{k=2}^{n-2} \nu(k, n-k).$$

Combining this identity with Theorem A.2.1 and (A.5) above, and making further simplifications, the task of computing $p(n)$ can be reduced to counting much smaller subsets of partitions of integers $\leq n-2$. These small subsets of partitions of integers up to $n-2$ completely encode the value of $p(n)$.

More generally, we might let $\nu_k(n)$ denote the number of partitions of n having no part equal to k — let us refer to these as "k-nuclear" partitions — setting $\nu_k(0) := 1$ for all $k \geq 1$; thus $\nu(n) = \nu_1(n)$. Let $\mathcal{N}^k$ denote the set of all k-nuclear partitions, and let $\mathcal{N}_n^k$ be k-nuclear partitions of n; thus $\mathcal{N} = \mathcal{N}^1$, $\mathcal{N}_n = \mathcal{N}_n^1$. Clearly we have $p(n) = \nu_k(n) + p(n-k)$, so $\nu_k(n)$ has the generating function $\frac{1-q^k}{(q;q)_\infty}$ and is subject to

produced by decay of some partition in $\mathcal{N}_n$. Then deleting all the 1's from ϕ and adding them to the largest part ϕ_1 produces a nuclear partition of n that decays into ϕ, a contradiction.

[4]In fact, much as nuclear partitions "control" the growth of $p(n)$, these nuclear partitions with first two parts equal — which the author thinks of as being in their "ground state" — control the growth of $\nu(n)$, thus appearing to fundamentally control $p(n)$.

[5]See Cor. 1.5 of [MS18] for a formula for $\nu(n)$ (there written as a backward difference $\nabla[p](n)$) involving the classical Möbius function.

essentially the same treatment as $\nu(n)$ above. Then by recursion, as previously,

$$p(n) = p\left(n - \left\lfloor \frac{n}{k} \right\rfloor k\right) + \sum_{j=1}^{\lfloor n/k \rfloor} \nu_k(n - jk), \tag{A.6}$$

where $\lfloor x \rfloor$ is the floor function, and by a similar proof (decay into k's instead of 1's) we generalize Theorem A.2.1, which represents the $k = 1$ case of the following identity.

Theorem A.2.3. *We have that*

$$p(n) = \left\lfloor \frac{n}{k} \right\rfloor^* + \nu_k(n) + \sum_{\mu \in \mathcal{N}_n^k \setminus (n)} \left\lfloor \frac{\mu_1 - \mu_2}{k} \right\rfloor,$$

where we set $\left\lfloor \frac{a}{b} \right\rfloor^* := \left\lfloor \frac{a}{b} \right\rfloor - 1$ *if* $b|a$ *and* $:= \left\lfloor \frac{a}{b} \right\rfloor$ *otherwise, and the right-hand sum is taken over* k*-nuclear partitions of* n *excluding the partition* (n)*.*

As in the remark at the end of the previous section, the Ramanujan congruences imply, for instance, that since $p(5n + 4) - p\,(5(n - 1) + 4) \equiv 0 \pmod 5$, then

$$\nu_5(5n + 4) \equiv 0 \pmod 5. \tag{A.7}$$

Similarly, we also have

$$\nu_7(7n + 5) \equiv 0 \pmod 7, \qquad \nu_{11}(11n + 6) \equiv 0 \pmod{11}. \tag{A.8}$$

If these congruences could be proved directly, it seems likely one could run this kind of argument (perhaps using Proposition A.1.3, as well) in reverse to prove Ramanujan-like congruences by induction.

Appendix B

Notes on Chapter 3: Applications and algebraic considerations

B.1 Ramanujan's tau function and k-color partitions

Here we give two immediate applications in number theory of the principle at the heart of Proposition 3.3.7 (the partition Cauchy product formula), extended to products of more than two series. The first example gives a formula for Ramanujan's tau function, an arithmetic function which appears as the coefficients of a weight-12, level 1 cusp form (see [Ono04]). As previously, let $m_i = m_i(\lambda)$ denote the multiplicity of i as a part of λ.

Example B.1.1. *Ramanujan's tau function $\tau(n)$, defined as the nth coefficient of $q(q;q)_\infty^{24}$, can be written*

$$\tau(n) = \sum_{\lambda \vdash (n-1)} (-1)^{\ell(\lambda)} \binom{24}{m_1(\lambda)} \binom{24}{m_2(\lambda)} \binom{24}{m_3(\lambda)} \cdots .$$

Proof. This follows by applying the binomial theorem to each factor $(1-q^i)^{24}$ of the

q-Pochhammer symbol to give

$$q(q;q)_\infty^{24} = q \prod_{n=0}^{\infty} \sum_{k=0}^{24} (-1)^k q^k \binom{24}{k},$$

then expanding the product and collecting coefficients of q^n as sums of the shape $\sum_{\lambda \vdash n}$. The extra factor of q produces the shift $\sum_{\lambda \vdash n} \mapsto \sum_{\lambda \vdash n-1}$ in the coefficients. □

Next we find a formula for the number $P_k(n)$ of k-color partitions of n, as studied by Agarwal and Andrews [AA87, Aga88] and other authors.

Example B.1.2. *The number $P_k(n)$ of k-color partitions of n, which is equal to the nth coefficient of $(q;q)_\infty^{-k}$ for $k \geq 1$, can be written*

$$P_k(n) = \sum_{\lambda \vdash n} \binom{k+m_1(\lambda)-1}{m_1(\lambda)} \binom{k+m_2(\lambda)-1}{m_2(\lambda)} \binom{k+m_3(\lambda)-1}{m_3(\lambda)} \cdots .$$

Proof. Just like the previous example, this follows by writing

$$(q;q)_\infty^{-k} = \prod_{n=0}^{\infty} \sum_{k=0}^{24} q^k \binom{n+k-1}{k},$$

expanding the product, and collecting coefficients. □

More generally, the same proofs extend to absolutely convergent products of the form $\prod_{n=1}^{\infty} (1-\phi(n)q^n)^k$ for any $k \in \mathbb{Z}$.

Theorem B.1.3. *For $\phi \colon \mathbb{N} \to \mathbb{C}$ and $q \in \mathbb{C}$ such that both sides converge, $k \geq 0$, we have the identities*

$$\prod_{n=1}^{\infty} (1-\phi(n)q^n)^k = \sum_{\lambda \in \mathcal{P}} q^{|\lambda|} (-1)^{\ell(\lambda)} \prod_{i=1}^{\infty} \phi(i)^{m_i(\lambda)} \binom{k}{m_i(\lambda)},$$

$$\prod_{n=1}^{\infty} (1-\phi(n)q^n)^{-k} = \sum_{\lambda \in \mathcal{P}} q^{|\lambda|} \prod_{i=1}^{\infty} \phi(i)^{m_i(\lambda)} \binom{k+m_i(\lambda)-1}{m_i(\lambda)}.$$

Combined with Theorems 4.1.1 and 4.2.8 in Chapter 4, and also with Faà di Bruno's formula as in Chapter 5 and Appendix D below, arbitrarily complicated products and quotients of q-Pochhammer symbols (and other product forms) can be evaluated similarly. Additional formulas for $\tau(n), P_k(n)$ are given in Appendix D.

B.2 q-bracket arithmetic

The q-bracket operator is reasonably well-behaved as an algebraic object; here we give a few formulas that may be useful for computation. From Definition 1.2.6 we have q-bracket addition

$$\langle f\rangle_q + \langle g\rangle_q = \langle f+g\rangle_q\,,$$

which is commutative, of course, and also associative:

$$\langle f+g\rangle_q + \langle h\rangle_q = \langle f\rangle_q + \langle g+h\rangle_q\,.$$

We have for a constant $c \in \mathbb{C}$ that $c\,\langle f\rangle_q = \langle cf\rangle_q$; other basic arithmetic relations such as $\langle 0\rangle_q = 0$ and $\langle f\rangle_q + \langle 0\rangle_q = \langle f\rangle_q$ follow easily as well.

Now, let

$$\widetilde{f}(n) := \sum_{\lambda\vdash n} f(\lambda).$$

We will define a convolution "$*$" of two such functions $\widetilde{f}, \widetilde{g}$ by

$$(\widetilde{f} * \widetilde{g})(\lambda) := \frac{1}{p(|\lambda|)} \sum_{k=0}^{|\lambda|} \widetilde{f}(k)\widetilde{g}(|\lambda|-k). \tag{B.1}$$

Note that, by symmetry, $\widetilde{f} * \widetilde{g} = \widetilde{g} * \widetilde{f}$.[1]

[1] The author is grateful to Alex Rice for a discussion about convolution that informed this section.

Let us also define a multiplication "$\star$" between q-brackets by

$$\langle f \rangle_q \star \langle g \rangle_q := \frac{\langle f \rangle_q \langle g \rangle_q}{(q;q)_\infty}, \tag{B.2}$$

where the product and quotient on the right are taken in $\mathbb{C}[[q]]$. It follows from B.1 and B.2 above that

$$\langle f \rangle_q \star \langle g \rangle_q = \left\langle \widetilde{f} * \widetilde{g} \right\rangle_q .$$

From here it is easy to establish a q-bracket arithmetic yielding a commutative ring structure, with familiar-looking relations such as

$$\langle f \rangle_q \star \left\langle \widetilde{g} * \widetilde{h} \right\rangle_q = \left\langle \widetilde{f} * \widetilde{g} \right\rangle_q \star \langle h \rangle_q ,$$

$$\langle f \rangle_q \star \langle g + h \rangle_q = \left\langle \widetilde{f} * \widetilde{g} \right\rangle + \left\langle \widetilde{f} * \widetilde{h} \right\rangle_q ,$$

and so on.

It is trivial to see that $\langle 1 \rangle_q = 1$; however, $\langle 1 \rangle_q \star \langle f \rangle_q = \frac{\langle f \rangle_q}{(q;q)_\infty} \neq \langle f \rangle_q$, so $\langle 1 \rangle_q$ is not the multiplicative identity in this arithmetic. In fact, as we note in Chapter 3, Section 1, the q-bracket of Dyson's rank function is

$$\langle \mathrm{rk} \rangle_q = (q;q)_\infty.$$

Then $\langle \mathrm{rk} \rangle_q$ may serve as the multiplicative identity in the q-bracket arithmetic above, by Equation (B.2).

B.3 Group theory and ring theory in $\mathcal{P}$

Based on joint work with Ian Wagner

B.3.1 Antipartitions and group theory

Here we will define the set $\mathcal{P}^-$ of *antipartitions*, in analogy with antiparticles in physics: partitions and antipartitions annihilate one another. Then we show that the set $\mathcal{P} \cup \mathcal{P}^-$ naturally forms a group structure.

Definition B.3.1. *For $\lambda = (\lambda_1, \lambda_2, ..., \lambda_r) \in \mathcal{P}$, we define an antipartition $\lambda^- = (\lambda_1^-, \lambda_2^-, ...,$ $\lambda_r^-) \in \mathcal{P}^-$ such that*

$$\lambda\lambda^- = \emptyset.$$

We refer to the $\lambda_i^- \in \lambda^-$ as "antiparts".

Let us adopt the convention $\lambda^{-a} := \lambda^-\lambda^-\lambda^- \cdots \lambda^-$ (a repetitions). Clearly we have that $(\lambda^-)^- = \lambda$; every partition is the antipartition of its own antipartition. We also have right away that $\emptyset^- = \emptyset$. For $a \in \mathbb{Z}^+$, corresponding parts and antiparts annihilate each other pair-wise in partitions (we adopt the convention of separating parts and antiparts with a semicolon, and putting the antiparts to the right in a partition):

$$(a; a^-) = (a)(a)^- = \emptyset.$$

But it is not necessary that partitions and antipartitions should cancel; in fact, we might have "mixed" partitions containing both parts and antiparts. We can compute, for example, that

$$(5, 4, 3, 3)(4, 3, 1, 1)^- = (5, 4, 3, 3; 4^-, 3^-, 1^-, 1^-) = (5, 3; 1^-, 1^-).$$

Note that parts and antiparts indexed by the same integer cancel. Mixed partitions may also be written in "rational" form, e.g.,

$$(5,3;1^-,1^-) = (5,3)(1,1)^- = (5,3)/(1,1).$$

Then we might refer to the set

$$\mathcal{Q} := \mathcal{P} \cup \mathcal{P}^- \tag{B.3}$$

as *rational partitions*. (In usage, however, we still refer to elements of $\mathcal{Q}$ as "partitions".)

