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Ha Ngoc Nguyen Date

Polynomials Nonnegative on Noncompact Subsets of the Plane

By

Ha Ngoc Nguyen
Doctor of Philosophy

Mathematics and Computer Science

Victoria Powers, Ph.D.
Advisor

Eric Brussel, Ph.D.
Committee Member

Raman Parimala, Ph.D.
Committee Member

Accepted:

Lisa A. Tedesco, Ph.D.
Dean of the James T. Laney School of Graduate Studies

Date

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Ha Ngoc Nguyen
M.S., Emory University, 2008
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A dissertation submitted to the Faculty of the
James T. Laney School of Graduate Studies of Emory University
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Abstract

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In 1991, Schmüdgen proved that if f is a polynomial in n variables with real coefficients such that $f > 0$ on a compact basic closed semialgebraic set $K \subseteq \mathbb{R}^n$, then there always exists an algebraic expression showing that f is positive on K. Then in 1999, Scheiderer showed that if K is not compact and its dimension is 3 or more, there is no analogue of Schmüdgen's Theorem. However, in the noncompact two-dimensional case, very little is known about when every f positive or nonnegative on a noncompact basic closed semialgebraic set $K \subseteq \mathbb{R}^2$ has an algebraic expression proving that f is nonnegative on K. Recently, M. Marshall answered a long-standing question in real algebraic geometry by showing that if $f \in \mathbb{R}[x, y]$ and $f \geq 0$ on the strip $[0, 1] \times \mathbb{R}$, then f has a representation $f = \sigma_0 + \sigma_1 x(1 - x)$, where $\sigma_0, \sigma_1 \in \mathbb{R}[x, y]$ are sums of squares.

This thesis gives some background to Marshall's result, which goes back to Hilbert's 17th problem, and our generalizations to other noncompact basic closed semialgebraic sets of $\mathbb{R}^2$ which are contained in strip. We also give some negative results.

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for Vu, my family, mentors, and friends

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Chapter 1

Introduction

In 1900, Hilbert posed his 17th problem, asking whether every real polynomial f in n variables that is nonnegative on $\mathbb{R}^n$ can be written as a sum of squares of rational functions. In 1927, Artin [1] gave an affirmative answer to Hilber's 17th Problem. A natural question to ask is what would happen if we change the condition "$f \geq 0$ on $\mathbb{R}^n$" by some other positivity condition, for example, $f \geq 0$, or $f > 0$ on some subset K of $\mathbb{R}^n$.

Given a finite subset $S = \{g_1, \dots, g_r\} \subseteq \mathbb{R}[x_1, \dots, x_n] =: \mathbb{R}[X]$, the **basic closed semialgebraic set** K_S associated to S is

$$K_S := \{\alpha \in \mathbb{R}^n \mid g_i(\alpha) \geq 0, i = 1, \dots, r\},$$

and the **preordering** T_S of $\mathbb{R}[X]$ generated by S is $T_S := \{ \sum_{\epsilon \in \{0,1\}^r} s_\epsilon \, g_1^{\epsilon_1} \dots g_r^{\epsilon_r} \}$, where s_ϵ is a sum of squares of polynomials in $\mathbb{R}[X]$ for all $\epsilon \in \{0,1\}^r$.

Since sums of squares of polynomials in $\mathbb{R}[X]$ are globally nonnegative, and the g_i are nonnegative on K_S, $f \in T_S$ implies that f is nonnegative on K_S, and a representation of f in T_S, i.e., $f = \sum_{\epsilon \in \{0,1\}^r} s_\epsilon \, g_1^{\epsilon_1} \dots g_r^{\epsilon_r}$, is an algebraic identity certifying that f is nonnegative on K_S.

Fix S as above. In 1991, Schmüdgen [17] proved that if K_S is compact and $f \in \mathbb{R}[X]$ such that $f > 0$ on K_S, then $f \in T_S$, i.e., there always exists an algebraic expression proving that the given polynomial f is positive on K_S. In general Schmüdgen's result does not hold if the condition "$f > 0$ on K_S" is replaced by "$f \geq 0$ on K_S".

An obvious question to ask is: What happens when K_S is not compact? In 1999, Scheiderer [15] showed that if K_S is not compact and its dimension is 3 or more, there is no analogue of Schmüdgen's Theorem. Then in 2002, Kuhlmann and Marshall [6] proved that there is a result similar to Schmüdgen's Theorem for a noncompact set $K_S \subseteq \mathbb{R}$, provided that S contains the "right" set of generators for K_S.

In the noncompact two-dimensional case, very little is known about when every f positive or nonnegative on a noncompact set $K_S \subseteq \mathbb{R}^2$ has an algebraic expression proving that f is nonnegative on K_S, i.e., $f > 0$ or $f \geq 0$ implies that $f \in T_S$. Recently, M. Marshall showed that if $f \in \mathbb{R}[x, y]$ is nonnegative on the strip $[0, 1] \times \mathbb{R} \subseteq \mathbb{R}^2$, then $f \in T_S$, where $S = \{x, 1 - x\}$. This is a stronger result than Schmüdgen's Theorem, as Marshall proved that the preordering in this case contains all polynomials nonnegative on K_S.

In this thesis, we explore representations of polynomials that are nonnegative on some noncompact subsets of the plane. Our work concerns generalizations of Marshall's result and an attempt to characterize noncompact semialgebraic sets for which there is a corresponding finitely generated preordering which contains all polynomials nonnegative on the set.

In Chapter 3, we generalize Marshall's theorem to the **half-strip** situation, by which we mean noncompact basic closed semialgebraic subsets of the form $\{(x, y) \mid 0 \leq x \leq 1, g(x, y) \geq 0\}$ which are bounded as $y \to -\infty$. We show that if $f \in \mathbb{R}[x, y]$ is nonnegative on certain types of half-strips in the plane, then $f \in T_S$, provided we choose the "right" set of generators S. The proof of the theorem involves two steps of reduction: first to the case $[0, 1] \times \mathbb{R}^+$ and secondly to the strip $[0, 1] \times \mathbb{R}$, and then using Marshall's theorem on the strip. Combining this half-strip result with a substitution technique from Scheiderer's work [16], we obtain more examples of half-strips for which the corresponding preorderings contain all nonnegative polynomials. Then we end this chapter with a family of examples of half-strips for which no corresponding finitely generated preordering contains all positive polynomials.

In Chapter 4, we give another generalization of Marshall's result by showing that if $f \in \mathbb{R}[x, y]$ such that $f(x, y) \geq 0$ on $U \times \mathbb{R}$, where $U \subseteq \mathbb{R}$ is compact, i.e., $U \times \mathbb{R}$ consists of multiple strips in the plane, then $f \in T_S$, again provided we choose the right set of generators. The proof uses generalizations of Marshall's arguments. The idea of the proof is to get representations of f on some small strips covering $U \times \mathbb{R}$, where the representations use the generators and sums of squares of polynomials in y whose coefficients are analytic functions of x defined in some open neighborhoods of these small strips. Then we apply a version of the Weierstrass Approximation Theorem to obtain a polynomial representation of $f(x, y)$ in T_S.

Finally we end this thesis with Chapter 5, where we summarize our work and propose a list of open problems.

Chapter 2

Preliminaries

Fix $n \in \mathbb{N}$ and let $\mathbb{R}[X] := \mathbb{R}[x_1, \ldots, x_n]$ be the ring of polynomials in n variables over $\mathbb{R}$. For the special cases $n = 1$ and $n = 2$, we use $\mathbb{R}[x]$ and $\mathbb{R}[x, y]$, respectively. Throughout, $\mathbb{R}^+$ denotes the nonnegative elements of $\mathbb{R}$, and $\mathbb{R}_{>0}$ denotes the strictly positive elements of $\mathbb{R}$.

2.1 Positivity and Sums of Squares

We say that a polynomial $f \in \mathbb{R}[X]$ is **positive semidefinite**, or **psd**, if $f(\alpha) \geq 0$ for all $\alpha \in \mathbb{R}^n$. A polynomial $f \in \mathbb{R}[X]$ is a **sum of squares**, or **sos**, if $f = \sum_{i=1}^{k} g_i^2$, for $g_1, \ldots, g_k \in \mathbb{R}[X]$. We write $\sum \mathbb{R}[X]^2$ for the set of sums of squares in $\mathbb{R}[X]$. Obviously, f sos implies that f is psd, since squares in $\mathbb{R}$ are nonnegative. The converse, in general, is not true. Also, writing f as a sum of squares gives an algebraic identity proving that f is psd.

It has been well-known since the late 19th century that in the one variable case, f psd implies f sos. This follows from the Fundamental Theorem of Algebra:

Theorem 2.1. *If $f(x) \in \mathbb{R}[x]$ is psd, then $f(x)$ is a sum of two squares in $\mathbb{R}[x]$.*

Proof. Factor $f(x)$ in $\mathbb{C}[x]$. Since $f \geq 0$ on $\mathbb{R}$, real roots appear to even degree, and complex roots appear in conjugate pairs. Thus, we have

$$f(x) = \prod c(x - z_j)(x - \bar{z}_j),$$

where $c \in \mathbb{R}^+$. Let $g = \prod(x - z_j)$ and write $g = g_1 + ig_2$, with $g_1, g_2 \in \mathbb{R}[x]$. Then $f = c(g_1^2 + g_2^2)$. $\square$

If $\deg f = 2$, it is easy to see that f is sos, using diagonalization of psd quadratic forms. In 1888, Hilbert [4] proved the following remarkable theorem:

Theorem 2.2 (Hilbert)**.** *Suppose f is psd of degree 4 in two variables, then f is sos. For all other cases, there exist psd f which are not sos.*

However, Hilbert did not give an explicit example of a psd polynomial that is not sos. The first published examples did not appear until the 1960s, and the most famous is the Motzkin polynomial [9] from 1967:

$$x^4y^2 + x^2y^4 - 3x^2y^2 + 1$$

In 1893, Hilbert [5] proved that for $n = 2$ every psd polynomial in $\mathbb{R}[X]$ can be written as a sum of squares of rational functions. Unable to answer the general question of whether every psd polynomial is a sum of squares of rational functions, it became the 17th problem on Hilbert's list of 23 problems he gave in his address to the International Congress of Mathematicians in 1900. In 1927, E. Artin [1] gave an affirmative solution to Hilbert's 17th problem.

Theorem 2.3 (Artin 1927)**.** *Suppose $f \in \mathbb{R}[X]$ is psd. Then there are polynomials g_i, $i = 1, \ldots, k$, and a nonzero $h \in \mathbb{R}[X]$ such that*

$$f = \left(\frac{g_1}{h}\right)^2 + \cdots + \left(\frac{g_k}{h}\right)^2$$

Note that an identity $f = \left(\frac{g_1}{h}\right)^2 + \cdots + \left(\frac{g_k}{h}\right)^2$ is an algebraic expression showing that f is psd.

What can be said if we replace the condition "$f \geq 0$ on $\mathbb{R}^n$" by $f \geq 0$ on some subset K of $\mathbb{R}^n$? In particular, we consider **semialgebraic** subsets, which are the sets of solutions of some finite system of polynomial equations and inequalities.

A subset of $\mathbb{R}^n$ is called **basic semialgebraic** if it is the set of solutions of a finite system of polynomial equations and inequalities, and **semialgebraic** if it is a finite union of basic semialgebraic sets. One checks easily that a subset of $\mathbb{R}$ is semialgebraic if and only if it is a finite union of points and intervals.

In classical algebraic geometry, the key idea is to associate algebraic objects – the ideal – with the geometric objects – varieties. In real algebraic geometry, the geometric objects are semialgebraic sets, and the corresponding algebraic objects are preorderings and quadratic modules.

We are interested in quadratic modules and preorderings in $\mathbb{R}[X]$ associated to basic closed semialgebraic sets. Given a finite subset $S = \{g_1, \ldots, g_s\}$ of $\mathbb{R}[X]$. Recall that the **basic closed semialgebraic** set K_S generated by S is

$$K_S := \{x \in \mathbb{R}^n \mid g_i(x) \geq 0, i = 1, \ldots, s\}.$$

The **quadratic module** M_S generated by S is

$$M_S := \{\sigma_0 + \sigma_1\, g_1 + \cdots + \sigma_s\, g_s \mid \sigma_i \in \sum \mathbb{R}[X]^2 \text{ for all } i = 0, \ldots, s\},$$

and the **preordering** T_S generated by S is

$$T_S := \left\{ \sum_{e \in \{0,1\}^s} \sigma_e g^e \mid \sigma_e \in \sum \mathbb{R}[X]^2 \text{ for all } e \in \{0,1\}^s \right\},$$

where $g^e := g_1^{e_1} \ldots g_s^{e_s}$, if $e = (e_1, \ldots, e_s)$. The preordering T_S is a quadratic module generated by products of the g_i's. Notice that an identity

$$f = \sum_{\epsilon \in \{0,1\}^r} s_\epsilon \, g_1^{\epsilon_1} \dots g_r^{\epsilon_r}$$

in T_S is an algebraic identity certifying that f is nonnegative on K_S.

