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Combinatorial Analysis of Go Endgame Positions

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Combinatorial Analysis of Go Endgame Positions

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Abstract

Combinatorial Analysis of Go Endgame Positions
By Jacob A. McMillen

Go is a two player skill game of Chinese origin. Although Go is praised for having a simple rule set, the game generates tremendous complexity. As such, programmers have been unable to design Go AI programs that exceed the level of intermediate human players. By using techniques of combinatorial theory, mathematicians have recently developed methods of determining optimal play on certain classes of Go positions. It is our goal to present a survey of these analytic methods. We will first provide an introduction on the rules of Go as well as relevant concepts of combinatorial game theory. We will then proceed to solve several categories of small Go positions and finally show how to determine perfect play on a full size Go endgame by way of partitioning into solvable subgames.

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S.D.G.

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Chapter 1

Overview

Go is generally considered to be the world's oldest board game. It is a skill game of Chinese origin and is thought to have existed as early as 3000-4000 years ago. Go remains an extremely popular game throughout much of East Asia and has experienced several surges of interest in the West during the past century. A thorough account of the lengthy history of Go can be found in [13].

Although Go is admired for having very simple and elegant rules, the game is enormously complex. In fact, the number of legal positions has been estimated to be $2.081681994 * 10^{170}$ by Tromp and Farneback [14]. Due to this enormous complexity, the level of computer Go artificial intelligence programs has lagged far behind that of many other skill games. In games such as Chess and Checkers, computers can match or exceed the level of play of the best human players in the world. In contrast to this, the strongest Go programs can be defeated by an intermediate to advanced amateur.

Due to advances made in combinatorial game theory in the 1980s, there has recently been tremendous progress in the analysis of one particular area of Go strategy, namely the endgame. In fact, using the mathematical tools that have been developed allows both computers and mathematicians alike to solve difficult endgame problems that have stumped the best professional Go players in the world.

It is our goal to provide a comprehensive survey of the mathematics used to analyze this ancient game. We assume that the reader has familiarity with the fundamentals of combinatorial game theory. Chapter 2 consists of a brief summary of the material that is particularly relevant to the study of Go positions. Readers who need additional background should see *Winning Ways for Your Mathematical Plays* [2] or *Lessons in Play* [1]. In Chapter 3, we analyze many different classes of Go positions by using combinatorial game theory and learn how this information can allow us to play optimally, even on very difficult endgame positions. Finally, in Chapter 4, we will examine new results and discuss the direction that current Go research is headed. For readers who are unfamiliar with the game of Go, Appendix A contains a complete summary of the rules.

Chapter 2

Combinatorial Games

2.1 What is a Combinatorial Game?

A combinatorial game typically satisfies the following:

1. Two Players - There are two players who alternate moves.
2. Perfect Information - Both players have complete knowledge of the state of the game and of all previous moves.
3. Deterministic - There are no chance elements such as dice, cards, or spinners.
4. Finite - There are finitely many legal positions and repeated positions are not allowed.
5. Zero-sum - Under normal play, the first player who cannot move loses.

The term 'position' refers to a particular state of the game at some time. For any given state of the game, there are rules that specify which positions are legal to move to, and often, a certain starting position is specified.

Formally, a *game* G, between players Left and Right, is defined inductively as a pair of sets of games,

$$G = \left\{ \left\{ G^{L_1}, G^{L_2}, \ldots, G^{L_m} \right\} \,\middle|\, \left\{ G^{R_1}, G^{R_2}, \ldots, G^{R_n} \right\} \right\}$$

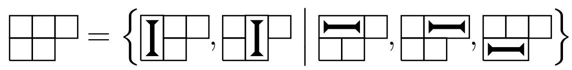
or simply

$$G = \left\{ G^L \,\middle|\, G^R \right\}.$$

If it is Left's turn, she may move to any game in the set G^L, and Right may move to any game in G^R on his turn. The sets G^L and G^R are called the left and right *options* of G respectively.

In order to illustrate various definitions and concepts, we will now introduce a simple combinatorial game called Domineering. The game is played on an $m \times n$ chessboard. Upon a player's turn, he or she lays a single domino on the board so that it covers two adjacent open squares. Left must place dominoes in a vertical orientation, and Right must place dominoes in a horizontal orientation. The normal play convention applies, so the first player who cannot make a legal move is the loser.

The following is an example of a simple Domineering position defined in terms of its left and right options. Left has two legal moves, whereas Right has three possible moves.



For notational convenience, braces will be often be omitted from the definitions of games and |'s will be used to indicate nested games. For example,

$$\{G_1 \mid \{G_2 \mid G_3\}\} = \left\{G_1 \,\|\, G_2 \mid G_3\right\}.$$

One highly important fact is stated in the Fundamental Theorem of Combinatorial Games:

Theorem 1 (Fundamental Theorem of Combinatorial Games) *Let G be a game between Left and Right where Left moves first. Either Left can win by moving first, or Right can win by moving second, but not both.*

Proof: Fix a game G. By reasoning inductively, any opening move on G by Left would send the game to a position where either

(i) Right can win by playing first, or

(ii) Left can win by playing second.

If there is an opening move that sends the game to a position of type (ii), then Left can begin with it and force a win. Otherwise all opening moves place the game in a state where Right can force a win by playing next. This means that Right can force a win on G by playing second. ■

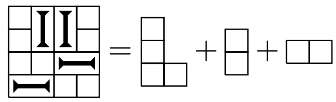
2.2 Algebra of Games

The sum of games G and H is given by

$$G + H = \left\{G^L + H, G + H^L \;\middle|\; G^R + H, G + H^R\right\}.$$

We may interpret this as the two games being played side by side. On a player's turn, he will choose to play on either G or H, and then make a legal move on that component.

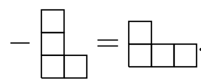
Using the previous definition allows us to treat a complex game as a sum of simple games. For example, we may break down the following Domineering endgame position as follows:



We may form the negative of a game by reversing the roles of the two players,

$$-G = \left\{-G^R \mid -G^L\right\}.$$

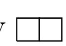
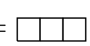
For example,



The difference of G and H is defined as

$$G - H = G + (-H).$$

2.3 Comparing Games and Game Equivalence

It will be necessary to define precisely when two games are equivalent, and when one game is preferable to Left over another. We define $G = H$ if, for any game X, $G + X$ has the same outcome as $H + X$ under optimal play. To play optimally means that whichever of the two players can force a win is employing a strategy to do so. Note that the Fundamental Theorem of Combinatorial Games guarentees that exactly one of the players will be able to force a win. This definition of equivalence states that G must behave like H in any sum. In Domineering we would say  =  since, in any game sum, they both provide exactly one move to Right.

We will also define $G \geq H$ if Left prefers G to H. Formally, $G \geq H$ if, for any game X, Left wins $G + X$ whenever Left wins $H + X$. Essentially, we can replace H by G in any game sum, and Left will have an outcome that is at least as good. Similarly, we say $G \leq H$ if, for any game X, Right wins $G + X$ whenever Right wins $H + X$.

It is possible to compare fixed games G and H by examining the difference game $G - H$. There are four possible outcomes.

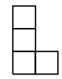
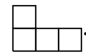
1. $G > H$ if $G - H$ is a win for Left

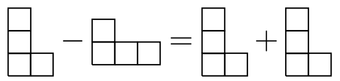
2. $G = H$ if $G - H$ is a win for the second player

3. $G < H$ if $G - H$ is a win for Right

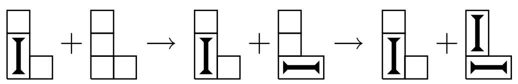
4. $G \parallel H$ is $G - H$ is a win for the first player

If $G = H$, then G and H are equivalent games. If $G > H$, then Left prefers G over H, and if $G < H$, then Right prefers G to H. Finally, $G \parallel H$ denotes that G and H are incomparable.

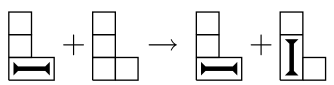
Suppose we wish to compare the two Domineering positions  and . We would examine the difference game

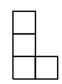
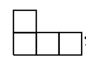
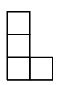
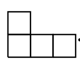


Now if Left moves first, we have



If Right moves first, we get



In both cases, Right is unable to move, so Left wins. This shows that  > , and hence Left prefers  over .

2.4 Canonical Forms

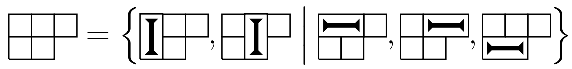
For any game G there is a unique smallest game that is equivalent to G. This is called the canonical form of G. The canonical form of any game can be obtained by performing two operations on G.