A few other relations are immediate.

Proposition B.3.1. *We have the following identities:*

$$\ell(\lambda^-) = -\ell(\lambda), \qquad |\lambda^-| = -|\lambda|, \qquad n_{\lambda^-} = \frac{1}{n_\lambda}, \qquad m_k(\lambda^-) = -m_k(\lambda).$$

Proof. The first identity follows from

$$\ell(\lambda) + \ell(\lambda^-) = \ell(\lambda\lambda^-) = \ell(\emptyset) = 0.$$

The second identity follows from

$$|\lambda| + |\lambda^-| = |\lambda\lambda^-| = |\emptyset| = 0.$$

The third identity follows from

$$n_\lambda n_{\lambda^-} = n_{\lambda\lambda^-} = n_\emptyset = 1.$$

The fourth identity is formally necessary if we want

$$\ell(\lambda^-) = -\ell(\lambda) = -(m_1(\lambda) + m_2(\lambda) + m_3(\lambda) + ...).$$

□

For the time being we can take negative lengths and multiplicities, as well as fractional norms, as just formal artifacts; but the second equation in Proposition B.3.1 admits the following interpretation: the antipartitions $\mathcal{P}^-$ are partitions of negative integers.

At this point the climax of the section will not be too surprising to the reader.

Theorem B.3.2. *Rational partitions $\mathcal{Q}$ form an abelian group under partition multiplication.*

Proof. Clearly under our multiplication operation " $\cdot$ " on $\mathcal{Q}$ we have the identity element $\emptyset$, the elements λ, λ^- are multiplicative inverses, associativity and commutativity are automatic from set-theoretic considerations, and $\mathcal{Q}$ is closed under multiplication, verifying $(\mathcal{Q}, \cdot)$ has the claimed group structure. □

Then $(\mathcal{Q}, \cdot)$ looks a lot like $\mathbb{Q}\backslash\{0\}$ as a multiplicative group. We hope that classical techniques of group theory may lead to new identities, congruences and bijections in the theory of partitions.

B.3.3 Partitions and diagonal matrices

For the sake of pointing toward future work in the algebraic vein, we also note a few connections to matrix algebra, a gold-mine of structural archetypes, although our study in this direction is incomplete. There is an obvious way to associate nonempty partitions to diagonal matrices, which are well known to enjoy beautiful algebraic properties.

Definition B.3.2. *For a nonempty partition $\lambda = (\lambda_1, \lambda_2, ..., \lambda_r) \in \mathcal{P}, \lambda_1 \geq \lambda_2 \geq ... \geq$*

$\lambda_r \geq 1$, we define the diagonal matrix

$$M_\lambda := \begin{pmatrix} \lambda_1 & 0 & 0 & \dots & 0 \\ 0 & \lambda_2 & 0 & \dots & 0 \\ 0 & 0 & \lambda_3 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 & \lambda_r \end{pmatrix},$$

which we might refer to as the "matrix of λ".

We have immediately an interpretation of dimension "dim", determinant "det", and trace "tr" in terms of partition-theoretic statistics:

$$\dim(M_\lambda) = \ell(\lambda), \quad \mathrm{tr}(M_\lambda) = |\lambda|, \quad \det(M_\lambda) = n_\lambda. \tag{B.4}$$

Then there are a natural addition and multiplication we might define on partitions of a fixed length r, as an extension of matrix operations. For $\lambda, \lambda' \in \mathcal{P}$ with $\ell(\lambda) = \ell(\lambda') = r$, we define $\lambda + \lambda',\ \lambda \times \lambda'$ to be the partitions whose parts are the diagonal entries of the matrices $M_\lambda + M_{\lambda'}, M_\lambda M_{\lambda'}$, respectively:

$$M_\lambda + M_{\lambda'} = M_{\lambda+\lambda'}, \quad M_\lambda M_{\lambda'} = M_{\lambda\times\lambda'}.$$

Our operations are given explicitly by

$$\lambda + \lambda' := (\lambda_1 + \lambda_1', \lambda_2 + \lambda_2', ..., \lambda_r + \lambda_r'), \quad \lambda \times \lambda' := (\lambda_1\lambda_1', \lambda_2\lambda_2', ..., \lambda_r\lambda_r').$$

Qualitatively, these operations are quite different from the partition multiplication introduced in Chapter 3, which is purely a set-theoretic operation and does not depend on any arithmetic taking place between the parts themselves, aside from putting them in weakly decreasing order. Now we see the parts adding and multiplying, to produce the parts of new partitions. (We discuss matrices involving antipartitions, as well, below.)

In any event, the "$\geq$" ordering on the entries on the diagonal ensure that the entries of $M_\lambda + M_{\lambda'}$, $M_\lambda M_{\lambda'}$ also obey the same ordering, so these entries do indeed comprise partitions of length r. Clearly the $r \times r$ zero matrix, which we identify with the empty partition, $\emptyset$, and $r \times r$ identity matrix, which we identify with the length-r partition into all 1's, viz.

$$I_0 := \emptyset, \quad I_r := (1, 1, ..., 1) \ \ (r \text{ repetitions}),$$

are, respectively, the additive and multiplicative identities:

$$\lambda + \emptyset = \lambda, \quad \lambda \times I_r = \lambda, \quad \lambda \times \emptyset = \emptyset. \tag{B.5}$$

Then all the machinery of linear algebra of diagonal matrices can be extended to partitions of length r (for any fixed r) under these operations.

We may also include partitions in $\mathcal{Q}$ if we define an arithmetic relating parts and antiparts. However, the arithmetic between parts and antiparts turns out to be a nontrivial question. Of course, antiparts are just positive integers decorated with minus signs, so we expect something like the usual arithmetic in $\mathbb{Z}$; for instance, if $a, b \in \mathbb{Z}^+$ we expect

$$ab^- = b^- + b^- + ... + b^- \ (a \text{ repetitions}) = (ab)^-,$$

because the antiparts should add together. On the other hand, considering the relations in Proposition B.3.1, we see that the antiparts sometimes act like negative numbers and sometimes act like fractions. These antiparts are, in fact, formal entities that arise naturally from partition-theoretic (as opposed to matrix-theoretic) considerations, and their arithmetic properties may well depend on context — indeed, this is what the author assumes to be the case.

In the case of the matrix-based operations above, a workable rule of thumb for the arithmetic between a part and an antipart is: *Antiparts act like negative integers under addition, and reciprocals under multiplication.*

In symbols, for $a, b \in \mathbb{Z}^+$ we might set

$$a \cdot b^- = ab^{-1} \in \mathbb{Q}, \quad a + b^- = a - b \in \mathbb{Z}. \tag{B.6}$$

Note that, when writing the partitions resulting from these operations, we will follow the convention of converting reciprocals and negative numbers back into the "minus" notation of antipartitions. (For this sketch of matrix-based ideas, we assume that in fact $b|a$ so the resulting part ab^- is still an integer; the question of partition-like objects whose parts come from other sets such as $\mathbb{Q}$, $\mathbb{F}_p$, ring ideals, etc., is beyond the scope of this thesis.)

The relations (B.6) fit intuitively with (B.4) above, and give natural-looking identities like

$$\lambda + \lambda^- = \emptyset, \quad \lambda \times \lambda^- = I_r, \tag{B.7}$$

where $r = \ell(\lambda)$. Encouraging as this matrix-analog structure is, there are points we have not followed through; we have not even proved the demands in (B.6) to be consistent.

B.3.4 Partition tensor product and ring theory

In this section we introduce a direct sum $\oplus$ and tensor product $\otimes$ between partitions, and prove that $(\mathcal{Q}, \oplus, \otimes)$ forms a commutative ring with identity.

Definition B.3.3. *For $\lambda, \lambda' \in \mathcal{P}$ we define the direct sum $\lambda \oplus \lambda' \in \mathcal{P}$ to be a rewriting of multiplication from Chapter 3:*

$$\lambda \oplus \lambda' := \lambda\lambda'.$$

Then we will also write $\lambda \oplus \lambda \oplus ... \oplus \lambda$ (n repetitions) $=: n\lambda$.

In fact, in [And98], Andrews uses the symbol $\oplus$ to define this exact operation, although he expresses the direct sum in terms of a sum of the multiplicities m_k (or "frequencies" f_k

in his terminology).

Remark. So, for example, we write (in a few alternative ways):

$$(3,1,1) = (3) \oplus (1,1) = (3,1) \oplus (1) = (3) \oplus (1) \oplus (1) = (3) \oplus 2(1).$$

In exploring operations between partitions, the author's collaborator Ian Wagner discovered — along the lines of the matrix analogy in the previous section — that the tensor product of two partition matrices suggests a very well-behaved "times" operation between partitions.

Definition B.3.4. *For $\lambda, \lambda' \in \mathcal{P}$ with $\ell(\lambda) = r, \ell(\lambda') = s$, we define the tensor product $\lambda \otimes \lambda' \in \mathcal{P}$ to be the partition whose parts are exactly the set*

$$\{\lambda_1\lambda'_1, \lambda_1\lambda'_2, ..., \lambda_1\lambda'_s, \lambda_2\lambda'_1, \lambda_2\lambda'_2, ..., \lambda_2\lambda'_s, ..., \lambda_r\lambda'_1, \lambda_r\lambda'_2, ..., \lambda_r\lambda'_s\} \subset \mathbb{Z}^+,$$

reorganized to be in canonical weakly decreasing order. Then we will also write $\lambda \otimes \lambda \otimes ... \otimes \lambda$ (n repetitions) $=: \lambda^{\otimes n}$.

Of course, the empty partition $\emptyset$ acts as the identity under $\oplus$, and in this setting, the length one partition $I_1 = (1)$ is the multiplicative identity. Thus the direct sum and tensor product of partitions lead to elementary identities like those in (B.5):

$$\lambda \oplus \emptyset = \lambda, \quad \lambda \otimes (1) = \lambda, \quad \lambda \otimes \emptyset = \emptyset. \tag{B.8}$$

Remark. Generalizing the middle equation in (B.8), we actually have that

$$\lambda \otimes I_n = \lambda \oplus \lambda \oplus \ ... \oplus \lambda \ (n \text{ repetitions}) = n\lambda.$$

Thus we might write any partition λ in a "split" form, i.e., some reordering of

$$\left[(1) \otimes I_{m_1(\lambda)}\right] \oplus \left[(2) \otimes I_{m_2(\lambda)}\right] \oplus \left[(3) \otimes I_{m_3(\lambda)}\right] \oplus ...$$

like

$$(4,4,3,2,2,2) = [(4) \otimes (1,1)] \oplus (3) \oplus [(2) \otimes (1,1,1)] .$$

This product $\otimes$ is in fact very similar to the Kronecker product in matrix algebra, a well-known case of the tensor product; and the trace of a Kronecker product is multiplicative. Analogous considerations give a very natural-looking pair of relations.

Proposition B.3.2. *For $\lambda, \lambda' \in \mathcal{P}$ we have that*

$$|\lambda \oplus \lambda'| = |\lambda| + |\lambda'|, \quad |\lambda \otimes \lambda'| = |\lambda||\lambda'|.$$

Proof. The sum identity is immediate from partition multiplication and the definition of $\oplus$. For the product identity in Proposition B.3.2, we simply rewrite the right-hand side as

$$|\lambda||\lambda'| = (\lambda_1 + \lambda_2 + ... + \lambda_r)(\lambda'_1 + \lambda'_2 + ... + \lambda'_s).$$

Directly expanding the product on the right and inspecting the resulting summands, we see term-by-term that they are the parts of $\lambda \otimes \lambda'$. $\square$

As in Chapter 3, we define $\mathrm{lg}(\lambda)$ and $\mathrm{sm}(\lambda)$ to denote the largest part and the smallest part of λ, respectively. Then these complementary identities follow from the definitions of $\oplus, \otimes$.

