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A Numerical Study on an Eigenvalue Conjecture

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An abstract of
a thesis submitted to the Faculty of Emory College of Arts and Sciences
of Emory University in partial fulfillment
of the requirements of the degree of
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Abstract

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This paper provides a brief introduction about spectral problem in mathematical study, and presents several important results related to spectral problem. Starting from Kac's question that whether we can use the full spectrum to determine the associated shape, the paper shows that the answer to Kac's question is "no" in general, but "yes" in the class of Euclidean triangles according to some related study. In the class of triangles, the paper produces a nice formula to calculate Dirichlet eigenvalues for equilateral triangles. For general triangles, the paper introduces a numerical method called finite-element method, to approximate the true eigenvalues. Moreover, the paper provides a numerical evidence for the eigenvalue conjecture on isosceles triangles: Given that $T$ is an isosceles triangle, $\lambda_1$ is the smallest eigenvalue associated with $T$ and $\lambda_2$ the next smallest, the ratio $\xi_{2,1}(T) = \frac{\lambda_2}{\lambda_1}$ on isosceles triangles is maximized if and only if $T$ is equilateral. The paper examines the behavior of the ratio $\xi_{2,1} = \frac{\lambda_2}{\lambda_1}$ in isosceles triangles, with different parameters that measure the triangle's "dissimilarity" to equilateral triangle. The numerical result suggests that $\xi_{2,1}$ can only be optimized when the triangle is equilateral, and $\xi_{2,1}$ decreases sharply as the triangle becomes dissimilar to equilateral triangle.

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# A NUMERICAL STUDY ON AN EIGENVALUE CONJECTURE

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# 1. Introduction

## Spectral Problem

In 1966, Marc Kac [9] raised a very interesting question that inspired many other mathematicians: ‘Can one hear the shape of a drum?’ In his paper, Kac explains his question more in details and leads us explore some related subjects leisurely. To explain more about this topic, let’s first imagine in our minds that the drum face is a thin membrane on a domain $\Omega \subset \mathbb{R}^2$. The membrane’s boundary, which is denoted as $\partial\Omega$, is fixed, in consistency with the fact that a drum face is normally fixed in reality. If we hit on the membrane, it will vibrate and we can write the displacement of any point on the membrane as a function on $\Omega$:

$$F(\mathrm{x}, \mathrm{y}; \mathrm{t}) \equiv F(\vec{p}; t)$$

From Physics (see [9]), we know that the function $F(\vec{p}; t)$ should obey the wave equation:

$$\frac{\partial^2 F}{\partial t^2} = c^2 \Delta F \tag{1}$$

where $\Delta$ is the Laplace Operator: $\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$, and $c$ is a constant, depending on the physical properties of the membrane and also the tension the membrane is held. For simplicity, we just assume $\mathrm{c} = 1$ for the rest of the paper.

Given the partial differential equation (1), there is a universe of different kinds of functions that can be the solution. However, among all possible solutions for $F$, it is the solutions of the form

$$F(\vec{p};t) = \phi(\vec{p})e^{i\omega t} \tag{2}$$

that are of special interest to mathematicians and musicians, because the values of ω represent the *pure tones* that the membrane can produce.

From now on, we should just focus on this special form of solutions $\phi(\vec{p})e^{i\omega t}$, and explore more about its properties. If we put the solution form in (2) into the wave equation (1), we can have a very neat equation:

$$\Delta\phi + \omega^2\phi = 0 \tag{3}$$

with boundary condition $\phi = 0$ on $\partial\Omega$.

If we let $\lambda = \omega^2$, then the equation (3) can be written as

$$-\Delta\phi = \lambda\phi \tag{4}$$

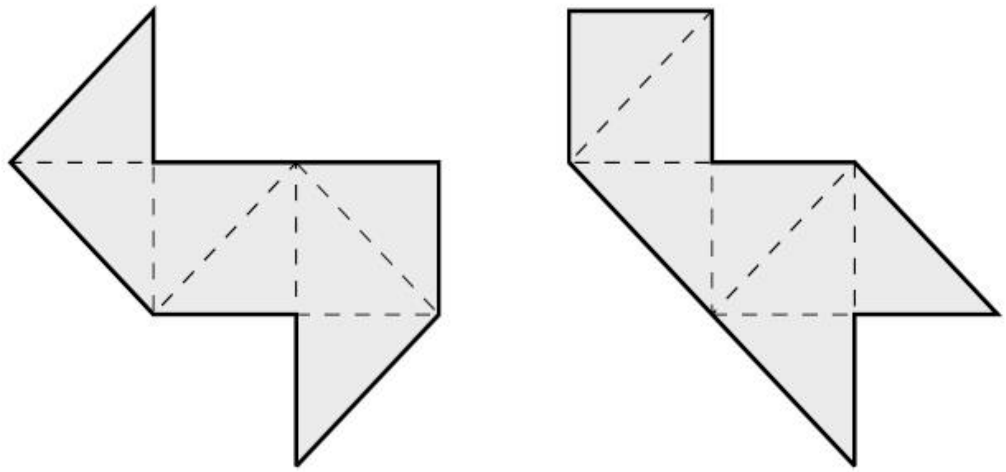
with the same boundary condition $\phi = 0$ on $\partial\Omega$. The value of λ is what we call a *Dirichlet eigenvalue* of Ω, if given that there exists a corresponding solution $\phi \not\equiv 0$. And $\phi$ is the corresponding *eigenfunction* or *mode*.

Using the techniques from PDE and functional analysis (see [5]), it is proven that there is an infinite sequence of positive eigenvalues$0 < \lambda_1 \leq \lambda_2 \leq \lambda_3 \cdots \to \infty$ associated with a given domain Ω.

Based on the above set up, we can now translate Kac's question to the following: If $\{\lambda_n\}$ is the set of eigenvalues associated with $\Omega_1$ and $\{\mu_n\}$ is the set of eigenvalues associated with $\Omega_2$, and $\lambda_n = \mu_n$for all $n$ (or we can say $\Omega_1$and $\Omega_2$ are *isospectral*), are the regions $\Omega_1$and $\Omega_2$ congruent?

Of course, Kac is not the first person who are interested in the relation between the domain and its associated Dirichlet eigenvalues. Many mathematicians have paid great efforts in studying how Dirichlet eigenvalues are dependent on its domain, much earlier than Kac's paper published. In fact, such an investigation has become a mathematical discipline called spectral geometry. Nevertheless, the problem that Kac brings up is special, because it is actually an inverse spectral problem. We use the word "inverse" in the sense that it is starting from the eigenvalues to try to tell something about the shape of the domain.

In recent decades, many exciting results were found from studies that are related to spectral problems. As for the answer to Kac's specific question, it is "No". Many counterexamples have been found to prove that two isospectral domains may not be congruent. The first planar counterexample [6] was given by C. Gordon, D. Webb and S. Wolpert in 1992. But still, there remains a lot of interesting questions related to the problem that are worth deeper explorations. For example, would that answer be different if we restrict the domain to be of some certain classes? Or can we calculate the exact eigenvalues for all kinds of domains? These questions will be addressed in the later parts of this paper.



A counter example for domains that
are isospectral, but not congruent

**The Class of Triangles**

Although the answer to Kac's question is no in a general case, there are positive results that encourage further study. In 2009, it is shown by Zelditch [14] that if one restricts the universe of all possible domains to those having an analytical boundary and some symmetry, the full spectrum (full set of eigenvalues, in other words) will be sufficient to determine the domain uniquely. Therefore, the result suggests that there is still hope in the inverse spectral problem.

It is quite natural that people then switch their attention to Euclidean triangles. As one of the simplest classes of domains, Euclidean triangles are likely to have some special properties that other domains do not have. Recently, Grieser and Maronna [7] examine the relation between the set of eigenvalues $\{\lambda_k\}$ and the shape of triangle from a quite unusual aspect. We shall briefly go through their paper, as the current paper is partially derived from it.

Grieser and Maronna start with the relation between heat trace and heat Kernel function.

In general, the heat trace is a function defined as

$$h(t) = \sum_{k=1}^{\infty} e^{-\lambda_k t}, t > 0 \tag{5}$$

where $\lambda_k$ is a Dirichlet eigenvalue as discussed before. Since $\lambda_k \to \infty$ as $k \to \infty$, $h(t)$ is a smooth function that converges for all $t > 0$.

In order to see why the heat trace function is useful in the proof, let's first look at the heat equation (see [4] for more reference):

$$(\partial_t - \Delta)v(t,x) = 0, t > 0, x \epsilon \Omega \tag{6}$$

where $\partial_t := \frac{\partial}{\partial t}$.

The heat equation (6) has a unique solution for any initial data $v(0,x) = f(x)$. Given the boundary condition that $v(t,x) = 0$ for all $t > 0$ and $x \in \partial\Omega$, and by separation of variables, we can solve for $v(t,x)$ and obtain

$$v(t,x) = \sum_{k=1}^{\infty} a_k e^{-\lambda_k t} \phi_k(x) \tag{7}$$

where $\phi_k$ forms an orthonormal basis of $L^2(\Omega)$ of real valued eigenfunctions corresponding to $\lambda_k$, and $a_k = \int_\Omega f(y)\phi_k(y)\,dy$.

In other words, $v(t,x)$ can be represented as

$$v(t,x) = \int_\Omega \sum_{k=1}^{\infty} f(y)\phi_k(y)e^{-\lambda_k t}\phi_k(x)\,dy \tag{8}$$

If we define

$$H(t,x,y) = \sum_{k=1}^{\infty} \phi_k(y)e^{-\lambda_k t}\phi_k(x) \tag{9}$$

where the function $H$ is called *the heat kernel* of $\Omega$,

then we can obtain

$$\int_\Omega H(t,y,y)\,dy = \int_\Omega \sum_{k=1}^{\infty} \phi_k(y)e^{-\lambda_k t}\phi_k(y)\,dy = \sum_{k=1}^{\infty} e^{-\lambda_k t}\int_\Omega {\phi_k}^2(y)\,dy \tag{10}$$

Since each of the $\phi_k$ is normalized in $L^2$, $\int_\Omega {\phi_k}^2(y)\,dy = 1$. Therefore,

$$\int_\Omega H(t,y,y)\,dy = \sum_{k=1}^{\infty} e^{-\lambda_k t} = h(t) \tag{11}$$

Grieser and Maronna then state that for a Riemannian surface $\Omega$ without boundary

$$H(t,y,y) \sim t^{-1}\sum_{j=0}^{\infty} a_j(y)t^j \quad \text{as } t \to 0 \tag{12}$$

where each $a_j(y)$ is a universal polynomial in derivatives of the Gauss curvature $K(y)$ of $\Omega$ at $\mathrm{y}$. If $\Omega$ has a boundary, then its influence is felt only when the distance of $\mathrm{y}$ to the boundary is of order at most $\sqrt{t}$, which contributes extra terms involving the curvature of the boundary and terms involving the powers $t^{-\frac{1}{2}+j}$. In the case of planar domains with polygonal boundary, there is no curvature, but the corners will give contribution to $h(t)$. Therefore, due to the influence caused by the boundary, the formula of $h(t)$ becomes

$$h(t) = a_0 t^{-1} + a_{1/2} t^{-\frac{1}{2}} + a_1 + O\left(e^{-\frac{c}{t}}\right) \quad \text{as } t \to 0 \tag{13}$$

for some constant $\mathrm{c} > 0$. And the coefficients are

$$a_0 = \frac{A}{4\pi}, a_{1/2} = -\frac{P}{8\sqrt{\pi}}, a_1 = \frac{1}{24}\sum_i \left(\frac{\pi}{\alpha_i} - \frac{\alpha_i}{\pi}\right)$$

where $A$ is the area, $P$ is the perimeter and $\alpha_i$ are the interior angles of the polygon.