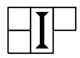
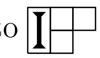
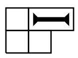
1. Removing Dominated Options
   Let
   $$G = \{A, B, C, \dots \mid X, Y, Z, \dots\}.$$
   If Left considers option B to be at least as good as A, that is, if $B \geq A$, then B is said to dominate A. In this case, A may be deleted from G without changing the game so that
   $$G = \{B, C, \dots \mid X, Y, Z, \dots\}.$$

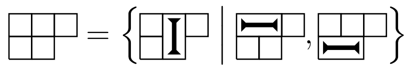
In a similar fashion, if $Y \leq X$ in G, then Y dominates X and option X may be deleted from G. That is

$$G = \{A, B, C, \ldots \mid Y, Z, \ldots\}.$$

Consider the following example:



Clearly Left's option  dominates her other possible move to  since the former does not leave a move for Right and the latter does. Similarly, Right's option  is dominated by his other two as well. We would rewrite the game as follows after removing the dominated options:



2. Removing Reversible Options

   Let

   $$G = \{A, B, C, \ldots \mid X, Y, Z, \ldots\}.$$

   Suppose that Left moves to A, and there is some right option of A, say A^R, that Right prefers to G, that is $A^R \leq G$. We can then replace option A in game G with the left options of A^R. More formally, if the left options of A^R are $\{L, M, N, \ldots\}$, then

   $$G = \{L, M, N, \ldots, B, C, \ldots \mid X, Y, Z, \ldots\}.$$

While removing dominated options certainly aligns with one's common sense, the process of reversing through options is not quite so clear. In the situation above, Left knows that if she moves to A, then Right will respond by moving to A^R. Left does not prefer A^R to the original position, so she will make the move to A only when she plans to immediately follow with a move to a left option of A^R. Thus it makes sense to simply replace option A in the original position by the left options of A^R. Figure 2.1 and Figure 2.2 show a before and after of this process.

2.5 Values of Games

We would like to assign numerical values to games to show what they are worth to each player. The most simple case is a game where one player cannot move while the

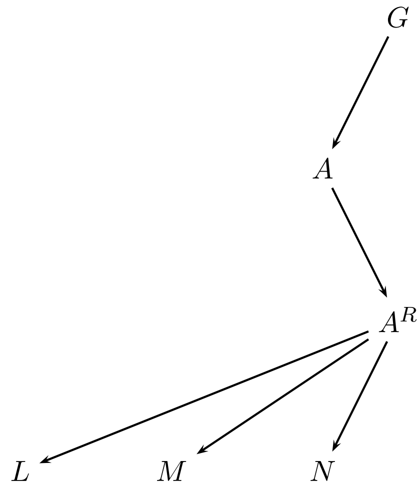


Figure 2.1: A is reversible

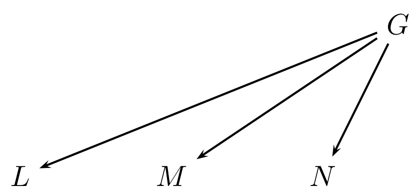
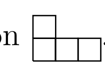
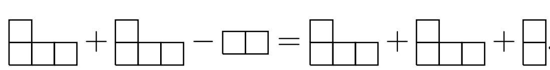


Figure 2.2: A has been reversed through

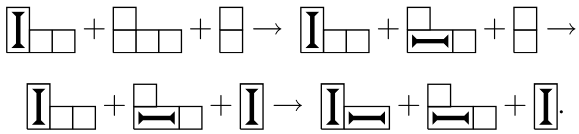
other player can move an integer number of times. If G is a game where Left can move n times, then we say G has value n. Similarly, if G is a game where Right can move a total of n times, then G has value $-n$. In Domineering, a $2n \times 1$ region would allow Left to place n dominoes while giving no moves to Right, so it would have value n. In a similar fashion, a $1 \times 2n$ region has value $-n$. In formal notation,

$$
\begin{aligned}
n &= \{n-1 \mid \} \text{ and} \\
0 &= \{ \mid \}.
\end{aligned}
$$

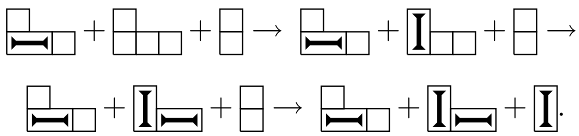
Many games are not worth an integer number of moves, so we will need to assign numbers to non-integer-valued games as well. Consider the position . If Left moves first, both players get one move. However, if Right moves first, then Left does not get to move at all. So it seems that this game should have a value between 0 and -1 since it's not quite worth a full move to Right but it is clearly better for Right than for Left. To determine its exact value, we play the following difference of games:

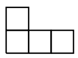
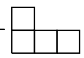
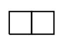
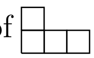


If Left moves first, we get



Right wins in this case. Now if Right moves first we have



Left wins here, so we see that the game is always a second player win under optimal play, and thus  +  =  = -1. Therefore, the value of  should be -1/2 since two copies of it are worth 1 move to Right.

For finite games, we need only to use the dyadic rational numbers. That is, we use only fractions of the form $m/2^n$ where m is odd and n is a positive integer. We define these valued games formally as follows:

For m odd and $n \geq 1$,

$$\frac{m}{2^n} = \left\{ \frac{m-1}{2^n} \,\middle|\, \frac{m+1}{2^n} \right\}.$$

There is also a large class of games with infinitesimal values. These will be defined formally for now and discussed in more detail later in tandem with particular Go positions. Table 2.1 contains many of these positions.

Below is a summary of how the values of various infinitesimal games compare to one another.

1. The value * is incomparable with 0, $\uparrow$, $\downarrow$, $\boldsymbol{+}_r$, and $\boldsymbol{-}_r$.

2. For $n \geq 2$, $n \cdot \uparrow > *$.

3. For $r_1 > r_2 > 0$ rational, $\uparrow > \boldsymbol{+}_{r_2} > \boldsymbol{+}_{r_1} > 0 > \boldsymbol{-}_{r_1} > \boldsymbol{-}_{r_2} > \downarrow$.

These rules will be highly useful later on when we are computing the value of entire Go endgames.

Lastly, we define

$$\begin{aligned} 0^n|G &= \begin{cases} 0||0^{n-1}|G & \text{if } n > 0, \\ G & \text{if } n = 0 \end{cases}, \\ G|0^n &= -\{0^n \mid -G\}. \end{aligned}$$

Symbol	Definition	Name
$*$	$= \{0 \mid 0\}$	star
$\uparrow$	$= \{0 \mid *\}$	up
$\downarrow$	$= -\uparrow$	down
$\uparrow*$	$= \uparrow + *$ $= \{0, * \mid 0\}$	up-star
$\Uparrow$	$= \uparrow + \uparrow$	double-up
$⤊$	$= \uparrow + \uparrow + \uparrow$	triple-up
$n \cdot \uparrow$	$= \uparrow + \uparrow + \cdots + \uparrow$ $= \{0 \mid (n-1) \cdot \uparrow *\}$	n-up
$n \cdot \uparrow*$	$= n \cdot \uparrow + *$ $= \{0 \mid (n-1) \cdot \uparrow\}$	n-up-star
$⧾_G$	$= \{0 \parallel 0 \mid -G\}$, $G > r$, $r > 0$ a number	tiny G
$⧿_G$	$= \{G \mid 0 \parallel 0\}$, $G > r$, $r > 0$ a number	miny G

Table 2.1: Some common infinitesimal games

Games of the form $\{0^n \mid G\}$ will arise often when looking at Go positions. We make the following basic observations that will be used later.

$$0^n|1 = \frac{1}{2^n}$$

$$0^n|0 = \begin{cases} (n-1) \cdot \uparrow & \text{if } n \text{ is even,} \\ (n-1) \cdot \uparrow* & \text{if } n \text{ is odd} \end{cases}$$

$$0^2| - G = ⧾_G \quad \text{if } G \text{ is greater than some positive number}$$

2.6 Game Trees and Birthdays

Games may be represented in full by a tree. Figure 2.3 shows the tree for 3/4. Moves made by Left correspond to edges moving to the left, while moves by Right are shown by edges moving towards the right. Note that we still draw in options that result from the same player making multiple moves in a row, even though moves would alternate in actual play. It is a common practice to omit dominated options when drawing the tree for a game.

The height of game tree is known as that game's *birthday*. For example, 3/4 has a birthday of 3, and we would say that 3/4 is *born on day* 3. Formally, the birthday of a game $G = \left\{G^L \mid G^R\right\}$, with G^L, G^R not both empty, is 1 more than the maximum

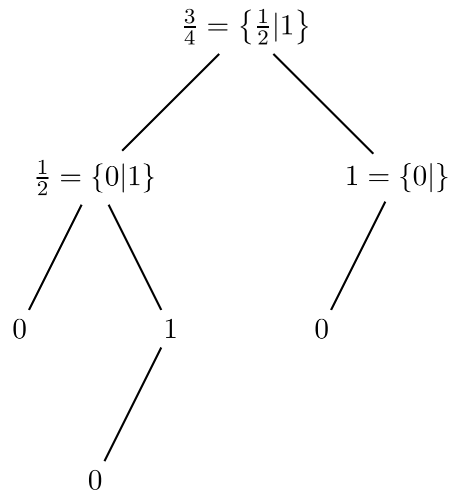


Figure 2.3: Game tree of 3/4

birthday of all the games in $G^L \cup G^R$. The birthday of the game $\{|\}$ is 0.

2.7 The Simplest Number and Number Avoidance

Suppose we have a game G that is not in canonical form. It is still possible to determine if G is a number. Given numbers $x^L < x^R$, we define the simplest number x between x^L and x^R as follows:

1. If there are one or more integers between x^L and x^R, then x is the one with the smallest absolute value.

2. If not, then x is the number of the form $\frac{i}{2^j}$ between x^L and x^R that minimizes j.

Finding the simplest number between two numbers is akin to finding the longest mark between two points on a ruler. By using this idea, we may find the number value for a game G in non-canonical form by means of Theorem 2.

