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By

Yining Cheng

Suresh Venapally, Ph.D.
Advisor

Department of Mathematics and Computer Science

Suresh Venapally, Ph.D.
Advisor

Raman Parimala, Ph.D.
Committee Member

Ruixuan Liu, Ph.D.
Committee Member

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An abstract of
a thesis submitted to the Faculty of Emory College of Arts and Sciences
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Abstract

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We first list and state some basic definitions and theorems of the Galois theory of finite extensions, as well as state and prove the Kummer theory and the Artin-Schreier extensions as prerequisites. The main part of this thesis is the proof of the Artin-Schreier Theorem, which states that an algebraic closed field having finite extension with its subfield F has degree at most two and F must have characteristic 0. After the proof, we will discuss the applications for the Artin-Schreier Theorem.

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Contents

1 Introduction

Since elementary schools we start to solve quadratic polynomials, and as we approach high school and college, the polynomials become more and more complex and the roots of a polynomial change from simple integers to complex numbers. In field theory, we say that a field C is an algebraic closure of a field F if C is algebraic over F and every polynomial $f(x) \in F[x]$ splits over C. Simply, it can be seen as that C contains all roots of every polynomial whose coefficients are in field F. If we look at the fields $\mathbb{R}$ and $\mathbb{C}$, $\mathbb{R}$ is not algebraic closed since there are polynomials which do not have any root in $\mathbb{R}$; whereas $\mathbb{C}$ is the algebraic closure of $\mathbb{R}$, for that every polynomial $f(x) \in \mathbb{R}$ has all its roots in $\mathbb{C}$. Moreover, the characteristics of $\mathbb{R}$ and $\mathbb{C}$ are 0, meaning that none of their elements have multiples equal to 0 (i.e.,$nx \neq 0$, for all $x \in \mathbb{R}$). The degree of $[\mathbb{C} : \mathbb{R}] = 2$ because we can write $\mathbb{C} = \{a + bi \mid a, b \in \mathbb{R}, i^2 = -1\}$. In general, is there any other example of such relationship between a non-algebraic closed field and its algebraic closure that is a finite extension? The Artin-Schreier Theorem will answer this question, demonstrating that if F is not algebraic closed, and C is its algebraic closure which is its finite extension, then F must have characteristic 0 and C is of the form $F(i)$.

In section 2, we will recall basic definitions, state and prove important theorems in field theory and Galois theory following [2] and [1] as well as some significant lemmas toward the proof of the Artin Schreier theorem. In section 3 and 4, we will state and prove the Artin Schreier extension theorem and Kummer theory. In section 5, we will reproduce the Artin-Schreier Theorem following the proof of Keith Conrad in [3] by adding more detailed explanations from section 2, 3 and 4. In section 6, we will state and prove three simple corollaries as consequences of the Artin-Schreier Theorem.

2 Basic Definitions, Theorems, And Some Lemmas

2.1 Field Theory

Definition 2.1. *The characteristic of a ring R is the least positive integer n such that $nx = 0$ for all x in R. If no such integer exists, we say that R has characteristic 0. The characteristic of R is denoted by char R.*

Theorem 2.2. *The characteristic of a field is 0 or a prime.*

Proof. Let F be a field with identity 1. If 1 has infinite order, then by definition there is no positive integer n such that $n \cdot 1 = 0$. Otherwise, suppose 1 has additive order n, then $n \cdot 1 = 0$, and n is the least positive integer with such property. So for any $x \in R$

$$\begin{aligned} n \cdot x &= (1 + 1 + \cdots + 1) \cdot x \quad (n \text{ summands}) \\ &= (n \cdot 1) \cdot x \\ &= 0x \\ &= 0 \end{aligned}$$

Thus, to show char F is a prime, it suffices to show that the additive order of 1 is finite and is a prime. Suppose 1 has additive order n and write $n = s \cdot t$. Then

$$0 = n \cdot 1 = (s \cdot t) \cdot 1 = (s \cdot 1) \cdot (t \cdot 1)$$

So either $s \cdot 1 = 0$ or $t \cdot 1 = 0$. But by assumption n is the least positive integer with $n \cdot 1 = 0$, so we must have either $s = n$ or $t = n$. Thus, n is a prime. □

Definition 2.3. *If K is a field containing a subfield F, then K is said to be an extension field (or simply an extension) of F, denoted as $K|F$.*

Definition 2.4. *A principal ideal domain is an integral domain R in which every ideal has the form $\langle a \rangle = \{ra \mid r \in R\}$ for some $a \in R$.*

Definition 2.5. *An integral domain D is a unique factorization domain if*

1. every nonzero element of D that is not a unit can be written as a product of irreducibles of D; and

2. the factorization into irreducibles is unique up to associates and the order in which the factors appear.

Theorem 2.6. *Let F be a field. Then F[x] is a principal ideal domain.*

Theorem 2.7. *Let F be a field and $p(x)$ be an irreducible polynomial. Then $\frac{F[x]}{<p(x)>}$ is a field.*

Theorem 2.8 (PID implies UFD)**.** *Every principal ideal domain is a unique factorization domain.*

Lemma 2.9. *Let p be a prime, then $x^{p^n} - y^{p^n} = (x-y)^{p^n}$.*

Proof. Proof by induction on n.

Base step: show $x^p - y^p = (x-y)^p$. Consider the following two cases:

Case 1: $p = 2$.

$$\begin{aligned}(x-y)^2 &= x^2 - 2xy + y^2 \\ &= x^2 + y^2 \\ &= x^2 - y^2 \ \ (\text{ since y}^2 = -\text{y}^2 \text{ in a field of characteristic 2})\end{aligned}$$

Case 2: $p \neq 2$ implies p is odd.

$$(x-y)^p = \sum_{k=1}^{p} \binom{p}{k} x^k y^{p-k} (-1)^k,$$

where $\binom{p}{k} = \frac{p!}{k!(p-k)!}$. Therefore, for $k \neq 1$ *or* p, the coefficient of each term is a multiple of p and thus 0 in a characteristic p field. Hence, $(x-y)^p = x^p - y^p$.

Induction step: Suppose the induction hypothesis is true for $n-1$. Then

$$(x-y)^{p^{n-1}} = x^{p^{n-1}} - y^{p^{n-1}}.$$

So

$$(x-y)^{p^n} = [(x-y)^{p^{n-1}}]^p = (x^{p^{n-1}} - y^{p^{n-1}})^p = x^{p^n} - y^{p^n}.$$

Therefore, by induction, we have shown that $x^{p^n} - y^{p^n} = (x-y)^{p^n}$.

□

Lemma 2.10. *Let F be a field of char $p > 0$ and $a \in F$. If $a \notin F^p$, then $x^{p^n} - a$ is irreducible in F[x] for every $n \geq 1$.*

Proof. Proof by contrapositive.

Suppose $x^{p^n} - a$ is reducible in F[x] for some $n \geq 1$, show that $a \in F^p$.

First, note that by 2.1, since F is a field and p is positive, then p must be a prime.

Let $p(x) = x^{p^n} - a = f(x)g(x)$ for some monic polynomials $f(x), g(x) \in F[x]$. Let E be an extension field of F containing a root α in $p(x)$. So $\alpha^{p^n} = a$. By 2.9, we have $x^{p^n} - a = x^{p^n} - \alpha^{p^n} = (x-\alpha)^{p^n}$. Since E is a field, by 2.6, we know that E[x] is a PID, and by 2.8, we have E[x] a UFD. Since $f(x)$ *and* $g(x)$ are monic, and by 2.5, we can write $f(x) = (x-\alpha)^r$, where $0 < r < p^n$.

Let $r = p^t s$, where s is a non-zero integer, $p \nmid s$ *and* $t < n$.

Thus, $f(x) = (x^{p^t} - \alpha^{p^t})^s = x^{p^t s} - s\alpha^{p^t} x^{p^t s - 1} +$ lower order terms. Since $f(x) \in F[x]$, so $-s\alpha^{p^t} \in F$. Hence, $\alpha^{p^t} \in F$, which implies $a = (\alpha^{p^t})^{p^{n-t}} \in F^{p^{n-t}} \subseteq F^p$. □

Lemma 2.11. *Let F be a field in which -1 is not a square, and every element of F(i) is a square in F(i), where $i^2 = -1$. Then any finite sum of squares in F is again a square* *in F and F has characteristic 0.*

Proof. This is enough to prove that the sum of two squares is a square. Let $a, b \in F$. Since every element in F(i) is a square, there exists $c, d \in F$ such that $a+bi = (c+di)^2$. Then $a + bi = c^2 - d^2 + 2cdi$. This implies that $a = c^2 - d^2, d = 2cd$. So

$$\begin{aligned} a^2 + b^2 &= (c^2 - d^2)^2 + 4c^2d^2 \\ &= c^4 - 2c^2d^2 + d^4 + 4c^2d^2 \\ &= c^4 + 2c^2d^2 + d^4 \\ &= (c^2 + d^2)^2 \end{aligned}$$

Therefore, we have shown that the sum of two square is again a square.
If $char\ F = p > 0$, then $-1 = \sum_{i=1}^{n} 1$, if $1 + 1 + 1 + \cdots + 1 = 0$ (p summands), then $1 + 1 + 1 + \cdots + 1 = -1$ ($p - 1$ summands). Since 1 is a square in F, then the sum -1 is a square in F, which is a contradiction. Therefore, $char\ F = 0$. □

Definition 2.12. *The degree(or relative degree or index) of a field extension $K|F$, denoted [K:F], is the dimension of K as a vector space of F. The extension is said to be finite if [K:F] is finite and is said to be infinite otherwise.*

Theorem 2.13. *Let $F \subset K \subset E$ be fields, then*

$$[E : F] = [E : K][K : F].$$

Proof. First, note that extension degrees are multiplicative, so if one side of the equation is infinite, then the other side is also infinite. Suppose $[E : F] < \infty$, then [F : K] $<$ ∞ and [E : K] $< \infty$.

