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A Hopf Theorem and Related Results for Pseduo-Riemannian Geometry

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B.A., University of Redlands, CA, 2020

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An abstract of
A dissertation submitted to the Faculty of the
James T. Laney School of Graduate Studies of Emory University
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Abstract

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We show that a complete Pseudo-Riemannian metric without conjugate points along time-like curves which is flat outside of a compact set must be flat inside that compact set. This type of result is called a Hopf theorem, and our result is a generalization of a result by Croke in the Riemannian case. We use a mixture of geometric methods from that work and methods used in showing boundary rigidity through integral geometry. During the course of this three part proof we show other related results, including that the geodesic ray transform of functions over time-like curves for separable Pseudo-Riemannian manifolds is injective.

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Chapter 1

Introduction and Main Results

1.1 The theorem of E. Hopf

This thesis concerns some geometric inverse problems. The main focus is to determine the detailed information about the interior of a compact set when we have universal information about the complement of that compact set, and minimal information about the interior. The case I wish to study and generalize is centered around the following question.

Question 1. If a Riemannian metric on $\mathbb{R}^n$ has no conjugate points and is flat outside of a compact set Ω, must it also be flat on Ω?

Here by flat we mean the usual Euclidean metric. It is clear that the Euclidean metric is one such Riemannian metric, and the question concerns the uniqueness of Riemannian metrics without conjugate points. This type of question started in Riemannian geometry from the work of E. Hopf [7] who proved that a Riemannian metric on a two-dimensional torus without conjugate points is flat. This result is generalized to high dimensional tori in [3]. Green and Gulliver [11] proved the result for the plane, namely that metric perturbations of the Euclidean metric on $\mathbb{R}^2$ in a compact set without conjugate points must be flat. Recently, this is proved for asymptotically Euclidean type metric perturbations in [6]. A Hopf type theorem has been studied in Loretnzian geometry, see [1]. In particular, the authors proved that metric perturbations of the Minkowski metric on $\mathbb{R}^2$ without conjugate points along time-like geodesics is flat. We discuss more details of these results in Section 1.4.1.

1.2 Main results

In this thesis, we are interested in Question 1 for Pseudo-Riemannian metrics which generalize the Riemannian one. Our goal is to develop some methods based on recent advances in geometric inverse problems to obtain Hopf type theorems. Along the way, we also study the related geodesic ray transform in pseudo-Riemannian geometry. We will now describe a concrete result for which our method works.

Let g be a smooth Pseudo-Riemannian metric on $\mathbb{R}^n \times \mathbb{R}^m$ where $n \geq 2, m \geq 1$. We refer the readers to Chapter 2 for more discussions on Pseudo-Riemannian geometry. We let e be the flat metric on $\mathbb{R}^n \times \mathbb{R}^m$ given by

$$e = \begin{pmatrix} -I_n & 0 \\ 0 & I_m \end{pmatrix},$$

where I_k denotes the $k \times k$ identity matrix. In Pseudo-Riemannian geometry, geodesics can be classified into time-like, null, and space-like geodesics according to

$$g(\dot{\gamma}, \dot{\gamma}) < 0, \quad g(\dot{\gamma}, \dot{\gamma}) = 0, \quad g(\dot{\gamma}, \dot{\gamma}) > 0,$$

respectively, where γ denotes a geodesic for g. For $n = 1$, the metric g is called Lorentzian and the classification has physical meanings. We refer to Chapter 2 for more discussions. In this work, we primarily consider the time-like geodesics.

Below we consider g conformal to e so $g = \phi e$ for some positive scalar function ϕ.

Theorem 1.2.1 (A.)**.** Let $g = \phi e$ be conformal to e and $g = e$ outside of a bounded open set Ω. Then there exists a open dense set Σ of C^2 functions such that for any $g = \phi e$ with $\phi \in \Sigma$, if there are no conjugate points along any time-like geodesic for g, then $g = e$.

In other words, the answer to Question 1 is affirmative for generic pseudo-Riemannian metrics that are conformal to the flat metric. In Chapter 3, we give a more general Theorem 3.3.1, which relies on certain geodesic ray transforms on $(\mathbb{R}^{n+m}, g)$ being injective. This naturally leads to the key part of the analysis: the geodesic ray transform.

Let Ω be a bounded open set of $\mathbb{R}^{n+m}$. Let f be a smooth function on $\mathbb{R}^n \times \mathbb{R}^m$ supported in Ω. Let γ_g be a time-like geodesic in $(\mathbb{R}^n \times \mathbb{R}^m, g)$. The geodesic ray transform for time-like geodesics is defined by

$$I_g f(\gamma_g) = \int f(\gamma_g(r))dr.$$

In Chapter 3, we will briefly discuss the transform for tensors. In the last two decades, various geodesic ray transforms in Riemannian and pseudo-Riemannian geometries have attracted lots of attention. For example, the transform played an important role in the study of the boundary rigidity problem. We will discuss more details in Section 1.4.2 and 1.4.3. Here, we prove the injectivity and stability for I_g.

Theorem 1.2.2 (A.)**.** There exists an open dense set of C^2 separable Pseudo-Riemannian metrics g on $\mathbb{R}^{n+m}$ such that the geodesic ray transform I_g for time-like geodesics is injective on $C_0^\infty(\Omega)$. That is, for any $f \in C_0^\infty(\Omega)$, if $I_g f(\gamma) = 0$ for all time-like geodesics, then $f = 0$. Moreover, there exists $C > 0$ such that the following stability holds

$$\|f\|_{H^s} \leq C\|N_g f\|_{H^{s+1}}.$$

Here, by separable, we mean that g can be written as

$$g(z) = \begin{pmatrix} -g_N(z) & 0 \\ 0 & g_M(z) \end{pmatrix}$$

for $z \in \mathbb{R}^{n+m}$ where g_N, g_M are symmetric non-degenerate matrices. However, we do not assume that $(\mathbb{R}^{n+m}, g)$ is a product manifold of $(\mathbb{R}^m, g_M)$ and $(\mathbb{R}^n, g_N)$. For a Lorentzian metric g, it can be written in this form under some global hyperbolic condition. We discuss these in Chapter 3.

1.3 Outline of the proof

The proof for Theorem 1.2.1 will consist of three parts, around each of which we will prove more general results. First, in Chapter 2, we will reduce from the purely geometric inverse problem to a problem about boundary distances. A modified version of the argument presented by Croke [2] and used by Anderson, Dahl, and Howard [1] is used to reduce this problem back down to a boundary rigidity problem.

After reduction to a geometric inverse problem, we will further reduce to a geodesic ray transform problem in Chapter 3. The methods we will adapt come from various historical sources. Since the case we care most about is the case with one metric being flat, we will primarily adapt the argument from [19]. From there we will work with conformal metrics to reduce the problem down to a problem about the injectivity of a geodesic ray transform over functions. Additionally, we briefly discuss some cases where the problem reduces to the injectivity of an geodesic ray transform over 2-tensors, which adds complications we do not consider here.

Next we will prove some injectivity results about geodesic ray transforms over functions in Pseudo-Riemannian geometry in Chapter 4. We will directly show the form of the normal operator for the geodesic ray transform for functions defined over time-like geodesics for a separable analytic Pseudo-Riemannian manifold. This will allow us to conclude that the normal operator is an elliptic pseudo-differential operator of order -1. To get a proper injectivity result we will use some tools for analytic pseudo-differential operators from [22, 20]. Here we will extend the results to generic separable Pseudo-Riemannian metrics to get the final result.

1.4 Historical results

There are three parts to the proof of Theorem 1.2.1, in Chapters 2, 3 and 4. We will briefly state the historical results for all three parts of the proof, as each can be viewed as a separate problem and have been worked on in very different contexts than the one presented here.

1.4.1 Historical Hopf-type theorems

The First result in the nature of Theorem 1.2.1 was shown by E. Hopf in [7]. Explicitly, he showed that if there is a C^3 surface S such that there are no conjugate points on S, and the total curvature is zero, then the curvature is zero locally. This implies that if S is topologically a two-dimensional torus without conjugate points, then it must be flat. The argument was done by explicitly considering curvature equations and clever integration.

The conjecture that such a result can be generalized was proven in [3]. They showed that if a Riemannian metric on the n-dimensional torus has no conjugate points, then it is a flat metric. This proof relies on a lift to a periodic metric on $\mathbb{R}^m$, and constructing Banach norms that relate properly to the Euclidean norm.

Before Croke's more general result, Green and Gulliver [11] proved the result for the plane. They proved that on $\mathbb{R}^2$, if a metric without conjugate points is equal to the flat metric outside a compact set, it must be isometric to the flat metric everywhere.

The result we are most interest is a result from Croke [2]. A crucial generalization is that instead of working with metrics that universally have no conjugate points, he worked with metrics that have no conjugate points on very specific geodesics over some compact subset $\Omega \subset \mathbb{R}^m$. This is described in 2.1, and is defined as strongly geodesic minimizing on Ω. He proved that if a metric is strongly geodesic minimizing over some compact subset $\Omega \subset \mathbb{R}^m$ and is equal to the flat metric outside of Ω, it is isometric to the flat metric inside of Ω. This proof comes in two parts. First it determines that the distance between boundary points on Ω for g and the flat metric must be the same, and it then uses a volumetric argument to show the isometry.

In recent work, a collection of authors [6] showed a similar result for asymptotically flat metrics. They showed that if g is a Riemannian metric without conjugate points on $\mathbb{R}^m$ that is asymptotically Euclidean to some order greater than 2, then that metric is diffeomorphic to the flat metric. This argument consists of four parts, which are beyond the scope here. The work is reminiscent of the arguments by Croke, and follows, very roughly, the idea of finding an equivalence between a map which is rigid in the Euclidean case.

Work in the more general Pseudo-Riemannian setting is sparse, but was studied in Lorentzian geometry in [1]. The full statement for their work is too ornery for this quick review. They define an equivalent to strongly geodesic minimizing, and enforce natural geometric conditions. In doing so, they get the result that if a metric that satisfies the appropriate conditions is equal to the flat metric outside a compact set, it must share the scattering relation with the flat metric. This argument is a direct adaption of the first argument used in [2].

It is here that the present work sits. Except for the asymptotically flat case, the work provides a generalization about as far as we can take the set-up for a Hopf type theorem. We prove the equivalence of distance functions, which provides an odd framework. Taking the standard distance function puts our work squarely in the settings used above and still gives the scattering relation.

1.4.2 Historical reductions to geodesic ray transforms

The work to reduce boundary rigidity problems to geodesic ray transforms is in truth part of many works about the geodesic ray transforms and rarely studied separately. There are a few novel or standard reductions, which we will mention and directly generalize.

The construction we are most interested in comes from the seminal work by Stefanov and Uhlmann in [19]. This work showed, roughly, that if there are two metrics that share boundary data on some compact $\Omega \subset \mathbb{R}^m$, and are sufficiently close to the Euclidean metric, those two metrics must be diffeomorphic to each other. The result is done using an explicit calculation through the construction and then integration of a clever function. This function is defined using the construction of geodesics from a Hamiltonian system and the equivalence of boundary points. The argument is simpler to work with if one of the metrics is flat (as it is in the case we consider). Using this simplicity, we can prove a similar result with much care to the differences in moving to Pseudo-Riemannian geometry. Another standard reference for this reduction is the work by Sharafutinov [18] for the one parameter family case.

1.4.3 Historical injectivity for geodesic ray transforms

There is a glut of results in the field of injectivity for geodesic ray transforms, and here we will very briefly describe a few that either are useful for our arguments or are of particular interest.

For an overview of the process of proving injectivity of geodesic ray transforms using pseudo-differential operators, see for example the process described by Stefanov and Uhlmann in [20]. This has the skeleton of most arguments: find the kernel of the normal operator in the correct sense, then use theory from pseudo-differential operators to get a result (often using some form of stability estimate). This is, in fact, the exact process we will use.

Much work has been done in the case of simple manifolds. For now, we take an intuitive understanding of simple manifolds to be a manifold that is diffeomorphic to ball where every point is connected to every other point by a unique geodesic. Under this, Stefanov and Uhlmann proved in [21] that the geodesic ray transform for 2-tensors on an analytic simple manifold is s-injective. Here s-injectivity is the best possible result (see section 3.4 for a brief discussion). This result used a much more in depth method that still resembles the process described above. This result is extended in the same work for a generic simple metric. A result that followed this work by Stefanov and Uhlmann [22], showed a similar result for a limited angle problem that helped inspire this work. Another major result using the method of micro-local analysis comes in a paper by Stefanov, Uhlmann and Vasy [23]. In this work, they show that, under a foliation condition, on a manifold with boundary the geodesic ray transform is s-injective.