Theorem 2 *Let G be a game whose options are all numbers and all $G^L < G^R$. Then G is the simplest number x that satisfies $G^L < x < G^R$.*

Proof: Let G be a game whose options are all numbers with all $G^L < G^R$, and let x be the simplest number such that $G^L < x < G^R$. To prove $G = x$ we show that the game $x - G$ is a second player win. Without loss of generality we may assume that Left moves first. There are two possibilities. If Left moves to $x - G^R$, then this game is negative, and hence is a win for Right since $x < G^R$ by assumption. For the other case, if Left moves to the game $x^L - G$, then Right may respond with a move to $x^L - G^L$. Now suppose that $x^L > G^L$. Then x^L is simpler than x but also satisfies $G^L < x^L < G^R$, contradicting our choice of x as the simplest number between G^L and G^R. Thus $x^L - G^L$ is non-positive and is a win for Right, hence $x - G$ is a second player win. ■

One important fact is that if x is a number and G is not, then the optimal move in the game $G + x$ is to play on G. This is stated formally in Theorem 3, the *Number Avoidance Theorem.*

Theorem 3 (Number Avoidance Theorem) *Let G be a game that is not a number, and let x be a number. Then, optimal play on the game $G+x$ is given by moving on G.*

Proof: Assume that the game x is in canonical form. We can equivalently say that if some $G + x^L \geq 0$, then some $G^L + x \geq 0$. Let's suppose that $G + x^L \geq 0$. G is not a number, thus it follows that $G + x^L > 0$. In this case Left will win if she moves first on $G + x^L$. Then, by reasoning inductively, we have $G^L + x^L \geq 0$. Since we must have $x > x^L$, it follows that $G^L + x \geq 0$. ■

2.8 Incentives of a Game

Incentives are used to show how much a game is worth to a player. The *left incentives* of a game G are the elements of

$$\begin{aligned} \Delta^L\{G\} &= G^L - G \\ &= \{H - G : H \in G^L\}. \end{aligned}$$

The *right incentives* of G are

$$\begin{aligned} \Delta^R\{G\} &= G - G^R \\ &= \{G - H : H \in G^R\}. \end{aligned}$$

The *incentives* of G, denoted $\Delta\{G\}$, are the elements of the union of the left and right incentives of G. That is,

$$\Delta\{G\} = \Delta^L\{G\} \cup \Delta^R\{G\}.$$

Incentives are defined in such a way that both players would prefer positive numbers instead of Right preferring negative. It is worth noting that the incentive for either player to move on a number $m/2^n$ (with m odd) is $-1/2^n$. This shows neither player would like to move on a number.

2.9 Stops of Games

Suppose two players are playing a game, and a number is reached. At this point, neither player wishes to move, and the outcome of the game is known. It makes sense for the players to stop at this point. Left would like the value of the stopped game to be as large as possible, and Right would like it to be as small as possible. If Left moves first, the value of the stopping position that is reached under optimal play is called the *left stop* of the original position, denoted $LS(G)$. The value reached if Right moves first is called the *right stop*, denoted $RS(G)$. They are defined together by

$$LS(G) = \begin{cases} G & \text{if } G \text{ is a number,} \\ \max(RS(G^L)) & \text{if } G \text{ is not a number,} \end{cases}$$

$$RS(G) = \begin{cases} G & \text{if } G \text{ is a number,} \\ \max(LS(G^R)) & \text{if } G \text{ is not a number.} \end{cases}$$

Although the standard play convention for combinatorial games does not involve a score, the concept of stops gives rise to a natural scoring system where the number of points received from a game is simply the value of its stop. This idea is of particular importance to us since Go employs a scoring system in which both players wish to maximize the number of points they have.

2.10 Cooling

All games can be classified as either hot, cold, or tepid. A game G is *hot* if $LS(G) > RS(G)$. A game is *cold* if it is a number. A game is *tepid* if it differs from a number by a non-zero infinitesimal. Both players would like to move first on a hot game, while neither player wishes to move on a cold game. A hot game can be made cold by the process of cooling, which makes the players pay a tax for the right to move first. For a game $G = \left\{G^L \mid G^R\right\}$, we define G_t, the game G cooled by t, as

$$G_t = \left\{G_t^L - t \mid G_t^R + t\right\}$$

unless there is some value $\tau < t$, so that G_τ is infinitesimally close to a number m. If this happens, then

$$G_t = m.$$

If the tax t on moving is made sufficiently large, then neither player wants to move, and the game is said to be frozen. (This occurs when G_t is infinitesimally close to a number m as above.) We say that m is the *mean* of G, and this can be thought of as the fair value for the game G independent of who gets to move first. The smallest value t such that G_t is frozen is called the *temperature* of G, and we say that G freezes to G_t. Conway proved that cooling is linear and order preserving in *ONAG* [4]. In particular, the following properties hold for games G, H and $t \geq 0$:

1. $G_t + H_t = (G + H)_t$
2. $G \geq H \Rightarrow G_t \geq H_t$
3. $\text{mean}(G + H) = \text{mean}(G) + \text{mean}(H)$
4. $\text{temp}(G + H) \leq \max\{\text{temp}(G), \text{temp}(H)\}$.

The process of cooling as well as reversing it are quite complex and have numerous applications to combinatorial games. For a more general treatment of the topic refer to [2].

2.11 Warming

In general, cooling is a many-to-one function, and as such, has no inverse. However, the process of cooling a game of Go by a value of 1 does have an inverse, which is known as *warming*. Let $G = \left\{G^L \mid G^R\right\}$ be a game. We define the operation of warming G, denoted $\int G$, by

$$\int G = \begin{cases} G & \text{if G is an even integer,} \\ G* & \text{if G is an odd integer,} \\ \left\{1 + \int G^L \mid -1 + \int G^R\right\} & \text{otherwise.} \end{cases}$$

We will demonstrate that warming is in fact the inverse of cooling by 1 point. That is, we will show that $G = \int G_1$ where G_1 is the result of chilling G by 1. This result, as well as the strategy we use for the proof, are due to Berlekamp and Wolfe in [3]. This proof will first require some additional terminology.

We say that a Go position is *even* (or *odd*) if the sum of the number of empty intersections and number of prisoners captured is even (or odd). These parities behave as expected in that the sum of two even (or two odd) games is even, and the sum of one even game and one odd game is odd. Additionally, the parity alternates during play. We say that a Go position is *elementary* if, when it is completely played out, every point on the board either contains a stone or is territory for one of the players. To use Go terminology, a position is elementary if it contains no kos or sekis. For the rest of the section, we let G be an even elementary Go position that is in canonical form.

In order to aid us in our proof that warming inverts cooling by one point, we will introduce a new function, $f(G)$ which is said to *chill* the game G. It is defined by

$$f(G) = \begin{cases} n & \text{if G has form } n \text{ or } n*, \\ \{f(G^L) - 1 | f(G^R) + 1\} & \text{otherwise.} \end{cases}$$

Note that applying f is the same as cooling by one point except that we ignore the case when G_τ is a number for some $\tau < 1$.

Lemma 1 *Either G is of the form n or $n*$ with n an integer, or $LS(G) > RS(G)$.*

Proof: Suppose that $LS(G) = n = RS(G)$. If n is even, then consider the difference game $G - n$. Neither player will play on n due to the Number Avoidance Theorem. Since G is even by assumption, it will take an even number of moves on $G - n$ to reach $n - n$. Thus $G - n$ is a second player win and $G = n$.

If n is odd, then consider $G - n*$. Given the choice, Left will always play on G, since allowing two consecutive moves by Right on G will cause the stop to be n or less. The second player can then win by always playing on G until it becomes n. This will take an odd number of moves, leaving Left to move on $n*$, which brings the game to $n - n$. Thus, $G = n*$.

The same argument applies to even subpositions of G, and a similar argument holds when considering odd subpositions of G. ■

Lemma 2 $G = \int f(G)$.

Proof: If G is of the form n or $n*$, then the result follows directly from the definitions of chilling and warming. If not, we wish to show that G is in canonical form, and hence warming will invert chilling.

To show that G has no dominated option, suppose for contradiction that G has two left options G^{L_1} and G^{L_2} such that $f(G^{L_1}) \geq f(G^{L_2})$. Lemma 1 asserts that the left stops of G^{L_1} and G^{L_2} exceed their right stops. Since this is the case, to play optimally on $G^{L_1} - G^{L_2}$ neither player will play on a leaf position of n or $n*$ until the games are settled. We want to find a winning strategy for Left on the game $G^{L_1} - G^{L_2}$.

Suppose that we are playing the game $f(G^{L_1}) - f(G^{L_2})$ and that it takes an even number of moves to reach a stop. In this case, any 1-point adjustments resulting from applying the function f cancel out. Also, since G is an even position, if $G = 0$, then the corresponding stopping position in $G^{L_1} - G^{L_2}$ is 0. Thus $G^{L_1} - G^{L_2} \geq 0$ and we get a Left win.

