Maass Forms and Quantum Modular Forms

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Abstract

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This thesis describes several new results in the theory of harmonic Maass forms and related objects. Maass forms have recently led to a flood of applications throughout number theory and combinatorics in recent years, especially following their development by the work of Bruinier and Funke [10] the modern understanding Ramanujan's mock theta functions due to Zwegers [36,37]. The first of three main theorems discussed in this thesis concerns the integrality properties of *singular moduli*. These are well-known to be algebraic integers, and they play a beautiful role in complex multiplication and explicit class field theory for imaginary quadratic fields. One can also study "singular moduli" for special non-holomorphic functions, which are algebraic but are not necessarily algebraic integers. Here we will explain the phenomenon of integrality properties and provide a sharp bound on denominators of symmetric functions in singular moduli. The second main theme of the thesis concerns Zagier's recent definition of a *quantum modular form*. Since their definition in 2010 by Zagier, quantum modular forms have been connected to numerous different topics such as strongly unimodal sequences, ranks, cranks, and asymptotics for mock theta functions. Motivated by Zagier's example of the quantum modularity of Kontsevich's "strange" function $F(q)$, we revisit work of Andrews, Jiménez-Urroz, and Ono to construct a natural vector-valued quantum modular form whose components. The final chapter of this thesis is devoted to a study of asymptotics of mock theta functions near roots of unity. In his famous deathbed letter, Ramanujan introduced the notion of a *mock theta function*, and he offered some alleged examples. The theory of mock theta functions has been brought to fruition using the framework of harmonic Maass forms, thanks to Zwegers [36,37]. Despite this understanding, little attention has been given to Ramanujan's original definition. Here we prove that Ramanujan's examples do indeed satisfy his original definition.

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Chapter 1

Introduction

In this chapter, we will describe modular forms and related objects, concluding with a statement of this thesis' main results.

1.1 Classical Modular Forms

We begin by defining *modular forms*. These are complex-analytic functions with strong transformation properties. More precisely, from now on we let $\mathbb{H}$ denote the *upper half-plane*

$$\mathbb{H} := \{z \in \mathbb{C} \colon \Im z > 0\}.$$

We will also write throughout $z = x + iy$. The upper half-plane comes endowed with a natural hyperbolic metric, $d\mu := \frac{dx\,dy}{y^2}$.

The key group that describes the symmetries of modular forms is $\mathrm{SL}_2(\mathbb{Z})$, the group of 2×2 matrices with integer entries and determinant 1. We have that $\mathrm{SL}_2(\mathbb{Z})$ acts on $\mathbb{H}$ by Möbius transformations

$$\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix} \right) \cdot z := \frac{az + b}{cz + d}.$$

That this is indeed a group action follows from a routine calculation which shows that

$$\Im(\gamma \cdot z) = \frac{\Im z}{|cz + d|^2}$$

for $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix} \right) \in \Gamma$. We can now define modular forms.

Definition 1.1. *For any $k \in \mathbb{Z}$ we say that a holomorphic function $f \colon \mathbb{H} \to \mathbb{C}$ is a weight k modular form on $\Gamma = \mathrm{SL}_2(\mathbb{Z})$ if*

  1. $f(\frac{az+b}{cz+d}) = (cz+d)^k f(z)$ for all $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$,

  2. $\lim\limits_{z \to i\infty} f(z) < \infty$.

The second condition, which states that f is holomorphic at the "cusp" infinity will later be weakened in more general spaces of "modular" objects, and we will also consider "higher level" examples later where the first condition is only required to hold for a subgroup of $\mathrm{SL}_2(\mathbb{Z})$. By considering the action of $\left(\begin{smallmatrix} -1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$ in the first condition, we see that a modular form of odd weight is automatically zero.

A very useful classical fact is that Γ is generated by just two matrices, namely

$$T := \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right), \qquad S := \left(\begin{smallmatrix} -1 & 0 \\ 0 & 1 \end{smallmatrix}\right).$$

Thus, the modular transformations are equivalent to the two transformations $f(z+1) = f(z)$ and $f(-1/z) = z^k f(z)$. Throughout, we will denote the $\mathbb{C}$-vector space of modular forms of weight k by M_k. In particular, modular forms are periodic, holomorphic functions on $\mathbb{H}$, which implies that they have Fourier expansions. Throughout, we denote $q := e^{2\pi i z}$. Thus, for any modular form f there are complex numbers a_n such that

$$f(z) = \sum_{n \geq 0} a_n q^n.$$

The fact that there are no negative powers of q in this sum comes from the growth condition (2) in the above definition of modular forms. Such a Fourier expansion will often be referred to as a *q-series*. Much of the classical interest in modular forms come from studying their Fourier expansions, providing an analytic tool for studying sequences which often encode number theoretic,

combinatorial, or geometric data.

It is not immediately clear from the definition that the spaces M_k are ever non-zero, but we can show this with our first canonical family of modular forms known as *Eisenstein series*. For $k > 2$, we define the series

$$G_k := \sum_{(0,0) \neq (m,n) \in \mathbb{Z}^2} \frac{1}{(m+nz)^k}$$

For $k > 2$, this series is absolutely convergent, and it is a simple exercise to show that G_k satisfies the modularity transformations by checking the transformations under S and T. Holomorphicity at ∞ is also clear from the definition. As noted above, the modularity transformations for k odd imply that $G_{2k+1} \equiv 0$.

For k even, G_k is non-zero and we can write down its Fourier expansion explicitly. When given a q-series, it is customary to "normalize" so that the leading coefficient is 1, and in this case it will even turn out that the Fourier coefficients become rational. Namely, we let $E_k := \frac{G_k}{2\zeta(k)}$ where $\zeta(s)$ is the Riemann-zeta function. Then a standard trick shows that

$$E_k = 1 + \frac{-2k}{B_k} \sum_{n \geq 1} \sigma_{k-1}(n) q^n,$$

where B_k is the kth Bernoulli number, and $\sigma_k(n) := \sum_{d|n} d^k$ is the kth divisor sum. Thus, for example, we have the following:

$$E_4 = 1 + 240 \sum_{n \geq 1} \sigma_3(n) q^n,$$

$$E_6 = 1 - 504 \sum_{n \geq 1} \sigma_5(n) q^n,$$

$$E_8 = 1 + 480 \sum_{n \geq 1} \sigma_7(n) q^n.$$

One of the salient features of modular forms is that M_k forms a finite-dimensional vector space. Thus, proving equalities (and it turns out, congruences) between modular forms only requires checking finitely many coefficients. In order to describe this result, we require the important notion of a *fundamental domain*. This is a subset $\mathcal{F}$ of the upper half-plane such that every point in $\mathbb{H}$ is $\mathrm{SL}_2(\mathbb{Z})$-equivalent to exactly one point of $\mathcal{F}$. Thus, by the modularity transformations, any modular form is uniquely determined by its values on a fundamental domain. The standard choice for $\mathcal{F}$ in this case is given by

$$\mathcal{F} = \{z \in \mathbb{H} : |z| \geq 1, \ -\frac{1}{2} \leq \Re z < \frac{1}{2}, \ \Re z \leq 0 \text{ for } |z| = 1\}.$$

The fundamental result we now state is known as the *valence formula*.

Proposition 1.2. *Let $0 \neq f(z) \in M_k(\Gamma)$. Let ν_∞ be the exponent of the first non-zero term in the Fourier expansion of $f(z)$, and let $\nu_z(f)$ for a point $z \in \mathbb{H}$ be the order of vanishing of f at z. Then we have that*

$$\nu_\infty(f) + \frac{1}{2}\nu_i(f) + \frac{1}{3}\nu_\omega(f) + \sum_{z \in \mathcal{F}, \ z \neq i, \omega} \nu_z(f) = \frac{k}{12},$$

where $\omega = e^{\frac{2\pi i}{3}}$.

This formula easily yields that M_k is finite dimensional as a $\mathbb{C}$-vector space, and moreover gives us explicit dimension formulas.

In particular, we find that if $k \in \{4, 6, 8, 10, 14\}$ then M_k is one-dimensional, and hence spanned by the Eisenstein series E_k. As an amusing consequence of finite dimensionality, we prove an otherwise highly non-obvious combinatorial divisor sum formula. Namely, it is clear from the definition of a modular form that if $f \in M_{k_1}$ and $g \in M_{k_2}$, then $f \cdot g \in M_{k_1+k_2}$. Thus, we have that $E_4^2 \in M_8$. But the dimension of M_8 is 1 and $E_8 \in M_8$ as well. Thus, E_4^2 and E_8 are scalar multiples of one another. But we normalized them

so their Fourier expansions started with 1, so in fact $E_4^2 = E_8$. Comparing Fourier expansions and rearranging yields the following curious combinatorial identity:

$$\sigma_7(n) = \sigma_3(n) + 120 \sum_{i=1}^{n-1} \sigma_3(i)\sigma_3(n-i).$$

We would also like to remark that although there are no modular forms of weight 2 and the series defining G_2 is not absolutely convergent, it is still convergent and gives rise to a function E_2 which has a Fourier expansion in terms of the divisor function $\sigma_{k-1}(n)$ as in the weight $k > 2$ case, namely

$$E_2 = 1 - 24 \sum_{n \geq 1} \sigma_1(n)q^n.$$

Thus, E_2 is still periodic, but it has a slightly more complicated transformation under S, namely

$$z^{-2}E_2(-1/z) = E_2(z) + \frac{12}{2\pi i z}.$$

We call E_2 "quasimodular", and we can "fix" the modularity by setting $E_2^*(z) := E_2(z) - \frac{3}{\pi y}$. Although it is nonholomorphic, $E_2^*(z)$ satisfies the modularity properties in condition (1) of the definition of a modular form. We will also see that one can build "higher level" weight 2 modular forms out of E_2.

Besides the space of Eisenstein series, there is another distinguished set of modular forms which we will often need.

Definition 1.3. *We say that a modular form* $f(z) = \sum_{n \geq 0} a_n q^n$ *is a* cusp form *if* $a_0 = 0$.

The term "cusp form" will become apparent when we discuss cusps and higher level forms below. We denote the space of cusp forms of weight k by $S_k(\Gamma) = S_k$. We can also compute the dimensions of the space of cusp forms. Namely, we have that $S_k = 0$ if $k < 12$ or $k = 14$, as the dimension of M_k is

1 and we have the Eisenstein series, which are not cusp forms. For $k = 12$, there is a distinguished cusp form called the discriminant modular form or delta function.

Proposition 1.4. *Let* $\Delta(z) := q \prod_{n \geq 1} (1 - q^n)^{24}$. *Then* $\Delta(z)$ *is a weight* 12 *cusp form.*

It is customary to denote the Fourier coefficients of $\Delta(z)$ by $\tau(n)$; i.e. $\Delta(z) := \sum_{n \geq 1} \tau(n) q^n$. These coefficients, which were studied by Ramanujan, have very special properties, as we will see in our discussion of Hecke operators below.

One of the key properties of $\Delta(z)$ is that it is non-vanishing on the upper-half plane, which is apparent from its product definition. Thus, the map $f \colon M_k \to S_{k+12}$ defined by $f \mapsto f \cdot \Delta$ is an isomorphism. In many general situations, we may decompose spacese of modular forms into Eisenstein series and cusp forms. The philosophy is that any modular form has an "Eisenstein part" which is easily describable, with a much smaller contribution to the coefficients coming from the "cusp part". The coefficients of cusp forms are much more mysterious. For example, a famous unsolved conjecture of Lehmer states that that $\tau(n) \neq 0$ for all n.

Ramanujan's study of the τ function leads us to an important family of operators on modular forms, the *Hecke operators*. They can be used to easily prove a conjecture of Ramanujan that the τ function is multiplicative, i.e. $\tau(mn) = \tau(m)\tau(n)$ for $(m,n) = 1$. For every $m > 1$, we define the mth Hecke operator T_m applied to a weight k modular form f by

$$T_m f := m^{k-1} \sum_{\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix} \right) \in \mathrm{SL}_2(\mathbb{Z}) \backslash \mathcal{M}_m} (cz + d)^{-k} f(\gamma \cdot z),$$

where $\mathcal{M}_m$ is the group of 2×2 integer matrices of determinant m. The constant m^{k-1} is chosen so that Hecke operators preserve integrality of co-

efficients. The quotient is under the action of Γ on M_m by left multiplication. For convenience, we define the *Petersson slash operator* $|_k\gamma$ by $f|_k\gamma := (\det \gamma)^{k/2}(cz + d)^{-k}f(\gamma \cdot z)$, so that $T_k f = m^{k-1}\sum_{\mathrm{SL}_2(\mathbb{Z})\backslash\mathcal{M}_m} f|_k\gamma$. It is elementary to show that $T_m \colon M_k \to M_k$. A very useful fact about the Hecke operators is that their action on the Fourier coefficients of f is easily described.

Lemma 1.5. *Let* $f(z) = \sum_{n\geq 0} a_n q^n \in M_k$. *Then the Fourier expansion of* $T_k f$ *is given by*

$$T_m f(z) = \sum_{n\geq 0}\left(\sum_{d|\gcd(m,n)} d^{k-1}a_{mn/d^2}\right) q^n. \tag{1.1}$$

It also turns out that for $(m, n) = 1$, the Hecke operators commute:

$$T_m T_n = T_n T_m = T_{mn}. \tag{1.2}$$

We define a Hecke eigenform to be a modular form which is an eigenfunction $T_m f = \lambda_m f$ for all $m > 1$. Although this may seem at first like a strong condition, there are many natural examples of Hecke eigenforms. It can be shown that every Eisenstein series E_k is a Hecke eigenform, and we will show momentarily that $\Delta(z)$ is one. The key arithmetic property of a Hecke eigenform is that its coefficients are multiplicative (up to a constant). This is because for a Hecke eigenform f whose Fourier coefficient of q^1 is 1, (1.1) implies that the the the mth Fourier coefficient of f is equal to the eigenvalue of f by T_m and by (1.2).

We are now in a position to prove Ramanujan's conjecture.

Corollary 1.6. *The τ function is multiplicative. That is, if $(m, n) = 1$, we have $\tau(mn) = \tau(m)\tau(n)$.*

Proof. By the discussion above, it suffices to show that Δ is a Hecke eigenform. By (1.1), it follows that T_m takes cusp forms to cusp forms. As the dimension of S_{12} is 1, $T_m\Delta$ is a multiple of Δ, as desired. $\square$

We conclude this section with a discussion of several generalizations of the definition of modular forms above. Firstly, we often do not require the full strength of modular transformations on all of $\mathrm{SL}_2(\mathbb{Z})$. Namely, we may only require that f has modular transformations for matrices γ in a *congruence subgroup* of $\mathrm{SL}_2(\mathbb{Z})$. The groups of primary interest for us are the following congruence subgroups of level N, for $0 < N \in \mathbb{Z}$:

$$\Gamma(N) := \left\{ \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix} \right) \in \mathrm{SL}_2(\mathbb{Z}) \mid a \equiv d \equiv 1,\, b \equiv c \equiv 0 \pmod{N} \right\},$$

$$\Gamma_1(N) := \left\{ \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix} \right) \in \mathrm{SL}_2(\mathbb{Z}) \mid a \equiv d \equiv 1,\, c \equiv 0 \pmod{N} \right\},$$

$$\Gamma_0(N) := \left\{ \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix} \right) \in \mathrm{SL}_2(\mathbb{Z}) \mid c \equiv 0 \pmod{N} \right\}.$$

In order to describe modularity on congruence subgroups, we first need to define *cusps*. We have $\mathbb{P}_1(\mathbb{Q}) = \mathbb{Q} \cup \{\infty\}$. Then $\mathrm{SL}_2(\mathbb{Z})$ clearly acts on $\mathbb{P}_1(\mathbb{Q})$ by Möbius transformations, and for any congruence subgroup Γ we call the set of Γ-orbits of $\mathbb{P}_1(\mathbb{Q})$ the set of cusps of Γ. For $\mathrm{SL}_2(\mathbb{Z})$, one can show there is only one cusp using the Euclidean algorithm, and we usually choose ∞ as a representative. The set of cusps of any congruence subgroup is always finite. We then modify the definition of modular forms given above for $\mathrm{SL}_2(\mathbb{Z})$ to modular forms f on an arbitrary congruence subgroup Γ by changing condition (1) to require modular transformations to hold only for $\gamma \in \Gamma$ and by requiring that f be holomorphic at all of the cusps. We say that f is a cusp form on Γ if f also vanishes at all the cusps. We denote the space of modular forms of weight k on Γ by $M_k(\Gamma)$, and the subspace of cusp forms by $S_k(\Gamma)$. It is also customary to denote $M_k(\Gamma_0(N)) = M_k(N)$ and $S_k(\Gamma_0(N)) = S_k(N)$. We note that as $T \in \Gamma_0(N)$ and $T \in \Gamma_1(N)$, modular forms on these congruence subgroups are periodic and hence still have Fourier expansions as above.

We will also frequently require the theory of half-integral weight modular forms as developed by Shimura. For the remainder of this section, we assume that $k \in \frac{1}{2} + \mathbb{Z}$. Due to the non-uniqueness of square roots, we need to be

more careful when defining modular forms of half-integral weight. We first define

$$\epsilon_d = \begin{cases} 1 & \text{if } d \equiv 1 \pmod 4 \\ i & \text{if } d \equiv 3 \pmod 4 \end{cases},$$

and we recall the notation for the *Kronecker symbol* $\left(\frac{c}{d}\right)$. We also choose a branch of the square root having argument in $(-\pi/2, \pi/2]$.

