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Jack Barlow Date

A local-global principle for adjoint groups over function fields of $p$-adic curves

By

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A local-global principle for adjoint groups over function fields of $p$-adic curves

By

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MMath, University of Warwick, 2018
M.S., Emory University, 2022

Advisor: R. Parimala, Ph.D.

An abstract of
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Abstract

A local-global principle for adjoint groups over function fields of $p$-adic curves
By Jack Barlow

Let $k$ be a number field and $G$ a semisimple simply connected linear algebraic group over $k$. The Kneser conjecture states that the Hasse principle holds for principal homogeneous spaces under $G$. Kneser's conjecture is a theorem due to Kneser for all classical groups, Harder for exceptional groups other than $E_8$, and Chernousov for $E_8$. It has also been proved by Sansuc that if $G$ is an adjoint linear algebraic group over $k$, then the Hasse principle holds for principal homogeneous spaces under $G$.

Now let $p \in \mathbb{N}$ be a prime with $p \neq 2$, and let $K$ be a $p$-adic field. Let $F$ be the function field of a curve over $K$. Let $\Omega_F$ be the set of all divisorial discrete valuations of $F$. It is a conjecture of Colliot-Thélène, Parimala and Suresh that if $G$ is a semisimple simply connected linear algebraic group over $F$, then the Hasse principle holds for principal homogeneous spaces under $G$. This conjecture has been proved for all groups of classical type. In this thesis, we ask whether the Hasse principle holds for adjoint groups over $F$, motivated by the number field case. We give a positive answer to this question for a class of adjoint classical groups.

A local-global principle for adjoint groups over function fields of $p$-adic curves

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# Contents

# Chapter 1

# Introduction

Let $k$ be a number field and $G$ a semisimple simply connected linear algebraic group over $k$. The Kneser conjecture states that the Hasse principle holds for principal homogeneous spaces under $G$ ( [13]). Kneser's conjecture is a theorem due to Kneser for all classical groups ( [14]), Harder for exceptional groups other than $E_8$ ( [10], [11], [12]) and Chernousov for $E_8$ ( [4]). It has also been proved that if $G$ is an adjoint linear algebraic group over $k$, then the Hasse principle holds for principal homogeneous spaces under $G$ ( [20, Corollary 5.4]).

Now let $p \in \mathbb{N}$ be a prime with $p \neq 2$, and let $K$ be a $p$-adic field. Let $F$ be the function field of a curve over $K$. Let $\Omega$ be the set of all divisorial discrete valuations of $F$. It is a conjecture of Colliot-Thélène, Parimala and Suresh that if $G$ is a semisimple simply connected linear algebraic group over $F$, then the Hasse principle holds for principal homogeneous spaces under $G$ ( [5]). This conjecture has been proved for all groups of classical type ( [19]). In this thesis we prove the following:

**Theorem 1.0.1.** *Let $p \in \mathbb{N}$ be a prime with $p \neq 2$, and let $K$ be a $p$-adic field. Let $F$ be a function field of a curve over $K$. Let $q$ be a quadratic form over $F$. Then the Hasse principle holds for principal homogeneous spaces under* $\mathrm{PSO}(q)$.

A by-product of Theorem 1.0.1 is the following:

**Theorem 1.0.2.** *Let $p \in \mathbb{N}$ be a prime with $p \neq 2$, and let $K$ be a $p$-adic field. Let $F$ be a function field of a curve over $K$. Let $L/F$ be a quadratic field extension, and let $\tau$ be the nontrivial automorphism of $L/F$. Then the Hasse principle holds for principal homogeneous spaces under* $\mathrm{PGU}(L, \tau)$.

We have the following theorem for the symplectic case:

**Theorem 1.0.3.** *Let $p \in \mathbb{N}$ be a prime with $p \neq 2$, and let $K$ be a $p$-adic field. Let $F$ be a function field of a curve over $K$. Let $G$ be an absolutely simple adjoint linear algebraic group over $F$ of classical type $C_n$. Then the Hasse principle holds for principal homogeneous spaces under $G$.*

Absolutely simple adjoint linear algebraic groups over $F$ of classical type $C_n$ are described by the group of similitudes of central simple algebras with symplectic involution. To prove Theorem 1.0.3, a main step is to prove the following:

**Theorem 1.0.4.** *Let $A$ and $B$ be central simple algebras over $F$ with $\deg(A) = \deg(B)$. Let $\sigma$ be a symplectic involution on $A$ and let $\tau$ be a symplectic involution on $B$. If $(A, \sigma) \simeq (B, \tau)$ locally over all divisorial discrete valuations of $F$, then $(A, \sigma) \simeq (B, \tau)$ over $F$.*

The proofs of Theorems 1.0.1, 1.0.2, 1.0.3 and 1.0.4 reduce to a Hasse principle for similarity of hermitian forms.

## 1.1 The Plan

Here we give a brief description of the contents of the thesis:

Chapters 2-4 consist of preliminaries. In chapter 2, we give an overview of the foundations of the theory of central simple algebras, involutions and hermitian forms. In Chapter 3, we study a few elementary notions in group cohomology and look at profinite groups and Galois cohomology. In Chapter 4, we introduce linear algebraic groups and give a classification of absolutely simple, adjoint, classical linear algebraic groups. We go on to review some of the patching techniques of Harbater, Hartmann and Krashen.

Chapter 5 consists of the proofs of our main theorems. In Section 5.1, we analyze similarities of quadratic forms over two dimensional complete fields and approximate similarity factors along branches. In Section 5.2, we use the results of Section 5.1 to prove a Hasse principle for similarity of quadratic forms over semi-global fields and deduce as a corollary a Hasse principle for similarity of hermitian forms over quadratic extensions of semi-global fields. In Section 5.3, we look at the case of a quaternion division algebra $D$ over a two dimensional complete field with canonical involution $\tau$. We use the results of Section 5.1 to approximate similarity factors of hermitian forms over $(D, \tau)$ along branches. In Section 5.4, we use the results of Section 5.3 to prove a Hasse principle for similarity of hermitian forms over central simple algebras over semi-global fields with symplectic involution. We combine all of the results of Sections 5.1-5.4 to prove our main theorems in Section 5.5.

# Chapter 2

# Central Simple Algebras, Involutions and Hermitian Forms

## 2.1 Central Simple Algebras

**Definition.** A *central simple algebra* over a field $F$ is a finite dimensional algebra $A \neq \{0\}$ with center $F = F \cdot 1$ which has no two-sided ideals except $\{0\}$ and $A$.

**Theorem 2.1.1.** (Wedderburn Structure Theorem). *Let $A$ be a central simple algebra over a field $F$. Then there exists an integer $n \geq 1$ and a central division algebra $D$ over $F$ such that $A \simeq M_n(D)$. Moreover, $D$ is uniquely determined up to isomorphism.*

*Proof.* See for instance [6, p. 22]. □

**Notation.** For any algebra $A$ over a field $F$ and any field extension $K/F$, we write $A_K$ for the $K$-algebra obtained from $A$ by extending scalars to $K$:

$$A_K = A \otimes_F K.$$

**Theorem 2.1.2.** (Wedderburn). *Let $A$ be an algebra over a field $F$. Then $A$ is central simple if and only if there is a field $K$ containing $F$ such that $A_K \simeq M_n(K)$ for some $n$.*

*Proof.* See for instance Scharlau [21, Chapter 8]. □

**Definition.** The fields $K$ containing $F$ such that $A_K \simeq M_n(K)$ for some $n$ are called *splitting fields* of $A$. If $K$ is a splitting field of $A$, we also say that *$A$ splits over $K$* or *$K$ splits $A$*.

**Definition.** Let $A$ be a central simple algebra over a field $F$ and let $K$ be a splitting field of $A$. Let $\phi\colon A_K \xrightarrow{\sim} M_n(K)$ be an isomorphism and let $a \in A$. We define the *reduced norm* of $a$, denoted $\mathrm{Nrd}_A(a)$, to be

$$\mathrm{Nrd}_A(a) := \det(\phi(a \otimes 1)) \in F.$$

**Remark.** The reduced norm $\mathrm{Nrd}_A(a) \in F$ is independent of the choices of the splitting field $K$ and isomorphism $\phi\colon A_K \xrightarrow{\sim} M_n(K)$.

**Theorem 2.1.3.** *If $A$ is a central simple $F$-algebra, its dimension over $F$ is a square.*

*Proof.* See for instance [6, p. 24]. □

**Definition.** Let $A$ be a central simple algebra over a field $F$. We define the *degree* of $A$, denoted $\deg_F(A)$ or simply $\deg(A)$, to be the integer $\deg(A) := \sqrt{\dim_F A}$.

### 2.1.1 The Brauer Group

It turns out that to gain an understanding of the finite dimensional central division algebras over a field $F$, it is best to consider the more general central simple algebras over $F$. Essentially, this is because central simple algebras are closed under the tensor product operation, whereas finite dimensional central division algebras in general are not (e.g. if $\mathbb{H}$ is Hamilton's quaternion algebra over $\mathbb{R}$, then $\mathbb{H} \otimes_{\mathbb{R}} \mathbb{H} \simeq M_4(\mathbb{R})$). Now if $A$ is a central simple algebra over $F$, then by the Wedderburn Structure Theorem, Theorem 2.1.1, we have an $F$-algebra isomorphism $A \simeq M_n(D)$, for some integer $n \geq 1$ and some finite dimensional central division algebra $D$ over $F$ which is uniquely determined up to $F$-algebra isomorphism. This promts the following definition.

**Definition.** Let $A \simeq M_{n_1}(D_1)$ and $B \simeq M_{n_2}(D_2)$ be two central simple algebras over a field $F$ (where $D_1$ and $D_2$ are finite dimensional central division algebras over $F$). We call $A$ and $B$ *similar*, and write $A \sim B$, if there is an $F$-algebra isomorphism $D_1 \simeq D_2$.

**Remark.** Since $D_1$ and $D_2$ above are uniquely determined up to $F$-algebra isomorphism, it is clear to see that $\sim$ is an equivalence relation on the set of central simple algebras over $F$. We write $[A]$ to denote the equivalence class of $A$ under the equivalence relation of similarity.

**Definition.** For any algebra $A$ over a field $F$, we define the *opposite algebra* $A^{\mathrm{op}}$ by

$$A^{\mathrm{op}} = \{a^{\mathrm{op}} \mid a \in A\},$$

with the operations defined as follows:

$$a^{\mathrm{op}} + b^{\mathrm{op}} = (a+b)^{\mathrm{op}},\ a^{\mathrm{op}}b^{\mathrm{op}} = (ba)^{\mathrm{op}},\ \alpha \cdot a^{\mathrm{op}} = (\alpha \cdot a)^{\mathrm{op}}$$

for $a, b \in A$ and $\alpha \in F$.

**Definition.** The *Brauer group* of a field $F$, denoted $\mathrm{Br}(F)$, is the set of equivalence classes of central simple $F$-algebras under the equivalence relation of similarity, with the tensor product acting as the group operation in the following way:

$$[A] \cdot [B] := [A \otimes_F B].$$

The pair $(\mathrm{Br}(F), \cdot)$ is an abelian group with $[F] = 1 \in \mathrm{Br}(F)$ and $[A]^{-1} = [A^{\mathrm{op}}]$ for all $[A] \in \mathrm{Br}(F)$.

**Remark.** If $D$ is a finite dimensional central division algebra over $F$, then $D \in [D] \in \mathrm{Br}(F)$. Conversely, if $A$ is a central simple algebra over $F$, then $A \simeq M_n(D')$ for some integer $n \geq 1$ and some finite dimensional central division algebra $D'$ over $F$, which is uniquely determined up to $F$-algebra isomorphism, and we have $[A] = [D'] \in \mathrm{Br}(F)$. Therefore there is a one-to-one correspondence between the set of finite dimensional central division algebras over $F$ (where two $F$-algebra isomorphic algebras are considered equal) and the set of elements of $\mathrm{Br}(F)$, the bijection taking a finite dimensional central division algebra $D$ over $F$ to its similarity class $[D] \in \mathrm{Br}(F)$.

**Theorem 2.1.4.** *The Brauer group is a torsion abelian group.*

*Proof.* See for instance [6, p. 54]. □

**Notation.** We denote the $n$-torsion subgroup of $\mathrm{Br}(F)$ by ${}_n\mathrm{Br}(F)$.

**Definition.** Let $A$ be a central simple algebra over a field $F$. Let $D$ be the central division algebra for which $A \simeq M_n(D)$. We define the *index* of $A$ over $F$, denoted $\mathrm{ind}_F(A)$ or simply $\mathrm{ind}(A)$, to be $\mathrm{ind}(A) := \deg(D)$.

**Definition.** The *period* (or *exponent*) of a central simple $F$-algebra $A$, denoted $\mathrm{per}(A)$, is the order of its class $[A]$ in $\mathrm{Br}(F)$.

**Theorem 2.1.5.** (Brauer). *Let $A$ be a central simple $F$-algebra. Then the period* $\mathrm{per}(A)$ *divides the index* $\mathrm{ind}(A)$. *Moreover, the period* $\mathrm{per}(A)$ *and the index* $\mathrm{ind}(A)$ *have the same prime factors.*

*Proof.* See for instance [6, pp. 54 and 55]. □

### 2.1.2 Quaternion Algebras

**Definition.** Let $F$ be a field with $\mathrm{char}(F) \neq 2$. Let $a, b \in F^*$. We define the *quaternion algebra* $A = (a,b)_F$ to be the $F$-algebra on two generators $i, j$ with the defining relations

$$i^2 = a,\ j^2 = b,\ ij = -ji.$$

**Proposition 2.1.6.** *Let* $k := ij \in A = (a,b)_F$. *Then* $\{1, i, j, k\}$ *form an* $F$*-basis for* $A$ *(so that* $\dim_F A = 4$*).*

*Proof.* See [16, p. 51, Proposition 1.0]. □

**Proposition 2.1.7.** $(a,b)_F$ *is a central simple algebra over* $F$*.*

*Proof.* See [16, p. 52, Proposition 1.1]. □

**Theorem 2.1.8.** *Let* $A$ *be a central simple* $F$*-algebra. Then* $\deg(A) = 2$ *if and only if* $A$ *is isomorphic to a quaternion algebra over* $F$*.*

*Proof.* See [16, p. 74, Theorem 5.1]. □

### 2.1.3 Ramifications of Central Simple Algebras

Let $R$ be a commutative regular local ring with field of fractions $F = \mathrm{ff}(R)$. Let $\mathfrak{m}$ be the maximal ideal of $R$.

**Definition.** An $R$-algebra $\mathscr{A}$ is called an *Azumaya algebra* over $R$ if $\mathscr{A}$ is free of positive finite rank as an $R$-module, and the algebra $\mathscr{A} \otimes_R (R/\mathfrak{m})$ is a central simple algebra over $R/\mathfrak{m}$.

**Definition.** Let $A$ be a central simple algebra over $F$. We say that $A$ is *unramified on* $R$ if there exists an Azumaya algebra $\mathscr{A}$ over $R$ such that $A \cong \mathscr{A} \otimes_R F$.

Now let $\mathscr{X}$ be a regular integral scheme with function field $F$, and $x \in \mathscr{X}$ a point. Let $\mathcal{O}_{\mathscr{X},x}$ be the local ring of $\mathscr{X}$ at $x$. Let $A$ be a central simple algebra over $F$.

**Definition.** We say that $A$ is *unramified at* $x \in \mathscr{X}$ if $A$ is unramified on $\mathcal{O}_{\mathscr{X},x}$. If $A$ is not unramified at $x$, we say that $A$ is *ramified at* $x$.

## 2.2 Involutions

**Definition.** An *involution* on a central simple algebra $A$ over a field $F$ is a map $\sigma\colon A \to A$ subject to the following conditions:

(a) $\sigma(x+y) = \sigma(x) + \sigma(y)$ for $x, y \in A$.

(b) $\sigma(xy) = \sigma(y)\sigma(x)$ for $x, y \in A$.

(c) $\sigma^2(x) = x$ for $x \in A$.

**Remark.** The center $F = F \cdot 1$ is preserved under $\sigma$. The restriction of $\sigma$ to $F$ is therefore an automorphism which is either the identity or of order 2.

**Definition.** Involutions which leave the center elementwise invariant are called *involutions of the first kind.*

**Remark.** If $A$ is a central simple algebra over a field $F$ with an involution $\sigma$ of the first kind, then $\sigma$ defines an isomorphism $A \simeq A^{\mathrm{op}}$. Hence $\mathrm{per}(A) \leq 2$ and $[A] \in {}_2\mathrm{Br}(F)$.

**Definition.** Involutions whose restriction to the center is an automorphism of order 2 are called *involutions of the second kind.* Involutions of the second kind are also called *of unitary type* (or simply *unitary*).

**Definition.** An *isomorphism of algebras with involution* $f\colon (A,\sigma) \xrightarrow{\sim} (A',\sigma')$ is an $F$-algebra isomorphism $f\colon A \xrightarrow{\sim} A'$ such that $\sigma' \circ f = f \circ \sigma$.

**Definition.** Let $A$ be a central simple algebra over a field $F$, and let $\sigma$ be an involution (of any kind) on $A$. An *automorphism* of $(A,\sigma)$ is an isomorphism of algebras with involution $f\colon (A,\sigma) \simeq (A,\sigma)$.

**Notation.** Let $A$ be a central simple algebra over a field $F$, and let $\sigma$ be an involution (of any kind) on $A$. The set of automorphisms of $(A,\sigma)$ is denoted by $\mathrm{Aut}_F(A,\sigma)$:

$$\mathrm{Aut}_F(A,\sigma) = \{f \in \mathrm{Aut}_F(A) \mid \sigma \circ f = f \circ \sigma\}.$$

**Definition.** Let $V$ be a finite dimensional vector space over a field $F$ with $\mathrm{char}(F) \neq 2$. A bilinear form $b\colon V \times V \to F$ is called *nonsingular* if the induced map

$$\hat{b}\colon V \to V^* = \mathrm{Hom}_F(V,F)$$

defined by

$$\hat{b}(x)(y) = b(x,y) \text{ for } x, y \in V$$

is an isomorphism of vector spaces.