Assume, on the other hand, that it takes an odd number of moves to reach a stop. Since $f(G^{L_1}) - f(G^{L_2})$ is a win for Left, then the stopping position has to be at least 1. If it is 1 and we look at the corresponding stopping position in $G^{L_1} - G^{L_2}$, the value is $*$, so Left can make the final move and win. If the value of $f(G^{L_1}) - f(G^{L_2})$ is greater than 1, Left can only do better.

In both cases, we have a win for Left, so $G^{L_1} - G^{L_2} \geq 0$. Thus, we conclude that $f(G)$ has no dominated options. We can use a parallel argument to demonstrate that $f(G)$ also is free of reversible options. It follows that $f(G)$ is in canonical form and that warming inverts chilling.

Lemma 3 *If the mean of G is $i/2^j$ where i is odd, then the temperature of G is at least $1 - 1/2^j$.*

Proof: First, consider the game tree of a game H and take H' to be any subtree of H. The distance from the mean value of H' to any of its left or right stops cannot exceed the temperature of H'. From this, we can conclude that if H is a game whose stopping positions are multiples of $2t$, with $t = 1/2^i$, then the temperature of H is either zero or is at least t. We can also say that the stopping positions of H_t are multiples of t.

To prove the lemma, we apply this idea repeatedly. We begin by cooling G by 1/2, then by 1/4, and so forth. We end this process when we cool G by $1/2^i$. ■

Lemma 4 $f(G) = G_1$.

Proof: Suppose that the statement does not hold for some G, and choose G to be a counterexample with a tree of minimal depth. We will show that the game $f(G) - G_1$ is a second player win. By our choice of G, any follower of G will satisfy $f(G) = G_1$. Additionally, the temperature t of G will be less than 1. Now, let $G_1 = i/2^j$ with i odd. From the previous lemma, it follows that $t \geq 1 - i/2^j$. There must be some left option in G_t that has a right stop of $i/2^j$. Consider the corresponding stopping positions in $f(G)$. They are either $i/2^j$ with Right moving last or at least $(i-1)/2^j$ with Left moving last. Thus if Right moves on the game $f(G)-G_1$ to $f(G)-(i-1)/2^j$, Left has a winning response. Since Left's options in $f(G)$ are no better than those in G_1, we have $f(G) = G_1$. ■

Theorem 4 $\int G_1 = G$.

Proof: The result is immediate from Lemmas 2 and 4. ■

Corollary 1 *Cooling by one point and warming are inverses up to a dame (neutral point) in elementary Go positions. That is,*

$$\int G_1 = G \quad \text{or} \quad \int G_1 = G * .$$

Proof: If G is an even position, then we simply apply the theorem. If G is odd, then $G + *$ is even. Now, since $\{G^L|G^R\}* = \{G^L * |G^R *\}$ in games with unequal left and right stops, we have that

$$\int G_1 = \int (G + *)_1 = G * . \qquad ■$$

Chapter 3

Analyzing Go Positions

The endgame of a game of Go can be treated as a sum of numerous subgames, each of which is hot (both players wish to move first). In Figure 3.1, we see an endgame that we would like to evaluate.

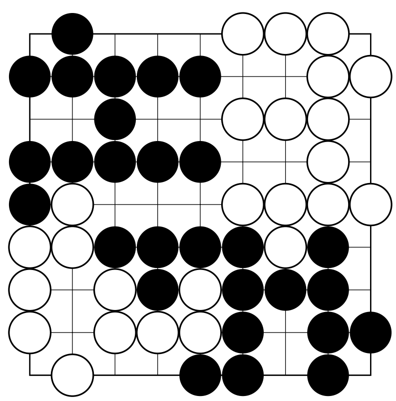


Figure 3.1: A typical Go endgame

We begin by setting aside all territory that is under control (completely surrounded by one color). We then partition the game by sectioning off each unresolved subgame and determining the value of each. This will allow us to determine who will win the game and the order in which moves should be played. This chapter will provide these values for a variety of common small positions. We will then use a technique called *thermography* to determine precisely how to play optimally on these complex sums of hot subgames.

3.1 Chilling and Marking

Before we attempt to compute the values of small games we must first discuss a few basic conventions and definitions. It is standard that Black is the Left player, and White is the Right player. In any Go diagram, stones that are connected to a line that runs off the diagram cannot ever be captured and are said to be *immortal*.

Black and white *markings* are drawn on the vertices of a diagram to denote that a certain integer number of points is being given to each player. A black or white dot indicates that a single point is being subtracted from the player of that color. The markings are generally applied in such a way that the value of an area is kept close to zero.

Chilling is used in conjunction with marking to simplify many results. We denote that a Go position has been chilled by either clipping around the outside edges of the diagram or by outlining and shading the chilled region. Recall that chilling involves placing a single point tax on each move. If we enforce this 1-point tax we refer to this as *playing the chilled game*. When playing the chilled game, we follow the rule that every time a player moves, they must add a marking of their color or remove one of the opponent's color. This corresponds to paying the tax for the move. Figure 3.2 shows the endgame from Figure 3.1 after it has been chilled and marked.

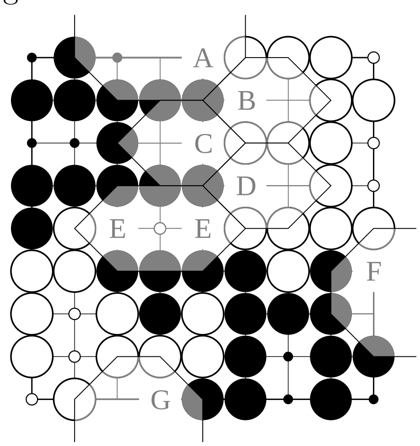


Figure 3.2: Endgame after marking and partitioning

The issue of which player gets to move next is highly important in the game of Go. There is a distinction between moves that keep initiative and moves that do not. Some moves make an overwhelming threat that would be disastrous if not answered immediately. The moves that require the opponent to make an immediate reply and thereby keep the initiative are called *sente* moves, while those that do not require an immediate local response are called *gote* moves.

3.2 Blocked and Unblocked Corridors

The first small positions we will analyze are called *blocked corridors* and *unblocked corridors*. In blocked corridors, the opponent threatens to invade the territory from just one end, and in unblocked corridors, an invasion is threatened at both ends. Table 3.1 shows these two types of positions for length $n \leq 5$.

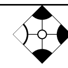
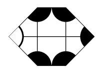
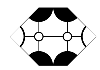
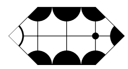
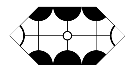
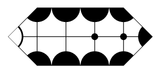
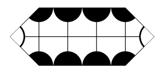
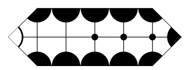
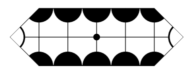
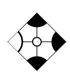
Length n	Blocked	Unblocked
$n = 1$		
$n = 2$		
$n = 3$		
$n = 4$		
$n = 5$		

Table 3.1: Blocked and unblocked corridors

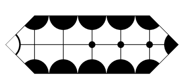
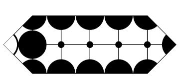
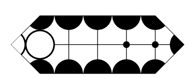
We may calculate the chilled, marked value of these positions by means of the following theorem and corollary.

Theorem 5 *A chilled blocked corridor of length* $n \geq 1$ *having* $n - 2$ *markings has a value of* 2^{1-n}.

Proof: We proceed by induction on the length n of the corridor. When $n = 1$ we have

 $= \{0|2\} = 1 = 2^{1-1}$

so the formula holds. Then a corridor of length $n \geq 2$ is equal to $\{0|2^{2-n}\} = 2^{1-n}$. Below, we illustrate the case when $n = 5$.

 $= \left\{\right.$  $\Big|$  $\left.\right\}$

$$= \left\{0 \;\middle|\; 2^{1-(n-1)}\right\} = 2^{1-n}$$

Thus the formula holds for all n. ■

Corollary 2 *A chilled unblocked corridor of length* $n \geq 2$ *having* $n - 4$ *markings has a value of* 2^{3-n}.

Proof: The two open points on either end of an unblocked corridor are equally good moves due to symmetry. We may assume that if White invades at one end, then Black will block the other. Thus an unblocked corridor of length n is equivalent to a blocked corridor of length $n-2$. We then apply the previous theorem to compute the desired value. ■

With these results we may now calculate the value of Figure 3.2. The value of each region is found using our formulas and then summed along with the marked points to get the total value of the game. We have

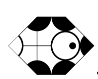
$$\begin{array}{ll} A = 1/4 & E = 1 \\ B = -1/2 & F = 1/2 \\ C = 1/2 & G = -1/2. \\ D = -1/2 & \end{array}$$

Both Black and White have 7 marked points, so the sum of this game is $7 - 7 + \frac{1}{4} - \frac{1}{2} + \frac{1}{2} - \frac{1}{2} + 1 + \frac{1}{2} - \frac{1}{2} = \frac{3}{4}$. Thus, if Black moves first, the 3/4 is rounded up and she wins by 1 point. If White moves first, the 3/4 is rounded down and the game is a tie.

Now that we know what the score should be under optimal play, our final task is to determine in what order the various subgames should be played on. To do this we simply compute the incentives of each subgame. The computation involves finding how much a player stands to lose by not moving on a particular game, so we simply subtract the value of the position where the opponent has been allowed to make one move from the value of the original position. Once we have the incentives, the remaining subgame with the highest incentive is always played on first. Table 3.2 shows the incentives for blocked and unblocked corridors. It is clear that if an endgame consists only of corridors, then it is ideal to move on the longest corridor first.