Proposition B.3.3. *For $\lambda, \lambda' \in \mathcal{P}$ we have the relations:*

$$\ell(\lambda \oplus \lambda') = \ell(\lambda) + \ell(\lambda'), \qquad \ell(\lambda \otimes \lambda') = \ell(\lambda)\ell(\lambda'),$$
$$\lg(\lambda \otimes \lambda') = \lg(\lambda)\lg(\lambda'), \qquad \mathrm{sm}(\lambda \otimes \lambda') = \mathrm{sm}(\lambda)\,\mathrm{sm}(\lambda'),$$
$$n_{\lambda\oplus\lambda'} = n_\lambda n_{\lambda'}, \qquad n_{\lambda\otimes\lambda'} = n_\lambda^{\ell(\lambda')} n_{\lambda'}^{\ell(\lambda)}.$$

We also have that

$$m_k(\lambda \oplus \lambda') = m_k(\lambda) + m_k(\lambda'), \qquad m_k(\lambda \otimes \lambda') = \sum_{d|k} m_d(\lambda)m_{k/d}(\lambda'),$$

where the final summation is taken over the divisors of k.

Remark. The next-to-last identity in Proposition B.3.3, giving $m_k(\lambda \oplus \lambda')$, is equivalent to the definition of the operation $\oplus$ given in Andrews [And98].

Proof. All of these identities but the last one are immediate. The final summation is clear if we write Definition B.3.4 in the alternative notation

$$\begin{aligned}\lambda \otimes \lambda' &= (1^{m_1(\lambda)}2^{m_2(\lambda)}3^{m_3(\lambda)}4^{m_4(\lambda)}...) \otimes (1^{m_1(\lambda')}2^{m_2(\lambda')}3^{m_3(\lambda')}4^{m_4(\lambda')}...)\\ &= (1^{m_1(\lambda\otimes\lambda')}2^{m_2(\lambda\otimes\lambda')}3^{m_3(\lambda\otimes\lambda')}4^{m_4(\lambda\otimes\lambda')}...).\end{aligned}$$

For every pair of divisors d, d' of a given part $k \in \lambda \otimes \lambda'$, the number of repetitions of k in $\lambda \otimes \lambda'$ produced by the pairing $d \in \lambda, d' \in \lambda'$ is $m_d(\lambda)m_{d'}(\lambda')$. Noting in the definition of $\otimes$ that for each k we sum over all pairings d, d' with $dd' = k$, finishes the proof. $\square$

We note that the final identity in Proposition B.3.3 gives the tensor product of the m_k's essentially as Dirichlet convolution (see [HW79]); to some extent, the arithmetic of partition multiplicities inherits the convenient algebra of convolutions. This observation, together with standard facts about convolutions, connects the tensor product to the algebra of classical Dirichlet series as well.

Corollary B.3.1. *For $\lambda, \lambda' \in \mathcal{P},\ s \in \mathbb{C}$, we have the multiplication identity*

$$\left(\sum_{k=1}^{\infty} \frac{m_k(\lambda)}{k^s}\right)\left(\sum_{k=1}^{\infty} \frac{m_k(\lambda')}{k^s}\right) = \sum_{k=1}^{\infty} \frac{m_k(\lambda \otimes \lambda')}{k^s}.$$

Remark. As these series have only finitely many nonzero m_k, they are well-defined for any $s \in \mathbb{C}$. For $s = 0$, equation (B.3.1) reduces to $\ell(\lambda)\ell(\lambda') = \ell(\lambda \otimes \lambda')$.

We would like to extend the preceding definitions and relations involving the operations $\oplus, \otimes$ to the larger class $\mathcal{Q} \supset \mathcal{P}$ of rational partitions. Of course, we can rewrite Definition B.3.1 in the form

$$\lambda \oplus \lambda^{-} = \emptyset. \tag{B.9}$$

But to extend all of the preceding relations to include antipartitions and partitions in $\mathcal{Q}$, we need to decide on an arithmetic for interactions between parts and antiparts. The bad news is, the part-antipart arithmetic that worked so well for our matrix analogy in the previous section is incompatible with Proposition B.3.2. Happily, in the present setting, we can in fact impose an even simpler rule than the relations in (B.6): *Antiparts act like negative integers in both addition and multiplication.*

Definition B.3.5. *For $a, b \in \mathbb{Z}^+$, in the context of the operations $\oplus, \otimes$ defined above, we set*

$$ab^{-} := -ab \in \mathbb{Z}, \qquad a + b^{-} := a - b \in \mathbb{Z}.$$

Remark. We note that these imply $a^{-}b = (ab)^{-} = -ab$ as well, and also $a^{-}b^{-} = ab$.

As in the previous section, we will translate negative numbers back to antiparts with the minus signs in the upper indices, when we write them inside partitions. Using Definition B.3.5 to give meaning to the products of parts and antiparts, we can immediately generalize the structure we have built in this section to rational partitions $\mathcal{Q}$, just by inserting antipartitions and antiparts appropriately.

Proposition B.3.4. *All of the definitions and relations given above in this section may* *be extended to hold for $\lambda, \lambda' \in \mathcal{Q}$.*

Now the real goal of this section is an easy deduction.

Theorem B.3.5. *The set $\mathcal{Q}$ of rational partitions is a commutative ring under the sum $\oplus$ and product $\otimes$.*

Proof. Theorem B.3.2 plus Definition B.3.3 already give that $(\mathcal{Q}, \oplus)$ is an abelian group with identity element $\emptyset$. That $\otimes$ is associative and commutative are automatic from its set-theoretic definition, and we noted above that $I_1 = (1) \in \mathcal{Q}$ is the tensor product identity. To see that associativity holds, we observe that for $\lambda, \alpha, \beta \in Q$ with $\ell(\lambda) = r, \ell(\alpha) = s, \ell(\beta) = t$, by the definitions of $\oplus$ and $\otimes$,

$$\lambda \otimes (\alpha \oplus \beta) = (\lambda_1, \lambda_2, ..., \lambda_r) \otimes (\alpha_1, \alpha_2, ..., \alpha_s, \beta_1, \beta_2, ..., \beta_t)$$

produces the following set of parts (which we then reorder to look like a partition):

$$\{\lambda_1\alpha_1, \lambda_2\alpha_2, ..., \lambda_1\alpha_s, \lambda_2\alpha_1, \lambda_2\alpha_2..., \lambda_2\alpha_s, ..., \lambda_r\alpha_1, \lambda_r\alpha_2..., \lambda_r\alpha_s,$$

$$\lambda_1\beta_1, \lambda_1\beta_2, ..., \lambda_1\beta_t, \lambda_2\beta_1, \lambda_2\beta_2, ..., \lambda_2\beta_t, ..., \lambda_r\beta_1, \lambda_r\beta_2..., \lambda_r\beta_t\}.$$

By noting that this set of parts (reordered) is also identically equal to

$$(\lambda \otimes \alpha) \oplus (\lambda \otimes \beta),$$

we prove the distributive property, verifying $(\mathcal{Q}, \oplus, \otimes)$ has the claimed ring structure. $\square$

Thus both the size $|\cdot|$ and length $\ell(\cdot)$ represent ring homomorphisms in $(\mathcal{Q}, \oplus, \otimes)$.

We note that if we restrict our attention to partitions of the shapes

$$I_n = (1, 1, ..., 1) \ \text{ and } \ I_n^- = (1^-, 1^-, ..., 1^-) \ \ (n \text{ repetitions in both cases}),$$

the set of all such partitions is isomorphic to the integers. Size and length are equal in these cases, and uniquely associate each partition I_n (resp. I_n^-) to the integer n (resp. $-n$). Moreover, from the $\oplus, \otimes$ operations we recover addition and multiplication on the integers in two different ways via the size map "$|\cdot|$". If we restrict the operations to partitions into all 1's and anti-1's as above, then we can write

$$a + b = |I_a \oplus I_b|, \qquad ab = |I_a \otimes I_b|, \tag{B.10}$$

and so on, with negative numbers coming into play when we involve antipartitions. (Note that we also have $a+b = \ell(I_a \oplus I_b),\ ab = \ell(I_a \otimes I_b)$.) Evidently, the partitions into all 1's and anti-1's is a subring of $\mathcal{Q}$.

In the next section we will explore further ring-theoretic aspects of these ideas.

B.3.6 Ring theory in $\mathcal{Q}$

Certainly many classical techniques from ring theory can be brought to bear on algebraic questions in $(\mathcal{Q}, \oplus, \otimes)$. In this section, we consider factorization and the nature of ideals in the ring of partitions.

First we address the question of which partitions in $\mathcal{Q}$ can in fact be written as a tensor product.

Definition B.3.6. *For $\delta, \lambda \in \mathcal{P}$, let us write $\delta \| \lambda$ to mean we can write λ as a tensor product of the form*

$$\lambda = \delta \otimes \delta'$$

for some partition δ'. If λ has a nontrivial factorization (i.e., both $\delta, \delta' \neq (1)$), we say it is reducible; otherwise we call it irreducible.

Note that $(1)\|\lambda$ and $\lambda\|\lambda$ for all partitions λ. Then if $\delta\|\lambda$, it must obey all of the divisibility relations

$$|\delta| \mid |\lambda|, \quad n_\delta | n_\lambda, \quad \ell(\delta)|\ell(\lambda), \quad \lg(\delta)|\lg(\lambda), \quad \mathrm{sm}(\delta)|\,\mathrm{sm}(\lambda). \tag{B.11}$$

That is a lot of information about what a "tensor divisor" of λ must look like. (These are just easy-to-check consequences of Proposition B.3.3: necessary, but not sufficient.)

Of course, there is the trivial factorization for every partition:

$$\lambda = \lambda \otimes (1).$$

It is also clear that if ab is composite $(a, b \neq 1)$, there is the factorization

$$(ab) = (a) \otimes (b).$$

Continuing in this direction, from the relation $|\lambda \otimes \lambda'| = |\lambda||\lambda'|$ in Proposition B.3.2 we can fully characterize the reducible partitions of any integer n, as well as their tensor divisors.

Let $\mathcal{R}_n \subset \mathcal{P}$ denote the set of reducible partitions of n. It turns out we can construct the set $\mathcal{R}_n$ by applying the tensor product to the partitions of pairs d, d' of nontrivial divisors of n.

Theorem B.3.7. *The reducible partitions of $n > 1$ are given by*

$$\mathcal{R}_n = \{\lambda \otimes \lambda' : \lambda \vdash d, \lambda' \vdash d' \text{ for all } dd' = n,\ d, d' \neq 1\}.$$

As usual, let $p(n)$ denote the classical partition function, i.e., the number of partitions of $n \geq 1$, and set $r(n)$ equal to the number of reducible partitions of n; then Theorem B.3.7 gives a natural upper bound on $r(n)$.

Corollary B.3.2. *The number of reducible partitions of size n obeys the inequality*

$$r(n) \leq \sum_{\substack{dd'=n \\ d,d'\neq 1}} p(d)p(d').$$

In lieu of using Theorem B.3.7 to generate a list of reducible partitions of n and then checking all the entries, in order to determine the reducibility or irreducibility of a given partition of size n, here are a few easy criteria for irreducibility that follow from Proposition B.3.3.

Proposition B.3.5. *We have the following rules-of-thumb for irreducibility:*

1. *If $|\lambda|$ is prime then λ is irreducible.*
2. *If all its parts are mutually coprime, then λ is irreducible.*
3. *If $\mathrm{lg}(\lambda)$ is prime with multiplicity 1, then λ is irreducible.*
4. *If $\ell(\lambda)$ is prime and there is no integer $d > 1$ dividing all the parts of λ, then λ is irreducible.*

Proof. All of these items are obvious from Proposition B.3.3 together with the definition of the tensor product $\otimes$. $\square$

There are also two easy rules in the affirmative direction, which the reader can easily check.

Proposition B.3.6. *We have the following rules-of-thumb for reducibility:*

1. *If all the parts of $\lambda \in \mathcal{P}$ are divisible by some $k > 1$, then λ is reducible and $(k)\|\lambda$.*
2. *If $k > 1$ divides the multiplicity of every part of λ, then λ is reducible and $I_k\|\lambda$.*

Of course, the subsets of partitions focused on in Proposition B.3.6, the set $\mathcal{P}_{k\mathbb{Z}}$ of partitions into parts divisible by $k > 1$, and the set $\mathcal{P}_{k|m_*}$ of partitions with multiplicities all divisible by k, are familiar to students of partition theory. Interestingly, both subsets are closed under the operations $\oplus$ and $\otimes$. In fact, $\mathcal{P}_{k\mathbb{Z}}$ and $\mathcal{P}_{k|m_*}$ are subgroups of $(\mathcal{Q}, \oplus)$ and subrings of $(\mathcal{Q}, \oplus, \otimes)$, and are also two-sided ideals in $(\mathcal{Q}, \oplus, \otimes)$ according to the standard usage in ring theory.