Definition 1.7. *Let N be a positive integer and $k \in \frac{1}{2} + \mathbb{Z}$. Then we say a holomorphic function f on $\mathbb{H}$ is a* holomorphic half-integral weight modular form *of weight k if*

1. $f\left(\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \cdot z\right) = \left(\frac{c}{d}\right)^{2k} \epsilon_d^{-2k} f(z)$ *for all* $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_0(4N)$,

2. *f is holomorphic at the cusps of $\Gamma_0(4N)$.*

Important examples of half-integral weight modular forms are provided by the theory of theta functions, which inspired much early interest in modular forms. These are modular forms defined by summing over a lattice. The prototypical example is given by Jacobi's theta function

$$\theta(z) := \sum_{n \in \mathbb{Z}} q^{n^2},$$

which was of classical interest as the Fourier coefficients of $\theta(z)^r$ count the number of representations of integers as the sum of r squares. In general, we have a theory of weight $1/2$ *unary theta functions*. For a Dirichlet character χ mod N (resp. $4N$) on a congruence subgroup Γ of level N, we define the space of modular forms of integer (resp. half-integral) weight k with *Nebentypus* χ by modifying the definition of the modular transformations as follows:

$$f(\gamma \cdot z) = \begin{cases} \chi(d)(cz+d)^k f(z) & \text{if } k \in \mathbb{Z} \\ \chi(d) \left(\frac{c}{d}\right)^{2k} \epsilon_d^{-2k} (cz+d)^k f(z) & \text{if } k \in \frac{1}{2} + \mathbb{Z}. \end{cases} \tag{1.3}$$

We denote the space of modular forms with Nebentypus χ and weight k on a subgroup Γ by $M_k(\Gamma, \chi)$, and similarly we denote the subspace of cusp forms by $S_k(\Gamma, \chi)$. For a Dirichlet character χ, we may define a theta function by

$$\theta_\chi(z) = \begin{cases} \sum_{n \in \mathbb{Z}} \chi(n) q^{n^2} & \text{if } \chi \text{ is even} \\ \sum_{n \in \mathbb{Z}} \chi(n) n q^{n^2} & \text{if } \chi \text{ is odd.} \end{cases} \tag{1.4}$$

As with the Jacobi theta function, one can show that these theta functions are modular forms. More specifically, we have the following.

Theorem 1.8 (See [29]). *Suppose that χ is a primitive Dirichlet character of conductor $r(\chi)$.*

1. *If χ is even, then $\theta_\chi(z) \in M_{\frac{1}{2}}(\Gamma_0(4 \cdot r(\chi)^2), \chi)$.*

2. *If χ is odd, then $\theta_\chi(z) \in S_{\frac{3}{2}}(\Gamma_0(4 \cdot r(\chi)^2, \chi\chi_{-4})$, where χ_{-4} is the non-trivial Dirichlet character modulo 4.*

Theta functions are particularly useful in the case of weight $1/2$ modular forms, thanks to the *Serre-Stark basis theorem* which states that any weight $1/2$ modular form is a linear combination of weight $1/2$ theta functions.

Another important modular form is the *Dedekind eta-function*, defined by the infinite product

$$\eta(z) := q^{\frac{1}{24}} \prod_{n \geq 1} (1 - q^n) \tag{1.5}$$

The eta function serves as a basic building block in the theory of classical modular forms, and it has the following modular transformation properties

$$\eta(z + 1) = e^{\frac{\pi i}{12}} \eta(z), \tag{1.6}$$

$$\eta(-1/z) = (-iz)^{\frac{1}{2}} \eta(z). \tag{1.7}$$

In particular, this implies that $\eta(24z) \in S_{\frac{1}{2}}(\Gamma_0(576), \chi_{12})$.

Our final generalization of the spaces of classical modular forms comes

from weakening the condition at the cusps. Namely, we say that f is a *weakly holomorphic modular form* of weight k for a congruence subgroup Γ if the same conditions as above hold, except that we only require f to be *meromorphic* at the cusps instead of holomorphic. We denote the space of such forms by $M_k^!(\Gamma)$. The first canonical example of a weakly holomorphic modular form is the j-invariant, given by

$$j(z) := \frac{E_4^3}{\Delta} = q^{-1} + 744 + 196884q + \ldots \tag{1.8}$$

Thus, $j(z) \in M_0^!(\mathrm{SL}_2(\mathbb{Z}))$. The j-function plays a beautiful role in class field theory, as its values at quadratic irrationalities describe the abelian extensions of imaginary quadratic fields, and it serves as a parameterization of elliptic curves over $\mathbb{C}$.

One of the combinatorial applications of the Dedekind eta-function stems from the fact that its reciprocal, which is essentially a modular form of weight $-\frac{1}{2}$, is closely related to the *partition function*. Recall that a partition of a positive integer n is a non-decreasing sequence of positive integers which sum to n, and we define $p(n)$ to be the number of partitions of n. If $n = 0$, by convention we write $p(0) = 1$. This function has many deep arithmetic properties which were in particular studied by Ramanujan. The role of modular forms in studying this function comes from the following elementary identity due to Euler:

$$\sum_{n \geq 0} p(n)q^n = \prod_{n \geq 1} \frac{1}{1 - q^n}. \tag{1.9}$$

1.2 Harmonic Maass Forms

We now turn our study to *harmonic Maass forms* and more generally *weak Maass forms*. These are also functions which map from $\mathbb{H}$ to $\mathbb{C}$, but they are no longer required to be holomorphic, or even meromorphic. Instead, we demand that they satisfy a certain differential equation. More specifically,

we define the weight k hyperbolic Laplacian by

$$\Delta_k := -y^2 \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) + iky \left(\frac{\partial}{\partial y} + i\frac{\partial}{\partial y} \right). \qquad (1.10)$$

We note that Δ_k is an elliptic differential operator, and hence its eigenfunctions are real-analytic. Then we define a weak Maass form as an eigenfunction of Δ_k which transforms like a modular form and has a suitable growth condition.

Definition 1.9. *Let $k \in \frac{1}{2}\mathbb{Z}$. We say that a $\mathcal{C}^2$ function $f \colon \mathbb{H} \to \mathbb{C}$ is a weak Maass form of weight k on a congruence subgroup Γ if the following are satisfied.*

1. *For all $\gamma \in \Gamma$, we have*

$$M(\gamma \cdot z) = \begin{cases} (cz + d)^k M(z) & \text{if } k \in \mathbb{Z} \\ \left(\frac{c}{d} \right)^{2k} \epsilon_d^{-2k} (cz + d)^k M(z) & \text{if } k \in \frac{1}{2} + \mathbb{Z}. \end{cases}$$

2. *There is a complex number λ for which $\Delta_k f(z) = \lambda f(z)$.*

3. *There is a finite Fourier polynomial $P_\infty = \sum_{n \leq 0} c^+(n)q^n$ and a constant $C > 0$ for which $f(z) - P_\infty = O(e^{-Cy})$ as $y \to \infty$. The analogous condition is required at all cusps of Γ.*

Note that the factor which appears in the transformations in half-integral weight is consistent with the transformation of classical modular forms of half-integral weight. We also remark that sometimes condition (3) is weakened to say only that $f(z) = O(e^{cy})$ as y approaches any cusp. The Fourier polynomial in (3) is called the *principal part* of $f(z)$ at the corresponding cusp.

In the case that the eigenvalue in (2) is zero, we call the form a (weak) harmonic Maass form. We denote the space of such forms by H_k. Any

weakly holomorphic modular form is automatically a harmonic Maass form. Maass forms are known to have deep analytic structure, and constructing explicit examples is a very important problem. In the sense of Maass cusp forms, which we do not consider here, this is a difficult problem; see [13] for an explicit example of a Maass cusp form constructed from q-hypergeometric series. In the next subsection, we will see that beautiful examples of Ramanujan provide nice examples of harmonic Maass forms, as shown by Zwegers [42, 43]. This has led to an explosion of research with numerous applications scattered throughout combinatorics, Lie theory, moonshine, and even black holes and theoretical physics.

1.3 Mock Theta Functions

Some of the most interesting examples of Maass forms comes to us from the *mock theta functions*. The story of the mock theta functions begins with the enigmatic "deathbed" letter of Ramanujan to Hardy, written just months before his untimely death. After a very successful several years in England with Hardy and Littlewood, Ramanujan became very ill and returned to his native India. Incredibly, in this state, and in mathematical isolation, Ramanujan began investigating a new class of strange q-series. Ramanujan had a strange notion of definition for his forms and could not even show that they satisfied his own definition, even classifying his forms into categories which he never defined. If Ramanujan's notion of what he was defining was vague, and his death came before he had time to develop any substantial theory of them, Ramanujan's legacy has shown him to be a great anticipator with remarkable intuition. To describe what Ramanujan was talking about, we first describe some of his examples of "Eulerian series". These are q-series which are built out of q-hypergeometric terms. We define the *q-Pochhammer*

symbol

$$(a;q)_n := (1-a)(1-aq)\ldots(1-aq^{n-1}),\ n \geq 1 \qquad (1.11)$$

and we set $(a;q)_0 := 1$ by convention. Here we consider a typical mock theta function offered by Ramanujan called $f(q)$:

$$f(q) := \sum_{n \geq 0} \frac{q^{n^2}}{(-q;q)_n^2}. \qquad (1.12)$$

Although the mock theta functions are not modular, Ramanujan noted that they have well-defined asymptotic expansions as q tends to roots of unity. He also noted that near roots of unity, they "behave" like modular forms (or theta functions, in Ramanujan's language). Ramanujan then asked a question: must an Eulerian series which looks like a modular form at roots of unity be simply a modular form plus a function which is bounded at roots of unity? We will return to this question in §1.5.

As it stood, little progress was made in proving Ramanujan's definition or in understanding where the mock theta functions come from or in putting them into a coherent theory, although many suspected that these functions should be important. A remarkable chain of events then led to the finding of the "lost notebook" by George Andrews, which had narrowly avoided destruction and been forgotten by history for decades (see [35], for example, for an account with personal interviews of Andrews and others on this delightful tale). This notebook showed that Ramanujan had been working diligently on the mock theta functions in his last year of life, and they provided much insight and inspiration for a theory yet to come. As Freeman Dyson remarked in a talk at the centennial of Ramanujan's birth:

"The mock theta-functions give us tantalizing hints of a grand synthesis still to be discovered....This remains a challenge for the future." In particular, Dyson called for a theory which would put the mock theta functions into a "group-theoretic" framework as for classical modular forms.

The future arrived in 2002, 82 years after Ramanujan's last letter to Hardy, when Zwegers proved that Ramanujan's mock theta functions can be explained and studied using the theory of harmonic Maass forms in his seminal Ph.D. thesis (see [42, 43]). Namely, Zwegers tied together clues from work of Watson and others, together with work by Appell and Lerch on certain generalized Lambert-type series, to prove the following.

Theorem 1.10 (Zwegers [42, 43]). *For any of Ramanujan's mock theta functions f, there exist integers γ and δ for which*

$$q^\gamma f(q^\delta)$$

is the holomorphic part of a weight $1/2$ harmonic Maass form on some $\Gamma_1(N)$.

This theorem led to a flood of applications to fields throughout number theory, combinatorics, topology, and even string theory and black holes. Thus, Watson was well-justified in proclaiming in [37] that *"Ramanujans discovery of the mock theta functions makes it obvious that his skill and ingenuity did not desert him at the oncoming of his untimely end. As much as any of his earlier work, the mock theta functions are an achievement sufficient to cause his name to be held in lasting remembrance. To his students such discoveries will be a source of delight and wonder until the time shall come when we too shall make our journey to that Garden of Proserpine (a.k.a. Persephone)..."*

1.4 Quantum Modular Forms

In a recent Clay lecture, Zagier defined a new type of automorphic object known as a "quantum modular form" [41]. These are functions which live not on $\mathbb{H}$ but on the cusps $\mathbb{P}_1(\mathbb{Q})$. As a congruence subgroup actions on $\mathbb{P}_1(\mathbb{Q})$ with only finitely many orbits, functions $\mathbb{P}_1(\mathbb{Q}) \to \mathbb{C}$ which transform like modular forms are not very interesting. However, we can obtain numerous

interesting examples by relaxing the definition of our transformations slightly. Zagier then makes the following:

Definition 1.11. *We say a function $f(z)$ from $\mathbb{P}_1(\mathbb{Q})$ (or possibly an infinite subset of $\mathbb{P}_1(\mathbb{Q})$) to $\mathbb{C}$ is a weight k* quantum modular form *if for all $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$ the function*

$$h_\gamma := f(z) - \epsilon(\gamma)(cz+d)^{-k} f\left(\frac{az+b}{cz+d}\right)$$

is "sufficiently nice".

Here, sufficiently nice means some appropriate analyticity condition such as differentiable, $\mathcal{C}^k$, etc, and the $\epsilon(\gamma)$ are suitable complex numbers, for example those occurring in the multiplier system in the definition of half-integral weight modular forms. The definition is intentionally vague to allow flexibility for different types of quantum behavior. Zagier constructs several examples of quantum modular forms related to period integrals, Dedekind sums, and q-series connected to Maass cusp forms.

One particularly interesting example of a quantum modular form comes from the Kontsevich "strange function"

$$F(q) := \sum_{n \geq 0} (q;q)_n = 1 + (1-q) + (1-q)(1-q^2) + \dots \qquad (1.13)$$

The reason this function is "strange" is that $F(q)$ is not convergent on any open subset of $\mathbb{C}$, and only makes sense when q is a root of unity. Zagier then shows a beautiful q-series identity connecting $F(q)$ to a "half-derivative" of the Dedekind eta-function using a "sum of tails" identity [38]. In particular, this implies that

$$F(q) = -\frac{1}{2} \sum_{n \geq 1} n\chi_{12}(n) q^{\frac{n^2-1}{24}}, \qquad (1.14)$$

where $\chi_{12}(n) = \left(\frac{12}{n}\right)$. Even stranger is the fact that neither side of this identity makes sense simultaneously, as the right-hand side only makes sense for

$|q| < 1$. By the identity, we mean that the left-hand side at roots of unity agrees with the radial limit of the right-hand side. In fact, the equality is true in the sense of full asymptotic expansions as well (this is what Zagier refers to as a *strong* quantum modular form).

Using the connection of the half-derivative to Eichler integrals, Zagier shows that if we define $\phi(z) := e^{\frac{2\pi iz}{24}}$, then ϕ is a weight $3/2$ quantum modular form, with the obstruction to modularity h_γ being a smooth function on $\mathbb{R}$.

One application of quantum modular forms is in finding asymptotic expansions for q-series at roots of unity. For example, using (1.14), Zagier uses a standard Mellin transform argument to show the following identity

$$e^{-\frac{t}{24}} \sum_{n \geq 0} (1 - e^{-t})(1 - e^{-2t}) \dots (1 - e^{-nt}) = \sum_{n \geq 0} \frac{T_n}{n!} \cdot \left(\frac{t}{24}\right)^n. \qquad (1.15)$$

Here T_n is the nth *Glaisher number*, which is also the algebraic part of $L(\chi_{12}, 2n + 2)$.

1.5   Main Results

Classically, the term "singular modulus" refers to a value of the modular j-invariant at quadratic irrational points in the upper half plane. These are well-known to be algebraic integers, and they play a beautiful role in complex multiplication and explicit class field theory for imaginary quadratic fields. In [39], Zagier initiated the study of "traces" of singular moduli. He proved that the generating function associated to these numbers is a modular form of weight $3/2$.

Recall that a *weakly holomorphic modular form* of weight k, where $k \in 2\mathbb{Z}$, is a holomorphic function on the upper half plane $\mathbb{H}$ such that

$$f\left(\frac{az + b}{cz + d}\right) = (cz + d)^k f(z)$$

for all $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) =: \Gamma$ and all $z \in \mathbb{H}$, which is meromorphic "at infinity". We let $M_k^!$ denote the space of all weakly holomorphic modular forms of weight k. One can also define the space of modular forms of weight k for any $k \in \frac{1}{2}\mathbb{Z}$. Recall also that any modular form has a q-expansion

$$f(z) = \sum_{n \gg -\infty}^{\infty} a(n)q^n,$$

where $q := e^{2\pi i z}$. Suppose $d \equiv 0, 1 \pmod 4$ and consider any fundamental discriminant D with $dD < 0$. Further suppose that $F \in M_0^!$. Zagier defined the *twisted trace of singular moduli* by

$$\mathrm{Tr}_{d,D}(F) := \sum_Q w_Q^{-1} \chi(Q) F(z_Q), \tag{1.16}$$

where the sum is indexed over a complete set of Γ-inequivalent positive-definite, integral quadratic forms $Q(x, y) = ax^2 + bxy + cy^2$ with discriminant $dD = b^2 - 4ac$. Here

$$z_Q := \frac{-b + \sqrt{dD}}{2a} \in \mathbb{H} \tag{1.17}$$

is the associated CM point, and $w_Q = 1$ unless $Q \sim a(x^2 + y^2)$ or $Q \sim a(x^2 + xy + y^2)$, in which case $w_Q = 2$ or 3 respectively. Here also

$$\chi(Q) := \chi(a, b, c) := \begin{cases} \chi_D(r) \text{ if } (a, b, c, D) = 1 \text{ and } Q \text{ represents } r, \\ \quad \text{where } (r, D) = 1; \\ 0 \text{if } (a, b, c, D) > 1, \end{cases}$$

where χ_D is the Kronecker symbol $\left(\frac{D}{\cdot}\right)$. It is a classical fact that χ is well-defined on Γ-classes of binary quadratic forms with fixed discriminant.

To illustrate Zagier's general theory, let $j(z)$ be the usual modular j-invariant and consider the Hauptmodul for $\mathrm{SL}_2(\mathbb{Z})$,

$$J(z) := j(z) - 744 = q^{-1} + 196884q + 21493760q^2 + \ldots$$

Then for the weight $3/2$ modular form g defined by

$$g(z) := \theta_1(z) \cdot \frac{E_4(4z)}{\eta(4z)^6} := \sum_{n \geq -1} B(n)q^n,$$

Zagier proved the following theorem.