A different method of showing injectivity of geodesic ray transforms comes from the Pestov identity. After the work by Muhometov in [13], the use of the Pestov identity, coined due to the work by Pestov and Sharafutdinov in [17] and discussed in detail in the book by Sharafutdinov [18], has proven an incredibly powerful tool. Under strict curvature conditions, this identity can provide injectivity results and stability estimates.

Of particular note is the result by Paternian, Salo and Uhlmann in [15], which used this method to show that for simple two-dimensional metrics the geodesic ray transform over tensor fields is injective in a sense analogous to s-injectivity. This work cleverly navigates around any curvature restrictions using the unique tools available in two dimensions.

The only work in this field done in Pseudo-Riemannian geometry is a recent work done by Ilmavirta in [10]. Using a Pestov identity, it is shown that the geodesic ray transform of functions or 1-tensors on a limited class of Pseudo-Riemannian metrics with non-positive sectional curvature in the appropriate sense over light-like geodesics is injective or injective up to the natural kernel, respectively. This result inspired the work done here.

Chapter 2

Reduction to Geometric Inverse Problems

2.1 Flat on a complement set in Riemannian geometry

We begin with defining most of the parts of Riemannian Geometry necessary for the proof. In displaying the proof in the Euclidean case, we will explain the necessary parts for the generalization, which otherwise may seem arbitrary.

Let $M = \mathbb{R}^m$, $m \geq 2$. Let e be the smooth $(0, 2)$ tensor field defined by the identity matrix. We call e the flat metric, which gives rise to the familiar Euclidean geometry. Suppose $x, y \in M$. Let γ be the geodesic, in the sense of standard Riemannian geometry, between x and y. Due to being in Euclidean geometry, this is simply a straight line.

We will heavily lean on this connection between the ability to describe the Euclidean structure in Riemannian geometry and the vast depth of knowledge available in Euclidean geometry. Each of the parts we define below will match their intuitive case in Euclidean geometry.

Define the length of a piecewise $C^1(M)$ curve $\alpha : [0,T] \to M$ to be

$$\int_0^T e(\dot{\alpha}(r), \dot{\alpha}(r))dr. \tag{2.1}$$

This coincides exactly with the length of the curve α in the standard sense. In Euclidean geometry, the geodesics are always the unique minimizers (up to parameterization) of the lengths over of the class of C^1 curves between x and y. That is, in this special Euclidean case, lines are the shortest paths between two points. The fact that we will be primarily working with geodesics in Euclidean geometry rather than geodesics in an unknown geometry is how we are able to connect this generalized theorem to the specialized tools at our disposal.

In this result we will have a metric equal to e outside of some set Ω. However, we need the fact that geodesics are the unique length minimizing curves between two points for this argument. To this end, we will adopt the definition used in [2] to try and keep the class of metrics we can consider as general as possible. It is this idea that we will adapt in the next section to get a very general result.

Now we explicitly state some Riemannian geometry to clarify what we need for the argument. Let g be a positive-definite smooth $(0,2)$ tensor field on M. We can view g as a Riemannian metric, which lets us construct the same definitions as we had before in Euclidean geometry. Exactly as above, we define the length of a piecewise $C^1(M)$ curve $\alpha : [0,T] \to M$ to be

$$\int_0^T g(\dot{\alpha}(r), \dot{\alpha}(r))dr. \tag{2.2}$$

We define the distance between x and y in M to be the minimum length of all C^1 curves between x and y.

We often phrase the ideas used in terms of the length of curves. Since each of the curves we consider will be piecewise geodesics, we could focus exclusively on the distance between points. This strategy, focusing solely on distances between points, will provide us with the general result in Theorem 2.2.2.

Let Ω be some compact set with non-empty interior such that $g = e$ outside of Ω. We desire for geodesics of g to minimize across points on the boundary of Ω. Define g to be strongly geodesic minimizing on Ω if for all $x, y \in \partial\Omega$ there exists a geodesic between x and y which is the unique minimizer of the length of C^1 curves between x and y. Alternatively, we can define g to be strongly geodesic minimizing on $\partial\Omega$ if for all $x, y \in \partial\Omega$ there is a geodesic between x and y which has no conjugate points.

In the work on the injectivity of geodesic ray transforms, it is common to work on simple manifolds. A metric g is simple if for all points $x, y \in \Omega$, there a is geodesic from x to y that has no conjugate points and every geodesics extends to intersect with the boundary of Ω in a non-tangential manner. In relation to the boundary this means that for $x \in \partial\Omega$, every geodesic issued into Ω from x intersects the boundary. The major difference between the conditions simple and strongly geodesic minimizing is that for strongly geodesic minimizing we do not necessarily consider all possible geodesics issued from x, rather only geodesics we already know go between two boundary points. This is a relatively minor difference, but when we generalize it, the natural fact that we are only looking at a specific class of geodesics will be helpful.

We define the boundary distance function for a metric g on Ω to be a map $d_g : \partial\Omega \times \partial\Omega \to \mathbb{R}_{\geq 0}$ where $d_g(x, y)$ is the distance between x and y in Ω. Now we finally have enough to properly state our theorem, which comes from Croke [2].

Theorem 2.1.1. Suppose g is a Riemannian metric on $\mathbb{R}^m$ such that there is some Ω such that $g = e$ outside of Ω, and that g is strongly geodesic minimizing on Ω. The boundary distance function for g and e on $\partial\Omega$ are the same.

Next we establish a set of conditions that are necessary for the proof, and which will be the basis of the generalization. We will notate a unit speed geodesic through x

and y according to a specified Riemannian metric as $\gamma_{x,y} : \mathbb{R} \rightarrowtail \mathbb{R}^m$ with $\gamma_{x,y}(0) = x$ and $\gamma_{x,y}(T) = y$ for some T. Since we use the same notation across different metrics, each time we define the geodesic we will say which metric we are defining it across. Notate the distance between x and y according to a metric as $d_g(x, y)$.

Lemma 2.1.2. For the flat metric e and a metric g described as above, the following hold.

1. If x, y and z are points in $\mathbb{R}^m$, and $\gamma_{xy} : \mathbb{R} \to \mathbb{R}^m$ is the unit speed geodesic according to g through x and y as described above, then $d_g(\gamma_{x,y}(-t), \gamma_{x,y}(T + t)) \leq d_g(\gamma_{x,y}(-t), x) + d_g(x + z, y + z) + d_g(y, \gamma_{x,y}(T + t))$.

2. If x, y are points in $\mathbb{R}^m$ such that in a neighborhood of the geodesic from x to y according to e $g = e$ then $d_e(x, y) = d_g(x, y)$.

3. If x, y and z are points in $\mathbb{R}^m$, then $d_e(x, y) = d_e(x + z, y + z)$.

4. If x, y and z are points in $\mathbb{R}^m$, and $\gamma_{x,y}$, $\gamma_{x,z}$, and $\gamma_{y,z}$ are unit speed geodesics according to e as described above such that $\gamma_{x,y}(0) = x$ and $\gamma_{x,z}(0) = x$ and $e(\dot{\gamma}_{x,y}(0), \dot{\gamma}_{x,z}(0)) = 0$, then, $\lim_{k\to\infty} d_e(\gamma_{x,z}(k), x) - d_e(\gamma_{x,z}(k), y) = 0$.

In more plain terms, the above are as follows. The first statement is a very complicated consequence of the triangle inequality, or it can be viewed as a consequence of the fact that the unique distance minimizing curve between x and y is a geodesic.

The second statement says that when two metrics are equal the distance between points is equal.

The third statement says that in the Euclidean space the length of geodesics, which are lines, are translation invariant.

The fourth statement says that if we take a right triangle and stretch out one of the sides with a right angle then the difference in the Euclidean length of the stretched hypotenuse and the length of the stretched side goes to zero.

Further, each statement is a simple conclusion from our assumptions. The first statements is a result of the strongly geodesic minimizing assumption, while the second statement is simply a truth about the length function as it is defined above. The last two statements are result from basic Euclidean geometry, and in fact the fourth statement can directly be computed using the Pythagorean theorem. It is primarily these simple facts that generate the proof.

Proof. Let $x, y \in \partial\Omega$. Let $\gamma_g : \mathbb{R} \to \mathbb{R}^m$ be the unit speed geodesic through x and y with regards to g with $\gamma_g(0) = x$ and $\gamma_g(T) = y$ for some T. Let $\gamma_e : \mathbb{R} \to \mathbb{R}^m$ be the unit speed geodesic through x and y with regards to e with $\gamma_e(0) = x$ and $\gamma_e(T') = y$ for some T'. Note that γ_e is just a line segment. Suppose that $\dot{\gamma}_g(0) \neq \dot{\gamma}_g(T)$, or that a line entering Ω, following the geodesic from g inside, comes out as a line that is not parallel to how it entered. Since these two lines are not parallel, there exist infinitely many lines that intersect the two original lines. So if the extension of γ_g outside of Γ does not form parallel lines, then there are geodesics between at least one pair of points on γ_g that are not γ_g. This implies that γ_g has conjugate points, which contradicts the assumption that g is strongly geodesic minimizing. Thus, $\dot{\gamma}_g(x) = \dot{\gamma}_g(y)$, or the geodesic according to g enters and exits Ω in the same direction.

The basic idea of the proof is that we wish to create right triangles using the geodesics of either e or g outside of Ω, and then use Lemma 2.1.2 to get inequalities purely in terms of d_g and d_e by stretching these right triangles out to infinity.

First we will show that $d_g(x, y) \leq d_e(x, y)$. Recall that x, y are points on the boundary of Ω. Recall the notation that $\gamma_g : \mathbb{R} \to \mathbb{R}^m$ is the unit speed geodesic with regards to g such that $\gamma_g(0) = x$ and $\gamma_g(T) = y$ for some T and that $\gamma_e : \mathbb{R} \to \mathbb{R}^m$ is the unit speed geodesic with regards to e with $\gamma_e(0) = x$ and $\gamma_e(T') = y$ for some T'. Due to the fact that Ω is compact, there is some k such that both $x + kv$ and $y + kv$ are far enough outside of Ω that the line between them does not intersect Ω. By the construction of γ_g, $\gamma_g(-t)$ is a point a distance of t away from x, and $\gamma_g(T + t)$ is a

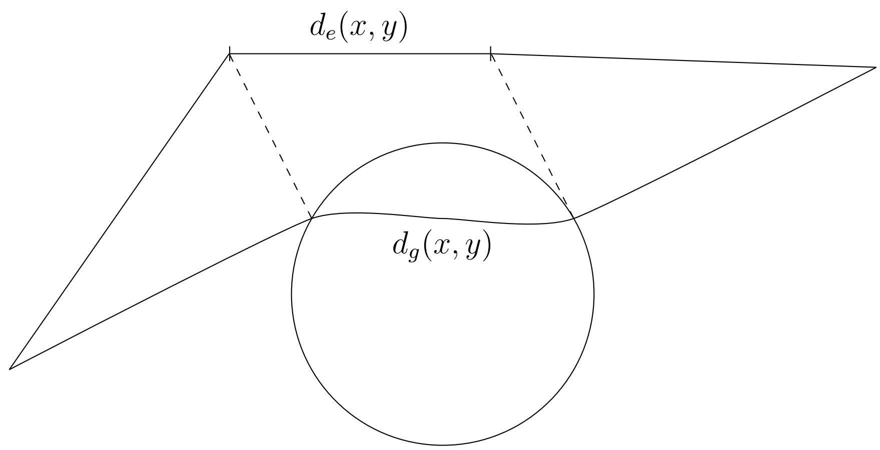

Figure 2.1: An illustration of the argument that $d_g(x, y) \leq d_e(x, y)$

point a distance of t away from y.

We now have three points on both sides of γ_g that form right triangles. Namely the triple x, $x+kv$ and $\gamma_g(-t)$, and the triple y, $y+kv$, and $\gamma_g(T+t)$. Now, the length of the geodesic from $\gamma_g(-t)$ to $\gamma_g(T+t)$ is $t+d_g(x, y)+t$. Let $l_{-t} = d_g(x+kv, \gamma_g(-t))$ and $l_t = d_g(x + kv, \gamma_g(T + t))$. Note that since the length of a line is translation invariant, that the length of the line from $x + kv$ to $y + kv$ is the same as the length of the line from x to y, or $d_e(x, y)$. Then the length of the curve given by the lines from $\gamma_g(-t)$ to $x + kv$ to $y + kv$ to $\gamma_g(T + t)$ is $l_{-t} + d_e(x + y) + l_t$.

Since γ_g is a geodesic without conjugate points, and thus the unique distance minimizing curve between any points on it, $t + d_g(x, y) + t \leq l_t + d_e(x, y) + l_{-t}$. Moreover, $d_g(x, y) \leq l_t - t + d_e(x, y) + l_{-t} - t$. We have constructed it so that l_t represents the length of the hypotenuse of a right triangle with one side of length t, and similarly l_{-t} represents the length of the hypotenuse of a right triangle with one side of length t. As we increase t, we can apply 2.1.2 to get that as $t \to \infty$, $l_t - t \to 0$ and $l_{-t} - t \to 0$.