In the case of triangle, since we have $\sum_i \alpha_i = \pi$,

$$a_1 = \frac{\pi}{24}\sum_i \frac{1}{\alpha_i} - \frac{1}{24}$$

Therefore, if we know all the $\lambda_k$ of a triangle, we know what $h(t)$ is exactly, and hence the coefficients $a_0, a_{1/2}$ and $a_1$. According to the definitions of those coefficients, we then have the area $A$, perimeter $P$, and the sum of the reciprocals of the interior angles $R$. In other words, we can “hear” all these quantities of a triangle.

Grieser and Maronna use the rest of their paper to prove that a triangle is determined uniquely up to congruence by its area $A$, perimeter $P$, and the sum of the reciprocals of the interior angles $R$. For more detail about the proof, one can refer to the original paper. As for now, we will skip the proof and take the result as granted.

So at last, we achieve the final result: given the full spectrum of a triangle, one can determine up to its unique shape.

## 2. The Eigenvalue Problems on Triangles

### Eigenvalues for Equilateral Triangles

One may notice that the above theorem about triangles will be meaningless, if we can't obtain the full spectrum of a triangle. Actually, it has been a very challenging task for a long time to calculate the values of $\lambda_k$. So far, we can do that in only very few cases (rectangles, disks, and certain triangles). For the purpose of the current paper, we will show how to calculate the eigenvalues for equilateral triangles. The following proof refers to Brian J. McCartin's paper [11].

First of all, let's have some preliminaries.

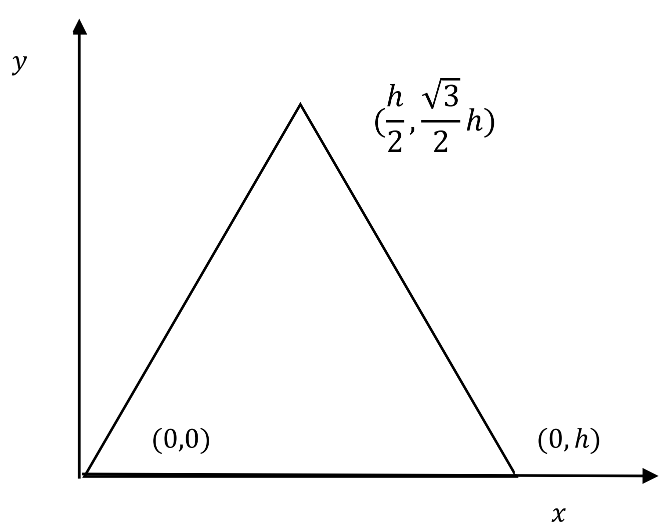
An equilateral triangle of side $h$ has the following properties:

| | |
|---|---|
| Perimeter ($p$) | $3h$ |

| | |
|---|---|
| Altitude($a$) | $\frac{\sqrt{3}}{2}h$ |
| Area($A$) | $\frac{\sqrt{3}}{4}h^2$ |
| Inradius($r$) | $\frac{\sqrt{3}}{6}h$ |
| Circumradius($R$) | $\frac{\sqrt{3}}{3}h$ |
| Incircle Area($A_r$) | $\frac{\pi}{12}h^2$ |
| Circumcircle Area($A_R$) | $\frac{\pi}{3}h^2$ |

Consider the equilateral triangle of side $h$ is positioned in the Cartesian coordinates system with vertices $(0,0), (0, h), (\frac{h}{2}, \frac{\sqrt{3}}{2}h)$ as Figure 1.

Figure 1. Equilateral Triangles in the Cartesian Coordinates System



For convenience of further calculation, we introduce Lamé's triangular coordinates system, where each point is denoted as $(u, v, w)$. The coordinates $u, v$ and $w$ describe the distances of the triangle center to the projections of the point onto the altitudes respectively, and we define the direction from center to the corresponding vertex to be negative. (see Figure 2 for illustration). According to the above definition, a point $P$ with Cartesian coordinates $(x, y)$ can be transformed into triangular coordinates $(u, v, w)$ in the following way:

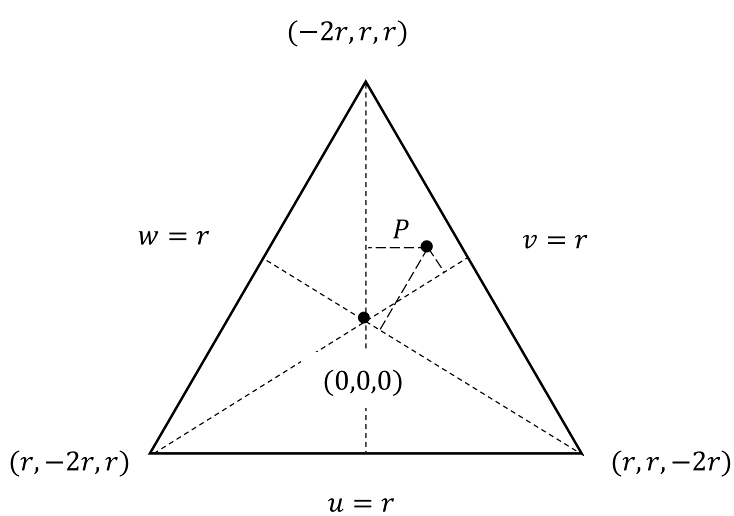
$$u = r - y$$

$$v = \frac{\sqrt{3}}{2}\left(x - \frac{h}{2}\right) + \frac{1}{2}(y - r)$$

$$w = \frac{\sqrt{3}}{2}\left(\frac{h}{2} - x\right) + \frac{1}{2}(y - r)$$

where $r = \frac{\sqrt{3}}{6}h$ is the inradius of the triangle.

Figure 2. Equilateral Triangle in Triangular Coordinates System

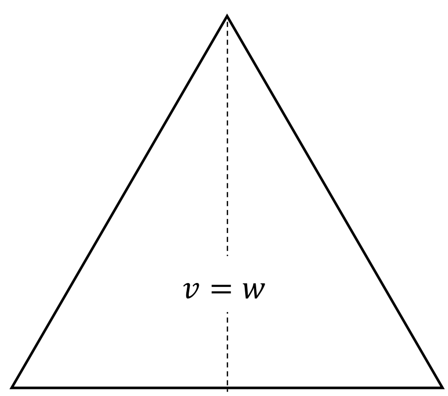


Thanks to the triangular coordinates, the three sides of the triangle now can be nicely represented as $u = r, v = r, w = r$ respectively, which simplifies the application of boundary condition at a later stage. Moreover, one may notice that for any point $P(u, v, w)$, the following equation is satisfied:

$$u + v + w = 0. \tag{14}$$

Now we may borrow the symmetric-antisymmetric decomposition of a function from signal processing. For any function $f$ on a domain of an equilateral triangle, we decompose it into a symmetric part and an antisymmetric part about the altitude $v = w$.

Figure 3. Line of Symmetry *v=w*




To be more precise, the function $f$ can be written as

$$f(u, v, w) = f_s(u, v, w) + f_a(u, v, w) \tag{15}$$

where

$$f_s(u,v,w) = \frac{f(u,v,w) + f(u,w,v)}{2}$$

and

$$f_a(u,v,w) = \frac{f(u,v,w) - f(u,w,v)}{2}$$

Now let's go back and look at our original problem

$$-\Delta\phi(x,y) = \lambda\phi(x,y) \tag{16}$$

with boundary condition $\phi(x,y) = 0$, for $(x,y) \in \partial T$.

If we define the orthogonal coordinates

$$\xi = u = r - y$$

and

$$\mu = v - w = \sqrt{3}\left(x - \frac{h}{2}\right),$$

then the original problem (16) is transformed to

$$\frac{\partial^2\phi}{\partial\xi^2} + 3\frac{\partial^2\phi}{\partial\mu^2} + \lambda\phi(\xi,\mu) = 0 \tag{17}$$

Considering separation of variables in solving partial differential equation, we seek a solution of the above equation in the form of

$$\phi(\xi,\mu) = f(\xi)g(\mu)$$

and substituting $f \cdot g$ for $\phi$ in the equation (17), we obtain

$$f''g + 3fg'' + \lambda fg = 0 \tag{18}$$

After dividing both sides of (18) by $f \cdot g$ (for $f \cdot g \neq 0$) ,

$$\frac{f''}{f} + 3\frac{g''}{g} + \lambda = 0 \tag{19}$$

Since $f$ and $g$ are independent, we have that

$$\frac{f''}{f} = A; \frac{g''}{g} = B \tag{20}$$

for some constant $A$ and $B$. Moreover, we can have the following relation, according to (18) and (19):

$$A + 3B = -\lambda \tag{21}$$

So far, the solution $\phi$ in a separable form seems to be promising, and therefore, it is worth more exploration.

In order to get more clues for eigenfunction $\phi$ and the corresponding eigenvalue $\lambda$, we may first look at Lamé's Fundamental Theorem:

*Suppose that $\psi(x, y)$ can be represented by the trigonometric series*

$$\psi(x, y) = \sum_{i=1}^{n} A_i \sin(c_i x + d_i y + \alpha_i) + B_i \cos(c_i x + d_i y + \beta_i)\,, \text{for } n \in \mathbb{N}^+$$

*with ${c_i}^2 + {d_i}^2 = \lambda$,*

*then*

1. *$\psi(x, y)$ is antisymmetric about any line along which it vanishes.*
2. *$\psi(x, y)$ is symmetric about any line along which its normal derivative $\frac{\partial \psi}{\partial \nu}$*

*vanishes*.

Lamé's theorem somehow suggests that one possible solution $\phi$ on an equilateral triangle might have trigonometric functions, because the theorem provides necessary but not sufficient conditions of $\phi$ (see [11] for more reference).