Theorem (Zagier [39], Theorem 1). *Let d be any positive integer such that $d \equiv 0, 3 \pmod 4$. Then*

$$\mathrm{Tr}_{-d,1}\left(J(z)\right) = -B(d).$$

Zagier also considered examples of trace generating functions associated to modular forms of negative even weight. In order to define traces of singular moduli, we need a modular *function*, and the theory of Maass forms provides us with the raising operators to increase the weight. Thus, we define the "traces of singular moduli" of a non-positive weight modular form $f \in M_{-2k}$ to be the traces of the modular function

$$\partial f := R^k f \tag{1.18}$$

in the sense of (1.16). Here R^k is the *iterated Maass raising operator* defined as the composition of k raising operators of the appropriate weights.

Remark. In general it seems that these "singular moduli" for a fixed d are all Galois conjugates, though this remains to be proven.

Zagier also showed that these traces are the coefficients of certain half-integral weight modular forms. Here we consider the general question of integrality for symmetric functions of these singular moduli. For a modular form F, define the *Hilbert class polynomial* as the product

$$H_d(F; x) := \prod_Q \left(x - \frac{F(z_Q)}{w_Q} \right). \tag{1.19}$$

Table 1.1: "Class Polynomials" of $K(z)$

d	$H_d(K;x)$
-23	$x^3 - 23 \cdot 141826x^2 - \frac{3945271661}{23}x - 7693330369871$
-31	$x^3 - 31 \cdot 1201149x^2 - \frac{61346290410}{31}x + 1143159756791823$
-39	$x^4 - 39 \cdot 8067588x^3 + \frac{8602826222178}{39}x^2 - 84029669803810035x$ $+ \frac{95749227855890319016073}{39^2}$

Zagier notes that although the traces are algebraic integers for a form F with integral coefficients, the other terms of $H_d(F;x)$ need not be. For example, consider the weight -2 modular form

$$F_2(z) := \frac{E_4(z)E_6(z)}{\Delta} = q^{-1} - 240 - 141444q - 85292800q^2 - 238758390q^3 + \ldots,$$

and define the weight 0 non-holomorphic derivative

$$K(z) := \partial F_2 = R_{-2}F_2 = \frac{E_2^*(z)E_4(z)E_6(z) + 3E_4^3(z) + 2E_6(z)^2}{6\Delta(z)}.$$

Then one can compute that $H_{-31}(K;x)$ does not have integral coefficients. To illustrate the general phenomenon, consider the above table for $H_d(K;x)$ (we have chosen the first few negative discriminants of class number at least 3).

Note that the third symmetric function is always integral. Based on numerics, it seems that the first (trace) and third symmetric functions are the only coefficients which are generically integral.

We explain this phenomenon and give a bound on all other denominators of the coefficients of the class polynomials. This bound appears to

be sharp in general. Recall that for a weakly holomorphic modular form $f(z) = \sum a(n)q^n$, we define the *principal part* of f to be the sum $\sum_{n \leq 0} a(n)q^n$. To state our theorem, we recall that a prime p is said to be *ordinary* for an cusp form $f = \sum a_n q^n$ which is an eigenform for the Hecke operators if $p \nmid a_p$ and is otherwise said to be *non-ordinary*. For convenience, will say that a prime p is *good* for a pair (k, n), with k a negative even integer and n a positive integer if all eigenforms ina a basis of cusp forms of weight $2 - \ell$ are p-ordinary for $\ell = -10, -14, \ldots, k \cdot n$. The following is our main result:

Theorem 1.12 ([22]). *Let $f(z) \in M_k^!$ be a modular form of negative, even weight with integral principal part. Denote the nth symmetric function in the singular moduli of negative fundamental discriminant d for the modular form ∂f by $\mathbf{e}_{n;f}(d)$. If $(p, d) = 1$, then $\mathbf{e}_{n;f}(d)$ is p-integral. Further let*

$$B(n, k) := \begin{cases} \frac{-nk}{4} & \text{if } nk \in 4\mathbb{Z} \\ \frac{1}{4}(-nk + 2k - 2) & \text{otherwise.} \end{cases}$$

Then if $d \neq -3$ or 4, $p \mid d$, and p is good for the pair (k, n), we have that

$$d^{B(n,k)} \cdot \mathbf{e}_{n;f}(d) \text{ is } p\text{-integral}$$

In particular, this explains the integrality pattern in the computed examples for $K(z)$. As there are no cusp forms of weight less than 12, we also have the following:

Corollary 1.13. *For any $f(z) \in M_{-2}^!$ with integral principal part, we have that*

$$\mathbf{e}_{3;f}(d) \in \mathbb{Z}.$$

Remarks.
(1) when $d = -3$ or -4, we have that $3\mathbf{e}_{n;f}(d)$ or $2\mathbf{e}_{n;f}(d)$ must be integral. See the discussion at the end of Section 3.2.

(2). Although the theorem is only stated for $D = 1$ and d negative, it is clear from the proof that an analogous result for arbitrary "twisted class polynomials" holds in general.

(3). By using the integrality results for $E_2^*(z)/\Delta(z)^{\frac{1}{6}}$, $E_4(z)/\Delta(z)^{\frac{1}{3}}$, and $E_6(z)/\sqrt{\Delta(z)}$ in [39], one can prove Theorem 1.12 directly in the case when $nk \in 4\mathbb{Z}$.

(4). Although the definition of good for the pair (k, n) above includes all even weights in the given range, Theorem 3.2 we will show that certain weights automatically do not arise in the spectral decomposition and the proof of the theorem shows that these weights do not need to be checked.

These results will be proven in Chapter 3 following a review of the basic theory of harmonic Maass forms.

Our second main result is the construction of a new "strange" vector-valued quantum modular form. We revisit Zagier's construction using work of Andrews, Jiménez-Urroz, and Ono on more general sums of tails formulae [2] (see also [1]). We construct a natural 3-dimensional vector-valued quantum modular form associated to tails of infinite products. Moreover, the components are analogous "strange" functions; they do not converge on any open subset of $\mathbb{C}$ but make sense for an infinite subset of $\mathbb{Q}$. We define:

$$H(q) = \begin{pmatrix} \theta_1 \\ \theta_2 \\ \theta_3 \end{pmatrix} := \begin{pmatrix} \eta(z)^2/\eta(2z) \\ \eta(z)^2/\eta(z/2) \\ \eta(z)^2/\eta(\frac{z}{2}+\frac{1}{2}) \end{pmatrix}. \tag{1.20}$$

We also note that $\theta_3 = \zeta_{48}^{-1} \cdot \frac{\eta(z/2)\eta(2z)}{\eta(z)}$ by the following identity which is easily seen by expanding the product definition of $\eta(z)$:

$$\eta(z + 1/2) = \zeta_{48} \cdot \frac{\eta(2z)^3}{\eta(z) \cdot \eta(4z)}, \tag{1.21}$$

where $\zeta_k := e^{2\pi i/k}$. From this it follows that if we let

$$F_9(z) := \eta(z)^2/\eta(2z), \qquad\qquad F_{10}(z) := \eta(16z)^2/\eta(8z)$$

then

$$H(q) = \left(F_9(q) \ F_{10}(q^{1/16}) \ \zeta_{12}^{-1} F_{10}(\zeta_{16} \cdot q^{1/16}) \right)^T \qquad (1.22)$$

(the notations F_9 and F_{10} come from [2]). For convenience, we recall the classical theta-series identities for F_9 and F_{10}:

$$F_9(q) = 1 + 2\sum_{n=1}^{\infty}(-1)^n q^{n^2}, \qquad\qquad F_{10}(q) = \sum_{n=0}^{\infty} q^{(2n+1)^2}. \quad (1.23)$$

It is simple to check that $H(z)$ is a 3-dimensional vector-valued modular form using basic properties of $\eta(z)$, as we describe in §4.3. To each component θ_i we associate for all $n \geq 0$ a finite product $\theta_{i,n}$:

$$\theta_{1,n} := \frac{(q;q)_n}{(-q;q)_n}, \quad \theta_{2,n} := q^{\frac{1}{16}} \cdot \frac{(q;q)_n}{(q^{\frac{1}{2}};q)_{n+1}}, \quad \theta_{3,n} := \frac{\zeta_{16}}{\zeta_{12}} \cdot q^{\frac{1}{16}} \cdot \frac{(q;q)_n}{(-q^{\frac{1}{2}};q)_{n+1}},$$
$$(1.24)$$

such that $\theta_{i,n} \to \theta_i$ as $n \to \infty$. Next, we construct corresponding "strange" functions $\theta_i^S := \sum_{n=0}^{\infty} \theta_{i,n}$. Note that these functions do not make sense on any open subset of $\mathbb{C}$, but that each θ_i^S is defined for an infinite set of roots of unity and, in particular, θ_2^S is defined for all roots of unity. Our primary object of study will then be the vector of "strange" series $H_Q(z) := \left(\theta_1^S(z) \ \theta_2^S(z) \ \theta_3^S(z) \right)^T$. In order to obtain a quantum modular form, we first define $\phi_i(x) := \theta_i^S(e^{2\pi i x})$ from a subset of $\mathbb{Q}$ to $\mathbb{C}$, and let $\phi(x) := \left(\phi_1(x) \ \phi_2(x) \ \phi_3(x) \right)^T$. We will then show the following, which is joint work with Robert Schneider.

Theorem 1.14 ([34]). *Assume the notation above. Then the following are true:*

1. *There exist q-series G_i (see §4.3) which are well-defined for $|q| < 1$, such that $\theta_i^S(q^{-1}) = G_i(q)$ for any root of unity for which θ_i^S converges.*

2. *We have that $\phi(x)$ is a weight $3/2$ vector-valued quantum modular form. In particular, we have that*

$$\phi(z+1) - \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & \zeta_{12} \\ 0 & \zeta_{24} & 0 \end{pmatrix} \phi(z) = 0,$$

and we also have that

$$\left(\frac{z}{-i}\right)^{-3/2}\phi(-1/z) + \begin{pmatrix} 0 & \sqrt{2} & 0 \\ 1/\sqrt{2} & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}\phi(z) = \begin{pmatrix} 0 & \sqrt{2} & 0 \\ 1/\sqrt{2} & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}g(z),$$

where $g(z)$ is a 3-dimensional vector of smooth functions defined as period integrals.

In addition, we deduce the following corollary regarding generating functions of special values of zeta functions from the sums of tails identities. Let

$$H_9(t, \zeta) := -\frac{1}{4}\sum_{n=0}^{\infty}\frac{(1 - \zeta e^{-t})(1 - \zeta^2 e^{-2t})\cdots(1 - \zeta^n e^{-nt})}{(1 + \zeta e^{-t})(1 + \zeta^2 e^{-2t})\cdots(1 + \zeta^n e^{-nt})}, \qquad (1.25)$$

$$H_{10}(t, \zeta) := -2(\zeta e^{-t})^{1/8}\sum_{n=0}^{\infty}\frac{(1 - \zeta e^{-2t})(1 - \zeta^2 e^{-4t})\cdots(1 - \zeta^n e^{-2nt})}{(1 - \zeta e^{-t})(1 - \zeta^2 e^{-3t})\cdots(1 - \zeta^n e^{-(2n+1)t})}.$$

$$(1.26)$$

Remark. Note that there are no rational numbers for which all three components of ϕ make sense simultaneously. To be specific, $\phi_1(z)$ makes sense for rational numbers which correspond to primitive odd order roots of unity, $\phi_2(z)$ makes sense for all rational numbers, and $\phi_3(z)$ converges at even order roots of unity. Hence, by the equation in (2) of 1.14, we understand that each of the six equations of the vector-valued transformation laws is true where the corresponding component in the equation is well-defined; as there are no equations in which ϕ_1 and ϕ_3 both appear, then for all the equations there is an infinite subset of rationals on which this is possible.

For a root of unity ζ, we define the following two L-functions

$$L_1(s, \zeta) := \sum_{n=1}^{\infty}\frac{(-\zeta)^{n^2}}{n^s},$$

$$L_2(s, \zeta) := \sum_{n=1}^{\infty}\left(\frac{2}{n}\right)^2 \cdot \frac{\zeta^{\frac{n^2}{8}}}{n^s}.$$

Then we have the following, which is also joint work with Robert Schneider.

Corollary 1.15 ([34])**.** *Let* $\zeta = e^{2\pi i \alpha}$ *be a primitive* k^{th} *order root of unity,* k *odd for* H_9 *and* k *even for* H_{10}. *Then as* $t \searrow 0$, *we have as power series in* t

$$H_9(t, \zeta) = \sum_{n=0}^{\infty} \frac{L_1(-2n-1, \zeta)(-t)^n}{n!}, \qquad (1.27)$$

$$H_{10}(t, \zeta) = \sum_{n=0}^{\infty} \frac{L_2(-2n-1, \zeta)(-t)^n}{8^n n!}. \qquad (1.28)$$

To illustrate our results by way of an application, we provide a numerical example which gives finite evaluations of seemingly complicated period integrals. First define

$$\Omega(x) := \int_x^{i\infty} \frac{\theta_1(z)}{(z-x)^{3/2}} \, dz$$

for $x \in \mathbb{Q}$, and consider $\theta_1^S(\zeta_k)$ for k odd, which is a finite sum of kth roots of unity. Then the proof of Theorem 1.14 will imply that $\Omega(1/k) = \pi i(1+i)\theta_1^S(\zeta_k)$ by showing that the period integral $\Omega(x)$ is a "half-derivative" which is related to θ_1^S at roots of unity by a sum of tails formula. Table 1.2 above gives finite evaluations of $\theta_1^S(\zeta_k)$ and numerical approximations to the integrals $\Omega(1/k)$.

We will prove these results in Chapter 4.

Our third main result addresses Ramanujan's deathbed letter [4], which gave tantalizing hints of his theory of *mock theta functions*. Thanks to Zwegers [42, 43], it is now known that these functions are essentially the holomorphic parts of weight $1/2$ *harmonic weak Maass forms* whose non-holomorphic parts are period integrals of weight $3/2$ unary theta functions. This realization has many applications (e.g. [30, 40]).

Here we revisit Ramanujan's original definition from his deathbed letter [4]. After a discussion of the asymptotics of certain modular forms which are

Table 1.2: Evaluation of $\theta_1^S(\zeta_k)$ and $\Omega(1/k)$

k	$\pi i(i+1)\theta_1^S(\zeta_k)$	$\int_{1/k+10^{-9}}^{10^9 i} \frac{\theta_1(z)}{(z-1/k)^{\frac{3}{2}}}\, dz$
3	$\pi i(i+1)(-2\zeta_3+3) \sim -7.1250+18.0078i$	$-7.1249+18.0078i$
5	$\pi i(i+1)(-2\zeta_5^3-2\zeta_5^2-8\zeta_5+3) \sim 12.078+35.7274i$	$12.078+35.7273i$
7	$\pi i(i+1)(6\zeta_7^4-2\zeta_7^2-10\zeta_7+7) \sim 52.0472+25.685i$	$52.0474+25.685i$
9	$\pi i(i+1)(8\zeta_9^4-16\zeta_9+3) \sim 76.4120-28.9837i$	$76.4116-28.9836i$

given as *Eulerian series*, he writes:

"...Suppose there is a function in the Eulerian form and suppose that all or an infinity of points $q = e^{2i\pi m/n}$ are exponential singularities and also suppose that at these points the asymptotic form of the function closes as neatly as in the cases of (A) and (B). The question is: - is the function taken the sum of two functions one of which is an ordinary theta function and the other a (trivial) function which is $O(1)$ at <u>all</u> the points $e^{2i\pi m/n}$? The answer is <u>it is not necessarily so.</u> When it is not so I call the function a mock ϑ-function. I have not proved rigorously that <u>it is not necessarily so</u>. But I have constructed a number of examples in which it is inconceivable to construct a ϑ-function to cut out the singularities of the original function."

Remark. By "ordinary theta function", Ramanujan meant a *weakly holomorphic modular form* with weight $k \in \frac{1}{2}\mathbb{Z}$ on some $\Gamma_1(N)$ (see [29] for background). Recall that a weakly holomorphic modular form is a meromorphic modular form whose poles (if any) are supported at cusps.

Little attention has been given to Ramanujan's original definition, prompting Berndt to remark [3] that *"it has not been proved that any of Ramanujan's mock theta functions are really mock theta functions according to his definition."* The following fact fills in this gap. This is joint work with Michael

Griffin and Ken Ono.

Theorem 1.16 ([21]). *Suppose that $f(z) = f^-(z) + f^+(z)$ is a harmonic weak Maass form of weight $k \in \frac{1}{2}\mathbb{Z}$ on $\Gamma_1(N)$, where $f^-(z)$ (resp. $f^+(z)$) is the nonholomorphic (resp. holomorphic) part of $f(z)$. If $f^-(z)$ is nonzero and $g(z)$ is a weight k weakly holomorphic modular form on any $\Gamma_1(N')$, then $f^+(z) - g(z)$ has exponential singularities as q approaches infinitely many roots of unity ζ.*

As a corollary, we obtain the following fitting conclusion to Ramanujan's enigmatic question by proving that his alleged examples indeed satisfy his original definition. More precisely, we prove the following.

Corollary 1.17 ([21]). *Suppose that $M(z)$ is one of Ramanujan's mock theta functions, and let γ and δ be integers for which $q^\gamma M(\delta z)$ is the holomorphic part of a weight 1/2 harmonic weak Maass form. Then there does not exist a weakly holomorphic modular form $g(z)$ of any weight $k \in \frac{1}{2}\mathbb{Z}$ on any congruence subgroup $\Gamma_1(N')$ such that for every root of unity ζ we have*

$$\lim_{q \to \zeta} (q^\gamma M(\delta z) - g(z)) = O(1).$$

Remark. The limits in Corollary 1.17 are *radial* limits taken from within the unit disk.