Applying the limit as t approaches infinity to the construction above yields that $d_g(x, y) \leq d_e(x, y)$.

Next we will show that $d_g(x, y) \geq d_e(x, y)$. This, again, will be done by cleverly

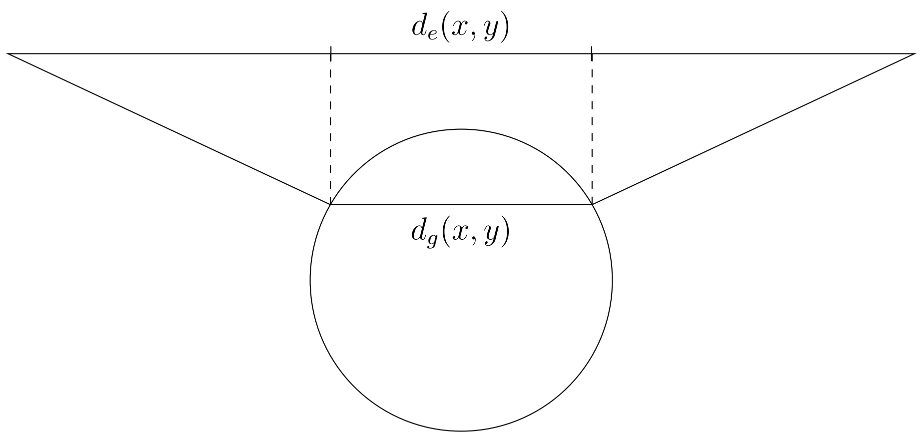

Figure 2.2: An illustration of the argument that $d_e(x,y) \leq d_g(x,y)$

constructing right triangles. Recall that x, y are points on the boundary of Ω. Recall the notation that $\gamma_g : \mathbb{R} \to \mathbb{R}^m$ is the unit speed geodesic with regards to g such that $\gamma_g(0) = x$ and $\gamma_g(T) = y$ for some T. There is some k such that $x+vk$ and $y+vk$ are both far enough outside of Ω that the line between them does not intersect Ω. Let $\alpha_e : \mathbb{R} \to \mathbb{R}^m$ be the unit speed geodesic with regards to e such that $\alpha_e(0) = x+vk$ and $\alpha_e(T'') = y + vk$ for some T''. Since α_e stays outside of Ω, that we could analogously define it in terms of g. Note that $\alpha_e(-t)$ is a distance of t away from $x + kv$ and $\alpha_e(T''+t)$ is a distance of t away from $y+kv$. Since the length of lines are translation invariant, then the length of the line from $x + kv$ to $y + kv$ is equal to the length of the line from x to y, or $d_e(x + kv, y + kx) = d_e(x, y)$. Note that since $g = e$ outside Ω then the distances between points connected by geodesics that stay outside of Ω are the same for e and g. The length along the geodesic from $\alpha_e(-t)$ to $\alpha_e(T'' + t)$ according to g is then $t + L_e(\gamma_e) + t$.

Once again, we now have three points on both sides of γ_g that form right triangles. Namely the triple x, $x + kv$ and $\alpha_e(-t)$, and the triple y, $y + kv$, and $\alpha_e(T' + t)$. Let $r_{-t} = d_e(x, \alpha_e(-t))$ and $r_{-t} = d_e(y, \alpha_e(T'' + t))$. Additionally, we note that since $g = e$ outside Ω then the distances between points are the same for e and g. Then the length of the curve given by the line from $\alpha_e(-t)$ to x, the geodesic according to

g from x to y and the line from y to $\alpha_e(-t)$ according to g is $r_{-t} + d_g(x, y) + r_t$.

Since $g = e$ outside of Ω, we have that γ_e is a geodesic of the metric g without conjugate points and as such is the unique distance minimizing curve. Thus, $t + d_e(x, y) + t \leq r_t + d_g(x, y) + r_{-t}$. Moreover, $d_e(x, y) \leq r_t - t + d_g(x, y) + r_{-t} - t$. As noted above, r_t is the hypotenuse of a right triangle with one side of length t and similarly, r_{-t} is the hypotenuse of a right triangle with one side of length t. The construction once again precisely fits the conditions for 2.1.2, and thus as $t \to \infty$, $r_t - t \to 0$ and $r_{-t} - t \to 0$. This yields that $d_g(x, y) \geq d_e(x, y)$.

Thus, since $d_g(x, y) \geq d_e(x, y)$ and $d_g(x, y) \leq d_e(x, y)$, then $d_g(x, y) = d_e(x, y)$.

□

2.2 Flat on a compliment set in Pseduo-Riemannian geometry

From the above, we now generalize. First we must define the new space we are working under.

Let K be a smooth manifold of dimension k. Let g be a symmetric nondegenerate smooth $(0, 2)$ tensor field on K of constant index $(\overbrace{-, \cdots, -}^{n}, \overbrace{+, \cdots, +}^{m})$ where $n + m = k$. Then g is called a Pseudo-Riemannian metric, (K, g) is called a Pseudo-Riemannian (or semi-Riemannian) manifold, and (n, m) is called the signature of g. We refer to [12, 14] for the background.

We quickly note that for much of the arguments presented the metrics need not be smooth, but for ease and simplicity we work with smooth metrics.

When $n = 0$, or (K, g) is a Pseudo-Riemannanian manifold of signature $(0, k)$, we call g a Riemannian metric. This coincides with the more traditional definition of a Riemannian metric discussed above. Much of the construction used in Riemannian geometry carry over to Pseudo-Riemannian geometry.

For a point $z \in K$, and a vector $v \in T_zK$ there are three cases for the sign of $g(v,v)$, each of which we refer to differently. If $g(v,v) < 0$ we call v time-like. If $g(v,v) > 0$ we call v space-like. If $g(v,v) = 0$ we call v light-like. For any given T_zK we can split it into time-like, space-like and light-like subspaces. Sometimes light-like vectors are called null vectors. This is language inherited from general relativity.

Geodesics on (K,g) can be defined in the usual way using the Levi-Civita connection. Alternatively, geodesics can defined using a Hamiltonian system, as described in section 3.1. We can define the length of a $C^1(K)$ curve $\alpha : [0,T] \to K$ as

$$\int_0^T g(\dot{\alpha}(r), \dot{\alpha}(r))dr. \tag{2.3}$$

This definition, however, has many flaws. Curves whose tangent vectors are all null have a length of zero, as do some curves with mixed time-like and space-like tangent vectors. Defining the distance between two points is thus very difficult. To try to make sense of this, we will work in a setting where we can exert more control and more easily separate out classes of curves we wish to work over.

Suppose that $K = N \times M$, $\dim(N) = n$, $\dim(M) = m$. Notate $z = (z_N, z_M) = (t,x) \in N \times M$. Suppose that g is of the form

$$g = \begin{pmatrix} -g_N & 0 \\ 0 & g_M \end{pmatrix}$$

where for any z, $g_N(z)$ is a positive definite matrix on T_tN and $g_M(z)$ is a positive definite matrix on T_xM, then we call g a separable Pseudo-Riemannian metric, and $(K,g) = (N \times M, g)$ a separable Pseudo-Riemannian manifold. The condition that (K,g) is time-orientable is equivalent to the fact that K is separable.

The ability to explicitly describe behaviors on N versus on M is immensely helpful. For a separable Pseudo-Riemannian manifold we refer to N as time and M as space.

We can separate points into time and space components, or $z = (z_N, z_M) \in K$ where $z_N \in N$ and $z_M \in M$. Similarly, we can split $v = (v_N, v_M) \in T_zK$ where $v_N \in T_tN$ and $v_M \in T_xM$.

Splitting in this way lets us more easily describe, and control, the types of geodesics. In this setting time-like vectors are vectors where $-g_N(v_N, v_N) > g_M(v_M, v_M)$, space-like vectors are vectors where $g_N(v_N, v_N) < g_M(v_M, v_M)$ and light-like vectors are vectors where $g_N(v_N, v_N) = g_M(v_M, v_M)$. Curves, including geodesics, with time-like, space-like, or light-like tangent vectors we call time-like, space-like, or light-like curves respectively. Since (K, g) is time orientable, this a complete classification of all geodesics.

When $n_1 = 1$, (M, g) is called a Lorentzian manifold, which is the arena for general relativity theory. In this case, time-like geodesics represent trajectories of particles of positive mass, see for example [27].

Similar to the Riemannian case, we can define a flat Pseudo-Riemannian metric by the $(0, 2)$ tensor field on $N \times M = \mathbb{R}^{n+m}$ by

$$e = \begin{pmatrix} -I_n & 0 \\ 0 & I_m \end{pmatrix}$$

where I_k denotes the $k \times k$ identity matrix. When $n = 1$, the space is the familiar Minkowski space and when $n = 0$ the space is the friendly Euclidean space. We denote the flat metric by e and call any such $(\mathbb{R}^{n+m}, e)$ the flat Pseudo-Riemannian space.

With all that setup, we can finally begin to narrow our focus down to the requisite parts for the generalization of the argument in Section 2.1. We must generalize the idea of a strongly geodesic minimizing manifold and Lemma 2.1.2. This lemma is about a distance function on a class of geodesics. We will phrase our construction in this setting. In application we will care about the integral that gives the length of curves which gives rise to a distance function, rather than the distance function itself. Still, the theorem is complicated in notation but simple in argument to generalize in terms of a distance function. First we will define a limited class of geodesics which we will consider. In application, this will be a class like the time-like or light-like geodesics.

Let g be a Pseudo-Riemannian metric with signature (n, m) on $\mathbb{R}^{n+m}$. Let $\Gamma \subset \{(\gamma, \dot{\gamma}) \in T\mathbb{R}^{n+m} | \gamma \text{ is a unit speed geodesic with respect to } g \text{ on } T\mathbb{R}^{n+m}\}$ such that for all $z \in \mathbb{R}^{n+m}$ there are infinitely many $\gamma \in \Gamma$ that pass through z, and there is a smooth curve in $T_z\mathbb{R}^{n+m}$ that is contained in Γ. In an abuse of notation, we refer to Γ as the collection of points (z, v) where there is some geodesic $\gamma \in \Gamma$ through z with tangent vector v. In fact, we typically refer to Γ as that subset of $T\mathbb{R}^{n+m}$.

This set Γ represents the geodesics over which we work. Due to the drawbacks of the length functional for C^1 curves, it is necessary to limit the scope we wish to consider. By forcing points to be connected only by a certain type of geodesic, we can cover for the shortcomings of the length functional. This is further shown before Corollary 2.2.3.

Suppose that g agrees with e outside a compact set Ω. Since $g = e$ on the complement of Ω, defining Γ with respect to g and with respect to e is identical outside Γ. With this construction we are looking at a class of geodesics called Γ, and we are trying to determine the relation of g and e on the boundary of Ω using information about Γ.

We now have to construct an analogue for strongly geodesic minimizing. The easiest generalization is the alternative definition given. We define a Pseudo-Riemannian metric g to be strongly geodesic minimizing on $\partial\Omega$ over Γ if Ω is simply connected and for all $x, y \in \partial\Omega$ such there is a geodesic between x and y in Γ that geodesic has no conjugate points (also referred to as cut points). The strength of this definition will force time-like geodesics to be distance maximizing (or their negative distances to be distance minimizing).

Now, finally, we can introduce the equivalent to the distance function in the Riemannian case. We limit the scope of our map to just the geodesics in Γ and put the equivalent requirements from Lemma 2.1.2 on our new map.

Definition 2.2.1. We call a continuous map defined for the flat metric e and some other metric g, notated by $d_g : \mathbb{R}^{n+m} \times \mathbb{R}^{n+m} \to \mathbb{R}$ distance-like on Γ if it satisfies the following conditions.

1. If x, y, a and b are points in $\mathbb{R}^{n+m}$, such that $a \neq b$, x and y are connected by a geodesic in Γ, and a and b are connected by a geodesic in Γ, then if we let α_{xy} be the extension of the parameterized geodesic according to g from $[0,T]$ to $\mathbb{R}$, then for all t greater than some number r dependent on x, y, a, b, $\gamma_{xy}(-t)$ and a are connected by a geodesic in Γ, $\gamma_{xy}(T+t)$ and b are connected by a geodesic Γ, and $d_g(\gamma_{xy}(-t), \gamma_{xy}(T+t)) \leq d_g(\gamma_{xy}(-t), a) + d_g(a,b) + d_g(b, \gamma_{xy}(T+t))$.

2. If x, y are points in $\mathbb{R}^{n+m}$ such that in a neighborhood of the geodesic from x to y according to e $g = e$, then $d_e(x,y) = d_g(x,y)$.