3.3 Ups, Downs, and Stars

We need not look very hard to discover small positions whose values cannot be given by numbers. They will require us to use infinitesimals such as $\uparrow, \downarrow$ and $*$. Consider the position


.

When we attempt to find its value by playing the chilled game we get

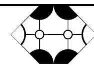
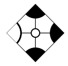
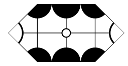
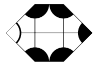
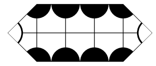
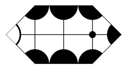
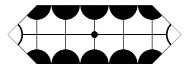
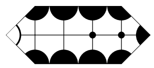
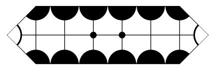
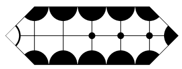
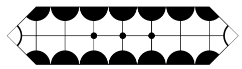
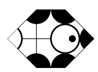
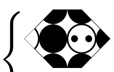
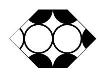
Blocked	Unblocked	Value	Incentive
		2	
		1	$1-2=-1$
		$\frac{1}{2}$	$\frac{1}{2}-1=-\frac{1}{2}$
		$\frac{1}{4}$	$\frac{1}{4}-\frac{1}{2}=-\frac{1}{4}$
		$\frac{1}{8}$	$\frac{1}{8}-\frac{1}{4}=-\frac{1}{8}$
		$\frac{1}{16}$	$\frac{1}{16}-\frac{1}{8}=-\frac{1}{16}$

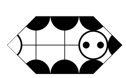
Table 3.2: Incentives of corridors

 $= \left\{\right.$  $\big|$  $\left.\right\}$

$$= \{0|0\}$$
$$= *.$$

Here, Black gets 2 points in the unchilled, unmarked game, but these are lost due to the chilling yielding the 0 game. Similarly, White saves his stone from capture, but this point is taxed away leaving 0. So, this chilled and marked position is equal to $*$. To compute sums of games involving $*$ we use the simple observation that $*+*=0$. Since $*$ is incomparable with 0, we must address this when we are looking at the game sums. Suppose we have a game of Go where the sum of all the subgames is $1\frac{1}{8}*$. Since $1\frac{1}{8}* > 0$, this would be rounded as before with Black winning by two points if she moves first and Black winning by one point if she moves second. But since $* \parallel 0$, in the case of a game value of $1*$, we would round * to 1 if Black moves first and round * to -1 if White moves first. Thus the game $1*$ is a win for Black by two points if she moves first, but a tie if White moves first.

If we look at longer corridors with stranded pieces inside, we will need even more infinitesimal values. For example,

 $= \{0|*\} = \uparrow$.

To compute game sums containing $\uparrow$ or $\downarrow$, we use the rules that $\uparrow$ is incomparable with $*$ and $\uparrow > 0$. Since $\uparrow = -\downarrow$ switching the black and white stones in the above position will give us a game with value $\downarrow$. Recall that although $\uparrow$ is infinitesimal like $*$, we have that $\uparrow > 0$. So for a game that sums to $1\uparrow$, Black would win by two points moving first and by one point moving second. But since $\uparrow *$ is incomparable with 0, Black would win the game $1 \uparrow *$ by two points moving first and tie if White moves first. Table 3.3 summarizes the values for positions of this type.

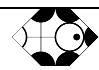
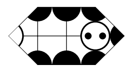
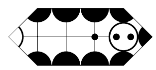
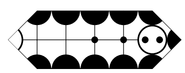
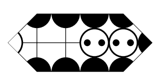
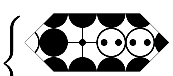
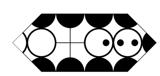
Length	Position	Value	Black Incentive	White Incentive
$n = 2$		$*$	$*$	$*$
$n = 3$		$\uparrow$	$\downarrow$	$\uparrow*$
$n = 4$		$\Uparrow*$	$\Downarrow*$	$\uparrow*$
$n = 5$		$\Uparrow\!\!\uparrow$	$\Downarrow\!\!\downarrow$	$\uparrow*$

Table 3.3: Values of stars and ups

3.4 Tinies and Minies

The next class of small positions we will examine are very similar to those giving rise to ups, downs, and stars with the difference that there will be multiple stones in danger of capture. Consider the position below.

 $=$ $\Bigg\{$  $\Bigg|$  $\Bigg\}$

$$= \{0 \,\|\, 0 \mid -2\}$$

This is precisely the game $+_2$. The general rule here is that if after one move White is threatening to make an n point gote play (one that does not need to be responded to by Black), then the game has value $+_{n-2}$. This result is summed up formally in the following theorem whose proof we omit.

Theorem 6 *A chilled corridor of length n that has $x + n - 1$ markings and results in an x point gote play has a value of* $0^{n+1}|2 - x$.

If we switch the black and white stones in $+_x$, then we get the position $-_x$. Recall that if $x > 0$, then $+_x$ is a positive infinitesimal that is less than $\uparrow = +_0$. In general, if $x > y$ then $+_x$ is infinitesimal with respect to $+_y$.

3.5 Sums of Hot Games

Sums of games involving tinies and minies with different subscripts quickly become too cumbersome to compute. In these cases it is best not to try to find the exact value of a game, but rather focus on how to win independent of this knowledge. If Black has a winning strategy, then she can play well enough to force a win by following the guidelines given below.

1. First, form pairs of any infinitesimal games that are negatives of each other. These positions can be ignored. If White plays on one of these positions, then Black would simply answer by playing on its negative.

2. Next, long corridors should be invaded and attacks on $+$'s should be defended against. If $0 < x < y$, Table 3.4 shows the order in which these positions should be played on. Moves that are higher up on the table should be played on first.

3. Any remaining infinitesimals should either be positive or $*$'s. If there are an odd number of $*$'s, then the best play is to move on one of them. If there are an even number of $*$'s, then any positive infinitesimal may be played upon. Table 3.5 summarizes the positions and play order.

4. At this point there should be only number values left, thus play is easily determined from their incentives. As before, the longest corridors are worth the most. Equivalently, the remaining number with the largest denominator should be played first. Once all numbers are resolved, any dame are played. Table 3.6 summarizes this step.

The full proof that following these steps provides a win is given by David Moews [8]. The proof contains a classification of the different possible sums of winning positions and then uses induction to show that the moves provided above are good enough for Black to win. In addition, Moews extends this result to include sums of games of the form $\{x|0^n\}^m$.

3.6 Rooms

Our next task is to analyze small subgames that are more complex than the simple corridors we have seen thus far. We examine invasions of small territories that are in the middle of the board. These positions are referred to as rooms. Due to the large number of various rooms that may arise in endgame play, we will not show the

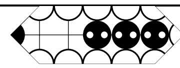
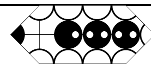
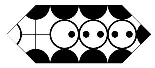
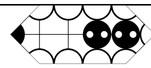
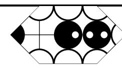
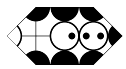
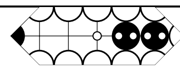
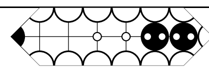
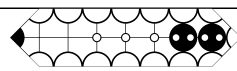
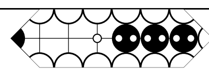
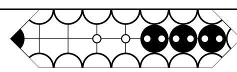
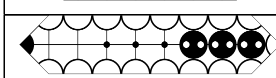
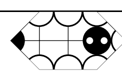
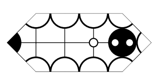
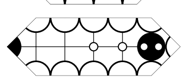
	⧿$_y$
 	$\{y\|0\}$ $\{0\|-y\}$
	⧿$_x$
 	$\{x\|0\}$ $\{0\|-x\}$
	⧿$_x\|0$
	⧿$_x\|0^2$
	⧿$_x\|0^3$
⋮	⋮
	⧿$_y\|0$
	⧿$_y\|0^2$
	⧿$_y\|0^3$
⋮	⋮
   ⋮	$\downarrow$ $\Downarrow *$ ⤋ ⋮

Table 3.4: Optimal play order

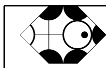
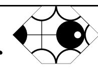
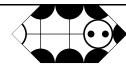
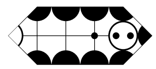
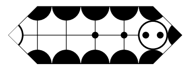
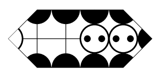
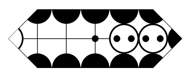
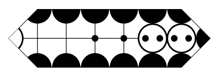
 **or** 	$*$ (possibly)
	$\uparrow$
	$\Uparrow *$
	$\Uparrow\!\!\uparrow$
⋮	⋮
or	**or**
	$+_x$
	$0 \mid +_x$
	$0^2 \mid +_x$
⋮	⋮

Table 3.5: Optimal play order continued

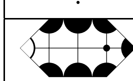
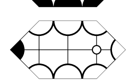
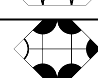
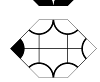
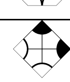
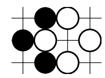
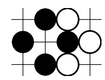
⋮	⋮
	$\frac{1}{4}$
	$-\frac{1}{4}$
	$\frac{1}{2}$
	$-\frac{1}{2}$
	0

Table 3.6: Optimal play order continued

computations involved in computing the values of each. Instead, we will provide a catalog of these small positions and their values.