As his letter indicates, Ramanujan was inspired by the intimate relationship between the exponential singularities of modular forms at roots of unity and the asymptotics of their corresponding Fourier coefficients. As a toy model of his question, we begin by considering the following question whose solution would have been clear to him: If $f(z)$ is a weight k_1 weakly holomorphic modular form which has some exponential singularities at cusps, then can there be another weakly holomorphic modular form of different weight k_2, say $g(z)$, that exactly cuts out its singularities at roots of unity? The answer

is *no.* If such a $g(z)$ exists, then both $f(z)$ and $g(z)$ must have the same principal parts at all cusps, and at least one of these must be nonconstant. Without loss of generality, suppose that the principal part at the cusp infnity is nonconstant, and then consider the function $h(z) := f(z) - g(z)$. By hypothesis, $h(z)$ has bounded radial limits as q approaches every root of unity. Now, since $f(z)$ and $g(z)$ are modular on some common subgroup $\Gamma_1(N')$, then if we take $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix} \right) \in \Gamma_1(N')$ with $cd \neq 0$, then we have

$$h\left(\tfrac{az+b}{cz+d}\right) = f\left(\tfrac{az+b}{cz+d}\right) - g\left(\tfrac{az+b}{cz+d}\right) = (cz+d)^{k_1} f(z) - (cz+d)^{k_2} g(z). \quad (1.29)$$

Letting $z \to i\infty$, we find that $f(z)$ and $g(z)$ cannot cut out the same exponential singularities at roots of unity because of the difference between the weights.

In the case of Ramanujan's examples, the situation is much more subtle, and this is the point of his last letter and this paper.

Although these functions cannot asymptotically match a modular form at all roots of unity, Ramanujan offers a tantalizing example of a "near miss". For his mock theta function

$$f(q) := 1 + \frac{q}{(1+q)^2} + \frac{q^4}{(1+q)^2(1+q^2)^2} + \dots,$$

he defines a function $b(q)$, which is modular up to multiplication by $q^{-\frac{1}{24}}$, and claims for a primitive even order $2k$ root of unity ζ, that as q approaches ζ radially inside the unit disk we have

$$f(q) - (-1)^k b(q) = O(1).$$

In [18] and [19], Folsom, Ono, and Rhoades give two proofs that this is indeed the case by giving explicit formulas for the $O(1)$ numbers. Here we show another "near miss" example using a similar proof as that in [19]. Namely, we consider the mock theta function

Table 1.3: Radial Limits for $\omega(q)$

q	$W(q)$	$W(q) - M(q)$	$N(\zeta(5))$
$\zeta_5 \cdot (0.99)$	$-3.78 + 12.35i$	$0.67 - 1.35i$	$0.81 - 1.76i$
$\zeta_5 \cdot (0.995)$	$-31.89 + 98.87i$	$0.72 - 1.51i$	$0.81 - 1.76i$
$\zeta_5 \cdot (0.9975)$	$-1236.9 + 3807.7i$	$0.76 - 1.62i$	$0.81 - 1.76i$
$\zeta_5 \cdot (0.999)$	$-3.78 \times 10^7 + 1.16 \times 10^8 i$	$0.79 - 1.70i$	$0.81 - 1.76i$

$$\omega(q) := \sum_{n \geq 0} \frac{q^{2n(n+1)}}{(q; q^2)_{n+1}^2}. \tag{1.30}$$

We let $\epsilon(k) := \begin{cases} -1 & \text{if } k \in 2\mathbb{Z} + 1 \\ 1 & \text{if } k \equiv 2 \pmod{4}. \end{cases}$ We prove a similar result for Ramanujan's ω function.

Theorem 1.18. *Let $0 < k$ be an integer not divisible by 4. Then for ζ a primitive k^{th} root of unity, we have*

$$\lim_{q \to \zeta} q\omega(q^2) + \epsilon(k)\frac{1}{2} \cdot \left(\sum_{n \in \mathbb{Z}} q^{n(n+1)}\right) \cdot \prod_{n \geq 0}(1 + q^{2n}) = \epsilon(k) \sum_{n \geq 0}(\epsilon(k)q)^n(-\epsilon(k)q; q^2)_n. \tag{1.31}$$

Note that the right-hand side is a finite sum for any such ζ. The proof uses a formula of Ramanujan and Watson, together with a result of Fine on hypergeometric series. Finally, we numerically illustrate this result. Define $W(q) := q\omega(q^2)$, $M(q) := \frac{1}{2} \cdot \left(\sum_{n \in \mathbb{Z}} q^{n(n+1)}\right) \cdot \prod_{n \geq 0}(1 + q^{2n})$, and $N(q) := -\sum_{n \geq 0}(-q)^n(q; q^2)_n$. Then for $\zeta_5 := e^{2\pi i/5}$, we consider Table 1.3 above.

These results will be proven in Chapter 5.

Chapter 2

Basic Facts from the Theory of Harmonic Maass Forms

2.1 Raising and Lowering Operators

In this section, we recall and sketch the proofs of some basic results from the theory of harmonic Maass forms. The proofs of these theorems follow those in [10]. Our first basic fact is that weakly holomorphic modular forms are harmonic Maass forms. To show this, we recall the Maass raising operator

$$R_k := 2i\frac{\partial}{\partial z} + ky^{-1}, \tag{2.1}$$

and the Maass lowering operator

$$L_k := -2iy^2\frac{\partial}{\partial \overline{z}}. \tag{2.2}$$

Our definition varies slightly from that found in, for example [12] and in particular L_k is independent of the weight as our definition shifts all the k-dependence into R_k. The terminology for these operators is apparent once we recall their key intertwining properties with the Petersson slash operator:

$$R_k(f|_k\gamma) = (R_kf)|_{k+2}\gamma, \tag{2.3}$$

$$L_k(f|_k\gamma) = (L_k f)|_{k-2}\gamma, \tag{2.4}$$

for any $\gamma \in \mathrm{SL}_2(\mathbb{R})$. Thus, the Maass raising operator takes a function which transforms like a modular form of weight k and gives a function which transforms like a modular form of weight $k + 2$, and the lowering operator lowers the weight by 2. To understand how these operators act on weak Maass forms, we need the following formulas.

Proposition 2.1. *Let f be an eigenfunction of Δ_k with eigenvalue λ. Then we have:*

1. $\Delta_{k+2} R_k f = (\lambda + k) R_k f$

2. $\Delta_{k-2} L_k f = (\lambda - k - 2) L_k f$.

Thus, the lowering and raising operators take weak Maass forms to weak Maass forms with shifted weight and eigenvalue. Now we can show the following.

Proposition 2.2. *For any $k \in \frac{1}{2}\mathbb{Z}$, we have that $M_k^! \subseteq H_k$.*

Proof. It suffices to show that an analytic function on $\mathbb{H}$ is harmonic, as the modularity transformation condition is the same and the growth condition is clear for weakly holomorphic modular forms as these have finitely many negative powers of q in their Fourier expansion at every cusp. To show this, we recall a further identity which expresses Δ_k in terms of the raising and lowering operators

$$\Delta_k = -R_{k-2} L_k.$$

But it is clear from the definition of L_k that this kills analytic functions. $\square$

2.2 Fourier Expansions

Perhaps the most useful feature of classical modular forms is their Fourier expansion. As nonanalytic functions, harmonic Maass forms may not have

q-series expansions, however they still have a canonical decomposition into two pieces, one of which is a q-series as for weakly holomorphic forms, and another Fourier expansion in terms of nonholomorphic functions.

Theorem 2.3 (Bruinier-Funke [10]). *If $f \in H_{2-k}(\Gamma_1(N))$ for $0 < N \in \mathbb{Z}$ and $2 - k > 0$, then f has a Fourier expansion*

$$f(z) = \sum_{n \gg -\infty} c^+(n)q^n + \sum_{n<0} c^-(n)\Gamma(k-1, 4\pi|y|n)q^n. \qquad (2.5)$$

Here $\Gamma(a; x) := \int_x^\infty e^{-t}t^{a-1}dt$ is the incomplete-gamma function. For $2-k \leq 0$, the proof will show that the same result is true but for a shifted argument inside the incomplete-gamma function. However, for our restricted growth condition, we will soon see that there are no non-holomorphic harmonic Maass forms of such weight, as there are no cusp forms of nonpositive weight.

The first sum in the Fourier expansion of a harmonic Maass form is called the *holomorphic part*, and is denoted f^+, and the second sum involving incomplete Gamma functions is called the *non-holomorphic part* and denoted f^-.

Proof. As the translation matrix $\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right) \in \Gamma_1(N)$ and any harmonic function is real-analytic, a harmonic Maass form f has a Fourier expansion of the form

$$f(z) = \sum_{n \in \mathbb{Z}} c(n, y)e(nx),$$

where $y = \Im z$ and $e(x) := e^{2\pi ix}$. As f is a harmonic function and functions are uniquely determined by their Fourier series, we have that for all n,

$$\Delta_k c(n, y)e(nx) = 0.$$

This is a second order differential equation, so to find the solution space we only need to write down two independent solutions. For $n = 0$, the equation

is
$$-y^2 \frac{\partial^2}{\partial y^2} c(0, y) - k y \frac{\partial}{\partial y} c(0, y),$$

and it is easily checked that two independent solutions are given by 1 and y^{1-k}. For $n \neq 0$, we set $c(n, y) = b(2\pi n y)$, and we find that $b(w)$ satisfies

$$\frac{\partial^2}{\partial w^2} b(w) + \frac{k}{w} \left(\frac{\partial}{\partial w} b(w) + b(w) \right) = 0. \tag{2.6}$$

One can check that e^{-w} is a particular solution, and a second, independent, solution is given by

$$H(w) = e^{-w} \int_{-2w}^{\infty} e^{-t} t^{-k} dt. \tag{2.7}$$

This integral converges for $k < 1$, but it can be analytically continued to $\mathbb{C}$ in a similar fashion as the Γ-function. We note that for $w < 0$, which is the case we are interested in, that

$$H(w) = \Gamma(1 - k, -2w).$$

Thus, we see that

$$c(n, y) = \begin{cases} c^+(0) + c^-(0) y^{1-k} & \text{if } n = 0 \\ c^+(n) e^{-2piy} + c^-(n) H(2\pi n y) e(nx). \end{cases} \tag{2.8}$$

Finally, we need to show that the correct Fourier coefficients vanish as per our theorem, namely that $c^+(n) = 0$ for all but finitely many n and that $c^-(n) = 0$ for $n \geq 0$. For the first condition, note that the growth condition (3) in our definition of Maass forms implies there can only be finitely many negative powers of q. In the non-holomorphic piece, we remark that by realizing $H(w)$ as a specialization of a Whittaker function, standard formulas on special functions imply the asymptotic

$$H(w) \sim (-2w)^{-k} e^w$$

as $w \to \infty$. Thus, the growth condition (3) again implies that the coefficients in the nonholomorphic part are all negative (although in the relaxed growth condition of Bruinier and Funke, there can be finitely many nonnegative terms).

$\square$

2.3 The ξ-operator and "Shadows"

Our next main result concerns the Fourier expansion of the nonholomorphic part, which shows that it is essentially the Fourier expansion of a cusp form of "dual weight". To describe this, we require the ξ-operator

$$\xi_k := 2iy^k \overline{\frac{\partial}{\partial \bar{z}}} \tag{2.9}$$

We note that a harmonic Maass form is killed by ξ_k if and only if $f \in M_k^!$. Remarkably, the ξ-operator sends Maass forms to modular forms of dual weight.

Theorem 2.4. *We have that*

$$\xi_{2-k} \, H_{2-k} \twoheadrightarrow S_k$$

Proof. We begin by computing the action of L_k on the Fourier expansion of a harmonic Maass form.

Lemma 2.5. *Let $f \in H_k$ have a Fourier expansion as in 2.3. Then*

$$L_k f = L_k f^- = \sum_{n<0} c^-(n)(-4\pi n)^{1-k} e(n\bar{z}).$$

Proof. The first equality which states that $L_k f^+ = 0$ is clear. For the second fact, we plug in the Fourier expansion. Note that

$$\frac{\partial}{\partial \bar{z}} = \frac{1}{2} \cdot \left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y} \right),$$

so that

$$\frac{\partial}{\partial \bar{z}} \Gamma(k-1, 4\pi|n|y)q^n = \frac{i}{2}\frac{\partial}{\partial y}\int_{4\pi|n|y}^{\infty} e^{-t}t^{-k}dt.$$

Using the Fundamental Theorem of Calculus and plugging in gives the result.

□

To prove that $\xi_k f$ is a cusp form, we first note that it is a holomorphic function on $\mathbb{H}$ as $\xi_k = y^{k-2}\overline{L_k}$ and the powers of y cancel to give an ordinary q-series. Specifically, we find that

$$\xi_{2-k}f^-(z) = -(4\pi)^{k-1}\sum_{n\geq 1}\overline{c^-(-n)}n^{k-1}q^n. \tag{2.10}$$

The modular transformation properties are routine to check, so we leave them to the reader. To show, surjectivity of the map ξ_{2-k} onto S_k, Bruinier and Funke use a geometric argument, or alternatively the theory of Poincaré series can be used to construct preimages.

□

Following Zagier, we call the image $\xi_k f$ the *shadow* of f. The computation in (2.10) allows us to realize the nonholomorphic part of f as a period integral of its shadow. Namely, a simple change of variables $\tau \to \tau - z$ shows that we have the following integral identity.

$$\int_{-\bar{z}}^{i\infty}\frac{e^{2\pi in\tau}}{(-i(\tau+z))^{2-k}}d\tau = i(2\pi n)^{1-k}\Gamma(k-1, 4\pi ny)q^{-n}. \tag{2.11}$$

Plugging the Fourier expansion for f^- into this formula, we find

$$f^-(z) = -\int_{-\bar{z}}^{i\infty}\frac{\xi_{2-k}f}{(-i(\tau+z))^{2-k}}d\tau. \tag{2.12}$$

This realization played an important role in Zwegers' work on Ramanujan's mock theta functions [42, 43], where we will see that the shadows of the mock theta functions are weight $3/2$ unary theta series.

2.4 The Bruinier-Funke Pairing

We conclude this section with a proof of a useful fact by Bruinier and Funke that harmonic Maass forms which are not holomorphic modular forms have an exponential singularity at some cusp.

Theorem 2.6 (Bruinier-Funke [10])**.** *Let f be a harmonic Maass form of weight k such that $f^- \neq 0$. Then f has a nonzero principal part at some cusp.*

Proof. The proof relies on an important paring defined by Bruinier and Funke. We first recall the Petersson inner product $(\cdot, \cdot)\, M_k \times S_k \to \mathbb{C}$ defined by

$$(f, g) := \int_{\Gamma \backslash \mathbb{H}} f(z) \overline{g(z)} y^k \frac{dx\, dy}{y^2}. \tag{2.13}$$

We then define the a pairing $\{\cdot, \cdot\}\, M_k \times H_{2-k} \to \mathbb{C}$ using the ξ-operator:

$$\{g, f\} := (g, \xi_{2-k}). \tag{2.14}$$

The idea is to consider $\{\xi_{2-k} f, f\} = (\xi_{2-k} f, \xi_{2-k} f)$, which is non-zero as $f^- \neq 0$ implies that $\xi_{2-k} f$ is a non-zero cusp form. The result then follows from an alternative expression for $\{\cdot, \cdot\}$ in terms of the Fourier expansions of f and g.

Proposition 2.7. *If h runs over all the cusps, and the nth Fourier coefficient of g at a cusp h is $b(h, n)$, and the nth Fourier coefficient of f^+ at a cusp h is $a^+(h, n)$, we have that*

$$\{g, f\} = \sum_h \sum_{n \leq 0} a^+(h, n) b(h, -n).$$

Proof. We consider the usual fundamental domain $\mathcal{F}$ for $\Gamma = \mathrm{SL}_2(\mathbb{Z})$, and we also consider the truncated fundamental domain:

$$\mathcal{F}_T := \{z \in \mathcal{F} \colon \Im z \leq T\}.$$

By a simple calculation, using Stokes' theorem we find that

$$\int_{\mathcal{F}_T} g(z) L_k f(z) \frac{dx\,dy}{y^2} = -\int_{\partial \mathcal{F}_T} g(z) f(z) dz.$$

By the usual relation between L_k and ξ_k, we see that the right-hand side is just $\int_{\mathcal{F}_T} g(z)\overline{\xi_{2-k} f(x)} y^k \frac{dx\,dy}{y^2}$. For the right-hand side, as f and g have dual weights, we have that $g(z)f(z)dz$ is an invariant 1-form, so that its integral on all the pieces of $\mathcal{F}_T$ except the top line from $-\frac{1}{2}+iT$ to $\frac{1}{2}+iT$ cancel out. Hence, the right-hand side is equal to

$$\int_{-\frac{1}{2}}^{\frac{1}{2}} g(x+iT) f(x+iT) dx.$$

By our growth assumption in our definition of harmonic Maass forms, and as for a q series such an integral just picks out the constant term, we have that this integral is equal to

$$\sum_h \sum_n a^+(h,n) b(h,-n) + O(e^{-CT}).$$

Taking the limit $T \to \infty$ gives the result.

□

□

Chapter 3

Integrality Properties of Singular Moduli

In this chapter, we prove Theorem 1.12, which is joint work with Michael Griffin in [22]. The chapter is organized as follows. In §3.1 we use the Maass lowering and raising operators to prove our "spectral decomposition". This has the effect of relating our problem to the study of *traces* of singular moduli for negative weight modular forms. In §3.2, we recall the important work of Duke-Jenkins on integrality for traces of modular forms of negative weight. In §3.3, we use Rankin-Cohen brackets and a basic binomial sum identity to study which forms in the decomposition actually appear. In §3.6 we use the theory of Poincaré series and develop a p-adic theory using special families of Hecke operators to deal with cusp forms arising in the "dual weight" and take care of complicated denominators introduced in the spectral decomposition of §3.1.