3. If e is the flat metric then for x, y, and z in $\mathbb{R}^{n+m}$ such that the geodesic from x to y is in Γ, then the geodesic from $x+z$ to $y+z$ is in Γ and $d_e(x,y) = d_e(x+z, y+z)$.

4. If e is the flat metric then for x, y and z in $\mathbb{R}^n$, where γ_{xy}, γ_{xz}, and γ_{yz} are geodesics in Γ such that $\gamma_{xy}(0) = x$, $\gamma_{xz}(0) = x$ and $e(\dot{\gamma}_{xy}(0), \dot{\gamma}_{xz}(0)) = 0$, then if we let $z_k = \exp_x(k\dot{\gamma}_{xz})$, $\lim_{k\to\infty} d_e(y, z_k) - d_e(\gamma_{x,z_k}) = 0$.

These conditions correspond exactly to the ones in Lemma 2.1.2. The first condition can be viewed as saying that when taking four points x, a, b, y that are connected pairwise by different geodesics in Γ, the distance between x and y is less than the sum of the distances between x and a, a and b, and b and y.

The second condition essentially says that if $g = e$ between two points x and y, then their distances according to g and e are the same.

The third condition says that the distance between points according to the flat metric is invariant under translation of those points.

The fourth conditions say the exact same thing as in Lemma 2.1.2. That is, if we take a right triangle and stretch out one of the sides with a right angle, then with respect to the flat metric the difference of the distance of the points that define the hypotenuse and the distance of the points that define the stretched side goes to zero.

We also say that if $-d_g$ is distance-like, then d_g is distance-like. However, we will stick to the statements provided in the original definition for the proof. This set of definitions serves to exactly generalize the conditions of Lemma 2.1.2, where d_g serves as our equivalent to a distance function.

With these assumptions the following theorem holds.

Theorem 2.2.2 (A.)**.** If g is a Pseudo-Riemannian metric that is equal to e outside of some Ω that is strongly geodesic minimizing on $\partial\Omega$ over some collection of geodesics Γ, and d_g is distance like over Γ, then for all $x, y \in \partial\Omega$ such that there exists a geodesic in Γ between x and y, $d_e(x, y) = d_g(x, y)$.

To see the utility of this approach, we state some examples which are shown more explicitly after the proof. If $n = 0$, $\Gamma = T\mathbb{R}^m$, and d_g is the distance function defined by the length of geodesics between points, then the construction exactly matches the conditions for the theorem above.

In Anderson, Dahl and Howard [1], the setting they use is that $n = 1$, g is time orientable, and Γ is the set of future (and past) pointing time-like vectors. In parallel d_g is the negative distance function defined by the length of geodesics between two points.

Proof. 2.2.2

Let $x, y \in \partial\Omega$ such that there exists a geodesic in Γ between x and y in Γ. Using the same argument as in 2.1.1, we see that the geodesic from x to y with respect to g must enter and leave Ω in the same direction. If the lines coming out were not parallel, we would be able to connect the two new ends by a straight line that is away from Ω. Thus, there is another geodesic connecting points along the geodesic from x to y with respect to g. This contradicts the strongly geodesic minimizing condition. Thus, the geodesic from x to y with respect to g must enter and leave Ω in the same direction.

We now show that $d_g(x, y) \leq d_e(x, y)$. Let $\gamma_g : \mathbb{R} \to \mathbb{R}^{n+m}$ be the unit speed geodesic through from x to y with respect to g, where $\gamma_g(0) = x$, $\gamma_g(T) = y$ for some T. For convenience, we will notate $\alpha_{a,b}$ for $a, b \in \mathbb{R}^{n+m}$ to be the geodesic between that a and b with respect to e. Let z be such that $x+z$ and $y+z$ are outside of Ω and $\alpha_{x+z,y+z}$ is an element of Γ that lays in the complement set of Ω. Additionally assume that $e(\dot{\gamma}(0), \dot{\alpha}_{x,x+z}(0)) = 0$. This is equivalent to saying that $\alpha_{x,x+z}$ is perpendicular to γ. By condition 3 of Definition 2.2.1, $d_e(x, y) = d_e(x + z, y + z)$. Note that since $\dot{\gamma}_g(0) = \dot{\gamma}_g(T)$, such a z must exist. Recall that γ is an element of Γ. By the fact that d_g is distance-like, and then by condition 1 for all sufficiently large t,

$$\begin{aligned} &d_g(\gamma(-t), x) + d_g(x, y) + d_g(y, \gamma(T + t)) \\ &\qquad \leq d_g(\gamma(-t), x + z) + d_g(x + z, y + z) + d_g(y + z, \gamma(T + t)). \end{aligned} \tag{2.4}$$

We note that by condition 4 of definition 2.2.1,

$$\lim_{t\to\infty} d_g(\gamma(-t), x + z) - d_g(\gamma(-t), x) = \lim_{t\to\infty} d_g(\gamma(T + t), y + z) - d_g(\gamma(T + t), y) = 0. \tag{2.5}$$

Hence,

$$d_g(x,y) \leq d_g(x+z, y+z). \tag{2.6}$$

As stated above, since $g = e$ outside Ω, and by condition 2 of definition 2.2.1, $d_g(x+z, y+z) = d_e(x+z, y+z) = d_e(x,y)$. Thus,

$$d_g(x,y) \leq d_e(x,y). \tag{2.7}$$

We now show that $d_e(x,y) \leq d_g(x,y)$. Let z be as above. Let $\gamma_e : \mathbb{R} \to \mathbb{R}^{n+m}$ be the unit speed geodesic in Γ that passes through $x+z$ to $y+z$ such that $\gamma_e(0) = x+z$ and $\gamma_e(T') = y+z$ for some T'. Using a similar approach to before, we find that by condition 1 in definition 2.2.1, for sufficiently large t,

$$\begin{aligned} d_g(\gamma_e(-t), x+z) + d_g(x+z, y+z) + d_g(y+z, \gamma_e(T'+t)) \\ \leq d_g(\gamma_e(-t), x) + d_g(x,y) + d_g(y, \gamma_e(T+t)). \end{aligned} \tag{2.8}$$

Note that outside of Ω, $g = e$, so by condition 2 of definition 2.2.1, $d_g = d_e$ outside of Ω, allowing us to phrase the above terms in terms of d_e with the sole exception of $d_g(x,y)$. Using condition 4 of definition 2.2.1 and taking a limit we find that

$$\begin{aligned} d_e(x+z, y+z) \leq \lim_{t\to\infty} d_e(\gamma_e(-t), x) - d_e(\gamma_e(-t), x+z) + d_g(x,y) + \\ \lim_{t\to\infty} d_e(\gamma_e(T'+t), y) - d_e(\gamma_e(T'+t), y+z) \end{aligned} \tag{2.9}$$

Hence, $d_e(x+z, y+z) \leq d_g(x,y)$. As stated above, since $g = e$ outside Ω and by conditions 2 and 3 of definition 2.2.1,

$$d_g(x+z, y+z) = d_e(x+z, y+z) = d_e(x,y). \tag{2.10}$$

Thus,

$$d_e(x, y) \leq d_g(x, y). \tag{2.11}$$

Combining these two inequalities together we get that

$$d_e(x, y) = d_g(x, y). \tag{2.12}$$

□

Theorem 2.2.2 is a very abstract result, and so we will narrow our focus down to the case we will work with most.

First, we will restate the Riemannian result in this new framework. Suppose $n = 0$, Γ is all geodesics, and $d_g(x, y)$ is the minimum length of C^1 curves between x and y.

As given in Lemma 2.1.2, this d_g satisfies the conditions for the theorem.

In a more complicated example, and the one we will consider for the next two chapters, let $n \geq 2$ and Γ be all time-like geodesics. Define

$$d_g(x, y) = -\sup\{\int_0^T g(\dot{\alpha}(r), \dot{\alpha}(r))dr \mid \alpha : [0, T] \to \mathbb{R}^{n+m} \text{ is a time-like piecewise } C^1 \text{ curve from } x \text{ to } y\} \tag{2.13}$$

This is, in truth, the negative of the standard definition for Pseudo-Riemannian distance, just phrased oddly. It is known that the negative length of time-like geodesics between two points is equal to this distance. If the geodesics of Γ uniquely connect points, then geodesics are the unique minimizers for the above.

To see that d_g is distance-like, we note that here $d_g(x, y)$ will be the negative length of the geodesic between x and y. For the first condition, let γ_g be the unit length geodesic through x and y where $\gamma_g(0) = x$ and $\gamma_g(T) = y$ for some T. Then since the space of time-like vectors is open at all points in TK, and eventually γ_g exits Ω, there will be some r such that for $t \geq r$ there is a time-like geodesic between $\gamma_g(-t)$ and a and a time-like geodesic between b and $\gamma_g(T+t)$. The curve described by the geodesics between $\gamma_g(t)$ to a, a to b, then b to $\gamma_g(T+t)$ has negative length greater than or equal to the geodesic between x and y, and the negative length of this curve is the three distances $d_g(\gamma_g(-t), a) + d_g(a, b) + d_g(b, \gamma_g(T+t)$.

Conditions 2 and 3 are direct consequences of the fact that distances are given by the negative length of geodesics between points in Γ. To see the last condition for Definition 2.2.1, it is easiest to separate $\mathbb{R}^{n+m}$ into $\mathbb{R}^n$ and $\mathbb{R}^m$, then apply the Pythagorean theorem on each.

Thus, Theorem 2.2.2 applies. To further push the result, we notice that if K is some set where $\Omega \subset K$ and K is convex with respect to Euclidean geometry, then for every $x, y \in \partial K$, $d_e(x, y) = d_g(x, y)$. This is due to the fact that K will be strongly geodesic minimizing since Ω is strongly geodesic minimizing.

Define $\delta_{-}T\Omega = \{(z, v) \in \partial\Omega \mid v \text{ is an inward pointing unit vector}\}$. Similarly, define $\delta_{+}T\Omega = \{(z, v) \in \partial\Omega \mid v \text{ is an outward pointing unit vector}\}$.

Next we will define the scattering map as a map $S_g(z, v) : \partial_{-}T\Omega \to \partial_{+}T\Omega$ to be where the geodesic issued from z in the v direction according to g exits Ω and the direction it exits in. Frequently, we call the scattering map the scattering relation. For a certain class of manifolds, including simple manifolds, it is known that two metrics have the same boundary distances if and only if they share the scattering map (see [16] for precise details). In addition, we will refer to [12] to state that for a complete manifold, there is a geodesic between any two points on that manifold. Meaning that a complete manifold without conjugate points is strongly geodesic minimizing.

This gives the following corollary which will be the starting point for chapter 3.

Corollary 2.2.3 (A.)**.** If g is a complete Pseudo-Riemannian metric of degree $n \geq$

$1, m \geq 1$ that is equal to e outside of some Ω and has no conjugate points over time-like geodesics, then, with d_g defined as the length of the geodesic between two points, for all $x, y \in \partial\Omega \cup \Omega^c$ such that x and y are connected by a time-like geodesic, $d_e(x, y) = d_g(x, y)$ and the scattering relation for e and g must be the same.

Proof. The first part of the statement was shown above. That is, for all $x, y \in \partial\Omega\cup\Omega^c$ such that x and y are connected by a time-like geodesic, $d_e(x, y) = d_g(x, y)$. Suppose that for some $(x, v) \in \delta_- T\Omega$, the scattering map according to g and e results in different points $(y_g, v) \neq (y_e, v) \in \delta_+ T\Omega$. The proof of Theorem 2.2.2 already showed that they must have the same v. Let γ be the geodesic from x in the v direction and T be a number such that $\gamma(T) = y_g$. Then T is the distance from x to y_g. Let ε be a number such that the line from x to $\gamma(T + \varepsilon)$ is time-like. This line is not equal to γ outside of Ω by assumption. Then $d(x, \gamma(T + \varepsilon) < d(x, y_g) + d(y_g, \gamma(T + \varepsilon)$. The latter term is $d(x, \gamma(T + \varepsilon)$, which is a contradiction. Thus e and g have the same scattering relation. □

Chapter 3

From Boundary Rigidity to Integral Geometry

3.1 Formulation of the problem

In this chapter, we reduce the boundary rigidity problem obtained in Chapter 2 to an integral geometry problem, which we then solve in Chapter 4. We begin with the formulation of the problem.