Note that if M_S is the quadratic module (respectively, preordering T_S) of $\mathbb{R}[X]$ generated by S, and I is the ideal of $\mathbb{R}[X]$ generated by $h_1, \dots, h_t$, then $M + I$ is the quadratic module (respectively, preordering) of $\mathbb{R}[X]$ generated by

$$g_1, \dots, g_s, h_1, -h_1, \dots, h_t, -h_t.$$

The preordering $\mathbb{R}[X]^2 + I$ of $\mathbb{R}[X]$ is generated (as a quadratic module or as a preordering) by $h_1, -h_1, \dots, h_t, -h_t$.

Fix S as above. In 1991, Schmüdgen [17] proved a remarkable theorem that created quite a stir in the community and gave rise to new directions in research.

Theorem 2.4 (Schmüdgen's Positivstellensatz)**.** *Given a finite set $S \subseteq \mathbb{R}[X]$. If K_S is compact, then for any $f \in \mathbb{R}[X]$,*

$$f > 0 \text{ on } K_S \Rightarrow f \in T_S.$$

In other words, the theorem says that if $f > 0$ on a compact basic closed semialgebraic set K_S, there always exists an algebraic expression proving the positivity condition. In general, Schmüdgen's result does not hold if the condition "$f > 0$ on K_S" is replaced by "$f \geq 0$ on K_S", as the following example shows.

Example 2.5. [7, 2.7.3]. Take $n = 1$ and $S = \{-x^2\}$. Then K_S is the singleton set $\{0\}$. Clearly, $x \geq 0$ on K_S. Assume that $x \in T_S$, so x can be written as

$$x = s_0 + s_1(-x^2),$$

where s_0, s_1 are sums of squares in $\mathbb{R}[x]$. Evaluating at $x = 0$ yields that $s_0(0) = 0$. As $s_0 \in \sum \mathbb{R}[x]^2$, write $s_0 = \sum g_i^2$, where $g_i \in \mathbb{R}[x]$. Then $s_0(0) = 0$ implies that $\sum g_i(0)^2 = 0$, which means $g_i(0) = 0$, for every i. Hence we can factor g_i as $g_i = h_i x$, with $h_i \in \mathbb{R}[x]$ and $\deg h_i \leq \deg g_i$. This implies that $s_0 = \sum g_i^2 = \sum (h_i x)^2 = \sum (h_i)^2 x^2$. Thus, we have

$$x = \sum (h_i)^2 x^2 + s_1(-x^2).$$

Dividing x on both side of the equation yields

$$1 = \sum (h_i)^2 x - s_1 x = (\sum h_i^2 - s_1)x,$$

which is not possible.

Hence, x is not in the preordering T_S.

An obvious question to ask is: What happens when K_S is not compact? It turns out that, unlike the compact case, the answer depends on choosing the right set of generators.

Definition 2.6. Given $U \subseteq \mathbb{R}$, a basic closed semialgebraic set. Then U is finite union of closed intervals and points. As in [6], we define the **natural set of generators** S for U as follows:

(1) If U is compact, then $U = [a_1, b_1] \cup \cdots \cup [a_k, b_k]$, where $a_i, b_i, \in \mathbb{R}$ with $i = 1, \ldots, k$, and $a_1 \leq b_1 < a_2 \leq b_2 < \cdots < a_k \leq b_k \in \mathbb{R}$. Let $S = \{x - a_1, (b_1 - x)(a_2 - x), \ldots, (b_{k-1} - x)(a_k - x), b_k - x\}$.

(2) If U is noncompact, then S is defined as follows:

- If $a \in U$ and $(-\infty, a) \cap U = \emptyset$, then $x - a \in S$.
- If $a \in U$ and $(a, \infty) \cap U = \emptyset$, then $a - x \in S$.
- If $a, b \in U, a < b$ and $(a, b) \cap U = \emptyset$, then $(x - a)(x - b) \in S$
- Other than the above, S has no other elements.

Clearly, in both case, $K_S = U$.

For example, the natural set of generators of $\{-1\} \cup [0,1] \cup [2,\infty)$ is $\{x+1, (x+1)x, (x-1)(x-2)\}$, and the natural set of generators for $[0,1]\cup[2,3]$ is $\{x, (1-x)(2-x), 3-x\}$.

Definition 2.7. (i) Set $T_S^{\text{alg}} = \{f \in \mathbb{R}[X] \mid f \geq 0 \text{ on } K_S\}$. The set T_S^{alg} is a preordering, called the **saturation** of T_S. We say T_S is **saturated** if $T_S = T_S^{\text{alg}}$.

(ii) The **closure** of a quadratic module $M_S \subseteq \mathbb{R}[X]$ is defined to be the closure of M_S in the unique finest locally convex topology on $\mathbb{R}[X]$, and M_S is said to be **closed** if $\overline{M_S} = M_S$.

(iii) We say that M_S has the **strong moment property**, or **(SMP)**, if $\overline{M_S} = M_S^{\text{alg}}$.

In 1999, Scheiderer [15] gave a negative result for the dim $K_S \geq 3$ case.

Theorem 2.8 (Scheiderer, 1999)**.** *Suppose K_S is not compact, and dim K_S is 3 or more. Then there always exists a polynomial that is strictly positive on K_S but not in the preordering T_S, regardless of the choice of generators S.*

Then in 2002, Kuhlmann and Marshall [6] settled the case where $K_S \subseteq \mathbb{R}$ is noncompact.

Theorem 2.9. *[6, Theorem 2.2] Suppose $S \subseteq \mathbb{R}[x]$, and K_S is a noncompact subset of $\mathbb{R}$. Then T_S is saturated if and only if S contains the natural set of generators.*

We begin by showing that if $S \subseteq \mathbb{R}[x]$ and K_S is compact, then T_S is saturated. We will need this result for our main theorem (Theorem 4.1) in Chapter 4. This result is probably well-known to experts, but we were unable to find a proof in the literature. The proof is a generalization of the proof of Theorem 2.9.

Proposition 2.10. *Suppose $U \subseteq \mathbb{R}$ is a compact set, and S is the natural set of generators for U. If $f(x) \in \mathbb{R}[x]$ is nonnegative on U, then f is in the preordering T_S. In other words, T_S is saturated.*

Proof. We have $U = [a_1, b_1] \cup \cdots \cup [a_k, b_k]$, where $a_i, b_i, \in \mathbb{R}$ with $i = 1, \ldots, k$, and $a_1 \leq b_1 < a_2 \leq b_2 < \cdots < a_k \leq b_k$. Recall by Definition 2.6 that the natural set of generators S for U is

$$S = \{x - a_1, (b_1 - x)(a_2 - x), \ldots, (b_{k-1} - x)(a_k - x), b_k - x\}$$

Suppose f is a polynomial in $\mathbb{R}[x]$ of degree d and $f \geq 0$ on K_S. Then f can be written as a product of psd quadratic polynomials times a product of linear polynomials in $\mathbb{R}[x]$. Since each psd quadratic polynomial is a sum of squares in $\mathbb{R}[x]$, without lost of generality, we can reduce to the case where f is a product of linear polynomials in $\mathbb{R}[x]$.

We will prove the proposition by induction on d, the degree of f. If $d = 0$, then $f = a \in \mathbb{R}$ with $a > 0$. In this case, f is clearly in T_S. Hence we may assume $d \geq 1$. If $f \geq 0$ on $\mathbb{R}$, then $f \in \sum \mathbb{R}[x]^2$ by Theorem 2.1. Thus we can assume that $f(c) < 0$ for some $c \in \mathbb{R}$ and consider the following 3 cases:

Case 1: $c < a_1$. In this case, as f changes sign in the interval $(c, a_1]$, there must be a least root r of f in $(c, a_1]$. Write $f = (x - r)f_1$, where $f_1 \in \mathbb{R}[x]$ is of degree $d - 1$. As $a_1 - r \geq 0$ and $x - a_1 \in T_S$, we have $x - r = (x - a_1) + (a_1 - r) \in T_S$. Since $f \geq 0$ on K_S and $x - r \in T_S$, this forces $f_1 \geq 0$ on K_S. Then $f_1 \in T_S$ by the induction hypothesis. Thus, $f \in T_S$.

Case 2: $b_i \leq c \leq a_{i+1}$. Since $f(c) < 0$ by assumption while $f \geq 0$ on K_S with $b_i, a_{i+1} \in K_S$, there must be a greatest root r_1 in the interval $[b_i, c)$ and a least root r_2 in the interval $(c, a_{i+1}]$. Thus $b_i \leq r_1 < c < r_2 \leq a_{i+1}$. Write $f = (x - r_1)(x - r_2)f_1$, where $f_1 \in \mathbb{R}[x]$ is of degree $d - 2$.

By [2, Lemma 4], since $b_i \leq r_1 < r_2 \leq a_{i+1}$, the product $(x - r_1)(x - r_2)$ is in the preordering generated by $(x - b_i)(x - a_{i+1})$. In particular, we have

$(x-r_1)(x-r_2) \in T_S$, as $(x-b_i)(x-a_{i+1})$ is in T_S.

Using an argument similar to that in Case 1, it follows that $f_1 \in K_S$, and and subsequently f is in T_S.

Case 3: $b_k < c$. By a similar argument in the above cases, there must exist a greatest root r in the interval $[b_k, c)$. Write $f = (r-x)f_1$, where $f_1(x) \in \mathbb{R}[x]$ is of degree $d-1$. As $r - b_k \geq 0$ and $b_k - x \in T_S$, it follows that $(r-x) = (b_k - x) + (r - b_k) \in T_S$. Since $f \geq 0$ on K_S and $x - r \in T_S$, this implies that $f_1 \geq 0$ on K_S. Then $f_1 \in T_S$ by the induction hypothesis. Thus, $f \in T_S$. □

Remark 2.11. Note that this is a stronger result than Schmüdgen's Positivstellensatz (Theorem 2.4), since Schmüdgen's Positivstellensatz tells us that f strictly positive on a compact set K_S holds in this case, but it does not imply that T_S is saturated.

2.2 Nonnegative Polynomials in $\mathbb{R}^2$

Fix $S = \{g_1, \ldots, g_r\} \subseteq \mathbb{R}[x, y]$. Recall that T_S is saturated if $f \geq 0$ on K_S implies $f \in T_S$. Define the following property of T_S:

(*) For all $f \in \mathbb{R}[x, y]$, $f > 0$ on K_S implies $f \in T_S$.

Schmüdgen's Theorem (Theorem 2.4)says that if K_S is compact, then (*) holds. Obviously, if T_S is saturated, then (*) holds for T_S. Example 2.5 shows that the converse is not true in general.

We focus on noncompact subsets of $\mathbb{R}^2$. In 2000, by work of Powers and Scheiderer [14], and independently proven by Kuhlmann and Marshall [6], if $K_S \subseteq \mathbb{R}^2$ is not compact and contains a 2-dimensional cone, then (*) never holds, regardless of the choice of generators S.

In [16] Scheiderer showed that if $S = \{x - x^2, y, 1 - xy\}$, then the preordering T_S is saturated. This was the first example known where (*) holds for a noncompact set in $\mathbb{R}^2$.

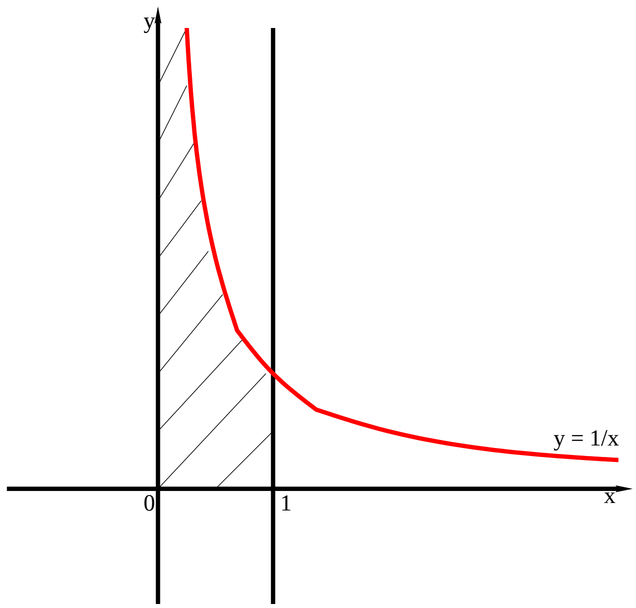

Figure 2.1: $S = \{x - x^2, y, 1 - xy\}$

Consider the case where $S = \{x - x^2\}$ so that K_S is a strip

$$[0, 1] \times \mathbb{R} = \{(x, y) \in \mathbb{R}^2 \mid x - x^2 \geq 0\}.$$

Recently, M. Marshall [8] settled this case.