**Definition.** Let $V$ be a finite dimensional vector space over a field $F$ with $\mathrm{char}(F) \neq 2$. For any $f \in \mathrm{End}_F(V)$, let $f^t \in \mathrm{End}_F(V^*)$ be defined by mapping $\phi \in V^*$ to $\phi \circ f$. The map $f^t$ is called the *transpose* of $f$.

**Definition.** Let $V$ be a finite dimensional vector space over a field $F$ with $\mathrm{char}(F) \neq 2$. Let $b\colon V \times V \to F$ be a nonsingular bilinear form. For any $f \in \mathrm{End}_F(V)$ we define $\sigma_b(f) \in \mathrm{End}_F(V)$ by

$$\sigma_b(f) = \hat{b}^{-1} \circ f^t \circ \hat{b}.$$

The map $\sigma_b\colon \mathrm{End}_F(V) \to \mathrm{End}_F(V)$ is then an involution of $\mathrm{End}_F(V)$ which is known as the *adjoint involution* with respect to the nonsingular bilinear form $b$.

**Notation.** Let $\sigma$ be an involution of the first kind on a central simple algebra $A$ over a field $F$ with $\mathrm{char}(F) \neq 2$. If $L$ is any field containing $F$, the involution $\sigma$ extends to an involution of the first kind $\sigma_L = \sigma \otimes \mathrm{Id}_L$ on $A_L = A \otimes_F L$. In particular, if $L$ is a splitting field of $A$, we may identify $A_L = \mathrm{End}_L(V)$ for some vector space $V$ over $L$ of dimension $n = \deg(A)$. The involution $\sigma_L$ is then the adjoint involution $\sigma_b$ with respect to some nonsingular symmetric or skew-symmetric bilinear form $b$ on $V$ ( [15, p. 13]).

**Definition.** An involution $\sigma$ of the first kind is said to be *of symplectic type* (or simply *symplectic*) if for any splitting field $L$ and any isomorphism $(A_L, \sigma_L) \simeq (\mathrm{End}_L(V), \sigma_b)$, the bilinear form $b$ is skew-symmetric.

**Definition.** An involution $\sigma$ of the first kind is said to be *of orthogonal type* (or simply *orthogonal*) if for any splitting field $L$ and any isomorphism $(A_L, \sigma_L) \simeq (\mathrm{End}_L(V), \sigma_b)$, the bilinear form $b$ is symmetric.

**Definition.** In a central simple $F$-algebra $A$ with involution of the first kind $\sigma$, the *set of symmetric elements* in $A$ is defined as

$$\mathrm{Sym}(A, \sigma) = \{a \in A \mid \sigma(a) = a\}.$$

**Definition.** In a central simple $F$-algebra $A$ with involution of the first kind $\sigma$, the *set of skew-symmetric elements* in $A$ is defined as

$$\mathrm{Skew}(A, \sigma) = \{a \in A \mid \sigma(a) = -a\}.$$

**Theorem 2.2.1.** *Let $F$ be a field with $\mathrm{char}(F) \neq 2$. Let $A$ be a central simple $F$-algebra of degree $n$, and let $\sigma$ be an involution on $A$ of the first kind. Then $\sigma$ is of symplectic type if and only if*

$$\dim_F(\mathrm{Sym}(A, \sigma)) = \frac{n(n-1)}{2} \quad \left(\textit{and thus } \dim_F(\mathrm{Skew}(A, \sigma)) = \frac{n(n+1)}{2}\right).$$

*$\sigma$ is of orthogonal type if and only if*

$$\dim_F(\mathrm{Sym}(A,\sigma)) = \frac{n(n+1)}{2} \quad \left( \textit{and thus } \dim_F(\mathrm{Skew}(A,\sigma)) = \frac{n(n-1)}{2} \right).$$

*Moreover, if $\sigma$ is of symplectic type, then $n$ is necessarily even.*

*Proof.* See [15, Proposition 2.6]. □

**Definition.** Let $F$ be a field with $\mathrm{char}(F) \neq 2$. Let $A = (a,b)_F$ be a quaternion algebra over $F$ with $a, b \in F^*$. Let $i, j \in A$ be the standard generators of $A$ with $i^2 = a,\ j^2 = b$ and $ij = -ji$. The unique involution $\tau$ on $A$ with $\tau(i) = -i$ and $\tau(j) = -j$ is called the *quaternion conjugation* or the *canonical involution* on $A$.

**Remark.** The canonical involution $\tau$ on $A = (a,b)_F$ is the only involution of the first kind of symplectic type.

### 2.2.1 Similitudes of Algebras with Involution

**Definition.** Let $A$ be a central simple algebra over a field $F$, and let $\sigma$ be an involution (of any kind) on $A$. A *similitude* of $(A,\sigma)$ is an element $g \in A$ such that

$$\sigma(g)g \in F^*.$$

**Definition.** Let $A$ be a central simple algebra over a field $F$, and let $\sigma$ be an involution (of any kind) on $A$. Let $g \in A$ be a similitude of $(A,\sigma)$. The scalar $\sigma(g)g \in F^*$ is called the *multiplier* of $g$ and is denoted $\mu(g)$.

**Notation.** Let $A$ be a central simple algebra over a field $F$, and let $\sigma$ be an involution (of any kind) on $A$. The set of all similitudes of $(A,\sigma)$ is a subgroup of $A^*$ which we call $\mathrm{Sim}(A,\sigma)$:

$$\mathrm{Sim}(A,\sigma) = \{g \in A \mid \sigma(g)g \in F^*\}.$$

**Remark.** The map $\mu\colon \mathrm{Sim}(A,\sigma) \to F^*$ given by $\mu(g) = \sigma(g)g$ for all $g \in \mathrm{Sim}(A,\sigma)$ is a group homomorphism.

**Definition.** Let $F$ be a field and let $A$ be an $F$-algebra. For any $b \in A^*$, we define the *inner automorphism of $A$ induced by $b$*, denoted $\mathrm{Int}(b)$, to be the $F$-algebra automorphism $\mathrm{Int}(b)\colon A \to A$ given by

$$\mathrm{Int}(b)(a) = bab^{-1} \text{ for } a \in A.$$

**Theorem 2.2.2.** *Let $A$ be a central simple algebra over a field $F$, and let $\sigma$ be an involution (of any kind) on $A$. Then*

$$\mathrm{Aut}_F(A,\sigma) = \{\mathrm{Int}(g) \mid g \in \mathrm{Sim}(A,\sigma)\}.$$

*There is therefore an exact sequence*

$$1 \to F^* \to \mathrm{Sim}(A,\sigma) \xrightarrow{\mathrm{Int}} \mathrm{Aut}_F(A,\sigma) \to 1.$$

*Proof.* See [15, Theorem 12.15]. □

**Definition.** Let $A$ be a central simple algebra over a field $F$, and let $\sigma$ be an involution (of any kind) on $A$. Let $\mathrm{PSim}(A,\sigma)$ be the group of *projective similitudes* of $(A,\sigma)$, defined as

$$\mathrm{PSim}(A,\sigma) = \mathrm{Sim}(A,\sigma)/F^*.$$

**Remark.** By Theorem 2.2.2, the map $\mathrm{Int}\colon \mathrm{Sim}(A,\sigma) \to \mathrm{Aut}_F(A,\sigma)$ induces a natural isomorphism $\mathrm{PSim}(A,\sigma) \simeq \mathrm{Aut}_F(A,\sigma)$.

**Notation.** Let $A$ be a central simple algebra over a field $F$, and let $\sigma$ be a symplectic involution on $A$. Then $\mathrm{GSp}(A,\sigma) := \mathrm{Sim}(A,\sigma)$ and $\mathrm{PGSp}(A,\sigma) := \mathrm{PSim}(A,\sigma)$.

**Notation.** Let $A$ be a central simple algebra over a field $F$, and let $\sigma$ be an orthogonal involution on $A$. Then $\mathrm{GO}(A,\sigma) := \mathrm{Sim}(A,\sigma)$ and $\mathrm{PGO}(A,\sigma) := \mathrm{PSim}(A,\sigma)$.

**Notation.** Let $A$ be a central simple algebra over a field $F$, and let $\sigma$ be a unitary involution on $A$. Then $\mathrm{GU}(A,\sigma) := \mathrm{Sim}(A,\sigma)$ and $\mathrm{PGU}(A,\sigma) := \mathrm{PSim}(A,\sigma)$.

**Definition.** Let $A$ be a central simple algebra over a field $F$, and let $\sigma$ be an involution (of any kind) on $A$. A similitude $g \in \mathrm{Sim}(A,\sigma)$ with multiplier $\mu(g) = 1$ is called an *isometry* of $(A,\sigma)$.

**Notation.** Let $A$ be a central simple algebra over a field $F$, and let $\sigma$ be an involution (of any kind) on $A$. The set of all isometries of $(A,\sigma)$ is a subgroup of $\mathrm{Sim}(A,\sigma)$ which we call $\mathrm{Iso}(A,\sigma)$:

$$\mathrm{Iso}(A,\sigma) = \{g \in A \mid \sigma(g)g = 1\}.$$

**Notation.** Let $A$ be a central simple algebra over a field $F$, and let $\sigma$ be a symplectic involution on $A$. Then $\mathrm{Sp}(A,\sigma) := \mathrm{Iso}(A,\sigma)$.

**Notation.** Let $A$ be a central simple algebra over a field $F$, and let $\sigma$ be an orthogonal involution on $A$. Then $\mathrm{O}(A,\sigma) := \mathrm{Iso}(A,\sigma)$.

**Notation.** Let $A$ be a central simple algebra over a field $F$, and let $\sigma$ be a unitary involution on $A$. Then $\mathrm{U}(A,\sigma) := \mathrm{Iso}(A,\sigma)$.

Now suppose $A$ is a central simple algebra over a field $F$ with even degree $\deg(A) = 2m$, and let $\sigma$ be an involution of the first kind on $A$. Let $g \in \mathrm{Sim}(A, \sigma)$, so that

$$\sigma(g)g = \mu(g) \in F^*. \tag{$*$}$$

Taking the reduced norm of both sides of $(*)$, we obtain

$$\mathrm{Nrd}_A(g) = \pm\mu(g)^m.$$

**Theorem 2.2.3.** *If $\sigma$ is a symplectic involution on $A$, then*

$$\mathrm{Nrd}_A(g) = \mu(g)^m \textit{ for all } g \in \mathrm{GSp}(A, \sigma).$$

*Proof.* See [15, Proposition 12.23]. □

**Definition.** Let $A$ be a central simple algebra over a field $F$ with even degree $\deg(A) = 2m$, and with an orthogonal involution $\sigma$. A similitude $g \in \mathrm{GO}(A, \sigma)$ is called *proper* if $\mathrm{Nrd}_A(g) = +\mu(g)^m$. A similitude $g \in \mathrm{GO}(A, \sigma)$ is called *improper* if $\mathrm{Nrd}_A(g) = -\mu(g)^m$.

**Notation.** Let $A$ be a central simple algebra over a field $F$ with even degree $\deg(A) = 2m$, and with an orthogonal involution $\sigma$. The set of all proper similitudes of $(A, \sigma)$ is a subgroup of $\mathrm{GO}(A, \sigma)$ which we call $\mathrm{GO}^+(A, \sigma)$:

$$\mathrm{GO}^+(A, \sigma) = \{g \in \mathrm{GO}(A, \sigma) \mid \mathrm{Nrd}_A(g) = +\mu(g)^m\}.$$

**Remark.** We have $[\mathrm{GO}(A, \sigma) \colon \mathrm{GO}^+(A, \sigma)] \leq 2$.

**Notation.** Let $A$ be a central simple algebra over a field $F$ with even degree $\deg(A) = 2m$, and with an orthogonal involution $\sigma$. The set of all improper similitudes of $(A, \sigma)$ is a coset of $\mathrm{GO}^+(A, \sigma)$ in $\mathrm{GO}(A, \sigma)$ which we call $\mathrm{GO}^-(A, \sigma)$:

$$\mathrm{GO}^-(A, \sigma) = \{g \in \mathrm{GO}(A, \sigma) \mid \mathrm{Nrd}_A(g) = -\mu(g)^m\}.$$

**Remark.** It is possible for $\mathrm{GO}^-(A, \sigma)$ to be empty.

**Definition.** Let $A$ be a central simple algebra over a field $F$ with even degree $\deg(A) = 2m$, and with an orthogonal involution $\sigma$. Let $\mathrm{PGO}^+(A, \sigma)$ be the group of *proper projective similitudes* of $(A, \sigma)$, defined as

$$\mathrm{PGO}^+(A, \sigma) = \mathrm{GO}^+(A, \sigma)/F^*.$$

**Definition.** Let $A$ be a central simple algebra over a field $F$ with even degree $\deg(A) = 2m$, and with an orthogonal involution $\sigma$. A proper similitude $g \in \mathrm{GO}^+(A, \sigma)$ with multiplier $\mu(g) = 1$ is called a *proper isometry* of $(A, \sigma)$.

**Notation.** Let $A$ be a central simple algebra over a field $F$ with even degree $\deg(A) = 2m$, and with an orthogonal involution $\sigma$. The set of all proper isometries of $(A, \sigma)$ is a subgroup of $\mathrm{GO}^+(A, \sigma)$ which we call $\mathrm{O}^+(A, \sigma)$:

$$\mathrm{O}^+(A, \sigma) = \mathrm{GO}^+(A, \sigma) \cap \mathrm{O}(A, \sigma) = \{g \in A \mid \mathrm{Nrd}_A(g) = \sigma(g)g = 1\}.$$

## 2.3 Hermitian Forms

**Definition.** Let $A$ be a central simple algebra over a field $F$ with an involution $\sigma$ (of any kind). Let $M$ be a finitely generated right $A$-module. A bi-additive map

$$h\colon M \times M \to A$$

is called an *hermitian form* over $(A, \sigma)$ if $h$ satisfies the following conditions:

(1) $h(m_1 a_1, m_2 a_2) = \sigma(a_1) h(m_1, m_2) a_2$ for all $m_1, m_2 \in M$ and $a_1, a_2 \in A$,

(2) $h(m_2, m_1) = \sigma(h(m_1, m_2))$ for all $m_1, m_2 \in M$.

If (2) is replaced by

(2′) $h(m_2, m_1) = -\sigma(h(m_1, m_2))$ for all $m_1, m_2 \in M$,

the map $h$ is called a *skew-hermitian form* over $(A, \sigma)$.

Let $A$ be a central simple algebra over a field $F$ with an involution $\sigma$ (of any kind). Let $M$ be a finitely generated right $A$-module. Let $M^* = \mathrm{Hom}_A(M, A)$ be the dual space of $M$. Then $M^*$ can be viewed as a right $A$-module given by $(f \cdot a)(m) = \sigma(a) f(m)$ for all $f \in M^*,\ m \in M$, $a \in A$. Let $h\colon M \times M \to A$ be an hermitian form over $(A, \sigma)$. Then $h$ induces a right $A$-module homomorphism $\tilde{h}\colon M \to M^*$ given by $\tilde{h}(m_1)(m_2) = h(m_1, m_2)$ for all $m_1, m_2 \in M$.

**Definition.** If the map $\tilde{h}$ above is a right $A$-module isomorphism, we say that $h$ is a *regular* (or *nonsingular*) hermitian form.

**Remark.** The hermitian form $h$ is nonsingular if and only if the only element $m \in M$ such that $h(m, m') = 0$ for all $m' \in M$ is $m = 0$. The same definition of nonsingular can be made for skew-hermitian forms.

**Theorem 2.3.1.** *Let $A$ be a central simple algebra over a field $F$ with an involution $\sigma$ (of any kind). Let $M$ be a finitely generated right $A$-module. Let $h\colon M \times M \to A$ be a nonsingular hermitian or skew-hermitian form over $(A, \sigma)$. Then there exists a unique involution $\sigma_h$ on $\mathrm{End}_A(M)$ such that $\sigma_h(\alpha) = \sigma(\alpha)$ for all $\alpha \in F$ and*

$$h(m_1, f(m_2)) = h(\sigma_h(f)(m_1), m_2) \text{ for } m_1, m_2 \in M,\ f \in \mathrm{End}_A(M).$$

*Proof.* See [15, Proposition 4.1]. □

**Definition.** The involution $\sigma_h$ in Theorem 2.3.1 is called the *adjoint involution* with respect to $h$.

**Definition.** Let $A$ be a central simple algebra over a field $F$ with an involution $\sigma$ (of any kind). Let $M$ and $M'$ be finitely generated right $A$-modules. Let $h\colon M \times M \to A$ and $h'\colon M' \times M' \to A$ be hermitian forms over $(A, \sigma)$. We say that $h$ and $h'$ are *equivalent*, denoted $h \simeq h'$, if there exists a bijective $A$-linear mapping $\phi\colon M \to M'$ such that

$$h'(\phi(m_1), \phi(m_2)) = h(m_1, m_2) \text{ for all } m_1, m_2 \in M.$$

**Definition.** Let $A$ be a central simple algebra over a field $F$ with an involution $\sigma$ of the first kind. Let $M$ be a finitely generated right $A$-module. Let $h\colon M \times M \to A$ be a nonsingular hermitian form over $(A, \sigma)$. A bijective $A$-linear mapping $\phi\colon M \to M$ for which there exists $\lambda \in F^*$ such that

$$h(\phi(m_1), \phi(m_2)) = \lambda h(m_1, m_2) \text{ for all } m_1, m_2 \in M$$

is called a *similitude* of $h$. The set of all similitudes of $h$ form a group which we call $\mathrm{Sim}(h)$.