To better see the symmetric/antisymmetric behavior of $\phi$, we decompose $\phi$ into symmetric and antisymmetric parts as (15). That is, define

$$\phi(u,v,w) = \phi_s(u,v,w) + \phi_a(u,v,w) \tag{22}$$

where

$$\phi_s(u,v,w) = \frac{\phi(u,v,w) + \phi(u,w,v)}{2}$$

and

$$\phi_a(u,v,w) = \frac{\phi(u,v,w) - \phi(u,w,v)}{2}$$

We will focus on solving the symmetric mode $\phi_s$ first. With the hope of finding a solution that is separable into $f(\xi)g(\mu)$ and is consisted of trigonometric functions, we consider the form

$$\sin(\beta_0\xi)\cos(\beta_1\mu)$$

And based on the fact that $\phi_s$ must vanish at $u = r$ and at $u = -2r$, and it must be an even function of $\eta$ (or $v - w$), we look for a solution of the form

$$\sin\left[\frac{\pi l}{3r}(u+2r)\right]\cos[\beta_1(v-w)] \tag{23}$$

where $l \neq 0$ is an integer and $(\frac{\pi l}{3r})^2 + 3{\beta_1}^2 = \lambda$ (according to (21)).

Along $v = r$, and according to (14), we have

$$v - w = v - (-v - u) = u + 2v = u + 2r$$

Substituting $v - w$ into $u + 2r$ in (23), we then have a form of solution as

$$\sin\left[\frac{\pi l}{3r}(u+2r)\right]\cos[\beta_1(u+2r)]$$

along $v = r$.

But since the solution above is not identically equal to zero, for all $u \in [-2r, r]$, it shows that just one single term does not suffice. Therefore, we try the form that is a sum of two such terms:

$$\sin\left[\frac{\pi l}{3r}(u+2r)\right]\cos[\beta_1(v-w)] + \sin\left[\frac{\pi m}{3r}(u+2r)\right]\cos[\beta_2(v-w)] \tag{24}$$

where $m \neq 0$ is an integer and $(\frac{\pi l}{3r})^2 + 3{\beta_1}^2 = (\frac{\pi m}{3r})^2 + 3{\beta_2}^2 = \lambda$

And along $v = r$, the solution (24) becomes

$$\sin\left[\frac{\pi l}{3r}(u+2r)\right]\cos[\beta_1(u+2r)] + \sin\left[\frac{\pi m}{3r}(u+2r)\right]\cos[\beta_2(u+2r)] \tag{25}$$

By using the trigonometric identity: $\sin(\theta_1)\cos(\theta_2) = \frac{1}{2}[\sin(\theta_1+\theta_2) + \sin(\theta_1-\theta_2)]$, the solution (25) can be rewritten as

$$\frac{1}{2}\{\sin\left[(\frac{\pi l}{3r}+\beta_1)(u+2r)\right]+\sin\left[(\frac{\pi l}{3r}-\beta_1)(u+2r)\right]+\sin\left[(\frac{\pi m}{3r}+\beta_2)(u+2r)\right]+\sin\left[(\frac{\pi m}{3r}-\beta_2)(u+2r)\right]\} \tag{26}$$

In order to make (26) identically equal to zero for all $u \in [-2r, r]$, we must have either

$$\left(\frac{\pi l}{3r}+\beta_1\right)=-\left(\frac{\pi m}{3r}+\beta_2\right);\ \left(\frac{\pi l}{3r}-\beta_1\right)=-\left(\frac{\pi m}{3r}-\beta_2\right)$$

or

$$\left(\frac{\pi l}{3r}+\beta_1\right)=-\left(\frac{\pi m}{3r}-\beta_2\right);\ \left(\frac{\pi l}{3r}-\beta_1\right)=-\left(\frac{\pi m}{3r}+\beta_2\right)$$

However, both of the above conditions lead to $l+m=0$ and $\beta_1+\beta_2=0$, which means that the solution vanishes everywhere and therefore, it is not valid.

Then we come to consider the sum of three terms

$$\begin{gathered}\sin\left[\frac{\pi l}{3r}(u+2r)\right]\cos[\beta_1(v-w)]+\sin\left[\frac{\pi m}{3r}(u+2r)\right]\cos[\beta_2(v-w)] \\ +\sin\left[\frac{\pi n}{3r}(u+2r)\right]\cos[\beta_3(v-w)]\end{gathered} \tag{27}$$

where $n \neq 0$ is an integer and $(\frac{\pi l}{3r})^2+3{\beta_1}^2=(\frac{\pi m}{3r})^2+3{\beta_2}^2=(\frac{\pi n}{3r})^2+3{\beta_3}^2=\lambda$.

After we did the same procedure on (27) as on the sum of two terms, we have the form

$$\frac{1}{2}\{\sin\left[(\frac{\pi l}{3r}+\beta_1)(u+2r)\right]+\sin\left[(\frac{\pi l}{3r}-\beta_1)(u+2r)\right]$$
$$+\sin\left[(\frac{\pi m}{3r}+\beta_2)(u+2r)\right]$$
$$+\sin\left[(\frac{\pi m}{3r}-\beta_2)(u+2r)\right]+\sin\left[(\frac{\pi n}{3r}+\beta_3)(u+2r)\right]$$
$$+\sin\left[(\frac{\pi n}{3r}-\beta_3)(u+2r)\right]\} \tag{28}$$

along $v = r$.

Now there are eight possible ways to make (28) fit the boundary condition. But in fact, all of the cases will lead to one essential conclusion. For here, we will just look at one possible case and skip the other:

$$\left(\frac{\pi l}{3r}+\beta_1\right)=-\left(\frac{\pi m}{3r}+\beta_2\right);\ \left(\frac{\pi l}{3r}-\beta_1\right)=-\left(\frac{\pi n}{3r}-\beta_3\right); \tag{29}$$
$$\left(\frac{nm}{3r}-\beta_2\right)=-(\frac{\pi n}{3r}+\beta_3)$$

By adding up all the three equations, we reach the important equation

$$l+m+n=0 \tag{30}$$

and based on (29) and (30), we can further calculate

$$\beta_1=\frac{\pi(m-n)}{9r};\beta_2=\frac{\pi(n-l)}{9r};\beta_3=\frac{\pi(l-m)}{9r} \tag{31}$$

Finally, putting (31) into (27), the symmetric mode is

$$\phi_s{}^{m,n} = -\sin\left[\frac{\pi(m+n)}{3r}(u+2r)\right]\cos\left[\frac{\pi(m-n)}{9r}(v-w)\right]$$

$$+\sin\left[\frac{\pi m}{3r}(u+2r)\right]\cos\left[\frac{\pi(2n+m)}{9r}(v-w)\right]$$

$$+\sin\left[\frac{\pi n}{3r}(u+2r)\right]\cos[\frac{\pi(2m+n)}{9r}(v-w)]$$

and the corresponding eigenvalue is

$$\lambda = (\frac{\pi l}{3r})^2 + 3{\beta_1}^2 = [\frac{\pi(m+n)}{3r}]^2 + 3\left[\frac{\pi(m-n)}{9r}\right]^2 = \frac{4\pi^2}{27r^2}(m^2+mn+n^2) \qquad (32)$$

An analogous way can be applied in looking for the antisymmetric mode, as one sees that $\phi_a$ is an odd function about $v-w$, and tries a similar development as $\phi_s$. We will just directly give the result here:

$$\phi_a{}^{m,n} = -\sin\left[\frac{\pi(m+n)}{3r}(u+2r)\right]\sin\left[\frac{\pi(m-n)}{9r}(v-w)\right]$$

$$+\sin\left[\frac{\pi m}{3r}(u+2r)\right]\sin\left[\frac{\pi(2n+m)}{9r}(v-w)\right]$$

$$-\sin\left[\frac{\pi n}{3r}(u+2r)\right]\sin[\frac{\pi(2m+n)}{9r}(v-w)]$$

and still, with corresponding eigenvalue

$$\lambda = \frac{4\pi^2}{27r^2}(m^2+mn+n^2)$$

After such great effort in looking for the eigenvalues on an equilateral triangle, we are now able to get as many eigenvalues as we want, based on (32). For example, the smallest eigenvalue is

$$\lambda_1 = \frac{4\pi^2}{27r^2}(1^2 + 1 + 1^2) = \frac{4\pi^2}{9r^2}$$

and the next smallest eigenvalue is

$$\lambda_2 = \frac{4\pi^2}{27r^2}(1^2 + 2 + 2^2) = \frac{28\pi^2}{27r^2}$$

Therefore,

$$\frac{\lambda_2}{\lambda_1} = \frac{7}{3}$$

for any equilateral triangle domain.

**Numerical Approach**

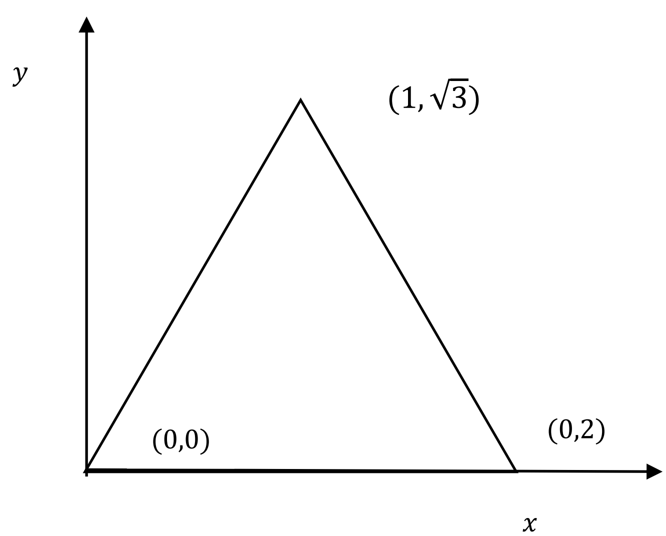
Except for equilateral triangles, eigenvalues on other triangles are hard to find. So we need a numerical approach to help the investigation. In this paper, we adopt the finite element method, or FEM in short. Broadly used in practical problems in engineering and scientific analysis, the finite element method is essentially a numerical method that produces a solution that approximates the true solution. The finite element method divides the domain of the problem into smaller subregions or elements. Usually, these elements are line segments in one dimension, triangles in two dimension, and prisms in three dimensions. Following the subdivision, the change in the unknown function with position is approximated by some simple functions. Then the original problem can be replaced by an equivalent integral formulation. There are two popular methods for constructing the integral formulation- one is called Galerkin's method, which is based on weighted residual methods, and the other one is the variational formulation, which is based on variational principles. For here, we use Galerkin's method. For more introduction about FEM, see [10].