Let e be the flat Pseudo-Riemannian metric on $\mathbb{R}^{n+m}$ with signature $(n,m), n \geq 2, m \geq 1$. Let $\Omega \subset \mathbb{R}^{n+m}$ be a bounded open set with smooth boundary $\partial\Omega$. We let g be a smooth Pseudo-Riemannian metric on $\mathbb{R}^{n+m}$ such that $g = e$ on Ω^c. For simplicity, we assume that g is separable and write $g = (-g_N, g_M)$ where g_N, g_M are positive definite matrices of degree n and m respectively. We remark that the argument of this chapter also works for general Pseudo-Riemannian metrics as long as the scattering relations can be reasonably defined. We are concerned with time-like geodesics. In this setting, these geodesics can be parametrized by

$$\gamma(t) = \exp^g_{(z_N, z_M)} t(-v_N, v_M) \tag{3.1}$$

where $\exp^g$ denotes the exponential map for g, $z_N \in \mathbb{R}^n, z_M \in \mathbb{R}^m, v_M \in S_{z_M} \doteq \{v \in T_{z_M}M : g_M(v,v) = 1\}$ and $v_N \in B_{z_N} \doteq \{v \in T_{z_N}N : g_N(v,v) > 1\}$.

Following from the setup in Chapter 2, we assume that the scattering relations for e and g for time-like curves are the same. Because of the product structure of the manifold, we can describe the scattering relation in more explicit terms. Without loss of generality, we assume that $\Omega \subset \mathbb{B}^n \times \mathbb{B}^m$ where $\mathbb{B}^n$ denotes the unit ball in $\mathbb{R}^n$. If we let Γ be the time-like geodesics, then we can parametrize time-like geodesics through $\mathbb{B}^n \times \mathbb{B}^m$ by using boundary points and inward pointing vectors as in the Riemannian case. We note that $\partial(\mathbb{B}^n \times \mathbb{B}^m) = \partial\mathbb{B}^n \times \mathbb{B}^m \cup \mathbb{B}^n \times \partial\mathbb{B}^m \cup \partial\mathbb{B}^n \times \partial\mathbb{B}^m$.

Let

$$\Gamma_- = \{(z,v) : z \in \partial(\mathbb{B}^m \times \mathbb{B}^n) \mid v \in T_z \text{ is inward pointing}, g(v,v) = -1\},$$

and

$$\Gamma_+ = \{(z,v) : z \in \partial(\mathbb{B}^m \times \mathbb{B}^n) \mid v \in T_z \text{ is outward pointing}, g(v,v) = -1\}.$$

The scattering relation for e can be seen as a map

$$\begin{aligned} S_e : \Gamma_- &\to \Gamma_+ \\ (z_N, v_N, z_M, v_M) &\to (\gamma(t_0), \dot{\gamma}(t_0)) \end{aligned} \tag{3.2}$$

where $\gamma(t)$ is the unique time-like geodesic (3.1) with

$$\gamma(0) = (z_N, z_M), \quad \dot{\gamma}(0) = (v_N, v_M),$$

which exits $\mathbb{B}^n \times \mathbb{B}^m$ at $t = t_0$.. If we assume that $\mathbb{B}^n \times \mathbb{B}^m$ is time-like geodesiccally convex then scattering relation can be defined for g. Note that this is a slightly stronger assumption than strongly geodesic maximizing. As in Section 2.2, we denote the scattering relation by S_g.

Now we introduce the geodesic ray transform for time-like geodesics. Consider $(\mathbb{R}^{n+m}, g)$. We can again parametrize time-like geodesics as in equation (3.1). Equivalently, given $(z_N, z_M) \in \mathbb{R}^{n+m}$ and $(v_N, v_M) \in T_{(z_N,z_M)}\mathbb{R}^{n+m}$ in Γ, we let γ_g be the unique geodesic for g satisfying the geodesic equation $\nabla_{\dot{\gamma}_g(t)}\dot{\gamma}_g(t) = 0$ with initial conditions

$$\gamma_g(0) = (z_N, z_M), \quad \dot{\gamma}_g(0) = (v_N, v_M).$$

Let h be a sufficiently regular 2-tensor field on $\mathbb{R}^{n+m}$ supported in Ω. We define a geodesic ray transform as

$$I_g h(z_N, z_M, v_N, v_M) = \int_{\mathbb{R}} \sum_{j,k=1}^{n+m} h_{jk}(\gamma_g(t))\dot{\gamma}_g^j(t)\dot{\gamma}_g^k(t)dt. \tag{3.3}$$

For proving the main results of this thesis, it suffices to consider the transform for scalar functions. However, the result for this chapter also applies to tensors. We discuss this possibility in Section 3.4.

Explicitly, the goal of the chapter is to prove the following

Theorem 3.1.1 (A.)**.** Assume $\mathbb{B}^n \times \mathbb{B}^m$ is time-like geodesically convex for g. If $S_e = S_g$ as defined in equation (3.2), then $I_g h = 0$ with $h = g - e$.

3.2 Derivation of the geodesic ray transform

To prove Theorem 3.1.1, we adapt the method in Stefanov and Uhlmann [19] to the Pseudo-Riemannian setting. Since we know one of the metrics is constant, we can simplify some of the calculations.

As stated in Chapter 2, we can define geodesics using the Levi-Civita connection. However for the argument below it will be useful to describe geodesics in the tangent bundle from the Hamiltonian point of view. The advantage to this method is that we have a clear relation between the vectors defining the geodesics and the metric. We will take great care to keep universal coordinates for $z = (z_N, z_M) \in \mathbb{R}^{n+m}$.

Define the Hamiltonian related to g to be

$$H_g(z, v) = \frac{1}{2}(\sum_{i,j=1}^{n+m} g^{ij}(z) v_i v_j + 1)$$

where $z \in \mathbb{R}^{n+m}$ and $v \in T_z\mathbb{R}^{n+m}$ is a vector in Γ_-. We remark that all time-like vectors can be rescaled to satisfy this condition. The scaling will be convenient for the analysis below, and it is the reason why we included $+1$ in the Hamiltonian. The Hamiltonian for e is denoted by H_e.

Suppose that $(z^{(0)}, v^{(0)}) \in T\mathbb{R}^{n+m}$ with $z^{(0)} = (z_N^{(0)}, z_M^{(0)}), v^{(0)} = (v_N^{(0)}, v_M^{(0)})$. For $r \geq 0$, we let $X_g(r, z^{(0)}, v^{(0)}) = (z(r), v(r))$ be the solution to the Hamiltonian system

$$\begin{gathered} \frac{\partial}{\partial r} z_l = \sum_{j=1}^{n+m} g^{lj} v_j, \quad \frac{\partial}{\partial r} v_l = \frac{-1}{2} \sum_{i,j=1}^{n+m} \frac{\partial g^{ij}}{\partial z_l} v_i v_j \\ z|_{r=0} = z^{(0)}, \quad v|_{r=0} = v^{(0)} \end{gathered} \tag{3.4}$$

where $l = 1, 2, \cdots, n + m$. It is wellknown that the projection of X_g to the base manifold $\mathbb{R}^{n+m}$ gives exactly the unit speed parameterization of the time-like geodesic issued from $z^{(0)}$ in the $v^{(0)}$ direction. The benefit to defining the geodesic in this manner is that there is an explicit relation between the formulas defining the geodesic and the metric. Of course, in the case of the flat metric e,

$$X_e(r, z_N^{(0)}, z_M^{(0)}, v_N^{(0)}, v_M^{(0)}) = (z_N^{(0)} + r v_N^{(0)}, z_M^{(0)} + r v_M^{(0)}, v_N^{(0)}, v_M^{(0)}). \tag{3.5}$$

Proof of Theorem 3.1.1. By the assumption that $\overline{\Omega} \subset \mathbb{B}^n \times \mathbb{B}^m$, we know that $g = e$ on the boundary of $\mathbb{B}^n \times \mathbb{B}^m$. Let $z^{(0)} = (z_N, z_M), v^{(0)} = (v_N, v_M)$ be such that $(z_N, v_N, z_M, v_M) \in \Gamma_-$. Then we consider the Hamiltonian flow for metrics g and e with initial condition $X^{(0)} = (z^{(0)}, v^{(0)})$, denoted by $X_g(r, X^{(0)}), X_e(r, X^{(0)})$. Accordinging to the assumption that $S_g = S_e$, we know that the flows exit $\mathbb{B}^n \times \mathbb{B}^m$ at the same time $r_0 > 0$ and

$$X_g(r_0, X^{(0)}) = X_e(r_0, X^{(0)}). \tag{3.6}$$

We define the function central to this proof as

$$F(r) := X_e(r_0 - r, X_g(r, X^{(0)})). \tag{3.7}$$

Because of (3.6), we know that $F(0) = F(r_0)$. Here, we remark that the z component of $F(r)$ may be outside of $\mathbb{B}^n \times \mathbb{B}^m$ for $r \in [0, r_0]$. However, the Hamiltonian flows X_g, X_e are well-defined even for non-time-like geodesics on $\mathbb{R}^n \times \mathbb{R}^m$, so this is not an issue. We find that

$$\int_0^{r_0} F'(r)dr = 0. \tag{3.8}$$

To compute the derivative, we let

$$V_g = (\partial H_g/\partial v, -\partial H_g/\partial z)$$

be the Hamilton vector field for g and V_e for e. Similarly to the calculation in (2.8) of [19], we have

$$F'(r) = -V_e(X_e(r_0 - r, X_g(r, X^{(0)}))) + \frac{\partial X_e}{\partial X^{(0)}}(r_0 - r, X_g(r, X^{(0)}))V_g(X_g(r, X^{(0)})) \tag{3.9}$$

The first term on the right hand side can be transformed by using the following observation. For any $R > 0$,

$$\begin{aligned} 0 &= \frac{d}{dr}|_{r=0} X(R - r, X(r, X^{(0)})) \\ &= -V(X(R, X^{(0)})) + \frac{\partial X}{\partial X^{(0)}}(R, X^{(0)})V(X^{(0)}) \end{aligned} \tag{3.10}$$

in which X, V can be either X_g, V_g or X_e, V_e as appropriate. Setting $R = r_0 - r$ and applying (3.10) to X_e, V_e, we get

$$V_e(X_e(r_0 - r, X_g(r, X^{(0)}))) = \frac{\partial X_e}{\partial X^{(0)}}(r_0 - r, X_g(r, X^{(0)}))V_e(X_g(r, X^{(0)})). \tag{3.11}$$

Combining (3.8), (3.9) and (3.11), we arrive at

$$\int_0^{r_0} \frac{\partial X_e}{\partial X^{(0)}}(r_0 - r, X_g(r, X^{(0)}))(V_g - V_e)(X_g(r, X^{(0)}))dr = 0. \tag{3.12}$$

By the fact that e is a constant metric, we can see from the expression of the Hamilton vector field (3.4) that

$$V_g - V_e = (\sum_{j=1}^{n+m} g^{lj} v_j, -\frac{1}{2}\sum_{i,j=1}^{n+m} \frac{\partial g^{ij}}{\partial z_l} v_i v_j).$$

Also, from (3.5), we see that $\partial X_e / \partial X^{(0)}$ is the identity matrix. Thus (3.12) can be simplified to

$$\int_0^{r_0} (V_g - V_e)(X_g(r, X^{(0)}))dr = 0 \tag{3.13}$$

and the last $(n+m)$ component gives for $l = 1, 2, \cdots, n+m$

$$\int_0^{r_0} -\frac{1}{2}\sum_{i,j=1}^{n+m} \frac{\partial g^{ij}(\gamma_g(r))}{\partial z_l} v_i(\gamma_g(r)) v_j(\gamma_g(r))dr = 0 \tag{3.14}$$

where $\gamma_g(r)$ is the projection of X_g to the base manifold, or in other words, the geodesic from $z^{(0)}$ in direction $v^{(0)}$. This completes the proof of Theorem 3.1.1.

□

Remark 3.2.1. If we switch the role of g and e in the proof, we would get an integral transform similar to (3.14) for the flat metric e but with a weight. The flow structure for e is well understood. For small metric perturbations studied in [19], the structure of the weight can be analyzed. However, in general, the weight is quite complicated. This is the reason that we choose to derive (3.14).

For metrics that are conformal to e, we can reduce (3.14) to a transform for scalar functions. Let ϕ be a positive smooth function on $\mathbb{R}^n \times \mathbb{R}^m$ and assume that $g = \phi e$. Then we find that for $l = 1, 2, \cdots, n+m$,

$$\frac{\partial g^{ij}}{\partial z_l} v_i v_j = \sum_{i,j=1}^{n+m} \frac{\partial (\phi(z) e)^{ij}}{\partial z_l} v_i v_j \tag{3.15}$$

$$= \sum_{i,j=1}^{n+m} \frac{\partial \phi(z)}{\partial z_l} e^{ij} v_i v_j + \phi(z) \sum_{i,j=1}^{n+m} \frac{\partial e^{ij}}{\partial z_l} v_i v_j \tag{3.16}$$

$$= \sum_{i,j=1}^{n+m} \frac{\partial \phi(z)}{\partial z_l} e^{ij} v_i v_j \tag{3.17}$$

$$= \frac{\partial \phi(z)}{\partial z_l} \phi(z)^{-1} \tag{3.18}$$

where in the last line, we used that $g(v, v) = -1$ so $\phi e(v, v) = -1$. Thus (3.14) becomes (ignoring the scalar factor $-1/2$)

$$0 = \int_0^{r_0} \frac{\partial g^{ij}}{\partial z_l} v_i v_j dr = \int_0^{r_0} \frac{\partial \phi(\gamma_g(r))}{\partial z_l} \phi(\gamma_g(r))^{-1} dr \tag{3.19}$$

for $l = 1, 2, \cdots, n+m$.