**Definition.** Let $A$ be a central simple algebra over a field $F$ with an involution $\sigma$ of the first kind. Let $h$ be an hermitian form over $(A, \sigma)$. An element $\lambda \in F^*$ satisfying $\lambda h \simeq h$ is called a *similarity factor* of $h$. The *group of similarity factors* of $h$ is defined to be the collection of all similarity factors of $h$:

$$G_F(h) \coloneqq \{\lambda \in F^* \mid \lambda h \simeq h\}.$$

**Definition.** Let $F/F_0$ be a quadratic field extension. Let $A$ be a central simple algebra over $F$ with an involution $\sigma$ of the second kind such that $\sigma(x) = x$ for all $x \in F_0$. Let $M$ be a finitely generated right $A$-module. Let $h\colon M \times M \to A$ be a nonsingular hermitian form over $(A, \sigma)$. A bijective $A$-linear mapping $\phi\colon M \to M$ for which there exists $\lambda \in F_0^*$ such that

$$h(\phi(m_1), \phi(m_2)) = \lambda h(m_1, m_2) \text{ for all } m_1, m_2 \in M$$

is called a *similitude* of $h$. The set of all similitudes of $h$ form a group which we call $\mathrm{Sim}(h)$.

**Definition.** Let $F/F_0$ be a quadratic field extension. Let $A$ be a central simple algebra over $F$ with an involution $\sigma$ of the second kind such that $\sigma(x) = x$ for all $x \in F_0$. Let $h$ be an hermitian form over $(A, \sigma)$. An element $\lambda \in F_0^*$ satisfying $\lambda h \simeq h$ is called a *similarity factor* of $h$. The *group of similarity factors* of $h$ is defined to be the collection of all similarity factors of $h$:

$$G_{F_0}(h) \coloneqq \{\lambda \in F_0^* \mid \lambda h \simeq h\}.$$

**Remark.** Let $A$ be a central simple algebra over a field $F$ with an involution $\sigma$ (of any kind). Let $M$ be a finitely generated right $A$-module. Let $h\colon M \times M \to A$ be a nonsingular hermitian over $(A, \sigma)$, and let $\sigma_h\colon \mathrm{End}_A(M) \to \mathrm{End}_A(M)$ be the adjoint involution with respect to $h$. Then we have

$$\mathrm{Sim}(\mathrm{End}_A(M), \sigma_h) = \mathrm{Sim}(h).$$

### 2.3.1 Hermitian Forms over Division Algebras and Quadratic Forms

Let $F$ be a field with $\mathrm{char}(F) \neq 2$. Let $D$ be a central division algebra over $F$ with an involution $\sigma$ (of any kind). Let $V \simeq D^n$ be a right $D$-vector space of dimension $n$. Let $h\colon V \times V \to D$ be an hermitian form over $(D, \sigma)$. Then there exist $a_1, \ldots, a_n \in D^*$ such that $\sigma(a_i) = a_i$ for $1 \leq i \leq n$ and for all $\mathbf{x} = (x_1, \ldots, x_n),\ \mathbf{y} = (y_1, \ldots, y_n) \in D^n$ we have

$$h(\mathbf{x}, \mathbf{y}) = \sum_{i=1}^{n} \sigma(x_i) a_i y_i.$$

In this case, we write $h = \langle a_1, \ldots, a_n \rangle$.

**Definition.** We define the *rank* of $h$, denoted $\mathrm{Rank}(h)$, to be the integer $\mathrm{Rank}(h) \coloneqq \dim_D V = n$.

**Example.** If $D = F$ and $\sigma = \mathrm{Id}_F$ is the identity map on $F$, then $h\colon V \times V \to F$ is a symmetric bilinear pairing on $V$ and the map $q_h\colon V \to F$ given by $q_h(x) = h(x, x)$ for all $x \in V$ is a quadratic form over $F$. Conversely, let $q\colon V \to F$ be a quadratic form over $F$. Then the associated symmetric bilinear pairing $B\colon V \times V \to F$ given by

$$B(x, y) = \frac{q(x+y) - q(x) - q(y)}{2} \quad \text{for all } x, y \in V$$

is an hermitian form over $(F, \mathrm{Id}_F)$.

**Definition.** The *determinant* of a nonsingular quadratic form $q$ over $F$, denoted $d(q)$, is defined to be $d(q) = \det(M_q) \cdot (F^*)^2 \in F^*/(F^*)^2$, where $M_q$ is the symmetric matrix associated with $q$.

**Definition.** Let $q$ be a nonsingular quadratic form over $F$ of dimension $n$. We define the *discriminant* of $q$ by

$$\mathrm{disc}(q) = (-1)^{\frac{n(n-1)}{2}} d(q) \in F^*/(F^*)^2.$$

**Definition.** Let $q\colon V \to F$ be a quadratic form over $F$. Let $v \in V$ with $v \neq 0$. We say that $v$ is an *isotropic vector* if $q(v) = 0$. We say that $v$ is *anisotropic* if $q(v) \neq 0$.

**Definition.** Let $q\colon V \to F$ be a quadratic form over $F$. We say that $q$ is *isotropic* if there exists an isotropic vector $v \in V$. Otherwise, we say that $q$ is *anisotropic*.

**Theorem 2.3.2.** *Let $q\colon V \to F$ be a quadratic form over $F$ with $\dim(q) = 2$. The following four statements are equivalent:*

(1) *$q$ is regular and isotropic.*

(2) *$q$ is regular, with $d(q) = -1 \cdot (F^*)^2$.*

(3) *$q \simeq \langle 1, -1 \rangle$.*

(4) *$q$ corresponds to the equivalence class of the binary quadratic form $xy$.*

*Proof.* See [16, p. 9, Theorem 3.2]. □

**Definition.** The isometry class of a quadratic form $q$ over $F$ with $\dim(q) = 2$ satisfying the conditions in Theorem 2.3.2 is called the *hyperbolic plane* and is denoted by $\mathbb{H}$.

**Definition.** Let $q_1 \colon V_1 \to F$ and $q_2 \colon V_2 \to F$ be quadratic forms over $F$. Define $q_1 \perp q_2 \colon V_1 \oplus V_2 \to F$ by setting

$$(q_1 \perp q_2)(v_1, v_2) = q_1(v_1) + q_2(v_2) \text{ for all } v_1 \in V_1,\ v_2 \in V_2.$$

Then $q_1 \perp q_2$ is a quadratic form over $F$, and we call $q_1 \perp q_2$ the *orthogonal sum* of $q_1$ and $q_2$.

**Definition.** Let $q$ be a quadratic form over $F$. We say that $q$ is *hyperbolic* if $q$ is isometric to an orthogonal sum of hyperbolic planes, that is, $q \simeq m \cdot \mathbb{H}$ for some $m \in \mathbb{N}$.

**Definition.** Let $q_1 \colon V_1 \to F$ and $q_2 \colon V_2 \to F$ be regular quadratic forms over $F$. We call $q_1$ and $q_2$ *Witt equivalent*, and write $q_1 \sim q_2$, if there exist $r, s \in \mathbb{N}$ such that $q_1 \perp r \cdot \mathbb{H} \simeq q_2 \perp s \cdot \mathbb{H}$.

**Remark.** It is clear to see that $\sim$ is an equivalence relation on the set of isometry classes of regular quadratic forms over $F$. For a regular quadratic form $q$ over $F$, we write $[q]$ to denote the equivalence class of (the isometry class of) $q$ under the equivalence relation of Witt equivalence.

**Definition.** The *Witt group* of $F$, denoted $W(F)$, is the set of equivalence classes of (isometry classes of) regular quadratic forms over $F$ under the equivalence relation of Witt equivalence, with the orthogonal sum acting as the group operation in the following way:

$$[q_1] + [q_2] \coloneqq [q_1 \perp q_2].$$

The pair $(W(F), +)$ is an abelian group with $[\mathbb{H}] = 0 \in W(F)$ and $-[q] = [-q]$ for all $[q] \in W(F)$.

**Theorem 2.3.3.** (Witt's Decomposition Theorem). *Any regular quadratic form $q$ over $F$ splits into an orthogonal sum*

$$q \simeq q_h \perp q_a,$$

*where $q_h$ is hyperbolic and $q_a$ is anisotropic. Furthermore, the isometry classes of $q_h$ and $q_a$ are uniquely determined.*

*Proof.* See [16, p. 12, Theorem 4.1]. □

**Definition.** The splitting $q \simeq q_h \perp q_a$ of Theorem 2.3.3 is called the *Witt decomposition* of $q$.

**Remark.** It follows from Witt's Decomposition Theorem, Theorem 2.3.3, that the elements of $W(F)$ are in one-to-one correspondence with the isometry classes of all anisotropic regular quadratic forms over $F$. If $q$ and $q'$ are regular quadratic forms over $F$, then $q$ and $q'$ represent the same element in $W(F)$ ($[q] = [q'] \in W(F)$) if and only if their anisotropic parts are equivalent ($q_a \simeq q'_a$). Thus $W(F)$ can be thought of as a group consisting of isometry classes of anisotropic regular quadratic forms over $F$.

### 2.3.2 Quadratic Forms over Complete Discretely Valuated Fields

Let $(F, v)$ be a nondyadic complete discretely valuated field with valuation ring $A = \{x \in F \mid v(x) \geq 0\} \cup \{0\}$. Let $\pi \in A$ be a uniformizer of $A$, and let the group of units of the ring $A$ be denoted by $U$. Then every element $y \in F^*$ can be written uniquely in the form $y = u\pi^{v(y)}$ for some $u \in U$. Thus any 1-dimensional reqular quadratic form over $F$ can be written as $\langle u \rangle$ or $\langle u\pi \rangle$ for some $u \in U$. Hence an arbitrary regular quadratic form $q$ over $F$ can be written as

$$q \simeq q_1 \perp q_2\pi$$

where $q_1 = \langle u_1, \ldots, u_{n_1} \rangle,\ q_2 = \langle v_1, \ldots, v_{n_2} \rangle$ with $u_i, v_i \in U$.

Let $\mathfrak{m} = \{x \in F \mid v(x) \geq 1\} \cup \{0\}$ be the unique maximal ideal of $A$, and let $\overline{F} = A/\mathfrak{m}$ be the residue class field of $A$. By assumption, $(F, v)$ is nondyadic and so char $\left(\overline{F}\right) \neq 2$. For $a \in A$, let $\overline{a} = a + \mathfrak{m} \in \overline{F}$. Let $\overline{q_1} = \langle \overline{u_1}, \ldots, \overline{u_{n_1}} \rangle$ and $\overline{q_2} = \langle \overline{v_1}, \ldots, \overline{v_{n_2}} \rangle$.

**Theorem 2.3.4.** (Springer). *We have a group isomorphism*

$$(\delta_1, \delta_2) \colon W(F) \to W\left(\overline{F}\right) \oplus W\left(\overline{F}\right),$$

*where* $\delta_1 \colon W(F) \to W\left(\overline{F}\right)$ *is given by* $\delta_1(q) = \overline{q_1}$ *and* $\delta_2 \colon W(F) \to W\left(\overline{F}\right)$ *is given by* $\delta_2(q) = \overline{q_2}$.

*Proof.* See [16, p. 147, Corollary 1.6]. □

**Definition.** The map $\delta_1 \colon W(F) \to W\left(\overline{F}\right)$ given by $\delta_1(q) = \overline{q_1}$ is called the *first residue homomorphism*, and $\overline{q_1}$ is called the *first residue form* of $q$. The map $\delta_2 \colon W(F) \to W\left(\overline{F}\right)$ given by $\delta_2(q) = \overline{q_2}$ is called the *second residue homomorphism*, and $\overline{q_2}$ is called the *second residue form* of $q$.

**Theorem 2.3.5.** *Suppose that* $q = q_1 \perp q_2\pi$, *where* $q_1 = \langle u_1, \ldots, u_{n_1}\rangle$, $q_2 = \langle v_1, \ldots, v_{n_2}\rangle$ *with* $u_i, v_i \in U$. *Then the following are equivalent:*

*(1)* $q$ *is isotropic;*

*(2)* $q_1$ *or* $q_2$ *is isotropic;*

*(3)* $\overline{q_1}$ *or* $\overline{q_2}$ *is isotropic.*

*Proof.* See [16, p. 148, Proposition 1.9]. □

# Chapter 3

# Galois Cohomology

## 3.1 Profinite Groups and Galois Groups

**Definition.** Let $(\Lambda, \leq)$ be a partially ordered set. We say that $(\Lambda, \leq)$ is *directed* if for all $\alpha, \beta \in \Lambda$ there exists $\gamma \in \Lambda$ such that $\alpha \leq \gamma$ and $\beta \leq \gamma$.

**Definition.** A *filtered inverse system* of groups $(G_\alpha, \phi_{\alpha\beta})$ consists of:

(a) a directed partially ordered set $(\Lambda, \leq)$;

(b) for all $\alpha \in \Lambda$ there exists a group $G_\alpha$;

(c) if $\alpha, \beta \in \Lambda$ with $\alpha \leq \beta$ then there exists a group homomorphism $\phi_{\alpha\beta} \colon G_\beta \to G_\alpha$;

(d) if $\alpha, \beta, \gamma \in \Lambda$ with $\alpha \leq \beta \leq \gamma$ then $\phi_{\alpha\gamma} = \phi_{\alpha\beta} \circ \phi_{\beta\gamma}$.

**Definition.** Let $(G_\alpha, \phi_{\alpha\beta})$ be a filtered inverse system of groups. The *inverse limit* of $(G_\alpha, \phi_{\alpha\beta})$, denoted $\varprojlim G_\alpha$, is defined to be

$$\varprojlim G_\alpha := \left\{ (g_\alpha) \in \prod_{\alpha \in \Lambda} G_\alpha \;\middle|\; \phi_{\alpha\beta}(g_\beta) = g_\alpha \text{ for all } \alpha \leq \beta \right\}.$$

**Definition.** A group $G$ is called *profinite* if $G = \varprojlim G_\alpha$ for some filtered inverse system of groups $(G_\alpha, \phi_{\alpha\beta})$ where $G_\alpha$ is a finite group for all $\alpha$.

**Remark.** A profinite group $G = \varprojlim G_\alpha$ has a natural topology: give $G_\alpha$ the discrete topology for all $\alpha$, give $\prod_\alpha G_\alpha$ the product topology and then the profinite group $G \subseteq \prod_\alpha G_\alpha$ is given the subspace topology.

Let $K/F$ be a Galois extension. Then the Galois groups of finite Galois subextensions of $K/F$ together with the group homomorphisms

$\phi_{LM}\colon \mathrm{Gal}(M/F) \to \mathrm{Gal}(L/F)$ (where $L/F$ and $M/F$ are finite Galois subextensions of $K/F$ such that $F \subseteq L \subseteq M \subseteq K$) form a filtered inverse system of groups $(\mathrm{Gal}(L/F), \phi_{LM})$.

**Proposition 3.1.1.** *Let $K/F$ be a Galois extension and let $(\mathrm{Gal}(L/F), \phi_{LM})$ be the filtered inverse system of groups defined above. Then*

$$\varprojlim \mathrm{Gal}(L/F) = \mathrm{Gal}(K/F).$$

*Proof.* See [6, Proposition 4.1.3]. □

**Remark.** Since $|\mathrm{Gal}(L/F)| = [L\colon F] < \infty$ for all finite Galois subextensions $L/F$ of $K/F$, it follows from Proposition 3.1.1 that $\mathrm{Gal}(K/F)$ is a profinite group.

## 3.2 Cohomology of Profinite Groups

**Definition.** Let $\Gamma$ be a profinite group and let $A$ be a discrete topological space. A left action by $\Gamma$ on $A$ is called *continuous* if for all $a \in A$, the stabilizer of $a$ in $\Gamma$

$$\mathrm{Stab}_\Gamma(a) = \{\sigma \in \Gamma \mid \sigma \cdot a = a\} \leq \Gamma$$

is an open subgroup of $\Gamma$.

**Definition.** Let $\Gamma$ be a profinite group and let $A$ be a discrete topological space. We call $A$ a $\Gamma$-*set* if $A$ is equipped with a continuous left action by $\Gamma$.

**Definition.** Let $\Gamma$ be a profinite group and let $A$ be a group which is also a $\Gamma$-set. We call $A$ a $\Gamma$-*group* if $\Gamma$ acts by group homomorphisms, that is,

$$\sigma(a_1 a_2) = \sigma(a_1)\sigma(a_2) \text{ for all } \sigma \in \Gamma,\ a_1, a_2 \in A.$$

**Definition.** Let $\Gamma$ be a profinite group. A $\Gamma$-*module* is an abelian $\Gamma$-group.

**Definition.** Let $\Gamma$ be a profinite group and let $A$ be a $\Gamma$-set. We define

$$H^0(\Gamma, A) := A^\Gamma = \{a \in A \mid \sigma a = a \text{ for all } \sigma \in \Gamma\}.$$

**Remark.** If $A$ is a $\Gamma$-group, then $H^0(\Gamma, A) \leq A$ is a subgroup of $A$.

**Definition.** Let $\Gamma$ be a profinite group and let $A$ be a $\Gamma$-group. Let $\alpha\colon \Gamma \to A$ be a continuous map and for $\sigma \in \Gamma$, let $\alpha_\sigma = \alpha(\sigma) \in A$. We call $\alpha$ a 1-*cocycle* of $\Gamma$ with values in $A$ if

$$\alpha_{\sigma\tau} = \alpha_\sigma \sigma(\alpha_\tau) \text{ for all } \sigma, \tau \in \Gamma.$$

**Notation.** Let $\Gamma$ be a profinite group and let $A$ be a $\Gamma$-group. The set of all 1-cocycles of $\Gamma$ with values in $A$ is denoted by $Z^1(\Gamma, A)$.

**Definition.** Let $\Gamma$ be a profinite group and let $A$ be a $\Gamma$-group. The 1-cocycle $\alpha\colon \Gamma \to A$ given by $\alpha_\sigma = 1$ for all $\sigma \in \Gamma$ is a distinguished element in $Z^1(\Gamma, A)$ which is called the *trivial* 1*-cocycle*.