Theorem 2.12 (Marshall, 2008)**.** *Let $S = \{x, 1 - x\}$. Then T_S is saturated.*

This settled a long-standing question, and certain weak versions of this result can be found in [6], [12] and [13]. The proof uses symbolic computation along with a careful analysis of the complex analytic branches of the curve $f = 0$, where $f \in \mathbb{R}[x, y]$ and $f \geq 0$ on K_S.

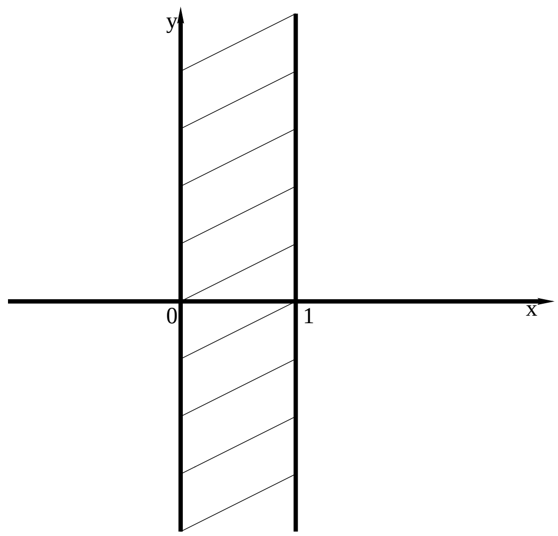

Figure 2.2: The strip $[0,1]\times\mathbb{R}$

Our work concerns generalizations of Marshall's result and an attempt to characterize noncompact semialgebraic sets in $\mathbb{R}^2$ for which there is a corresponding finitely generated preordering which is saturated.

Chapter 3

Polynomials Nonnegative on Half-strips in the Plane

3.1 Introduction

In this section we look at some generalizations of Marshall's theorem (Theorem 2.12). We are interested in basic closed semialgebraic subsets of $\mathbb{R}^2$ which are contained in the strip $[0,1] \times \mathbb{R}$, noncompact, and are bounded as $y \to -\infty$. We work with the strip $[0,1] \times \mathbb{R}$ for ease of exposition; Marshall's theorem and our results generalize immediately to the corresponding semialgebraic subset contained in any strip $[a,b] \times \mathbb{R}$.

Remarks 3.1. 1. It is well-known that the preordering generated by x and $1-x$ is the same as the quadratic module generated by x and $1-x$. This follows from the identity

$$x(1-x) = (1-x)^2 x + x^2(1-x).$$

This means Marshall's theorem could be stated with "preordering" or "quadratic module". However, in [13, Theorem 2] it is shown that the quadratic module generated by $\{x, 1-x, y\}$ is not saturated and is strictly smaller than the preordering T_S. Hence our results in general only hold for preorderings.

2. Marshall's result is stated for the preordering generated by $\{x, 1-x\}$. For ease of exposition, we replace $\{x, 1-x\}$ by $\{x-x^2\}$. It makes no difference in our results since $T_{\{x,1-x\}} = T_{\{x-x^2\}}$, using the identities

$$x = x^2 + (x - x^2)$$

and

$$1 - x = (1-x)^2 + (x - x^2)$$

3.2 Half-strips

In this section we look at noncompact basic closed semialgebraic subsets of the form $\{(x, y) \mid 0 \leq x \leq 1, g(x, y) \geq 0\}$ which are bounded as $y \to -\infty$. We refer to such a set as a **half-strip** in $\mathbb{R}^2$. Suppose $S = \{x - x^2, y - q(x)\}$, where $q(x) \in \mathbb{R}[x]$ with $q(x) \geq 0$ on $[0, 1]$, then K_S is the half-strip

$$\{(x, y) \in \mathbb{R}^2 \mid 0 \leq x \leq 1, y \geq q(x)\}.$$

Our first result is that in this case the preordering T_S is saturated. This follows from Marshall's theorem by an elementary argument.

Theorem 3.2. *Suppose $S = \{x - x^2, y - q(x)\}$, where $q(x) \in \mathbb{R}[x]$ with $q(x) \geq 0$ on $[0, 1]$. Set $K = K_S$ and $T = T_S$. Then T is saturated.*

Proof. We first claim that it is enough to prove the theorem for $q(x) = 0$, i.e., for the half-strip $[0, 1] \times \mathbb{R}^+$.

Suppose that the preordering $T_{\{u-u^2,v\}} \subseteq \mathbb{R}[u, v]$ is saturated and that $f(x, y) \geq 0$ on K. Write f as a polynomial in y, say $f(x, y) = \sum_{i=0}^{k} a_i(x)y^i$, and define g in $\mathbb{R}[u, v]$ by $g(u, v) := \sum a_i(u)(q(u) + v)^j$. Then $f(x, y) \geq 0$ on K implies $g(u, v) \geq 0$ on $[0, 1] \times \mathbb{R}^+$. Hence, as $T_{\{u-u^2,v\}}$ is saturated by our assumption, there exist sums of squares $\sigma_0, \sigma_1, \sigma_2, \sigma_3 \in \mathbb{R}[u, v]$ such that

$$g = \sigma_0 + \sigma_1(u - u^2) + \sigma_2 v + \sigma_3 v(u - u^2).$$

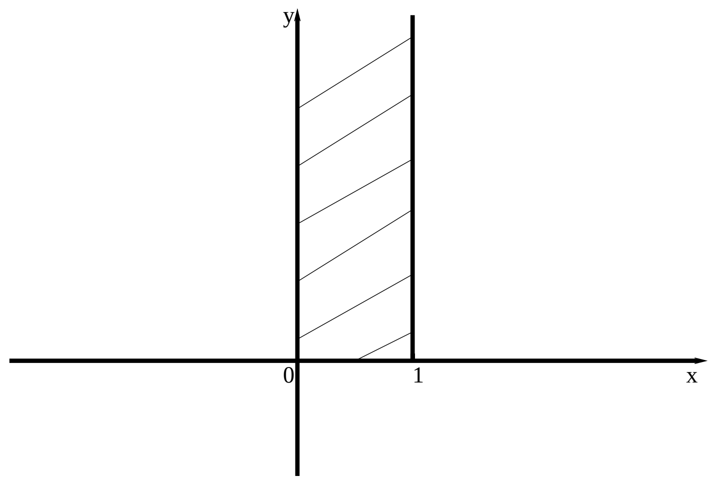

Figure 3.1: Half-strip $[0,1] \times \mathbb{R}^+$

Substituting $u = x, v = y - q(x)$, we obtain a representation of $f(x, y)$ in T.

We are reduced to proving the theorem for $S = \{x - x^2, y\}$. If $f(x, y) \geq 0$ on $[0,1] \times \mathbb{R}^+$, then $f(x, y^2) \geq 0$ on $[0,1] \times \mathbb{R}$. Thus, by Theorem 2.12, there are $g_1, \ldots, g_k$ and $h_1, \ldots, h_l$ in $\mathbb{R}[x, y]$ such that

$$f(x, y^2) = \sum_{i=1}^{k} g_i^2 + (x - x^2) \sum_{i=1}^{l} h_i^2.$$

Replacing y by $-y$, adding and dividing by 2, we obtain

$$f(x, y^2) = \sum_{i=1}^{k} \frac{1}{2} \left(g_i(x, y)^2 + g_i(x, -y)^2\right) + (x - x^2) \sum_{j=1}^{l} \frac{1}{2} \left(h_j(x, y)^2 + h_j(x, -y)^2\right)$$

Using the standard identity

$$\frac{1}{2} \left(\sum_i a_i y^i\right)^2 + \frac{1}{2} \left(\sum_i a_i (-y)^i\right)^2 = \left(\sum_j a_{2j} y^{2j}\right)^2 + \left(\sum_j a_{2j+1} y^{2j}\right)^2 \cdot y^2$$

we have

$$\begin{aligned} f(x,y^2) &= \sum_{i=1}^{k}\left(\sigma_i(x,y^2)^2+\tau_i(x,y^2)^2\cdot y^2\right) \\ &+ \sum_{j=1}^{l}\left(\gamma_j(x,y^2)^2+\delta_j(x,y^2)^2.y^2\right)(x-x^2), \end{aligned}$$

where $\sigma_i,\tau_i,\gamma_j,\delta_j \in \mathbb{R}[x,y]$. Replacing y^2 by y yields a representation of $f(x,y)$ in T. □

Consequently, the halfstrip result gives an infinite family of new examples of saturated preorderings corresponding to noncompact semialgebraic sets in $\mathbb{R}^2$.

Example 3.3. The proof of the theorem gives a method for finding a representation in the general case by reducing to the case $[0,1]\times\mathbb{R}^+$. For example, let $S=\{x-x^2, y-x^2\}$ and $f(x,y)=x^4-x^3+x^2-x^2y-xy+y^2$. We claim that $f(x,y)\geq 0$ on K_S.

Proof. Proceeding as in the proof, we define

$$\begin{aligned} g(u,v) &= u^4-2u^3+u^2+u^2v-uv+v^2 \\ &= (u^2-u+v)^2+v(u-u^2) \end{aligned}$$

Then $f(x,y)=g(x,y-x^2)$ which yields $f(x,y)=(x-y)^2+(y-x^2)(x-x^2)$. Hence $f\in T_S$, which implies the claim. □

Example 3.4. let $S=\{x-x^2, y+x^3-x^2\}$ and

$$f(x,y)=-x^5-2x^4+4x^3-2x^3y+x^3y^2-x^2+x^2y-2x^2y^2+xy+xy^2+y+2y^2+y^3.$$

We claim that $f(x,y)\geq 0$ on K_S.

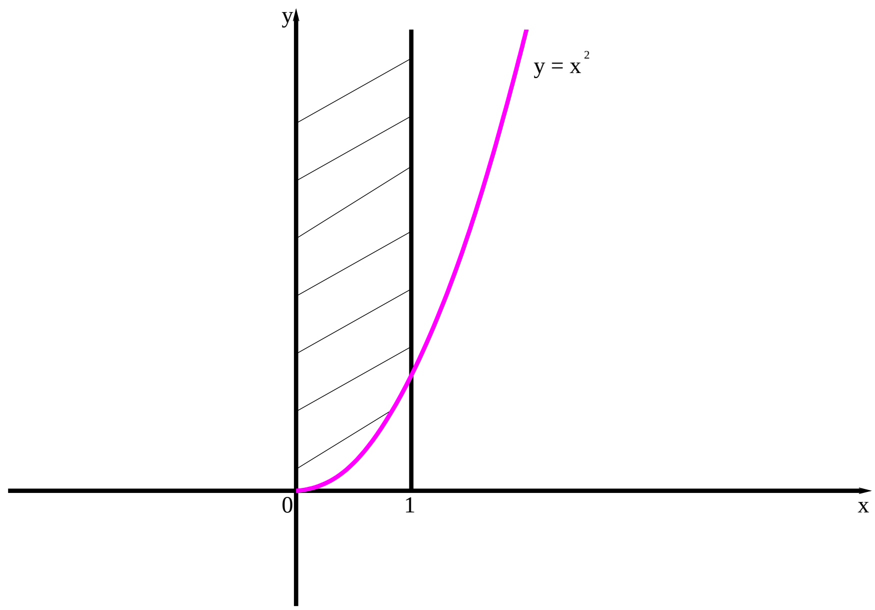

Figure 3.2: Half-strip cut by parabola $y = x^2$

Proof. As in the proof, we define

$$\begin{aligned} g(u,v) &:= f(u, q(u)+v) \\ &= -u^8+3u^7+u^6+u^6v-7u^5-3u^4v+4u^3-4u^3v-2u^3v^2+5u^2v \\ &+ u^2v^2+uv+uv^2+v+2v^2+v^3 \\ &= (u^3-u^2-2u-v)^2u(1-u)+(u^3-u^2-v-1)^2v+u(1-u)v \end{aligned}$$

Substituting back, we obtain

$$f(x,y) = (2x+y)^2(x-x^2)+(y+1)^2(y+x^3-x^2)+(y+x^3-x^2)(x-x^2)$$

Hence, $f \in T_S$. $\square$

Combining Theorem 3.2 with a substitution technique from work of Scheiderer [16], we can obtain more examples of half-strips for which the corresponding preordering is saturated.