The author and his collaborator would like to see if standard theorems from classical ring theory extend to this partition-theoretic scheme. Moreover, it is our goal to use this ring structure to seek alternative proofs of partition bijections, Ramanujan-like congruences and other classical partition theorems, as well as to seek applications in Andrews's theory of partition ideals (see [And98]).

Appendix C

Notes on Chapter 4: Further observations

C.1 Sequentially congruent partitions

We consider a somewhat exotic subset $\mathcal{S} \subset \mathcal{P}$ suggested by the indices of Corollary 4.2.9, which we refer to as "sequentially congruent partitions", the parts of which obey abnormally strict congruence conditions. We find sequentially congruent partitions are in bijective correspondence with the set of all partitions, and yield explicit expressions for the coefficients of the expansions of a broad class of infinite products. Somehow these complicated-looking objects are embedded in a natural way in partition theory.

Definition C.1.1. *We define a partition λ to be sequentially congruent if the following congruences between the parts are all satisfied:*

$$\lambda_1 \equiv \lambda_2 \pmod{1},\ \lambda_2 \equiv \lambda_3 \pmod{2},\ \lambda_3 \equiv \lambda_4 \pmod{3},\ \ldots\ ,\ \lambda_{r-1} \equiv \lambda_r \pmod{r-1},$$

and for the smallest part, $\lambda_r \equiv 0 \pmod{r}$.

For example, the partition $(20, 17, 15, 9, 5)$ is sequentially congruent, because $20 \equiv$

$17 \pmod 1$ trivially, $17 \equiv 15 \pmod 2$, $15 \equiv 9 \pmod 3$, $9 \equiv 5 \pmod 4$, and $5 \equiv 0 \pmod 5$. On the other hand, $(21, 18, 16, 10, 6)$ is *not* sequentially congruent, for while the first four congruences still hold, clearly $6 \not\equiv 0 \pmod 5$. Note that increasing the largest part λ_1 of any $\lambda \in \mathcal{S}$ yields another partition in $\mathcal{S}$, as does adding or subtracting a fixed integer multiple of the length r to all its parts, so long as the resulting parts are still positive.

No doubt, this congruence restriction on the parts hardly appears natural. However, it turns out sequentially congruent partitions are in one-to-one correspondence with the entire set of partitions.

Let $\mathcal{P}_n$ denote the set of partitions of n, let $\mathcal{S}_{\mathrm{lg}=n}$ denote sequential partitions λ whose largest part λ_1 is equal to n, and let $\#Q$ be the cardinality of a set Q as usual.

Theorem C.1.1. *There is an explicit bijection π between the set $\mathcal{S}$ and the set $\mathcal{P}$. Furthermore, it is the case that*

$$\pi(\mathcal{S}_{\mathrm{lg}=n}) = \mathcal{P}_n, \qquad \pi^{-1}(\mathcal{P}_n) = \mathcal{S}_{\mathrm{lg}=n}.$$

We prove these bijections directly, by construction.

Proof. For any partition $\lambda = (\lambda_1, \lambda_2, ..., \lambda_i, ..., \lambda_r)$, one can construct its sequentially congruent dual

$$\lambda' = (\lambda'_1, \lambda'_2, ..., \lambda'_i, ..., \lambda'_r)$$

by taking the parts equal to

$$\lambda'_i = i\lambda_i + \sum_{j=i+1}^{r} \lambda_j. \tag{C.1}$$

Note that $\lambda'_r \equiv 0 \pmod r$ as $\sum_{j=r+1}^{r}$ is empty; the other congruences between successive parts of λ' are also immediate from equation (C.1).

Let us call this map π:

$$\pi\colon \mathcal{P} \to \mathcal{S}$$
$$\lambda \mapsto \lambda'$$

The above argument establishes, in fact, that we have more strongly $\pi\colon \mathcal{P}_n \to \mathcal{S}_{\mathrm{lg}=n}$.

Conversely, given a sequentially congruent partition λ', one can recover the dual partition λ by working from right-to-left. Begin by computing the smallest part

$$\lambda_r = \frac{\lambda'_r}{r}, \tag{C.2}$$

then compute $\lambda_{r-1}, \lambda_{r-2}, ..., \lambda_1$ in this order by taking

$$\lambda_i = \frac{1}{r}\left(\lambda'_i - \sum_{j=i+1}^{r} \lambda_j\right). \tag{C.3}$$

We let this construction define the inverse map $\pi^{-1}\colon \mathcal{S} \to \mathcal{P}$. Noting that the uniqueness of λ implies the uniqueness of λ', and vice versa, the bijection between $\mathcal{S}$ and $\mathcal{P}$ follows from this two-way construction. Furthermore, observe that $\lambda'_1 = |\lambda|$, thus every partition λ of n corresponds to a sequentially congruent partition λ' with largest part n, and vice versa. $\square$

We see by construction that $\pi(\lambda) = \lambda'$ "looks similar" to λ in terms of length and distribution of the parts. For example, taking $\lambda = (2)$, $(3,1,1)$, $(2,2,2,2)$, respectively, and writing $\pi(\lambda) = \pi(\lambda_1, \lambda_2, ..., \lambda_r)$ instead of $\pi\left((\lambda_1, \lambda_2, ..., \lambda_r)\right)$ for notational ease:

$$\pi\,(2) = (2), \quad \pi\,(3,1,1) = (5,3,3), \quad \pi\,(2,2,2,2) = (8,8,8,8).$$

Thus the set $\pi(\mathcal{P}_n) = \mathcal{S}_{\mathrm{lg}=n}$ "looks like" the set $\mathcal{P}_n$, even up to "similar-looking" partitions being in the same positions if we consider their ordering within each set (the map π does not permute the sequentially congruent images within the set). Of course, the same is true for $\pi^{-1}(\mathcal{S}_{\mathrm{lg}=n}) = \mathcal{P}_n$ "looking like" $S_{\mathrm{lg}=n}$.

The sets $\mathcal{P}$ and $\mathcal{S}$ enjoy another interrelation that can be used to compute the coefficients of infinite products. Now, it is the first equality of Theorem 4.1.1 in Chapter 4 (and equivalent to Equation 22.16 in Fine [Fin88]) that for a function $f\colon \mathbb{N} \to \mathbb{C}$ and $q \in \mathbb{C}$ with f, q chosen such that the product converges absolutely, we can write

$$\prod_{n=1}^{\infty}(1 - f(n)q^n)^{-1} = \sum_{\lambda\in\mathcal{P}} q^{|\lambda|} \prod_{i\geq 1} f(i)^{m_i}, \tag{C.4}$$

where $m_i = m_i(\lambda)$ is the multiplicity of i as a part of partition λ. In fact, it follows from another formula in Chapter 4 that the product on the left-hand side of (C.4) can also be written as a sum over sequentially congruent partitions.

Let $\mathrm{lg}(\lambda) = \lambda_1$ denote the *largest part* of partition λ, and set $\lambda_k = 0$ if $k > \ell(\lambda)$.

Theorem C.1.2. *For $f\colon \mathbb{N} \to \mathbb{C}, q \in \mathbb{C}$ such that the product converges absolutely, we have*

$$\prod_{n=1}^{\infty}(1 - f(n)q^n)^{-1} = \sum_{\lambda\in\mathcal{S}} q^{\mathrm{lg}(\lambda)} \prod_{i\geq 1} f(i)^{(\lambda_i-\lambda_{i+1})/i}.$$

Proof. Theorem C.1.2 results from Corollary 4.2.9 in Chapter 4. For every $j \in \mathbb{N}$ take $\mathbb{X}_j = \{j\}$, fix $f_j = f$, and set $\pm$ equal to a minus sign,. In this case, $\lambda \in \mathcal{P}_{\mathbb{X}_j}$ means if $\lambda \neq \emptyset$ that $\lambda = (j, j, ..., j)$, so we must have $j|(k_j - k_{j+1})$ in any nonempty partition sum on the right side above. Then every summand comprising c_k vanishes unless all the $k_i \leq k$ are parts of a sequentially congruent partition having length $\leq n$: each sum over partitions is empty (i.e., equal to zero) if j does not divide $k_j - k_{j+1}$; is equal to 1 if $k_j - k_{j+1} = 0$ as then $\lambda = \emptyset$ and $\prod_{\lambda_i\in\emptyset}$ is an empty product; or else has one term $f(j)^{m_j} = f(j)^{(k_j-k_{j+1})/j}$ as there is exactly one $\lambda = (j, j, ..., j)$ with $|\lambda| = m_j j = k_j - k_{j+1} > 0$. Finally, let $n \to \infty$ so this argument encompasses partitions in $\mathcal{S}$ of unrestricted length. $\square$

Remark. We note that setting $f = 1$, then comparing equation (C.4) to Theorem C.1.2, gives another proof of Theorem C.1.1: the sets $\mathcal{S}_{\mathrm{lg}=n}$ and $\mathcal{P}_n$ (and thus, the sets $\mathcal{S}$ and $\mathcal{P}$) have the same product generating function.

Remark. If we instead take every $\pm$ equal to plus in Corollary 4.2.9, we see there is also a bijection between partitions into *distinct* parts and a subset of $\mathcal{S}$, viz. partitions into parts with differences $\lambda_i - \lambda_{i+1} = i$ exactly.

Comparing Theorem C.1.2 with (C.4) above, we have two quite different-looking decompositions of the coefficients of $\prod_{n\geq 1}(1 - f(n)q^n)^{-1}$ as sums over partitions. One observes that these decompositions of $\sum_{\lambda\in\mathcal{P}_n}$ and $\sum_{\lambda\in\mathcal{S}_{\mathrm{lg}=n}}$ have identical summands, that is, the sums do not just involve different numbers that add up to the same coefficient of q^n, but rather involve the same set of terms in seemingly a different order. Then one wonders: for given $\phi \in \mathcal{S}_{\mathrm{lg}=n}$, precisely *which* partition $\gamma \in \mathcal{P}_n$ is such that

$$\prod_{i\geq 1} f(i)^{(\phi_i - \phi_{i+1})/i} = \prod_{j\geq 1} f(j)^{m_j(\gamma)}?$$

This γ is generally not the same partition $\lambda = \pi^{-1}(\phi)$ as above. Then the set $\mathcal{S}$ evidently enjoys a second map (beside π^{-1}) to $\mathcal{P}$, which we will call σ:

$$\sigma\colon \mathcal{S} \to \mathcal{P}$$
$$\phi \mapsto \gamma$$

We can easily write down this map by comparing the forms of the products above, using parts-multiplicity notation:

$$\sigma(\phi) := (1^{\phi_1-\phi_2}\ 2^{(\phi_2-\phi_3)/2}\ 3^{(\phi_3-\phi_4)/3}...) = \gamma.$$

For example, $\sigma(5,3,3) = (1^{5-3}\ 2^{(3-3)/2}\ 3^{(3-0)/3}) = (3,1,1)$, which in this case turns out to be the pre-image of $(5,3,3)$ over π (but this is not generally the case, as we will see).