**Definition.** Let $\Gamma$ be a profinite group and let $A$ be a $\Gamma$-group. Let $\alpha\colon \Gamma \to A$ and $\alpha'\colon \Gamma \to A$ be 1-cocycles. We say that $\alpha$ and $\alpha'$ are *cohomologous* or *equivalent* if there exists $a \in A$ such that

$$\alpha'_\sigma = a\alpha_\sigma\sigma(a)^{-1} \text{ for all } \sigma \in \Gamma.$$

**Notation.** Let $\Gamma$ be a profinite group and let $A$ be a $\Gamma$-group. The set of equivalence classes of 1-cocycles of $\Gamma$ with values in $A$ is denoted by $H^1(\Gamma, A)$. Then $H^1(\Gamma, A)$ is a pointed set whose distinguished element is the cohomology class of the trivial 1-cocycle.

**Remark.** If $A$ is a $\Gamma$-module, then $Z^1(\Gamma, A)$ is an abelian group, where the group operation is given by $(\alpha \cdot \beta)_\sigma = \alpha_\sigma\beta_\sigma$ for all $\alpha, \beta \in Z^1(\Gamma, A)$ and $\sigma \in \Gamma$. This group operation is compatible with the equivalence relation on 1-cocycles and thus makes $H^1(\Gamma, A)$ an abelian group.

## 3.3 Principal Homogeneous Spaces

**Definition.** Let $\Gamma$ be a profinite group and let $A$ be a $\Gamma$-group. Let $P$ be a nonempty $\Gamma$-set equipped with a right action by $A$. We call $P$ a $(\Gamma, A)$*-set* if

$$\sigma(pa) = \sigma(p)\sigma(a) \text{ for all } \sigma \in \Gamma,\ p \in P \text{ and } a \in A.$$

**Definition.** Let $\Gamma$ be a profinite group, let $A$ be a $\Gamma$-group and let $P$ be a $(\Gamma, A)$-set. We say that $P$ is a *principal homogeneous space under* $A$ (or an $A$*-torsor*) if the action of $A$ on $P$ is simply transitive, that is, for all $p, q \in P$ there exists a unique $a \in A$ such that $q = pa$.

**Notation.** Let $\Gamma$ be a profinite group and let $A$ be a $\Gamma$-group. We will denote the collection of all principal homogeneous spaces under $A$ by $\mathrm{PHS}(\Gamma, A)$.

**Example.** Let $\Gamma$ be a profinite group and let $A$ be a $\Gamma$-group. Given any 1-cocycle of $\Gamma$ with values in $A$, we may construct a corresponding principal homogeneous space under $A$ by defining a map $\psi\colon Z^1(\Gamma, A) \to \mathrm{PHS}(\Gamma, A)$ given by $\psi(\alpha) = P_\alpha$ for all $\alpha \in Z^1(\Gamma, A)$, where $P_\alpha$ is the set $A$ equipped with a left action $\bullet$ by $\Gamma$ given by

$$\sigma \bullet a = \alpha_\sigma\sigma(a) \text{ for all } \sigma \in \Gamma,\ a \in A$$

and a right action $*$ by $A$ given by

$$a * b = ab \text{ for all } a, b \in A.$$

**Definition.** Let $\Gamma$ be a profinite group and let $A$ be a $\Gamma$-group. Let $P$ and $Q$ be principal homogeneous spaces under $A$. A map $\phi\colon P \to Q$ is called a *morphism* of principal homogeneous spaces under $A$ if

(1) $\phi(\sigma p) = \sigma\phi(p)$ for all $\sigma \in \Gamma,\ p \in P$ and

(2) $\phi(pa) = \phi(p)a$ for all $p \in P,\ a \in A$.

**Theorem 3.3.1.** *Let $\Gamma$ be a profinite group and let $A$ be a $\Gamma$-group. Let $\psi\colon Z^1(\Gamma, A) \to \mathrm{PHS}(\Gamma, A)$ be the map defined in the example above. Then $\psi$ induces a bijection between $H^1(\Gamma, A)$ and the set of isomorphism classes of principal homogeneous spaces under $A$.*

*Proof.* See for instance [15, Proposition 28.14]. □

# Chapter 4

# Linear Algebraic Groups and Patching Techniques

## 4.1 First Definitions

Some general references for the contents of this section are
( [15, Chapters VI and VII]) and ( [2]).

**Definition.** Let $F$ be a field. A *linear algebraic group* over $F$ is an affine algebraic variety $G$ over $F$ endowed with the structure of a group such that the multiplication map

$$\mu\colon G \times G \to G$$
$$(g_1, g_2) \mapsto g_1 g_2$$

and the inverse map

$$i\colon G \to G$$
$$g \mapsto g^{-1}$$

are morphisms of varieties.

**Example.** Let $F$ be a field. The *additive group* $\mathbb{G}_a$ over $F$ is the affine line $\mathbb{A}^1_F$ endowed with the group operation $\mu(x, y) = x + y$, the identity element 0 and the inverse map $i(x) = -x$.

**Example.** Let $F$ be a field. The *multiplicative group* $\mathbb{G}_m$ over $F$ is the affine open set $F^* \subseteq \mathbb{A}^1$ endowed with the multiplication map $\mu(x, y) = xy$, the identity element 1 and the inverse map $i(x) = x^{-1}$.

**Example.** Let $F$ be a field and let $n \in \mathbb{N}$. The *general linear group* $\mathrm{GL}_n$ over $F$ is the set of invertible $n \times n$ matrices over $F$ endowed with the multiplcation map given by matrix multiplication $\mu(A, B) = AB$, the identity element being the $n \times n$ identity matrix $I_n$ and the inverse map given by the inverse matrix $i(A) = A^{-1}$.

**Notation.** Let $F$ be a field and let $F_s$ be the separable closure of $F$. Let $G$ be a linear algebraic group over $F$. Then $G(F_s)$ is a $\mathrm{Gal}(F_s/F)$-group, and we define

$$H^1(F, G) := H^1(F, G(F_s)) = H^1(\mathrm{Gal}(F_s/F), G(F_s)).$$

**Definition.** Let $F$ be a field. Let $G_1$ and $G_2$ be linear algebraic groups over $F$. A *morphism of linear algebraic groups* $\phi\colon G_1 \to G_2$ is a group homomorphism which is also a morphism of varieties.

**Definition.** Let $F$ be a field and let $F_s$ be the separable closure of $F$. A linear algebraic group $T$ over $F$ is called a *torus* if there exists $n \in \mathbb{N}$ such that

$$T(F_s) \simeq \mathbb{G}_m^n.$$

Let $K/F$ be a field extension. We say that the torus $T$ is *split* over $K$ if

$$T(K) \simeq \mathbb{G}_m^n.$$

**Definition.** Let $F$ be a field and let $G$ be a linear algebraic group over $F$. A subtorus $T \subseteq G$ is said to be *maximal* if $T$ is not contained in a larger subtorus of $G$.

**Definition.** Let $F$ be a field and let $\overline{F}$ be the algebraic closure of $F$. Let $G$ be a linear algebraic group over $F$. We say that $G$ is *semisimple* if $G \neq \{1\}$ and $G \times_F \overline{F}$ has no nontrivial solvable connected normal subgroups.

**Definition.** Let $F$ be a field and let $G$ be a semisimple linear algebraic group over $F$. We say that $G$ is *split* if it contains a split maximal torus.

**Definition.** Let $V$ be a finite-dimensional $\mathbb{R}$-vector space, let $\alpha \in V$ with $\alpha \neq 0$ and let $s \in \mathrm{End}(V)$. We say that $s$ is a *reflection with respect to* $\alpha$ if

(1) $s(\alpha) = -\alpha$ and

(2) there exists a hyperplane $W \subseteq V$ such that $s|_W = \mathrm{Id}$.

**Remark.** If $s \in \mathrm{End}(V)$ is a reflection with respect to $\alpha \in V$, then there exists a unique $f \in V^*$ with $f|_W = 0$ and $f(\alpha) = 2$ such that

$$s(v) = v - f(v)\alpha \quad \text{for all } v \in V.$$

**Definition.** Let $V$ be a finite-dimensional $\mathbb{R}$-vector space with $V \neq 0$, and let $\Phi \subseteq V$ be a finite subset of $V$. We call $\Phi$ a *root system* if the following conditions hold:

(a) $0 \notin \Phi$.

(b) $\Phi$ spans $V$.

(c) If $\alpha \in \Phi$ and $x\alpha \in \Phi$ for $x \in \mathbb{R}$, then $x = \pm 1$.

(d) For each $\alpha \in \Phi$ there exists a reflection $s_\alpha \in \mathrm{End}(V)$ with respect to $\alpha$ such that $s_\alpha(\Phi) = \Phi$.

(e) For all $\alpha, \beta \in \Phi$ we have $s_\alpha(\beta) - \beta = n_{\alpha,\beta} \cdot \alpha$ for some $n_{\alpha,\beta} \in \mathbb{Z}$.

**Remark.** The reflection $s_\alpha \in \mathrm{End}(V)$ with respect to $\alpha \in \Phi$ in (d) is uniquely determined by $\alpha$ (see Bourbaki [3, Chapter VI, §1, Lemme 1]).

**Definition.** Let $V$ be a finite-dimensional $\mathbb{R}$-vector space with $V \neq 0$, and let $\Phi \subseteq V$ be a root system. The elements of $\Phi$ are called *roots*.

**Definition.** Let $V$ be a finite-dimensional $\mathbb{R}$-vector space with $V \neq 0$, and let $\Phi \subseteq V$ be a root system. For $\alpha \in \Phi$, we define $\alpha^* \in V^*$ by

$$s_\alpha(v) = v - \alpha^*(v)\alpha \quad \text{for all } v \in V.$$

Such $\alpha^*$ are called *coroots*.

**Definition.** Let $V$ be a finite-dimensional $\mathbb{R}$-vector space with $V \neq 0$, and let $\Phi \subseteq V$ be a root system. We define the *root lattice*, denoted $\Lambda_r$, to be the additive subgroup of $V$ generated by all roots $\alpha \in \Phi$.

**Definition.** Let $V$ be a finite-dimensional $\mathbb{R}$-vector space with $V \neq 0$, and let $\Phi \subseteq V$ be a root system. We define the *weight lattice*, denoted $\Lambda$, to be

$$\Lambda := \{v \in V \mid \alpha^*(v) \in \mathbb{Z} \text{ for } \alpha \in \Phi\}.$$

**Remark.** By definition, we have $\Lambda_r \subseteq \Lambda$.

Let $F$ be a field and let $G$ be a split semisimple linear algebraic group over $F$ with a split maximal torus $T$ over $F$. Using the adjoint representation $\mathrm{Ad}\colon G \to \mathrm{GL}(\mathrm{Lie}(G))$, one can define a root system $\Phi(G) \subseteq T^* \otimes_{\mathbb{Z}} \mathbb{R}$ such that $\Lambda_r \subseteq T^* \subseteq \Lambda$, where $\Lambda_r$ is the root lattice, $T^*$ is the character group of $T$ and $\Lambda$ is the weight lattice ( [15, Theorem 25.1 and Proposition 25.2]).

**Definition.** We say that $G$ above is *simply connected* if the character group $T^* = \Lambda$. We say that $G$ is *adjoint* if the character group $T^* = \Lambda_r$.

**Definition.** Let $F$ be a field and let $F_s$ be the separable closure of $F$. Let $G$ be a semisimple linear algebraic group over $F$. We say that $G$ is *simply connected* if the split group $G \times_F F_s$ is simply connected. We say that $G$ is *adjoint* if the split group $G \times_F F_s$ is adjoint.

**Definition.** Let $F$ be a field and let $F_s$ be the separable closure of $F$. Let $G$ be a semisimple linear algebraic group over $F$. We say that $G$ is *absolutely simple* if $G \times_F F_s$ has no nontrivial connected normal subgroups.

## 4.2 Classification of Absolutely Simple, Adjoint, Classical Linear Algebraic Groups

Let $F$ be a field with $\mathrm{char}(F) \neq 2$, and let $G$ be an absolutely simple linear algebraic group over $F$ of classical type. Then, for an arbitrary integer $n \geq 1$, to the group $G$ is associated a central simple algebra with possible additional structure:

(1) ${}^1A_n$: Central simple $F$-algebras of degree $n+1$;

(2) ${}^2A_n$: Central simple algebras of degree $n+1$ over a quadratic extension of $F$ with involution of the second kind leaving $F$ elementwise invariant;

(3) $B_n$: Quadratic forms over $F$ of dimension $2n+1$;

(4) $C_n$: Central simple $F$-algebras of degree $2n$ with symplectic involution;

(5) $D_n$: Central simple $F$-algebras of degree $2n$ with orthogonal involution.

### Case 1: Linear Algebraic Groups Of Type ${}^1A_n$

Let $G$ be an absolutely simple, simply connected linear algebraic group of type ${}^1A_n$ over $F$, and let $\overline{G}$ be the corresponding absolutely simple, adjoint linear algebraic group of type ${}^1A_n$ over $F$. Then $G = \mathrm{SL}(A)$ and $\overline{G} = \mathrm{PGL}(A)$ for some central simple $F$-algebra $A$ of degree $n+1$. Then $H^1(F, \mathrm{PGL}(A))$ classifies $F$-isomorphism classes of central simple $F$-algebras $B$ such that $\deg(A) = \deg(B)$.

### Case 2: Linear Algebraic Groups Of Type ${}^2A_n$

Let $G$ be an absolutely simple, simply connected linear algebraic group of type ${}^2A_n$ over $F$, and let $\overline{G}$ be the corresponding absolutely simple, adjoint linear algebraic group of type ${}^2A_n$ over $F$. Then $G = \mathrm{SU}(A, \sigma)$ and $\overline{G} = \mathrm{PGU}(A, \sigma)$ for some central simple algebra $A$ of degree $n+1$ whose center $Z(A)$ is a quadratic extension of $F$, with involution $\sigma$ of the second kind such that $\sigma(x) = x$ for all $x \in F$.

Now $H^1(F, \mathrm{PGU}(A, \sigma))$ classifies $F$-isomorphism classes of tuples $(B, \tau)$ consisting of a central simple algebra $B$ whose center $Z(B) \cong Z(A)$ is a quadratic extension of $F$ such that $\deg(A) = \deg(B)$, with involution $\tau$ of the second kind such that $\tau(x) = x$ for all $x \in F$. The trivial element in this set is the class of $(A, \sigma)$.

Now suppose $[(A, \tau)] = 1 \in H^1(F, \mathrm{PGU}(A, \sigma))$, so that $(A, \tau) \simeq (A, \sigma)$. Write $A$ as $A \cong M_m(D)$ for some $m \in \mathbb{N}$ and $D$ a central division algebra over $Z(A)$. Let $h_1$ be the hermitian form on $D$ corresponding to $\sigma$, so that $\sigma$ is the adjoint involution with respect to $h_1$, and let $h_2$ be the hermitian form on $D$ corresponding to $\tau$, so that $\tau$ is the adjoint involution with respect to $h_2$. Then the condition that $(A, \tau) \simeq (A, \sigma)$ is equivalent to the condition that $h_1 \simeq \lambda h_2$ for some $\lambda \in F$.

### Case 3: Linear Algebraic Groups Of Type $B_n$

Let $G$ be an absolutely simple, simply connected linear algebraic group of type $B_n$ over $F$, and let $\overline{G}$ be the corresponding absolutely simple, adjoint linear algebraic group of type $B_n$ over $F$. Then $G = \mathrm{Spin}(q)$ and $\overline{G} = \mathrm{O}^+(q)$ for some quadratic form $q$ over $F$ of dimension $2n + 1$. Then $H^1(F, \mathrm{O}^+(q)) \cong H^1(F, \mathrm{SO}(q))$ classifies isometry classes of quadratic forms $q'$ over $F$ such that $\dim(q) = \dim(q')$ and $\mathrm{disc}(q) = \mathrm{disc}(q')$.

### Case 4: Linear Algebraic Groups Of Type $C_n$

Let $G$ be an absolutely simple, simply connected linear algebraic group of type $C_n$ over $F$, and let $\overline{G}$ be the corresponding absolutely simple, adjoint linear algebraic group of type $C_n$ over $F$. Then $G = \mathrm{Sp}(A, \sigma)$ and $\overline{G} = \mathrm{PGSp}(A, \sigma)$ for some central simple $F$-algebra $A$ of degree $2n$ with symplectic involution $\sigma$.

Now $H^1(F, \mathrm{PGSp}(A, \sigma))$ classifies $F$-isomorphism classes of central simple $F$-algebras $B$ such that $\deg(A) = \deg(B)$, with symplectic involution $\tau$. The trivial element in this set is the class of $(A, \sigma)$.

Now suppose $[(A, \tau)] = 1 \in H^1(F, \mathrm{PGSp}(A, \sigma))$, so that $(A, \tau) \simeq (A, \sigma)$. Write $A$ as $A \cong M_m(D)$ for some $m \in \mathbb{N}$ and $D$ a central division algebra over $F$. Let $h_1$ be the hermitian form on $D$ corresponding to $\sigma$, so that $\sigma$ is the adjoint involution with respect to $h_1$, and let $h_2$ be the hermitian form on $D$ corresponding to $\tau$, so that $\tau$ is the adjoint involution with respect to $h_2$. Then the condition that $(A, \tau) \simeq (A, \sigma)$ is equivalent to the condition that $h_1 \simeq \lambda h_2$ for some $\lambda \in F$.

### Case 5: Linear Algebraic Groups Of Type $D_n$

Let $G$ be an absolutely simple, simply connected linear algebraic group of type $D_n$ over $F$, and let $\overline{G}$ be the corresponding absolutely simple, adjoint linear algebraic group of type $D_n$ over $F$. Then $G = \mathrm{Spin}(A, \sigma)$ and $\overline{G} = \mathrm{PGO}^+(A, \sigma)$ for some central simple $F$-algebra $A$ of degree $2n$ with orthogonal involution $\sigma$.

Now $H^1(F, \mathrm{PGO}^+(A,\sigma))$ classifies $F$-isomorphism classes of triples $(B,\tau,\eta)$ consisting of a central simple $F$-algebra $B$ with orthogonal involution $\tau$ such that $\deg(A) = \deg(B)$, with an $F$-algebra isomorphism $\eta\colon Z(C(A,\sigma)) \to Z(C(B,\tau))$ of the centers of the Clifford algebras. The trivial element in this set is the class of $(A,\sigma,\mathrm{Id})$.