Proposition 3.5. *Let $S = \{x - x^2, xy - 1\}$, then we claim T_S is saturated.*

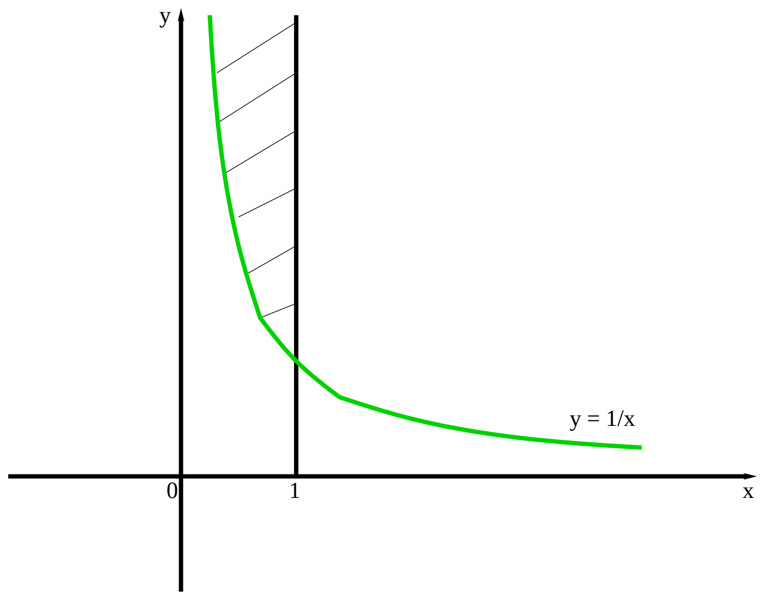

Figure 3.3: Half-strip cut by $xy = 1$

Proof. Suppose $f(x, y) \geq 0$ on K_S. Pick an integer $n \geq 0$ large enough so that $x^{2n} f \in \mathbb{R}[x, xy]$, and hence $f\left(x, \frac{y}{x}\right) \in \mathbb{R}[x, y]$. Define g in $\mathbb{R}[u, v]$ by $g(u, v) := u^{2n} f\left(u, \frac{v}{u}\right)$ so that $g(x, xy) = x^{2n} f(x, y)$. As $f(x, y) \geq 0$ on K_S, it implies that $g(u, v) \geq 0$ on $[0, 1] \times [1, \infty)$. Then by Theorem 3.2 there exist sums of squares $\sigma_0, \sigma_1, \sigma_2, \sigma_3 \in \mathbb{R}[u, v]$ such that

$$g(u, v) = \sigma_0 + \sigma_1(u - u^2) + \sigma_2(v - 1) + \sigma_3(u - u^2)(v - 1).$$

Then $x^{2n} f(x, y) =$

$$\sigma_0(x, xy) + \sigma_1(x, xy)(x - x^2) + \sigma_2(x, xy)(xy - 1) + \sigma_3(x, xy)(x - x^2)(xy - 1). \tag{3.1}$$

Define $s_m(x, y) := \dfrac{\sigma_m(x, xy)}{x^{2n}}$, for $m = 0, \ldots, 3$. We want to show that the s_m's are in $\sum \mathbb{R}[x, y]^2$. If $n = 0$, we are done. If $n \geq 1$, then x^{2n} doesn't divide $x, 1 - x$, or $xy - 1$. Since x^{2n} divide the RHS of (3.1), it follows that x^{2n} must divide each of the σ_m. Thus $s_m \in \mathbb{R}[x, y]$, and since each σ_m is a sos in $\mathbb{R}[x,y]$, so is each s_m. Then f can be written as

$$f(x,y) = s_0(x,y) + s_1(x,y)(xy-1) + [s_2(x,y) + s_3(x,y)(xy-1)]x(1-x)$$

Hence $f \in T_S$. $\square$

Here are some more examples to show how Proposition 3.5 works:

Example 3.6. Set $S = \{x - x^2, xy - 1\}$, and let

$$f = -x^4 + x^3 - 3x^3y + x^2 + 3x^2y - x^2y^2 - x + xy^2,$$

then $f \geq 0$ on K_S. We write

$$f\left(u, \frac{v}{u}\right) = -u^4 + u^3 + u^2 - 3u^2v - u + 3uv - v^2 + \frac{v^2}{u}$$

Define $g(u,v) \in \mathbb{R}[u,v]$ by $g(u,v) := u^2 f\left(u, \frac{v}{u}\right)$. In particular,

$$\begin{aligned} g(u,v) &= -u^6 + u^5 + u^4 - 3u^4v - u^3 + 3u^3v - u^2v^2 + uv^2 \\ &= (u^2+v)^2(u-u^2) + u^2(v-1)(u-u^2) \end{aligned} \tag{3.2}$$

Clearly, $g(u,v) \in T_{S'}$ where $S' = \{u - u^2, v - 1\}$. Substituting $u = x$ and $v = xy$ back in (3.2), we get

$$x^2 f(x,y) = (x^2 + xy)^2(x - x^2) + x^2(xy-1)(x - x^2)$$

and hence,

$$f(x,y) = (x+y)^2(x-x^2) + (xy-1)(x-x^2),$$

which implies that $f \in T_S$.

Example 3.7. Set $S = \{x - x^2, xy - 1\}$.
Let $f(x,y) = x^9y^{11} - x^8 - x^8y^{10} + x^7 + 2x^7y^6 - 2x^6y^5 + x^5y - 2x^5y^2 + 2x^5y^9 - x^4 + 2x^4y^2 - 2x^4y^8 - x^3y + 2x^3y^4 + x^2 + x^2y - 2x^2y^3 - x^2y^4 + x + xy^4 + xy^7 - y^6$.

Then $f \geq 0$ on K_S and $f\left(u, \frac{v}{u}\right) =$

$$\begin{aligned} & \frac{v^{11}}{u^2} - u^8 + \frac{v^{10}}{u^2} + u^7 + 2uv^6 - 2uv^5 + u^4v - 2u^3v^2 + 2\frac{v^9}{u^4} - u^4 \\ + \quad & u^2v^2 - 2\frac{v^8}{u^4} - u^2v + 2\frac{v^4}{u} + u^2 + uv - 2\frac{v^3}{u} - \frac{v^4}{u^2} + u + \frac{v^4}{u^3} + \frac{v^7}{u^6} - \frac{v^6}{u^6} \\ = \quad & \left(u^3 + \frac{v^2}{u^2}\right)^2 (u - u^2) + \left(u^2 + \frac{v^3}{u^3} + \frac{v^5}{u}\right)^2 (v-1) + (v-1)(u-u^2) \\ = \quad & \frac{1}{u^4}(u^5 + v^2)^2(u - u^2) + \frac{1}{u^6}(u^5 + u^2v^5 + v^3)^2(v-1) + (v-1)(u-u^2) \end{aligned}$$

In this case, choose $n = 3$ and define $g(u, v) \in \mathbb{R}[u, v]$ by $g(u, v) := u^6 f(u, \frac{v}{u})$. We then obtain

$$\begin{aligned} u^6 f(u, \tfrac{v}{u}) &= g(u, v) \\ &= u^2(u^5 + v^2)^2(u - u^2) + (u^5 + u^2v^5 + v^3)^2(v-1) \qquad (3.3) \\ &+ u^6(v-1)(u-u^2) \end{aligned}$$

So $g(u, v) \in T_{S'}$ with $S' = \{u - u^2, v - 1\}$ and by substituting $u = x$ and $v = xy$ back in (3.3), we get $x^6 f_2(x, y) =$

$$x^2(x^5 + x^2y^2)^2(x - x^2) + (x^5 + x^3y^3 + x^7y^5)^2(xy - 1) + x^6(xy - 1)(x - x^2).$$

Since x^6 divides each summand on the right hand right of the equation, we have

$$f(x, y) = (x^3 + y^2)^2(x - x^2) + (x^2 + y^3 + x^4y^5)^2(xy - 1) + (xy - 1)(x - x^2).$$

Thus, $f \in T_S$.

3.3 Further Examples in $[0, 1] \times \mathbb{R}^+$

Example 3.8. Suppose $S = \{x, 2 - x, xy - 1, 2 - xy\}$. Then the preordering T_S is saturated.

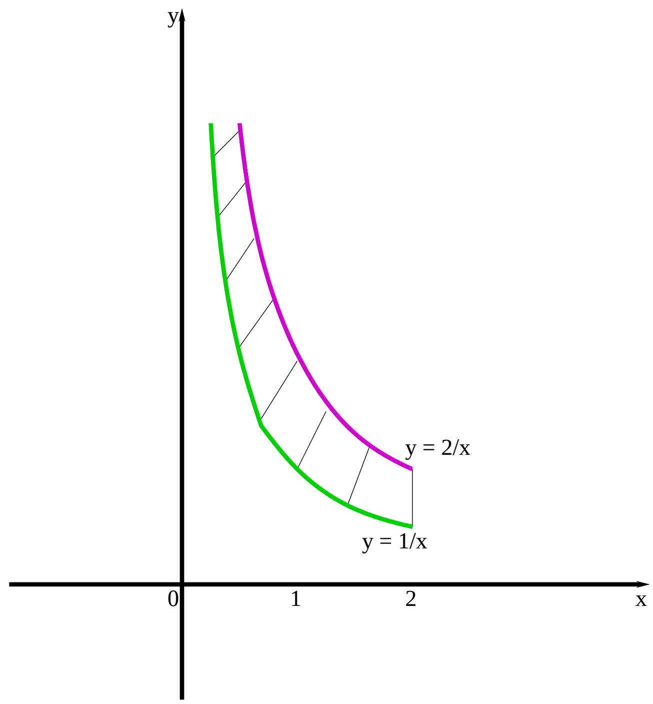

Figure 3.4: Half-strip cut by $xy = 1$ and $xy = 2$

Recall that a quadratic module M has (SMP), if $\overline{M_S} = M_S^{\text{alg}}$ (Definition 2.7). In [11, Example 5.3], it is shown that T_S satisfies the following property that is weaker than saturation: Every finitely generated preordering describing this set S has (SMP). We will prove the stronger result that T_S is saturated.

Proof. Suppose $f(x, y) \in \mathbb{R}[x, y] \geq 0$ on K_S. As in Proposition 3.5, we choose an integer $n \geq 0$ large enough so that $x^{2n} f \in \mathbb{R}[x, xy]$. Define g in $\mathbb{R}[u, v]$ by $g(u, v) := u^{2n} f(u, \frac{v}{u})$ so that $g(x, xy) = x^{2n} f(x, y)$. Since $f(x, y) \geq 0$ on K_S, it follows that $g(u, v) \geq 0$ on the closed rectangle $K_{S'} = [0, 2] \times [1, 2]$ where $S' = \{u(2 - u), (v - 1)(2 - v)\}$. By [16, Theorem 3.2], $T_{S'}$ is saturated, and hence $g(u, v)$ has a representation

$$g(u, v) = \sigma_0 + \sigma_1 u(2 - u) + \sigma_2 (v - 1)(2 - v) + \sigma_3 u(2 - u)(v - 1)(2 - v),$$

where σ_i are sos in $\mathbb{R}[u, v], i = 0, \ldots, 3$.

Since $g(x, xy) = x^{2n} f(x, y)$, we have

$$x^{2n} f(x, y) = \tau_0 + \tau_1 x(2 - x) + \tau_2(xy - 1)(2 - xy) + \tau_3 x(2 - x)(xy - 1)(2 - xy),$$

where $\tau_i = \sigma_i(x, xy)$ are the sos in $\mathbb{R}[x, y], i = 0, \ldots, 3$.