The best way to understand the finite element method might be looking at an example. Let's say we have an equilateral triangular domain $T$ as shown in Figure 4. And we are now trying to solve for a trial solution $\tilde{\phi}$ on $T$, which is a numerical approximation of the true solution $\phi$ on $T$ that satisfies

$$\Delta\phi + \lambda\phi = 0 \tag{33}$$

with boundary condition $\phi = 0$ on $\partial T$.

Figure 4. The Triangle Domain in Example



In general, the trial solution $\tilde{\phi}$ should satisfy that (i) it is simple in form and (ii) the boundary conditions of the problem is fulfilled. In particular, $\tilde{\phi}$ is assumed to take the form:

$$\tilde{\phi}(x,y) = \theta_0(x,y) + \sum_{i=1}^{n} \alpha_i \theta_i(x,y) \tag{34}$$

In that particular form, the function $\theta_0(x,y)$ is chosen to satisfy all the boundary conditions of the problem, and the functions $\theta_i(x,y)$, which are also called coordinate functions, are chosen to satisfy the corresponding homogeneous form of the boundary conditions. For example, if we have boundary conditions

$$\tilde{\phi}(x,y) = k \;\text{ and }\; \frac{\partial \tilde{\phi}(x,y)}{\partial x} = k' \;\text{ on a boundary } \Gamma$$

then $\theta_0(x,y)$ is chosen to satisfy

$$\theta_0(x,y) = \mathrm{k} \;\text{ and }\; \frac{\partial \theta_0(x,y)}{\partial x} = k' \quad \text{on } \Gamma$$

and each $\theta_i(x,y)$ is chosen to satisfy

$$\theta_i(x,y) = 0 \;\text{ and }\; \frac{\partial \theta_i(x,y)}{\partial x} = 0 \quad \text{on } \Gamma$$

In our case, since the boundary condition is already in a homogeneous form, we don't need the function $\theta_0(x,y)$. Therefore, we simply have

$$\theta_0(x,y) \equiv 0$$

and

$$\theta_i(x,y) = 0 \text{ on } \partial T \qquad i = 1,2,\dots,n$$

It follows that $\tilde{\phi}(x,y)$ satisfies all the boundary conditions of the problem regardless of the parameters $\alpha_i$. The functions $\theta_0(x,y)$ and $\theta_i(x,y)$, $i = 1,2,\dots,n$ are determined by the individual user so that $\tilde{\phi}(x,y)$ will be completed, once the parameters $\alpha_i$ are determined. We will discuss how to choose $\theta_0(x,y)$ and $\theta_i(x,y)$ in a later part.

Since $\tilde{\phi}$ is just an approximation of the true function $\phi$, $\tilde{\phi}$ will not satisfy the partial differential equation (33). Instead, we have

$$\Delta\tilde{\phi} + \lambda\tilde{\phi} = \varepsilon(x,y) \tag{35}$$

where $\varepsilon(x,y)$ is called the residual function.

The equation (35) is so called the “strong” form of *Helholtz equation*. To make it adaptable for a numerical solution, we need to convert it into the “weak” form, which means the integral form. One way we can do it is to multiply both sides with a testing function $T_s$ and integrate it over the surface of $T$, and then set it equal zero:

$$\iint_T (\Delta\tilde{\phi} + \lambda\tilde{\phi}) T_s dxdy = 0 \tag{36}$$

According to Galerkin’s method, the testing function $T_s$ has the following definition:

$$T_s = \theta_i(x,y) \qquad i = 1,2,\dots,n$$

We can write the integral in (36) as a sum of two integrals:

$$\iint_T \Delta\tilde{\phi}\, T_s dxdy + \lambda \iint_T \tilde{\phi} T_s dxdy = 0 \qquad (37)$$

In order to reduce the order of the partial derivative terms in the integrand, we need to incorporate *Green's Theorem in the Plane:*

*If $A$ is a region in the xy plane bounded by a closed curve $\Gamma$ and if $F(x,y)$ and $G(x,y)$ are suitably 'smooth' functions then*

$$\iint_A \{\frac{\partial G}{\partial x} - \frac{\partial F}{\partial y}\} dxdy = \oint_\Gamma (Fdx + Gdy) \qquad (38)$$

In our case, we need to choose

$$F = -\frac{\partial\tilde{\phi}}{\partial y} T_s; G = \frac{\partial\tilde{\phi}}{\partial x} T_s$$

so that, by (38)

$$\iint_T \{\frac{\partial^2\tilde{\phi}}{\partial x^2} T_s + \frac{\partial T_s}{\partial x}\frac{\partial\tilde{\phi}}{\partial x} + \frac{\partial^2\tilde{\phi}}{\partial y^2} T_s + \frac{\partial T_s}{\partial y}\frac{\partial\tilde{\phi}}{\partial y}\} dxdy = \oint_{\partial T} T_s(-\frac{\partial\tilde{\phi}}{\partial y} dx + \frac{\partial\tilde{\phi}}{\partial x} dy)$$

After some transformations, we have

$$\iint_T \{\frac{\partial^2\tilde{\phi}}{\partial x^2} + \frac{\partial^2\tilde{\phi}}{\partial y^2}\} T_s dxdy = -\iint_T \{\frac{\partial T_s}{\partial x}\frac{\partial\tilde{\phi}}{\partial x} + \frac{\partial T_s}{\partial y}\frac{\partial\tilde{\phi}}{\partial y}\} dxdy + \oint_{\partial T} T_s(-\frac{\partial\tilde{\phi}}{\partial y} dx + \frac{\partial\tilde{\phi}}{\partial x} dy)$$

(39)

The second integral on the right of (39) vanishes as $T_s$ vanishes on the boundary $\partial T$.

Hence,

$$\iint_T \{\frac{\partial^2\tilde{\phi}}{\partial x^2} + \frac{\partial^2\tilde{\phi}}{\partial y^2}\} T_s dxdy = -\iint_T \{\frac{\partial T_s}{\partial x}\frac{\partial\tilde{\phi}}{\partial x} + \frac{\partial T_s}{\partial y}\frac{\partial\tilde{\phi}}{\partial y}\} dxdy$$

Putting this result back into (37), we have

$$\lambda \iint_T \tilde{\phi} T_s dxdy = \iint_T \{\frac{\partial T_s}{\partial x}\frac{\partial\tilde{\phi}}{\partial x} + \frac{\partial T_s}{\partial y}\frac{\partial\tilde{\phi}}{\partial y}\} dxdy$$

Now if we substituting $\theta_i(x,y)$ for $T_s$, then we have

$$\lambda \iint_T \tilde{\phi}\theta_i dxdy = \iint_T \{\frac{\partial \theta_i}{\partial x}\frac{\partial\tilde{\phi}}{\partial x} + \frac{\partial \theta_i}{\partial y}\frac{\partial\tilde{\phi}}{\partial y}\} dxdy \qquad i = 1,2,\dots,n \tag{40}$$

In setting up the finite element equations that are suitable for treatment on a computer, we need to proceed in a different manner from the above equations, which are used in hand calculations.

So the next step is to mesh up the region with elements of our choice. For convenience of illustration, we divide the region into three triangular elements and make a mark for each node and each element (see Figure 5).

The finite element approximation should always be of the form

$$\tilde{\phi}(x,y) = \sum_{i=1}^{4} \tilde{\phi}_i R_i(x,y) \tag{41}$$

where $\tilde{\phi}_i$ are the nodal values of the unknown function $\tilde{\phi}(x,y)$ and $R_i(x,y)$ are roof functions. A nodal value $\tilde{\phi}_i$ simply means the value of the function $\tilde{\phi}$ at node $i$. A roof function $R_i(x, y)$ has value equal to 1 at node $i$, is zero at other nodes, and varies linearly with $x$ and $y$. We take $R_4(x, y)$ for example and it is shown in Figure 6.

Figure 5. Discretization of the Domain

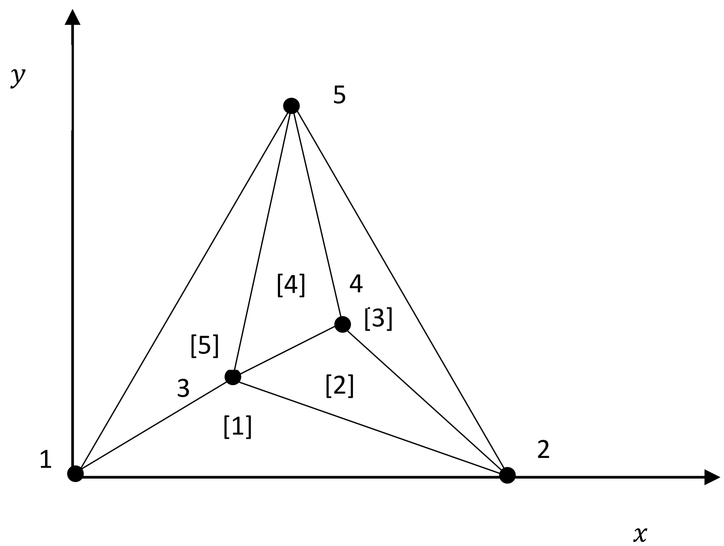
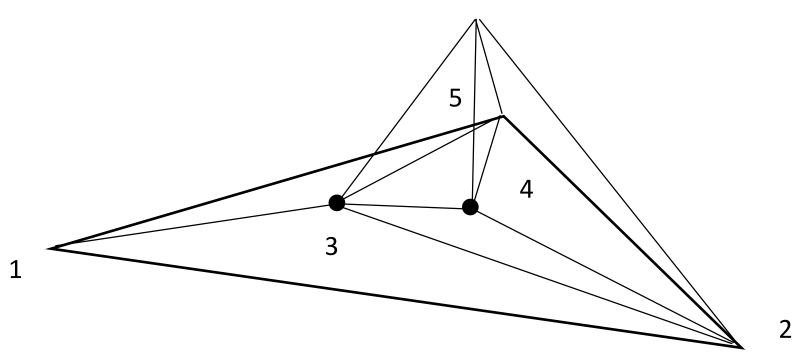



Figure 6. Roof function $R_4(x, y)$




Although the two expressions of $\tilde{\phi}$ are similar, they are not identical, because they are consisted of components that have different meanings. We should also note that, some of the nodal values might not been unknown, since they might be fixed by the boundary conditions. Take our case for example, since we have $\tilde{\phi}_1 = \tilde{\phi}_2 = \tilde{\phi}_5 = 0$ from the boundary condition, and by (41)

$$\begin{aligned}\tilde{\phi}(x,y) &= 0 \cdot R_1(x,y) + 0 \cdot R_2(x,y) + \tilde{\phi}_3 R_3(x,y) + \tilde{\phi}_4 R_4(x,y) + 0 \cdot R_5(x,y) \\ &= 0 \cdot [R_1(x,y) + R_2(x,y) + R_5(x,y)] + \tilde{\phi}_3 R_3(x,y) + \tilde{\phi}_4 R_4(x,y)\end{aligned}$$

According to the requirements (see equation (34)) for $\theta_0(x,y)$ and $\theta_i(x,y)$,

$$\theta_0(x,y) = 0 \cdot [R_1(x,y) + R_2(x,y) + R_5(x,y)] = 0$$

and we only need two coordinate functions

$$\theta_1(x,y) = R_3(x,y)$$

$$\theta_2(x,y) = R_4(x,y)$$

However, we will always formulate the finite element equations by taking $\theta_i \equiv R_i$, because we want to make the structure works for more general cases, where the $\tilde{\phi}_i$ may all be unknown. We note that, the roof functions $R_i$ do not really satisfy equation (40), because they are not differentiable over the edges between elements. Therefore, taking $\theta_i \equiv R_i$ produces error in the approximation of $\lambda$. But as the domain is subdivided into more and more elements, we shall see that the roof functions become smoother and the approximated value of $\lambda$ should converge to its true value. We will be discuss more about the error in a later part. And since $R_i$ are not always zero on $\partial T$, we must rectify any inconsistencies due to the boundary conditions in the matrices formulation. The specific way to rectify inconsistencies will be shown in our example.