Now suppose $[(A,\tau,\eta)] = 1 \in H^1(F, \mathrm{PGO}^+(A,\sigma))$, so that $(A,\tau,\eta) \simeq (A,\sigma,\mathrm{Id})$. Write $A$ as $A \cong M_m(D)$ for some $m \in \mathbb{N}$ and $D$ a central division algebra over $F$. Let $h_1$ be the hermitian form on $D$ corresponding to $\sigma$, so that $\sigma$ is the adjoint involution with respect to $h_1$, and let $h_2$ be the hermitian form on $D$ corresponding to $\tau$, so that $\tau$ is the adjoint involution with respect to $h_2$. Then the condition that $(A,\tau,\eta) \simeq (A,\sigma,\mathrm{Id})$ is equivalent to the condition that there is a similitude $\phi\colon h_1 \simeq \lambda h_2$ for some $\lambda \in F^*$ such that $\phi\restriction_{Z(C(A,\sigma))}\colon Z(C(A,\sigma)) \to Z(C(A,\tau))$ coincides with $\eta$.

## 4.3 Semi-Global Fields and Patching

**Definition.** A *semi-global field* is the function field of a smooth, projective, geometrically integral curve over a complete discretely valuated field.

Let $K$ be a complete discretely valuated field with valuation ring $T$ and a parameter $t \in T$. Let $X$ be a smooth, projective, geometrically integral curve over $K$, and let $F = K(X)$ be the function field of the curve $X$ (so that $F$ is a semi-global field).

**Definition.** A regular two dimensional integral scheme $\mathscr{X}$ which is proper over $T$ with function field $F$ is called a *regular proper model* of $F$.

By Abhyankar ( [1]) and Lipman ( [17]), there exists a regular proper model $\mathscr{X}$ of $F$ with special fibre $X_0$ such that $X_0$ is a union of regular curves with normal crossings. Let $\mathscr{P} \subseteq X_0$ be a finite set of closed points of $\mathscr{X}$ containing all the nodal points of $X_0$ and at least one point on each component. Let $\mathscr{U}$ be the set of irreducible components of $X_0 \setminus \mathscr{P}$. Then $\mathscr{U} = \{U_1, U_2, \ldots, U_l\}$ is a finite set.

**Notation.** For $P \in \mathscr{P}$, let $\mathcal{O}_{\mathscr{X},P}$ be the local ring at $P$. So $\mathcal{O}_{\mathscr{X},P}$ is a two dimensional regular local ring. Let $\mathfrak{m}_P$ be the maximal ideal of $\mathcal{O}_{\mathscr{X},P}$, and let $\widehat{\mathcal{O}_{\mathscr{X},P}}$ denote the completion of $\mathcal{O}_{\mathscr{X},P}$ at the maximal ideal $\mathfrak{m}_P$. Define $F_P := \mathrm{ff}(\widehat{\mathcal{O}_{\mathscr{X},P}})$.

**Notation.** For $U \in \mathscr{U}$, let $R_U$ be the set of rational functions which are regular on $U$:

$$R_U := \{f \in F \mid f \text{ is regular on } U\}.$$

Let $\widehat{R_U}$ be the $(t)$-adic completion of $R_U$. Define $F_U := \mathrm{ff}\left(\widehat{R_U}\right)$.

**Notation.** For $P \in \mathscr{P}$, each height one prime ideal $\rho$ of $\widehat{\mathcal{O}_{\mathscr{X},P}}$ that contains $t$ determines a *branch* of $X_0$ at $P$ (i.e. an irreducible component of the pullback of $X_0$ to Spec $\widehat{\mathcal{O}_{\mathscr{X},P}}$). We let $\hat{R}_\rho$ denote the completion of the local ring $\widehat{\mathcal{O}_{\mathscr{X},P}}$ at $\rho$. Define $F_\rho \coloneqq \mathrm{ff}\left(\hat{R}_\rho\right)$. Since $t \in \rho$, the contraction of $\rho \subseteq \widehat{\mathcal{O}_{\mathscr{X},P}}$ to the local ring $\mathcal{O}_{\mathscr{X},P}$ defines an irreducible component of Spec $\mathcal{O}_{X_0,P}$ and hence an irreducible component of $X_0$ containing $P$. This in turn is the closure of a unique connected component $U$ of $X_0 \setminus \mathscr{P}$, and we say that $\rho$ *lies on* $U$. We call $F_{U,P} \coloneqq F_\rho$ a *branch field*.

**Remark.** For $P$ and $U$ as above, there are natural inclusions $F_P \hookrightarrow F_{U,P}$ and $F_U \hookrightarrow F_{U,P}$.

## 4.4 Local-Global Principles for Linear Algebraic Groups

**Notation.** Let $F, \mathscr{X}, \mathscr{P}, \mathscr{U}$ be as in Section 4.3, and let $G$ be a linear algebraic group over $F$. We define

$$\text{Ш}_{\mathscr{X},\mathscr{P},\mathscr{U}}(F,G) \coloneqq \ker\left(H^1(F,G) \to \prod_{x \in \mathscr{P} \cup \mathscr{U}} H^1(F_x,G)\right).$$

**Theorem 4.4.1.** *Let $F, \mathscr{X}, \mathscr{P}, \mathscr{U}$ be as in Section 4.3, and let $\mathscr{B} = \{(P,U) \in \mathscr{P} \times \mathscr{U} \mid P$ is in the closure of $U\}$. Let $G$ be a linear algebraic group over $F$. Then we have a bijection*

$$\prod_{U \in \mathscr{U}} G(F_U) \Big\backslash \prod_{(P,U) \in \mathscr{B}} G(F_{U,P}) \Big/ \prod_{P \in \mathscr{P}} G(F_P) \xrightarrow{\sim} \text{Ш}_{\mathscr{X},\mathscr{P},\mathscr{U}}(F,G).$$

*Proof.* See [9, Corollary 3.6]. □

**Notation.** Let $F$ be any field and let $\Omega_F$ be the set of all discrete valuations on $F$. For $v \in \Omega_F$, let $\hat{F}_v$ denote the completion of $F$ at $v$. Let $G$ be a linear algebraic group over $F$. We define

$$\text{Ш}(F,G) \coloneqq \ker\left(H^1(F,G) \to \prod_{v \in \Omega_F} H^1\left(\hat{F}_v, G\right)\right).$$

**Definition.** Let $F$ be any field and let $G$ be a linear algebraic group over $F$. We say that the *Hasse principle* holds for $G$ if $\text{Ш}(F,G)$ is trivial.

**Theorem 4.4.2.** *Let $F, \mathscr{X}, \mathscr{P}, \mathscr{U}$ be as in Section 4.3, and let $G$ be a linear algebraic group over $F$. Then we have an injection*

$$\text{Ш}_{\mathscr{X},\mathscr{P},\mathscr{U}}(F,G) \hookrightarrow \text{Ш}(F,G).$$

*Proof.* See [9, Proposition 8.2]. □

# Chapter 5

# Main Theorems

## 5.1 Quadratic Forms Over Two Dimensional Complete Fields

Let $R$ be a complete two dimensional regular local ring, let $F = \mathrm{ff}(R)$, and suppose $2 \in R^*$. Let $\mathfrak{m} = (\pi, \delta)$ be the maximal ideal of $R$. Let $\widehat{R_{(\pi)}}$ denote the completion of the localization of $R$ at the prime ideal $(\pi)$, and let $\widehat{R_{(\delta)}}$ denote the completion of the localization of $R$ at the prime ideal $(\delta)$. Define $F_\pi := \mathrm{ff}\left(\widehat{R_{(\pi)}}\right)$ and $F_\delta := \mathrm{ff}\left(\widehat{R_{(\delta)}}\right)$. Then $F_\pi$ and $F_\delta$ are complete discretely valued fields. Further the residue field $\kappa(\pi)$ of $F_\pi$ is the field of fractions of $R/(\pi)$ and hence a local field. Similarly the residue field $\kappa(\delta)$ of $F_\delta$ is the field of fractions of $R/(\delta)$ and hence a local field.

Let $q$ be a quadratic form over $F$. Suppose

$$q \simeq q_1 \perp q_2\pi \perp q_3\delta \perp q_4\pi\delta,$$

where $q_1 = \langle u_1, \ldots, u_{n_1}\rangle,\ q_2 = \langle v_1, \ldots, v_{n_2}\rangle,\ q_3 = \langle w_1, \ldots, w_{n_3}\rangle$, $q_4 = \langle \theta_1, \ldots, \theta_{n_4}\rangle$ with $u_i, v_i, w_i, \theta_i \in R^*$. In this section we analyze elements $\lambda$ in $F$ with $\lambda q \simeq q$.

Suppose $k = R/\mathfrak{m}$ is a finite field. Then the order of $k^*/k^{*2}$ is 2. For any $\theta \in R$, let $\bar{\theta}$ denote the image of $\theta$ in $k$.

We begin with the following

**Lemma 5.1.1.** *There exists $\beta \in F$ such that $\beta(F_\pi^*)^2 = (F_\pi^*)^2$, $\beta(F_\delta^*)^2 = t(F_\delta^*)^2$ and $\beta\langle 1, -t\rangle \simeq \langle 1, -t\rangle$ over $F$.*

*Proof.* Consider the quadratic field extension $k\left(\sqrt{(\bar{t})}\right)/k$. Since $k$ is a finite field, the field norm map $N_{k\left(\sqrt{(\bar{t})}\right)/k} \colon k\left(\sqrt{(\bar{t})}\right) \to k$ is surjective.

In particular, there exists $s \in k\left(\sqrt{(\bar{t})}\right)$ such that $N_{k\left(\sqrt{(\bar{t})}\right)/k}(s) = \bar{t}$. Let $\overline{P}(z) = z^2 + \bar{r}z + \bar{t} \in k[z]$ be the minimal polynomial of $s$ over $k$. Then $\overline{P}(z)$ splits over $k\left(\sqrt{(\bar{t})}\right)$ as $\overline{P}(z) = (z-s)(z-s_0)$ for some $s_0 \in k\left(\sqrt{(\bar{t})}\right)$. Note that $s \neq s_0$, for if $s = s_0$, then $s \in k\left(\sqrt{(\bar{t})}\right)$ is equal to its $k$-conjugate, so $s \in k$ and $N_{k\left(\sqrt{(\bar{t})}\right)/k}(s) = s^2 \in (k^*)^2$, which contradicts the fact that $N_{k\left(\sqrt{(\bar{t})}\right)/k}(s) = \bar{t} \notin (k^*)^2$.

Let $\tilde{R}$ be the integral closure of $R$ in $F\left(\sqrt{t}\right)$. Since $R$ is a complete two dimensional local ring, so is $\tilde{R}$. Let $\tilde{\mathfrak{m}}$ be the maximal ideal of $\tilde{R}$. Now $t \in R^*$, and by assumption, $2 \in R^*$. Hence $\tilde{R} = R\left[\left(\sqrt{t}\right)\right]$ and $\tilde{R}/\tilde{\mathfrak{m}} = k\left(\sqrt{(\bar{t})}\right)$. Let $P(z) = z^2 + rz + t \in R[z]$ be a lift of $\overline{P}(z)$. Since $\tilde{R}$ is Henselian and $P(z)$ is monic, the factorization $\overline{P}(z) = (z-s)(z-s_0) \in k\left(\sqrt{(\bar{t})}\right)[z]$ can be lifted to a factorization $P(z) = (z-\tilde{s})(z-\tilde{s_0}) \in R\left[\left(\sqrt{t}\right)\right][z]$, where $\tilde{s} \in R\left[\left(\sqrt{t}\right)\right]$ is a lift of $s \in k\left(\sqrt{(\bar{t})}\right)$ and $\tilde{s_0} \in R\left[\left(\sqrt{t}\right)\right]$ is a lift of $s_0 \in k\left(\sqrt{(\bar{t})}\right)$. Then $P(z) = z^2 + rz + t = (z-\tilde{s})(z-\tilde{s_0})$, so $t = \tilde{s}\tilde{s_0} \in N_{F\left(\sqrt{t}\right)/F}\left(F\left(\sqrt{t}\right)\right)$.

Let $\beta = \delta^2 + r\pi\delta + t\pi^2 \in F$. Then $\beta = \delta^2(1 + \delta^{-2}r\pi\delta + \delta^{-2}t\pi^2) \in F_\pi$. But $1 + \delta^{-2}r\pi\delta + \delta^{-2}t\pi^2 \in (F_\pi^*)^2$. Therefore $\beta(F_\pi^*)^2 = (F_\pi^*)^2$. Similarly, $\beta = t\pi^2(1 + t^{-1}\pi^{-1}r\delta + t^{-1}\pi^{-2}\delta^2) \in F_\delta$. But $1 + t^{-1}\pi^{-1}r\delta + t^{-1}\pi^{-2}\delta^2 \in (F_\delta^*)^2$. Therefore $\beta(F_\delta^*)^2 = t(F_\delta^*)^2$.

It remains to show that $\beta\langle 1, -t\rangle \simeq \langle 1, -t\rangle$. To this end, let

$$\beta' = \frac{\beta}{\pi^2} = \left(\frac{\delta}{\pi}\right)^2 + r\left(\frac{\delta}{\pi}\right) + t \in F.$$

Let $\alpha = \delta/\pi \in F$, so that $\beta' = \alpha^2 + r\alpha + t$. Then $\beta' = P(\alpha) = (\alpha-\tilde{s})(\alpha-\tilde{s_0})$. Now $\alpha \in F$ and $\tilde{s}, \tilde{s_0} \in F\left(\sqrt{t}\right)$ are $F$-conjugates. Thus $\alpha - \tilde{s},\ \alpha - \tilde{s_0} \in F\left(\sqrt{t}\right)$ are $F$-conjugates, and $\beta' = (\alpha-\tilde{s})(\alpha-\tilde{s_0}) \in N_{F\left(\sqrt{t}\right)/F}\left(F\left(\sqrt{t}\right)\right) = D_F(\langle 1, -t\rangle) = G_F(\langle 1, -t\rangle)$ (since $\langle 1, -t\rangle$ is a Pfister form over $F$). So $\beta' \in G_F(\langle 1, -t\rangle)$, and thus $\beta = \beta'\pi^2 \in G_F(\langle 1, -t\rangle)$ also. Therefore $\beta\langle 1, -t\rangle \simeq \langle 1, -t\rangle$ as required. □

Now let $q$ be a quadratic form over $F$. Suppose

$$q \simeq q_1 \perp q_2\pi \perp q_3\delta \perp q_4\pi\delta,$$

where $q_1 = \langle u_1, \ldots, u_{n_1}\rangle,\ q_2 = \langle v_1, \ldots, v_{n_2}\rangle,\ q_3 = \langle w_1, \ldots, w_{n_3}\rangle$, $q_4 = \langle \theta_1, \ldots, \theta_{n_4}\rangle$ with $u_i, v_i, w_i, \theta_i \in R^*$.

**Lemma 5.1.2.** *Let* $q_1' = \langle u_1', \ldots, u_{n_1}'\rangle,\ q_2' = \langle v_1', \ldots, v_{n_2}'\rangle$, $q_3' = \langle w_1', \ldots, w_{n_3}'\rangle,\ q_4' = \langle \theta_1', \ldots, \theta_{n_4}'\rangle$ *be quadratic forms over* $F$ *with*

$u_i', v_i', w_i', \theta_i' \in R^*$. *Then*

$$q_1 \perp q_2\pi \perp q_3\delta \perp q_4\pi\delta \simeq q_1' \perp q_2'\pi \perp q_3'\delta \perp q_4'\pi\delta \text{ over } F$$

*if and only if* $q_i \simeq q_i'$ *over* $F$ *for all* $i$.

*Proof.* The "if" part is clear. For the converse, first note that $F_\pi$ is a complete discretely valued field with parameter $\pi$. Let $\kappa(\pi) := \widehat{R_{(\pi)}}/(\pi)$ be the residue field of $F_\pi$. For $a \in \widehat{R_{(\pi)}}$, let $\overline{a} = a + (\pi) \in \widehat{R_{(\pi)}}/(\pi) = \kappa(\pi)$. For $1 \le i \le 4$, let $\overline{q_i}$ be the residue form of $q_i$ over $F_\pi$, so that $\overline{q_i}$ is a quadratic form over $\kappa(\pi)$. Now

$$\begin{aligned} &q_1 \perp q_2\pi \perp q_3\delta \perp q_4\pi\delta \simeq q_1' \perp q_2'\pi \perp q_3'\delta \perp q_4'\pi\delta \text{ over } F \\ &\implies (q_1 \perp q_3\delta) \perp (q_2 \perp q_4\delta)\pi \simeq (q_1' \perp q_3'\delta) \perp (q_2' \perp q_4'\delta)\pi \text{ over } F_\pi. \end{aligned}$$

By Springer's theorem ( [16, p. 147, Corollary 1.6]), we obtain

$$\overline{q_1} \perp \overline{q_3}\ \overline{\delta} \simeq \overline{q_1'} \perp \overline{q_3'}\ \overline{\delta} \text{ and } \overline{q_2} \perp \overline{q_4}\ \overline{\delta} \simeq \overline{q_2'} \perp \overline{q_4'}\ \overline{\delta} \text{ over } \kappa(\pi).$$

Now $\kappa(\pi)$ is a complete discretely valuated field with parameter $\overline{\delta}$ and residue field $k = F/\mathfrak{m}$. For $1 \le i \le 4$, let $\tilde{\overline{q_i}}$ be the residue form of $\overline{q_i}$ over $\kappa(\pi)$, so that $\tilde{\overline{q_i}}$ is a quadratic form over $k$. Then we can apply Springer's theorem again to obtain $\tilde{\overline{q_i}} \simeq \tilde{\overline{q_i'}}$ over $k$ for all $i$. Since $k = F/\mathfrak{m}$, it follows that $q_i \simeq q_i'$ over $F$ for all $i$ as required. ∎

**Remark 5.1.3.** For $1 \le i \le 4$, let $q_i$ and $q_i'$ be as above. The proof of Lemma 5.1.2 shows that

$$q_1 \perp q_2\pi \perp q_3\delta \perp q_4\pi\delta \simeq q_1' \perp q_2'\pi \perp q_3'\delta \perp q_4'\pi\delta \text{ over } F_\pi$$

if and only if $q_i \simeq q_i'$ over $F$ for all $i$. Thus

$$q_1 \perp q_2\pi \perp q_3\delta \perp q_4\pi\delta \simeq q_1' \perp q_2'\pi \perp q_3'\delta \perp q_4'\pi\delta \text{ over } F$$

if and only if

$$q_1 \perp q_2\pi \perp q_3\delta \perp q_4\pi\delta \simeq q_1' \perp q_2'\pi \perp q_3'\delta \perp q_4'\pi\delta \text{ over } F_\pi.$$

As a consequence, for $w \in R^*$ and $r, s \in \mathbb{Z}$, if $\theta = w\pi^r\delta^s$ satisfies $\theta q \simeq q$ over $F_\pi$, then $\theta q \simeq q$ over $F$. Similarly, if $\theta = w\pi^r\delta^s$ satisfies $\theta q \simeq q$ over $F_\delta$, then $\theta q \simeq q$ over $F$.