It will be useful to now introduce the ko at this point since it may arise in some of the possible rooms. Even though the rules prevent the entire Go board from ever repeating a position, it is possible that a subgame could repeat locally. For more information on ko rules, see Appendix A.4. We define the basic, single-point ko positions as follows:

Ⓚ = 

$\overline{Ⓚ}$ =  .

Because of the loopy nature of the ko position, the game tree, shown in Figure 3.3, is somewhat unique.

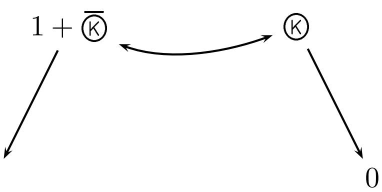
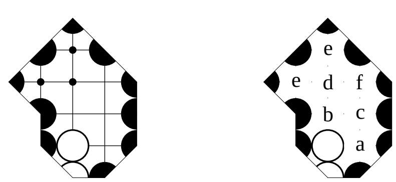


Figure 3.3: Game tree of simple ko

In order to be able to easily look up a position, we associate a grid of numbers with each room. For a given position, we examine the graph consisting of the empty vertices. The number of vertices in the graph that are at distance i from the invading stone and with degree j is put in the ith row and jth position from the right. Rooms are then listed in our catalog lexicographically by their number grids. While it is possible for different rooms to have the same number grid, this method sufficiently simplifies the task of locating a given room. To better understand this method, consider the following example:



11
1100
12

Of the vertices that are distance 1 from the invading stone, one has degree 1 (at a) and one has degree 2 (at b). Thus the first row of our grid is 11. Of the vertices with distance 2 from the invading stone, one has degree 3 (at c) and one has degree 4 (at d) giving us 1100 as the second row of our grid. Finally, of vertices at distance 3, two have degree 1 (at e) and one has degree 2 (at f), so the final row of the grid is 12.

Table 3.7 shows all rooms having 3, 4, or 5 empty vertices enclosed. The value given is the chilled, marked value of the corresponding room.

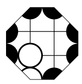
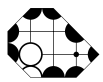
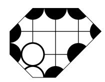
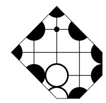
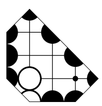
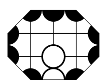
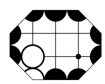
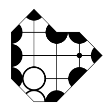
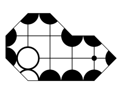
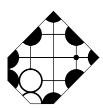
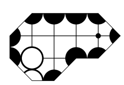
	0		$-\frac{1}{4}$		$\frac{1}{2}$		$\frac{1}{2}\|0$
2 10		11 11		2 100 1		101 12	
	0		$\frac{1}{2}$		$\frac{1}{8}\|0$		$\frac{3}{8}$
20 12		12 20		11 110 10		11 101 1	
	$\frac{1}{2}*$		$*$		$\frac{1}{4}$		
11 20 1		2 1000 2		2 100 10 1			

Table 3.7: Rooms with 3, 4, and 5 vertices

3.7 Multiple Invasions

The last situation that we wish to investigate is that of an unconnected group that is simultaneously threatening to invade enemy territory at multiple points. Consider the situation in Figure 3.4.

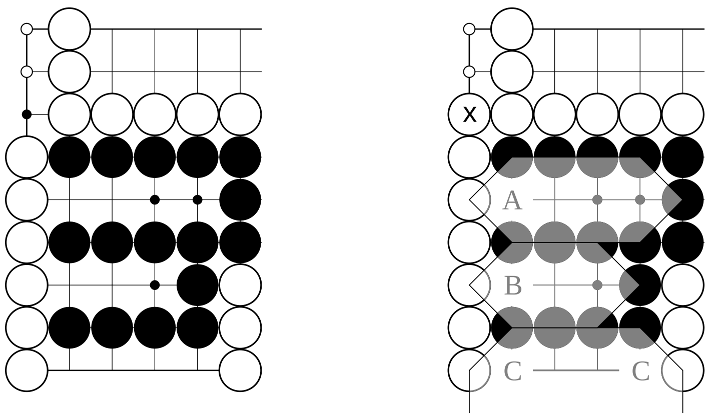


Figure 3.4: Multiple invasions by white group on left

Here, the white group invading on the left is not connected as it is in the marked position on the right diagram. This potential point of connection is refered to as a *socket*. In order to compute the value of the whole position, it is necessary to compute the value for each one of the corridors. The values of the corridors are then added together, and we determine the total value of the game as follows:

1. If the sum is greater than or equal to 1, then this is the actual value of the position.

2. If the sum is less than 1 and there is an unblocked corridor with value m then the shortest unblocked corridor has value $m/2$, and other unblocked corridors are unaffected. The value of any blocked corridors is altered as well and now has value $bm/2$, where b was the previous value.

3. If the sum is less than 1 and there is no unblocked corridor, then the value of the position has the form $⧿_x|0^n$ where n is one less than the number of blocked corridors being invaded. The value x hinges on the size of the invading group.

3.8 Example Endgame

Now we will apply the results given so far in this chapter to actual Go endgames. As before, we will partition each endgame position into simple subgames, determine the value of the subgames, and finally identify the best move based on these values. We will assume that Black is to play next. We begin be analyzing the 13×13 game shown in Figure 3.5.

Note that we have multiple invasions by the unconnected black group on the top left and by the white group on the bottom right. We will need to apply the techniques

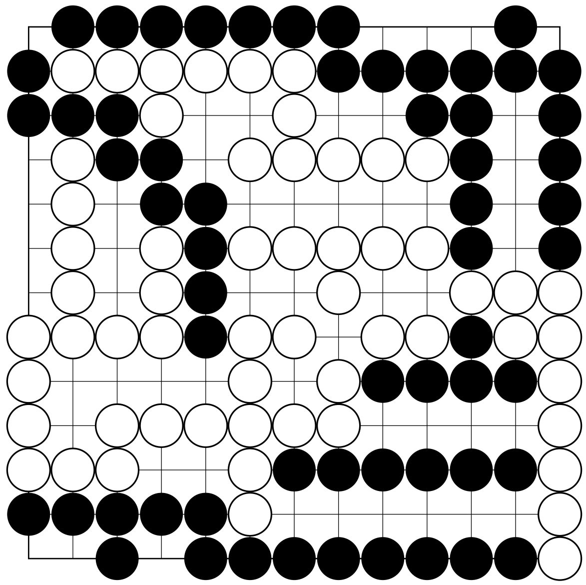


Figure 3.5: Example endgame 1

from Section 3.7 accordingly. Figure 3.6 shows the game after chilling and marking.

The Black invading group borders on the six blocked corridors A=-1/4, C=-1/8, D=-1/4, E=-1/4, F=-1/2, and G=-1/16. The absolute value of the sum of these is 23/16, which exceeds 1. In this case, the values of these positions are unchanged. The White invading group on the left connects to B=1/8, H=1/2, and I=1/8. Since the sum is less than 1, the shortest unblocked corridor at H becomes $\frac{1/2}{2} = 1/4$, I stays unchanged, and B becomes 1/8*1/4=1/32. We summarize the values below.

$$A = -1/4 \qquad D = -1/4 \qquad G = -1/16$$
$$B = 1/32 \qquad E = -1/4 \qquad H = 1/4$$
$$C = -1/8 \qquad F = -1/2 \qquad I = 1/8$$

By combining these values with the 12 marked black points and 12 marked white points, we find the sum of the game to be $12 - 12 - 1/4 + 1/32 - 1/8 - 1/4 - 1/4 - 1/2 - 1/16 + 1/4 + 1/8 = -1/32$. Thus, if Black moves first, the -1/32 is rounded up to 0 and the game is a tie. Since there are only number values, the best play is on the number with the highest incentive. Hence the optimal move is to block the corridor at I.

We now turn our attention to the full-size 19×19 game in Figure 3.7. After chilling and marking, the game appears as shown in Figure 3.8.

Many of the subgames here are familiar and easily dealt with, but there are a couple situations which require special attention. The position in area B is new to us, so we compute its value.