Define $\alpha_i(x, y) := \dfrac{\tau_i(x, y)}{x^{2n}}$, with $i = 0, \ldots, 3$. We want to show that α_i's are sos in $\mathbb{R}[x, y]$. If $n = 0$, we are done. If $n \geq 1$, then x^{2n} doesn't divide $x, 2 - x, xy - 1$ or $2 - xy$. Thus, x^{2n} must divide each of the $\tau_i, i = 0, \ldots, 3$. It follows that $\alpha_i \in \mathbb{R}[x, y]$, and since each τ_i is a sos in $\mathbb{R}[x, y]$, so is each α_i. Then f can be written as $f(x, y) =$

$$\alpha_0 + \alpha_1 x(2 - x) + \alpha_2(xy - 1)(2 - xy) + \alpha_3 x(2 - x)(xy - 1)(2 - xy)$$

Hence $f \in T_S$. $\square$

Remark 3.9. Suppose $S \subseteq \mathbb{R}[x]$, and $f, -f \in S$, for some $f \in \mathbb{R}[x]$. Then the ideal $R[X]f \subseteq T_S$

Proof. Using the identity

$$a = \left(\frac{a+1}{2}\right)^2 - \left(\frac{a-1}{2}\right)^2,$$

we have $\sum \mathbb{R}[x] = \sum \mathbb{R}[x]^2 - \sum \mathbb{R}[x]^2$. Subsequently, we get

$$\mathbb{R}[x]f = \left(\sum \mathbb{R}[x]^2 - \sum \mathbb{R}[x]^2\right) f = \sum \mathbb{R}[x]^2 f + \sum \mathbb{R}[x]^2(-f).$$

Then $f, -f,$ and $\sum \mathbb{R}[x]^2 \in S$ implies that $\mathbb{R}[x]f \subseteq T_S$. $\square$

Next we give an example of $S \subseteq \mathbb{R}[x, y, z]$ such that K_S is noncompact of dimension 2, and T_S is saturated.

Example 3.10. Suppose $S = \{1 - x^2, z - x^2, x^2 - z\}$ so that

$$K := K_S = \{(x, y, z) \in \mathbb{R}^3 \mid -1 \leq x \leq 1, z = x^2\}.$$

Then T_S is saturated.

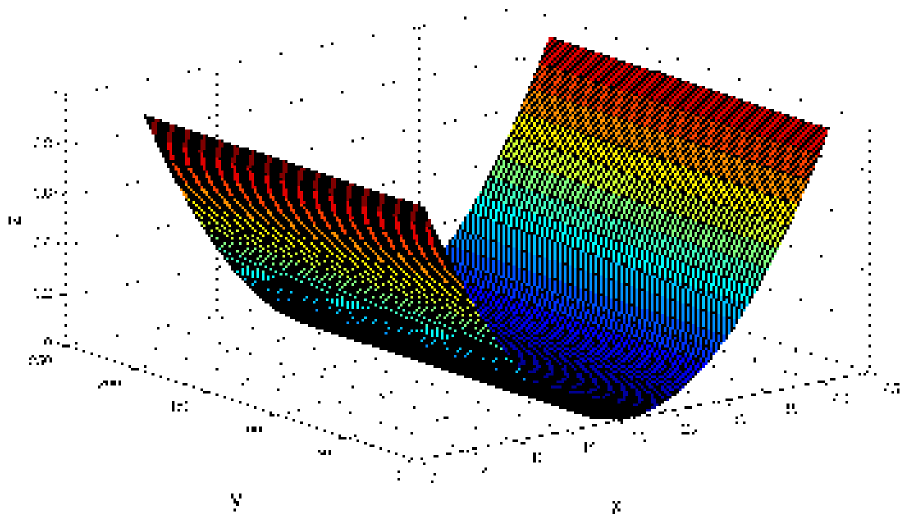

Figure 3.5: Half-strip in $\mathbb{R}^3$

Proof. Given $f(x, y, z) \geq 0$ on K. Write

$$f = \sum g_i(x, y) z^i = \sum g_i(x, y)(z^i - x^{2i}) + \sum g_i(x, y) x^{2i}, \tag{3.4}$$

where $g_i(x, y) \in \mathbb{R}[x, y]$. Then $\sum g_i(x, y)(z^i - x^{2i})$ is in the ideal generated by $z - x^2$. Thus, $\sum g_i(x, y)(z^i - x^{2i})$ is in T_S by the above remark.

Let $g(x, y) = \sum g_i(x, y) x^{2i} = f(x, y, x^2)$. Since $f(x, y, z) \geq 0$ on K, this implies that $g(x, y) \geq 0$ on $[-1, 1] \times \mathbb{R}$. By Marshall's result [8], we obtain a representation

$$g(x, y) = \sigma(x, y) + \tau(x, y)(1 - x^2),$$

where σ, τ are sums of squares in $\mathbb{R}[x, y]$. Thus $g(x, y) \in T_S$.

Since each summand of the RHS in (3.4) is in T_S, it follows that $f \in T_S$. $\square$

We end with a family of examples of half-strips for which no corresponding finitely generated preordering is saturated. This is a generalization of an example due to Netzer, see [3, Lemma 7.4].

Proposition 3.11. *Let* $K = \{(x,y) \mid x - x^2 \geq 0, y^m - q(x) \geq 0, y \geq 0\}$*, where* m *is even,* $q(x) \in \mathbb{R}[x]$ *with* $\deg q$ *odd, and* $q(x) \geq 0$ *on* $[0,1]$*. Then no finitely generated preordering describing* K *is saturated.*

Proof. Suppose there exist a finite set of polynomials $S = \{g_1, \ldots, g_s\}$ such that $K_S = K$ and the preordering T_S is saturated. For $c \in [0,1]$, let T_c be the preordering in $\mathbb{R}[x]$ generated by $\{g_1(c,y), \ldots, g_s(c,y)\}$, then T saturated implies that T_c is saturated. Since $\{g_1(c,y) \geq 0, \ldots, g_s(c,y) \geq 0\} = [q(c)^{\frac{1}{m}}, \infty)$, by Theorem 2.1 and 2.2 in [6], $y - q(c)^{\frac{1}{m}}$ must be among the $g_i(c,y)$ up to a constant factor. Without lost of generality, we can assume

$$g_1(c,y) = r(c)\left(y - q(c)^{\frac{1}{m}}\right),$$

for infinitely many $c \in [0,1]$, where $r(c) \in \mathbb{R}_{>0}$. Let d be the degree of $g_1(x,y)$ in y, and write $g_1(x,y) = \sum_{i=0}^{d} a_i(x) y^i$ with $a_i(x) \in \mathbb{R}[x]$. Then

$$g_1(c,y) = r(c)\left(y - q(c)^{\frac{1}{m}}\right) = a_0(c) + a_1(c)y + \cdots + a_d(c)y^d$$

for infinitely many $c \in [0,1]$. Comparing coefficients on both sides of the above equation, this implies that $a_0(c) = -r(c)q(c)^{\frac{1}{m}}$ and $a_1(c) = r(c)$ for infinitely many $c \in [0,1]$. Hence, $a_0(x)^m = a_1(x)^m q(x) \in \mathbb{R}[x]$, since a_0, a_1 are polynomials. But this is a contradiction, since the degree of $a_0(x)^m$ is $m\cdot$ deg $a_0(x)$ while the degree of the $a_1(x)^m q(x)$ is $m \cdot$ deg $a_1(x) + \deg q(x)$, which implies that one is even and one is odd, respectively. $\square$

Example 3.12. Suppose $S = \{x - x^2, y^2 - x, y\}$ so that K_S is the half-strip $\{(x,y) \mid x - x^2 \geq 0, y^2 - x, y \geq 0\}$. Then no finitely generated preordering describing K_S is saturated.

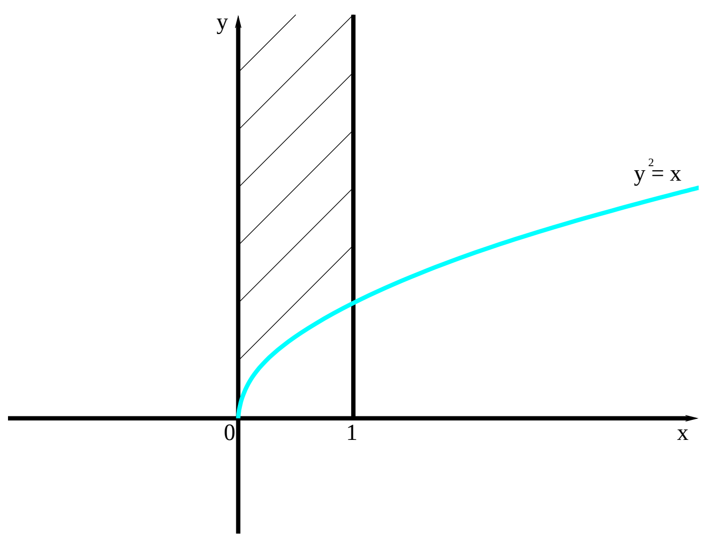

Figure 3.6: Half-strip cut by $y^2 = x$

Chapter 4

Polynomials Nonnegative on Strips in the Plane

In this section, we generalize Marshall's result (Theorem 2.12) to the case $U \times \mathbb{R}$, where $U \subseteq \mathbb{R}$ is compact. More precisely, we show that if $S \subseteq \mathbb{R}[x]$ is the set of natural generators for U, so that in $\mathbb{R}^2$, $K_S = U \times \mathbb{R}$, then T_S is saturated.

For the rest of this section, fix $U \subseteq \mathbb{R}$ compact, say $U = [a_1, b_1] \cup \cdots \cup [a_k, b_k]$, where $a_1 \leq b_1 < a_2 \leq b_2 < \cdots < a_k \leq b_k$. Let $K = U \times \mathbb{R}$ and $S \subseteq \mathbb{R}[x]$ be the natural set of generators for U, i.e.,

$$S = \{x - a_1, (b_1 - x)(a_2 - x), \ldots, (b_{k-1} - x)(a_k - x), b_k - x\}.$$

Then in $\mathbb{R}^2$, we have $K_S = K$. Let T denote the preordering in $\mathbb{R}[x, y]$ generated by S.

Our main theorem in this section is the following:

Theorem 4.1. *Let U, K and T be as above, then T is saturated. In other words, if $f(x, y) \in \mathbb{R}[x, y]$ is nonnegative on $U \times \mathbb{R}$, then $f \in T$.*

First we show that we can reduce Theorem 4.1 to the case where the leading coefficient of f as a polynomial in y is strictly positive on U. All steps are generalizations of results from [8].

4.1 Reduction to a Positive Leading Coefficient

Our first step is to reduce to the case where the leading coefficient of f as a polynomial in y is strictly positive on U.

Fix $f(x,y) \in \mathbb{R}[x,y]$ with $f \geq 0$ on $U \times \mathbb{R}$. If f is a polynomial in x only, then by Proposition 2.10, $f \in T$. Hence we assume $\deg_y f \geq 1$. We first show that $\deg_y f$ is even and that the leading coefficient of f as a polynomial in y is nonnegative on U.

Lemma 4.2. *Suppose $f(x,y) = \sum_{j=0}^{d} a_j(x)y^j$, with $a_j(x) \in \mathbb{R}[x]$, and $f \geq 0$ on $U \times \mathbb{R}$. Then d is even, and $a_d(x) \geq 0$ on U.*

Proof. Suppose $u \in U$ with $a_d(u) < 0$ and consider $f(u,y) \in \mathbb{R}[y]$. Since the leading coefficient of $f(u,y)$ is negative, $\lim\limits_{y\to\infty} f(u,y) = -\infty$, which contradicts f nonnegative on $U \times \mathbb{R}$. Hence, $a_d(x) \geq 0$ on U.

If $\deg_y f$ is odd, pick $u \in U$ with $a_d(u) > 0$. Then $\lim\limits_{y\to-\infty} f(u,y) = -\infty$, which contradicts the assumption that $f \geq 0$ on $U \times \mathbb{R}$. Therefore, $\deg_y f$ is even. □

The following lemma, a generalization of [8, Lemma 2.1], is the key idea needed for our reduction.