We can use Lemma 5.1.2 to analyze when $\lambda \in F^*$ satisfies $\lambda q \simeq q$ over $F$ for the three cases $\lambda = w,\ \lambda = \pi$ and $\lambda = \delta$, where $w \in R^*$.

**Proposition 5.1.4.** *Let* $w \in R^*$. *We have* *(i)* $wq \simeq q$ *over* $F \iff wq_i \simeq q_i$ *over* $F$ *for all* $i$*;*

*(ii)* $\pi q \simeq q$ *over* $F \iff q_1 \simeq q_2$ *and* $q_3 \simeq q_4$ *over* $F$*;*

*(iii)* $\delta q \simeq q$ *over* $F \iff q_1 \simeq q_3$ *and* $q_2 \simeq q_4$ *over* $F$*.*

*Proof.* (i) We have

$$\begin{aligned} wq \simeq q &\iff wq_1 \perp wq_2\pi \perp wq_3\delta \perp wq_4\pi\delta \simeq q_1 \perp q_2\pi \perp q_3\delta \perp q_4\pi\delta \\ &\iff wq_i \simeq q_i \text{ for all } i, \end{aligned}$$

where the second equivalence follows from Lemma 5.1.2.

(ii) We have

$$\begin{aligned} \pi q \simeq q &\iff q_2 \perp q_1\pi \perp q_4\delta \perp q_3\pi\delta \simeq q_1 \perp q_2\pi \perp q_3\delta \perp q_4\pi\delta \\ &\iff q_1 \simeq q_2 \text{ and } q_3 \simeq q_4, \end{aligned}$$

where the second equivalence follows from Lemma 5.1.2.

(iii) We have

$$\begin{aligned} \delta q \simeq q &\iff wq_3 \perp wq_4\pi \perp wq_1\delta \perp wq_2\pi\delta \simeq q_1 \perp q_2\pi \perp q_3\delta \perp q_4\pi\delta \\ &\iff wq_1 \simeq q_3 \text{ and } wq_2 \simeq q_4, \end{aligned}$$

where the second equivalence follows from Lemma 5.1.2.

□

The goal of this section is to prove the following:

**Proposition 5.1.5.** *Suppose there exists* $\lambda_\pi \in F_\pi$ *such that* $\lambda_\pi q \simeq q$ *over* $F_\pi$*, and suppose there exists* $\lambda_\delta \in F_\delta$ *such that* $\lambda_\delta q \simeq q$ *over* $F_\delta$*. Then there exists* $\beta \in F$ *such that* $\beta(F_\pi^*)^2 = \lambda_\pi(F_\pi^*)^2$*,* $\beta(F_\delta^*)^2 = \lambda_\delta(F_\delta^*)^2$ *and* $\beta q \simeq q$ *over* $F$*.*

*Proof.* From the unit structure of $\widehat{R_{(\pi)}}$ and $\widehat{R_{(\delta)}}$ (cf. [19, Remark 7.1]), we have $\lambda_\pi = w'\pi^{r_1}\delta^{s_1}$ and $\lambda_\delta = w\pi^{r_2}\delta^{s_2}$, where $w, w' \in R^*$ and $r_1, r_2, s_1, s_2 \in \mathbb{Z}$. Since we are interested in the square classes, we assume that $r_1, r_2, s_1, s_2 \in \{0, 1\}$.

Suppose there exists $\beta' \in F$ such that $\beta'(F_\pi^*)^2 = (F_\pi^*)^2$, $\beta'(F_\delta^*)^2 = \lambda_\pi^{-1}\lambda_\delta(F_\delta^*)^2$ and $\beta' q \simeq q$ over $F$. Let $\beta = \beta'\lambda_\pi \in F$. Then

$$\beta(F_\pi^*)^2 = \beta'\lambda_\pi(F_\pi^*)^2 = \lambda_\pi(F_\pi^*)^2,$$

$$\beta(F_\delta^*)^2 = \beta'\lambda_\pi(F_\delta^*)^2 = \lambda_\pi^{-1}\lambda_\delta\lambda_\pi(F_\delta^*)^2 = \lambda_\delta(F_\delta^*)^2$$

and $\beta q \simeq \beta' \lambda_\pi q \simeq \lambda_\pi q \simeq q$ over $F$ by Remark 5.1.3. Therefore, we may assume that $\lambda_\pi = 1$. By multiplicativity, it is enough to consider the cases $\lambda_\delta = w$, $\lambda_\delta = \pi$ and $\lambda_\delta = \delta$.

**Case 1:** $\lambda_\delta = w$.

For $a \in R$, let $\overline{a} = a + \mathfrak{m} \in R/\mathfrak{m} = k$. Let $\overline{q_1} = \langle \overline{u_1}, \dots, \overline{u_{n_1}} \rangle$, $\overline{q_2} = \langle \overline{v_1}, \dots, \overline{v_{n_2}} \rangle$, $\overline{q_3} = \langle \overline{w_1}, \dots, \overline{w_{n_3}} \rangle$ and $\overline{q_4} = \langle \overline{\theta_1}, \dots, \overline{\theta_{n_4}} \rangle$. If $\lambda_\delta = w \in (R^*)^2$, then $\beta = 1 \in F$ has the required properties. So suppose that $\lambda_\delta = w \notin (R^*)^2$. Since $k = R/\mathfrak{m}$ is a finite field, $|k^*/(k^*)^2| = 2$. So $k^*/(k^*)^2 = \{\overline{1}, \overline{t}\}$. By lifting from $k$ to $R$, we may assume that $\lambda_\delta = tu^2$ for some $u \in R^*$.

For $1 \leq i \leq 4$, let $(q_i)_a$ denote the anisotropic part of $q_i$. If $q_i$ is hyperbolic over $F$, the group of similarity factors of $q_i$ is $G_F(q_i) = F^*$. Thus for $\lambda \in R^*$, by Proposition 5.1.4 (i),

$$\begin{aligned}\lambda \in G_F(q) &\iff \lambda \in G_F(q_1) \cap G_F(q_2) \cap G_F(q_3) \cap G_F(q_4)\\ &\iff \lambda \in G_F((q_1)_a) \cap G_F((q_2)_a) \cap G_F((q_3)_a) \cap G_F((q_4)_a)\\ &\iff \overline{\lambda} \in G_k((\overline{q_1})_a) \cap G_k((\overline{q_2})_a) \cap G_k((\overline{q_3})_a) \cap G_k((\overline{q_4})_a),\end{aligned}$$

where the third equivalence follows because $R$ is complete. But the only anisotropic forms over the finite field $k$ are $\langle \overline{1} \rangle, \langle \overline{t} \rangle$ and $\langle \overline{1}, -\overline{t} \rangle$ ( [16, p. 37]). Thus, for $1 \leq i \leq 4$, $(\overline{q_i})_a = \langle \overline{1} \rangle, \langle \overline{t} \rangle$ or $\langle \overline{1}, -\overline{t} \rangle$, or $\dim((\overline{q_i})_a) = 0$. Now $\lambda_\delta \in G_F(q)$ by Remark 5.1.3. Hence, by the above equivalences, $\overline{\lambda_\delta} = \overline{t} \in G_k((\overline{q_i})_a)$ for all $i$. But $\overline{t} \notin (k^*)^2$, so $\overline{t} \notin G_k(\langle \overline{1} \rangle)$ and $\overline{t} \notin G_k(\langle \overline{t} \rangle)$. Therefore, for each $i$, either $(\overline{q_i})_a = \langle \overline{1}, -\overline{t} \rangle$ or $\dim((\overline{q_i})_a) = 0$.

By Lemma 5.1.1, there exists $\beta \in F$ such that $\beta(F_\pi^*)^2 = (F_\pi^*)^2 = \lambda_\pi(F_\pi^*)^2$, $\beta(F_\delta^*)^2 = t(F_\delta^*)^2 = \lambda_\delta(F_\delta^*)^2$ and $\beta \in G_F(\langle 1, -t \rangle) = G_F((q_i)_a)$ for $1 \leq i \leq 4$. It follows from the above equivalences that $\beta \in G_F(q)$, so $\beta q \simeq q$ as required.

**Case 2:** $\lambda_\delta = \pi$.

Let $\beta = \delta^2 + \pi \in F$. Then $\beta = \delta^2(1 + \delta^{-2}\pi) \in F_\pi$. But $1 + \delta^{-2}\pi \in (F_\pi^*)^2$. Therefore $\beta(F_\pi^*)^2 = (F_\pi^*)^2 = \lambda_\pi(F_\pi^*)^2$. Similarly, $\beta = \pi(1 + \pi^{-1}\delta^2) \in F_\delta$. But $1 + \pi^{-1}\delta^2 \in (F_\delta^*)^2$. Therefore $\beta(F_\delta^*)^2 = \pi(F_\delta^*)^2 = \lambda_\delta(F_\delta^*)^2$.

It remains to show that $\beta q \simeq q$. To this end, first note that since $\lambda_\delta q = \pi q \simeq q$, we have $q_1 \simeq q_2$ and $q_3 \simeq q_4$. Then

$$\begin{aligned}q &\simeq q_1 \perp q_2\pi \perp q_3\delta \perp q_4\pi\delta\\ &\simeq q_1 \perp q_1\pi \perp q_3\delta \perp q_3\pi\delta\\ &\simeq \langle 1, \pi \rangle q_1 \perp \langle 1, \pi \rangle q_3\delta.\end{aligned}$$

Now $\beta = \delta^2 + \pi \in D_F(\langle 1, \pi \rangle) = G_F(\langle 1, \pi \rangle)$ (since $\langle 1, \pi \rangle$ is a Pfister form over $F$ ( [16, p. 319, Theorem 1.8])). Hence $\beta \in G_F(q)$, so $\beta q \simeq q$ as required.

**Case 3:** $\lambda_\delta = \delta$.

Let $\beta = \delta^2 + \delta\pi^2 \in F$. Then $\beta = \delta^2(1 + \delta^{-1}\pi^2) \in F_\pi$. But $1 + \delta^{-1}\pi^2 \in (F_\pi^*)^2$. Therefore $\beta(F_\pi^*)^2 = (F_\pi^*)^2 = \lambda_\pi(F_\pi^*)^2$. Similarly, $\beta = \delta\pi^2(1 + \pi^{-2}\delta) \in F_\delta$. But $1 + \pi^{-2}\delta \in (F_\delta^*)^2$. Therefore $\beta(F_\delta^*)^2 = \delta(F_\delta^*)^2 = \lambda_\delta(F_\delta^*)^2$.

It remains to show that $\beta q \simeq q$. To this end, first note that since $\lambda_\delta q = \delta q \simeq q$, we have $q_1 \simeq q_3$ and $q_2 \simeq q_4$. Then

$$\begin{aligned} q &\simeq q_1 \perp q_2\pi \perp q_3\delta \perp q_4\pi\delta \\ &\simeq q_1 \perp q_2\pi \perp q_1\delta \perp q_2\pi\delta \\ &\simeq \langle 1, \delta\rangle q_1 \perp \langle 1, \delta\rangle q_2\pi. \end{aligned}$$

Now $\beta = \delta^2 + \delta\pi^2 \in D_F(\langle 1, \delta\rangle) = G_F(\langle 1, \delta\rangle)$ (since $\langle 1, \delta\rangle$ is a Pfister form over $F$ ( [16, p. 319, Theorem 1.8])). Hence $\beta \in G_F(q)$, so $\beta q \simeq q$ as required. □

## 5.2 Semi-Global Fields - Quadratic Forms Case

Let $p \in \mathbb{N}$ be a prime with $p \neq 2$, and let $K$ be a $p$-adic field. Let $X$ be a geometrically integral curve over $K$, and let $F = K(X)$ be the function field of the curve $X$. Suppose $q$ and $q'$ are quadratic forms over $F$ with $\dim(q) = \dim(q')$ and $\mathrm{disc}(q) = \mathrm{disc}(q')$. Write $q = \langle a_1, a_2, \ldots, a_n\rangle$ and $q' = \langle b_1, b_2, \ldots, b_n\rangle$ with $a_i, b_i \in F^*$. By Abhyankar ( [1]) and Lipman ( [17]), there exists a regular integral model $\mathscr{X}$ with special fibre $X_0$ such that for all $i$, $\sup(a_i) \cup \sup(b_i) \cup X_0$ is a union of regular curves with normal crossings. Let $\mathscr{P} \subseteq X_0$ be a finite set of closed points of $\mathscr{X}$ containing all the nodal points of $\sup(a_i) \cup \sup(b_i) \cup X_0$ and at least one point on each component. Let $\mathscr{U}$ be the set of irreducible components of $X_0 \setminus \mathscr{P}$. Then $\mathscr{U} = \{U_1, U_2, \ldots, U_l\}$ is a finite set.

**Notation.** For $P \in \mathscr{P}$, let $\mathcal{O}_{\mathscr{X},P}$ be the local ring at $P$. So $\mathcal{O}_{\mathscr{X},P}$ is a two dimensional regular local ring. Let $\mathfrak{m}_P$ be the maximal ideal of $\mathcal{O}_{\mathscr{X},P}$, and let $\widehat{\mathcal{O}_{\mathscr{X},P}}$ denote the completion of $\mathcal{O}_{\mathscr{X},P}$ at the maximal ideal $\mathfrak{m}_P$. Define $F_P \coloneqq \mathrm{ff}(\widehat{\mathcal{O}_{\mathscr{X},P}})$.

**Notation.** For $U \in \mathscr{U}$, let $R_U$ be the set of rational functions which are regular on $U$:

$$R_U \coloneqq \{f \in F \mid f \text{ is regular on } U\}.$$

Let $t \in K$ be a parameter and let $\widehat{R_U}$ be the $(t)$-adic completion of $R_U$. Define $F_U \coloneqq \mathrm{ff}\left(\widehat{R_U}\right)$.

**Notation.** For $P \in \mathscr{P}$, each height one prime ideal $\rho$ of $\widehat{\mathcal{O}_{\mathscr{X},P}}$ that contains $t$ determines a *branch* of $X_0$ at $P$ (i.e. an irreducible component of the pullback of $X_0$ to Spec $\widehat{\mathcal{O}_{\mathscr{X},P}}$). We let $\hat{R}_\rho$ denote the complete local ring of $\widehat{\mathcal{O}_{\mathscr{X},P}}$ at $\rho$. Define $F_\rho := \mathrm{ff}\left(\hat{R}_\rho\right)$. Since $t \in \rho$, the contraction of $\rho \subseteq \widehat{\mathcal{O}_{\mathscr{X},P}}$ to the local ring $\mathcal{O}_{\mathscr{X},P}$ defines an irreducible component of Spec $\mathcal{O}_{X_0,P}$ and hence an irreducible component of $X_0$ containing $P$. This in turn is the closure of a unique connected component $U$ of $X_0 \setminus \mathscr{P}$, and we say that $\rho$ *lies on* $U$. We call $F_{U,P} := F_\rho$ a *branch field.*

We begin by proving the following local-global principle for similarities in the patching set up:

**Proposition 5.2.1.** *Suppose for all* $U \in \mathscr{U}$ *there exists* $\lambda_U \in F_U^*$ *such that* $q \simeq \lambda_U q'$ *over* $F_U$, *and suppose for all* $P \in \mathscr{P}$ *there exists* $\lambda_P \in F_P^*$ *such that* $q \simeq \lambda_P q'$ *over* $F_P$. *Then there exists* $\lambda \in F$ *such that* $q \simeq \lambda q'$ *over* $F$.

*Proof.* By assumption, for all $U \in \mathscr{U}$ there exists $\lambda_U \in F_U^*$ such that $q \simeq \lambda_U q'$ over $F_U$, and for all $P \in \mathscr{P}$ there exists $\lambda_P \in F_P^*$ such that $q \simeq \lambda_P q'$ over $F_P$. So for all $U \in \mathscr{U}$ we have an isomorphism $\phi_U \colon q \simeq \lambda_U q'$ over $F_U$, and for all $P \in \mathscr{P}$ we have an isomorphism $\phi_P \colon q \simeq \lambda_P q'$ over $F_P$. Then for all $P \in \mathscr{P}$, $U \in \mathscr{U}$ the map $\phi_P^{-1}\phi_U \colon q \simeq \lambda_P^{-1}\lambda_U q$ is a similitude of $q$ over the branch field $F_{U,P}$ with similarity factor $\lambda_P^{-1}\lambda_U \in F_{U,P}$. For each $P \in \mathscr{P}$, $U \in \mathscr{U}$ define $\lambda_{U,P} := \lambda_P^{-1}\lambda_U \in F_{U,P}$.