In fact, under this map we also have $\sigma(\mathcal{S}_{\mathrm{lg}=n}) = \mathcal{P}_n$, but a fact hidden by the preceding example and differing from the map π is that σ *does* permute the images in $\mathcal{P}_n$, so $\sigma(\mathcal{S}_{\mathrm{lg}=n})$ and $\mathcal{P}_n$ do not "look similar". Then we also have that the composite map

$$\sigma\pi\colon \mathcal{P}_n \to \mathcal{P}_n$$

permutes the set of partitions of n. Similarly, the map $\pi\sigma\colon \mathcal{S}_{\mathrm{lg}=n} \to \mathcal{S}_{\mathrm{lg}=n}$ permutes the elements of $\mathcal{S}_{\mathrm{lg}=n}$. A natural question one might ask is: what if we apply $\sigma\pi\sigma\pi...\sigma\pi$ to a partition of n? Let's see some examples for $n = 1, 2, 3, 4$. For $n = 1$,

$$(1) \xmapsto{\pi} (1) \xmapsto{\sigma} (1)$$

stabilizes right away as there is only one such partition. For $n = 2$:

$$(2) \xmapsto{\pi} (2) \xmapsto{\sigma} (1,1) \xmapsto{\pi} (2,2) \xmapsto{\sigma} (2),$$

$$(1,1) \xmapsto{\pi} (2,2) \xmapsto{\sigma} (2) \xmapsto{\pi} (2) \xmapsto{\sigma} (1,1).$$

For $n = 3$:

$$(3) \xmapsto{\pi} (3) \xmapsto{\sigma} (1,1,1) \xmapsto{\pi} (3,3,3) \xmapsto{\sigma} (3),$$

$$(2,1) \xmapsto{\pi} (3,2) \xmapsto{\sigma} (2,1),$$

$$(1,1,1) \xmapsto{\pi} (3,3,3) \xmapsto{\sigma} (3) \xmapsto{\pi} (3) \xmapsto{\sigma} (1,1,1).$$

Finally, for $n = 4$:

$$(4) \xmapsto{\pi} (4) \xmapsto{\sigma} (1,1,1,1) \xmapsto{\pi} (4,4,4,4) \xmapsto{\sigma} (4),$$

$$(3,1) \xmapsto{\pi} (4,2) \xmapsto{\sigma} (2,1,1) \xmapsto{\pi} (4,3,3) \xmapsto{\sigma} (3,1),$$

$$(2,2) \xmapsto{\pi} (4,4) \xmapsto{\sigma} (2,2),$$

$$(2,1,1) \xmapsto{\pi} (4,3,3) \xmapsto{\sigma} (3,1) \xmapsto{\pi} (4,2) \xmapsto{\sigma} (2,1,1),$$

$$(1,1,1,1) \xmapsto{\pi} (4,4,4,4) \xmapsto{\sigma} (4) \xmapsto{\pi} (4) \xmapsto{\sigma} (1,1,1,1).$$

At this point the following fact is apparent.

Theorem C.1.3. *The composite map $\sigma\pi \colon \mathcal{P}_n \to \mathcal{P}_n$ takes partitions to their conjugates.*

Proof. If we write

$$\lambda = (a_1^{m_{a_1}} a_2^{m_{a_2}} a_3^{m_{a_3}} \ldots\, a_r^{m_{a_r}}),\ a_1 > a_2 > \ldots > a_r \geq 1,$$

then we can compute the parts and multiplicities of the conjugate partition

$$\lambda^* = (b_1^{m_{b_1}} b_2^{m_{b_2}} b_3^{m_{b_3}} \ldots\, b_s^{m_{b_s}}),\ b_1 > b_2 > \ldots > b_s \geq 1,$$

directly from the parts and multiplicities of λ by comparing the Ferrers-Young diagrams of λ, λ^*.

Lemma C.1.1. *The conjugate λ^* of partition λ has largest part b_1 given by*

$$b_1 = \ell(\lambda) = m_{a_1} + m_{a_2} + \ldots + m_{a_r}, \quad \textit{with} \ \ m_{b_1}(\lambda^*) = a_r,$$

and for $1 < i \leq s$, the parts and their multiplicities are given by

$$b_i = m_{a_1} + m_{a_2} + \ldots + m_{a_{r-i+1}}, \qquad m_{b_i}(\lambda^*) = a_{r-i+1} - a_{r-i+2}.$$

Moreover, we have that $s = r$.

The theorem results from using the definitions of the maps π and σ, keeping track of the parts in the transformation $\lambda \mapsto \sigma\pi(\lambda)$, then comparing the parts of $\sigma\pi(\lambda)$ with the parts of λ^* in Lemma C.1.1 above to see they are the same. $\square$

Thus $\sigma\pi(\lambda) = \lambda$ if λ is self-conjugate, and $(\sigma\pi)^2(\lambda) = \lambda$ for all $\lambda \in \mathcal{P}$, as we can see in the examples above. For ϕ sequentially congruent, we also have $\pi\sigma(\phi) = \phi$ if $\sigma(\phi)$ is self-conjugate, and $(\pi\sigma)^2(\phi) = \phi$ for all ϕ. Interestingly, the composite map $\pi\sigma$ defines a duality analogous to conjugation of partitions in $\mathcal{P}_n$, that instead connects partitions ϕ and $\pi\sigma(\phi)$ in $S_{\mathrm{lg}=n}$. For instance, from the examples above, we have that $(2, 1, 1)$ and $(3, 1) = \sigma\pi(2, 1, 1)$ are conjugates in $\mathcal{P}_4$, while $(4, 3, 3)$ and $(4, 2) = \pi\sigma(4, 3, 3)$ are paired under the analogous duality in $\mathcal{S}_{\mathrm{lg}=4}$.

These phenomena give further partition-theoretic examples resembling structures in abstract algebra. One more fact is also evident by considering Ferrers-Young diagrams.

Theorem C.1.4. *A sequentially congruent partition ϕ is mapped by conjugation to a partition ϕ^* whose multiplicities $m_i = m_i(\phi^*)$ obey the congruence condition*

$$m_i \equiv 0 \pmod{i}.$$

Conversely, any partition with parts obeying this congruence condition has a sequentially congruent partition as its conjugate.

Appendix D

Notes on Chapter 5: Faà di Bruno's formula in partition theory

D.1 Faà di Bruno's formula with product version

Francesco Faà di Bruno was an Italian priest and mathematician active in the mid-nineteenth century. For the convenience of the reader, we record an easy proof of a useful variant of the formula that bears his name [FdB55], and also adjoin an infinite product representation to the usual statement of the identity, based on elementary ideas. We follow up with a few examples related to topics studied in this thesis[1].

We will write Faà di Bruno's identity in a slightly simplified, equivalent form to that given in (5.7), as a sum over all partitions λ — making it amenable to techniques developed in this dissertation such as application of the q-bracket — and add in a product representation as well, using other classical facts.

Proposition D.1.1 (Faà di Bruno's formula with product representation). *For $a(n)$ an*

[1] See [And98], Chapter 12, for more about this useful formula.

arithmetic function and $a_n \in \mathbb{C}$*, we have*

$$\exp\left(\sum_{n=1}^{\infty} a_n q^n\right) = \sum_{\lambda \in \mathcal{P}} q^{|\lambda|} \frac{a_1^{m_1} a_2^{m_2} a_3^{m_3} \dots}{m_1!\ m_2!\ m_3!\ \dots} = \prod_{n=1}^{\infty} (1-q^n)^{a(n)},$$

where a_n *and* $a(n)$ *are related by*

$$a_n = -\frac{1}{n} \sum_{d|n} a(d)d, \qquad a(n) \ = \ -\frac{1}{n} \sum_{d|n} \mu(n/d) a_d d$$

with the sums taken over divisors of n*, and* μ *being the classical Möbius function.*

Proof. To prove the first equality, we begin with the well-known *multinomial theorem*, re-written as a sum over partitions λ in the set $\mathcal{P}_{[k]} \subset \mathcal{P}$ whose parts are all $\leq k$, having length $\ell(\lambda) = n$:

$$(a_1 + a_2 + a_3 + \dots + a_k)^n = n! \sum_{\substack{\lambda \in \mathcal{P}_{[k]} \\ \ell(\lambda)=n}} \frac{a_1^{m_1} a_2^{m_2} a_3^{m_3} \dots a_k^{m_k}}{m_1!\ m_2!\ m_3!\ \dots\ m_k!}. \tag{D.1}$$

If we let k tend to infinity, assuming the infinite sum $a_1 + a_2 + a_3 + \dots$ converges, the series on the right becomes a sum over all partitions of length n. Then dividing both sides of (D.1) by $n!$ and summing over $n \geq 0$, the left-hand side yields the Maclaurin series expansion for $\exp(a_1 + a_2 + a_3 + \dots)$, and the right side can be rewritten as a sum over all partitions:

$$\exp(a_1 + a_2 + a_3 + \dots) = \sum_{\lambda \in \mathcal{P}} \frac{a_1^{m_1} a_2^{m_2} a_3^{m_3} \dots}{m_1!\ m_2!\ m_3!\ \dots}. \tag{D.2}$$

Now, taking $a_k \mapsto a_k q^k$ in (D.2), we can write

$$\begin{aligned} a_1^{m_1} a_2^{m_2} a_3^{m_3} \dots \mapsto \ & (a_1 q)^{m_1} (a_2 q^2)^{m_2} (a_3 q^3)^{m_3} \dots \\ & = q^{m_1 + 2m_2 + 3m_3 + \dots} a_1^{m_1} a_2^{m_2} a_3^{m_3} \dots = q^{|\lambda|} a_1^{m_1} a_2^{m_2} a_3^{m_3} \dots \end{aligned}$$

in the summands on the right-hand side, which completes the series aspect of the proof.

The product representation follows from Bruinier, Kohnen and Ono [BKO04], and also is immediate from the proofs in [Sch14] if we replace $-f(n)/n$ with an arithmetic function $a(n)$ in the final equation of that paper. For a given $a(n)$, if we set

$$a_n \;=\; -\frac{1}{n}\sum_{d|n} d \cdot a(d), \tag{D.3}$$

we have

$$\prod_{n=1}^{\infty}(1-q^n)^{a(n)} = \exp\left(\sum_{n=1}^{\infty} a_n q^n\right).$$

Applying Möbius inversion to (D.3) gives the converse divisor sum identity for $a(n)$.

□

D.2 Further examples

So we can view Faà di Bruno's formula as a generating function for coefficients of certain partition-theoretic sums involving the form $(a_1^{m_1}a_2^{m_2}a_3^{m_3}...)/(m_1!\, m_2!\, m_3!\, ...)$. As examples, we give a few simple substitutions that lead to interesting partition sum identities.

Example D.2.1. *Setting $a_i = i^{-s}, \mathrm{Re}(s) > 1$, and $q = 1$ in Proposition D.2, gives*

$$\exp\left(\zeta(s)\right) = \sum_{\lambda \in \mathcal{P}} \frac{1}{n_\lambda^s \; m_1! \; m_2! \; m_3! \; ...}$$

with $\zeta(s) := \sum_{n\geq 1} n^{-s}$ the Riemann zeta function.

We note that the right-hand side of this example is a type of partition Dirichlet series. More generally, if we exponentiate a convergent classical Dirichlet series $\sum_{n=1}^{\infty} a(n)n^{-s}$ we arrive at a partition Dirichlet series of the form introduced at the end of Chapter 5, viz.

$$\sum_{\lambda \in \mathcal{P}} A(\lambda) n_\lambda^{-s} = \exp\left(\sum_{n=1}^{\infty} a(n)n^{-s}\right), \tag{D.4}$$

where

$$A(\lambda) := \frac{a(1)^{m_1} a(2)^{m_2} a(3)^{m_3} \cdots a(i)^{m_i} \cdots}{m_1!\ m_2!\ m_3!\ \cdots\ m_i!\ \cdots}. \tag{D.5}$$

In fact, $A(\lambda)$ is multiplicative in the partition sense (see Definition 3.3.3), that is, $A(\lambda\gamma) = A(\lambda)A(\gamma)$ when $\gcd(\lambda, \gamma) = \emptyset$.[2]

Next, we give alternative evaluations of functions evaluated in Appendix B, (B.1.1) and (B.1.2).

Example D.2.2. *Setting $a \equiv 24$ in Proposition D.2 yields $a_i = -24\sigma(i)$, where $\sigma(i) = \sum_{d|i} d$ as usual. Then Ramanujan's tau function $\tau(n)$ can be written*

$$\tau(n) = \sum_{\lambda \vdash (n-1)} (-24)^{\ell(\lambda)} \frac{\sigma(1)^{m_1} \sigma(2)^{m_2} \sigma(3)^{m_3} \ldots}{n_\lambda\ m_1!\ m_2!\ m_3!\ \ldots}.$$

Example D.2.3. *Setting $f \equiv -k$ with $k \geq 1$ in Proposition D.2 yields $a_i = k\sigma(i)$. Then the number $P_k(n)$ of k-color partitions of n can be written*

$$P_k(n) = \sum_{\lambda \vdash n} k^{\ell(\lambda)} \frac{\sigma(1)^{m_1} \sigma(2)^{m_2} \sigma(3)^{m_3} \ldots}{n_\lambda\ m_1!\ m_2!\ m_3!\ \ldots}.$$

Let $\varphi = \frac{1+\sqrt{5}}{2}$ denote the *golden ratio*, a number that makes connections throughout the sciences, nature and the arts. The reciprocal of the golden ratio is similarly "golden": the two constants are intertwined in classical relations like

$$\varphi = 1 + \frac{1}{\varphi}. \tag{D.6}$$

Then we can write down formulas to compute φ and $1/\varphi$ in terms of π and $\zeta(s)$.