3.3 Proof of the Hopf-type theorem

At this point, we can prove some Hopf-type theorems in Pseudo-Riemannian geometry, assuming that the geodesic ray transform I_g is injective. More precisely, we consider the transform on scalar functions. Let f be a smooth function on $\mathbb{R}^n \times \mathbb{R}^m$ supported in Ω. Let γ_g be any time-like geodesic in $(\mathbb{R}^n \times \mathbb{R}^m, g)$. The injectivity means that if

$$I_g f(\gamma_g) = \int f(\gamma_g(r)) dr = 0,$$

then $f = 0$. We will prove in Chapter 4 that this is true for generic separable metrics.

Theorem 3.3.1. Let g be a smooth Pseudo-Riemannian metric on $\mathbb{R}^n \times \mathbb{R}^m, n \geq 2, m \geq 1$ conformal to e and $g = e$ outside of a bounded open set Ω. Suppose that

1. There are no conjugate points along any time-like geodesic.
2. The metric g is geodesically convex on Ω
3. The geodesic ray transform I_g along time-like curves is injective on scalar functions.

Then $g = e$.

Proof. Let $g = \phi e$ with $\phi > 0$ a smooth function. First, from Chapter 2, we can derive from the assumptions 1 and 2 that $S_e = S_g$. Then Theorem 3.1.1 implies that for $l = 1, 2, \cdots, n+m$

$$\int \frac{\partial \phi(\gamma_g(r))}{\partial z_l} \phi(\gamma_g(r))^{-1} dr = 0$$

for any time-like geodesic γ_g for $(\mathbb{R}^n \times \mathbb{R}^m, g)$. By assumption 3, we know that

$$\frac{\partial \phi(z)}{\partial z_l} \phi(z)^{-1} = 0, \quad l = 1, 2, \cdots, n+m.$$

Thus, $\frac{\partial \phi(z)}{\partial z_l} = 0$. Because $\phi = 1$ on Ω^c, we deduce that $\phi = 1$ is a constant function on $\mathbb{R}^n \times \mathbb{R}^m$. Thus $g = e$. □

3.4 Remarks on tensor problems

From the proof of Theorem 3.1.1, we can see that the argument works for a much larger class of Pseudo-Riemannian metrics, not necessarily just separable metrics. If we can show that the geodesic ray transform $I_g h$ defined in (3.3) is "injective", then it is possible to get a Hopf-type theorem for g. However, we expect that the geodesic ray transforms acting on 2-tensors has a kernel. So the injectivity only makes sense modulo the kernel.

We explain the issue in the Riemannian setting which is somewhat well-understood. We refer to the book [18] of Sharafutidnov for details. Let Ω be a bounded domain of $\mathbb{R}^m$ with smooth boundary. For ease, we let g be a simple Riemannian metric on Ω. That is, Ω is strictly convex with respect to g, and for any $x \in \overline{\Omega}$ the exponential map is a diffeomorphism. Given a symmetric 2-tensor $f = f_{ij}$, we define the 1-tensor $\delta^s f$, called the divergence of f by

$$(\delta^s f)_i = g^{jk} \nabla_k f_{ij}$$

where ∇_i are the covariant derivatives for g. Given a 1-tensor field v, we denote by $d^s v$ the 2-tensor called the symmetric differential of v

$$(d^s v)_{ij} = \frac{1}{2}(\nabla_i v_j + \nabla_j v_i).$$

It is known that for symmetric 2-tensor field f in $L^2(\Omega)$, we can decompose

$$f = f^s + d^s v \tag{3.20}$$

such that

1. $f^s \in L^2(\Omega)$ is solenoidal, that is $\delta^s f^s = 0$.
2. $v \in H_0^1(\Omega)$ so $v = 0$ on $\partial\Omega$.

Now we consider the geodesic ray transform defined by

$$I_g f(\gamma) = \int f_{ij}(\gamma(t))\dot{\gamma}^i(t)\dot{\gamma}^j(t)dt$$

where γ is any geodesic on Ω. We can easily check that if $f = d^s v$ for some $v \in H_0^1(\Omega)$, then $I_g(f) = 0$ so $d^s v$ belongs to the kernel or null space of I_g. It is generally believed that this is the full null space. In the literature, this is often called the s-injectivity of I_g. For example, Sharafutidnov proved that this is true for metrics with specific and explicit upper bounds of the curvature, see [18].

In the Pseudo-Riemannian setting, the kernel problem is more complicated. For example, let's consider a Lorentzian metric g on $\mathbb{R}^{1+n}, n \geq 1$. The geodesic ray transform along light-like geodesics, also called the light ray transform, must have conformal metrics ϕg in the kernel when acting on 2-tensors. This can be verified easily via

$$I_g(\phi g)(\gamma) = \int \phi g_{ij} \dot{\gamma}^i(t) \dot{\gamma}^j(t) dt = 0$$

because γ is a null geodesic. The full characterization of the kernel has been established for some cases, see e.g. [4]. For the geodesic ray transform along time-like geodesics as we study in this thesis, conformal metrics will not be in the kernel. We expect that a decomposition similar to (3.20) holds, but we will not be going much further with this issue. We only wish to emphasize its role in proving the Hopf type theorem in our approach.

Chapter 4

Analysis of Geodesic Ray Transforms

In this chapter, we study the mapping properties of geodesic ray transforms for time-like geodesics. In particular, we let $g = (-g_N, g_M)$ be a separable metric on $\mathbb{R}^n \times \mathbb{R}^m$, and we analyze

$$I_g f(\gamma_g) = \int f(\gamma_g(t))dt$$

where γ_g is a time-like geodesic and f is a scalar function. We are interested in determining the injectivity of the transform and finding stability estimates. For a somewhat similar problem, namely the integral transform for light-like geodesics, Ilmavirta [10] investigated the injectivity question by using the Pestov energy method. It is well known (see for example [18]) that the method requires certain curvature conditions, so in the setting of [10], even though the metric is product type, there are curvature constraints on g_N and g_M. Besides this, there are requirements on the signature. We also remark that Pestov's energy method can yield stability estimate even though this is not done in [10]. The stability estimates are often referred to as conditional type because some a priori bound on f is needed.

In this chapter, we will take another approach similar to [20] for the Riemannian problem. The method does not require any curvature condition, and it produces injectivity of the ray transform for metrics in some open and dense subset of simple metrics. This type of injectivity result is often referred to as generic injectivity. The method relies on the analysis of the normal operator of the ray transform. Under proper signature conditions, namely $n \geq 2, m \geq 1$, we can show that the normal operator for I_g is an elliptic pseudo-differential operator. Besides the injectivity, the method allows us to obtain stability estimates in the conventional form.

In this chapter, we start with a brief summary of the tools from microlocal analysis in Section 4.1. Then we adapt the method from [22] to the Pseudo-Riemannian setting. Roughly speaking, the method has two components. The first one is the injectivity for analytic metrics. This only relies on a property of the geodesic ray transform near a fixed geodesic, which applies to our setting with minor changes. We discuss it in Section 4.4. The second component is the microlocal analysis of the normal operator. Instead of trying to adapt the results from [22], we calculate the Schwartz kernel and show directly that the normal operators are elliptic pseudo-differential operators under the signature condition $n \geq 2$. These are done in Sections 4.2 and 4.3. Before we delve into the analysis, we elaborate on the necessity of the signature condition.

It has been known for a long time that the inversion of the geodesic ray transform on Riemannian manifolds with dimension ≥ 3 is an over-determined problem, see for instance [18]. Thus, it is possible to study the inversion problem with less geodesics. For example, in the seminal work [26], the local geodesic ray transform was studied from this point of view. In an interesting paper [22], Stefanov and Uhlmann studied the tensor tomography problem on non-simple Riemannian manifolds. Roughly speaking, they showed that for any $(x, \xi) \in T^*X$ for some Riemannian manifold (X, h), as long as there is a geodesic curve normal to ξ at x, the normal operator of the geodesic ray transform will be elliptic (as a pseudo-differential operator) at (x, ξ).

We expect the same principle to be true for geodesic ray transform in Pseudo-Riemannian geometry. Now let's consider the flat metric e on $N \times M$ with $N = \mathbb{R}^n, M = \mathbb{R}^m$. Suppose ζ is a non-zero co-vector at $z = (t, x)$. Then we can split $\zeta = (\tau, \xi)$ so that $\tau \in T_t^*N$ and $\xi \in T_x^*M$. We claim that there is time-like vector v at z such that the Euclidean inner product $\zeta \cdot v = 0$. In fact, we can write $v = (v_N, v_M)$ with $v_N \in T_tN, v_M \in T_xM$. We know that $-|v_N| + |v_M| < 0$. We also have

$$\tau \cdot v_N + \xi \cdot v_M = 0. \tag{4.1}$$

If $\tau = 0$, this is clearly true. Suppose $\tau \neq 0$. Without loss of generality, we write $\tau = (\tau_1, \tau_2, \cdots, \tau_n)$ and assume that $(\tau_1, \tau_2) \neq 0$. Here, we used $n \geq 2$. Then we can choose $(a_1, a_2) \neq 0$ such that $(a_1, a_2) \cdot (\tau_1, \tau_2) = 0$. With $v_N = (a_1, a_2, 0, \cdots, 0)$, we see that (4.1) is satisfied with any v_M. By rescaling v_N, we can arrange $|v_M| < |v_N|$. Thus if $n \geq 2$, we can easily find v_N, v_M to satisfy $|v_M| < |v_N|$ so $v = (v_N, v_M)$ is a time-like vector. It is here that the importance of having signature (n, m) with $n \geq 2, m \geq 1$ becomes clear. This allows us to have the freedom to have time-like covectors normal to all other covectors. Following [10], it seems that this is a crucial condition for such X-ray transforms to be injective.

4.1 Basics of pseudo-differential operators

We collect some facts about pseudo-differential operators that we will use in the following sections. We also take this opportunity to introduce the notation. We refer the readers to [5, 9, 24] for the detailed discussions on the general topics of microlocal analysis.

We start with symbols. Let $X \subset \mathbb{R}^n$ be an open set and $m \in \mathbb{R}, N \in \mathbb{Z}_+$.

Definition 4.1.1. $S^m(X \times \mathbb{R}^N)$ is the space of all $a \in C^\infty(X \times \mathbb{R}^N)$ such that for all compact set K of X and all multi-index $\alpha = (\alpha_1, \cdots, \alpha_n) \in \mathbb{N}^n$ and multi-index

$\beta \in \mathbb{N}^N$, there is a constant $C = C_{K,\alpha,\beta}(a)$ such that

$$|\partial_x^\alpha \partial_\theta^\beta a(x,\theta)| \leq C(1+|\theta|)^{m-|\beta|}, \quad (x,\theta) \in K \times \mathbb{R}^N.$$

We say S^m is the space of symbols of order m.

The space of symbols of order $-\infty$ is useful. This is defined as

$$\begin{gathered} S^{-\infty}(X \times \mathbb{R}^N) = \{a \in C^\infty(X \times \mathbb{R}^N) : \text{for every compact } K \subset X, \\ \text{all multi-index } \alpha \in \mathbb{N}^n, \text{ multi-index } \beta \in \mathbb{N}^N \text{ and } M \in \mathbb{R}, \\ \text{there is a constant } C = C_{K,\alpha,\beta,M}(a) \text{ such that} \\ |\partial_x^\alpha \partial_\theta^\beta a(x,\theta)| \leq C(1+|\theta|)^{-M}, \quad (x,\theta) \in K \times \mathbb{R}^N\}. \end{gathered}$$

In fact, $S^{-\infty}(X \times \mathbb{R}^N) = \bigcap_{m\in\mathbb{R}} S^m(X \times \mathbb{R}^N)$.

Definition 4.1.2. A pseudo-differential operator $A : C_0^\infty(X) \to \mathcal{D}'(X)$ is of the form

$$Au(x) = \frac{1}{(2\pi)^n} \int\int e^{i(x-y)\theta} a(x,\theta) u(y) dy d\theta, \quad u \in C_0^\infty(X)$$

where $a \in S^m(X \times \mathbb{R}^n)$ is called the complete symbol of A. We also write $a = \sigma_A$. Here $\mathcal{D}'(X)$ denotes the set of distributions on X. We denote by $\Psi^m(X)$ the space of pseudo-differential operators of order $\leq m$.