Let $P \in \mathscr{X}$ be a closed point. Let $R_P := \widehat{\mathcal{O}_{\mathscr{X},P}}$. Then $R_P$ is a complete two dimensional regular local ring with $F_P = \mathrm{ff}(R_P)$. Then, by the choice of $\mathscr{X}$, the maximal ideal $\mathfrak{m}_P = (\pi_P, \delta_P)$ for some $\pi_P, \delta_P$ such that $a_i = u_{iP}\pi_P^{r_{iP}}\delta_P^{s_{iP}}$ and $b_i = w_{iP}\pi_P^{r'_{iP}}\delta_P^{s'_{iP}}$ for some units $u_{iP}, w_{iP} \in R_P^*$ and $st_{iP}, s_{iP}, r'_{iP}, s'_{iP} \in \mathbb{Z}$. In particular we have

$$q \simeq q_1 \perp q_2\pi \perp q_3\delta \perp q_4\pi\delta,$$

$$q' \simeq q'_1 \perp q'_2\pi \perp q'_3\delta \perp q'_4\pi\delta,$$

where $q_1 = \langle u_1, \dots, u_{n_1}\rangle$, $q_2 = \langle v_1, \dots, v_{n_2}\rangle$, $q_3 = \langle w_1, \dots, w_{n_3}\rangle$, $q_4 = \langle \theta_1, \dots, \theta_{n_4}\rangle$, $q'_1 = \langle u'_1, \dots, u'_{n_1}\rangle$, $q'_2 = \langle v'_1, \dots, v'_{n_2}\rangle$, $q'_3 = \langle w'_1, \dots, w'_{n_3}\rangle$, $q'_4 = \langle \theta'_1, \dots, \theta'_{n_4}\rangle$ with $u_i, v_i, w_i, \theta_i, u'_i, v'_i, w'_i, \theta'_i \in R^*$.

Let $\widehat{(R_P)_{(\pi_P)}}$ denote the completion of the localization of $R_P$ at the prime ideal $(\pi_P)$, and let $\widehat{(R_P)_{(\delta_P)}}$ denote the completion of the localization of $R_P$ at the prime ideal $(\delta_P)$. Define $(F_P)_{\pi_P} := \mathrm{ff}\left(\widehat{(R_P)_{(\pi_P)}}\right)$ and $(F_P)_{\delta_P} := \mathrm{ff}\left(\widehat{(R_P)_{(\delta_P)}}\right)$.

**Claim.** For all $P \in \mathscr{P}$, $U \in \mathscr{U}$ we may write $\lambda_{U,P} = \beta_P z^2$ where $\beta_P \in F_P$ is such that $q \simeq \beta_P q'$ over $F_P$ and $z \in F_{U,P}^*$.

*Proof of Claim.* Fix $P \in \mathscr{P}$. There are two cases:

**Case 1:** There is only one $U \in \mathscr{U}$ with $P$ in the closure of $U$. Then either $F_{U,P} = (F_P)_{\pi_P}$ or $F_{U,P} = (F_P)_{\delta_P}$. From the unit structure of $\widehat{(R_P)_{(\pi_P)}}$ and $\widehat{(R_P)_{(\delta_P)}}$ (cf. [19, Remark 7.1]), we have $\lambda_{U,P} = w\pi_P^r\delta_P^s z^2$ where $w \in R_P^*$, $r, s \in \mathbb{Z}$ and $z \in F_{U,P}^*$. Let $\beta_P = w\pi_P^r\delta_P^s \in F_P$. Then $\lambda_{U,P} = \beta_P z^2$. Since $\lambda_{U,P}$ is a similarity for $q$ over $F_{U,P}$, we have that $\beta_P$ is a similarity for $q$ over $F_{U,P}$ also. Thus, by (5.1.3), we have that $\beta_P$ is a similarity for $q$ over $F_P$ and the claim is proved in this case.

**Case 2:** There exist $U_1, U_2 \in \mathscr{U}$ with $U_1 \neq U_2$ such that $P$ is in the closure of $U_1$ and $P$ is in the closure of $U_2$. Then by reordering the $U_i$ if necessary, we have $F_{U_1,P} = (F_P)_{\pi_P}$ and $F_{U_2,P} = (F_P)_{\delta_P}$. Then $\lambda_{U_1,P} \in (F_P)_{\pi_P}$ is such that $(\lambda_{U_1,P})q \simeq q$ over $(F_P)_{\pi_P}$ and $\lambda_{U_2,P} \in (F_P)_{\delta_P}$ is such that $(\lambda_{U_2,P})q \simeq q$ over $(F_P)_{\delta_P}$. Thus, by Proposition 5.1.5, there exists $\beta_P \in F_P$ such that $\beta_P((F_P)_{\pi_P}^*)^2 = \lambda_{U_1,P}((F_P)_{\pi_P}^*)^2, \beta_P((F_P)_{\delta_P}^*)^2 = \lambda_{U_2,P}((F_P)_{\delta_P}^*)^2$ and $\beta_P q \simeq q$ over $F_P$. Hence $\lambda_{U_1,P} = \beta_P z_1^2$ for some $z_1 \in F_{U_1,P}^*$ and $\lambda_{U_2,P} = \beta_P z_2^2$ for some $z_2 \in F_{U_2,P}^*$. This completes the proof of the claim.

By the claim, for all $P \in \mathscr{P}$ we have an isomorphism $\alpha_P \colon q \simeq \beta_P q$ over $F_P$. By [9, Corollary 3.4], for all $P \in \mathscr{P}$, $U \in \mathscr{U}$ we can factorize $z \in F_{U,P}^*$ as $z = z_P z_U$ for some $z_P \in F_P^*$ and $z_U \in F_U^*$. Then for all $P \in \mathscr{P}$, $U \in \mathscr{U}$ we have $\lambda_{U,P} = \beta_P z_P^2 z_U^2$. Then for all $U \in \mathscr{U}$ we have an isomorphism $\phi'_U := \phi_U \circ m_{z_U^{-1}} \colon q \simeq \lambda_U z_U^{-2} q'$ over $F_U$, and for all $P \in \mathscr{P}$ we have an isomorphism $\phi'_P := \phi_P \circ m_{z_P} \circ \alpha_P \colon q \simeq \lambda_P z_P^2 \beta_P q'$ over $F_P$. Then for all $P \in \mathscr{P}$, $U \in \mathscr{U}$ the map $(\phi'_P)^{-1}\phi'_U \colon q \simeq (\lambda_P z_P^2 \beta_P)^{-1}\lambda_U z_U^{-2} q$ is a similitude of $q$ over the branch field $F_{U,P}$ with similarity factor

$$(\lambda_P z_P^2 \beta_P)^{-1}\lambda_U z_U^{-2} = \lambda_P^{-1}\lambda_U z_U^{-2} z_P^{-2}\beta_P^{-1} = \lambda_{U,P}\lambda_{U,P}^{-1} = 1 \in F_{U,P}. \quad (*)$$

Therefore for all $P \in \mathscr{P}$, $U \in \mathscr{U}$ the map $(\phi'_P)^{-1}\phi'_U$ is an isometry of $q$ over $F_{U,P}$. Now by rearranging $(*)$, for all $P \in \mathscr{P}$, $U \in \mathscr{U}$ we have $\lambda_U z_U^{-2} = \lambda_P z_P^2 \beta_P \in F_U \cap F_P = F$. For each $P \in \mathscr{P}$, $U \in \mathscr{U}$ define $\lambda := \lambda_U z_U^{-2} = \lambda_P z_P^2 \beta_P \in F$. Then for all $U \in \mathscr{U}$ the map $\phi'_U \colon q \simeq \lambda q'$ over $F_U$ is an isomorphism, and for all $P \in \mathscr{P}$ the map $\phi'_P \colon q \simeq \lambda q'$ over $F_P$ is an isomorphism.

**Case 1:** $\dim(q)$ is even. Then $\mathrm{disc}(q) = \mathrm{disc}(q') = \mathrm{disc}(\lambda q')$, so $[\lambda q'] \in H^1(F, \mathrm{SO}(q))$. Now $\mathrm{SO}(q)$ is a rational, connected group and therefore the map

$$\Psi\colon H^1(F,\mathrm{SO}(q)) \to \prod_{U\in\mathcal{U}} H^1(F_U,\mathrm{SO}(q)) \prod_{P\in\mathcal{P}} H^1(F_P,\mathrm{SO}(q))$$

has trivial kernel ( [8]). Since for all $P \in \mathscr{P}$, $U \in \mathscr{U}$ the maps $\phi'_U\colon q \simeq \lambda q'$ over $F_U$ and $\phi'_P\colon q \simeq \lambda q'$ over $F_P$ are isomorphisms, we have $\Psi([\lambda q']) = 0$, and thus $[\lambda q'] = 0 = [q]$. Therefore $q \simeq \lambda q'$ over $F$ as required.

**Case 2:** $\dim(q)$ is odd. For all $P \in \mathscr{P}$, $U \in \mathscr{U}$ we have $q \simeq \lambda q'$ over $F_U$ and $q \simeq \lambda q'$ over $F_P$. Since $\dim(q) = \dim(q')$ is odd and $\mathrm{disc}(q) = \mathrm{disc}(q')$, it follows that for all $P \in \mathscr{P}$, $U \in \mathscr{U}$ we have $q \simeq q'$ over $F_U$ and $q \simeq q'$ over $F_P$. Hence $q \simeq q'$ over $F$ ( [8]). □

Let $\Omega_F$ be the set of all divisorial discrete valuations of $F$. For $v \in \Omega_F$, let $\hat{F}_v$ denote the completion of $F$ at $v$.

**Theorem 5.2.2.** *Suppose for all divisorial discrete valuations $v \in \Omega_F$ there exists $\lambda_v \in \hat{F}_v$ such that $q \simeq \lambda_v q'$ over $\hat{F}_v$. Then there exists $\lambda \in F$ such that $q \simeq \lambda q'$ over $F$.*

*Proof.* Choose a regular integral model $\mathscr{X}$ with special fibre $X_0$ such that for all $j$, $\sup(a_j) \cup \sup(b_j) \cup X_0$ is a union of regular curves with normal crossings. Write $X_0 = \bigcup_{i=1}^{d} X_i$ where the $X_i$ are irreducible components. For $1 \le i \le d$, let $v_i$ be the discrete valuation on $F$ corresponding to $X_i$. So for $1 \le i \le d$, we have $q \simeq \lambda_{v_i} q'$ over $\hat{F}_{v_i}$.

Since $F_{v_i}$ is the completion of $F$ at the discrete valuation $v_i$, we have $\lambda_{v_i} = \lambda'_{v_i} x_i^2$ for some $\lambda'_{v_i} \in F^*$. Hence replacing $\lambda_{v_i}$ by $\lambda'_{v_i}$, we assume that $\lambda_{v_i} \in F^*$.

Since $q \simeq \lambda_{v_i} q'$ over $\hat{F}_{v_i}$, by [9, Proposition 5.8], there exists a nonempty open set $U_i \subsetneq X_i$ such that $q \simeq \lambda'_{v_i} q'$ over $F_{U_i}$. Let $\mathscr{U} = \{U_1, \ldots, U_d\}$ and let $\mathscr{P} = X_0 \setminus \bigcup_{i=1}^{d} U_i$. Then for each $P \in \mathscr{P}$, by (5.1.5), there exists $\lambda_P \in F_P$ such that $q \simeq \lambda_P q'$ over $F_P$. Then applying Proposition 5.2.1 to the patch $\{\mathscr{U}, \mathscr{P}\}$, it follows that there exists $\lambda \in F$ such that $q \simeq \lambda q'$ over $F$ as required. □

Let $L/F$ be a quadratic field extension, and let $\tau$ be the nontrivial automorphism of $L/F$. Let $h_1$ and $h_2$ be hermitian forms over $(L, \tau)$.

**Corollary 5.2.3.** *Suppose for all divisorial discrete valuations $v \in \Omega_F$ there exists $\lambda_v \in \hat{F}_v$ such that $h_1 \simeq \lambda_v h_2$. Then there exists $\lambda \in F$ such that $h_1 \simeq \lambda h_2$ over $L$.*

*Proof.* Let $q_h$ denote the trace form of $h$ (cf. [21, p. 348]). By assumption, for all divisorial discrete valuations $v \in \Omega_F$ there exists $\lambda_v \in \hat{F}_v$ such that $h_1 \simeq \lambda_v h_2$. Then, by Jacobson, for all $v \in \Omega_F$ we have $q_{h_1} \simeq \lambda_v q_{h_2}$ over $\hat{F}_v$ ( [21, p. 348, Theorem 1.1]). Then, by Theorem 5.2.2, there exists $\lambda \in F$ such that $q_{h_1} \simeq \lambda q_{h_2}$ over $F$. Thus, by Jacobson, we have $h_1 \simeq \lambda h_2$ over $L$ as required ( [21, p. 348, Theorem 1.1]). □

## 5.3 Quaternion Division Algebras Over Two Dimensional Complete Fields

Let $R$ be a complete two dimensional regular local ring, let $F = \mathrm{ff}(R)$, and suppose $2 \in R^*$. Let $\mathfrak{m} = (\pi, \delta)$ be the maximal ideal of $R$. Suppose $k = R/\mathfrak{m}$ is a finite field with $\mathrm{char}(k) \neq 2$. Let $D$ be a quaternion division algebra over $F$ which is unramified on $R$ except possibly at $(\pi)$ and $(\delta)$. Let $\tau$ be the canonical involution on $D$. Let $h$ be an hermitian form over $(D, \tau)$. Then $h = \langle a_1, \ldots, a_n \rangle$ where $a_i \in F^*$. Suppose $a_i = u_i \pi^{r_i} \delta^{s_i}$ with $u_i \in R^*$, $r_i, s_i \in \mathbb{Z}$ for $1 \leq i \leq n$. Let $\widehat{R_{(\pi)}}$ denote the completion of the localization of $R$ at the prime ideal $(\pi)$, and let $\widehat{R_{(\delta)}}$ denote the completion of the localization of $R$ at the prime ideal $(\delta)$. Define $F_\pi \coloneqq \mathrm{ff}\left(\widehat{R_{(\pi)}}\right)$ and $F_\delta \coloneqq \mathrm{ff}\left(\widehat{R_{(\delta)}}\right)$. Then $F_\pi$ and $F_\delta$ are complete discretely valued fields.

**Proposition 5.3.1.** *Suppose there exists $\lambda_\pi \in F_\pi$ such that $\lambda_\pi h \simeq h$ over $F_\pi$, and suppose there exists $\lambda_\delta \in F_\delta$ such that $\lambda_\delta h \simeq h$ over $F_\delta$. Then there exists $\beta \in F$ such that $\beta (F_\pi^*)^2 = \lambda_\pi (F_\pi^*)^2$, $\beta (F_\delta^*)^2 = \lambda_\delta (F_\delta^*)^2$ and $\beta h \simeq h$ over $F$.*

*Proof.* First note that since $D$ is a division algebra over $F$, it follows that $D \otimes_F F_\pi$ is a division algebra over $F_\pi$ and $D \otimes_F F_\delta$ is a division algebra over $F_\delta$ ( [18, Proposition 5.8]). Now since $D$ is unramified on $R$ except possibly at $(\pi)$ and $(\delta)$, we have that $D = (u, v\pi), (u, v\delta), (u\pi, v\delta)$ or $(u, v\pi\delta)$ where $u, v \in R^*$ [22, Lemma 3.6]. Let $N = \langle 1, -a, -b, ab \rangle$ be the norm form of $D$, so that $a \in \{u, u\pi\}$ and $b \in \{v\pi, v\delta, v\pi\delta\}$. Let $q_h = \langle a_1, \ldots, a_n \rangle \otimes N$ denote the trace form of $h$ (cf. [21, p. 352]). Then $q_h = \langle b_1, \ldots, b_{4n} \rangle$ where $b_i \in F^*$, $b_i = v_i \pi^{x_i} \delta^{y_i}$ with $v_i \in R^*$, $x_i, y_i \in \{0, 1\}$ for $1 \leq i \leq 4n$.

By assumption, there exists $\lambda_\pi \in F_\pi$ such that $\lambda_\pi h \simeq h$ over $F_\pi$. Then, by Jacobson, we have $\lambda_\pi (q_h)_{F_\pi} \simeq (q_h)_{F_\pi}$ over $F_\pi$ ( [21, p. 352, Theorem 1.7]). By assumption, there exists $\lambda_\delta \in F_\delta$ such that $\lambda_\delta h \simeq h$ over $F_\delta$. Then, by Jacobson, we have $\lambda_\delta (q_h)_{F_\delta} \simeq (q_h)_{F_\delta}$ over $F_\delta$ ( [21, p. 352, Theorem 1.7]). Therefore, by Proposition 5.1.5, there exists $\beta \in F$ such that $\beta (F_\pi^*)^2 = \lambda_\pi (F_\pi^*)^2$, $\beta (F_\delta^*)^2 = \lambda_\delta (F_\delta^*)^2$ and $\beta q_h \simeq q_h$ over $F$. Then, by Jacobson, we have $\beta h \simeq h$ over $F$ as required ( [21, p. 352, Theorem

1.7]). □

## 5.4 Semi-Global Fields - Symplectic Involution Case

Let $p \in \mathbb{N}$ be a prime with $p \neq 2$, and let $K$ be a $p$-adic field. Let $X$ be a geometrically integral curve over $K$, and let $F = K(X)$ be the function field of the curve $X$. Let $A$ be a central simple algebra over $F$, and let $\sigma$ be a symplectic involution on $A$. Let $h_1$ and $h_2$ be two hermitian forms over $(A, \sigma)$. Choose a regular integral model $\mathscr{X}$ with special fibre $X_0$ with the following properties:

(1) $\operatorname{ram}_{\mathscr{X}}(A) \cup X_0$ is a union of regular curves with normal crossings.