Now, we can assume $[E:K]=m<\infty,[K:F]=n<\infty$. Since $[E:K]=m$, then E is a vector space of K with dimension m. So $\exists\ \{\beta_1,\beta_2,\cdots,\beta_m\}\subset E$ a basis of $E|K$. Similarly, $\exists\ \{\alpha_1,\alpha_2,\cdots,\alpha_n\}\subset K$ a basis of $K|F$. We claim that

$$\{\alpha_i\beta_j\ |1\leq i\leq n, 1\leq j\leq m\}\ \text{ is a basis of E|F.}$$

Suppose

$$\sum_{i,j}a_{ij}\alpha_i\beta_j=0 \text{ for some } a_{ij}\in F.$$

Then

$$\sum_{j=1}^{m}(\sum_{i=1}^{n}a_{ij}\alpha_i)\beta_j=0,\ \text{ and }\ \sum_{i=1}^{n}a_{ij}\alpha_i\in K.$$

Therefore, because $\{\beta_1,\beta_2,\cdots,\beta_m\}$ is a basis of $E|K$ and thus linearly independent, we have that

$$\sum_{i=1}^{n}a_{ij}\alpha_i=0,\forall j=1,2,\cdots,m.$$

Since $\{\alpha_1,\alpha_2,\cdots,\alpha_n\}$ is a basis of $K|F$, and so is linearly independent. Also, since $a_{ij}\in F$, we have that

$$a_{ij}=0,\forall i,j.$$

Therefore, $\{\alpha_i\beta_j\}$ is linearly independent over F.

Let $x\in E$. Since $\{\beta_1,\beta_2,\cdots,\beta_m\}$ is a basis of $E|K$, we can write $x=\sum_{j=1}^{n}\lambda_j\beta_j$ for some $\lambda_j\in K$. Since $\{\alpha_1,\alpha_2,\cdots,\alpha_n\}$ is a basis of $K|F$, $\lambda_j=\sum_{i=1}^{n}a_{ij}\alpha_i$ for some $a_{ij}\in F$. Thus, we can write

$$x=\sum_{j=1}^{m}(\sum_{i=1}^{n}a_{ij}\alpha_i)\beta_j=\sum_{i,j}a_{ij}\alpha_i\beta_j.$$

Hence,$\{\alpha_i\beta_j\}$ spans E. Therefore, $\{\alpha_i\beta_j\ |1\leq i\leq n,1\leq j\leq m\}$ is a basis of $E|F$. Thus, $[E:F]=mn=[E:K][K:F]$. □

Definition 2.14. *If a field K is generated by a single element α over F, $K = F(\alpha)$, then K is said to be a simple extension of F and the element α is called a primitive element for the extension.*

Definition 2.15. *The element $\alpha \in K$ is said to be algebraic over F if α is a root of some nonzero polynomial $f(x) \in F[x]$. If α is not algebraic over F (i.e., is not the root of any nonzero polynomial with coefficients in F) then α is said to be transcendental over F. The extension $K|F$ is said to be algebraic if every element of K is algebraic over F.*

Proposition 2.16. *If $K|F$ is a finite extension, then $K|F$ is algebraic.*

Proof. Suppose $[K : F] = n$, Let $\alpha \in K$, then $\{1, \alpha, \alpha^2, \cdots, \alpha^n\}$ are linearly dependent (since the dimension is n, but the set has n+1 elements). Therefore,

$$b_0 + b_1\alpha + \cdots + b_n\alpha^n = 0,$$

with $b_i's \in F$ not all 0. Thus, α is a root of the polynomial $b_0 + b_1x + \cdots + b_nx^n$. So α is algebraic over F. □

Definition 2.17. *Let α be algebraic over F. Then there is a unique monic irreducible polynomial $m_{\alpha,F}(x) \in F[x]$ which has α as a root, and this polynomial is called the minimal polynomial for α over F.*

Definition 2.18. *The extension field K of F is called a splitting field for the polynomial $f(x) \in F[x]$ factors completely into linear factors (or splits completely) in K[x] and f(x) does not factor completely into linear factors over any proper subfield of K containing F.*

Theorem 2.19. *Let $\phi : F \simeq F'$ be isomorphisms of fields. Let $f(x) \in F[x]$ be a polynomial and let $f'(x) \in F'[x]$ be the polynomial obtained by applying ϕ to the coefficients* *of $f(x)$. Let E be the splitting field for $f(x)$ over F and E' be the splitting field of $f'(x)$ over F'. Then there exists an extension of ϕ isomorphism $\sigma : E \longrightarrow E'$. In diagram*

$$\begin{array}{ccc} E & \xrightarrow{\exists\ \sigma} & E' \\ \Big| & & \Big| \\ F & \xrightarrow{\phi} & F' \end{array}$$

Proof. We will proceed the proof by induction on the degree n of $f(x)$.

Base step: n=1. Then $E = F, E' = F'$, so $\sigma = \phi$.

Induction step: Suppose the induction hypothesis is true for degree n-1. Let $p(x)$ be an irreducible factor of $f(x)$ in $F[x]$ of degree at least 2, and $p'(x)$ be the corresponding irreducible factor of $f'(x)$ in $F'[x]$ of degree at least 2.

If $\alpha \in E$ is a root to $p(x)$, and $\beta \in E'$ is a root to $p'(x)$, then we claim that $F(\alpha) \simeq \frac{F[x]}{<p(x)>}$ and $F(\beta) \simeq \frac{F'[x]}{<p'(x)>}$. Without loss of Generality, we will show $F(\alpha) \simeq \frac{F[x]}{<p(x)>}$.

First, note that $\frac{F[x]}{<p(x)>}$ is a field since $p(x)$ is irreducible, by 2.7. Let $\gamma : F[x] \longrightarrow F(\alpha)$. Since $p(\alpha) = 0$, then $p(x) \in ker(\gamma)$. Then by the First Theorem of Isomorphism, $\exists\ \psi : \frac{F[x]}{<p(x)>} \longrightarrow F(\alpha)$ a homomorphism. Furthermore, we know that $\frac{F[x]}{<p(x)>}$ is a field and $\psi \neq 0$, then ψ is an isomorphism since $F(\alpha)$ contains α and F implies that ψ is also surjective. Therefore, ψ is an isomorphism and $F(\alpha) \simeq \frac{F[x]}{<p(x)>}$.

Once we have $F(\alpha) \simeq \frac{F[x]}{<p(x)>}$ and $F(\beta) \simeq \frac{F'[x]}{<p'(x)>}$, since ϕ induces a natural isomorphism from $F[x]$ to $F'[x]$ which maps $< p(x) > \longrightarrow < p'(x) >$. Then we have the following diagram

$$\begin{array}{ccc} \frac{F[x]}{<p(x)>} & \xrightarrow{\phi} & \frac{F'[x]}{<p'(x)>} \\ \downarrow & & \downarrow \\ F(\alpha) & \xrightarrow{\exists\ \sigma'} & F(\beta) \end{array}$$

Therefore, $\exists$ an isomorphism σ' such that $\sigma' : F(\alpha) \simeq F(\beta)$.