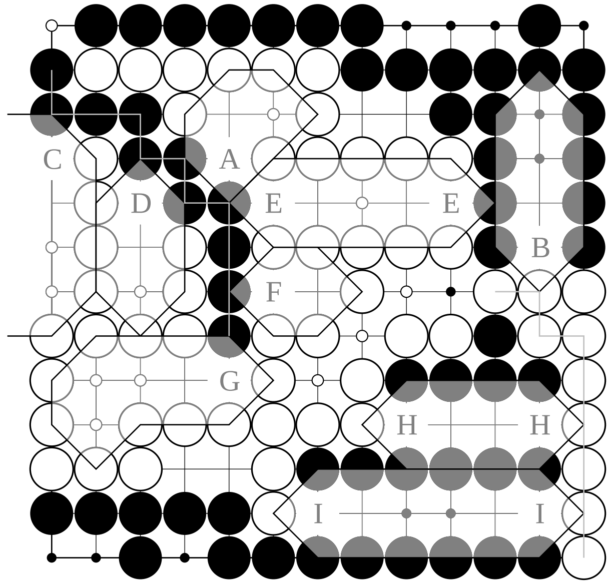


Figure 3.6: Example endgame 1 chilled

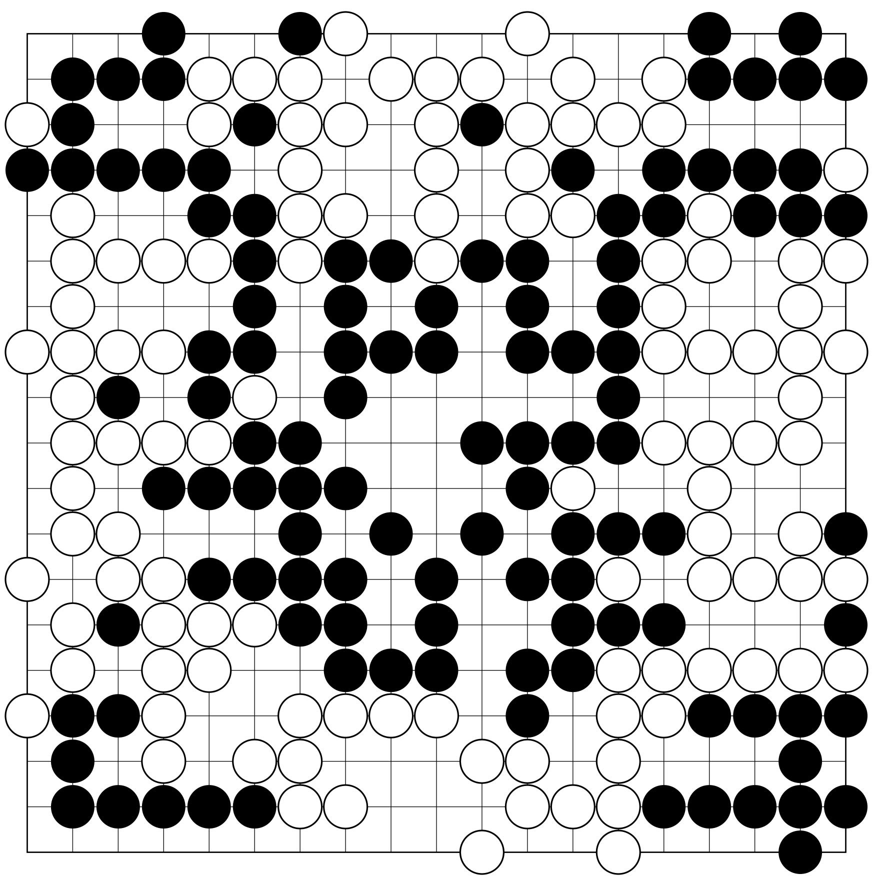


Figure 3.7: Example endgame 2

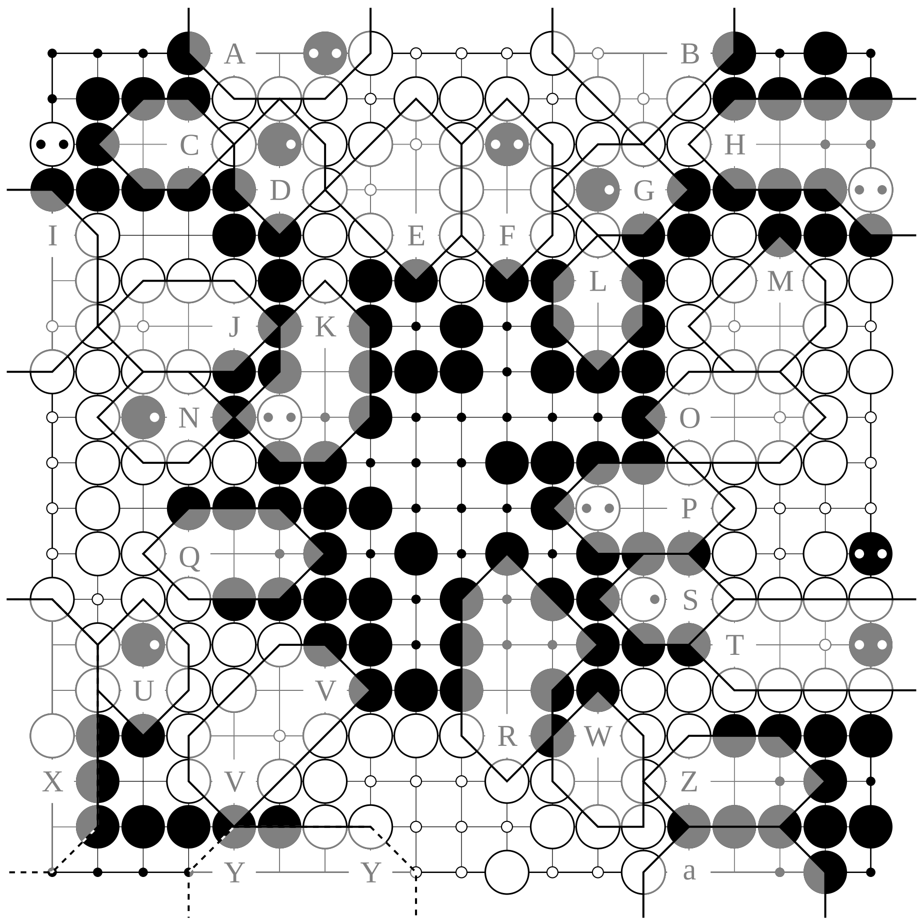
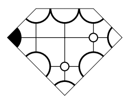
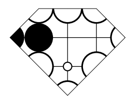
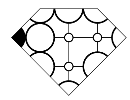
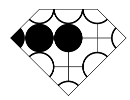
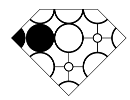


Figure 3.8: Example endgame 2 chilled



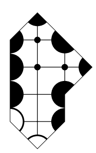
= {  |  }

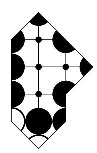
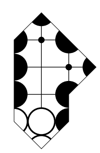
= { {  |  } | 0 }

$= \{\{0 \mid 0\} \mid 0\}$

$= \downarrow$

Using this fact, we can also compute the value of region R. We have



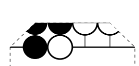
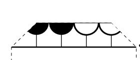
= {  |  }

$= 0|\uparrow$

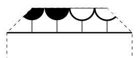
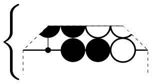
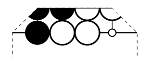
$= 0^3|0$

$= \Uparrow *$

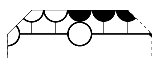
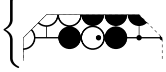
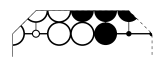
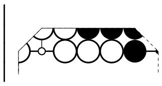
We have not previously encountered positions X or Y, so we now turn our attention to them. We first want to verify that the two regions are independent of one another. A check reveals that none of the stones that we treated as immortal can be captured during the course of play and that reversible options in both the regions still apply. For region Y, we have

 $\geq$ .

This means that moves by both Black and White will reverse. Thus

 $= \Big\{$  $\Big|$  $\Big\} = \{0|0\} = *.$

Similarly, in position X we have

 $= \Big\{\Big\{$  $\Big|$  $\Big\}\Big|$  $\Big\} = \{*|0\} = \downarrow.$

We now have values for all of the subgames, which we summarize below.

$A = \downarrow$	$J = -1/4$	$S = *$
$B = \downarrow$	$K = \Uparrow *$	$T = \Downarrow *$
$C = 1/2$	$L = 1/2$	$U = *$
$D = *$	$M = -1/4$	$V = -1/4$
$E = \downarrow$	$N = *$	$W = -1/2$
$F = \downarrow$	$O = -1/4$	$X = \downarrow$
$G = *$	$P = \uparrow$	$Y = *$
$H = \Uparrow$	$Q = 1/4$	$Z = 1/4$
$I = -1/4$	$R = \Uparrow *$	$a = 1/4$

The sum of the subgames is $\uparrow *$. So, Black can win by moving first. Since there are no minies, the best move is to play on $\Downarrow *$ in region T.

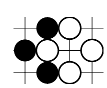
Chapter 4

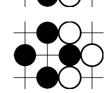
Current and Future Research

We will now examine some potential further applications of the techniques covered in Chapter 3. For example, the methods we have discussed so far are directly applicable only to endgame positions. It would be natural to attempt to use these techniques in order to simplify positions that occur earlier in the game as well. There are also situations we have not fully examined, such as ko, that can arise in endgames and which can be analyzed by extending previous results. In addition to discussing situations such as these, we will look at the results given in some recent papers.

4.1 Ko Positions

We have avoided a full treatment of ko positions thus far because we have wished to treat Go as a finite game. Allowing for kos means that Go positions can potentially repeat locally in a particular subgame we are examining. Thus the subgames we are analyzing may become become loopy (non-finite) games. Fortunately, we can modify our techniques to include simple repeated positions without losing too much. As in Chapter 3, we define the basic, single-point ko positions below.

$\textcircled{K}$ = 

$\overline{\textcircled{K}}$ = 

If we assume that Left can never win the ko, then we get $* \leq \textcircled{K}$. On the other hand, if we assume that Left can always win the ko, then we have $\textcircled{K} \leq \{1* \mid 0\}$. By chilling and combining these results we get the following bound on the chilled value of a single point ko:

$$0 \leq \textcircled{K}_1 \leq 1/2.$$

Thus the maximum incentive to move on a single point ko is $\int -1/2$, and it follows that the only worse move is to play on a dame. As such, kos have little value and will not be resolved until after other positions in the endgame.