We recall a fact from distribution theory. If $X \subset \mathbb{R}^{n_X}, Y \subset \mathbb{R}^{n_Y}$ are open sets and $A : C_0^\infty(Y) \to \mathcal{D}'(X)$ is continuous and linear with distributional kernel $K_A(x,y) \in \mathcal{D}'(X \times Y)$, then A can be extended to a continuous operator $\mathcal{E}'(Y) \to C^\infty(X)$ equivalent to $K_A \in C^\infty(X \times Y)$. Such operators A are called smoothing. We let $\Psi^{-\infty}(X)$ be the set of operators with $a \in S^{-\infty}(X \times \mathbb{R}^n)$. One can verify that such operators have smooth kernels and $\Psi^{-\infty}(X)$ is the space of smoothing operators $\mathcal{E}'(X) \to C^\infty(X)$.

Let $X \subset \mathbb{R}^{n_X}, Y \subset \mathbb{R}^{n_Y}$ be open sets. If C is a closed subset of $X \times Y$, we say C is proper if the two projections

$$\pi_X : (x, y) \in C \to X,$$

$$\pi_Y : (x, y) \in C \to Y$$

are proper. This means the pre-image of every compact set of X or Y respectively is compact. An operator $A \in \Psi^m(X)$ is called properly supported if $\text{supp}(K_A) \subset X \times X$ is proper. It is a useful fact that any $A \in \Psi^m(X)$ can be decomposed as $A = A' + A''$ where $A' \in \Psi^m(X)$ is properly supported and $A'' \in \Psi^{-\infty}(X)$.

Next, we discuss elliptic operators.

Definition 4.1.3. If $A \in \Psi^m(X)$, we define the principal symbol of A as the image of the complete symbol σ_A in $S^m(X \times \mathbb{R}^n)/S^{m-1}(X \times \mathbb{R}^n)$.

Let $P \in \Psi^m(X)$. We say P is elliptic at $(x_0, \xi_0) \in X \times \mathbb{R}^n \backslash \{0\}$ if there is a conic neighborhood V of (x_0, ξ_0) and $C > 0$ such that

$$|\sigma_P(x, \xi)| \geq \frac{1}{C}(1 + |\xi|)^m$$

for $(x, \xi) \in V$ and $|\xi| \geq C$. We say P is elliptic at $x_0 \in X$ if P is elliptic at (x_0, ξ_0) for every $\xi_0 \in \mathbb{R}^n \backslash \{0\}$. We say P is elliptic on $Y \subset X$ if P is elliptic for every $x_0 \in Y$. A key result for elliptic operator is the following.

Theorem 4.1.4. If $P \in \Psi^m(X)$ is elliptic, then there exists $Q \in \Psi^{-m}(X)$ properly supported such that

$$P \circ Q = Q \circ P = I \text{ mod } \Psi^{-\infty}.$$

Here, I denotes the identity operator. Also, Q is unique modulo $\Psi^{-\infty}(X)$.

Finally, we consider mapping properties of pseudo-differential operators on Sobolev spaces. We recall that $H^s(\mathbb{R}^n)$ is the space of tempered distributions $u \in \mathcal{S}'(\mathbb{R}^n)$ such that $\hat{u}(\xi)$ is locally square integrable and

$$\|u\|^2_{H^s(\mathbb{R}^n)} = \|u\|^2_s = \frac{1}{2\pi}\int |\hat{u}(\xi)|^2(1+|\xi|^2)^s d\xi < \infty.$$

It is well known that $H^s(\mathbb{R}^n)$ is a Hilbert space. For $s \in \mathbb{N}$, $H^s(\mathbb{R}^n)$ is also the space of $u \in L^2(\mathbb{R}^n)$ such that $D^\alpha u \in L^2(\mathbb{R}^n)$ for $|\alpha| \le s$. Then we define

$$H^s_{loc}(X) = \{u \in \mathcal{D}'(X) : \phi u \in H^s(\mathbb{R}^n), \forall \phi \in C_0^\infty(X)\}.$$

Theorem 4.1.5. Let $A \in \Psi^m(X)$ be properly supported. Then $A : H^s_{loc}(X) \to H^{s-m}_{loc}(X)$ is continuous for every $s \in \mathbb{R}$. If A is elliptic, then for every $u \in \mathcal{D}'(X)$, we have $u \in H^s_{loc}(X)$ if and only if $Au \in H^{s-m}_{loc}(X)$.

4.2 The normal operator for the flat metric

In this section, we use the flat Pseudo-Riemannian metrics to demonstrate the structure of the normal operator. Consider $\mathbb{R}^{2+2}$ with metric

$$g = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

We use $z = (t, x) = (t_1, t_2, x_1, x_2)$ as coordinates for $\mathbb{R}^{2+2}$. At any $(t, x) \in \mathbb{R}^2$, we consider time-like vectors of the form (v, w) where $v \in \mathbb{S}^1, w \in \mathbb{B}^1$ where $\mathbb{B}^1 = \{v \in \mathbb{R}^2 : |v| < 1\}$. It is easy to see that all time-like vectors at (t, x) are scalar multiples of such vectors. Then all time-like geodesics passing through (t, x) can be written as

$$\gamma_{t,x,v,w}(s) = (t + sv, x + sw), \quad s \in \mathbb{R}.$$

Note that there is some redundancy in this parametrization. In this case, for $f \in C_0^\infty(\mathbb{R}^{2+2})$, the time-like ray transform can be defined as

$$I_g f(t, x, v, w) = \int_{\mathbb{R}} f(t + sv, x + sw) ds,$$

which is a smooth function on $\mathbb{R}^4 \times \mathbb{S}^1 \times \mathbb{B}^1$. Note that recovery of f (which is a function of four variables) from $I_g f$ (which is a function of seven variables) is formally over-determined. To avoid some singularities in the analysis, we introduce a smooth weight function in I_g. Let $\chi \in C^\infty(\mathbb{R})$ be a cut-off function such that $\chi(\tau) = 1$ for $|\tau| > 3/2$ and $\chi(\tau) = 0$ for $|\tau| < 1$. Then by abusing the notation, we consider

$$I_g f(t, x, v, w) = \int_{\mathbb{R}} \chi(|v|/|w|) f(t + sv, x + sw) ds$$

Note that $\chi(|v|/|w|)$ is supported in $|v|/|w| > 1$, which is the set of time-like vectors. But it vanishes at the boundary $|v| = |w|$. We also assume that f is compactly supported in an open set $\Omega \subset \mathbb{R}^4$.

Lemma 4.2.1. Let I_g^* be the L^2 adjoint of I_g. Then for $(t', x') \in \mathbb{R}^2 \times \mathbb{R}^2$

$$I_g^* h(t', x') = \int_{\mathbb{B}^1} \int_{\mathbb{S}^1} \int_{\mathbb{R}} \chi(|v|/|w|) h(t' - sv, x' - sw, v, w) ds dv dw.$$

Proof. We start with the L^2 pairing

$$\langle I_g^* h, f\rangle = \langle h, I_g f\rangle$$
$$= \int_{\mathbb{R}^2}\int_{\mathbb{B}^1}\int_{\mathbb{S}^1}\int_{\mathbb{R}}\int_{\mathbb{R}^2} h(t,x,v,w)\chi(|v|/|w|)f(t+sv,x+sw)dtdsdvdwdx$$
$$= \int_{\mathbb{R}^2}\int_{\mathbb{B}^1}\int_{\mathbb{S}^1}\int_{\mathbb{R}}\int_{\mathbb{R}^2} h(t-sv,x-sw,v,w)\chi(|v|/|w|)f(t,x)dtdsdvdwdx.$$

□

Lemma 4.2.2. Let $N_g = I_g^* I_g$ be the normal operator. Then the Schwartz kernel of N_g is given by

$$K(t,x,t',x') = C\chi(\frac{|t-t'|}{|x-x'|})^2\frac{1}{|x-x'|^3}$$

where $C > 0$ depends on Ω. Additionally, $N_g \in \Psi^{-1}(\mathbb{R}^{2+2})$ is elliptic.

Proof. We start from

$$I_g^* I_g f(t',x') = \int_{\mathbb{R}}\int_{\mathbb{B}^1}\int_{\mathbb{S}^1}\chi(|v|/|w|)I_g f(t'-sv,x'-sw,v,w)dsdvdw$$
$$= \int_{\mathbb{R}}\int_{\mathbb{R}}\int_{\mathbb{B}^1}\int_{\mathbb{S}^1}\chi(|v|/|w|)^2 f(t'-sv+rv,x'-sw+rw)drdsdvdw.$$

We split the integral to $r-s \geq 0$ and $r-s \leq 0$. For the first case, we let $\alpha = r-s \geq 0$ and for the second case, we let $\beta = s-r \geq 0$. We have

$$I_g^* I_g f(t',x') = N_+ f(t',x') + N_- f(t',x')$$

where

$$N_+ f(t',x') = C\int_0^\infty\int_{\mathbb{B}^1}\int_{\mathbb{S}^1}\chi(|v|/|w|)^2 f(t'+\alpha v,x'+\alpha w)d\alpha dvdw,$$
$$N_- f(t',x') = C\int_0^\infty\int_{\mathbb{B}^1}\int_{\mathbb{S}^1}\chi(|v|/|w|)^2 f(t'+\beta v,x'+\beta w)d\beta dvdw.$$

Here, we used that the integration in s happens on a finite interval depending on Ω. Let $t'' = t' + \alpha t$. We get

$$N_+(t', x') = C \int_{\mathbb{B}^1} \int_{\mathbb{R}^2} \chi(\frac{|t'' - t'|}{|w|}) f(t'', x' + |t'' - t'|w) \frac{1}{|t'' - t'|} dt'' dw.$$

Then we let $x'' = x' + |t'' - t'|w$ and see that $w = (x'' - x')/|t'' - t'|$. We get

$$N_+(t', x') = C \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \chi(\frac{|t'' - t'|}{|x'' - x'|}) f(t'', x'') \frac{1}{|t'' - t'|^3} dt'' dx''. \tag{4.2}$$

The calculation for N_- is the same. So we get

$$K(t', x', t, x) = C\chi(\frac{|t - t'|}{|x - x'|})^2 \frac{1}{|t - t'|^3}.$$

We observe that the kernel is only singular at $x = x', t = t'$. Moreover, we can prove that N_g is a pseudo-differential operator. First, we observe that K is a convolution kernel. It suffices to compute the Fourier transform of

$$G(t, x) = \chi(\frac{|t|}{|x|})^2 \frac{1}{|t|^3}.$$

Now we define

$$F(t, x) = \frac{1}{(|t|^2 + |x|^2)^{3/2}},$$

and we observe that $G(t, x) = F(t, x)h(t, x)$ where $h \in C^\infty$ and $h(0, 0) = 1$. Thus for $|t| + |x|$ sufficiently small, we have the expansion

$$G(t, x) = \frac{1}{(|t|^2 + |x|^2)^{3/2}} (1 + \sum_{|\gamma| \geq 1} a_\gamma ((t, x) - 0)^\gamma)$$

where $\gamma \in \mathbb{Z}^4$ is a multi-index and $a_\gamma \in \mathbb{R}$. Note that each term in the expansion can be regarded as homogeneous distributions of degree $-3 + |\gamma|$, see for example Section

3.2 of [8]. The Fourier transforms of such distributions are homogeneous of degree $-4+3-|\gamma| = -1-|\gamma|$. This can also be found in Section 7.1 of [8]. For $\tau, \xi \in \mathbb{R}^2$, the Fourier transform of $G(t,x)$ have the following expansion

$$\hat{G}(\tau,\xi) \sim \sum_{j\geq -1} b_j(\tau,\xi),$$

where b_j is homogeneous of degree $-j$ and $b_{-1} \neq 0$. This proves that N_g is an elliptic pseudo-differential operator of order -1. In fact, this is an operator of the classical type, namely the symbol has asymptotic expansion in terms of homogeneous symbols. □

4.3 The normal operator for general metrics

For this section, we assume that g is a smooth Pseudo-Riemannian metric on $N \times M, N = \mathbb{R}^n, M = \mathbb{R}^m$ of the form

$$g = \begin{pmatrix} -g_N(z) & 0 \\ 0 & g_M(z) \end{pmatrix} \tag{4.3}$$

where $z \in \mathbb{R}^{n+m}$. Also, we assume that for each z, g_N, g_M are positive definite on T_zN, T_zM. Note that g is a separable Pseudo-Riemannian metric. We make the following geometric assumptions:

1. Time-like geodesics for g are non-trapping on Ω.

2. There are no conjugate points along time-like geodesics.

We remark that in Lorentzian geometry, it is known that under the globally hyperbolicity condition that any Lorentzian metric can be written in the form of (4.3). This might be true for Pseudo-Riemannian metrics with general signatures, but the relevant notions of hyperbolicity are only studied recently in [25].