Let $F_1 = F(\alpha)$ and $F_1' = F(\beta)$, so that we have the isomorphism $\sigma' : F_1 \longrightarrow F_1'$. By factoring, we have that $f(x) = (x - \alpha)f_1(x)$ over F_1 and similarly $f'(x) = (x - \beta)f_1'(x)$ over F_1', where $f_1(x), f_1'(x)$ have degree $n - 1$. Then E is a splitting field of $f_1(x)$ and similarly E' is also the splitting field of $f_1'(x)$. By induction hypothesis, there exists a map $\sigma : E \longrightarrow E'$ extending the isomorphism $\sigma' : F_1 \longrightarrow F_1'$. This gives the following diagram

$$\begin{array}{ccc} E & \xrightarrow{\ \sigma\ } & E' \\ | & & | \\ F_1 & \xrightarrow{\ \sigma'\ } & F_1' \\ | & & | \\ F & \xrightarrow{\ \phi\ } & F' \end{array}$$

□

Theorem 2.20 (Uniqueness of Splitting Fields). *Any two splitting fields for a polynomial $f(x) \in F[x]$ over a field F are isomorphic.*

Proof. By previous theorem, let F maps to itself and E, E' be two splitting fields of $f(x) \in F[x]$ will do the proof. □

Definition 2.21. *An algebraic extension $K|F$ is called a normal extension if an irreducible polynomial $f(x) \in F[x]$ has a root in K and f(x) splits completely over K.*

Lemma 2.22. *Let F be a field of characteristic not equal to 2. Let $K|F$ be a quadratic extension (i.e. $[K : F] = 2$). Then $K = F(\sqrt{a})$ for some $a \in F$, which is not a square in F.*

Proof. Since $[K : F] = 2$, by 2.16, the field extension $K|F$ is algebraic. So we can let $\alpha \in K \backslash F$. By 2.17, $\exists$ a minimal polynomial $m_\alpha(x) = x^2 + bx + c$. Using quadratic formula, we obtain the roots $\alpha = \frac{-b \pm \sqrt{b^2-4c}}{2}$. Note that $b^2 - 4c$ is not a square in F since $\alpha \notin F$. Then $\alpha \in F(\sqrt{b^2-4c})$ implies that $F(\alpha) \subseteq F(\sqrt{b^2-4c})$. Also, since $\sqrt{b^2-4c} = \mp(b+2\alpha)$, we have that $F(\sqrt{b^2-4c}) \subseteq F(\alpha)$. Therefore, $F(\alpha) = F(\sqrt{b^2-4c})$. Choose $a = b^2 - 4c$, we reach the conclusion. □

Definition 2.23. *A field K is called algebraically closed if every non-constant polynomial with coefficients in K has a root in K.*

Definition 2.24. *A polynomial over F is called separable if it does not have multiple roots (i.e. all its roots are distinct). A polynomial which is not separable is called inseparable.*

Definition 2.25. *A root α of $f(x) \in K$ is called a simple root if $(x-\alpha)^2 \nmid f(x)$.*

Lemma 2.26. *Let K be a field and let $f(x) \in K[x], \alpha \in K$, then α is a simple root of f(x) if and only if $f(\alpha) = 0$, $f'(\alpha) \neq 0$.*

Proof. Let $\alpha \in K$ *and* $f(\alpha) = 0$.
Suppose α is not a simple root of f(x), then $(x-\alpha)^2 | f(x)$, so we can write

$$f(x) = (x-\alpha)^2 g(x),$$

for some $g(x) \in F[x]$, Thus, $f'(x) = (x-\alpha)^2 g'(x) + 2(x-\alpha)g(x)$. so $f'(\alpha) = 0$.
Conversely, suppose α is a root of both $f(x)$ and $f'(x)$. Then we can write

$$f(x) = (x-\alpha)h(x),$$

for some $h(x) \in F[x]$. Take the derivative of f(x):

$$f'(x) = h(x) + (x-\alpha)h'(x).$$

Since α is a root to $f'(x)$, the equation above implies that $h(\alpha) = 0$, so we can write

$$h(x) = (x - \alpha)k(x),$$

for some $k(x) \in F[x]$. Thus,

$$f(x) = (x - \alpha)^2 k(x).$$

Therefore, α is not a simple root of f(x). □

Definition 2.27. *A field K of characteristic p is called perfect if every element of K is a p^{th} power in K, i.e. $K = K^p$. Any field of characteristic 0 is also called perfect.*

Proposition 2.28. *Every finite extension of a perfect field is separable.*

Proof. Let F be a finite field of *char* $p > 0$, and E be a finite extension of F with $[E : F] = n$. Then $|E| = p^n$. Therefore, $|E^*| = p^n - 1$, which implies that E is cyclic. Thus, for any nonzero element $\alpha \in E$, we have

$$\alpha^{p^n-1} = 1, \longrightarrow \alpha^{p^n} = \alpha, \longrightarrow \alpha^{p^n} - \alpha = 0.$$

Hence, any element in E is a root of the polynomial $f(x) = x^{p^n} - x$. Finally, to show f(x) is separable, note that $f'(x) = p^n x^{p^n-1} - 1 = -1 \neq 0$ (i.e. By 2.26 $f'(x)$ has no roots at all so it has no multiple roots). Therefore, $E|F$ is separable. □

2.2 Galois Theory

Definition 2.29. *Let K/F be a field extension. Let Aut(K/F) denote the set of all F-automorphisms of K, that is, $Aut(K|F) = \{\phi \in Aut(K) : \phi_{|F} = id_F\}$. Then $Aut(K|F)$ is called the automorphism group of $K|F$ or the Galois group of $K|F$.*

Definition 2.30. *A finite extension $K|F$ is called Galois if it is normal and separable.*

Definition 2.31. *If H is a subgroup of the automorphism group of K, the subfield of K fixed by all elements of H is called the fixed field of H, and is denoted as K^H (i.e. $K^H = \{\alpha \in K | \sigma(\alpha) = \alpha\}$).*

Proposition 2.32. *Let $K|F$ be a finite extension, then there exists a finite normal extension $N|F$ such that $K \subset N$.*

Proof. Since $K|F$ is finite, we can write $K = F(\alpha_1, \alpha_2, \cdots, \alpha_n)$ for some $\alpha_i \in K$. Let $f_i(x) \in F[x]$ be the minimal polynomial of α_i over F. Let $f(x) = \prod_{i=1}^{n} f_i(x) \in F[x]$ and let $N|F$ be the splitting field of this $f(x)$ over F. Then $\alpha_1, \cdots, \alpha_n \in N$ implies that $K = F(\alpha_1, \cdots, \alpha_n) \subset N$, and $N|F$ is the normal extension. □

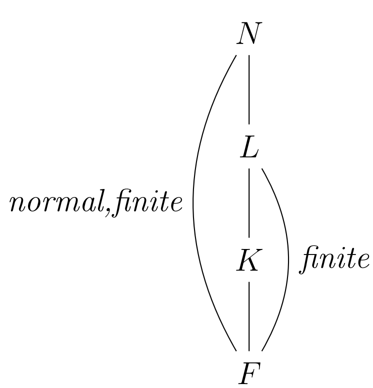
Theorem 2.33. *Let $N|F$ be a finite and normal extension, and let $L|F$ be a finite extension, then*

$$|\{\tau : K \longrightarrow N \mid \tau_F = id\}| * |\{\sigma : L \longrightarrow N \mid \sigma_K = id\}| = |\{\psi : L \longrightarrow N \mid \psi_F = id\}|,$$

denoted as the following:

$$|Hom_F(K, N)| * |Hom_K(L, N)| = |Hom_F(L, N)|.$$

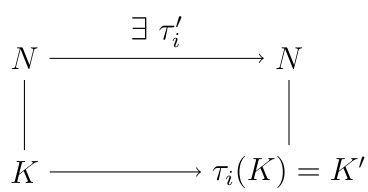
We can see more clearly by a diagram:



Proof. Since $N|F$ is finite, we have that $N|L$ and $N|K$ both finite. Then we suppose $Hom_F(K,N) = \{\tau_1, \cdots, \tau_m\}$ for some $m \in \mathbb{N}$, and $Hom_K(L,N) = \{\sigma_1, \cdots, \sigma_n\}$ for some $n \in \mathbb{N}$. We divide the proof into two parts.

Part (1): We show that $\forall\ \tau_i : K \longrightarrow N$ an F-isomorphism, $\exists\ \tau_i' : N \longrightarrow N$ such that $\tau_i'|_K = \tau_i$.

Since $N|F$ is normal and finite, then by 2.21, $N|F$ is a splitting field of some $f(x) \in F[x]$. Since $F \subset K \subset N$, the same polynomial $f(x)$ is also in $K[x]$, which implies that $N|K$ is a splitting field of $f(x)$. Then we denote $K' = \tau_i(K)$. Since $\tau_i(f(x)) = f(x)$ (note that τ_i is an F-isomorphism), then $N|K'$ is also a splitting field of $f(x)$. By uniqueness of splitting fields, $\exists\ \tau_i' : N \longrightarrow N$ an isomorphism such that $\tau_i'|_K = \tau_i$.



Part (2): Let $\phi : Hom_F(K,N) * Hom_K(L,N) \longrightarrow Hom_F(L,N)$ be a map such that $\phi(\tau_i, \sigma_j) = \tau_i' \circ \sigma_j$ for some $\tau_i \in Hom_F(K,N)$ and some $\sigma_j \in Hom_K(L,N)$. We claim that ϕ is a bijection.