So now, let's continue our example. Since

$$\frac{\partial\tilde{\phi}}{\partial x} = \sum_{j=1}^{5} \tilde{\phi}_j \frac{\partial R_j}{\partial x}$$

$$\frac{\partial\tilde{\phi}}{\partial y} = \sum_{j=1}^{5} \tilde{\phi}_j \frac{\partial R_j}{\partial y} \tag{42}$$

Putting (42) into (40),

$$\lambda \sum_{j=1}^{5} \tilde{\phi}_j \iint_T R_i R_j dxdy = \sum_{j=1}^{5} \tilde{\phi}_j \iint_T \{\frac{\partial R_i}{\partial x}\frac{\partial R_j}{\partial x} + \frac{\partial R_i}{\partial y}\frac{\partial R_j}{\partial y}\} dxdy \qquad i = 1,2,3,4,5 \tag{43}$$

The system of equations (43) can be represented in a matrix form as following:

$$\lambda[\mathbf{T}][\tilde{\boldsymbol{\phi}}] = [\mathbf{K}][\tilde{\boldsymbol{\phi}}] \tag{44}$$

The entries of $[\mathbf{K}]$ ($5 \times 5$ matrix) are:

$$K_{ij} = \iint_T \{\frac{\partial R_i}{\partial x}\frac{\partial R_j}{\partial x} + \frac{\partial R_i}{\partial y}\frac{\partial R_j}{\partial y}\} dxdy \tag{45}$$

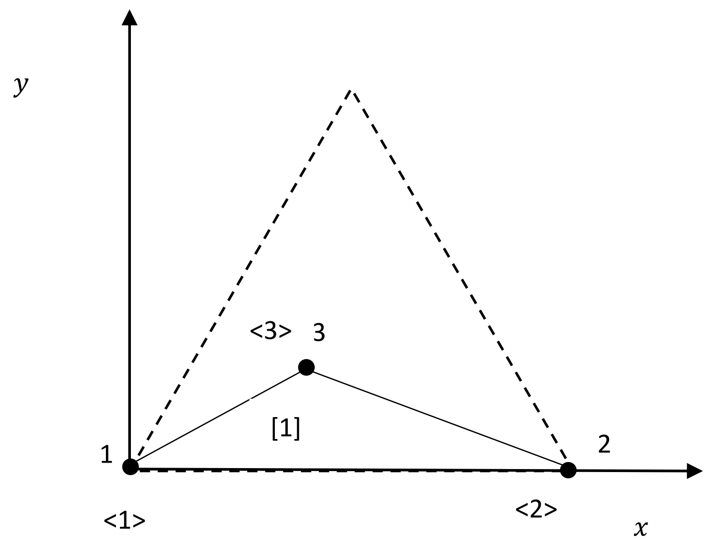
The entries of $[\mathbf{T}]$ ($5 \times 5$ matrix) are:

$$T_{ij} = \iint_T R_i R_j dxdy$$

And $[\tilde{\boldsymbol{\phi}}]$ ($5 \times 1$ matrix) is $\begin{bmatrix} \tilde{\phi}_1 \\ \tilde{\phi}_2 \\ \tilde{\phi}_3 \\ \tilde{\phi}_4 \\ \tilde{\phi}_5 \end{bmatrix}$

After the system (44) is set up, the next step is the construction of $[\mathbf{K}]$ and $[\mathbf{T}]$ based on elements. For here, we just pick up one triangular element (shown in Figure 6) to show how it works.

Figure 6. Element [1] for Example




We may now clarify the meanings of global node numbers and local node numbers.  Each node on the domain has a unique global node number. So in our example, global node numbers can be denoted by 1, 2, 3, 4, 5 since there are five nodes in total. In one single element, each node has a local node number, which is denoted as <1>, <2>, <3>. Local node numbers then repeat in another element. The relation between global node numbers and local node numbers in the same order, denoted by <1>, <2>, <3> is shown in Table 2. In a different element, such a relation may vary.

Table 2. Relation of Global and Local Nodes in Element [1]

| Element [1] | |
|---|---|
| global | local |
| 1 | <1> |
| 2 | <2> |
| 3 | <3> |

Let's first consider node 1.

By its definition from (45),

$$K_{1j} = \iint_{[1]} \left\{ \frac{\partial R_1}{\partial x}\frac{\partial R_j}{\partial x} + \frac{\partial R_1}{\partial y}\frac{\partial R_j}{\partial y} \right\} dxdy + \iint_{[2]} \left\{ \frac{\partial R_1}{\partial x}\frac{\partial R_j}{\partial x} + \frac{\partial R_1}{\partial y}\frac{\partial R_j}{\partial y} \right\} dxdy$$

$$+ \iint_{[3]} \left\{ \frac{\partial R_1}{\partial x}\frac{\partial R_j}{\partial x} + \frac{\partial R_1}{\partial y}\frac{\partial R_j}{\partial y} \right\} dxdy + \iint_{[4]} \left\{ \frac{\partial R_1}{\partial x}\frac{\partial R_j}{\partial x} + \frac{\partial R_1}{\partial y}\frac{\partial R_j}{\partial y} \right\} dxdy$$

$$+ \iint_{[5]} \left\{ \frac{\partial R_1}{\partial x}\frac{\partial R_j}{\partial x} + \frac{\partial R_1}{\partial y}\frac{\partial R_j}{\partial y} \right\} dxdy$$

Since $R_1$ is zero in element [2], [3] and [4], by properties of roof functions, the only non-zero contributions to $K_{1j}$ are $\iint_{[1]} \left\{ \frac{\partial R_1}{\partial x}\frac{\partial R_j}{\partial x} + \frac{\partial R_1}{\partial y}\frac{\partial R_j}{\partial y} \right\} dxdy$ and $\iint_{[5]} \left\{ \frac{\partial R_1}{\partial x}\frac{\partial R_j}{\partial x} + \frac{\partial R_1}{\partial y}\frac{\partial R_j}{\partial y} \right\} dxdy$

Furthermore, note that when $j = 4$, and 5

$$\iint_{[1]} \left\{ \frac{\partial R_1}{\partial x} \frac{\partial R_j}{\partial x} + \frac{\partial R_1}{\partial y} \frac{\partial R_j}{\partial y} \right\} dxdy = 0$$

because $R_4$ and $R_5$ are zero in element [1].

Similar considerations can apply to the other two nodes 2 and 3, which are attached to element [1]. And after that, we can come up with a summary of $K_{ij}$ terms to which element [1] makes non-zero distribution:

$$K_{11}, K_{12}, K_{13}, K_{21}, K_{22}, K_{23}, K_{31}, K_{32}, K_{33}$$

Based on the summary, we can construct a local element matrix $[\mathrm{K}]^{[1]}$ $(3 \times 3)$, of which the entries are:

$$K_{ab}^{[1]} = \iint_{[1]} \left\{ \frac{\partial R_{<a>}}{\partial x} \frac{\partial R_{<b>}}{\partial x} + \frac{\partial R_{<a>}}{\partial y} \frac{\partial R_{<b>}}{\partial y} \right\} dxdy$$

Note that we switch from global notation to local notation in the roof functions. And then, we need to look up the relation between global and local node numbers, and then make substitutions. That is, we put $R_{<1>} = R_1, R_{<2>} = R_2, R_{<3>} = R_3$, in element [1]. The reason we are doing this is to avoid confusion while constructing other element matrices.

Using the same method, we can also construct an element matrix $[\mathbf{T}]^{[1]}$ $(3 \times 3)$, of which the entries are:

$$T_{ab}^{[1]} = \iint\limits_{[1]} R_{<a>} R_{<b>} dxdy$$

Finishing constructing all the element matrices $[\mathrm{K}]^{[1]}$, $[\mathrm{K}]^{[2]}$, $[\mathrm{K}]^{[3]}$ and $[\mathbf{T}]^{[1]}$, $[\mathbf{T}]^{[2]}$, $[\mathbf{T}]^{[3]}$, we are now able to assemble the element matrices to construct the global matrix $[\mathbf{K}]$ and $[\mathbf{T}]$. Of course, the word "assemble" does not mean simply adding up. For example,

$$K_{11} = K_{11}^{[1]} + K_{11}^{[5]}$$

if in element [5], the global number 1 is related to local number <1>.

After construction of $[\mathbf{K}]$ and $[\mathbf{T}]$, we now need to address the inconsistencies caused by the boundary conditions. The system of equations we have is

$$\lambda \begin{bmatrix} \mathrm{T}_{11} & \cdots & \mathrm{T}_{15} \\ \vdots & \ddots & \vdots \\ \mathrm{T}_{51} & \dots & \mathrm{T}_{55} \end{bmatrix} \begin{bmatrix} \tilde{\phi}_1 \\ \tilde{\phi}_2 \\ \tilde{\phi}_3 \\ \tilde{\phi}_4 \\ \tilde{\phi}_5 \end{bmatrix} = \begin{bmatrix} \mathrm{K}_{11} & \cdots & \mathrm{K}_{15} \\ \vdots & \ddots & \vdots \\ \mathrm{K}_{51} & \dots & \mathrm{K}_{55} \end{bmatrix} \begin{bmatrix} \tilde{\phi}_1 \\ \tilde{\phi}_2 \\ \tilde{\phi}_3 \\ \tilde{\phi}_4 \\ \tilde{\phi}_5 \end{bmatrix}$$

Among all these five equations, only two are valid (equation 3 and 4). The other equations are invalid, because they don't satisfy equation (40) at the beginning. These equations are carried through the process just to maintain the structure of the system.