Lemma 4.3. *Suppose $h \in \mathbb{R}[x]$ with $h \geq 0$ on U, and h is a constant or a product of linear factors $x - r$ with $r \in U$. If $f \in \mathbb{R}[x,y]$ such that $hf \in T$, then $f \in T$.*

Proof. If $\deg h = 0$, this is trivial. Thus we assume $\deg h \geq 1$, and proceed by induction on $\deg h$. Since $hf \in T$, we have $hf =$

$$\sum_{e\in\{0,1\}^{k+1}} s_e (x-a_1)^{e_1}[(b_1-x)(a_2-x)]^{e_2}\dots(b_k-x)^{e_{k+1}},$$

where each $s_e \in \sum \mathbb{R}[x,y]^2$. Let $x-r$ be a factor of h, then these are several cases to consider:

Case 1: Suppose $r \in (a_1, b_1) \cup \cdots \cup (a_k, b_k)$ and $(x-r)^2$ divides h. Write $h = c(x-r)^2$ with $c \in \mathbb{R}[x], \deg c < \deg h$, and $c \geq 0$ on U. We have $hf = c(x-r)^2 f =$

$$\sum_{e \in \{0,1\}^{k+1}} s_e (x-a_1)^{e_1} [(b_1 - x)(a_2 - x)]^{e_2} \ldots (b_k - x)^{e_{k+1}} \tag{4.1}$$

Substitute $x = r$ into (4.1) to obtain

$$0 = \sum_{e \in \{0,1\}^{k+1}} s_e(r,y)(r-a_1)^{e_1} [(b_1 - r)(a_2 - r)]^{e_2} \ldots (b_k - r)^{e_{k+1}}.$$

For a fixed $y \in \mathbb{R}$, as $s_e(r,y) \geq 0, r - a_1 > 0, (b_i - r)(a_{i+1} - r) > 0,$ and $b_k - r > 0$, this implies that $s_e(r,y) = 0$ for all e. Since this is true for infinitely many y, it follows that $s_e(r,y) = 0$ in $\mathbb{R}[y]$. Hence $x - r$ divides every coefficient of $s_e(x,y)$, and consequently $x - r$ divides $s_e(x,y)$. As $s_e(x,y) \in \sum \mathbb{R}[x,y]^2$ with $x - r$ dividing $s_e(x,y)$, it follows that $(x-r)^2$ divides $s_e(x,y)$. Then we can write $s_e = t_e(x-r)^2$, with $t_e \in \sum \mathbb{R}[x,y]^2$, and $hf = c(x-r)^2 f =$

$$\sum_{e \in \{0,1\}^{k+1}} t_e (x-r)^2 (x-a_1)^{e_1} [(b_1 - x)(a_2 - x)]^{e_2} \ldots (b_k - x)^{e_{k+1}},$$

By canceling $(x-r)^2$ on both sides of the equation, we obtain

$$cf = \sum_{e \in \{0,1\}^{k+1}} t_e (x-a_1)^{e_1} [(b_1 - x)(a_2 - x)]^{e_2} \ldots (b_k - x)^{e_{k+1}}$$

Hence, $cf \in T$, and we are done by induction.

Case 2: $r \in (a_1, b_1) \cup \cdots \cup (a_k, b_k)$ and $x - r$ divides h. Then since $h \geq 0$ on U, h cannot change sign at r, and it follows that $(x-r)^2 | h$. Thus we are done by Case 1.

Case 3: Suppose $x - a_1$, $a_j - x$, or $b_i - x$ divides h for some i, j, with $j = 2, \ldots, k$ and $i = 1, \ldots, k$. We will give a proof for $x - a_1$; the other cases are similar.

If $x - a_1$ divides h, write $hf = c(x - a_1)f$, with $c \in \mathbb{R}[x], c \geq 0$ on U, and $\deg c < \deg h$. Decompose hf as $hf = c(x - a_1)f =$

$$\begin{aligned} &\sum_{e \in \{0,1\}^k} \alpha_e [(b_1 - x)(a_2 - x)]^{e_1} \ldots (b_k - x)^{e_k} + \\ &\sum_{e \in \{0,1\}^k} \beta_e (x - a_1)[(b_1 - x)(a_2 - x)]^{e_1} \ldots (b_k - x)^{e_k} \end{aligned} \tag{4.2}$$

Substitute $x = a_1$ into (4.4), we get

$$0 = \sum_{e \in \{0,1\}^k} \alpha_e(a_1, y)[(b_1 - a_1)(a_2 - a_1)]^{e_1} \ldots (b_k - a_1)^{e_k}$$

Using an argument similar to that in Case 1, this implies that $(x - a_1)^2$ divides α_e for all e. Thus we can write $\alpha_e = \bar{\alpha_e}(x - a_1)^2$, with $\bar{\alpha_e} \in \sum \mathbb{R}[x, y]^2$, and substitute back into the above equation to obtain $hf = c(x - a_1)f =$

$$\sum_{e \in \{0,1\}^k} \bar{\alpha_e}(x - a_1)^2 [(b_1 - x)(a_2 - x)]^{e_1} \ldots (b_k - x)^{e_k} +$$

$$\sum_{e \in \{0,1\}^k} \beta_e (x - a_1)[(b_1 - x)(a_2 - x)]^{e_1} \ldots (b_k - x)^{e_k}$$

By canceling $x - a_1$ on both sides of the above equation, we obtain $cf =$

$$\sum_{e \in \{0,1\}^k} \bar{\alpha_e}(x - a_1)[(b_1 - x)(a_2 - x)]^{e_1} \ldots (b_k - x)^{e_k} +$$

$$\sum_{e \in \{0,1\}^k} \beta_e [(b_1 - x)(a_2 - x)]^{e_1} \ldots (b_k - x)^{e_k}$$

Thus $cf \in T$, and we are done by induction.

Case 4: Suppose $(x-a_1)^2$, $(a_j-x)^2$, or $(b_i-x)^2$ divides h for some i,j, with $j=2,\dots,k$ and $i=1,\dots,k$. We will give a proof for $(x-a_1)^2$; the other cases are similar.

If $(x-a_1)^2$ divides h, write $hf=c(x-a_1)^2f$, with $c\in\mathbb{R}[x], c\geq 0$ on U, and $\deg c<\deg h$. Decompose hf as $hf=c(x-a_1)^2f=$

$$\sum_{e\in\{0,1\}^k}\alpha_e[(b_1-x)(a_2-x)]^{e_1}\dots(b_k-x)^{e_k}+$$
$$\sum_{e\in\{0,1\}^k}\beta_e(x-a_1)[(b_1-x)(a_2-x)]^{e_1}\dots(b_k-x)^{e_k}$$

By an argument similar to that in Case 1, we conclude that $(x-a_1)^2$ divides α_e for all e. Then write $\alpha_e=\bar{\alpha_e}(x-a_1)^2$, with $\bar{\alpha_e}\in\sum\mathbb{R}[x,y]^2$, and substitute back into the above equation to obtain $hf=c(x-a_1)^2f=$

$$\sum_{e\in\{0,1\}^k}\bar{\alpha_e}(x-a_1)^2[(b_1-x)(a_2-x)]^{e_1}\dots(b_k-x)^{e_k}+$$
$$\sum_{e\in\{0,1\}^k}\beta_e(x-a_1)[(b_1-x)(a_2-x)]^{e_1}\dots(b_k-x)^{e_k}$$

Cancelling $(x-a_1)$ on both sides of the equation, we get $c(x-a_1)f=$

$$\begin{aligned}&\sum_{e\in\{0,1\}^k}\bar{\alpha_e}(x-a_1)[(b_1-x)(a_2-x)]^{e_1}\dots(b_k-x)^{e_k}+\\&\sum_{e\in\{0,1\}^k}\beta_e[(b_1-x)(a_2-x)]^{e_1}\dots(b_k-x)^{e_k}\end{aligned}\tag{4.3}$$

Applying the argument of Case 2 to (4.3), it follows that $(x-a_1)^2|\beta_e$ for all e. Write $\beta_e=\bar{\beta}_e(x-a_1)^2$, with $\bar{\beta}_e\in\sum\mathbb{R}[x,y]^2$. Then plug back into (4.3) and cancel $x-a_1$ from both sides of the equation to get

$$cf = \sum_{e\in\{0,1\}^k} \bar{\alpha}_e[(b_1-x)(a_2-x)]^{e_1}\dots(b_k-x)^{e_k}+$$

$$\sum_{e\in\{0,1\}^k} \bar{\beta}_e(x-a_1)[(b_1-x)(a_2-x)]^{e_1}\dots(b_k-x)^{e_k}$$

Hence, $cf \in T$, and we are done by induction.

A similar proof works for any other a_i, b_i such that $(a_i-x)^2|h$ or $(b_i-x)^2|h$.

Case 5: Suppose none of the above cases hold, then

$$h = (x-a_1)^{d_1}(b_1-x)^{d_2}(a_2-x)^{d_3}\dots(a_k-x)^{d_{2k-1}}(b_k-x)^{d_{2k}},$$

with $d_i \in \{0,1\}$.

(i) If $d_1 = 1$ or $d_{2k} = 1$, then a proof similar to the proof of Case 3 will work for each of these two cases.

(ii) Suppose $d_2 = 1$. Then as $h \geq 0$ on U while $b_1 - x < 0$ on $U\backslash[a_1,b_1]$ and $x-a_1, b_2-x, \dots, a_k-x, b_k-x$ are nonnegative on $U\backslash[a_1,b_1]$, d_3 must be 1. Hence, $h = c(b_1-x)(a_2-x)$, with $c \in \mathbb{R}[x], \deg c < \deg h,$ and $c \geq 0$ on U.

Now decompose hf as $hf = c(b_1-x)(a_2-x)f =$

$$\sum_{e\in\{0,1\}^k} \alpha_e(x-a_1)^{e_1}[(b_2-x)(a_3-x)]^{e_2}\dots(b_k-x)^{e_k}+$$

$$\sum_{e\in\{0,1\}^k} \beta_e(b_1-x)(a_2-x)(x-a_1)^{e_1}[(b_2-x)(a_3-x)]^{e_2}\dots(b_k-x)^{e_k},$$

where $\alpha_e, \beta_e \in \sum\mathbb{R}[x,y]^2$. Using the same argument as in Case 1, it follows that (b_1-x) and (a_2-x) divide each α_e, which implies that the product $(b_1-x)(a_2-x)$ divides each α_e. As $s_e(x,y) \in \sum\mathbb{R}[x,y]^2$ with $(b_1-x)(a_2-x)$ dividing $s_e(x,y)$, it follows that $[(b_1-x)(a_2-x)]^2$ divides α_e. Thus we can write $\alpha_e = \bar{\alpha}_e[(b_1-x)(a_2-x)]^2$, where $\bar{\alpha}_e$'s are sums of squares in $\mathbb{R}[x,y]$. Then by canceling $(b_1-x)(a_2-x)$ on both sides of the equation, we get $cf =$

$$\sum_{e\in\{0,1\}^k} \bar{\alpha}_e(b_1-x)(a_2-x)(x-a_1)^{e_1}[(b_2-x)(a_3-x)]^{e_2}\dots(b_k-x)^{e_k}+$$

$$\sum_{e\in\{0,1\}^k} \beta_e(x-a_1)^{e_1}[(b_2-x)(a_3-x)]^{e_2}\dots(b_k-x)^{e_k}$$

This shows that $cf \in T$, and we are done by induction.

A similar proof works for $d_2, \dots, d_{2k-1}$.

Hence, in each case, we show that $f \in T$ by induction. $\square$

Proposition 4.4. *It is enough to prove Theorem 4.1 for $f \in \mathbb{R}[x, y]$ such that the leading coefficient of f as a polynomial in y is strictly positive on U.*

Proof. By Lemma 4.2, if $f(x, y) \geq 0$ on $U \times \mathbb{R}$, then $f(x, y) = \sum_{j=0}^{2d} a_j(x)y^j$ with $a_{2d} \geq 0$ on U. Factor the leading coefficient a_{2d} as $a_{2d} = \bar{a}h$, where $\bar{a}, h \in \mathbb{R}[x]$, with $\bar{a} > 0$ on U, $h = \pm$ a product of linear factors of the form $(x - r)$ with $r \in U$, and $h \geq 0$ on U.

Assume Theorem 4.1 is true if the leading coefficient $a_{2d} > 0$ on U. Let

$$g(x, y) := (h)^{2d-1} f\left(x, \frac{y}{h}\right) \in \mathbb{R}[x, y].$$

Then $g(x, y) \geq 0$ on $U \times \mathbb{R}$, and the leading coefficient of g is $\bar{a}$, which is strictly positive on U. By assumption, it follows that $g \in T$, i.e., g can be written as

$$g(x, y) = \sum_{e\in\{0,1\}^{k+1}} t_e(x-a_1)^{e_1}[(b_1-x)(a_2-x)]^{e_2}\dots(b_k-x)^{e_{k+1}},$$

where the t_e's are sums of squares in $\mathbb{R}[x, y]$. Then $g(x, hy) = h^{2d-1}f(x, y) =$

$$\sum_{e\in\{0,1\}^{k+1}} t_e(x, hy)(x-a_1)^{e_1}[(b_1-x)(a_2-x)]^{e_2}\dots(b_k-x)^{e_{k+1}}$$

Since $t_e(x, y) \in \sum \mathbb{R}[x, y]^2$, $t_e(x, hy) \in \sum \mathbb{R}[x, y]^2$. Thus

$$h^{2d-1}f(x, y) = g(x, hy) \in T.$$

By Lemma 4.3, this implies $f \in T$. $\square$

4.2 Additional Results

We establish the following additional results that we will need to prove our main theorem. We fix $f = \sum_{j=0}^{2d} a_j(x)y^j$ and assume $a_{2d} > 0$ on U.