[2] We note for the subset $\mathcal{P}^*$ of partitions into distinct parts there is the simpler Euler product generating function $\prod_{n=1}^{\infty}(1 + a(n)n^{-s}) = \sum_{\lambda \in \mathcal{P}^*} a(1)a(2)a(3)\cdots n_\lambda^{-s} = \sum_{\lambda \in \mathcal{P}^*} A(\lambda) n_\lambda^{-s}$.

Example D.2.4. *We have the following identities for the golden ratio and its reciprocal:*

$$\varphi \ = \ \frac{5}{\pi}\sum_{\lambda\in\mathcal{P}} \frac{\zeta(2)^{m_1}\zeta(4)^{m_2}\zeta(6)^{m_3}\zeta(8)^{m_4}\cdots}{n_\lambda\ 100^{|\lambda|}\ m_1!\ m_2!\ m_3!\ m_4!\ \cdots}, \tag{D.7}$$

$$\frac{1}{\varphi} \ = \ \frac{\pi}{5}\sum_{\lambda\in\mathcal{P}} \frac{(-1)^{\ell(\lambda)}\zeta(2)^{m_1}\zeta(4)^{m_2}\zeta(6)^{m_3}\zeta(8)^{m_4}\cdots}{n_\lambda\ 100^{|\lambda|}\ m_1!\ m_2!\ m_3!\ m_4!\ \cdots}. \tag{D.8}$$

Set $b_{2j} := (-1)^{j+1}B_{2j}2^{2j-1}/(2j)!$ with $B_k \in \mathbb{Q}$ the kth Bernoulli number. Then $\zeta(2j) = \pi^{2j}b_{2j}$ for $j \in \mathbb{Z}^+$, by Euler. Comparing this fact with equations (D.7) and (D.8) implies additional expressions giving φ and $1/\varphi$ in terms of π.

Example D.2.5. *We have the following identities for the golden ratio and its reciprocal:*

$$\varphi \ = \ 5\sum_{\lambda\in\mathcal{P}} \frac{\pi^{2|\lambda|-1}b_2^{m_1}b_4^{m_2}b_6^{m_3}b_8^{m_4}\cdots}{n_\lambda\ 100^{|\lambda|}\ m_1!\ m_2!\ m_3!\ m_4!\ \cdots}, \tag{D.9}$$

$$\frac{1}{\varphi} \ = \ \frac{1}{5}\sum_{\lambda\in\mathcal{P}} \frac{(-1)^{\ell(\lambda)}\pi^{2|\lambda|+1}b_2^{m_1}b_4^{m_2}b_6^{m_3}b_8^{m_4}\cdots}{n_\lambda\ 100^{|\lambda|}\ m_1!\ m_2!\ m_3!\ m_4!\ \cdots}. \tag{D.10}$$

Then by the classical relation (D.6), further formulas for φ may be obtained from adding 1 to both sides of equations (D.8) and (D.10).

Proof. It is a straightforward deduction from geometry (see [Liv08]) that we can write

$$\frac{1}{\varphi} := \frac{-1+\sqrt{5}}{2} = 2\ \sin\left(\frac{\pi}{10}\right).$$

Comparing this result to Euler's formula $\sin(x) = x\prod_{n=1}^{\infty}\left(1-\frac{x^2}{\pi^2 n^2}\right)$ with $x = \pi/10$, then gives

$$\varphi = \frac{1}{2\ \sin(\frac{\pi}{10})} = \frac{5}{\pi}\prod_{n=1}^{\infty}\left(1-\frac{1}{100\ n^2}\right)^{-1}. \tag{D.11}$$

At this stage, we note it follows immediately from Theorems 4.1.1 and 4.2.8 in Chapter 4 that

$$\varphi \ = \ \frac{5}{\pi}\sum_{\lambda\in\mathcal{P}}\frac{1}{n_\lambda^2\ 100^{\ell(\lambda)}}, \qquad \frac{1}{\varphi} \ = \ \frac{\pi}{5}\sum_{\lambda\in\mathcal{P}}\frac{\mu_{\mathcal{P}}(\lambda)}{n_\lambda^2\ 100^{\ell(\lambda)}}, \tag{D.12}$$

which are further examples of partition Dirichlet series. Now, to prove (D.7) and (D.8), begin with the Maclaurin expansion of the natural logarithm

$$-\ln(1-x) = \sum_{j=1}^{\infty} \frac{x^j}{j}, \qquad |x| < 1.$$

Setting $\exp(x) := e^x$, we then take $x = 1/100n^2 < 1$ for each $n = 1, 2, 3, ...$ to write

$$\begin{aligned}\prod_{n=1}^{\infty}\left(1-\frac{1}{100\ n^2}\right)^{-1} &= \prod_{n=1}^{\infty}\exp\left(-\ln\left(1-\frac{1}{100n^2}\right)\right) = \exp\left(\sum_{n=1}^{\infty}\sum_{j=1}^{\infty}\frac{1}{n^{2j}100^j j}\right)\\ &= \exp\left(\sum_{j=1}^{\infty}\frac{1}{100^j j}\sum_{n=1}^{\infty}\frac{1}{n^{2j}}\right) = \exp\left(\sum_{j=1}^{\infty}\frac{\zeta(2j)}{100^j j}\right).\end{aligned}$$

Then setting $q = 1, a_j = \zeta(2j)/j100^j$ in Proposition and comparing all this to (D.11), plus some algebra, proves (D.7). Setting $a_j = -\zeta(2j)/j100^j$ (a minus sign is introduced by taking reciprocals) gives (D.8).

Example D.2.5 follows easily from (D.7) and (D.8) by making the substitution $\zeta(2j) \mapsto \pi^{2j} b_{2j}$ in each summand. □

Using the Maclaurin expansion of the natural logarithm from the above proof plus a little algebra using a summation swap and geometric series in the exponential, we have

$$(z;q)_{\infty}^{-1} = \prod_{k=0}^{\infty}\exp\left(\sum_{n=1}^{\infty}\frac{z^n q^{nk}}{n}\right) = \exp\left(\sum_{n=1}^{\infty}\frac{z^n}{n(1-q^n)}\right).$$

It is a case of the q-binomial theorem (see Lemma 6.2.1) that $(z;q)_{\infty}^{-1} = \sum_{n=0}^{\infty}\frac{z^n}{(q;q)_n}$. Combining these formulas with Faà di Bruno's formula gives our next example.

Example D.2.6. *We have that*

$$(z;q)_{\infty}^{-1} = \sum_{n=0}^{\infty}\frac{z^n}{(q;q)_n} = \sum_{\lambda\in\mathcal{P}}\frac{z^{|\lambda|}}{n_{\lambda}\ m_1!\ m_2!\ m_3!\ \cdots\ (1-q)^{m_1}(1-q^2)^{m_2}(1-q^3)^{m_3}\ \cdots}.$$

Comparing coefficients on both sides of this identity gives Chapter 12, Example 1 of [And98], which Andrews attributes to Cayley, but Sills argues in [Sil17a] is due to MacMahon [Mac60]. Noting from Example D.2.6 (and Lambert series) that

$$(q;q)_\infty^{-1} = \exp\left(\sum_{n=1}^{\infty} \frac{q^n}{n(1-q^n)}\right) = \exp\left(\sum_{n=1}^{\infty} \frac{\sigma(n)q^n}{n}\right), \tag{D.13}$$

as a final example we show the q-bracket of $A(\lambda)$ from (D.5) above takes a nice form.

Example D.2.7. *We have that*

$$\langle A\rangle_q = \sum_{\lambda\in\mathcal{P}} q^{|\lambda|} \frac{(a(1)-\sigma(1))^{m_1}(a(2)-\sigma(2)/2)^{m_2}\cdots(a(i)-\sigma(i)/i)^{m_i}\cdots}{m_1!\ m_2!\ \cdots\ m_i!\ \cdots}. \tag{D.14}$$

More generally, using the notation of Definitions 3.4.1 and 3.4.2, it is the case that

$$\langle A\rangle_q^{(\pm k)} = \sum_{\lambda\in\mathcal{P}} q^{|\lambda|} \frac{(a(1)\mp k\sigma(1))^{m_1}(a(2)\mp k\sigma(2)/2)^{m_2}\cdots(a(i)\mp k\sigma(i)/i)^{m_i}\cdots}{m_1!\ m_2!\ \cdots\ m_i!\ \cdots}, \tag{D.15}$$

where with regard to "$\pm$, $\mp$", a plus on the left gives minus on the right, and vice versa.

It is interesting that multiplication and division by $(q;q)_\infty$ produces this homogenous shift in the values of the coefficients in the numerator, by terms involving $\sigma(n)$.

Proof. This follows from writing $(q;q)_\infty$ as the reciprocal of (D.13) (noting a minus sign is introduced inside the exponential) and using $\exp\left(\sum_{n=1}^{\infty} a(n)q^n\right) = \sum_{\lambda\in\mathcal{P}} A(\lambda)q^{|\lambda|}$. $\square$

In addition to applications in number theory, the author and his collaborators in the Emory Working Group in Number Theory and Molecular Simulation (an interdisciplinary research group run by Professor James Kindt in Emory's Chemistry Department) make extensive use of Faà di Bruno's formula in theoretical chemistry to develop simulation algorithms and probe classical laws like the Law of Mass Action from partition-theoretic first principles (see, for example, [ZPBSea17]).

Appendix E

Notes on Chapter 6: Further relations involving $F_{\mathbb{S}_{r,t}}$

E.1 Classical series and arithmetic functions

In this note we essentially use the left-hand side of Theorem 6.1.3, viz. the limit

$$\lim_{q\to\zeta} F_{\mathbb{S}_{r,t}}(q) = -\lim_{q\to\zeta} \sum_{\substack{\lambda\in\mathcal{P}\\ \mathrm{sm}(\lambda)\in\mathbb{S}_{r,t}}} \mu_{\mathcal{P}}(\lambda) q^{|\lambda|} = -\lim_{q\to\zeta} (q;q)_\infty \sum_{\substack{\lambda\in\mathcal{P}\\ \mathrm{lg}(\lambda)\in\mathbb{S}_{r,t}}} q^{|\lambda|} \tag{E.1}$$

from inside the unit circle, as an elaborate way to write $1/t$. Then trivially, we can rewrite many classical series as limits of this type. For instance, if we set $r = 0$ to satisfy $r < t$ for all $t \geq 1$, we can rewrite the zeta function as the limit of a Dirichlet series

$$\zeta(s) = \lim_{q\to\zeta} \sum_{t=1}^{\infty} F_{\mathbb{S}_{0,t}}(q) t^{s-1} \quad (\mathrm{Re}(s) > 1). \tag{E.2}$$

Another elementary observation is that if $A(t) := \sum_{d|t} a(d)$ for $a\colon \mathbb{N} \to \mathbb{C}$, we have

$$\sum_{t=1}^{\infty} A(t) \frac{q^t}{(q;q)_t} = \sum_{t=1}^{\infty} a(t) \sum_{k=1}^{\infty} \frac{q^{tk}}{(q;q)_{tk}} = -\sum_{t=1}^{\infty} a(t) \left(\frac{F_{\mathbb{S}_{0,t}}(q)}{(q;q)_\infty} + 1 \right).$$

Reorganizing gives the following identity.

Proposition E.1.1. *Using the above notation, if* $\sum_{t=1}^{\infty} a(t)$ *converges we have*

$$-(q;q)_\infty^{-1} \sum_{t=1}^{\infty} a(t) F_{\mathbb{S}_{0,t}}(q) = \sum_{t=1}^{\infty} a(t) + \sum_{t=1}^{\infty} A(t) \frac{q^t}{(q;q)_t}.$$

In terms of partitions we can write

$$\sum_{t=1}^{\infty} \sum_{\substack{\emptyset \neq \lambda \in \mathcal{P} \\ \lg(\lambda) \in \mathbb{S}_{0,t}}} a(t) q^{|\lambda|} = \sum_{\emptyset \neq \lambda \in \mathcal{P}} A(\lg(\lambda)) q^{|\lambda|}.$$

Remark. We note by conjugation that $\lg(\lambda) = \ell(\lambda^*)$ and $|\lambda| = |\lambda^*|$, thus for any $f \colon \mathbb{N} \to \mathbb{C}$ we have $\sum_{\lambda \in \mathcal{P}} f(\lg(\lambda)) q^{|\lambda|} = \sum_{\lambda \in \mathcal{P}} f(\ell(\lambda)) q^{|\lambda|}$ (which also holds for sums $\sum_{\emptyset \neq \lambda \in \mathcal{P}}$ above).