Selecting only the valid equations and imposing the boundary conditions:

$$\tilde{\phi}_1 = \tilde{\phi}_2 = \tilde{\phi}_5 = 0$$

We then have,

$$\lambda\left(T_{33}\tilde{\phi}_3 + T_{34}\tilde{\phi}_4\right) = K_{33}\tilde{\phi}_3 + K_{34}\tilde{\phi}_4$$

$$\lambda\left(T_{43}\tilde{\phi}_3 + T_{44}\tilde{\phi}_4\right) = K_{43}\tilde{\phi}_3 + K_{44}\tilde{\phi}_4$$

The rest of the work is left to the computer to compute $\lambda$. Note that since we have only two valid equations, we can only solve for two eigenvalues. However, as the number of nodes within the domain increases, there will be more valid equations so that we can compute more eigenvalues. Our example is an extreme and simple case that is only for illustration on how FEM works in solving for $\lambda$.

As a summary, we refer to a list of basic steps for how to use FEM to solve a two-dimensional boundary condition problem in the book [10]:

1. Reformulate the original boundary value problem by using the modified Galerkin's method.
2. Mesh the domain of the problem using appropriately chosen elements, numbering the nodes, the interior elements and the boundary elements.
3. Approximate the unknown function (or surface function) according to the chosen elements.
4. For each element note the relation between global node numbers and local node numbers on the standard element.
5. Note the relation between roof functions and shape functions.
6. Formulate the approximation over the complete domain in terms of roof functions

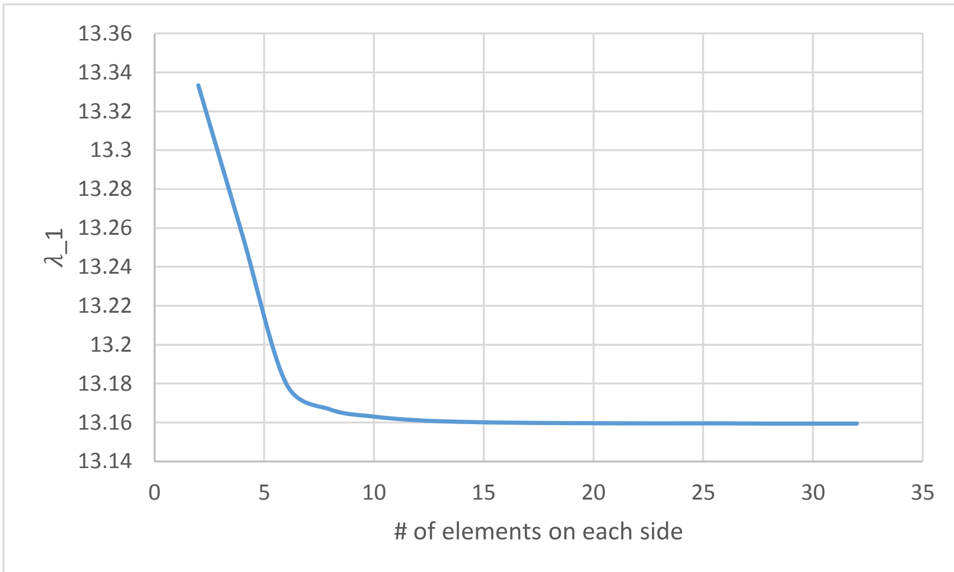
7. Form the overall matrix components and define corresponding element matrix components.
8. Evaluate each integral. In practical problems this is normally done numerically.
9. Impose boundary conditions.
10. Solve the resulting system of equations for the unknown nodal values.

Now, let's discuss the convergence of the approximation of $\lambda$. Since we already know the exact value of the smallest eigenvalue on equilateral triangle with its side $h = 2$:

$$\lambda_1 = \frac{4\pi^2}{3} \approx 13.1595$$

We calculate $\lambda_1$ using the FEM method with 2, 4, 6, …,32 elements on each side, to see how the approximated value converges to the true value. The result is shown in Figure 6. From the result, we see that the approximated value of $\lambda_1$ converges fast to its true value. Therefore, we think the approximation of $\lambda$ using FEM method is valid.

Figure 6. First Eigenvalue Approximation




### 3. A Numerical Study on $\frac{\lambda_2}{\lambda_1}$ on Isosceles Triangles

In 1991, Ashbaugh and Benguria [3] provided a proof of Payne-Pólya-Weinberger conjecture, suggesting that the ratio of the second smallest eigenvalue $\lambda_2$ over the smallest one $\lambda_1$ is optimized where the domain is a disk, the most regular shape in $\mathbb{R}^2$. And in 2010, Siudeja [13] that in the class of triangles, the ratio $\frac{\lambda_2}{\lambda_1}$ is optimized on an equilateral triangle, which is the regular case among triangles. In this session, we will perform a numerical study on a conjecture about eigenvalues on triangles:

Given that $T$ is an isosceles triangle, $\lambda_1$ is the smallest eigenvalue associated with $T$ and $\lambda_2$ the next smallest, the ratio $\xi_{2,1}(T) = \frac{\lambda_2}{\lambda_1}$ on isosceles triangles is maximized if and only if $T$ is equilateral.

As we have seen in the previous section, on equilateral triangles $T_e$,

$$\xi_{2,1}(T_e) = \frac{7}{3}$$

according to the exact calculation for eigenvalues on equilateral triangles.

From a numerical point of view, we get a similar result from the finite-element method. For example, for an equilateral triangle of side $h = 2$, we use *FreeFem*, a computer software that uses finite-element method for calculation, and it gives the result of first two eigenvalues as following:

Figure 7 First Eigenvalue and Eigenfunction

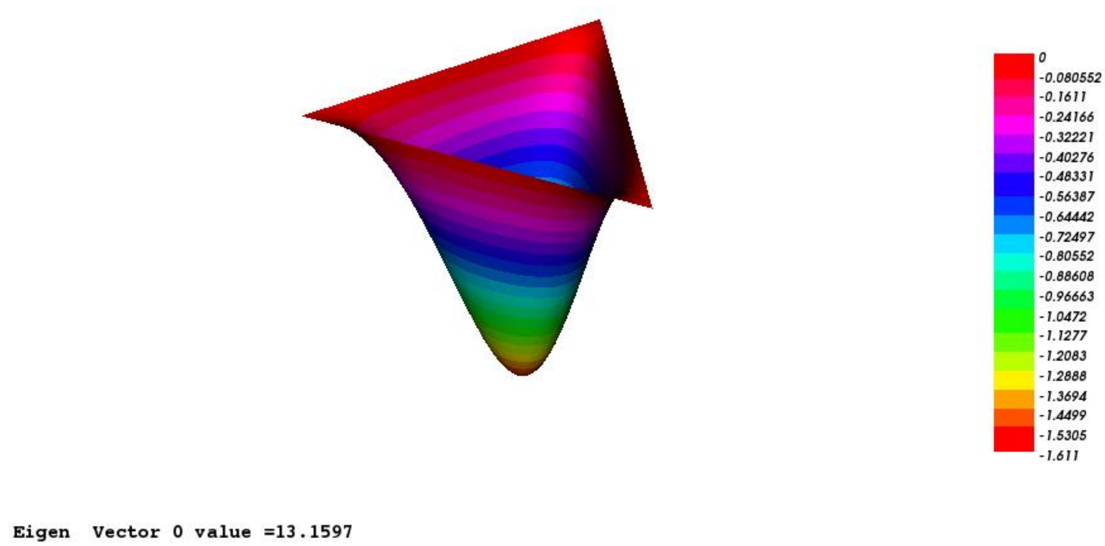
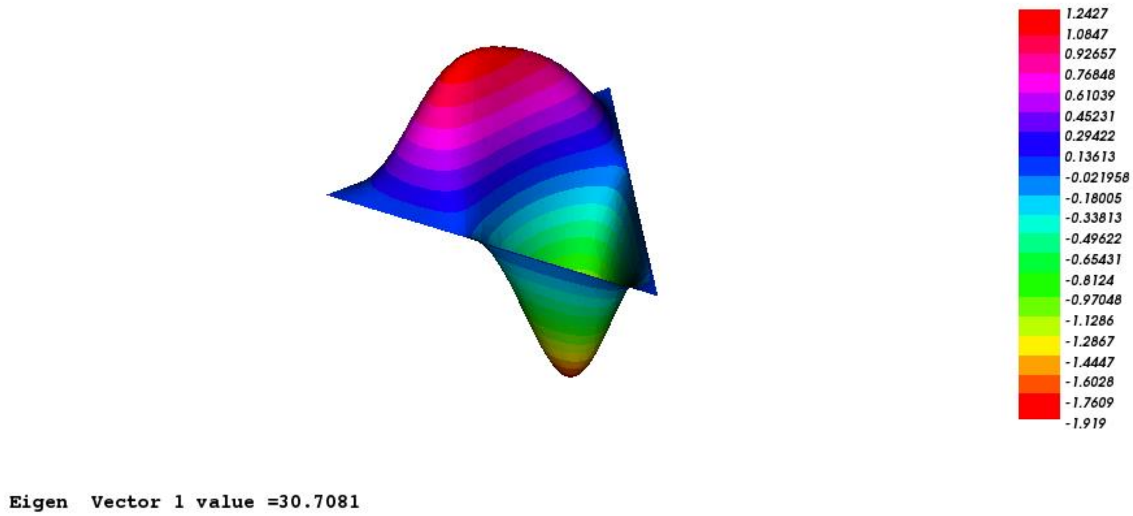