Lemma 4.5. *We may assume that f has finitely many zeros on $U \times \mathbb{R}$.*

Proof. The proof is essentially the same as the proof of [8, Lemma 2.2], and we include it for completeness.

First we show that we can assume f is square free. Suppose $f = g^2 h$, where $g, h \in \mathbb{R}[x, y]$. Note that if $h \in T$, then $f \in T$ as well. By [7, Proposition 1.1.2], if $g \neq 0$, then the set $\{(a, b) \in U \times \mathbb{R} \mid g(a, b) \neq 0\}$ is dense in $U \times \mathbb{R}$. Thus we get $h \geq 0$ on $U \times \mathbb{R}$, and it suffices to show the result for h, i.e., assuming f is square free.

Since the leading coefficient a_{2d} is strictly positive on U, all factors $x - a_i$ and $x - b_i$ do not divide a_{2d}, hence $x - a_i$ and $x - b_i$ do not divide f. Thus f has only finitely many zeros on the boundary of the strip $(a_i, b_i) \times \mathbb{R}$.

If f has infinitely many zeros in the interior of the strip $(a_i, b_i) \times \mathbb{R}$, then some irreducible factor p of f has infinitely many zeros in the interior. Then by [7, Lemma 9.4.1], p has a non-singular zero in the interior, which is not a zero of any other irreducible factor of f. Then f changes sign at this non-singular zero while all other irreducible factors of f have constant sign in a neighborhood of this non-singular zero. This contradicts the assumption that $f \geq 0$ on $(a_i, b_i) \times \mathbb{R}$. Subsequently, f has only finitely many zeros in $(a_i, b_i) \times \mathbb{R}$ for all i; therefore, f has finitely many zeros on $U \times \mathbb{R}$. $\square$

Lemma 4.6. *Suppose f has only finitely many zeros in $U \times \mathbb{R}$. Then there exists $\epsilon(x) \in \mathbb{R}[x]$, with $\epsilon(x) \geq 0$ on U, such that $f(x, y) \geq \epsilon(x)(1 + y^2)^d$ holds on $U \times \mathbb{R}$, and for each $x \in U$, $\epsilon(x) = 0$ if and only if there exists $y \in \mathbb{R}$ such that $f(x, y) = 0$.*

Proof. By [8, Lemma 4.2] and its proof, for $i = 1, \ldots, k$, there exists a polynomial $\epsilon_i(x) \in \mathbb{R}[x]$, with $\epsilon_i(x) \geq 0$ on $[a_i, b_i]$, such that $f(x, y) \geq \epsilon_i(x)(1+y^2)^d$ holds on $[a_i, b_i] \times \mathbb{R}$, $\epsilon_i(x) = 0$ for $x \in [a_i, b_i]$ if and only if there exists $y \in \mathbb{R}$ such that $f(x, y) = 0$, and $\epsilon_i(x) \neq 0$ for $x \in \mathbb{R} \backslash [a_i, b_i]$.

Dividing by the maximum of $\epsilon_i(x)$ on U, we may assume that each $\epsilon_i(x) \leq 1$ on U. Let $\epsilon(x) = \left(\prod_{i=1}^{k} \epsilon_i(x)\right)^2$, then $\epsilon(x) \geq 0$ on U, and

$$f(x, y) \geq \epsilon(x)(1 + y^2)^d$$

holds on $U \times \mathbb{R}$. For each $x \in U$, the polynomial $\epsilon(x) = 0$ if and only if some $\epsilon_i(x) = 0$, hence $\epsilon(x) = 0$ if and only if there exists $y \in \mathbb{R}$ such that $f(x, y) = 0$. $\square$

4.3 Representations of f by Analytic Functions

In [8], it is shown that if $f \in \mathbb{R}[x, y]$ such that $f \geq 0$ on $[0, 1] \times \mathbb{R}$ and the leading coefficients of f is positive on the interval $[0, 1]$, then for each $r \in [0, 1]$ there is a representation of f involving generators x and $1 - x$ and sums of g_i^2, where the g_i are polynomials in y with coefficients analytic functions of x in some neighborhood of r.

Next we want to generalize this result to attain similar representations of f involving the generators in S, for each $r \in U$. Then we will "glue" together these representations of f and apply the Weierstrass Approximation Theorem to obtain a polynomial representation of $f(x, y)$ in $\mathbb{R}[x, y]$. We can use the results from [8]; however, we need an extra step in order to handle the cases where r is an a_i or b_i.

Lemma 4.7. *Suppose $f \in \mathbb{R}[x, y]$ is nonnegative on $U \times \mathbb{R}$, and the leading coefficient of f as a polynomial in y is strictly positive on U. Then:*

1. *For each $r \in (a_i, b_i)$, for $i = 1, \dots, k$, there exist g_1, g_2 polynomials in y with coefficients analytic functions of x in some open neighborhood $V(r)$ of r, such that $f = g_1^2 + g_2^2$ on $V(r) \times \mathbb{R}$.*

2. *There exist g_l, h_l, with $l = 1, 2$, polynomials in y with coefficients analytic functions of x in some open neighborhood $V(a_1)$ of a_1 such that $f = \sum_{l=1}^2 g_l^2 + \sum_{l=1}^2 h_l^2 (x - a_1)$ on $V(a_1) \times \mathbb{R}$.*

3. *For $i = 1, \dots, k-1$, there exist g_l, h_l, with $l = 1, 2$, polynomials in y with coefficients analytic functions of x in some open neighborhood $V(b_i)$ of b_i such that $f = \sum_{l=1}^2 g_l^2 + \sum_{l=1}^2 h_l^2 (b_i - x)(a_{i+1} - x)$ on $V(b_i) \times \mathbb{R}$.*

4. *For $i = 1, \dots, k-1$, there exist $g_l, h_l, l = 1, 2$, polynomials in y with coefficients analytic functions of x in some open neighborhood $V(a_{i+1})$ of a_{i+1} such that $f = \sum_{l=1}^2 g_l^2 + \sum_{l=1}^2 h_l^2 (b_i - x)(a_{i+1} - x)$ on $V(a_{i+1}) \times \mathbb{R}$.*

5. *There exist g_l, h_l, with $l = 1, 2$, polynomials in y with coefficients analytic functions of x in some open neighborhood $V(b_k)$ of b_k, such that $f = \sum_{l=1}^2 g_l^2 + \sum_{l=1}^2 h_l^2 (b_k - x)$ on $V(b_k) \times \mathbb{R}$.*

Proof. (1), (2) and (5) follow from [8, Lemma 4.4], using a change of variables, if necessary.

For (3), if x is sufficiently close to b_i, by [8, Lemma 4.4], there exist $\varphi_l(x, y), \psi_l(x, y), l = 1, 2$, polynomials in y with coefficients analytic functions of x in some open neighborhood $V(b_i)$ of b_i, such that

$$f = \sum_{l=1}^{2} \varphi_l^2 + \sum_{l=1}^{2} \psi_l^2 \, (b_i - x).$$

We have

$$\begin{aligned} f &= \sum_{l=1}^{2} \varphi_l^2 + \sum_{l=1}^{2} \frac{\psi_l^2}{(a_{i+1} - x)} \, (b_i - x)(a_{i+1} - x) \\ &= \sum_{l=1}^{2} \varphi_l^2 + \sum_{l=1}^{2} \left(\frac{\psi_l}{\sqrt{a_{i+1} - x}} \right)^2 \, (b_i - x)(a_{i+1} - x). \end{aligned}$$

As $\dfrac{1}{\sqrt{a_{i+1}-x}}$ is analytic for x close to b_i, by taking $g_l = \varphi_l$ and $h_l = \dfrac{\psi_l}{\sqrt{a_{i+1}-x}}$, we get the desired result.

For (4), if x is sufficiently close to a_{i+1}, by [8, Lemma 4.4] and a change of variable, we get $f = \sum_{l=1}^{2} \varphi_l^2 + \sum_{l=1}^{2} \psi_l^2 (x - a_{i+1})$, where $\varphi_l, \psi_l, l = 1, 2,$ are polynomials in y with coefficients analytic functions of x in some open neighborhood $V(a_{i+1})$ of a_{i+1}. As in (3), we have

$$f = \sum_{l=1}^{2} \varphi_l^2 + \sum_{l=1}^{2} \left(\frac{\psi_l}{\sqrt{x-b_i}}\right)^2 (x-b_i)(x-a_{i+1})$$

and taking $g_l = \varphi_l$ and $h_l = \dfrac{\psi_l}{\sqrt{x-b_i}}$, we obtain the result. □

We need the following version of the Weierstrass Approximation Theorem, which is a generalization of [8, Proposition 4.5]

Proposition 4.8. *Suppose $\phi, \psi : U \to \mathbb{R}$ are continuous functions, where $U \subseteq \mathbb{R}$ is compact, $\phi(x) \le \psi(x)$ for all $x \in U$, and $\phi(x) < \psi(x)$ for all but finitely many $x \in U$. If ϕ and ψ are analytic at each point $a \in U$ where $\phi(a) = \psi(a)$ then there exists a polynomial $p(x) \in \mathbb{R}[x]$ such that $\phi(x) \le p(x) \le \psi(x)$ holds for all $x \in U$.*

Proof. This is proven for $U = [0, 1]$ in [8, Proposition 4.5]. The proof for U compact is identical. □

4.4 Proof of Theorem 4.1

We are now ready to prove Theorem 4.1. For ease of exposition, denote the natural set of generators S for U by $\{s_1, \ldots, s_{k+1}\}$, i.e.,

$$s_1 = x - a_1, s_2 = (b_1 - x)(a_2 - x), \ldots, s_{k+1} = b_k - x.$$

Let $f(x,y) = \sum_{j=0}^{2d} a_j(x)y^j$, where $d \geq 1$. By Proposition 4.4 and Lemma 4.5, we can assume that $a_{2d}(x) > 0$ on U and $f(x,y)$ has only finitely many zeros in $U \times \mathbb{R}$. By Lemma 4.6, there exists $\epsilon(x) \in \mathbb{R}[x]$ such that $\epsilon(x) \geq 0$ on U, $f(x,y) \geq \epsilon(x)(1+y^2)^d$, and $\epsilon(x) = 0$ if and only if there exists $y \in U$ such that $f(x,y) = 0$. Let $f_1(x,y) := f(x,y) - \epsilon(x)(1+y^2)^d$, then $f_1 \geq 0$ on $U \times \mathbb{R}$. Replacing $\epsilon(x)$ by $\frac{\epsilon(x)}{N}$, $N > 1$, if necessary, we can assume f_1 has degree $2d$ as a polynomial in y, and the leading coefficient of f_1 is positive on U.

By Lemma 4.7, for each $r \in U$, there exists an open neighborhood $V(r)$ of r so that

$$f_1 = \sum_{j=1}^{2} g_{0,j,r}(x,y)^2 + \sum_{j=1}^{2} g_{1,j,r}(x,y)^2\, s_1 + \cdots + \sum_{j=1}^{2} g_{k+1,j,r}(x,y)^2\, s_{k+1} \quad (4.4)$$

on $V(r) \times \mathbb{R}$, where $g_{i,j,r}(x,y)$ are polynomials in y of degree $\leq d$ with coefficients analytic functions of x in $V(r)$, for $i = 0, ..., k+1$ and $j = 1, 2$. If r is in the interior of U, note that $g_{i,j,r} = 0$ for $i \neq 0$. If $r = a_1$, then $g_{i,j,r} = 0$ for $i \neq 1$, etc.

Since U is compact, there are finitely many $V(r_1), \ldots, V(r_p)$ which cover U. Since $\epsilon(x)$ has only finitely many roots on U, we can choose the open cover so that no $V(r_l)$ contains more than one root of $\epsilon(x)$, and no root is in more than one $V(r_l)$. By [10, Theorem 36.1], there exists a partition of unity corresponding to the open cover of $\{V(r_l)\}$, i.e., we have $1 = \nu_1 + ... + \nu_p$, where $\nu_1, \ldots, \nu_p$ are continuous functions on U with $0 \leq \nu_l \leq 1$ on U, and $\overline{\operatorname{supp}(\nu_l)} \subseteq V(r_l)$ for $l = 1, ..., p$. Note that by construction, if a root u of $\epsilon(x)$ is in $V(r_l)$, then $\nu_l(x) = 1$ for x close to u.