First, we show that ϕ is one to one. Suppose $\phi(\tau_i, \sigma_j) = \phi(\tau_s, \sigma_k)$ for some $\tau_i, \tau_s \in Hom_F(K,N)$ and some $\sigma_j, \sigma_k \in Hom_K(L,N)$, then $\tau_i' \circ \sigma_j = \tau_s' \circ \sigma_k$. Let $\alpha \in K$, then since $\sigma_j|_K = \sigma_k|_K = id$, we have that $\sigma_j(\alpha) = \sigma_k(\alpha) = \alpha$. Then

$$\begin{aligned}
&\Rightarrow \quad (\tau'_i \circ \sigma_j)(\alpha) = (\tau'_s \circ \sigma_k)(\alpha) && (1)\\
&\Rightarrow \quad \tau'_i(\alpha) = \tau'_s(\alpha) && (2)\\
&\Rightarrow \quad \tau_i(\alpha) = \tau_s(\alpha) \text{ (since } \tau'_i|_K = \tau_i) && (3)\\
&\Rightarrow \quad \tau_i = \tau_s && (4)\\
&\Rightarrow \quad \tau'_i = \tau'_s && (5)\\
&\Rightarrow \quad \sigma_j = \sigma_k && (6)
\end{aligned}$$

Therefore, ϕ is one to one.

Now we will show that ϕ is surjective. Let $\theta \in Hom_F(L, N)$, then $\theta|_K \in Hom_F(K, N)$. This implies that $\theta|_K = \tau_i$ for some i. Consider the element $\tau_i'^{-1} \circ \theta$, then we will show that this element is in $Hom_K(L, N)$. Let $\alpha \in K$, then

$$(\tau_i'^{-1} \circ \theta)(\alpha) = \tau_i'^{-1}(\theta(\alpha)) = \tau_i'^{-1}(\tau_i(\alpha)) = \alpha.$$

Therefore, $\tau_i'^{-1} \circ \theta$ fixes any element in K, and thus $\tau_i'^{-1} \circ \theta \in Hom_K(L, N)$. So

$$\tau_i'^{-1} \circ \theta = \sigma_j \text{ for some } \sigma_j \in Hom_K(L, N).$$

Hence, ϕ is bijective and therefore

$$|Hom_F(K, N)| * |Hom_K(L, N)| = |Hom_F(L, N)|.$$

□

Theorem 2.34. *Let $\phi : F \simeq F'$ be isomorphisms of fields. Let E be a splitting field of $f(x)$ over F and E' be a splitting field of $f'(x) = \phi(f(x))$. Then the number of* *extensions satisfying that there exists an isomorphism between E and E' is at most* $[E : F]$*, with equality if* $f(x)$ *is separable over* F*.*

Proof. By 2.19, we know that $\exists$ an isomorphism $\sigma : E \longrightarrow E'$. So we will proceed by induction on $n = [E : F]$.

Base step: $n = 1$. Then $E = F, E' = F', \sigma = \phi$ and the number of such extension is 1.

Induction step: Suppose $n > 1$, then $f(x)$ has at least an irreducible factor $p(x)$ with degree at least 1 with corresponding irreducible factor $p'(x) \in F'[x]$ with degree at least 1. Then, by the proof of 2.19, if α is a root to $p(x)$, then $\exists\ \tau : F(\alpha) \simeq F'(\beta)$ with $\tau(\alpha) = \beta$. Then we have a diagram

$$\begin{array}{ccc} E & \xrightarrow{\sigma} & E' \\ | & & | \\ F(\alpha) & \xrightarrow{\tau} & F'(\beta) \\ | & & | \\ F & \xrightarrow{\phi} & F' \end{array}$$

Then we need only to count the number of such diagrams. The number of ϕ to τ is equal to the number of distinct roots of $p(x)$. Thus, since $deg(p(x)) = deg(p'(x)) = [F(\alpha) : F]$, we see that the number of such extensions is at most $[F(\alpha) : F]$, with equality if $p(x)$ is separable.

Since E is a splitting field of $f(x)$ over $F(\alpha)$ and E' is also a splitting field of $f'(x)$ over $F'(\beta)$. Then $[E : F(\alpha)] < [E : F]$ and by induction hypothesis, the number of such extensions is $\leq [E : F(\alpha)]$, with equality if $f(x)$ has distinct roots. Since $[E : F] = [E : F(\alpha)][F(\alpha) : F]$, the number of such extensions is $\leq [E : F]$, with equality if $f(x)$ has distinct roots. $\square$

Theorem 2.35. *Let* $K|F$ *be a finite extension and* $N|F$ *be a normal finite extension with* $K \subset N$*, then* $|Hom_F(K, N)| = [K : F]$ *if and only if* $K|F$ *is separable.*

Proof. $\Longleftarrow$ Suppose $K|F$ is separable. We will proceed by induction on the degree $[K : F] = n$.

Base step: n=1. Then $K = F$ and $|Hom_F(F, N)| = 1 = [F : F]$.

Induction step: Suppose the hypothesis is true for $n - 1$, where $n \geq 2$. Let $\alpha \in K \backslash F$. Let $m_\alpha(x)$ be the minimal polynomial of α over F. Since $N|F$ is normal and finite, by 2.16, $N|F$ is also algebraic. Thus, since $\alpha \in K \subset N$, $m_\alpha(x) \in F[x]$ is irreducible and $m_\alpha(\alpha) = 0$, we have that $m_\alpha(x)$ splits completely in N. Therefore, by 2.34,

$$|Hom_F(F(\alpha), N)| = \text{ number of distinct roots in } m_\alpha(x)$$

Since α is separable over F, then

$$\text{number of distinct roots in } m_\alpha(x) = deg(m_\alpha(x)) = [F(\alpha) : F].$$

Therefore, $|Hom_F(F(\alpha), N)| = [F(\alpha) : F]$. Since $[K : F(\alpha)] \leq [K : F]$, by induction hypothesis,

$$|Hom_{F(\alpha)}(K, N)| = [K : F(\alpha)].$$

By 2.33,

$$|Hom_F(F(\alpha), N)| * |Hom_{F(\alpha)}(K, N)| = |Hom_F(K, N)|.$$

Since $|Hom_F(F(\alpha), N)| = [F(\alpha) : F]$, and $|Hom_{F(\alpha)}(K, N)| = [K : F(\alpha)]$. then

$$[F(\alpha) : F] * [K : F(\alpha)] = |Hom_F(K, N)| = [K : F].$$

$\Longrightarrow$ Now suppose $|Hom_F(K,N)| = [K:F]$. Let $\alpha \in K \backslash F$. By 2.33,

$$|Hom_F(F(\alpha),N)| * |Hom_{F(\alpha)}(K,N)| = |Hom_F(K,N)| = [K:F].$$

Then

$$[K:F] = |Hom_F(F(\alpha),N)| * |Hom_{F(\alpha)}(K,N)| \leq [F(\alpha):F][K:F(\alpha)] = [K:F].$$

Therefore, α is separable over F. $\square$

Theorem 2.36 (Existence of Primitive Element). *Let $K|F$ be a finite separable extension. Then $K = F(\alpha)$ for some $\alpha \in K$.*

Proof. Let $N|K$ be an extension such that $N|F$ is normal (We know such extension exists from 2.32). Since $K|F$ is separable, by 2.35, we have that $|Hom_F(K,N) = [K:F]$.

Let $[K:F] = n$ and $Hom_F(K,N) = \{\sigma_1, \cdots, \sigma_n\}$ such that $\sigma_i \neq \sigma_j$ for $i \neq j$. Let $V_{ij} = \{\alpha \in K \ |\sigma_i(\alpha) = \sigma_j(\alpha)\}$. Since $\sigma_i \neq \sigma_j$, then V_{ij} is a proper subset of K. Since F is infinite,

$$\cup_{i \neq j} V_{ij} \subset K.$$

Let $\alpha \in K \backslash \cup_{i \neq j} V_{ij}$. We claim that $K = F(\alpha)$.

Let $m_\alpha(x)$ be the minimal polynomial of α over F, then $m_\alpha(\sigma_i(\alpha)) = 0, \forall\, i \in \{1, 2, \cdots, n\}$ (since σ_i only permutes the roots of $m_\alpha(x)$). Since $\alpha \notin \cup_{i \neq j} V_{ij}$, then $\forall i \neq j,\ \sigma_i(\alpha) \neq \sigma_j(\alpha)$. Therefore, $m_\alpha(x)$ has distinct roots, this implies that $deg(m_\alpha(x)) \geq n$. But $[K:F] = n$ and $\alpha \in K$ implies that $deg(m_\alpha(x)) \leq n$. Hence, $deg(m_\alpha(x)) = n$. So $[K:F] = [F(\alpha):F]$, and thus $K = F(\alpha)$.

$\square$

Theorem 2.37. *If $G \subseteq Aut(K)$ is a finite subgroup, then $K|K^G$ is a Galois extension with $Gal(K|K^G) = G$*

Proof. First note that by 2.31, $K^G = \{\alpha \in K \ |\sigma(\alpha) = \alpha \ , \ \forall \sigma \in G\}$. Let $G = \{\sigma_1 = id, \sigma_2, \cdots, \sigma_n\}$ and denote $F = K^G \subseteq K$. Let $\alpha \in K$. We can write

$$\{\sigma_i(\alpha) \ |\sigma_i \in G\} = \{\beta_1 = \alpha, \cdots, \beta_d\},$$

where $\beta_i \neq \beta_j, \ \forall i \neq j$. Let $f(x) = \prod_{i=1}^{d}(x - \beta_j) \in K[x]$. We claim that $f(x) \in F[x]$ and is irreducible.