Figure 8. Second Eigenvalue and Eigenfunction




Therefore, the ratio is

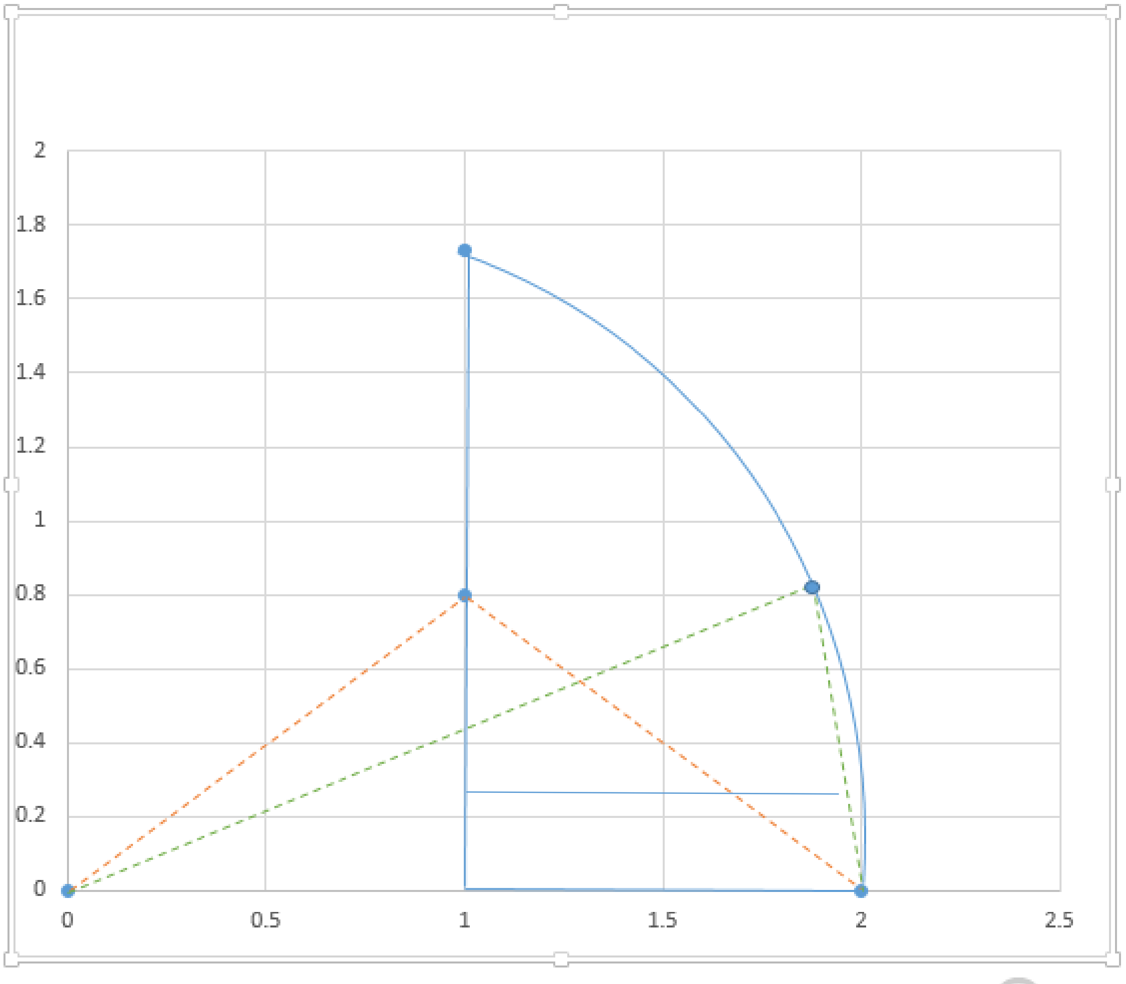
$$\xi_{2,1} = \frac{\lambda_2}{\lambda_1} = \frac{30.7081}{13.1597} \approx 2.333 \approx \frac{7}{3}$$

which is consistent with the result from exact calculation.

In order to see how $\xi_{2,1}$ is dependent on the shape of a triangle, we define the region $R$ as

$$R = \{(x, y) \in \mathbb{R}^2 : x \geq 1, y > 0, x^2 + y^2 \leq 4\}$$

Figure 9. The Region *R*




The boundary of $R$ consists of three parts, $\Gamma_1, \Gamma_2$ and $\Gamma_3$, where

$$\Gamma_1 = \{(x, y) \in \mathbb{R}^2 : 1 \leq x \leq 2, y = 0\}$$

$$\Gamma_2 = \left\{(x, y) \in \mathbb{R}^2 : 1 \leq x \leq 2, y = \sqrt{4 - x^2}\right\}$$

$$\Gamma_3 = \left\{(x, y) \in \mathbb{R}^2 : x = 0, 0 < y \leq \sqrt{3}\right\}$$

If we pick an arbitrary point $(x, y) \in R$, we can have a triangle, which is defined by vertices $(0,0), (2,0)$ and $(x, y)$. We let $S$ to be the set that contains all such triangles.

It is easy to see that for any arbitrary triangle $T$, there is a unique triangle $\hat{T} \in S$, such that $T$ is similar to $\hat{T}$. For our purpose, the set of triangles $S$ is enough for investigation, since $\xi_{2,1}(T) = \xi_{2,1}(\hat{T})$ if $T$and $\hat{T}$ are similar. To see why the equation is true, suppose we have a domain $T$ and we find an eigenvalue $\lambda$ and corresponding eigenfunction $\phi$ such that

$$\frac{\partial^2\phi(x_1,y_1)}{\partial {x_1}^2} + \frac{\partial^2\phi(x_1,y_1)}{\partial {y_1}^2} = -\lambda\phi(x_1,y_1)$$

with $\phi(x_1,y_1) = 0 \text{ on } \partial T$.

Let $\hat{T}$ be a similar triangle to $T$ and $\frac{\text{area of } T}{\text{area of } \hat{T}} = q^2$, then for any point $(x_1,y_1)$ on $T$, there is a point $(x_2,y_2)$ on $T$ such that $(x_1,y_1)$ is equivalent to $(qx_2,qy_2)$, and vice versa. Hence, the relation is one-to-one.

Substituting new coordinates $(qx_2,qy_2)$ for $(x_1,y_1)$,

$$\frac{\partial^2\phi(qx_2,qy_2)}{q^2\partial {x_2}^2} + \frac{\partial^2\phi(qx_2,qy_2)}{q^2\partial {y_1}^2} = -\lambda\phi(qx_2,qy_2)$$

Then multiplying both sides by $q^2$, we get $q^2\lambda$ as an eigenvalue and $\phi(qx_2,qy_2)$ the corresponding eigenfunction, associated with domain $\hat{T}$.

As a result,

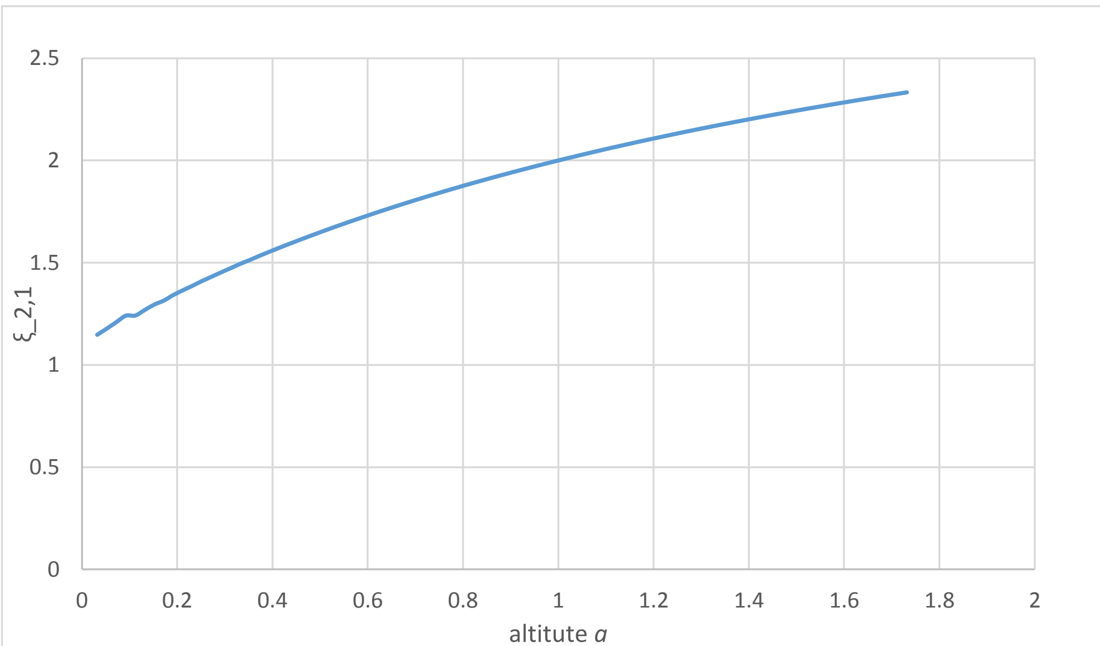
$$\xi_{2,1}(T) = \frac{\lambda_2(T)}{\lambda_1(T)} = \frac{q^2\lambda_2(T)}{q^2\lambda_1(T)} = \frac{\lambda_2(\hat{T})}{\lambda_1(\hat{T})} = \xi_{2,1}(\hat{T})$$

Also, we note that a triangle $T \in S$ is isosceles, if and only if its vertex is on the boundaries $\Gamma_2$ and $\Gamma_3$.

Let's first look at $\Gamma_2$ and see how $\xi_{2,1}$ is dependent on the shape of triangle. We start from altitude $a = \sqrt{3}$, calculate $\xi_{2,1}$, subtract it by 0.02 and repeat. Table 3 shows the result.

Plotting the result on a graph, we see a nice relation between altitute $a$ and $\xi_{2,1}$(see Figure 10).

Figure 10. Graph of ξ_2,1 v.s. *a*

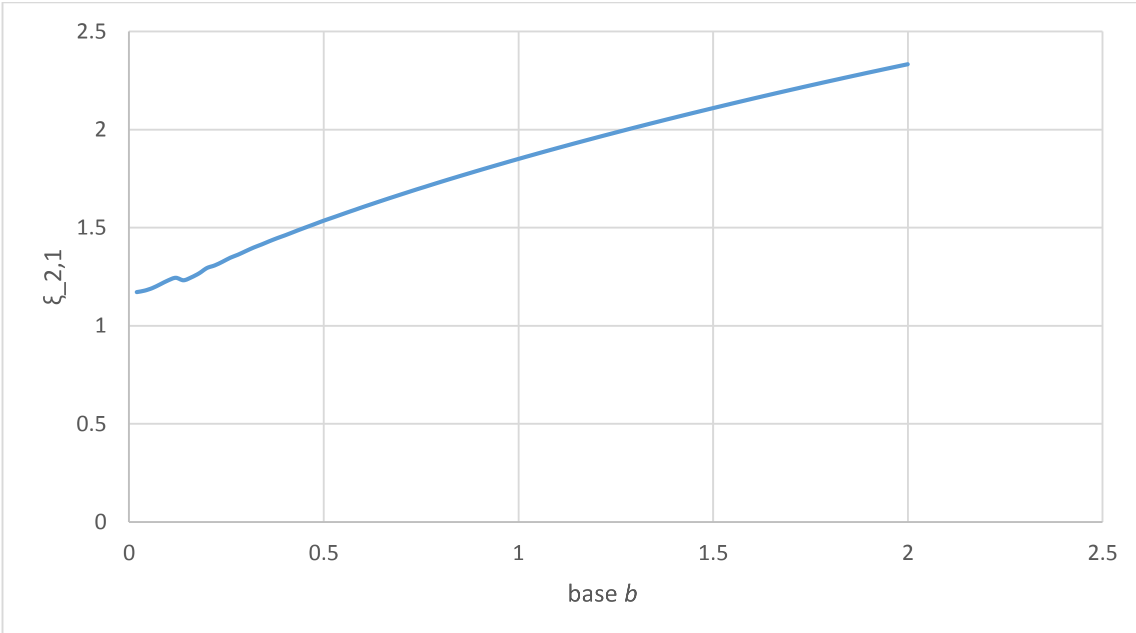



The result suggests that for $0 < a \leq \sqrt{3}$, $\xi_{2,1}$ is monotonically increasing. And $\xi_{2,1}$ reaches its maximum value $\frac{7}{3}$ if and only if $a = \sqrt{3}$, which is equivalent with $T$ is equilateral.