(2) There exists a finite set of closed points $\mathscr{P} \subseteq X_0$ containing all the nodal points of $\operatorname{ram}_{\mathscr{X}}(A) \cup X_0$ and at least one point on each component, such that for each $P \in \mathscr{P}$, we have $A \otimes_F F_P \cong M_n(D_P)$, where $D_P$ is a central division algebra over $F_P$ and for $1 \leq i \leq 2$, $(h_i)_{F_P}$ corresponds under Morita equivalence to $(\tilde{h_i})_P$ over $D_P$, where $(\tilde{h_i})_P$ is an hermitian form for the canonical involution such that $(\tilde{h_i})_P = \langle a_{i_1}, \ldots, a_{i_n} \rangle$ where $a_{i_j} \in F_P^*$, $a_{i_j} = u_{P_{i_j}} \pi_P^{r_{i_j}} \delta_P^{s_{i_j}}$ with $u_{P_{i_j}} \in \mathcal{O}^*_{\mathscr{X},P}$, $r_{i_j}, s_{i_j} \in \mathbb{Z}$ for $1 \leq i \leq 2$, $1 \leq j \leq n$ where $\mathfrak{m}_P = (\pi_P, \delta_P)$ is the maximal ideal of $\mathcal{O}_{\mathscr{X},P}$, the local ring at $P$.

Let $\mathscr{U}$ be the set of irreducible components of $X_0 \setminus \mathscr{P}$. Then $\mathscr{U} = \{U_1, U_2, \ldots, U_l\}$ is a finite set.

**Proposition 5.4.1.** *Suppose for all $U \in \mathscr{U}$ there exists $\lambda_U \in F_U^*$ such that $h_1 \simeq \lambda_U h_2$ over $F_U$, and suppose for all $P \in \mathscr{P}$ there exists $\lambda_P \in F_P^*$ such that $h_1 \simeq \lambda_P h_2$ over $F_P$. Then there exists $\lambda \in F$ such that $h_1 \simeq \lambda h_2$ over $F$.*

*Proof.* By assumption, for all $U \in \mathscr{U}$ there exists $\lambda_U \in F_U^*$ such that $h_1 \simeq \lambda_U h_2$ over $F_U$, and for all $P \in \mathscr{P}$ there exists $\lambda_P \in F_P^*$ such that $h_1 \simeq \lambda_P h_2$ over $F_P$. So for all $U \in \mathscr{U}$ we have an isomorphism $\phi_U \colon h_1 \simeq \lambda_U h_2$ over $F_U$, and for all $P \in \mathscr{P}$ we have an isomorphism $\phi_P \colon h_1 \simeq \lambda_P h_2$ over $F_P$. Then for all $P \in \mathscr{P}$, $U \in \mathscr{U}$ the map $\phi_P^{-1}\phi_U \colon h_1 \simeq \lambda_P^{-1}\lambda_U h_1$ is a similitude of $h_1$ over the branch field $F_{U,P}$ with similarity factor $\lambda_P^{-1}\lambda_U \in F_{U,P}$. For each $P \in \mathscr{P}$, $U \in \mathscr{U}$ define $\lambda_{U,P} \coloneqq \lambda_P^{-1}\lambda_U \in F_{U,P}$.

**Claim.** For all $P \in \mathscr{P}$, $U \in \mathscr{U}$ we have $\lambda_{U,P} = \beta_P z^2$ for some $\beta_P \in F_P$ is such that $h_1 \simeq \beta_P h_1$ over $F_P$ and $z \in F_{U,P}^*$.

*Proof of Claim.* Fix $P \in \mathscr{P}$. Let $R_P := \widehat{\mathcal{O}_{\mathscr{X},P}}$. Then $R_P$ is a complete two dimensional regular local ring with $F_P = \mathrm{ff}(R_P)$. By the choice of $\mathscr{X}$, the maximal ideal $\mathfrak{m}_P$ at $P$ is generated by $(\pi_P, \delta_P)$ such that $A \otimes F_P \simeq M_n(D_P)$ for some division algebra $D_P$ over $F_P$ of index at most 2 which is unramified at $P$ except possibly at $(\pi)$ and $(\delta)$, and under Moirta equivalence $h_i$ corresponds to hermitian forms $(\tilde{h}_i)_P$ over $D_P$ such that $(\tilde{h}_i)_P = \langle a_{i_1}, \ldots, a_{i_n} \rangle$ where $a_{i_j} \in F_P^*$, $a_{i_j} = u_{P_{i_j}} \pi_P^{r_{i_j}} \delta_P^{s_{i_j}}$ with $u_{P_{i_j}} \in \mathcal{O}^*_{\mathscr{X},P}$, $r_{i_j}, s_{i_j} \in \mathbb{Z}$ for $1 \leq i \leq 2$, $1 \leq j \leq n$.

Let $\widehat{(R_P)_{(\pi_P)}}$ denote the completion of the localization of $R_P$ at the prime ideal $(\pi_P)$, and let $\widehat{(R_P)_{(\delta_P)}}$ denote the completion of the localization of $R_P$ at the prime ideal $(\delta_P)$. Define $(F_P)_{\pi_P} := \mathrm{ff}\left(\widehat{(R_P)_{(\pi_P)}}\right)$ and $(F_P)_{\delta_P} := \mathrm{ff}\left(\widehat{(R_P)_{(\delta_P)}}\right)$. There are two cases:

**Case 1:** There is only one $U \in \mathscr{U}$ with $P$ in the closure of $U$. Then either $F_{U,P} = (F_P)_{\pi_P}$ or $F_{U,P} = (F_P)_{\delta_P}$. From the unit structure of $\widehat{(R_P)_{(\pi_P)}}$ and $\widehat{(R_P)_{(\delta_P)}}$ (cf. [19, Remark 7.1]), we have $\lambda_{U,P} = w\pi_P^r\delta_P^s z^2$ where $w \in R_P^*$, $r, s \in \mathbb{Z}$ and $z \in F_{U,P}^*$. Let $\beta_P = w\pi_P^r\delta_P^s \in F_P$. Then $\lambda_{U,P} = \beta_P z^2$. Since $\lambda_{U,P}$ is a similarity for $h_1$ over $F_{U,P}$, we have that $\beta_P$ is a similarity for $h_1$ over $F_{U,P}$ also. Let $q_{h_1}$ denote the trace form of $h_1$. Then $\beta_P$ is a similarity for $q_{h_1}$ over $F_{U,P}$. Thus, (5.1.3), we have that $\beta_P$ is a similarity for $q_{h_1}$ over $F_P$, and hence $\beta_P$ is a similarity for $h_1$ over $F_P$, which proves the claim in this case.

**Case 2:** There exist $U_1, U_2 \in \mathscr{U}$ with $U_1 \neq U_2$ such that $P$ is in the closure of $U_1$ and $P$ is in the closure of $U_2$. Then by reordering the $U_i$ if necessary, we have $F_{U_1,P} = (F_P)_{\pi_P}$ and $F_{U_2,P} = (F_P)_{\delta_P}$. Then $\lambda_{U_1,P} \in (F_P)_{\pi_P}$ is such that $(\lambda_{U_1,P})h_1 \simeq h_1$ over $(F_P)_{\pi_P}$ and $\lambda_{U_2,P} \in (F_P)_{\delta_P}$ is such that $(\lambda_{U_2,P})h_1 \simeq h_1$ over $(F_P)_{\delta_P}$. Thus, by Proposition 5.3.1, there exists $\beta_P \in F_P$ such that $\beta_P((F_P)^*_{\pi_P})^2 = \lambda_{U_1,P}((F_P)^*_{\pi_P})^2$, $\beta_P((F_P)^*_{\delta_P})^2 = \lambda_{U_2,P}((F_P)^*_{\delta_P})^2$ and $\beta_P h_1 \simeq h_1$ over $F_P$. Hence $\lambda_{U_1,P} = \beta_P z_1^2$ for some $z_1 \in F^*_{U_1,P}$ and $\lambda_{U_2,P} = \beta_P z_2^2$ for some $z_2 \in F^*_{U_2,P}$. This completes the proof of the claim.

By the claim, for all $P \in \mathscr{P}$ we have an isomorphism $\alpha_P \colon h_1 \simeq \beta_P h_1$ over $F_P$. By [9, Corollary 3.4], for all $P \in \mathscr{P}$, $U \in \mathscr{U}$ we can factorize $z \in F^*_{U,P}$ as $z = z_P z_U$ for some $z_P \in F_P^*$ and $z_U \in F_U^*$. Then for all $P \in \mathscr{P}$, $U \in \mathscr{U}$ we have $\lambda_{U,P} = \beta_P z_P^2 z_U^2$. Then for all $U \in \mathscr{U}$ we have an isomorphism $\phi'_U := \phi_U \circ m_{z_U^{-1}} \colon h_1 \simeq \lambda_U z_U^{-2} h_2$ over $F_U$, and for all $P \in \mathscr{P}$ we have an isomorphism $\phi'_P := \phi_P \circ m_{z_P} \circ \alpha_P \colon h_1 \simeq \lambda_P z_P^2 \beta_P h_2$ over $F_P$. Then for all $P \in \mathscr{P}$, $U \in \mathscr{U}$ the map $(\phi'_P)^{-1}\phi'_U \colon h_1 \simeq (\lambda_P z_P^2 \beta_P)^{-1} \lambda_U z_U^{-2} h_1$

is a similitude of $h_1$ over the branch field $F_{U,P}$ with similarity factor

$$(\lambda_P z_P^2 \beta_P)^{-1} \lambda_U z_U^{-2} = \lambda_P^{-1} \lambda_U z_U^{-2} z_P^{-2} \beta_P^{-1} = \lambda_{U,P} \lambda_{U,P}^{-1} = 1 \in F_{U,P}. \quad (*)$$

Therefore for all $P \in \mathscr{P}$, $U \in \mathscr{U}$ the map $(\phi'_P)^{-1}\phi'_U$ is an isometry of $h_1$ over $F_{U,P}$. Now by rearranging $(*)$, for all $P \in \mathscr{P}$, $U \in \mathscr{U}$ we have $\lambda_U z_U^{-2} = \lambda_P z_P^2 \beta_P \in F_U \cap F_P = F$. For each $P \in \mathscr{P}$, $U \in \mathscr{U}$ define $\lambda := \lambda_U z_U^{-2} = \lambda_P z_P^2 \beta_P \in F$. Then for all $U \in \mathscr{U}$ the map $\phi'_U \colon h_1 \simeq \lambda h_2$ over $F_U$ is an isomorphism, and for all $P \in \mathscr{P}$ the map $\phi'_P \colon h_1 \simeq \lambda h_2$ over $F_P$ is an isomorphism. Thus $h_1 \simeq \lambda h_2$ over $F$ as required. □

Let $\Omega_F$ be the set of all divisorial discrete valuations of $F$. For $v \in \Omega_F$, let $\hat{F}_v$ denote the completion of $F$ at $v$.

**Theorem 5.4.2.** *Suppose for all divisorial discrete valuations $v \in \Omega_F$ there exists $\lambda_v \in \hat{F}_v$ such that $h_1 \simeq \lambda_v h_2$ over $\hat{F}_v$. Then there exists $\lambda \in F$ such that $h_1 \simeq \lambda h_2$ over $F$.*

*Proof.* Choose a regular integral model $\mathscr{X}$ with special fibre $X_0$ such that $\mathrm{ram}_{\mathscr{X}}(A) \cup X_0$ is a union of regular curves with normal crossings. Write $X_0 = \bigcup_{i=1}^{d} X_i$ where the $X_i$ are irreducible components. For $1 \le i \le d$, let $v_i$ be the discrete valuation on $F$ corresponding to $X_i$. So for $1 \le i \le d$, we have $h_1 \simeq \lambda_{v_i} h_2$ over $\hat{F}_{v_i}$.

Since for any $\lambda \in F_v$, $\lambda = \lambda' a^2$ for some $a \in F_v$, without loss of generality we assume that $\lambda_{v_i} \in F^*$ for all $i$. Hence by [9, Proposition 5.8], for each $i$, there exists a proper nonempty set $U_i$ of $X_i$ such that $h_1 \simeq \lambda_{v_i} h_2$ over $\hat{F}_{U_i}$.

Let $\mathscr{P} = X_0 \setminus \cup U_i$. Let $P \in \mathscr{P}$. By the choice of $\mathscr{X}$, $A$ is unramified at $P$ which is unramified at $P$ except possibly at $(\pi_P)$ and hence $A \otimes F_P \simeq M_n(D_P)$ for some division algebra $D_P$ over $F_P$ which is unramified at $P$ except possibly at $(\pi_P)$ and $(\delta_P)$. Since $A$ is of period at most 2, by [19, Proposition 5.7], $\mathrm{ind}(D_P)$ is at most 2. Let $(\tilde{h}_i) =< a_{i_1}, \cdots, a_{i_n} >$ for some $a_{i_j} \in F_P^*$. By blowing up if necessary $\mathscr{X}$ at the closed points in $\mathscr{P}$, we may assume that $a_{i_j} = u_{P_{i_j}} \pi_P^{r_{i_j}} \delta_P^{s_{i_j}}$ with $u_{P_{i_j}} \in \mathcal{O}^*_{\mathscr{X},P}$, $r_{i_j}, s_{i_j} \in \mathbb{Z}$ for $1 \le i \le 2$, $1 \le j \le n$. Then for each $P \in \mathscr{P}$, by (5.3.1), there exists $\lambda_P \in F_P$ such that $h_1 \simeq \lambda_P h_2$ over $F_P$. Then applying Proposition 5.4.1 to the patch $\{\mathscr{U}, \mathscr{P}\}$, it follows that there exists $\lambda \in F$ such that $h_1 \simeq \lambda h_2$ over $F$ as required. □

## 5.5 The Main Theorems

Let $p \in \mathbb{N}$ be a prime with $p \neq 2$, and let $K$ be a $p$-adic field. Let $X$ be a geometrically integral curve over $K$, and let $F = K(X)$ be the function field of the curve $X$.

**Theorem 5.5.1.** *Let $G$ be an absolutely simple, adjoint linear algebraic group over $F$ of classical type ${}^2A_n$, so that $G = \mathrm{PGU}(A, \sigma)$ for some central simple algebra $A$ of degree $n+1$ whose center $Z(A)$ is a quadratic extension of $F$, with involution $\sigma$ of the second kind such that $\sigma(x) = x$ for all $x \in F$. Then the Hasse principle holds for principal homogeneous spaces under $G$ in the case when $(A, \sigma) = (L, \tau)$ where $L/F$ is a quadratic field extension and $\tau$ is the nontrivial automorphism of $L/F$.*

*Proof.* First note that in the case when $(A, \sigma) = (L, \tau)$ where $L/F$ is a quadratic field extension and $\tau$ is the nontrivial automorphism of $L/F$, the tuple $(A, \sigma)$ reduces to an hermitian form $h_1$ over $(L, \tau)$ and $G = \mathrm{PGU}(h_1)$. Now $H^1(F, \mathrm{PGU}(h_1))$ classifies similarity classes of nonsingular hermitian forms over $(L, \tau)$. The trivial element in this set is the similarity class of $h_1$. Then the condition that $[h_2] = 1 \in H^1(F, \mathrm{PGU}(h_1))$ is equivalent to the condition that $h_1 \simeq \lambda h_2$ for some $\lambda \in F$. So by Corollary 5.2.3, the Hasse principle holds for principal homogeneous spaces under $\mathrm{PGU}(h_1)$. □

**Theorem 5.5.2.** *Let $G$ be an absolutely simple, adjoint linear algebraic group over $F$ of classical type $C_n$. Then the Hasse principle holds for principal homogeneous spaces under $G$.*

*Proof.* Let $G$ be an absolutely simple, adjoint linear algebraic group of type $C_n$ over $F$. Then $G = \mathrm{PGSp}(A, \sigma)$ for some central simple $F$-algebra $A$ of degree $2n$ with symplectic involution $\sigma$.

Now $H^1(F, \mathrm{PGSp}(A, \sigma))$ classifies $F$-isomorphism classes of central simple $F$-algebras $B$ such that $\deg(A) = \deg(B)$, with symplectic involution $\tau$. The trivial element in this set is the class of $(A, \sigma)$.

Now suppose $[(A, \tau)] = 1 \in H^1(F, \mathrm{PGSp}(A, \sigma))$, so that $(A, \tau) \simeq (A, \sigma)$. Write $A$ as $A \cong M_m(D)$ for some $m \in \mathbb{N}$ and $D$ a central division algebra over $F$. Let $h_1$ be the hermitian form on $D$ corresponding to $\sigma$, so that $\sigma$ is the adjoint involution with respect to $h_1$, and let $h_2$ be the hermitian form on $D$ corresponding to $\tau$, so that $\tau$ is the adjoint involution with respect to $h_2$. Then the condition that $(A, \tau) \simeq (A, \sigma)$ is equivalent to the condition that $h_1 \simeq \lambda h_2$ for some $\lambda \in F$. So by Theorem 5.4.2, the Hasse principle holds for principal homogeneous spaces under $\mathrm{PGSp}(A, \sigma)$. □

**Theorem 5.5.3.** *Let $G$ be an absolutely simple, adjoint linear algebraic group over $F$ of classical type $D_n$, so that $G = \mathrm{PGO}^+(A, \sigma)$ for some central simple $F$-algebra $A$ of degree $2n$ with orthogonal involution $\sigma$. Then the Hasse principle holds for principal homogeneous spaces under $G$ in the case when $A$ is split.*

*Proof.* First note that in the case when $A$ is split, the tuple $(A, \sigma)$ reduces to a quadratic form $q$ over $F$ and $G = \mathrm{PSO}(q)$. Now $H^1(F, \mathrm{PSO}(q))$ clas-

sifies similarity classes of nonsingular quadratic forms $q'$ over $F$ such that $\dim(q') = \dim(q)$ and $\mathrm{disc}(q') = \mathrm{disc}(q)$. The trivial element in this set is the similarity class of $q$. Then the condition that
$[q'] = 1 \in H^1(F, \mathrm{PSO}(q))$ is equivalent to the condition that $q \simeq \lambda q'$ for some $\lambda \in F$. So by Theorem 5.2.2, the Hasse principle holds for principal homogeneous spaces under $\mathrm{PSO}(q)$. $\square$

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