Let $\tau \in G$, then the elements $\{\tau, \tau\sigma, \cdots, \tau\sigma_n\}$ are the same elements of $\{\sigma_1, \sigma_2, \cdots, \sigma_n\}$. Then it follows that applying τ to $\{\beta_1, \cdots, \beta_d\}$ simply permutes them. Therefore, $f(x)$ has coefficients which are fixed by all elements in G, so they are all in $K^G = F$. Hence, $f(x) \in F[x]$. Since $f(\alpha) = 0$, then α is algebraic over F. So we can let $m_\alpha(x)$ be the minimal polynomial of α over F. Therefore, $m_\alpha(\alpha) = m_\alpha(\sigma(\alpha)) = m_\alpha(\beta_i) = 0, \ \forall \ 1 \leq i \leq d$. Thus, $deg(m_\alpha(x)) \geq d$, implying that $m_\alpha(x) = f(x)$. Therefore, $f(x)$ is irreducible in $F[x]$. Since $f(x) = \prod_{i=1}^{d}(x - \beta_j)$ where $\beta_i \neq \beta_j, \ \forall \ i \neq j$. Hence, α is separable over F.

Now let $g(x) \in F[x]$ be an irreducible polynomial, and suppose $\exists \ \alpha \in K$ such that $g(\alpha) = 0$, then $g(x) = \lambda f(x) = \lambda \prod_{i=1}^{d}(x - \beta_j)$. Therefore, $f(x)$ splits completely in $K[x]$. By 2.21, $K|F$ is normal.

We know that for all $\beta \in K$, $deg_F\beta \leq |G|$. Let $\alpha \in K$ be such that $deg_F\alpha$ is maximal among all $deg_F\beta, \ \forall \ \beta \in K$. We claim that $K = F(\alpha)$.

Suppose in contrary that $K \neq F(\alpha)$, then $\exists \ \beta \in K \backslash F(\alpha)$. Hence, $F(\alpha, \beta)$ is a finite separable extension. By 2.36,

$$F(\alpha, \beta) = F(\gamma)$$

for some $\gamma \in K$. By 2.13,

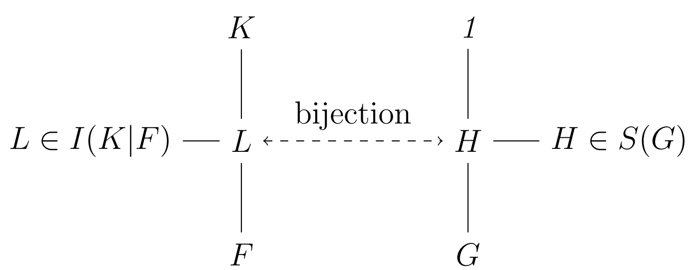
$$deg_F\gamma = [F(\gamma) : F] = [F(\alpha, \beta) : F(\alpha)][F(\alpha) : F].$$

Since $\beta \notin F(\alpha)$, we have that $[F(\alpha, \beta) : F(\alpha)] \geq 2$. Therefore, $deg_F\gamma \geq 2[F(\alpha) : F] = 2deg_F(\alpha) > deg_F(\alpha)$, which is a contradiction to $deg_F\alpha$ being maximal. So $K = F(\alpha)$. Hence, we have that $K|F$ is finite, separable and normal, so is Galois. Moreover, $[K : F] = [F(\alpha) : F] = deg_F\alpha \leq |G|$. Since $G \subseteq Gal(K|F)$, we have that

$$|G| \leq |Gal(K|F)| = [K : F] \leq |G|.$$

Therefore, $|G| = |Gal(K|F)|$, then $G = Gal(K|F)$. □

Theorem 2.38 (Fundamental Theorem of Galois Theory)**.** *Let $K|F$ be a Galois extension, and let $G = Gal(K|F)$. Define $S(G) =$ set of subgroups of G, and $I(K|F) =$ set of intermediate fields (i.e. $I(K|F) = \{L \mid F \subset L \subset K\}$). Then there is a bijection*



given by the correspondence

$$L \Longrightarrow \{\textit{the elements of } G \textit{ fixing } L\}$$

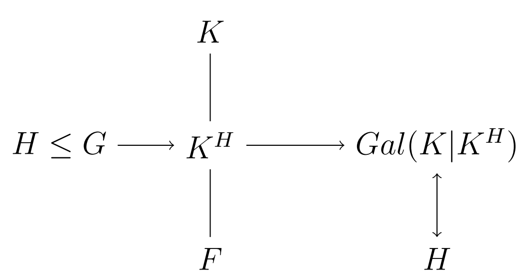
$$K^H \Longleftarrow H$$

which are inverses of each other. Under this correspondence:

(1) (inclusion reversing) If L_1, L_2 corresponds to H_1, H_2, respectively, then $L_1 \subset L_2$ if and only if $H_2 \leq H_1$.
(2)Let $H \in S(G)$, then H is a normal group in G if and only if $K^H|F$ is Galois (normal).
(3) If $F \subset L \subset K$ and $L|F$ is normal, then the natural map $Gal(K|F) \longrightarrow Gal(L|F)$ is onto with kernel $Gal(K|L)$.

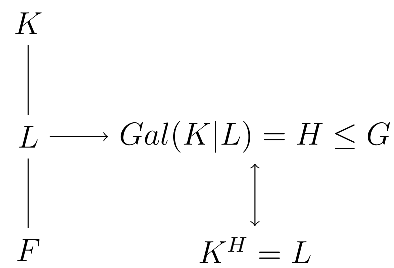
$$\frac{Gal(K|F)}{Gal(K|L)} \cong Gal(L|F).$$

Proof. For the purpose of this thesis, we will only prove that there is a bijection between the subfields L of K containing F and the subgroups H of G.
Define the map $\phi : S(G) \longrightarrow I(K|F)$ and $\psi : I(K|F) \longrightarrow S(G)$. To show there is a bijection, it is enough to show that $\phi \circ \psi = id_{I(K|F)}$ and $\psi \circ \phi = \mathrm{id}_{\mathrm{S(G)}}$. Since $H \leq G$, we have that $H \leq G = Gal(K|F) \subset Aut(K)$. Therefore, by 2.37, we have that $K|K^H$ is Galois with $H = Gal(K|K^H)$. Displaying an explicit diagram below,



hence, $\psi(\phi(H)) = H \Rightarrow \psi \circ \phi = id_{S(G)}$.
Now let $L \in I(K|F)$, then $F \subset L \subset K$. By 2.30, K is a splitting field of the separable polynomial $f(x) \in F[x]$, then we may also view $f(x)$ as an element of $L(x)$. Then K is also a splitting field of $f(x)$ over L, and thus the extension $K|L$ is also Galois. Let $H = Gal(K|L)$, then $L \subset K^H \subset K$ and $[K : L] = |H|$. Thus, $K|K^H$

is Galois with $Gal(K|K^H) = H$. So $[K : K^H] = |H| = [K : L]$, which implies $[K^H : L] = 1$, and thus $K^H = L$. Displaying an explicit diagram below,



therefore, $\phi \circ \psi = id_{I(K|F)}$.

□

Definition 2.39. *The extension $K|F$ is said to be cyclic if it is Galois with a cyclic Galois group.*

3 The Artin-Schreier Extension

Definition 3.1. *Let $K|F$ be a Galois extension and let $\alpha \in K$, define the trace of α from K to F to be $Tr_{K|F}(\alpha) = \sum_{\sigma \in Gal(K|F)} \sigma(\alpha)$.*

Lemma 3.2. *$Tr : K \longrightarrow F$ is an F-linear map.*

Proof. To show $Tr : K \longrightarrow F$ is an F-linear map, we need to show its additive and scalar multiplicative properties hold.

Let $\alpha, \beta \in K$, then we need to show that $Tr_{K|F}(\alpha+\beta) = Tr_{K|F}(\alpha) + Tr_{K|F}(\beta)$

$$\begin{aligned} Tr_{K|F}(\alpha+\beta) &= \sum_{\sigma \in Gal(K|F)} \sigma(\alpha+\beta) \\ &= \sum_{\sigma \in Gal(K|F)} (\sigma(\alpha)+\sigma(\beta)) \\ &= \sum_{\sigma \in Gal(K|F)} \sigma(\alpha) + \sum_{\sigma \in Gal(K|F)} \sigma(\beta) \\ &= Tr_{K|F}(\alpha) + Tr_{K|F}(\beta) \end{aligned}$$

Let $a \in F$, then

$$\begin{aligned} Tr_{K|F}(a\alpha) &= \sum_{\sigma \in Gal(K|F)} \sigma(a\alpha) \\ &= a \sum_{\sigma \in Gal(K|F)} \sigma(\alpha) \\ &= aTr_{K|F}(\alpha) \end{aligned}$$

□

Definition 3.3. *A character χ of a group G with values in a field L is a homomorphism from G to the multiplicative group of L:*

$$\chi : G \to L^x$$

i.e., $\chi(g_1g_2) = \chi(g_1)\chi(g_2)$ for all $g_1, g_2 \in G$ and $\chi(g)$ is a nonzero element of L for all $g \in G$.