Next, we look at boundary $\Gamma_2$. We start with base side $b = 2$ and subtract it by 0.02 for each step. Table 4 shows the result.

The same as what we do for boundary $\Gamma_3$, we plot the result on a graph shown in Figure 11.

Figure 11. Graph of ξ_2,1 v.s. *b*




As one can see, the result on $\Gamma_2$ is quite similar to the result on $\Gamma_3$. Again, $\xi_{2,1}$ is monotonically increasing, as its base *b* increases, $b \in (0,2]$. $\xi_{2,1}$ reaches its maximum value $\frac{7}{3}$ if and only if $T$ is equilateral.

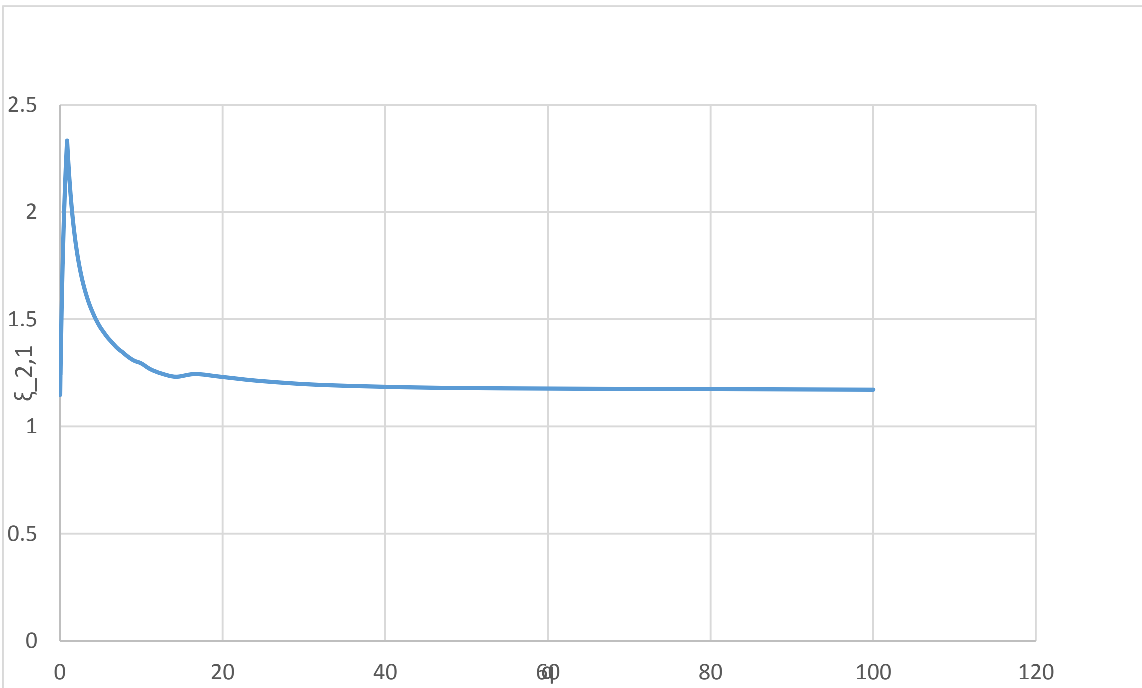
Summarizing the result of triangles on boundaries $\Gamma_2$ and $\Gamma_3$, we can see that among all isosceles triangles, $\xi_{2,1}$ is optimized when the triangle is equilateral. And the result seems to suggest that as the triangle gets less similar to an equilateral triangle, $\xi_{2,1}$ is getting smaller. We use the phrase "less similar" in the sense that the vertex on boundaries $\Gamma_2$ or $\Gamma_3$ moves in the direction that is not towards the point of equilateral triangle.

To better see the behavior of $\xi_{2,1}$, dependent on the triangle's "dissimilarity" to equilateral triangle, we consider a parameter

$$q = \frac{a}{b}$$

where $b$ is the base of the isosceles triangle, and $a$ is the altitude perpendicular to the base. And the result is shown in Figure 12.

Figure 12. Graph of ξ_2,1 v.s. *q*




We see that the graph changes rapidly. It climbs up fast to the peak and declines fast after the peak. The sharp peak occurs at $q = \frac{\sqrt{3}}{2}$, which represent the point of equilateral triangle. One can infer from the graph that the optimization of $\xi_{2,1}$ is very unstable with respect to $q = \frac{a}{b}$, in the sense that a small deviation from $q = \frac{\sqrt{3}}{2}$ will lead to a large decline of $\xi_{2,1}$.

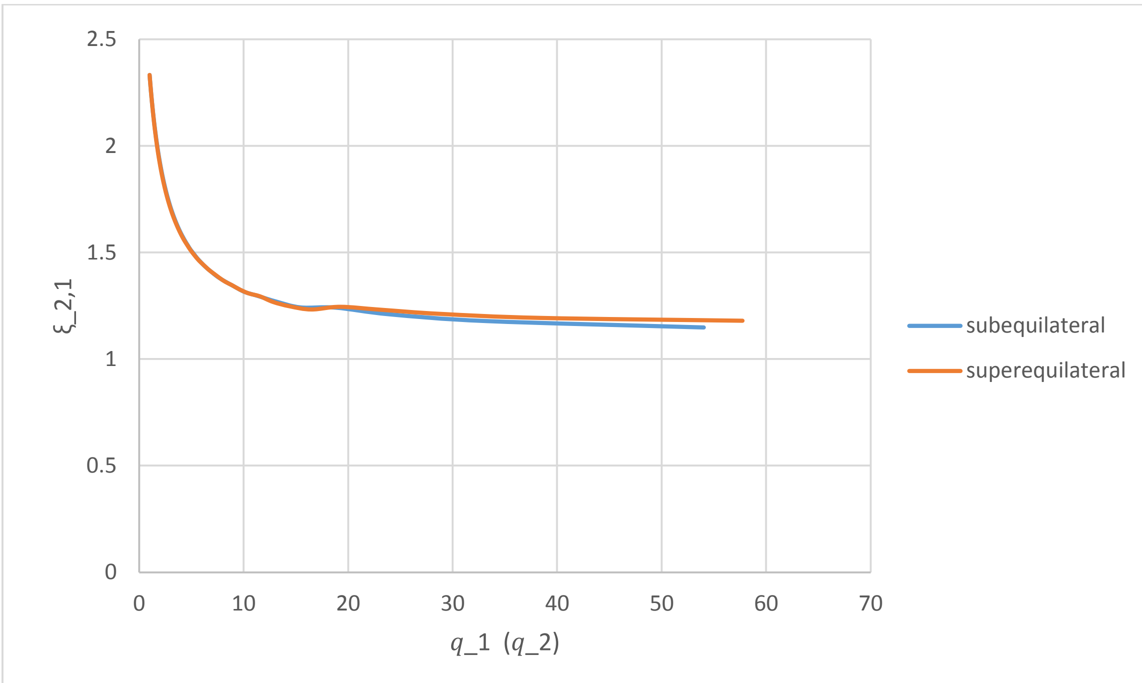
However, the result shows that using a single parameter $q = \frac{a}{b}$ may not be a good choice to see the behavior of $\xi_{2,1}$, since the change of $\xi_{2,1}$ is too rapid, especially in the part before the peak.

We then think of using parameter $q_1 = (\frac{b}{a})/(\frac{2}{\sqrt{3}})$ for subequilateral triangles (isosceles with aperture less than $\frac{\pi}{3}$) and $q_2 = (\frac{a}{b})/(\frac{\sqrt{3}}{2})$ for superequilateral triangles (isosceles with aperture greater than $\frac{\pi}{3}$). One may easily see that all subequilateral triangles are on the boundary $\Gamma_2$, and all superequilateral triangles are on the boundary $\Gamma_3$.

It is quite reasonable to set up parameters like this, because both $q_1$ and $q_2$ measure the "dissimilarity" between the isosceles triangle and equilateral triangle. According to the definitions of parameters $q_1$ and $q_2$, a subequilateral (superequilateral) triangle is most similar to equilateral (or say, it is actually equilateral) when $q_1(q_2)$ is equal to one, and is less similar to equilateral as $q_1$ $(q_2)$ increases.

The result, shown in Figure 13, is quite interesting, because the two curves almost overlap each other. From the graph, we see that $\xi_{2,1}$ is monotonically decreasing as $q_1$or $q_2$ increases. The optimization of $\xi_{2,1}$ occurs if and only if $q_1$ or $q_2$ is equal to one.

Figure 13. Graph of ξ_2,1 v.s. $q$_1 ($q$_2)




Since the two curves are so closed to identical, we may ignore the difference between subequilateral triangles and superequilateral triangles, and generally state that: for all isosceles triangles, $\xi_{2,1}$ is dependent on the triangle's "dissimilarity", which is measured by $q_1$or $q_2$, and $\xi_{2,1}$ is optimized when "dissimilarity" is minimized, and $\xi_{2,1}$ decreases as "dissimilarity" becomes larger.

## 4. Conclusion

Spectral problem has been a hot research topic in Mathematics and Physics in recent decades. Despite the fact that the answer to Kac's question is "no" in general, there are still a lot of mysteries related to this problem remains unconcealed. In the class of triangles, so far, we can have a very nice formula to calculate Dirichlet eigenvalues for equilateral triangles. However, for general triangles, we need to use some numerical method, like FEM, to approximate the true eigenvalues. Moreover, we examine the behavior of the ratio $\xi_{2,1} = \frac{\lambda_2}{\lambda_1}$ in isosceles triangles. The numerical result suggests that $\xi_{2,1}$ can only be optimized when the triangle is equilateral, and $\xi_{2,1}$ decreases sharply as the triangle becomes dissimilar to equilateral triangle.

For open questions, what is the exact function $\xi_{2,1}$ in terms of the "dissimilarity" we defined in previous section? Instead of $\xi_{2,1}$, what is the behavior of other ratios, like $\xi_{3,1} = \frac{\lambda_3}{\lambda_1}$ ? And what is the error in approximation, when using the FEM method? The current paper doesn't address these questions and more work should be done for answers.

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