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Shuo Li March 23rd, 2017

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An abstract of
a thesis submitted to the Faculty of Emory College of Arts and Sciences
of Emory University in partial fulfillment
of the requirements of the degree of
Bachelor of Sciences with Honors

Department of Mathematics and Computer Science

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Abstract

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The sphere packing problem has a long history. A sphere packing problem refers to the problem of finding arrangements of equal-sized nonoverlapping spheres that can fill a given space with maximized density. In the 17th century, German mathematician, Johannes Kepler proposed the so-called Kepler's Conjecture which is about the sphere packing problem in three-dimensional Euclidean space. This mathematical conjecture bothered mathematicians for more than 400 years. Moreover, since every dimension has its own version of sphere packing problem, this classic problem will continue to be a hot topic for mathematicians.

In this paper, we will first briefly introduce the sphere packing problem in dimension 2 and 3. Since we can visualize these packings, our intuition can help us to understand the problem better. Then we will discuss the recent breakthrough in some higher dimensions such as dimension 8 and 24. Then we will focus on sphere packings in dimension 2 and we will give an exposition of the theorem that the best lattice packing in dimension 2 is given by hexagonal lattice. This theorem is due to Axel Thue.

To achieve our main goal in this paper, we will use the definitions of well-rounded lattice and successive minima. Moreover, we will break down the proof of main theorem into several parts. In other words, we will prove some preliminary lemmas before proceeding to the main proof.

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Chapter 1 *Introduction*

1.1 Introduction

Imagine UPS has to fill an empty warehouse with equally sized basketballs. How can they fill the warehouse with as many basketballs as they can? In other words, they have to find an arrangement of basketballs such that the "density" of basketballs in that warehouse is maximized. We will define density later but here intuitively, density refers to the ratio of the space occupied by all the non-overlapping basketballs over the entire space of the warehouse. Back to the 17th century, Johannes Kepler, German mathematician and astronomer, proposed a famous mathematical conjecture that no arrangements of equally sized non-overlapping spheres is denser than face-centered cubic and hexagonal close packing. That is, each sphere touches with 12 others.

For many years, the upper bound for sphere packings in dimension 3 is 0.7796 by Roger's in 1958 [1]. However, in 1998, Thomas Hales followed the idea proposed by Fejes Tóth (1953) [2] and gave the proof of Kepler Conjecture in dimension 3 and concluded that the maximum possible density for all arrangements of spheres in dimension 3 is $\frac{\pi}{\sqrt{18}} \approx 0.7405$ [3]. In other words, the best possible arrangement of equally sized basketballs can occupy 74.05% of the space in the empty warehouse.

In addition to dimension 3, every dimension has its own version of Sphere Packing problem and its own upper bound of density. For example, In dimension 1, spheres become points and warehouse becomes an interval. So, intuitively, we can line up points and fill the interval completely. Therefore, the maximum possible density is 1. How-

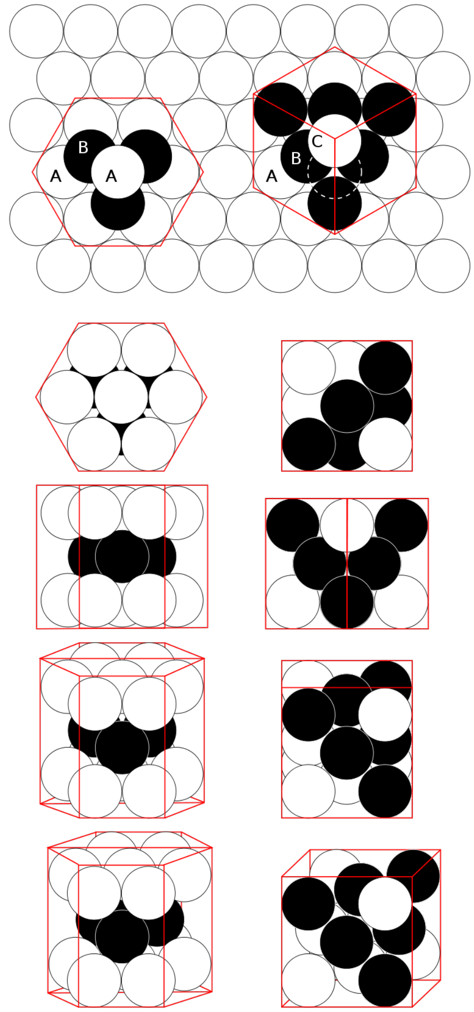

Figure 1.1: Face-Centered Cubic (fcc, right) and Hexgonal Close-Packed (hcp, left) [3]

ever, in dimension 2, hexagonal packing is the densest possible arrangement and the maximum possible density is $\frac{\pi}{\sqrt{12}} \approx 0.906809$. As what we have discussed above, the maximum density in dimension 3 is $\frac{\pi}{\sqrt{18}}$.