Theorem 3.4 (Dedekind Theorem)**.** *Let $\chi_1, \chi_2, \cdots, \chi_n$ be distinct characters of a group G with values in a field L. If $a_1\chi_1 + a_2\chi_2 + \cdots + a_n\chi_n = 0$, where $a_1, a_2, \cdots a_n \in L$, then $a_i = 0$ for all i.*

Proof. We will prove by induction on n.

Base step: $n = 1$, then $a_1 = 0$. The statement is true.

Induction step: Suppose the theorem is true for $n-1$, $n \geq 2$. Suppose $a_1\chi_1 + a_2\chi_2 + \cdots + a_n\chi_n = 0$, Since $\chi_i's$ are all distinct. So $\exists\, g_0 \in G$ such that $\chi_1(g_0) \neq \chi_n(g_0)$. Then

$$a_1\chi_1(g) + a_2\chi_2(g) + \cdots + a_n\chi_n(g) = 0 \tag{7}$$

Multiply g by g_0, we have:

$$a_1\chi_1(gg_0) + a_2\chi_2(gg_0) + \cdots + a_n\chi_n(gg_0) = 0 \tag{8}$$

Since χ_i is a homomorphism, we have:

$$a_1\chi_1(g_0)\chi_1(g) + a_2\chi_2(g_0)\chi_2(g) + \cdots + a_n\chi_n(g_0)\chi_n(g) = 0 \tag{9}$$

Multiply equation (7) by $\chi_1(g_0)$ on the left, we have:

$$a_1\chi_1(g_0)\chi_1(g) + a_2\chi_1(g_0)\chi_2(g) + \cdots + a_n\chi_1(g_0)\chi_n(g) = 0 \tag{10}$$

Then equation (9) - equation (10), we have:

$$a_2\chi_2(g)(\chi_2(g_0) - \chi_1(g_0)) + \cdots + a_n\chi_n(g)(\chi_n(g_0) - \chi_1(g_0)) = 0 \tag{11}$$

By induction hypothesis, $a_2(\chi_2(g_0) - \chi_1(g_0)), \cdots, a_n(\chi_n(g_0) - \chi_1(g_0)) = 0$. Since $\chi_1(g_0) - \chi_n(g_0) \neq 0$, then $a_n = 0$. Therefore,

$$a_1\chi_1 + a_2\chi_2 + \cdots + a_{n-1}\chi_{n-1} = 0$$

By induction hypothesis, $a_1 = a_2 = \cdots = a_{n-1} = a_n = 0$. □

Theorem 3.5 (Additive Hilbert's Theorem 90)**.** *Let $K|F$ be a cyclic extension of degree n with Galois group $G = Gal(K|F) =< \sigma >$. Then for $\beta \in K$*

$$Tr(\beta) = 0 \ if \ and \ only \ if \ \beta = \alpha - \sigma(\alpha) \ for \ some \ \alpha \in K.$$

Proof. $\Longleftarrow$ Let $\beta = \alpha - \sigma(\alpha)$, then

$$\begin{aligned} Tr(\beta) &= Tr(\alpha - \sigma(\alpha)) \\ &= Tr(\alpha) - Tr(\sigma(\alpha)) \\ &= \sum_{\sigma} \sigma(\alpha) - \sum_{\sigma} \sigma(\sigma(\alpha)) \\ &= 0 \end{aligned}$$

$\Longrightarrow$ Let $Tr(\beta) = 0$. By Dedekind Theorem, $Tr : K \longrightarrow F$ is a nonzero map, since $Tr = id + \sigma + \sigma^2 + \cdots + \sigma^{n-1}$ is nonzero (with coefficients of each term being 1). Therefore, $\exists \ \theta \in K^*$ such that $Tr(\theta) \neq 0$.

Consider the function

$$\chi = \beta + (\beta + \sigma(\beta))\sigma + \cdots + (\sum_{i=0}^{n-2} \sigma^i(\beta))\sigma^{n-2}$$

Let $\alpha = \frac{\chi(\theta)}{Tr(\theta)}$, then

$$\alpha = \frac{1}{Tr(\theta)}(\beta\theta + (\beta + \sigma(\beta))\sigma(\theta) + \cdots + (\beta + \sigma(\beta) + \cdots + \sigma^{n-2}(\beta))\sigma^{n-2}(\theta) \qquad (12)$$

$$\sigma(\alpha) = \frac{1}{\sigma(Tr(\theta))}(\sigma(\beta)\sigma(\theta) + (\sigma(\beta) + \sigma^2(\beta))\sigma^2(\theta) + \cdots + (\sigma(\beta) + \sigma^2(\beta) + \cdots + \sigma^{n-1}(\beta))\sigma^{n-1}(\theta) \qquad (13)$$

Note that $\frac{1}{\sigma(Tr(\theta))} = \frac{1}{Tr(\theta)}$. Then equation (12) - equation (13), we have

$$\begin{aligned}
\alpha - \sigma(\alpha) &= \frac{1}{Tr(\theta)}[\beta\theta + \beta\sigma(\theta) + \cdots + \beta\sigma^{n-2}(\theta) - (\beta + \cdots + \sigma^{n-1}(\beta))\sigma^{n-1}(\theta) + \beta\sigma^{n-1}(\theta)] \\
&= \frac{1}{Tr(\theta)}(\beta\theta + \beta\sigma(\theta) + \cdots + \beta\sigma^{n-1}(\theta) - Tr(\beta)\sigma^{n-1}(\theta)) \\
&= \frac{1}{Tr(\theta)}\beta Tr(\theta) \text{ (since Tr}(\beta) = 0 \text{ by assumption)} \\
&= \beta
\end{aligned}$$

□

Lemma 3.6. *Let $K|F$ be a cyclic extension of degree n. Let $\alpha \in K$, and $m_\alpha(x) = x^d + \alpha_{d-1}x^{d-1} + \cdots + \alpha_1 x + \alpha_0$ be the minimal polynomial of α over F, then $Tr_{K|F}(\alpha) = -\frac{n}{d}a_{d-1}$.*

Proof. Let $G = Gal(K|F) =< \sigma >$, consider:

$$\begin{aligned}
\prod_{\sigma \in G}(x - \sigma(\alpha)) &= x^n - (\sum_{\sigma \in G}\sigma(\alpha))x^{n-1} + \cdots \\
&= x^n - Tr_{K|F}(\alpha) + \cdots
\end{aligned}$$

Also,

$$\begin{aligned}
m_\alpha(x)^{\frac{n}{d}} &= (x^d + \alpha_{d-1}x^{d-1} + \cdots + \alpha_1 x + \alpha_0)^{\frac{n}{d}} \\
&= x^n + \frac{n}{d}a_{d-1}x^{n-1} + \cdots
\end{aligned}$$

Since we know that $\prod_{\sigma \in G}(x - \sigma(\alpha)) = m_\alpha(x)^{\frac{n}{d}}$, by equating the coefficients of x^{n-1}, we have that $Tr_{K|F}(\alpha) = -\frac{n}{d}a_{d-1}$. □

Theorem 3.7 (Artin-Schreier Extension)**.** *Let F be a field with characteristic $p > 0$ and let K be a cyclic extension of F of degree p. Then $K = F(\alpha)$, where α is a root of the polynomial $x^p - x - a$ for some $a \in F$.*

Proof. Let $K|F$ be a cyclic extension of degree p, and let $G = Gal(K|F) =< \sigma >$ for some $\sigma \in G^*$. Then by 3.6, $Tr(-1) = -p(1) = 0$ since $char\ F = p$. From Additive Hilbert's Theorem 90, we have that $-1 = \alpha - \sigma(\alpha)$, so $\sigma(\alpha) = \alpha + 1$. Moreover, $\sigma^2(\alpha) = \sigma(\sigma(\alpha)) = \sigma(\alpha + 1) = \alpha + 2$. Hence, generally we have that $\sigma^i = \alpha + i$, for $i = 1, 2, \cdots, p$. Since $char\ F = p$, the elements $\alpha, \alpha + 1, \cdots, \alpha + p - 1$ are all distinct conjugates. Hence, $[F(\alpha) : F] = p = [K : F]$. So $K = F(\alpha)$. Furthermore, consider the element $\alpha^p - \alpha \in K$,

$$\sigma(\alpha^p - \alpha) = \sigma^p(\alpha) - \sigma(\alpha) = (\alpha + 1)^p - \alpha - 1 = \alpha^p + 1 - \alpha - 1 = \alpha^p - \alpha.$$

Thus, the element $\alpha^p - \alpha$ is fixed by σ, which implies that $\alpha^p - \alpha \in F$. Hence, let $a = \alpha^p - \alpha \in F$, then α is a root to the polynomial $x^p - x - a$.