Proposition E.1.1 is useful in further applying the ideas of Chapter 6. Here is an example that gives another q-series relation to arithmetic densities.

Example E.1.1. *Set* $a(n) = \mu(n)/n$ *in Proposition E.1.1 with* μ *the Möbius function. Then as* $A(n) = \sum_{d|n} \mu(d)/d = \varphi(n)/n$*, using the classical facts* $\sum_{n\geq 1} \mu(n)/n = 0$ *and* $\sum_{n \geq 1} \mu(n)/n^2 = \zeta(2)^{-1}$ *together with Theorem 6.1.3 and a little algebra, we have*

$$-\lim_{q \to \zeta} (q;q)_\infty \sum_{n=1}^{\infty} \frac{\varphi(n) q^n}{n (q;q)_n} = \frac{6}{\pi^2}, \tag{E.3}$$

which is well known to be $\lim_{n\to\infty} \varphi(n)/n$.

Remark. One wonders more generally if there are classes of arithmetic functions $f(n)$ with

$$-\lim_{q \to \zeta} (q;q)_\infty \sum_{n=1}^{\infty} \frac{f(n)}{n} \cdot \frac{q^n}{(q;q)_n} = \lim_{n \to \infty} \frac{f(n)}{n}.$$

With $(q;q)_\infty$ floating around in these formulas, we could apply q-bracket ideas from Chapter 3 for further relations. Moreover, a finite version of the above order-of-summation swapping holds with respect to partial sums. Let $F_{\mathbb{S}_{r,t}}(q, N)$ denote the following partial sum, with $F_{\mathbb{S}_{r,t}}(q, N) \to F_{\mathbb{S}_{r,t}}(q)$ as $N \to \infty$ per the proof of Lemma 6.2.3:

$$F_{\mathbb{S}_{r,t}}(q, N) := -(q;q)_\infty \sum_{n=0}^{N} \frac{q^{nt+r}}{(q;q)_{nt+r}} = -(q;q)_\infty \sum_{\substack{\lambda \in \mathcal{P} \\ \lg(\lambda) \in \mathbb{S}_{r,t} \\ \lg(\lambda) \leq Nt+r}} q^{|\lambda|}. \tag{E.4}$$

Proposition E.1.2. *Using the above notation, we have that*

$$-(q;q)_\infty^{-1} \sum_{t=1}^{N} a(n) F_{\mathbb{S}_{0,t}}\left(q, \left\lfloor \frac{N}{t} \right\rfloor\right) = \sum_{t=1}^{N} a(t) + \sum_{t=1}^{N} A(t) \frac{q^t}{(q;q)_t}.$$

In terms of partitions we can write

$$\sum_{t=1}^{N} \sum_{\substack{\emptyset \neq \lambda \in \mathcal{P} \\ \lg(\lambda) \in \mathbb{S}_{0,t} \\ \lg(\lambda) \leq \lfloor N/t \rfloor t}} a(t) q^{|\lambda|} = \sum_{\substack{\emptyset \neq \lambda \in \mathcal{P} \\ \lg(\lambda) \leq N}} A(\lg(\lambda)) q^{|\lambda|}.$$

Here is an example involving Mertens's function, the summatory function of the Möbius function[1], viz. $M(x) := \sum_{1 \leq n \leq x} \mu(n)$.

Example E.1.2. *Set $a(n) = \mu(n)$ in Proposition E.1.2. Then as $A(n) = \sum_{d|n} \mu(d) = 1$ if $n = 1$ and $= 0$ otherwise, a little algebra gives*

$$-(q;q)_\infty M(N) \ = \ q(q^2;q)_\infty + \sum_{n=1}^{N} \mu(n) F_{\mathbb{S}_{0,n}}\left(q, \left\lfloor \frac{N}{n} \right\rfloor\right). \tag{E.5}$$

One notes heuristically in the double limit $q \to \zeta, N \to \infty$ (for instance, consider $q = e^{2\pi \mathrm{i} z}, z = \mathrm{i}/N$ as $N \to \infty$), the right-hand side of (E.5) appears to vanish (both $(\zeta^2;\zeta)_\infty$ and $\sum_{n=1}^{\infty} \mu(n)/n$ equal zero) while the left side is indeterminate ($(\zeta;\zeta)_\infty = 0$ and $M(N)$ oscillates but grows asymptotically without bound in absolute value). Can facts about $(q;q)_\infty$ tell us something about the growth of Mertens's function?[2]

[1] It is a famous fact that the statement $M(x) = \mathcal{O}(x^{1/2+\epsilon})$ is equivalent to the Riemann Hypothesis.

[2] For instance, for $q = e^{2\pi \mathrm{i} z}, z \in \mathbb{H}$ the upper half-plane, the modularity relation $\eta(z) := q^{1/24}(q;q)_\infty = \sqrt{-\mathrm{i}z} \cdot \eta(-1/z)$ yields $\eta(\mathrm{i}/N) = \eta(\mathrm{i}N)/\sqrt{N}$ in the case $z = \mathrm{i}/N$ suggested above.

Appendix F

Notes on Chapter 7: Alternating "strange" functions

Adapted from [Sch18]

F.1 Further "strange" connections to quantum and mock modular forms

Recall the "strange" function $F(q)$ of Kontsevich (see Definition 1.1.5) studied in Chapter 7, which has been studied deeply by Zagier [Zag01] — it was one of his prototypes for quantum modular forms — as well as by other authors [BFR15, BOPR12] in connection to quantum modularity, unimodal sequences, and other topics.

For the sake of this appendix, we remind the reader that $\sum_{n\geq 0}(q;q)_n$ converges almost nowhere in the complex plane. However, at $q = \zeta_m$ an mth order root of unity, F is suddenly very well-behaved: because $(\zeta_m;\zeta_m)_n = 0$ for $n \geq m$, then as $q \to \zeta_m$ radially from within the unit disk, $F(\zeta_m) := \lim_{q\to\zeta_m} F(q)$ is just a polynomial in $\mathbb{Z}[\zeta_m]$. (We generalize this phenomenon in Chapter 8.)

Now let us turn our attention to the alternating case of this series, viz.

$$\widetilde{F}(q) := \sum_{n=0}^{\infty}(-1)^n(q;q)_n, \tag{F.1}$$

a summation that has been studied by Cohen [BOPR12], which is similarly "strange": it doesn't converge anywhere in $\mathbb{C}$ except at roots of unity, where it is a polynomial. In fact, computational examples suggest the odd and even partial sums of $\widetilde{F}(q)$ oscillate asymptotically between two convergent q-series.

To capture this oscillatory behavior, let us adopt a notation we will use throughout this appendix. If S is an infinite series, we will write S_+ to denote the limit of the sequence of odd partial sums, and S_- for the limit of the even partial sums, if these limits exist (clearly if S converges, then $S_+ = S_- = S$).

Interestingly, like $F(q)$, the "strange" series $\widetilde{F}(q)$ is closely connected to a sum Zagier provided as another prototype for quantum modularity (when multiplied by $q^{1/24}$) [Zag10], the function

$$\sigma(q) := \sum_{n=0}^{\infty}\frac{q^{n(n+1)/2}}{(-q;q)_n} = 1 + \sum_{n=0}^{\infty}(-1)^n q^{n+1}(q;q)_n \tag{F.2}$$

from Ramanujan's "lost" notebook, with the right-hand equality due to Andrews [AJUO01]. If we use the convention introduced above and write $\widetilde{F}_+(q)$ (resp. $\widetilde{F}_-(q)$) to denote the limit of the odd (resp. even) partial sums of $\widetilde{F}$, we can state this connection explicitly, depending on the choice of "+" or "−".

Theorem F.1.1. *For $0 < |q| < 1$ we have*

$$\sigma(q) = 2\widetilde{F}_{\pm}(q) \pm (q;q)_{\infty}.$$

We can make further sense of alternating "strange" series such as this using Cesàro summation, a well-known alternative definition of the limits of infinite series (see [Har00]).

Definition F.1.1. *The Cesàro sum of an infinite series is the limit of the arithmetic* *mean of successive partial sums, if the limit exists.*

In particular, it follows immediately that the Cesàro sum of the series S is the average $\frac{1}{2}(S_+ + S_-)$ if the limits S_+, S_- exist. Then Theorem F.1.1 leads to the following fact.

Corollary F.1.1. *We have that $\frac{1}{2}\sigma(q)$ is the Cesàro sum of the "strange" function $\widetilde{F}(q)$.*

A similar relation to Theorem F.1.1 involves Ramanujan's prototype $f(q)$ for a mock theta function

$$f(q) := \sum_{n=0}^{\infty} \frac{q^{n^2}}{(-q;q)_n^2} = 1 - \sum_{n=1}^{\infty} \frac{(-1)^n q^n}{(-q;q)_n}, \tag{F.3}$$

the right-hand side of which is due to Fine (see (26.22) in [Fin88], Ch. 3). Now, if we define

$$\widetilde{\phi}(q) := \sum_{n=0}^{\infty} \frac{(-1)^n}{(-q;q)_n}, \tag{F.4}$$

which is easily seen to be "strange" like the previous cases, and write $\widetilde{\phi}_\pm$ for limits of the odd/even partial sums as above, we can write $f(q)$ in terms of the "strange" series and an infinite product.

Theorem F.1.2. *For $0 < |q| < 1$ we have*

$$f(q) = 2\widetilde{\phi}_\pm(q) \pm \frac{1}{(-q;q)_\infty}.$$

Again, the Cesàro sum results easily from this theorem.

Corollary F.1.2. *We have that $\frac{1}{2}f(q)$ is the Cesàro sum of the "strange" function $\widetilde{\phi}(q)$.*

Theorems F.1.1 and F.1.2 typify a general phenomenon: the combination of an alternating Kontsevich-style "strange" function with a related infinite product is a convergent q-series when we fix the $\pm$ sign in this modified definition of limits. Let us fix a few more notations in order to discuss this succinctly. As usual, for n a non-negative integer, define

$$(a_1, a_2, ..., a_r; q)_n := (a_1;q)_n(a_2;q)_n \cdots (a_r;q)_n,$$

along with the limiting case $(a_1, a_2, ..., a_r; q)_\infty$ as $n \to \infty$. Associated to the sequence $a_1, a_2, ..., a_r$ of complex coefficients, we will define a polynomial $\alpha_r(X)$ by the relation

$$(1 - a_1X)(1 - a_2X) \cdots (1 - a_rX) =: 1 - \alpha_r(X)X, \tag{F.5}$$

thus

$$(a_1q, a_2q, ..., a_rq; q)_n = \prod_{j=1}^{n}(1 - \alpha_r(q^j)q^j), \tag{F.6}$$

and we follow this convention in also writing $(1 - b_1X)(1 - b_2X) \cdots (1 - b_sX) =: 1 - \beta_s(X)X$ for complex coefficients $b_1, b_2, ..., b_s$. Moreover, we define a generalized alternating "strange" series:

$$\widetilde{\Phi}(a_1, a_2, ..., a_r; b_1, b_2, ..., b_s; q) := \sum_{n=0}^{\infty}(-1)^n \frac{(a_1q, a_2q, ..., a_rq; q)_n}{(b_1q, b_2q, ..., b_sq; q)_n}. \tag{F.7}$$

Thus $\widetilde{F}(q)$ is the case $\widetilde{\Phi}(1; 0; q)$, and $\widetilde{\phi}(q)$ is the case $\widetilde{\Phi}(0; -1; q)$. We note that if q is a kth root of $1/a_i$ for some i, then $\widetilde{\Phi}$ truncates after k terms like F and $\widetilde{F}$. As above, let $\widetilde{\Phi}_\pm$ denote the limit of the odd/even partial sums; then we can encapsulate the preceding theorems in the following statement.