Thanks to *Newton's identities*, one can express elementary symmetric functions in terms of sums of powers and vice versa. We recall that if $e_k(x_1, \ldots, x_n)$ is the usual elementary symmetric polynomial in $x_1, \ldots, x_n$ and

$p_k(x_1, \ldots, x_n) := \sum_{i=1}^{n} x_i^k$ is the kth *power sum*, then the Newton-Girard for-

mulae state that

$$ke_k(x_1, \ldots, x_n) = \sum_{i=1}^{k} (-1)^{i-1} e_{k-i}(x_1, \ldots, x_n) p_i(x_1, \ldots, x_n). \qquad (3.1)$$

Thus, our problem is reduced to the of study of traces of singular moduli for powers of Maass forms. Unfortunately, products and powers of Maass forms typically are not Maass forms, or even finite sums of Maass forms. However we are only studying a special class of Maass forms which are simply derivatives of modular forms. In this case, the problem has a straightforward solution.

We note that products and sums of raisings of modular forms must be in the kernal of a finite power of L. This allows us to decompose such forms as per the following theorem. This theorem is originally due to Shimura (see [36] Proposition 3.4, or [23] Section 10.1), however we give a short proof which gives explicit components which we will need later.

Theorem 3.1 (Shimura [36])**.** *Suppose F smooth function n $\mathbb{H}$ which is in the kernal of L^{E+1} (and not in the kernal of L^E). Then there exist uniquely determined modular forms $g_j \in M_{k-2j}^!$ such that*

$$F = \sum_{j=0}^{E} R^j g_j. \qquad (3.2)$$

3.1   The Spectral Decomposition

Here we prove Theorem 3.1.

Proof. Since $L^{E+1}F = 0$, we have that $L^E F$ is weakly holomorphic. We define the g_i recursively beginning with g_E. Let

$$g_E = \frac{L^E F}{c_{E,E}},$$

and for each i with $0 \le i < E$, let

$$g_i := \frac{1}{c_{i,i}} \left(L^i F - \sum_{j=i+1}^{E} c_{i,j} R^{j-i} g_j \right), \tag{3.3}$$

where

$$c_{i,j} := \frac{j!(-k+j+i)!}{(j-i)!(-k+j)!}. \tag{3.4}$$

By assumption, $k \le 0$, so $c_{i,j}$ is defined for all $j \ge i$. Note that each g_i is modular of weight $k - 2i$. By rearranging the definition of g_0, we see that $F = \sum_{i=0}^{E} R_i g_i$. Therefore we need only prove that each g_i is weakly holomorphic. We do so by inductively showing that $Lg_i = 0$ for each g_i.

We have already seen that $Lg_E = 0$. Suppose that $i < E$ is fixed, and that it is known that g_j is weakly holomorphic for each $i < j \le E$. By construction, we have that for any fixed i,

$$L^i F = \sum_{j=i}^{E} c_{i,j} R^{j-i} g_j.$$

Applying the lowering operator gives

$$L^{i+1} F = \sum_{j=i}^{E} c_{i,j} L R^{j-i} g_j = \sum_{j=i+1}^{E} c_{i+1,j} R^{j-i-1} g_j.$$

This rearranges to

$$c_{i,i} Lg_i = \sum_{j=i+1}^{E} \left(c_{i+1,j} R^{j-i-1} g_j - c_{i,j} (LR) R^{j-i-1} g_j \right).$$

Since for $j > i$ we have that g_i is weakly holomorphic, a short calculation shows that

$$(LR) R^{j-i-1} g_j = (j-i)(-k+j+i+1) R^{j-i-1} g_j.$$

Hence we have that

$$c_{i,i} Lg_i = \sum_{j=i+1}^{E} \left(c_{i+1,j} - c_{i,j}(j-i)(-k+j+i+1) \right) R^{j-i-1} g_j.$$

By the definition of the $c_{i,j}$, we see that this sum is zero. $\qquad \square$

3.2 Integrality Results of Duke and Jenkins

In this section we describe the important work of Duke and Jenkins on integrality of traces of singular moduli in [14]. In particular, their results allow us to predict the correct denominators of the traces of singular moduli of each summand arising in the relevant case of Theorem 3.1 in our spectral decomposition. Following their paper, consider any $f = \sum_{n \gg -\infty} a(n)q^n \in M^!_{2-2s}$ for $1 \leq s \in \mathbb{Z}$. For convenience, set

$$\widehat{s} := \begin{cases} s & \text{if } (-1)^s D > 0 \\ 1 - s & \text{otherwise.} \end{cases} \tag{3.5}$$

They also let

$$\mathrm{Tr}^*_{d,D}(f) := (-1)^{\lfloor \frac{\widehat{s}-1}{2} \rfloor} |d|^{\frac{-\widehat{s}}{2}} |D|^{\frac{\widehat{s}-1}{2}} \mathrm{Tr}_{d,D}((-1)^{s-1} \partial f). \tag{3.6}$$

Then for any fundamental discriminant D, they define the Dth *Zagier lift* of f to be:

$$\mathbf{3}_D(f) := \sum_{m>0} a(-m)m^{s-\widehat{s}} \sum_{n|m} \chi_D(n)n^{\widehat{s}-1} q^{-\frac{m^2|D|}{n^2}} + \frac{1}{2}L(1-s,\chi_D)a(0)$$
$$+ \sum_{dD<0} \mathrm{Tr}^*_{d,D}(f)q^{|d|}. \tag{3.7}$$

Their main theorem states that the linear map $\mathbf{3}(\cdot)$ preserves integrality of Fourier coefficients. More specifically, they show the following.

Theorem (Duke-Jenkins [14]). *Suppose that $f \in M^!_{2-2s}$ for an integer $s \geq 2$. If D is a fundamental discriminant with $(-1)^s D > 0$, then we have that $\mathbf{3}_D(f) \in M^!_{3/2-s}$, whereas if $(-1)^s D < 0$, then $\mathbf{3}_D(f) \in M^!_{s+1/2}$. Furthermore, if f has integral Fourier coefficients, so does $\mathbf{3}_D(f)$.*

This builds on Zagier's original work for $s = 1$. In that case the theorem and (3.7) hold, as long as the constant term of f is 0. Note that $\mathrm{Tr}_{d,1}(1)$ is the Hurwitz-Kronecker class number for d. It is well known that this is integral for d fundamental, unless $d = -3$ or -4. In these cases we have that $\mathrm{Tr}_{-3,1}(1) = \frac{1}{3}$ and $\mathrm{Tr}_{-4,1}(1) = \frac{1}{2}$. Therefore for the remainder of this paper we will assume that the constant term of the weight 0 component in future decompositions is zero.

3.3　A Useful Vanishing Condition

Using the Newton identities, Theorem 3.1, and the theorem of Duke and Jenkins, one can prove a bound on the denominators of symmetric functions in our singular moduli by bounding each summand in Theorem 3.1 individually. This falls short of Theorem 1.12. There are two obstructions. Firstly, the bounds for the denominators given by Duke-Jenkins applied to forms of certain weights in the allowed range exceed the bound given in Theorem 1.12. Secondly, the coefficients $c_{i,j}$ which appear in Theorem 3.1 could potentially contribute to the denominators. Here we address the first obstruction by showing that forms of certain "bad weight" in the decomposition must be identically zero. When $k = 0$, there are no bad weights. Otherwise, we make the following definition:

Definition 1. *Let k be a negative, even integer and $n \in \mathbb{N}$. Then we say that m is a* bad weight *for the pair (k, n) if m is an integer of the form $kn + 4i + 2$ for $0 \leq i \leq -\frac{k}{2} - 1$.*

We have the following theorem:

Theorem 3.2. *Let $f \in M_k^!$ and consider the product $F = (\partial f)^n$. Write $F = \sum \partial(g_i)$, where each g_i is a weakly holomorphic modular form as in Theorem 3.1. Then if m is a bad weight for k, n, we have that $g_m \equiv 0$.*

This theorem arises from a combinatorial fact which can be applied to arbitrary power series. In the next subsection, we define a family of bilinear operators on modular forms known as the *Rankin-Cohen brackets*. These will provide a convenient basis for expressing the combinatorics of the spectral decomposition of the product of two forms.

3.3.1   Rankin-Cohen Brackets

In [33] and [13], Rankin and Cohen studied the general question of which polynomials in derivatives of two modular forms are again modular. In doing so, they described the so-called *Rankin-Cohen Brackets*, which are defined as follows. Let f be a modular form of weight k, g a modular form of weight ℓ, and n a non-negative integer. Then the nth Rankin-Cohen bracket is defined as:

$$[f,g]_n^{(k,\ell)} := \sum_{r+s=n} (-1)^r \binom{n+k-1}{s} \binom{n+\ell-1}{r} f^{(r)} \cdot g^{(s)}. \qquad (3.8)$$

Here $f^{(n)} := \left(\frac{1}{2\pi i} \frac{d}{dz}\right)^n f$. We will also suppress the dependence on the weights k and ℓ and write simply $[f,g]_n$ when the dependence is clear from context. The key fact is that $[f,g]_n$ gives a map:

$$[\cdot,\cdot]_n^{(k),(\ell)} : \ M_k^! \otimes M_\ell^! \to M_{k+\ell+2n}^!. \qquad (3.9)$$

This can be seen, for example, using the Cohen-Kuznetsov lifting to Jacobi-like forms. Moreover, it can be shown that these operators are essentially the *unique* universal bilinear differential operators between these spaces of modular forms. That is, given a polynomial in the derivatives of f and g of degree at most 2 which is modular of weight $k+\ell+2n$, it must be a multiple of $[f,g]_n$.

3.4 The Vanishing Lemma

Using the Rankin-Cohen brackets as defined above, we are now in position to prove Theorem 3.2. Using an inductive argument and the spectral decomposition of Theorem 3.1, it suffices to prove the following lemma for the product of raisings of just two forms of possibly different weights.

Lemma 3.3. *Let $f \in M_k^!$ and $g \in M_\ell^!$ have negative even weight. Set $F := \partial f \cdot \partial g$ and write $F = \sum \partial(g_i)$ for the modular forms g_i defined in Theorem 3.1. Suppose $m = k + \ell + 4i + 2$ where $0 \leq i \leq -\frac{\min\{k,\ell\}}{2} - 1$. Then if g_i has weight m, we have that $g_i \equiv 0$.*

Proof. In Proposition 2.3 of [5], the authors consider a similar combinatorial expansion which is given in terms of Rankin-Cohen brackets. Using their proposition, it suffices to prove the following (setting $k = -2r$, $\ell = -2s$) whenever $j < r$ and j is odd:

$$S(j) := \sum_{m=0}^{s} (-1)^{(j+m)} \cdot \frac{\binom{m+r}{j}\binom{s}{m}\binom{m-r-1}{r+m-j}}{\binom{-r-2s+m+j-1}{m+r-j}} = 0. \qquad (3.10)$$

Using the Wilf-Zeilberger method [32], one finds that the function $S(j)$ satisfies the following recursion in the range $j < r$:

$$(2+j)(1+j-2r)(1+j-2s)(j-2r-2s) \cdot S(j+2)$$

$$-4(1+2j-2r-2s)(3+2j-2r-2s)(j-r-s)(1+j-r-s) \cdot S(j) = 0.$$

For the base case, $j = 1$, we must show that g_{E-1} vanishes in the notation of Theorem 3.1. A calculation shows that $L^{E-1}[(\partial f) \cdot (\partial g)]$ is some nonzero multiple of $R(f \cdot g)$, so that by (3.2), we have that g_{E-1} is a multiple of Rg_E. But g_{E-1} is holomorphic, whereas Rg_E is not, so in fact we must have $g_{E-1} \equiv 0$. Alternatively, the identity can be proven automatically for $j = 1$ with another application of the Wilf-Zeilberger method. $\qquad \square$

3.5 Hecke Structure of the Zagier Lift, and a Special Family of Operators.

In this section we will complete the proof of Theorem 1.12. We will need to define a notion of integrality for coefficients of the nonholomorphic modular forms we have been studying. We say that a nonholomorphic modular form F of integral weight (resp. half-integral weight) of level 1 (resp. level 4) has integral coefficients if $F \in \mathbb{Z}((q)) \left[\frac{1}{4\pi y}\right]$ $\left(\text{resp. } F \in \mathbb{Z}((q)) \left[\frac{1}{16\pi y}\right]\right)$. We may also refer to the coefficient of $\left(\frac{1}{4\pi y}\right)^0$ as the holomorphic part of F. We define rational coefficients for such forms similarly. Given this notation, we have the following:

Theorem 3.4. *Suppose*

$$F = \sum_{n=0}^{N} R^n f_{-2n}$$

has integral coefficients, where $f_{-2n} \in M_{-2n}^!$ If $d < 0$ is a fundamental discriminant, which is not divisible by any bad primes for the pair (k, n), then $d^e \operatorname{Tr}_{d,1}(F) \in \mathbb{Z}$, where $e = \min\{\ell \mid f_{-4\ell} \not\equiv 0\}$ or 0 if no such ℓ exist.

The proof of this theorem, requires an analog of the Zagier lift for the function F. This may be done by taking the Zagier lift of each of the f_{-2n} and combining them in an appropriate way. Let $e_1 = \max\{0\} \cup \{\ell \mid f_{-4\ell} \not\equiv 0\}$, and $e_2 = \max\{0\} \cup \{\ell \mid f_{2-4\ell} \not\equiv 0\}$. Then we define

$$\mathcal{Z}_D(F) := \sum R^{i+e_1} \mathfrak{Z}_D(f_{2-4i}) + \sum R^{e_1-i} \mathfrak{Z}_D(f_{-4i}), \tag{3.11}$$

which has weight $\frac{3}{2} + 2e_1$. If d is a negative fundamental discriminant, then the coefficient of $q^{|d|}$ in $\mathcal{Z}_1(F)$ is

$$(-1)^{\left\lfloor \frac{\hat{s}-1}{2} \right\rfloor} |d|^{-\frac{\hat{s}}{2}} \operatorname{Tr}_{d,1}(F), \tag{3.12}$$

where $\widehat{s} = -2e_1$. Thus, Theorem 3.4 is equivalent to the statement that the holomorphic part of $\mathcal{Z}_1(F)$ has integral coefficients. Note that, up to a sign, the coefficient of q^d in $\mathcal{Z}_1(F)$ is the coefficient of q^1 in $\mathcal{Z}_d(F)$. We prove this in two parts using the following lemmas:

Lemma 3.5. *Suppose F is a nonholomorphic modular function with rational coefficients and whose holomorphic part has p-integral principal part for some good prime p. Then F has p-integral coefficients.*

As noted in the remark following Theorem 1.12, we need only worry about the case where $e_2 > e_1$. Therefore, we can conclude the proof with the following:

Proposition 3.6. *Suppose F has integral coefficients for a good prime p and a decomposition with $e_2 > e_1$. Then for any negative discriminant d, the principal part of $\mathcal{Z}_d(F)$ is p-integral.*

In the following subsection, we set-up the structure of the Hecke algebra acting on Zagier lifts of certain families of Poincaré series in preparation for the proof of Lemma 3.5. This is used in Section 3.6 to construct a certain family of special family of Hecke operators. The final sections complete the proofs of Lemma 3.5 and Proposition 3.6 respectively.

3.6 Poincaré Series and the Hecke Structure of Zagier Lifts

The work in this paper relies heavily on the construction of the Zagier lift given by Duke and Jenkins in their proof of integrality of Zagier lifts mentioned above [14]. Their original construction of the lift is given as explicit linear combinations of Maass-Poincaré series, which are then expressed via the Hecke algebra.

The Maass-Poincaré series we need were studied by Fay [15] in the integral weight case, and by Bringmann and Ono [7] in the half-integral weight case. In the half-integral weight case, we restrict our attention to series in the Kohnen plus-space. That is, we assume they are supported on coefficients q^n where $n \equiv 0$, or $(-1)^{k+\frac{1}{2}} \pmod 4$. For each $k \in \frac{1}{2} + \mathbb{Z}$, Let $f_{k,m}$ be the unique Maass-Poincaré series in one of these families, of weight k and of the form $q^{-m} + O(1)$ if $k < 0$ or $\frac{3}{2}$ or of the form $q^{-m} + O(q)$ otherwise. The main results we need from these works may be summarized as follows:

Proposition 3.7. *If M_k is empty, then any form $f \in M_k^!$ may be expressed as a linear combination of the (harmonic) Poincaré series $\{f_{k,m}\}$. If S_{2-k} is empty, $\{f_{k,m}\} \subset M_k^!$*

Here, if $k \in \frac{1}{2} + \mathbb{Z}$, then $M_k^!$ denotes the subspace of $M_k^!(\Gamma_0(4))$ in the Kohnen plus-space. The Hecke operators preserve theses spaces of Maass-Poincaré series. Since $\{f_k\}$ spans $M_k^!$ for negative k, we have the following proposition.

Proposition 3.8. *Let k be a negative even integer, and let $F = \sum\limits_{n \in \mathbb{Z}} a(n)q^n \in M_k^!$. Then we have that*

$$F = \sum_{n<0} a(n)n^{1-k} f_{k,1}|_k T(n). \tag{3.13}$$

The same is true if $k = 0$ and F has no constant term. The n^{k-1} in the formula above is necessary to clear denominators introduced by the standard Hecke operator on non-positive weights.