□

4 Kummer Theory

Theorem 4.1 (Kummer)**.** *Let $K|F$ be a cyclic field extension of degree n, where char F does not divide n and F contains the n^{th} roots of unity, then $K = F(\sqrt[n]{a})$, for some $a \in F$.*

Proof. Let $K|F$ be a cyclic field extension of degree n and let $\rho \in F$ be the n^{th} root of unity. Since $(n, char(F)) = 1$, the elements $1, \rho, \rho^2, \cdots, \rho^{n-1}$ are all distinct. Suppose $G = Gal(K|F) =< \sigma >$, for some $\sigma \in G$, then $|G| = n$. Then for any $\sigma^i \in G,\ i \in \{1, 2, \cdots, n\}$, $\sigma^i : K^* \longrightarrow K^*$ is a homomorphism since $\sigma^i : K \longrightarrow K$ is a field automorphism. Therefore, σ^i is a characteristic of K^* with values in K. Hence, $\{id, \sigma, \cdots, \sigma^{n-1}\}$ are distinct characters of K^* with values in K. By Dedekind Theorem,

$$1 \cdot id + \rho\sigma + \cdots + \rho^{n-1}\sigma^{n-1} \neq 0.$$

Hence, $\exists\, \theta \in K^*$ such that $\theta + \rho\sigma(\theta) + \cdots + \rho^{n-1}\sigma^{n-1}(\theta) \neq 0$. Let $\beta = \theta + \rho\sigma(\theta) + \cdots + \rho^{n-1}\sigma^{n-1}(\theta)$. Then,

$$\sigma(\beta) = \sigma(\theta) + \rho\sigma^2(\theta) + \cdots + \rho^{n-1}\sigma^{n-1}(\theta). \tag{14}$$

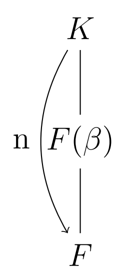
Multiply equation(14) by ρ, we have:

$$\rho\sigma(\beta) = \rho\sigma(\beta) + \rho^2\sigma^2(\theta) + \cdots + \rho^{n-1}\sigma^{n-1}(\theta) + \rho^n\sigma^n(\theta). \tag{15}$$

Since $\rho^n = 1$, and $\sigma^n = id$, then $\rho^n\sigma^n(\theta) = \theta$. Therefore, we have that $\rho\sigma(\beta) = \beta$. Hence, $\sigma(\beta) = \rho^{-1}\beta$. Then

$$\sigma(\beta^n) = \sigma^n(\beta) = \rho^{-n}\beta^n = \beta^n.$$

Therefore, $\sigma^i(\beta^n) = \beta^n$, for all $i \in \{1, 2, \cdots, n\}$, which implies that β^n is fixed by all elements of G. So $\beta^n \in F^*$. Write $a = \beta^n \in F^*$. We claim that $K = F(\sqrt[n]{a}) = F(\beta)$. Consider the diagram below:



We know that $[K : F] = n$, we need to show that $[F(\beta) : F] = n$. Since $\sigma(\beta) = \rho^{-i}(\beta)$, and ρ^{-i} are distinct for $i = 1, 2, \cdots, n$, then $\beta, \sigma(\beta), ..., \sigma^{n-1}(\beta)$ are all conjugates of β and are distinct. Therefore,

$$[F(\beta) : F] \geq n = [K : F] \geq [F(\beta) : F],$$

$$\Longrightarrow [F(\beta) : F] = n = [K : F].$$

Hence, $K = F(\beta) = F(\sqrt[n]{a})$. $\square$

5 The Artin-Schreier Theorem

Theorem 5.1 (The Artin-Schreier Theorem)**.** *Let C be algebraically closed with F a subfield such that $1 < [C : F] < \infty$. Then $C = F(i)$ where $i^2 = -1$, and F has characteristic 0. Moreover, for $a \in F$, exactly one of a or $-a$ is a square in F, and any finite sum of nonzero squares in F is again a nonzero square in F.*

Proof. We will divide the proof of the theorem to three steps.

Step I: Show that $C|F$ is Galois.

By 2.29, since we already have $[C : F] < \infty$, it is enough to show that $C|F$ is normal and separable. Since C is algebraic closed, by 2.21, we know that every nonconstant polynomial in C[x] has a root in C. Since every polynomial can be factored into irreducible factors and each factor has a root in C, this implies that every polynomial in C[x] has all roots in C. Therefore, every polynomial in C[x] splits completely in C and is normal.

To show that $C|F$ is separable, suppose $char\ F = p > 0$. It is suffices to show that F is perfect (i.e., $F = F^p$). If so, by 2.28, $C|F$ is separable. Suppose in contrary that $F \neq F^p$, then $\exists\ \alpha \in F \backslash F^p$. By 2.10, we know that $f(x) = x^{p^n} - \alpha$ is irreducible in F[x] for any $n \geq 1$, this implies that f(x) has a very large degree and thus has a very large algebraic extension, call it F_n such that $F_n \subseteq C$, which contradicts to that $[C : F] < \infty$.

Step II: Show [C:F]=2

Let $G = Gal(C|F)$, then $|G| = [C : F]$. Suppose in contrary that $|G| > 2$, then $|G|$

is divisible by 4 or by an odd prime. If $|G|$ is divisible by an odd prime, by Cauchy's theorem, G has a subgroup whose size is an odd prime; otherwise, if $|G|$ is not divisible by an odd prime, then $G = 2^r$ where $r \geq 2$, and thus is a p-group which has a subgroup of size 4. By Fundamental Theorem of Galois Theory, C has a subfield K containing F such that [C:K] is equal to 4 or an odd prime. Now replace K with F, we will show that [C:F] cannot be equal to 4 or an odd prime. Let's consider the following 2 cases:

Case 1: Suppose [C:F]=p, then $C|F$ is cyclic and so G is cyclic of order p. Let $G =< \sigma >$ for some $\sigma \in G^*$. So for any $a \in F$, $\sigma(a) = a$. Our goal is to show that p=2.

First, we will show that $char\ F \neq p$. Suppose in contrary that $char\ F = p$. By assumption $C|F$ is cyclic of order p and F has characteristic equal to p. Thus, by Artin-Schreier extension, $C = F(\alpha)$, where α is a root to the polynomial $x^p - x - a \in F[x]$. Since C is a simple extension of F, C has an F-basis $\{1, \alpha, \alpha^2, \cdots, \alpha^{p-1}\}$. Therefore, for any element $b \in C$, we can write

$$b = b_0 + b_1\alpha + b_2\alpha^2 + \cdots + b_{p-1}\alpha^{p-1},$$

where $b_i's \in F$. Then

$$\begin{aligned} b^p - b &= \sum_{i=0}^{p-1}(b_i\alpha^i)^p - \sum_{i=0}^{p-1} b_i\alpha^i \\ &= \sum_{i=0}^{p-1} b_i^p(\alpha + a)^i - b_i\alpha^i \ (Since\ \alpha\ is\ a\ root\ to\ x^p - x - a, \alpha^p = \alpha + a) \\ &= (b_{p-1}^p - b_{p-1})\alpha^{p-1} + \text{ lower degree terms} \end{aligned}$$

Since C is algebraically closed, every nonconstant polynomial has a root in C. Consider the polynomial $x^p - x - a\alpha^{p-1}$, then it has a root b so that $b^p - b = a\alpha^{p-1}$.

Compare the right side of this equation to the equation above, the coefficients of α^{p-1} implies that

$$b_{p-1}^p - b_{p-1} = a, \Rightarrow b_{p-1}^p - b_{p-1} - a = 0, \Rightarrow b_{p-1} \text{ is a root of } x^p - x - a.$$

But $b_{p-1} \in F$ and $x^p - x - a$ is the minimal polynomial of α, and thus irreducible. So this is a contradiction. Therefore, $char\ F \neq p$.

Since $char\ F \neq p$, and C is an extension of F, $char\ C \neq p$. And since C is algebraically closed, C contains a primitive p^{th} root of unity, call it ρ. We will show that $[F(\rho) : F] \leq p - 1$. First, note that we have the factorization

$$x^p - 1 = (x-1)(x^{p-1} + \cdots + x + 1).$$

Since $\rho \neq 1$, it follows that ρ is a root of the polynomial:

$$\Phi_p(x) = \frac{x^p - 1}{x - 1} = x^{p-1} + x^{p-2} + \cdots + 1$$

So

$$[F(\rho) : F] \leq p - 1.$$

Since $[C : F] = p$, and $[C : F] = [C : F(\rho)][F(\rho) : F]$, so $p \leq [C : F(\rho)](p-1)$. Since p is a prime so either $[C : F(\rho)] = p$ or $p - 1 = p$, and the latter is clearly impossible. So $[C : F(\rho)] = p$ and $[F(\rho) : F] = 1$. Therefore, $\rho \in F$. Thus, we have that $C|F$ a cyclic extension of degree p and F contains a primitive p^{th} root of unity. By Kummer theory, we can write $C = F(\gamma)$, where $\gamma^p \in F$.