Theorem F.1.3. *For $0 < |q| < 1$ we have*

$$\begin{aligned}2\widetilde{\Phi}_\pm(a_1, a_2, ..., a_r; b_1, b_2, ..., b_s; q) &\pm \frac{(a_1q, a_2q, ..., a_rq; q)_\infty}{(b_1q, b_2q, ..., b_sq; q)_\infty} \\ &= 1 - \sum_{n=1}^{\infty} \frac{(-1)^n q^n \left(\alpha_r(q^n) - \beta_s(q^n)\right)(a_1q, a_2q, ..., a_rq; q)_{n-1}}{(b_1q, b_2q, ..., b_sq; q)_n}.\end{aligned}$$

From this identity we can fully generalize the previous corollaries.

Corollary F.1.3. *We have that $1/2$ times the right-hand side of Theorem F.1.3 is the Cesàro sum of the "strange" function $\widetilde{\Phi}(a_1, ..., a_r; b_1, ..., b_s; q)$.*

The takeaway is that the Nth partial sum of an alternating "strange" series oscillates asymptotically as $N \to \infty$ between $\frac{1}{2}(S(q) + (-1)^N P(q))$, where S is an Eulerian infinite series and P is an infinite product as given in Theorem F.1.3. We recover Theorem F.1.1 from Theorem F.1.3 as the case $a_1 = 1,\ a_i = b_j = 0$ for all $i > 1, j \geq 1$. Theorem F.1.2 is the case $b_1 = -1,\ a_i = b_j = 0$ for all $i \geq 1, j > 1$.

Considering these connections together with diverse connections made by Kontsevich's $F(q)$ to important objects of study [BFR15, BOPR12, Zag01], it seems the ephemeral "strange" functions almost "enter into mathematics as beautifully"[1] as their convergent relatives, mock theta functions. We note that considerations of finiteness at roots of unity and renormalization phenomena studied in Chapter 8 apply to Theorem F.1.3 as well.

F.2 Proofs of results

In this section we quickly prove the preceding theorems, and justify the corollaries.

Proof of Theorem F.1.1. Using telescoping series to find for $|q| < 1$ that

$$(q;q)_\infty = 1 - \sum_{n=0}^{\infty} (q;q)_n \left(1 - (1 - q^{n+1})\right) = 1 - \sum_{n=0}^{\infty} q^{n+1} (q;q)_n,$$

and combining this functional equation with the right side of (F.2) above, easily gives

$$\sigma(q) - (q;q)_\infty = 2 \sum_{n=0}^{\infty} q^{2n+1} (q;q)_{2n}.$$

On the other hand, manipulating symbols heuristically (for we are working with a divergent series $\widetilde{F}$) suggests we can rewrite

$$\widetilde{F}(q) = \sum_{n=0}^{\infty} \left((q;q)_{2n} - (q;q)_{2n+1}\right) = \sum_{n=0}^{\infty} (q;q)_{2n} \left(1 - (1 - q^{2n+1})\right) = \sum_{n=0}^{\infty} q^{2n+1} (q;q)_{2n},$$

which is a rigorous statement if by convergence on the left we mean the limit as $N \to$

[1] To redirect Ramanujan's words

∞ of partial sums $\sum_{n=0}^{2N-1}(-1)^n(q;q)_n$. We can also choose the alternate coupling of summands to similar effect, e.g. considering here the partial sums $1+\sum_{n=1}^{N-1}[(q;q)_{2n}-(q;q)_{2n-1}]-(q;q)_{2N-1}$ as $N\to\infty$. Combining the above considerations proves the theorem for $|q|<1$, which one finds to agree with computational examples. □

Proof of Theorem F.1.2. Following the formal steps that prove Theorem F.1.1 above, we can use

$$\frac{1}{(-q;q)_\infty}=1-\sum_{n=0}^{\infty}\frac{1}{(-q;q)_n}\left(1-\frac{1}{1+q^{n+1}}\right)=1-\sum_{n=1}^{\infty}\frac{q^n}{(-q;q)_n}$$

and rewrite the related "strange" series

$$\widetilde{\phi}(q)=\sum_{n=0}^{\infty}\frac{1}{(-q;q)_{2n}}\left(1-\frac{1}{1+q^{2n+1}}\right)=\sum_{n=0}^{\infty}\frac{q^{2n+1}}{(-q;q)_{2n+1}},$$

which of course fails to converge for $0<|q|<1$ on the left-hand side but makes sense if we use the modified definition of convergence used above, to yield the identity in the theorem (which is, again, borne out by computational examples). □

Proof of Theorem F.1.3. Using the definitions of the polynomials $\alpha_r(X),\beta_s(X)$, then following the exact steps that yield Theorems F.1.1 and F.1.2, i.e., manipulating and comparing telescoping-type series with the same modified definition of convergence, gives the theorem. □

Proof of Corollaries. Clearly, for an alternating "strange" series in which the odd and even partial sums each approach a different limit, the average of these two limits will equal the Cesàro sum of the series. □

Remark. It follows from Euler's continued fraction formula [Eul85] that alternating "strange" functions have representations such as

$$\widetilde{F}(q)=\cfrac{1}{1+\cfrac{1-q}{q+\cfrac{1-q^2}{q^2+\cfrac{1-q^3}{q^3+\dots}}}},\qquad \widetilde{\phi}(q)=\cfrac{1}{1+\cfrac{1}{q+\cfrac{1+q}{q^2+\cfrac{1+q^2}{q^3+\dots}}}}.$$

These "strange" continued fractions diverge on $0 < |q| < 1$ with successive convergents equal to the corresponding partial sums of their series representations. Then we can substitute continued fractions for the Kontsevich-style summations in the theorems above using a similarly modified definition of convergence: we take the $\pm$ sign to be positive when we define the limit of the continued fraction to be the limit of the even convergents, and negative if instead we use odd convergents.

Appendix G

Notes on Chapter 8: Results from a computational study of $f(q)$

Based on joint work with Amanda Clemm

G.1 Cyclotomic-type structures at certain roots of unity

Here we record some relations the author and Amanda Clemm observed computationally during a study at Emory University (September–December, 2013) of the mock theta function $f(q)$[1] at roots of unity. In our study, we programmed SageMath using the finite formula for $f(\zeta_m)$ given in Example 8.3.3 and we looked for patterns in our numerics. We saw traces of cyclic group theory related to the values $f(\zeta_m^i)$ for odd m. The algebraic structure appears most transparently if we use the normalized version

$$\widetilde{f}(\zeta_m^i) := \frac{3}{4} f(\zeta_m^i) \tag{G.1}$$

for m an odd number, which is just the summation on the right-hand side of Example 8.3.3. We note $\widetilde{f}(1) = 1$. These evaluations of $\widetilde{f}$ enjoy surprisingly nice combinations.

[1] Recall from (1.4)

We observe computationally (without proof) that the coefficients of the cyclotomic-type polynomial

$$\widetilde{F}_m(X) := \prod_{\substack{1\leq i<m \\ \gcd(m,i)=1}} \left(X - \widetilde{f}(\zeta_m^i)\right) \tag{G.2}$$

are integers; in other words,

$$\sum_{\gcd(m,i)=1} \widetilde{f}(\zeta_m^i), \qquad \sum_{\substack{i\neq j \\ \gcd(m,i)=\gcd(m,j)=1}} \widetilde{f}(\zeta_m^i)\widetilde{f}(\zeta_m^j), \qquad \sum_{\substack{i\neq j\neq k \\ \gcd(m,i)=\gcd(m,j)=\gcd(m,k)=1}} \widetilde{f}(\zeta_m^i)\widetilde{f}(\zeta_m^j)\widetilde{f}(\zeta_m^k), \tag{G.3}$$

and so on up to

$$\prod_{\substack{1\leq i<m \\ \gcd(m,i)=1}} \widetilde{f}(\zeta_m^i), \tag{G.4}$$

are all integers. To simplify calculations with respect to the gcd, take $m = p$ a prime number. We observe computationally that for the first few primes p, the coefficients indicated in (G.3), (G.4) appear to be congruent to 1 modulo p, though we do not know if this holds for all primes. It also appears that the $\widetilde{f}(\zeta_p^i)$ are cyclic of order p, modulo p:

$$\widetilde{f}(\zeta_p^i)^n \equiv \widetilde{f}(\zeta_p^i)^{n+pk} \pmod p \text{ for all } i,k,n \in \mathbb{Z}. \tag{G.5}$$

Following up on these observations, we computed examples for $p = 5$ and found many linear combinations of the forms $\widetilde{f}(\zeta_5^i)$ yield nice evaluations, such as this infinite system, which is not hard to prove from facts about polynomials at roots of unity [DF04]:

$$\begin{aligned}
\widetilde{f}(\zeta_5) + \widetilde{f}(\zeta_5^2) + \widetilde{f}(\zeta_5^3) + \widetilde{f}(\zeta_5^4) &= 4, \\
\widetilde{f}(\zeta_5)^2 + \widetilde{f}(\zeta_5^2)^2 + \widetilde{f}(\zeta_5^3)^2 + \widetilde{f}(\zeta_5^4)^2 &= 4, \\
\widetilde{f}(\zeta_5)^3 + \widetilde{f}(\zeta_5^2)^3 + \widetilde{f}(\zeta_5^3)^3 + \widetilde{f}(\zeta_5^4)^3 &= -11, \\
\widetilde{f}(\zeta_5)^4 + \widetilde{f}(\zeta_5^2)^4 + \widetilde{f}(\zeta_5^3)^4 + \widetilde{f}(\zeta_5^4)^4 &= -76, \ldots
\end{aligned} \tag{G.6}$$

More strikingly, we see these $\widetilde{f}(\zeta_5^i)$ involved in cyclotomic-like structures. Noting $\widetilde{f}(1) = 1$, by direct computation we find this simple identity as the $m = 5$ case of (G.4):

$$\widetilde{f}(\zeta_5)\widetilde{f}(\zeta_5^2)\widetilde{f}(\zeta_5^3)\widetilde{f}(\zeta_5^4) = 1. \tag{G.7}$$

Direct calculation verifies further identities, which we did not prove formally.

Proposition G.1.1. *Certain products $\widetilde{f}(\zeta_5^i)\widetilde{f}(\zeta_5^j), i \neq j$, are equal to roots of unity:*

$$\begin{aligned}
\widetilde{f}(\zeta_5)\widetilde{f}(\zeta_5^3) &= \zeta_5, \\
\widetilde{f}(\zeta_5)\widetilde{f}(\zeta_5^2) &= \zeta_5^2, \\
\widetilde{f}(\zeta_5^3)\widetilde{f}(\zeta_5^4) &= \zeta_5^3, \\
\widetilde{f}(\zeta_5^2)\widetilde{f}(\zeta_5^4) &= \zeta_5^4.
\end{aligned}$$

At this point it is easy to derive any number of identities algebraically, e.g.,

$$\widetilde{f}(\zeta_5)^3\widetilde{f}(\zeta_5^2)^2\widetilde{f}(\zeta_5^3) = 1, \quad \left(\widetilde{f}(\zeta_5) + \widetilde{f}(\zeta_5^4)\right)\left(\widetilde{f}(\zeta_5^2) + \widetilde{f}(\zeta_5^3)\right) = -1.$$

From the preceding formulas and (G.1), we also derive a very tidy relation for $f(\zeta_5^i)$.

Proposition G.1.2. *At fifth-order roots of unity, we have the symmetric relation*

$$\zeta_5^i f(\zeta_5^i) \;=\; \zeta_5^{-i} f(\zeta_5^{-i}).$$

The empirical conjecture that (G.4) is an integer[2] suggests an equation like (G.7) exists for every odd-order root of unity ($f(q)$ diverges at even-order roots of unity, thus the finite formula in Example 8.3.3 does not represent its limit). Now, we computed (G.7) and Proposition G.1.1 directly from the formula in Example 8.3.3, letting m be the prime $p = 5$; we have not proved these by algebraic methods, so we don't have a clear intuition

[2]In fact, computations suggest (G.3), (G.4) may be integers even without conditions on the gcd's.

as to how the propositions generalize. One expects there to be analogs of Propositions G.1.1 and G.1.2 above (but perhaps more complicated) for $f(q)$ at other odd-order roots of unity ζ_m, as presumably these propositions depend in the end on properties of Example 8.3.3 and facts about polynomials at roots of unity, not on the choice of the order m.

Are there cyclotomic-type relations like (G.7) and Propositions G.1.1 and G.1.2 for other mock theta functions — or other q-hypergeometric series — at roots of unity?

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