Proposition 3.8 does not extend directly to half-integral weights. In particular, the Hecke operators with non-square index map half-integral weight forms identically to 0. The action of $T(p^2)$ on half-integral forms is given by

$$F|_k T(p^2) = F|U(p^2) + \left(\frac{(-1)^{k-\frac{1}{2}}}{p}\right) p^{k-\frac{3}{2}} F + p^{2k-2} F|V(p^2). \tag{3.14}$$

(In the case $p = 2$, we have taken an involution of the standard Hecke operator which would just be U_4 as we are working in level 4).

Duke and Jenkins showed that if d is a fundamental discriminant and k a negative even integer, then

$$\mathbf{3}_d(f_{k,1}) = f_{\widehat{k},d}, \tag{3.15}$$

where $\widehat{k} := \frac{k+1}{2}$ or $\frac{-k+3}{2}$. They also show that the Zagier lift commutes with the Hecke algebra. More precisely, we have

$$\mathbf{3}_D\left(n^{1-k}f|_k T(n)\right) = n^{1-k}(\mathbf{3}_D f)|_{\frac{k+1}{2}} T\left(n^2\right) \tag{3.16}$$

or

$$\mathbf{3}_D\left(n^{1-k}f|_k T(n)\right) = (\mathbf{3}_D f)|_{\frac{-k+3}{2}} T\left(n^2\right), \tag{3.17}$$

depending on the sign of D.

To ease notation, we introduce a few conventions. Suppose H is a Hecke operator

$$H = \sum_n a_n T(n). \tag{3.18}$$

Then we define

$$\widehat{H} := \sum_n a_n T(n^2). \tag{3.19}$$

$$H^{(m)} := \sum_n a_n n^{-m} T(n). \tag{3.20}$$

This notation is chosen so that if m is a positive integer, then

$$D^m(F|_k H) = D^m(F)|_{k+2m} H^{(m)}. \tag{3.21}$$

In this notation, we do not require m to be a positive integer. For convenience, if the weight k is clear from context, then we also define $H^* := H^{(2-k)}$.

Then (3.16) and (3.17) may be extended to compound Hecke operators as follows:

$$\mathbf{3}_D\left(f|_k H\right) = (\mathbf{3}_D f)|_{\frac{k+1}{2}} \widehat{H} \tag{3.22}$$

or

$$\mathbf{3}_D\left(f|_k H\right) = (\mathbf{3}_D f)|_{\frac{-k+3}{2}} \widehat{H}^*, \tag{3.23}$$

again depending on the sign of D.

We will require the following theorem and corollary, which is proved using standard techniques in the theory of Maass forms.

Theorem 3.9. *Let k be a negative even integer. If H is a compound Hecke operator, then $f_{k,1}|_k H \in M_k^!$ if and only if $S|_{2-k}H^* = 0$ for every $S \in S_{2-k}$.*

Corollary 3.10. *If k and H are as in Theorem 3.11, then $f_{\frac{k+1}{2},m}|_{\frac{k+1}{2}} H \in M_{\frac{k+1}{2}}^!$ for all $m \equiv 0$, or $1 \pmod 4$ and $S_{\frac{-k+3}{2}}$ is in the kernel of $\widehat{H}^*$.*

Proof. For the proof, we require the ξ-operator which is given by

$$\xi_k := 2iy^k \overline{\frac{\partial}{\partial \overline{z}}} \tag{3.24}$$

(see Section 7.3 of [30]). As before, we drop the dependence on k when it is clear from context. We recall that a harmonic Maass form is actually weakly holomorphic if and only if it is in the kernel of the ξ-operator. It is not difficult to see that the Hecke operator H commutes with the period integral, becoming H^*. Therefore $f_{k,1}|_k H \in M_k^!$ if and only if $\xi f_{k,1}|_{2-k}H^* = 0$. Moreover, $\{\xi f_{k,n}\}$ spans S_{2-k}, and $\xi(f_{k,n})|H^* = \xi(f_{k,1})|T(n)H^*$. Since Hecke operators commute, this is 0 if $\xi(f_{k,1})|H^* = 0$.

The proof of the corollary is similar, however we first note that from Duke and Jenkins we have that any Zagier lift takes weakly holomorphic forms

to weakly holomorphic forms. Thus $f_{\frac{k+1}{2},D}|_{\frac{k+1}{2}}\widehat{H} \in M^{!}_{\frac{k+1}{2}}$. This gives us $\xi(f_{\frac{k+1}{2},D})|_{\frac{-k+3}{2}}\widehat{H}^* = 0$ for all D of appropriate sign. As these span $S_{\frac{-k+3}{2}}$, we are finished. □

We will use this theorem and corollary in the following section to describe a p-adic structure for Poincaré series under the action of certain Hecke operators.

3.7   A Special Family of Hecke Operators

In this section, we construct a family of Hecke operators and demonstrate some basic properties. These will allow us to prove Theorem 3.5 by studying the p-adic properties of forms coming from Poincaré series via the Hecke algebra.

Theorem 3.11. *Let $k \in \frac{1}{2}\mathbb{Z}$ and let p be ordinary for all eigenforms in a basis of S_k. Then there is a compound Hecke operator $\mathcal{H}_{k,N}$ such that $f_{2-k,1}|\mathcal{H}_{k,N}$ is weakly holomorphic with integral coefficients, and $f_{k,1}|\mathcal{H}^*_{k,N}$ has integral coefficients with $f_{k,1}|\mathcal{H}_{k,N} \equiv q^{-1} + O(q) \pmod{p^N}$. Moreover, any such $\mathcal{H}_{k,N}$ satisfies the following properties:*

1. *If H is a compound Hecke operator such that $f_{2-k,1}|H^*$ is weakly holomorphic and $f_{k,1}|H$ has integer coefficients, then*
$$(f_{k,1}|\mathcal{H}_{k,N})\,|H \equiv f_{k,1}|H \pmod{p^N}.$$

2. *If $\mathcal{H}_{k,N}$ and $\mathcal{H}_{k,N'}$ are two such operators, then*
$$f_{k,1}|\mathcal{H}_{k,N} \equiv f_{k,1}|\mathcal{H}_{k,N'} \pmod{p^N}.$$

3. *If H is as above and $(f_{k,1}|\mathcal{H}_{k,N})\,|H \equiv 0 + O(q) \pmod{p^M}$ for some $M \leq N$, then*
$$(f_{k,1}|\mathcal{H}_{k,N})\,|H \equiv 0 \pmod{p^M}.$$

For our purposes, we are especially interested in the following:

Corollary 3.12. *If H is a compound Hecke operator such that $f_{k,1}|H$ has integer coefficients, p is ordinary for all eigenforms in a basis of S_k, and $f_{k,1}|H \equiv 0 + O(q) \pmod{p^N}$, then $f_{k,1}|H \equiv 0 \pmod{p^N}$*

As before, when k is clear from context, we may omit it from the notation, and write $\mathcal{H}_N$.

Construction of the Hecke operator. Let k be a fixed, positive, even integer. If S_k is empty, we can let $\mathcal{H}_N = T(1)$. Otherwise, let ℓ be the dimension of S_k. Let $F_{k,m}$ be the elements of the Miller basis for S_k, so that $F_{k,m}$ is the unique cusp form in S_k with Fourier expansion

$$F_{k,m} = q^m + \sum_{n > \ell} c_k(m, n) q^n$$

where $c_k(m, n)$ are integers. We represent this form by the m-th elementary column vector b_m which has a 1 in the mth row, and all other entries are 0. The Hecke operators act on S_k linearly with trivial kernel, so we may represent the action of $T_k(N)$ by left multiplication by a matrix $\mathbb{T}_N$.

Without loss of generality, assume that $p^n > \ell$. We then define the matrix

$$C_n := \begin{pmatrix} c_k(1, p^n) & c_k(2, p^n) & \dots & c_k(\ell, p^n) \\ c_k(1, 2p^n) & c_k(2, 2p^n) & \dots & c_k(\ell, 2p^n) \\ \vdots & \vdots & & \vdots \\ c_k(1, \ell p^n) & c_k(2, \ell p^n) & \dots & c_k(\ell, \ell p^n) \end{pmatrix}. \tag{3.25}$$

Considering the action of $\mathbb{T}_{p^n}$ on the identity, which represents the elements of the basis, we see that $\mathbb{T}_{p^n} \equiv C_n \pmod{p^{k-1}}$. The determinant of $\mathbb{T}_p$ is the product of the eigenvalues of $T_k(p)$, which by assumption, is not divisible p. Moreover, $\mathbb{T}_{p^n} \equiv \mathbb{T}_p^n \pmod{p^{k-1}}$, so $\mathbb{T}_{p^n}$, and hence C_n has an inverse with p-integral coefficients. Take $A \equiv C_n^{-1}$ such that A has integral coefficients.

By the Zagier duality studied by Duke and Jenkins [14], the rows of the matrix C_n give coefficients of the forms in an integral basis of $M_{2-k}^!$. The elements of this basis have the form

$$F_{2-k,m} = q^{-m} + \sum_{i=-\ell}^{\infty} c_{2-k}(m,i)q^{-i}, \tag{3.26}$$

where ℓ is as above. Using this notation, Zagier duality gives us that

$$c_{2-k}(a,b) = -c_k(b,-a). \tag{3.27}$$

If we now let C_n represent these modular forms of weight $2 - k$, and multiplication on the left by A to take linear combinations of these forms, we find see there is a form

$$\mathcal{F}_{k,n} = \sum_{m=0}^{\ell} \left(-a(m)q^{-p^n m} \right) - q^{-1} + O(1), \tag{3.28}$$

where $a(m)$ is the coefficient $a_{(1,m)}$ from A. As this is a integral linear combination of elements from an integral basis of $M_{2-k}^!$, we see that $\mathcal{F}_{k,n}$ has integral coefficients. We then take

$$\mathcal{H}_n = \sum_{m=0}^{\ell} \left(a(m)T(mp^n) \right) + T(1). \tag{3.29}$$

A short calculation shows that $f_{2-k,1}|\mathcal{H}_n^* = \mathcal{F}_{k,n}$, and that $f_{k,1}|\mathcal{H}_n = D^{k-1}(f_{2-k,1}|\mathcal{H}_n^*) \equiv q^{-1} + O(q) \pmod{p^n}$. Another short calculation shows that $\widehat{\mathcal{H}_N}$ satisfies the properties for dual weights $\frac{k+1}{2}$ and $\frac{3-k}{2}$. $\quad\square$

Proof of Property (1). We can write

$$f_{k,1}|\mathcal{H}_n = p^n F + f_{k,1} + s, \tag{3.30}$$

where F is a linear combination of elements from the integral basis. Thus by Theorem 3.9,

$$f_{k,1}|\mathcal{H}_n|H = p^n F|H + f_{1,k}|H \equiv f_{1,k}|H \pmod{p^n}. \tag{3.31}$$

Proof of Property (2). From Part (1), we have

$$f_{k,1}|\mathcal{H}_n \equiv f_{k,1}|\mathcal{H}_n|\mathcal{H}'_n = f_{k,1}|\mathcal{H}'_n|\mathcal{H}_n \equiv f_{k,1}|\mathcal{H}'_n. \tag{3.32}$$

Proof of Property (3) (and corollary). Suppose $f_{k,1}|H \equiv s$ where $s \in S^!_k$. Then,

$$s \equiv f_{k,1}|H \equiv f_{k,1}|\mathcal{H}_n|H = f|H|\mathcal{H}_n \equiv s|\mathcal{H}_n = 0. \tag{3.33}$$

3.8   Integrality of the Coefficients

We now prove Proposition 3.6, which is an immediate corollary of the following:

Proposition 3.13. *Suppose $F = \sum a(m,n)q^n \left(\frac{1}{4\pi y}\right)^m$ has a decomposition with $e_2 > e_1$. If $D > 0$, then the principal part of $\mathcal{Z}_D(F)$ (up to constant multiples of $\frac{1}{4\pi y}$) is given by*

$$\sum R^{e_2-\lceil m/2 \rceil}_{1/2+2\lceil m/2 \rceil} \left(q^{-1}\big|_{1/2-2\lfloor m/2 \rfloor} H_m \left(\frac{1}{16\pi y}\right)^m \right) \tag{3.34}$$

where $H_m = \sum_n a(m,n)n^{1-4\lfloor m/2 \rfloor}T(n^2)$.

Proof. The coefficients of the Hecke operators H_m above come from the coefficients of the form F, and thus are the sum of contributions from each piece in the decomposition of F. We consider the contribution from each piece independently, beginning with those of weight divisible by 4. Standard formulas for the iterated raising operator (for example see [11, Ch. 1]) give us that the contribution of f_{-4s} to F is

$$R^{2s}f_{-4s} = \sum_{j=0}^{2s} \binom{2s}{j} \frac{(2s+j)!}{(2s)!} \left(\frac{1}{4\pi y}\right)^j \mathcal{D}^{2s-j}f_{-4s}. \tag{3.35}$$

We have that $\widehat{f} = \mathbf{3}(f_{-4s})$ is weight $\frac{-4s+1}{2}$. When k is half-integral, the iterated raising operator can be written as

$$R^r \widehat{f} = \sum_{j=0}^{r} \binom{r}{j} \frac{\Gamma\left(\frac{-4s+1}{2} + r\right)}{\Gamma\left(\frac{-4s+1}{2} + r - j\right)} \left(\frac{-4}{16\pi y}\right)^j \mathcal{D}^{r-j}\widehat{f}. \qquad (3.36)$$

If we set $r = s+m$ and consider the coefficient $\left(\frac{1}{16\pi y}\right)^{2m}$ above, this reduces to

$$\frac{(s+m)!(2s+2m)!(s-m)!}{(2m)!(s-m)!(s+m)!(2s-2m)!} = \binom{2s}{2m} \frac{(2s+2m)!}{(2s)!}, \qquad (3.37)$$

which is the coefficient of $\left(\frac{1}{4\pi y}\right)^{2m}$ in (3.35). Similarly, the coefficient of $\left(\frac{1}{16\pi y}\right)^{2m+1}$ reduces to

$$\frac{(s+m)!(2s+2m+2)!(s-m)!}{(2m+1)!(s-m-1)!(s+m+1)!(2s-2m)!} = \binom{2s}{2m+1} \frac{(2s+2m+1)!}{(2s)!}, \qquad (3.38)$$

which is the coefficient of $\left(\frac{1}{4\pi y}\right)^{2m+1}$ in (3.35).

This, along with the action of the differential operator suffices to prove the contributions to (3.34) from the coefficients of forms of weight $-4s$. The contribution from the forms f_{2-4s} require more careful consideration. Note that

$$R^{2s-1} f_{2-4s} = \sum_{j=0}^{2s-1} \binom{2s-1}{j} \frac{(2s-1+j)!}{(2s-1)!} \left(\frac{-1}{4\pi y}\right)^j \mathcal{D}^{2s-1-j} f_{2-4s}. \qquad (3.39)$$

Let $\widehat{f} = \mathbf{3}_D(f_{2-4s})$, which is a modular form of weight $\frac{1+4s}{2}$. In order to describe the contribution from $\widehat{f}$ to (3.34), we define I to be an inverse operator to the derivative operator $\mathcal{D}$, whose action is on formal q-series $f = \sum_{n \neq 0} a_n q^n$, is given by

$$I(f) := \sum_{n \neq 0} a_n n^{-1} q^n. \qquad (3.40)$$

Using this notation, the contribution from the coefficients of $\widehat{f}$ to (3.34) is given by the principal part of

$$\sum_{m=0}^{2s-1} R_{1/2+2\lceil m/2\rceil}^{e_2-\lceil m/2\rceil}\left(a_m\left(I^{s+\lfloor m/2\rfloor}\widehat{f}(z)\right)\left(\frac{1}{16\pi y}\right)^m\right), \tag{3.41}$$

where $a_m := \begin{pmatrix} 2s-1 \\ m \end{pmatrix}\frac{(2s-1+m)!}{(2s-1)!}$. Using (3.11), the contribution to $\mathbb{Z}_D(F)$ is

$$R^{e_2-s}\widehat{f}.$$

We claim these are identical. It suffices to prove that the cancellation occurs in the non-holomorphic parts, giving

$$\sum_{m=0}^{2s-1} R_{1/2+2\lceil m/2\rceil}^{s-\lceil m/2\rceil}\left(a_m\left(I^{s+\lfloor m/2\rfloor}\widehat{f}(z)\right)\left(\frac{1}{16\pi y}\right)^m\right) = \widehat{f}(z). \tag{3.42}$$

Using the formulas for the iterated raising operator, and that

$$R_{2m}\left(\frac{1}{16\pi y}\right)^m = -4m\left(\frac{1}{16\pi y}\right)^{m+1}, \tag{3.43}$$

we may rewrite the left hand side of (3.42) as

$$\sum_{m=0}^{2s-1}\sum_{\ell=0}^{s-\lceil\frac{m}{2}\rceil}\begin{pmatrix} s-\lceil\frac{m}{2}\rceil \\ \ell \end{pmatrix}\left[\sum_{j=0}^{\ell}\begin{pmatrix}\ell \\ j\end{pmatrix}\frac{\Gamma\left(\frac{1}{2}-2\lfloor\frac{m}{2}\rfloor+\ell\right)}{\Gamma\left(\frac{1}{2}-2\lfloor\frac{m}{2}\rfloor+\ell-j\right)}\left(\frac{-4}{16\pi y}\right)^j\right]$$
$$\frac{(2s-1+m)!(s+\lfloor\frac{m}{2}\rfloor-\ell-1)!}{m!(2s-1-m)!(m-1)!}\frac{(-4)^{s-\lceil\frac{m}{2}\rceil}}{(16\pi y)^{s+\lfloor\frac{m}{2}\rfloor-\ell}}I^{s+\lfloor\frac{m}{2}\rfloor-\ell+j}\widehat{f}(z). \tag{3.44}$$