We again use $z = (t, x), t \in \mathbb{R}^n, x \in \mathbb{R}^m$ as coordinates. At any z, we consider time-like vectors ζ at z in the following set

$$\zeta \in S_z\mathbb{R}^{n+m} = \{\zeta : g(\zeta, \zeta) = -1\}.$$

Note that all time-like vectors can be rescaled to a vector in $S_z\mathbb{R}^{n+m}$. Then we use these vectors to parametrize time-like geodesics passing through z as

$$\gamma^g_{z,\zeta}(s) = \exp^g_z(s\zeta), \quad s \in \mathbb{R} \tag{4.4}$$

where $\exp^g$ denotes the exponential map for g.

As in the flat metric case, we need to introduce a cut-off function, but this time it varies as the base points changes. Let $z \in \mathbb{R}^{n+m}$ and $\zeta = (v, w), v \in T_t\mathbb{R}^n, w \in T_x\mathbb{R}^m$. Let χ be the cut-off function we used before. We define

$$\chi^g(z, \zeta) = \chi(g_N(v, v)/g_M(w, w)).$$

Again, the role of χ^g is to stay away from the boundary of time-like vectors. Now we consider the time-like geodesic ray transform as

$$I_g f(z, \zeta) = \int_{\mathbb{R}} \chi^g(z, \zeta) f(\gamma^g_{z,\zeta}(s)) ds. \tag{4.5}$$

We are ready to analyze the normal operator. Below, to choose the measure on $S\mathbb{R}^{n+m}$, we first note that $\det g = (-1)^n \det g_N \det g_M$. We will use the restriction of the measure $\sqrt{(-1)^n \det g}\, dz d\zeta$ on $S\mathbb{R}^{n+m}$, denoted by $d\mu(z, \zeta)$ in the following. Note that the measure corresponds to the measure for the Riemannian metric $\tilde{g} = (g_N, g_M)$ on $\mathbb{R}^{n+m}$. This is convenient for the following reason. Consider the Hamiltonian $H_g(z, \zeta) = (g(\zeta, \zeta) + 1)/2$. Let $\Phi_g(s)$ be the corresponding flow. On the energy level

$H_g = 0$, we know that the measure is preserved as in the Riemannian case.

Lemma 4.3.1. Let $N_g = I_g^* I_g$ be the normal operator, and $n \geq 2, m \geq 1$. Then $N_g \in \Psi^{-1}(\mathbb{R}^{n+m})$ is elliptic.

Proof. Let I_g^* be the L^2 adjoint of I_g in (4.5). We first compute the expression of I_g^* from the L^2 pairing

$$\langle I_g^* h, f\rangle = \langle h, I_g f\rangle = \int_{S\mathbb{R}^{n+m}} \int_{\mathbb{R}} h(z,\zeta)\chi^g(z,\zeta) f(\exp_z^g(s\zeta)) ds d\mu(z,\zeta). \tag{4.6}$$

By the assumption that there is no conjugate points along any time-like curves, and we know that the exponential map is a local diffeomorphism. It is convenient to use the Hamiltonian flow $\Phi_g(s)$ and write $(z', \zeta') = \Phi_g(s)(z,\zeta)$. Then $(z,\zeta) = \Phi_g(-s)(z',\zeta')$. Now making a change of variable in (4.6), we get

$$\langle I_g^* h, f\rangle = \int_{S\mathbb{R}^{n+m}} \int_{\mathbb{R}} h(\exp_{z'}^g(-s\zeta'))\chi^g(\Phi_g(-s)(z',\zeta')) f(z') ds d\mu(z',\zeta').$$

Therefore, for $z' \in \mathbb{R}^n \times \mathbb{R}^m$

$$I_g^* h(z') = \int_{S_{z'}\mathbb{R}^{n+m}} \int_{\mathbb{R}} h(\exp_{z'}^g(-s\zeta'))\chi^g(\Phi_g(-s)(z',\zeta')) ds d\mu(z',\zeta').$$

Next, we compute

$$\begin{aligned} I_g^* I_g f(z') &= \int_{S_{z'}\mathbb{R}^{n+m}} \int_{\mathbb{R}} I_g f(\exp_{z'}^g(-s\zeta'))\chi^g(\Phi_g(-s)(z',\zeta')) ds d\mu(z',\zeta') \\ &= \int_{\mathbb{R}} \int_{S_{z'}\mathbb{R}^{n+m}} \int_{\mathbb{R}} f(\exp_{z'}^g((r-s)\zeta'))\chi^g(\Phi_g(-s)(z',\zeta'))^2 ds d\mu(z',\zeta') dr \end{aligned}$$

We split the integral to $r - s \geq 0$ and $r - s \leq 0$. For the first case, we let

$\alpha = r - s \geq 0$ and for the second case, we let $\beta = s - r \geq 0$. We have

$$I_g^* I_g f(z') = N_+ f(z') + N_- f(z')$$

where

$$N_+ f(z') = \int_{\mathbb{R}} \int_{S_{z'}\mathbb{R}^{n+m}} \int_{\mathbb{R}} f(\exp^g_{z'}(\alpha\zeta'))\chi^g(\Phi_g(\alpha - r)(z', \zeta'))^2 d\alpha d\mu(z', \zeta') dr,$$
$$N_- f(z') = \int_{\mathbb{R}} \int_{S_{z'}\mathbb{R}^{n+m}} \int_{\mathbb{R}} f(\exp^g_{z'}(-\beta\zeta'))\chi^g(\Phi_g(-\beta - r)(z', \zeta'))^2 d\beta d\mu(z', \zeta') dr.$$

At this point, one can compute the kernel explicitly. We will not carry out the calculation. Instead, we make use of our knowledge of the flat case. We make the following observation and we use N_+ for example. First, for z' away from z, the kernel of N_+ is smooth. Let $z = \exp^g_{z'}(\alpha\zeta')$. If ζ' is time-like, we know that $\alpha = \text{dist}^g(z, z')$. Here dist^g is the Pseudo-Riemannian distance which is the negative of the length of the time-like geodesic from z' to z. Also, note that $\alpha^{n+m-1} d\alpha d\mu(z', \zeta') = dz$. We thus can write

$$\begin{aligned} N_+ f(z') &= \int_{\mathbb{R}^{n+m}} \int_{\mathbb{R}} f(z)\chi^g(\Phi_g(\text{dist}^g(z, z') - r)(z', \zeta'))^2 (\text{dist}^g(z, z'))^{-n-m+1} dz dr \\ &= \int_{\mathbb{R}^{n+m}} f(z)\tilde{\chi}^g(z, z')(\text{dist}^g(z, z'))^{-n-m+1} dz \end{aligned}$$

where

$$\tilde{\chi}^g(z, z') = \int_{\mathbb{R}} \chi^g(\Phi_g(\text{dist}^g(z, z') - r)(z', \zeta'))^2 dr,$$

and we need to keep in mind that we only consider f supported in Ω so the above integral makes sense. So for z' away from z, the kernel is smooth. If ζ' is not time-like, then the kernel is also smooth because of the cut off. Thus, the analysis of the kernel can be reduced to the case when z, z' are close. Now we can work in a small neighborhood of z such that g is a small perturbation of the constant metric equal to $g(z)$ in that neighborhood. The calculation in Section 4.2 can be slightly modified (for general signature and small perturbation) to show that the kernel has the same structure as (4.2). This argument shows that N_g is also an elliptic pseudo-differential operator of order -1. $\square$

4.4 The generic injectivity and stability

We start with the injectivity for analytic metrics. Note that in this chapter, we have assumed that g is a separable Pseudo-Riemannian metric on $\mathbb{R}^{n+m}$, but we do not assume $g-e$ is compactly supported. Instead, we assume that the scalar function f is supported in a fixed bounded open set Ω. Thus, the geodesic ray transform $I_g f(\gamma)$ is well-defined, and there is no issue working with analytic metrics on $\mathbb{R}^{n+m}$.

We first prove the following.

Proposition 4.4.1. Let g be a separable analytic metric on $\mathbb{R}^{n+m}, n, m \geq 1$. Let $f \in L^2$ be supported in Ω. If $I_g f(\gamma) = 0$ for all time-like geodesics, then f is analytic on $\mathbb{R}^{n+m}$. Hence $f = 0$.

Proof. This result can be established by using the results in [22]. Below, we use $\mathrm{WF}_A(f)$ to denote the analytic wave front set of f. We refer to [24] for the discussion on analytic microlocal analysis.

In Proposition 2 of [22], it is proved that the local geodesic ray transform determines the analytic wave front set. We can apply it to time-like geodesic and scalar functions as follows. Let $(z_0, \zeta^0) \in T^*(\mathbb{R}^{n+m})\backslash 0$, and let γ_0 be a fixed time-like geodesic through z_0 normal to ζ^0. Let $I_g f(\gamma) = 0$ for $f \in L^2(\mathbb{R}^{n+m})$ and all γ close to γ_0. Note that because γ_0 is a time-like geodesic, γ are also time-like if they are sufficiently close to γ_0. Under the above assumptions, we have

$$(z_0, \zeta^0) \notin \mathrm{WF}_A(f).$$

Because we know $I_g f(\gamma) = 0$ for all time-like geodesics, we can conclude that f is analytic on $\mathbb{R}^{n+m}$. Also, because f is compactly supported, $f = 0$. □

Now we can use Proposition 4.4.1 and the stability result to prove the generic injectivity result.

Theorem 4.4.2. Let Ω be a bounded open set of $\mathbb{R}^{n+m}$. Then there exists an open dense set of C^k separable Pseudo-Riemannian metrics g on $\mathbb{R}^{n+m}$ such that the geodesic ray transform I_g for time-like geodesics is injective on $C_0^\infty(\Omega)$. That is for any $f \in C_0^\infty(\Omega)$, if $I_g f(\gamma) = 0$ for all time-like geodesics, then $f = 0$. Moreover, there exists $C > 0$ such that the following stability holds

$$\|f\|_{H^s} \leq C\|N_g f\|_{H^{s+1}}.$$

Proof. Let g_0 be a separable analytic Pseudo-Riemannian metric on $\mathbb{R}^{n+m}$. We know that I_{g_0} is injective on $C_0^\infty(\Omega)$ from Proposition 4.4.1.

Now let g be a smooth separable Pseudo-Riemannian metric on $\mathbb{R}^{n+m}$. Consider I_g acting on $C_0^\infty(\Omega)$. We know that $N_g = I_g^* I_g$ is an elliptic pseudo-differential operator of order -1. According to Theorem 4.1.4, we can find $Q_g \in \Psi^1(\mathbb{R}^{n+m})$ such that $Q_g N_g = I$ modulo smoothing operators. Thus, by Theorem 4.1.5, we get for any $s, \rho \in \mathbb{R}$ that

$$\begin{aligned} \|f\|_{H^s} &\leq \|Q_g N_g f\|_{H^s} + C_\rho \|f\|_{H^\rho} \\ &\leq C\|N_g f\|_{H^{s+1}} + C_\rho \|f\|_{H^\rho}. \end{aligned} \tag{4.7}$$

To remove the last term, we will use a argument based on Lemma 2 of [20]. Let $\epsilon > 0$ be small and consider a separable smooth Pseudo-Riemannian metric g on $\mathbb{R}^{n+m}$ such that

$$\|g - g_0\|_{C^k} < \epsilon \tag{4.8}$$

We claim that there is $\epsilon > 0$ and $C > 0$ such that for all metrics g satisfying (4.8), we have

$$\|f\|_{H^s} \le C\|N_g f\|_{H^{s+1}}. \tag{4.9}$$

This implies the desired injectivity result.

We prove (4.9) by contradiction and assume that for $n = 1, 2, \cdots$, there are

1. Metrics g_n such that $\|g_n - g_0\|_{C^k} < 1/n$;

2. $f_n \in C_0^\infty(\Omega)$ such that $\|f_n\|_{H^s} \ge n\|N_{g_n} f_n\|_{H^{s+1}}$. Without loss of generality, we can assume $\|f_n\|_{H^s} = 1$.

In (4.7), we choose $\rho < s$. Note that f_n are supported in a fixed compact set $\overline{\Omega}$. We know that $H^s(\Omega)$ is compactly embedded in $H^\rho(\Omega)$. By going to a subsequence, still denoted by f_n, we can assume that f_n converges to f_0 in H^ρ. By using (4.7), we can show f_n is Cauchy in H^s and thus converges to f_0 in H^s. Using point 2 of the contradiction assumption, we get that $\|N_g f_0\|_{H^{s+1}} = 0$ so that $I_g f_0 = 0$. Because we know I_g is injective by Proposition 4.4.1 so $f_0 = 0$. But this contradicts to $\|f_0\|_{H^s} = 1$. Thus (4.9) holds. □

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