Define $\varphi_{i,j,l}$, polynomials in y with coefficients functions of x as follows: The coefficient of y^q in $\varphi_{i,j,l}$ is $\sqrt{\nu_l(x)}$ times the coefficient of y^q in g_{i,j,r_l}. Since ν_l is continuous on U, the coefficients of $\varphi_{i,j,l}$ as a polynomial in y are continuous functions of x on U, and they are 0 outside of $V(r_l)$ since ν_l is. Suppose $\epsilon(x)$

has a zero at $u \in V(r_l)$, then by construction of the open covering, $u \notin V(r_q)$ for any $q \neq l$, hence $\nu_l(u) \neq 0$. Subsequently, the coefficients of the $\varphi_{i,j,l}$ are analytic whenever $\epsilon(x) = 0$. Since $\deg_y f = 2d$, $\deg_y \varphi_{i,j,l} \leq d$. Thus $\varphi_{i,j,l}$ are polynomials of degree $\leq d$ in y whose coefficients are continuous on U and analytic at each of the roots of $\epsilon(x)$ in U. Further, f_1 satisfies

$$f_1 = \sum_{l=1}^{p} \nu_l f_1 = \sum_{l=1}^{p} \left(\sum_{j=1}^{2} \varphi_{0,j,l}^2 + \sum_{j=1}^{2} \varphi_{1,j,l}^2\, s_1 + \cdots + \sum_{j=1}^{2} \varphi_{k+1,j,l}^2\, s_{k+1} \right) \quad (4.5)$$

on $U \times \mathbb{R}$.

We want to approximate the coefficients of the $\varphi_{i,j,l}$'s by polynomials, using Proposition 4.8. Fix $\varphi_{i,j,l}$ and a coefficient $u(x)$. Then by construction, $u(x)+\frac{\epsilon(x)}{N} = u(x)-\frac{\epsilon(x)}{N}$ for only finitely many x in U, and $u(x)-\frac{\epsilon(x)}{N}$, $u(x)+\frac{\epsilon(x)}{N}$ are analytic at each point in U where they are equal.

Define $\phi, \psi : U \to \mathbb{R}$ by $\phi(x) = u(x) - \dfrac{2}{5}\epsilon(x)$, and $\psi(x) = u(x) + \dfrac{2}{5}\epsilon(x)$. Then $\phi(x) \leq \psi(x)$ for $x \in U$, with $\phi(x) < \psi(x)$ for all but finitely many $x \in U$, and ϕ, ψ are analytic at each point $x \in U$ where $\phi(x) = \psi(x)$. Hence, by Proposition 4.8, $\exists w \in \mathbb{R}[x]$ such that

$$u(x) - \frac{2}{5}\epsilon(x) \leq w(x) \leq u(x) + \frac{2}{5}\epsilon(x), \text{ for each } x \in U. \quad (4.6)$$

Now we use these $w(x)$'s to define, for each triple i, j, l, a polynomial $h_{i,j,l}$, where $\deg_y h_{i,j,l} = \deg_y \varphi_{i,j,l}$, and, for all q, if $u(x)$ is the coefficient of y^q in φ, and $w(x)$ is the coefficient of y^q in h, then (4.6) holds. Finally, let

$$h_l(x,y) := \sum_{j=1}^{2} h_{0,j,l}(x,y)^2 + \sum_{j=1}^{2} h_{1,j,l}(x,y)^2\, s_1 + \cdots + \sum_{j=1}^{2} h_{k+1,j,l}(x,y)^2\, s_{k+1}$$

Hence we have polynomials h_l and $\delta \in \mathbb{R}[x,y]$ such that we can write f_1 as follows:

$$f_1 = \left(\sum_{l=1}^{p} h_l(x,y) \right) + \delta(x,y),$$

where $\delta(x,y) = \sum_{i=0}^{2d} c_i(x)y^i$. By the construction of polynomials $h_{i,j,l}$ and (4.6), $|c_i(x)| \leq \frac{2}{5}\epsilon(x)$ on U, for all i.

The rest of the proof is identical to [8, The End Of The Proof]. We decompose

$$f(x,y) = f_1(x,y) + \epsilon(x)(1+y^2)^d$$

into $f(x,y) = s_1(x,y) + s_2(x,y) + s_3(x,y)$, where

$$s_1(x,y) := \sum_{l=1}^{p} h_l(x,y),$$

$$s_2(x,y) := \frac{2}{5}\epsilon(x)(2 + y + 3y^2 + y^3 + 3y^4 + ... + y^{2d-1} + 2y^{2d}) + \sum_{i=0}^{2d} c_i(x)y^i,$$

$$s_3(x,y) := \epsilon(x)[(1+y^2)^d - \frac{2}{5}(2 + y + 3y^2 + y^3 + 3y^4 + ... + y^{2d-1} + 2y^{2d})].$$

We are done if we show $s_1, s_2, s_3 \in T$. Clearly $s_1 \in T$. Since $|c_i(x)| \leq \frac{2}{5}\epsilon(x)$ on U, by Proposition 2.10 we get $\frac{2}{5}\epsilon(x) \pm c_i(x) \in T$ for $i = 0, \ldots, 2d$. Thus,

$$\frac{2}{5}\epsilon(x) + c_i(x) \in T, \text{ for } i \text{ even.}$$

Also, as $\frac{2}{5}\epsilon(x)y^{2m}(y+1)^2 + c_i(x)y^{2m}(y+1)^2$, $\frac{2}{5}\epsilon(x)y^{2m}y^2 - c_i(x)y^{2m}y^2$, $\frac{2}{5}\epsilon(x)y^{2m} - c_i(x)y^{2m}$ are all in T, and T is closed under addition, this implies

$$\frac{2}{5}\epsilon(x)y^{2m}\left((y+1)^2 + y^2 + 1\right) + c_i(x)y^{2m}\left((y+1)^2 - y^2 - 1\right) \in T.$$

Then, by simplifying, this yields

$$\frac{2}{5}\epsilon(x)y^{2m}2(y^2 + y + 1) + c_i(x)y^{2m}2y^2 =$$

$$\frac{2}{5}\epsilon(x)2(y^{2m+2} + y^{2m+1} + y^{2m}) + c_i(x)2y^{2m+1} \in T$$

Thus,

$$\frac{2}{5}\epsilon(x)(y^{i+1} + y^i + y^{i-1}) + c_i(x)y^i \in T, \text{ for } i \text{ odd .}$$

Hence $s_2(x,y) \in T$. Using the identity in [8]

$$(1+y^2)^d - \tfrac{2}{5}(2+y+3y^2+y^3+3y^4+\cdots+y^{2d-1}+2y^{2d})$$

$$= \tfrac{1}{5}(1+y^2+\cdots+y^{2d-2})(1-y)^2 + \sum_{i=1}^{d-1}\left(\binom{d}{i} - \tfrac{8}{5}\right) y^{2i},$$

it follows that $s_3(x,y) \in T$. Hence, this implies that $f(x,y) = s_1(x,y) + s_2(x,y) + s_3(x,y) \in T$. $\square$

Theorem 4.1 yields many more examples of finitely generated saturated preorderings in the two-dimensional noncompact case.

For example, let $K = [0,1] \cup [2,3] \times \mathbb{R}$ and $S = \{x, (1-x)(2-x), 3-x\}$ be the natural set of generators for K. If $f \in \mathbb{R}[x,y]$ such that $f \geq 0$ on K, then $f \in T_S$.

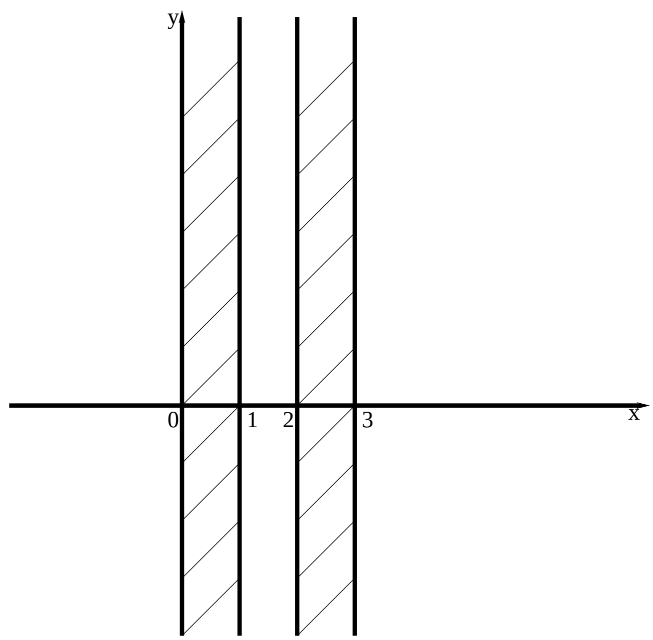

Figure 4.1: Multiple Strips $K = [0,1] \cup [2,3] \times \mathbb{R}$

Chapter 5

Conclusion and Future Work

In this thesis, we explored representations of polynomials that are nonnegative on some subsets of the plane. We gave generalizations of Marshall's strip theorem [8] to half-strips and multiple strips in the plane. Our work helped generate many more examples of finitely generated saturated preordering in the two-dimensional noncompact case. The remainder of this chapter is devoted to explain some future projects:

1. Suppose $S = \{x - x^2, y^2 - q(x)^2\}$, where $q(x) \in \mathbb{R}[x]$ with $q(x) \geq 0$ on $[0, 1]$ so that K_S is a "strip" $\{(x, y) \in \mathbb{R}^2 \mid x - x^2 \geq 0, y^2 - q^2(x) \geq 0\}$. Does $f \in \mathbb{R}[x, y] \geq 0$ on K_S imply $f \in T_S$? See Figure 5.1

2. Generalize question (1) to the case of a strip "cut" by finitely many $q(x)$ of the given form, where $q(x) \in \mathbb{R}[x]$. See Figure 5.2

3. Generalize question (1) to the case where the noncompact semialgebraic set is of the form $[0, 1] \times U$, where U is any noncompact closed subset of $\mathbb{R}$.

4. Find a general theory which would explain all the known results for noncompact semialgebraic sets in $\mathbb{R}^2$.

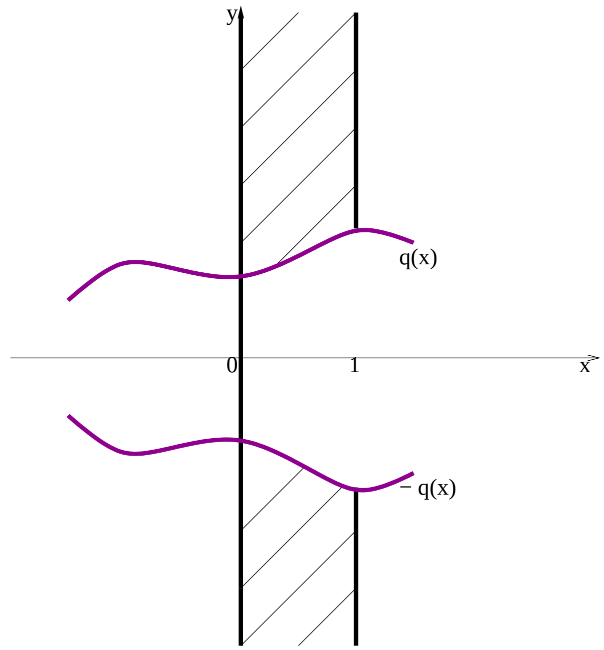

Figure 5.1: "strip" $\{(x, y) \in \mathbb{R}^2 \mid x - x^2 \geq 0, y^2 - q^2(x) \geq 0\}$

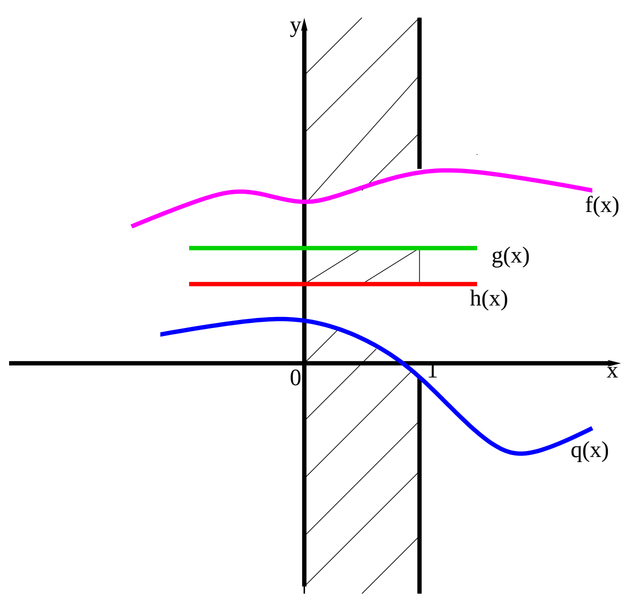

Figure 5.2: "strip" cut by finitely many polynomials in $\mathbb{R}[x]$

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