Setting $n = s+\lfloor\frac{m}{2}\rfloor-\ell+j$, we may rearrange the summations to get

$$\sum_{n=0}^{2s-1}\sum_{m=2\mathrm{Max}(0,n-s)}^{n}\frac{(2s-1+m)!}{m!(2s-1-m)!(m-1)!}(-4)^{n-m}\left(\frac{1}{16\pi y}\right)^n I^n\widehat{f}(z)$$
$$\times \sum_{\ell=s-n+\lfloor\frac{m}{2}\rfloor}^{s-\lceil\frac{m}{2}\rceil}\begin{pmatrix} s-\lceil\frac{m}{2}\rceil \\ \ell \end{pmatrix}\begin{pmatrix}\ell \\ j\end{pmatrix}\frac{\Gamma\left(\frac{1}{2}-2\lfloor\frac{m}{2}\rfloor+\ell\right)(s+\lfloor\frac{m}{2}\rfloor-\ell-1)!}{\Gamma\left(\frac{1}{2}+s-n-\lfloor\frac{m}{2}\rfloor\right)}. \tag{3.45}$$

Using `Mathematica` software [25], we find the inner sum collapses algebraically into a closed form. The full expression becomes

$$\sum_{n=0}^{2s-1} \left(\frac{1}{16\pi y}\right)^n I^n \widehat{f}(z) \cdot \sum_{m=2\max(0,n-s)}^{n} \frac{(2s-1+m)!(2s-m)!(-1)^m}{m!(2s-1-m)!(2s-2n+m)!(n-m)!}$$

(3.46)

Let $X_{s,n}$ be the inner sum. It is easy to see that $X_{s,0} = 1$. Using the W-Z method as before, we find for $0 \leq n \leq 2s-1$, that the $X_{s,n}$ satisfy the recursion relation

$$(n+1)X_{s,n+1} = n(3+4n-4s)X_{s,n}.$$

(3.47)

It follows that $X_{s,n} = 0$ if $0 < n \leq 2s-1$, and hence (3.46) is equal to $\widehat{f}(z)$. □

3.8.1   Integrality of $\mathcal{Z}_D(F)$

Here we complete the proof of Theorem 1.12 by proving cancellation of the denominators in the spectral decomposition. As before, assume we have cleared denominators, and we wish to show divisibility mod p^e for a good prime p. Let $\widehat{F} = \mathcal{Z}_D(F)$, and suppose $\widehat{F} \not\equiv 0 \pmod{p^e}$. Then let g_m be the coefficient of $\left(\frac{1}{16\pi y}\right)^m$ in $\widehat{F}$. We will refer to this as the component of $\widehat{F}$ at *depth* m. Note that each g_m is a quasi-modular form. If $\widehat{F}$ has weight k, and M is maximal such that g_M is not 0, then g_M is modular of weight $k - 2M$ since it is some multiple of $L^M \widehat{F}$. If $g_M \equiv 0 \pmod{p^e}$, and p is not 2 or 3, we may replace $\widehat{F}$ with

$$\widehat{F} - \left(\frac{E_2^*(4z)}{12}\right)^M \cdot g_M \equiv \widehat{F} \pmod{p^e},$$

(3.48)

where $E_2^* := E_2(z) - \frac{3}{\pi y}$. Therefore, if m is chosen such that $g_{m'} \equiv 0 \pmod{p^e}$ for all $m' > m$, then g_m is congruent to a modular form of weight $k - 2m$. A similar construction can be shown to work when p is 2 or 3.

Now return to $\mathcal{Z}_D(F)$. By Proposition 3.6, the principal part of $\mathcal{Z}_D(F)$ is congruent to 0 (mod p^e), so g_m is equivalent to a cusp form of weight $k-2m$. Suppose now that we replace each component f of weight $\ell < k - 2m$ in the decomposition of $\mathcal{Z}(F)$ with $f|\mathcal{H}_{k-2m,e}^{(\ell-k+2m)}$. These are precisely the forms which contribute to the component at depth m. In fact, its contribution to g_m is a multiple of $\mathcal{D}^{(k-2m-\ell)}\left(f|\mathcal{H}_{k-2m,e}^{(\ell-k+2m)}\right)$. By considering the action of the Hecke operators, we observe that this does not change the contribution to g_m modulo p^e (although it may alter components at other depths).

Now by the definition of $\mathcal{H}_{k-2m,e}^{(\ell-k+2m)}$, we have that

$$\mathcal{D}^{(k-2m-\ell)}\left(f|\mathcal{H}_{k-2m,e}^{(\ell-k+2m)}\right) = \left(\mathcal{D}^{(k-2m-\ell)}f\right)|\mathcal{H}_{k-2m,e}. \qquad (3.49)$$

Therefore, we have that $g_m \equiv g_m|\mathcal{H}_{k-2M',e}$ (mod p^e). As $\mathcal{H}_{k-2m,e}$ sends cusp forms of weight $k-m$ to zero without introducing denominators, we see that $g_m \equiv 0$ (mod p^e). Therefore, we have that the components of $\widehat{F}$ at every depth (and in particular at depth 0) are congruent to 0 (mod p^e). From the discussion above, this concudes the proof of Theorem 1.12.

Chapter 4

A New Quantum Modular Form

In this chapter, we prove Theorem 1.14. This theorem, along with Corollary 1.15, is joint work with Robert Schneider in [34]. The chapter is organized as follows. In §4.1 we recall the identities of [2], and in §4.2 we describe the modularity properties of Eichler integrals of half-integral weight modular forms. In §4.3 we complete the proof of Theorem 1.14. We finish with the proof of Corollary 1.15 in §4.4.

4.1 Preliminaries

In this section, we describe some of the machinery needed to prove Theorem 1.14.

4.1.1 Sums of Tails Identities

Here we recall the work of Andrews, Jiménez-Urroz, and Ono on sums of tails identities. To state their results for F_9 and F_{10} and connect θ_i^S to quantum modular objects, we formally define a "half-derivative operator" by

$$\sqrt{\theta}\left(\sum_{n=0}^{\infty} a(n)q^n\right) := \sum_{n=1}^{\infty} \sqrt{n}a(n)q^n. \qquad (4.1)$$

If we have a generic q-series $f(q)$, we will also denote $\sqrt{\theta}f(q) := \widetilde{f}(q)$. Then Andrews, Jiménez-Urroz, and Ono show [2] that for finite versions $F_{9,i}, F_{10,i}$ associated to F_9, F_{10} the following holds true:

Theorem 4.1 (Andrews-Jiménez-Urroz-Ono). *As formal power series, we have that*

$$\sum_{n=0}^{\infty} (F_9(z) - F_{9,n}(z)) = 2F_9(z)E_1(z) + 2\sqrt{\theta}(F_9(z)), \qquad (4.2)$$

$$\sum_{n=0}^{\infty} (F_{10}(z) - F_{10,n}(z)) = F_{10}(z)E_2(z) + \frac{1}{2}\sqrt{\theta}(F_{10}(z)), \qquad (4.3)$$

where the $E_i(z)$ are holomorphic Eisenstein-type series. In particular, as F_9, F_{10} vanish to infinite order while E_1, E_2 are holomorphic at all cusps where the "strange" functions are well-defined, we have for q an appropriate root of unity that the "strange" function associated to F_i equals $\widetilde{F}_i$ to infinite order. As the series θ_2, θ_3 do not have integral coefficients, we make the definitions $\widetilde{\theta}_2(z) := \widetilde{F}_{10}(z/16)$ and $\widetilde{\theta}_3(z) := \widetilde{F}_{10}(z/16 + 1/16)$. By the definition of the strange series, we obtain the following.

Corollary 4.2. *At appropriate roots of unity where each "strange" series is defined, we have that*

$$\theta_1^S(q) = 2\widetilde{\theta}_1(q), \qquad \theta_2^S(q) = \frac{1}{2}\widetilde{\theta}_2(q), \qquad \theta_3^S(q) = \frac{1}{2}\widetilde{\theta}_3(q). \qquad (4.4)$$

4.2 Properties of Eichler Integrals

In the previous section we have seen that at a rational point x, each component of $\phi(x)$ agrees up to a constant with a "half-derivative" of the corresponding theta function at $q = e^{2\pi ix}$. Thus, we can reduce part (2) of Theorem 1.14 to a study of modularity of such half-derivatives. We do so

following the outline given in [38], which is further explained in the weight $3/2$ case in [26]. Recall that in the classical setting of weight $2k$ cusp forms, $1 \leq k \in \mathbb{Z}$, we define the *Eichler integral* of $f(z) = \sum_{n=1}^{\infty} a(n)q^n$ as a formal $(k-1)^{\text{st}}$ antiderivative $\widetilde{f}(z) := \sum_{n=1}^{\infty} n^{1-k}a(n)q^n$. Then $\widetilde{f}$ is nearly modular of weight $2 - k$, as the differentiation operator $\frac{d}{dq}$ does not preserve modularity but preserves near-modularity. More specifically, $\widetilde{f}(z + 1) = \widetilde{f}(z)$ and $z^{k-2}\widetilde{f}(-1/z) - \widetilde{f}(z) = g(z)$ where $g(z)$ is the *period polynomial*. This polynomial encodes deep analytic information about f and can also be written as $g(x) = c_k \int_0^{i\infty} f(z)(z - x)^{k-2}\, dz$ for a constant c_k depending on k. Suppose we now begin with a weight $1/2$ vector-valued modular form f with n components f_i such that and $f(-1/z) = M_S f(z)$, for M_S both $n \times n$ matrices (the transformation under translation is routine).

In this case, of course, it does not make sense to speak of a half-integral degree polynomial, and the integral above does not even converge. However, we may remedy the situation so that the analysis becomes similar to the classical case. We formally define $\widetilde{f}$ by taking a formal antiderivative (in the classical sense) on each component. As $1 - k = 1/2$, we have in fact $\widetilde{f_i} = \sqrt{\theta} f_i$. We would like to determine an alternative way to write the Eichler integral as an actual integral, so that we may use substitution and derive modularity properties of $\widetilde{f}$ from f. However, the integral $g(z) = c_{1/2} \int_0^{i\infty} f(z)(z - x)^{-3/2}\, dz$ no longer makes sense. To remedy this in the weight $3/2$ case, Lawrence and Zagier define another integral $f^*(x) := c_k \int_{\bar{x}}^{\infty} \frac{f(z)}{(z-x)^{\frac{1}{2}}}\, dz$, which is meaningful for x in the lower half plane $\mathbb{H}^-$.

Here we sketch their argument in the weight $1/2$ case for completeness, and as the analysis involved in our own work differs slightly. Returning to our vector-valued form f, recall that the definition of the Eichler integral of f corresponds with $\sqrt{\theta} f$. For $x \in \mathbb{H}^-$, we define

$$f^*(x) = \left(\frac{-i}{\pi(1+i)}\right) \cdot \int_{\bar{x}}^{i\infty} \frac{f(z)}{(z - x)^{\frac{3}{2}}}\, dz. \tag{4.5}$$

To evaluate this integral, use absolute convergence to exchange the integral and the sum, and note that for $q_z = e^{2\pi i z}$,

$$\int_{\bar{x}}^{i\infty} \frac{q_z^n}{(z-x)^{\frac{3}{2}}}\, dz = \left((2+2i)\pi\sqrt{n}q_z^n \operatorname{erfi}\left((1+i)\sqrt{\pi n(z-x)} \right) - \frac{2q_x^n}{(z-x)^{\frac{1}{2}}} \right) \Bigg|_{z=\bar{x}}^{i\infty},$$
$$(4.6)$$

where $\operatorname{erfi}(x)$ is the *imaginary error function*. As in [26], we have that $\widetilde{f}(x + iy) = f^*(x - iy)$ as full asymptotic expansions for $x \in \mathbb{Q}$, $0 < y \in \mathbb{R}$. To see this, note that at the lower limit, the antiderivative vanishes as $y \to 0$ as $\operatorname{erfi}(0) = 0$ and although the square root in the denominator goes to zero, for each rational at which we are evaluating our "strange" series, the corresponding theta functions vanish to infinite order, which makes this term converge. For the upper limit, the square root term immediately vanishes, and we use the fact that $\lim_{x\to\infty} \operatorname{erfi}(1+i)\sqrt{ix+y} = i$ for $x, y \in \mathbb{R}$.

Thus, as in [26], we have that $\widetilde{f}(x) = f^*(x)$ to infinite order at rational points. In the case of θ_1, we have that $\widetilde{\theta}_1(x) = \theta^*(x)$, but for θ_2 and θ_3 we have to divide by $4 = \sqrt{16}$ due to the non-integrality of the powers of q in order to agree with the definition of $\widetilde{\theta}_i$. Using this together with Corollary 4.2, in all cases we find that $\theta_i^S(q) = \theta_i^*(q)$ at roots of unity where both sides are defined. Now, the modularity properties for the integral follow *mutatis mutandis* from [26] using the modularity of f and a standard u-substitution. More precisely, suppose $f(-1/z)(z)^{-\frac{1}{2}} = M_S f(z)$. Then we have shown that the following modularity properties hold for $f^*(z)$ when $z \in \mathbb{H}^-$, and hence also hold for $\widetilde{f}(z)$ for each component at appropriate roots of unity where each "strange" function is defined. By this, we mean that the modularity conditions in the following proposition can be expressed as six equations, and each of these equations is true precisely where the corresponding "strange" series make sense.

Proposition 4.3. *If* $g(x) := \left(\frac{-i}{\pi(1+i)} \right) \cdot \int_0^{i\infty} \frac{f(z)}{(z-x)^{\frac{3}{2}}} \, dz$, *then*

$$\left(\frac{x}{-i} \right)^{-\frac{3}{2}} f(-1/x) + M_S f(x) = M_S g(x).$$

It is also explained in [26] why $g_\alpha(z)$ is a smooth function for $\alpha \in \mathbb{R}$. Although $g(x)$ is *a priori* only defined in $\mathbb{H}^-$, we may take any path L connecting 0 to $i\infty$. Then we can holomorphically continue $g(x)$ to all of $\mathbb{C} - L$. Thus, we obtain a continuation of g which is smooth on $\mathbb{R}$ and analytic on $\mathbb{R} - \{0\}$.

4.3 Proof of Theorem 1.14

Here we complete the proofs of parts (1) and (2) of Theorem 1.14.

4.3.1 Proof of Theorem 1.14 (1)

We show that at appropriate roots of unity, our "strange" functions θ_i^S are reflections of q-series which are convergent on $\mathbb{H}$. Using (1.22), it suffices to show for θ_1^S that $\sum_{n=0}^\infty \frac{(q^{-1};q^{-1})_n}{(-q^{-1};q^{-1})_n}$ agrees at odd roots of unity with a q-series convergent when $|q| < 1$. To factor out inverse powers of q, we observe that

$$(a^{-1};q^{-\alpha})_n = (-1)^n a^n q^{\frac{\alpha n(n-1)}{2}} (a;q^\alpha)_n. \tag{4.7}$$

Applying this identity to the numerator and denominator term-by-term, we have at odd order roots of unity

$$\theta_1^S(q^{-1}) = \sum_{n=0}^\infty (-1)^n \frac{(q;q)_n}{(-q;q)_n} = 2 \sum_{n=0}^\infty \frac{q^{2n+1}(q;q)_{2n}}{(1+q^{2n+1})(-q;q)_{2n}}. \tag{4.8}$$

The series on the right-hand side is clearly convergent for $|q| < 1$, and results from pairing consecutive terms of the left-hand series as follows:

$$\frac{(q;q)_{2n}}{(-q;q)_{2n}} - \frac{(q;q)_{2n+1}}{(-q;q)_{2n+1}} = \frac{(q;q)_{2n}}{(-q;q)_{2n}} \left(1 - \frac{1-q^{2n+1}}{1+q^{2n+1}} \right) = \frac{2q^{2n+1}(q;q)_{2n}}{(1+q^{2n+1})(-q;q)_{2n}}.$$

Remark. Alternatively, one can show the convergence of $\theta_1^S(q^{-1})$ by letting $a = 1, b = -1, t = -1$ in Fine's identity [16]

$$\sum_{n=0}^{\infty} \frac{(aq;q)_n}{(bq;q)_n}(t)^n = \frac{1-b}{1-t} + \frac{b-atq}{1-t} \sum_{n=0}^{\infty} \frac{(aq;q)_n}{(bq;q)_n}(tq)^n, \qquad (4.9)$$

giving

$$\theta_1^S(q^{-1}) = 1 + \frac{q-1}{2} \sum_{n=0}^{\infty} \frac{(q;q)_n}{(-q;q)_n}(-q)^n \qquad (4.10)$$

which also converges for $|q| < 1$.

Similarly, we use (4.1) to study θ_2^S, θ_3^S. Note that it suffices by (1.22) to study $\sum_{n=0}^{\infty} \frac{(q^{-2};q^{-2})_n}{(q^{-3};q^{-2})_n}$. Factorizing as above, we find that

$$\sum_{n=0}^{\infty} \frac{(q^{-2};q^{-2})_n}{(q^{-3};q^{-2})_n} = \sum_{n=0}^{\infty} \frac{q^n(q^2;q^2)_n}{(q^3;q^2)_n}, \qquad (4.11)$$

the right-hand side of which is clearly convergent on $\mathbb{H}$. We note that in general, similar inversion formulae result from applying (4.7) to diverse q-series and other expressions involving eta functions, q-Pochhammer symbols and the like.