If we wish to include more complex kos in addition to the single point positions, we require extensive alterations to our techniques. In general, complex kos are not contained locally in a single small subgame. Thus when analyzing a complex ko, it is not reasonable to treat a Go board as a sum of games, and this tends to undermine the entire analytic process we have been using.

4.2 Life and Death Problems

One of the most important game play elements in Go is that groups of stones can be made invulnerable to capture if they are given certain structural characteristics. In general, if a single group contains two or more separate enclosed vertices, called eyes, then the group cannot be captured. On the other hand, if a group has fewer than two eyes, then it is still vulnerable to capture later in the game. For additional information, consult Appendix A.5.

Life and death problems involve the issue of judging whether a particular group has the potential to form two eyes or not. It is also important to determine how a player can create said eyes or how his opponent can prevent him from doing so. Life and death problems generally arise in the midgame and thus we are not able to apply our endgame analysis techniques directly. However, since life and death problems fundamentally involve counting, we are still able to apply combinatorial game theory to them.

Howard Landman looks at the values of eyes in [6]. Landman examined this issue by attempting to determine which game-theoretic values can possibly occur in life and death situations. In order to simplify matters, a modified rule set is used where the number of points a player receives is equal to the number of eyes that she forms. Single group positions in this game are then cataloged, and their values are computed. Landman also investigated life and death problems that involve two or more groups and discovered that the theory quickly grows more complex.

4.3 Hardness Problems

There have recently been numerous investigations into the hardness of various classes of Go positions. We provide a very brief list of some of the major results. Lichtenstein and Sipser [7] first proved that Go is PSPACE-hard, and Robson [12] later showed it to be EXPTIME-complete under the standard Japanese rules. Most recently, Wolfe [15] used a series of reductions to demonstrate that Go endgames are PSPACE-hard. Here, we are defining Go endgames to be those in which any local area where play may occur on the board has a game tree of polynomial size, and each such area is isolated from other areas by live stones.

4.4 Computer Go

Naturally, programmers would like to incorporate the new endgame analyzation techniques into Go AI programs. Muller and Gasser discuss their work in this area in [9]. They used the techniques that we described in Chapter 3, as well as extensions to include kos, to write a computer program called Explorer which determines optimal play on a certain class of endgames.

The program works by first partitioning a board into safe areas (those requiring no further play), and endgame areas (those that have not been fully settled). Once this is accomplished, the game tree of each endgame area is determined, and they are subsequently converted to mathematical games and simplified. Finally, the game is calculated as a sum of these various endgames, and the optimal move is determined using our mathematical techniques.

Muller and Gasser wish to extend their program to play a larger number of endgames and also offer several ideas as to how an endgame algorithm could do so. Some possibilities include relaxing the requirements for a subgame to be considered 'safe' or trying to simply determine how to play for a win on a given endgame rather than play optimally.

Appendix A

Rules of Go

A.1 Overview

The game of Go is played between two players, Black and White, on a 19x19 square grid of intersecting lines. Play begins with Black, and players alternate turns thereafter. Upon a player's turn, he places a stone of his color on an open vertex of the board. Once a stone is placed, it does not move unless the stone is captured, in which case it is removed from the board and kept as a prisoner. The main goal of the game is to control the most territory on the board as possible by surrounding it with one's own stones.

A.2 Capturing

Multiple stones that are connected via adjacent vertices function as a single unit called a *chain*. Every open intersection that is adjacent to a chain is called a *liberty* of that chain. It is possible for a player to capture an opponent's chain by occupying all of its liberties with his own pieces. In Figure A.1, Black can capture each of the white pieces or chains by playing on any of the marked vertices.

White can temporarily avoid capture by playing in the same vertices, thereby raising the number of liberties of the corresponding chains. Any pieces belonging to a chain that is captured are removed from the board and kept as prisoners.

A.3 Suicides

It is forbidden for either player to place a stone such that the resulting chain would have no liberties. Black cannot play in any of the marked positions in Figure A.2.

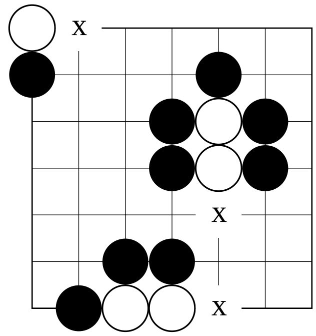


Figure A.1: Black may capture white

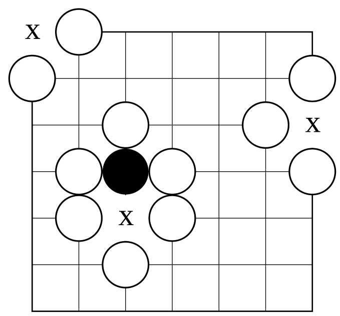


Figure A.2: Moves at x are forbidden

However, if a player places a stone that would result in capturing one or more of his opponent's pieces, then that capture takes precedence. The captured pieces are removed from the board first, and afterwards the piece just played must have at least one liberty. It is legal for Black to play on the marked vertex in each of the situations shown in Figure A.3.

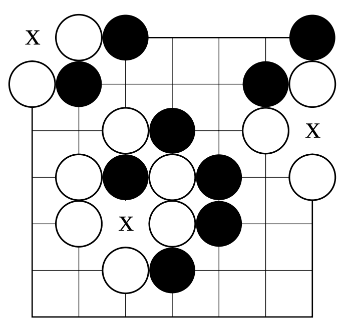


Figure A.3: Moves at x are allowed

A.4 The Rule of Ko

Consider the positions in Figure A.4. If Black were to capture a white stone by playing at any of the marked locations, then White would be in a position to immediately recapture, and an endless loop could ensue.

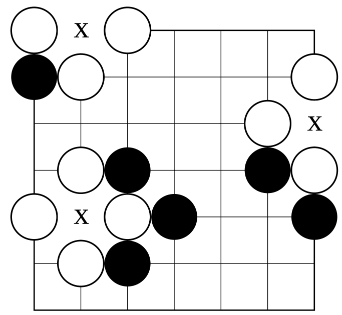


Figure A.4: Basic ko positions

The rule of *ko* exists in order to avoid these situations. It states that no player may make a move that would make the board look identical to any state it has previously been in during the game. This ensures that if a single stone is captured and the capturing stone could be recaptured on the following move, the player whose stone was captured must play elsewhere before recapturing the capturing stone.

A.5 Life and Death

Any group in play can be classified as either *alive* or *dead*. In order to understand this, it is first necessary to discuss the concept of an *eye*. An eye is a single point surrounded by a chain. A chain with an eye is difficult to capture since every other liberty of the chain must be occupied before the eye can be played in. In Figure A.5, the black group has an eye. Thus, it must be completely surrounded on the outside by white stones as shown before White can strike the final blow by playing in the eye.

A chain that has two or more eyes such as that in Figure A.6 cannot ever be captured. Even if all other liberties of a chain are occupied, none of the eyes can be occupied, since playing in any of them individually would be a suicide.

A group that has two or more eyes or has the potential to form them is said to be alive while a group that cannot form two eyes is said to be dead.

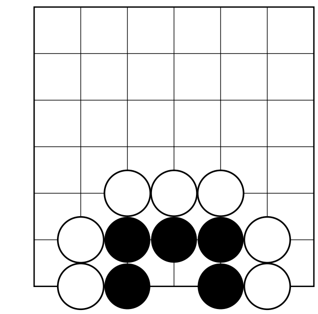


Figure A.5: Black has one eye

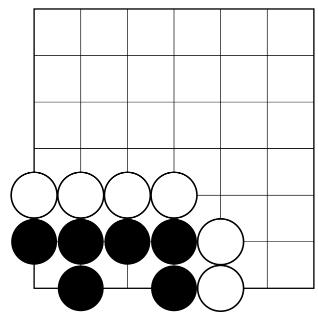


Figure A.6: Black is alive

A.6 Seki

There is one case in which a group can fail to have two eyes and still be immune to capture. This is known as *seki* and is seen in Figure A.7.

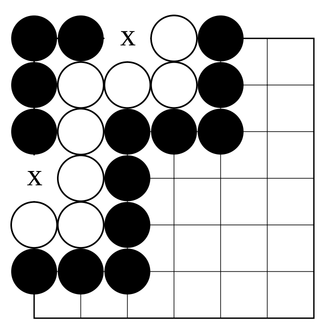


Figure A.7: Stones in seki

Observe that if Black plays either of the marked points, then White responds by capturing the inner Black chain on the following move. However, if White plays either of the marked points, then Black will capture the entire White chain on the next turn. In the case of seki, the stones are not dead, and the territory enclosed is not given to either player.

A.7 The Endgame and Scoring

On each turn, a player has the option of placing a stone or passing. A player would pass when there is nothing left to be gained by moving. The game of Go ends when both players pass consecutively. Once the game ends, scoring is performed as follows:

1. Any neutral points on the board that are not surrounded by either player are filled in with stones of either color. These neutral points are called *dame* by Go players.

2. All dead stones are removed from the board and treated as prisoners.

3. Each empty intersection that is surrounded by a living group of one player's stones counts as a point for that player.

4. Each player's total score is the sum of the number of points of territory that he controls and the number of prisoners that he has captured.

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