Let $\eta \in C$ be such that $\eta^p = \gamma$. So $\eta^{p^2} = \gamma^p \in F$. Therefore, $\sigma(\eta^{p^2}) = \sigma(\eta)^{p^2} = \eta^{p^2}$, which implies $\sigma(\eta) = \theta\eta$, where $\theta^{p^2} = 1$. Then, θ^p is either a primitive p^{th} root of unity or $\theta^p = 1$. First, let's consider the case when $\theta^p = 1$. Then

$$\sigma(\eta)^p = \eta^p, \Rightarrow \sigma(\eta^p) = \eta^p, \Rightarrow \eta^p = \gamma \in F$$

However, by assumption we have $\gamma \notin F$, which is a contradiction. Therefore, $\theta^p \neq 1$. Hence, θ^p has to be the primitive p^{th} root of unity and we have that $\theta^p \in F$. Therefore, $\sigma(\theta^p) = \theta^p = (\sigma(\theta))^p$. Since $\theta \in C$ and $char\ C \neq p$, we have

$$(\sigma(\theta))^p = \theta^p,$$

$$\sigma(\theta) = (\theta^p)^k \cdot \theta = \theta^{1+pk}$$

for some $k \in \mathbb{Z}$. Since $G =< \sigma >$, we have that $\sigma^p = id$. Then

$$\begin{aligned}
\eta &= \sigma^p(\eta) \\
&= \sigma^{p-1}(\sigma(\eta)) \\
&= \sigma^{p-1}(\theta\eta) \\
&= \sigma^{p-1}(\theta)\sigma^{p-1}(\eta) \\
&= \sigma^{p-1}\sigma^{p-2}(\sigma(\eta)) \\
&= \sigma^{p-1}\sigma^{p-2}(\theta)\cdots\sigma(\theta)\sigma(\eta) \\
&= \sigma^{p-1}\sigma^{p-2}(\theta)\cdots\sigma(\theta)\theta\eta \\
&= \theta\sigma(\theta)\cdots\sigma^{p-1}(\theta)\eta \\
&= \theta^{1+(1+pk)+\cdots+(1+pk)^{p-1}}\eta \text{ (since } \sigma(\theta) = \theta^{pk+1})
\end{aligned}$$

Since $\theta^{p^2} = 1$, we have the following sequence of congruence:

$$\begin{aligned}
1+(1+pk)+\cdots+(1+pk)^{p-1} &\equiv 0 \bmod p^2\\
\sum_{j=0}^{p-1}(1+pk)^j &\equiv 0 \bmod p^2\\
p+\sum_{j=0}^{p-1} jpk + (\text{ terms divisible by } p^2) &\equiv 0 \bmod p^2\\
p+\sum_{j=0}^{p-1} jpk &\equiv 0 \bmod p^2\\
p+pk\frac{(p-1)p}{2} &\equiv 0 \bmod p^2\\
1+k\frac{(p-1)p}{2} &\equiv 0 \bmod p.
\end{aligned}$$

Now we discuss the parity of p. Suppose p is odd, then $1+pkn \equiv 0 \ mod \ p$, for some $n \in \mathbb{N}$, which is impossible. Therefore, p is even (i.e., $p=2$), and k is odd.
Hence, the order of θ is $2^2 = 4$. So $\theta^4 = 1$ and $\sigma(\theta) = \theta^{1+2k} \neq \theta$. Thus, $\theta \notin F$, and we can write $\theta = i$. We then reach a conclusion that if $[C:F] = p$, then $[C:F] = 2$, $char\ F \neq 2, char\ C \neq 2$, and $C = F(i)$.

Case 2: Suppose $[C:F] = 4$, then $|G| = 4$. By Cauchy's Theorem, G has a subgroup of order 2. From the Fundamental Theorem of Galois Theory, there exists a subfield K of C such that $[C:K] = 2$. From above arguments we know that $i \notin K$. However, $F(i)$ is a subfield of C with $[C:F(i)] = 2$, and $i \in F(i)$, which is a contradiction to i not belong to a subfield of C. Therefore, $[C:F] \neq 4$.

Therefore, the above two cases have reached the following conclusion: If C is an algebraic closed field with F a subfield such that the extension degree is finite, then

$[C : F] = 2$, F and C do not have characteristic 2 and $i \notin F$. Hence, $C = F(i)$. By 2.11, we have that $char\ F = 0$.

Step III: Show that for $a \in F$, exactly one of a or $-a$ is a square in F, and any finite sum of nonzero squares in F is again a nonzero square in F.

We will prove by contradiction. Suppose that neither a nor $-a$ is a square in F, then $C|F(\sqrt{a})$ and $C|F(\sqrt{-a})$ are quadratic extensions by 2.22. Therefore, $C = F(\sqrt{a}) = F(\sqrt{-a})$. Hence, the ratio $\frac{a}{-a} = -1$ must be a square, otherwise $F(\sqrt{a}) \neq F(\sqrt{-a})$. As a consequence, $i \in F$, which contradicts to the previous conclusion in Step II. Therefore, exactly one of a or $-a$ is a square in F.

Let $b_1, b_2, \cdots, b_n$ be nonzero elements in F. Then by 2.11, $b_1^2 + b_2^2 + \cdots + b_n^2$ is again a square in F. Suppose in contrary that the sum is zero. Then

$$b_1^2 + b_2^2 + \cdots + b_n^2 = 0.$$

Divide each side by b_1^2 and rearrange the terms, we have

$$-1 = \frac{b_2^2}{b_1^2} + \cdots + \frac{b_n^2}{b_1^2}$$

This implies that -1 is the sum of squares and thus again is a square in F, which is a contradiction. So the sum of finite nonzero squares in F is a nonzero square. □

6 Applications

The Artin-Schreier theorem tells us that for an algebraic closed field C with a proper subfield F whose field extension is finite, the degree of such finite extension must be 2 and C is of the form $F(i)$ and F must have characteristic 0. We have seen a very common example of $\mathbb{C}|\mathbb{R}$, where we write $\mathbb{C} = \{a + bi \ |a, b \in \mathbb{R}, i^2 = -1\}$. For this example, $[\mathbb{C} : \mathbb{R}] = 2$ and $char\ \mathbb{R} = 0$, satisfying the main part of the Artin-Schreier theorem. Is there another example other than $\mathbb{C}|\mathbb{R}$? Yes, and in order to show another example, we will first state and prove two simple corollaries of the Artin-Schreier Theorem.

Corollary 6.1. *Let C be an algebraically closed field, and let $G \subseteq Aut(C)$ be a finite subgroup, then $|G| = 1$ or 2.*

Proof. Let $F = C^G$, then by theorem 2.37, $C|F$ is Galois and $G = Gal(C|F)$. By Artin-Schreier theorem, $|G| = [C : F] = 2$. □

Corollary 6.2. *Let C be an algebraically closed field, and let $\sigma \in Aut(C)$ and σ has finite order. Then $o(\sigma) = 1$ or 2.*

Proof. Let $G =< \sigma >$, then the result follows from Corollary 1. □

Now we can consider the following example. Consider the field $\bar{\mathbb{Q}} = \{\alpha \in \mathbb{C} \ |\alpha \text{ algebraic over } \mathbb{Q}\}$. Then $\bar{\mathbb{Q}}$ is algebraically closed. Consider the automorphism $\sigma : \bar{\mathbb{Q}} \longrightarrow \bar{\mathbb{Q}}$ givien by $\sigma(a + ib) = a - ib$. Then $o(\sigma) = 2$, by Corollary 2. Now we let $C = \bar{\mathbb{Q}}$ and $F = \bar{\mathbb{Q}}^{<\sigma>}$. Then $[C : F] = 2$.

Let's consider the last application of the Artin-Schreier Theorem.

Corollary 6.3. *Let C be an algebraically closed field. Let $\sigma_1, \sigma_2 \in Aut(C)$ be finite order elements such that $\sigma_1, \sigma_2 \neq id$ and $\sigma_1\sigma_2 \neq \sigma_2\sigma_1$. Then $o(\sigma_1\sigma_2) = \infty$.*

Proof. Since $\sigma_1, \sigma_2 \neq id$ and has finite order, by Corollary 2, we know that $o(\sigma_1) = o(\sigma_2) = 2$. Suppose in contrary that $o(\sigma_1\sigma_2) < \infty$, then $o(\sigma_1\sigma_2) \leq 2$. Let's consider the following two cases:

Case 1: $o(\sigma_1\sigma_2) = 1$.

Then $\sigma_1 = \sigma_2^{-1}$. But since $o(\sigma_1) = o(\sigma_2) = 2$,

$$\sigma_1 = \sigma_1^{-1}, \sigma_2 = \sigma_2^{-1}.$$

Hence, $\sigma_1 = \sigma_2$, which implies $\sigma_1\sigma_2 = \sigma_2\sigma_1$, contradicting to the assumption.

Case 2: $o(\sigma_1\sigma_2) = 2$. Then

$$\sigma_1\sigma_2 = (\sigma_1\sigma_2)^{-1} \Longrightarrow \sigma_1\sigma_2 = \sigma_2^{-1}\sigma_1^{-1} \Longrightarrow \sigma_1\sigma_2 = \sigma_2\sigma_1.$$

Thus, both cases imply that $\sigma_1\sigma_2 = \sigma_2\sigma_1$, which is a contradiction. So $o(\sigma_1\sigma_2) = \infty$. $\square$

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