4.3.2   Proof of Theorem 1.14 (2)

Proof. Here we complete the proof of Theorem 1.14. Note that by the Corollary 4.2 to the sums of tails formulae of Andrews, Jiménez-Urroz, and Ono [2], each component of $H(q)$ agrees to infinite order at rational numbers with a multiple of the corresponding Eichler integral. By the discussion of Eichler integrals above, the value of each $\widetilde{\theta}_i$ agrees at rationals with the value of the corresponding θ_i^*. Therefore, by the discussion of the modularity properties of θ_i^*, we need only to describe the modularity of $H(q)$. This is simple to check using the usual transformation laws

$$\eta(z+1) = \zeta_{24}\eta(z), \qquad (4.12)$$

$$\eta(-1/z) = \left(\frac{z}{i}\right)^{\frac{1}{2}} \eta(z), \tag{4.13}$$

and (1.21). Hence we see that

$$H(z+1) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & \zeta_{12} \\ 0 & \zeta_{24} & 0 \end{pmatrix} H(z), \tag{4.14}$$

$$H(-1/z) = \left(\frac{z}{i}\right)^{\frac{1}{2}} \begin{pmatrix} 0 & \sqrt{2} & 0 \\ 1/\sqrt{2} & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} H(z), \tag{4.15}$$

and the corresponding transformations of θ_i^* follow. $\qquad\square$

4.4 Proof of Corollary 1.15

Proof. The proof of Corollary 1.15 is a generalization of and proceeds similarly to the proofs of Theorems 4 and 5 of [2]. As the sums of tails identities in Theorem 2.1 show that the "strange" functions F_9 and F_{10} agree *to infinite order* with the half-derivatives of F_9 and F_{10} at the roots of unity under condsideration, the coefficients in the asymptotic expansion of $H_i(t, \zeta)$ for $i = 9, 10$ agree up to a constant factor with the coefficients of the asymptotic expansion of $\sqrt{\theta} F_i(\zeta e^{-t})$. Recalling the classical theta series expansions for F_i in (1.6), the first part of Corollary 1.15 follows immediately from the following well-known fact:

Lemma 4.4 (Proposition 5 of [24])**.** *Let $\chi(n)$ be a periodic function with mean value zero and $L(s, \chi) := \sum_{n=0}^{\infty} \chi(n) n^{-s}$. As $t \searrow 0$, we have*

$$\sum_{n=0}^{\infty} n\chi(n) e^{-n^2 t} \sim \sum_{n=0}^{\infty} L(-2n-1, \chi) \frac{(-t)^n}{n!}.$$

The proof follows from taking a Mellin transform, making a change of variables, and picking up residues at negative integers. The assumption on the coefficients $\chi(n)$ assures that $L(s, \chi)$ can be analytically continued to $\mathbb{C}$. The mean value zero condition is easily checked in our case; for example for

F_9 one needs to verify that $\{(-\zeta)^{n^2}\}_{n\geq 0}$ is mean value zero for ζ a primitive order $2k+1$ root of unity, and for F_{10} one must check that $\{\zeta^{\frac{(2n+1)^2}{8}}\}_{n\geq 0}$ is mean value zero for an even order root of unity ζ. These may both be checked using well-known results for the generalized quadratic Gauss sum

$$G(a,b,c) := \sum_{n=0}^{c-1} e\left(\frac{an^2 + bn}{c}\right). \tag{4.16}$$

In particular, for F_9, for an odd order root of unity ζ, $-\zeta$ is primitive of order k where $k \equiv 2 \pmod 4$, so we need that $G(a,0,k) = 0$ when $k \equiv 2 \pmod 4$, which fact is well known. For F_{10}, we may use the standard fact that $G(a,b,c) = 0$ whenever $4|c$, $(a,c) = 1$, and $0 < b \in 2\mathbb{Z}+1$ to obtain our result. This Gauss sum calculation follows, for instance, by using the multiplicative property of Gauss sums together with an application of Hensel's lemma.

In the case of F_{10}, note that the formula for $H_{10}(t,\alpha)$ is obtained by substituting $q = \zeta e^{-t}$ into the "strange" function for F_{10} after letting $q \to q^{\frac{1}{8}}$. A simple change of variables in the Mellin transform in the foregoing proof of the present Lemma adjusts for the $1/8$ powers by giving an extra factor of 8^s before taking residues. $\qquad\square$

Chapter 5

Ramanujan's Mock Theta Functions

In this chapter, we prove Theorem 1.16 and Corollary 1.17, which are joint work with Michael Griffin and Ken Ono in [21]. We then prove Theorem 1.18. We begin by stating the following result, which follows from the work of Zwegers [42, 43].

Fact 5.1. *Suppose that $M(z)$ is one of Ramanujan's alleged examples of a mock theta function. Thanks to Zwegers [42, 43], there are integers γ and δ for which $q^{\gamma} M(\delta z) =: f^+(z)$ is the holomorphic part of a weight 1/2 harmonic weak Maass form $f(z)$ on a congruence subgroup $\Gamma_1(N)$. Moreover, the nonholomorphic part of this form is the period integral of a weight 3/2 unary theta function. In particular, there are finitely many positive integers $\delta_1, \ldots, \delta_s$ for which $c_f^-(n) = 0$ unless $n = -\delta_i m^2$ for some $1 \le i \le s$ and some integer m.*

The rest of the chapter is organized as follows. In §5.1 we describe the construction of the Poincaré series. In §5.2 we give with the proof of Theorem 1.16 and Corollary 1.17. We conclude with the proof of Theorem 1.18 in §5.3.

5.1 Poincaré Series

We require Maass-Poincaré series, which were considered previously in work of Niebur [27, 28]. Their principal parts will serve as a basis for the principal parts of the mock theta functions. For $s \in \mathbb{C}$ and $y \in \mathbb{R} - \{0\}$ we let $\mathcal{M}_s(y) := |y|^{-\frac{k}{2}} M_{\frac{k}{2} \operatorname{sgn}(y), s - \frac{1}{2}}(|y|)$, where $M_{\nu, \mu}$ is the usual M-Whittaker function which satisfies

$$\frac{\partial^2 u}{\partial z^2} + \left(-\frac{1}{4} + \frac{\nu}{z} + \frac{\frac{1}{4} - \mu^2}{z^2} \right) u = 0.$$

Since spaces of forms on $\Gamma_1(N)$ are a direct sum over the spaces of Maass forms on $\Gamma_0(N)$ with Nebentypus, it suffices to construct Poincaré series on $\Gamma_0(N)$ with arbitrary Nebentypus χ. For a positive integer m, we define $\phi_{-m,s}(z) := \mathcal{M}_s(-4\pi m y) e(-mx)$, and we define the Poincaré series on $\Gamma_0(N)$ with Nebentypus χ and weight $k \in \frac{1}{2} + \mathbb{Z}$ by

$$\mathcal{F}_k(-m, s, z) := \sum_{\gamma \in \Gamma_\infty \backslash \Gamma_0(N)} \left(\frac{c}{d} \right)^{-2k} \epsilon_d^{2k} \chi(d)^{-1} (\phi_{-m,s}|_k \gamma)(z). \tag{5.1}$$

It turns out that $\phi_{-m,s}(z)$ is an eigenfunction of Δ_k with eigenvalue $s(1 - s) + (k^2 - 2k)/4$. Therefore $\mathcal{F}_k(-m, s, z)$ is a weak Maass form of weight k on Γ with character χ whenever the series is absolutely convergent. This is clear if $\Re(s) > 1$ as $\phi_{-m,s}(z) = O(y^{\Re(s) - \frac{k}{2}})$ as $y \to 0$. To obtain a *harmonic* Maass form, we choose $s = \frac{k}{2}$ (or $s = 1 - \frac{k}{2}$ if $k < 1$). Convergence for this choice of s for weight $k \in \frac{1}{2} + \mathbb{Z}$ Poincaré series is only questionable if $k = 1/2$ or $k = 3/2$. We are primarily interested in the case when $k = 1/2$.

The Fourier expansion of such series is well known (for example, see [27, 15, 9, 6, 8]). We recall the *Kloosterman sum* of weight $k \in \frac{1}{2} + \mathbb{Z}$ for $\Gamma_0(N)$ with Nebentypus χ.

$$K_k(m, n, c, \chi) := \sum_{d \,(\mathrm{mod}\ c)^\times} \left(\frac{c}{d} \right)^{-2k} \epsilon_d^{2k} \chi(d) e\left(\frac{m\bar{d} + nd}{c} \right), \tag{5.2}$$

where d runs through primitive residue classes mod c and $\bar{d}$ is the multiplicative inverse of d mod c. We then have the following.

Proposition 5.2. *If m is a positive integer, then the Poincaré series $\mathcal{F}_k(-m, z, s)$ for $\Gamma_0(N)$ with Nebentypus χ has the Fourier expansion*

$$\mathcal{F}_k(-m, z, s) = \mathcal{M}_s(-4\pi my)e(-mx) + \sum_{n \in \mathbb{Z}} c(n, y, s)e(nx),$$

where the coefficients $c(n, y, s)$ are given by

$$
\begin{cases}
\frac{2\pi i^{-k}\Gamma(2s)}{\Gamma(s-k/2)}\left|\frac{n}{m}\right|^{\frac{k-1}{2}} \displaystyle\sum_{c>0, N|c} \frac{K_k(-m, n, c, \chi)}{c} J_{2s-1}\left(\frac{4\pi\sqrt{|mn|}}{c}\right) \mathcal{W}_s(4\pi ny), & n < 0 \\[2ex]
\frac{2\pi i^{-k}\Gamma(2s)}{\Gamma(s+k/2)}\left|\frac{n}{m}\right|^{\frac{k-1}{2}} \displaystyle\sum_{c>0, N|c} \frac{K_k(-m, n, c, \chi)}{c} I_{2s-1}\left(\frac{4\pi\sqrt{|mn|}}{c}\right) \mathcal{W}_s(4\pi ny), & n > 0 \\[2ex]
\frac{4^{1-k/2}\pi^{1+s-k/2}i^{-k}|m|^{s-k/2}y^{1-s-k/2}\Gamma(2s-1)}{\Gamma(s+k/2)\Gamma(s-k/2)} \displaystyle\sum_{c>0, N|c} \frac{K_k(-m, 0, c, \chi)}{c^{2s}}, & n = 0
\end{cases}
$$

$$\tag{5.3}$$

In the proposition above, I_k is the usual modified Bessel function and J_k is the Bessel function of the first kind. If $s \geq 1$ and equals $k/2$ or $1 - k/2$, then these Poincaré series converge and are harmonic weak Maass forms. For $k = 1/2$ it is known that the formulas still hold. For completeness, we shall give brief remarks below concerning the convergence.

Before we discuss the weight $1/2$ case, we stress that this proposition allows us to easily determine the asymptotics of the coefficients of holomorphic parts of harmonic weak Maass forms. This follows from the well-known asymptotic

$$I_k(x) \sim \frac{e^x}{\sqrt{2\pi x}}\left(1 - \frac{4k^2-1}{8x} + \ldots\right). \tag{5.4}$$

The Poincaré series constructed above have nonconstant principal parts only at the cusp infinity. We may similarly construct Poincaré series at any cusp h. We let $\mathcal{F}_k(-m, s, z, h)$ denote the Poincaré series which is defined by modifying (5.1) as

$$\mathcal{F}_k(-m, s, z, h) := \sum_{\gamma \in \Gamma_h \backslash \Gamma_0(N)} \left(\frac{c}{d}\right)^{-2k} \epsilon_d^{2k}\chi(d)^{-1}(\phi_{-m,s}|_k\gamma)(z),$$

where Γ_h is the stabilizer of h. As in the case of the cusp at infinity, we obtain a weak Maass form with order $-m$ principal part at the cusp h and constant principal parts at all other cusps.

These facts allow us to conclude with the following crucial fact.

Fact 5.3. *Suppose that $f(z)$ is a weight 1/2 harmonic weak Maass form with a nonconstant principal part at some cusp. Let $f_P(z)$ be the weight 1/2 harmonic weak Maass form that is a linear combination of Maass-Poincaré series which matches, up to constants, the principal parts of $f(z)$ at all cusps. By Theorem 1.16, it follows that $f(z) - f_P(z)$ is a weight 1/2 holomorphic modular form, which, by the Serre-Stark Basis Theorem (for example, see [29]), implies that $f(z) - f_P(z)$ is a linear combination of weight 1/2 unary theta functions. Therefore, the subexponential growth of the I-Bessel function, combined with the periodicity of the Kloosterman sums in n, when m and c are fixed, then implies that a positive proportion of the coefficients of the holomorphic part of $f^+(z)$ are nonzero. Indeed, this gives arithmetic progressions of coefficients with smooth asymptotic subexponential growth.*

Remark. We briefly discuss the convergence in Proposition 5.2 for weight 1/2 harmonic weak Maass forms. To show this, we need similar estimates for sums of the Kloosterman sums as in Theorem 4.1 of [6]. In that work the Kloosterman sums were rewritten as Salie-type sums, which were then estimated using the equidistibution of CM-points. It is clear that the shape of the Salie-type sums do not depend on the multiplier system in a crucial way. Alternatively, the more general case, results of Goldfeld and Sarnak in [20] and the spectral theory of automorphic forms apply. By the asymptotics for Bessel functions, it suffices to consider the continuation of the Selberg-Kloosterman zeta function

$$Z_{n,m}(s,\chi) := \sum_{c>0} \frac{K_k(-m,n,c,\chi)}{c^{2s}}. \tag{5.5}$$

Namely, for $k = 1/2$ we need to show convergence at $s = 3/4$. The convergence we require was shown for a special case in Theorem 2.1 of [17]. The general case follows *mutatis mutandis*.

Theorem 5.4. *If m is a positive integer, then $Z_{n,m}(s, \chi)$ is convergent at $s = 3/4$.*

.

5.2 The Proof of Theorem 1.16 and Corollary 1.17

Here we prove Theorem 1.16 and Corollary 1.17.

5.2.1 Proof of Theorem 1.16

Suppose that $g(z)$ is a weakly holomorphic modular form on $\Gamma_1(N')$, for some N', which cuts out the exponential singularities of $f(z)$ as q approaches roots of unity. Then $h(z) := f(z) - g(z)$ is a harmonic weak Maass form of weight k on $\Gamma_1(\mathrm{lcm}(N, N'))$ with nonconstant nonholomorphic part. By Theorem 2.6, $h(z)$ has a nonconstant principal part at some cusp. Since the nonholomorphic part $f^-(z)$ exhibits exponential decay at cusps, it follows that $h(z)$ is also $O(1)$ as cusps. Suppose that $h(z)$ has a nonconstant principal part at infinity (a similar argument applies at other cusps). By choosing matrices $\left(\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix} \right) \right) \in \Gamma_1(\mathrm{lcm}(N, N'))$, combined with the fact that

$$\lim_{z \to i\infty} h\left(\tfrac{az+b}{cz+d}\right) = \lim_{z \to i\infty} (cz + d)^k h(z),$$

we find that infinitely many roots of unity are exponential singularities for $h(z)$.

5.2.2 Proof of Corollary 1.17

Suppose that $M(z)$ is one of Ramanujan's alleged examples of a mock theta function. Then there are integers γ and δ for which $q^\gamma M(\delta z) =: f^+(z)$ is the holomorphic part of the weight $1/2$ harmonic weak Maass form. Now suppose that $g(z)$ is a weakly holomorphic modular form of some weight k which cuts out the exponential singularities of $f(z)$. Following the proof of Theorem 1.4 of [31], we can use Fact 5.1, Fact 5.3, and the theory of quadratic (and trivial) twists to obtain a weight $1/2$ weakly holomorphic modular form $\widehat{f}(z)$. By Fact 5.3, this can be done so that $\widehat{f}(z)$ is nontrivial and has nonconstant principal parts at some cusp. Applying the same procedure to $g(z)$ gives a weakly holomorphic modular form $\widehat{g}(z)$. We then have that $\widehat{f}(z)$ and $\widehat{g}(z)$ cut out exactly the same exponential singularities at all roots of unity. By the discussion after Corollary 1.17, it then follows that $k = 1/2$. Therefore, if there is such a $g(z)$, then $f(z) - g(z)$ is a weight $1/2$ harmonic weak Maass form which has a nonvanishing nonholomorphic part, which also has the property that $f^+(z) - g(z)$ has no exponential singularities at any roots of unity. This contradicts Theorem 1.16.

5.3 Proof of Theorem 1.18

Ramanujan found, and Watson proved, a number of relations between mock theta functions [37]. In particular, Watson showed that for the mock theta function

$$\nu(q) := \sum_{n \geq 0} \frac{q^{n(n+1)}}{(-q; q^2)_{n+1}} \tag{5.6}$$

we have the relation

$$\nu(\pm q) \pm q\omega(q^2) = \frac{1}{2} \cdot \left(\sum_{n \in \mathbb{Z}} q^{n(n+1)} \right) \cdot \prod_{n \geq 0} (1 + q^{2n}). \tag{5.7}$$

The idea is then that $\nu(q)$ may be expressed in a different form due to a result of Fine. Namely, the following is a limiting case of the famous Rogers-Fine identity (see (7.31) of [16]):

$$F(b/t, 0; t) = \frac{1}{1-t} \sum_{n \geq 0} \frac{(-b)^n q^{\frac{n^2+n}{2}}}{(tq; q)_n}, \tag{5.8}$$

where $F(a, b; t) := \sum_{n \geq 0} t^n \frac{(aq;q)_n}{(bq;q)_n}$. Taking $b = -1$, $t = -q^{\frac{1}{2}}$ and letting $q \to q^2$, we find by (5.8):

$$\nu(q) = \sum_{n \geq 0} (-q)^n (q; q^2)_n. \tag{5.9}$$

It is thus apparent that at odd order roots of unity, the sum on the right-hand side is finite and $\nu(q)$ converges, whereas for a primitive order k root of unity with $k \equiv 2 \pmod 4$, it is clear that $\nu(-q)$ becomes a finite sum. Choosing the sign appropriately in (5.7) gives Theorem 1.18.

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