However, higher dimensional sphere packing problems are extremely hard, especially beyond dimension 3, because they are hard to visualize. Moreover, each added dimension means more packings to consider. Therefore, up until now, we even only had conjectures for the best packings for only two dimensions beyond dimension 3 which are

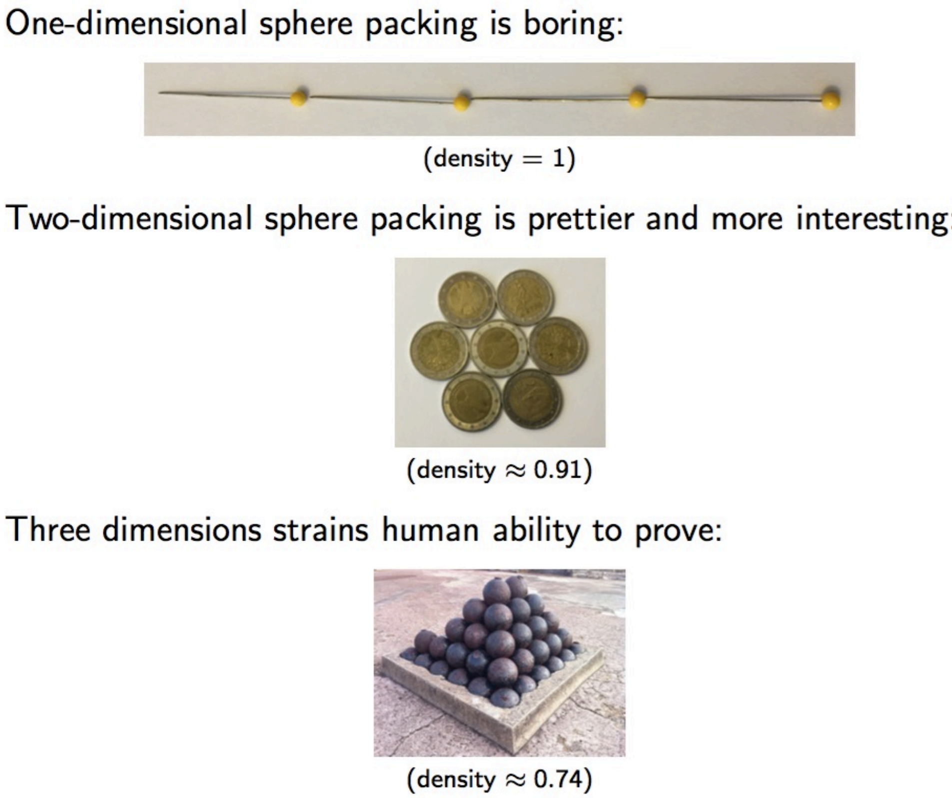

Figure 1.2: Visualization of Sphere Packings in Lower Dimensions [5]

dimension 8 and dimension 24. For a long time, mathematicians knew the maximum possible packings for dimension 8 and dimension 24 should be given by two symmetric sphere packings called E_8 and Leech lattice, respectively [4]. However, these conjectures remained open for a long time. For more than a decade, mathematicians knew that the only missing part of the proof should be some types of “auxiliary” functions but they just failed to find them until last year. In March 14th, 2016, Maryna Viazovska, a postdoctoral researcher at the Berlin Mathematical School and the Humboldt University of Berlin, found the mysterious missing ingredient of the proof. She used the idea of the theory of modular forms and proved that E_8 is the best sphere packing in dimension 8 which gives the maximum density $\frac{\pi^4}{384} \approx 0.25367$ [6]. Only one week later, Viazovska and some other mathematicians mimicked the proof for E_8 and successfully proved the similar argument related to Leech lattice in dimension 24 which gives the maximum density $\frac{\pi^{12}}{12!} \approx 0.0019295743$ [7]. As one may notice, the maximum density decreases as dimension increases.

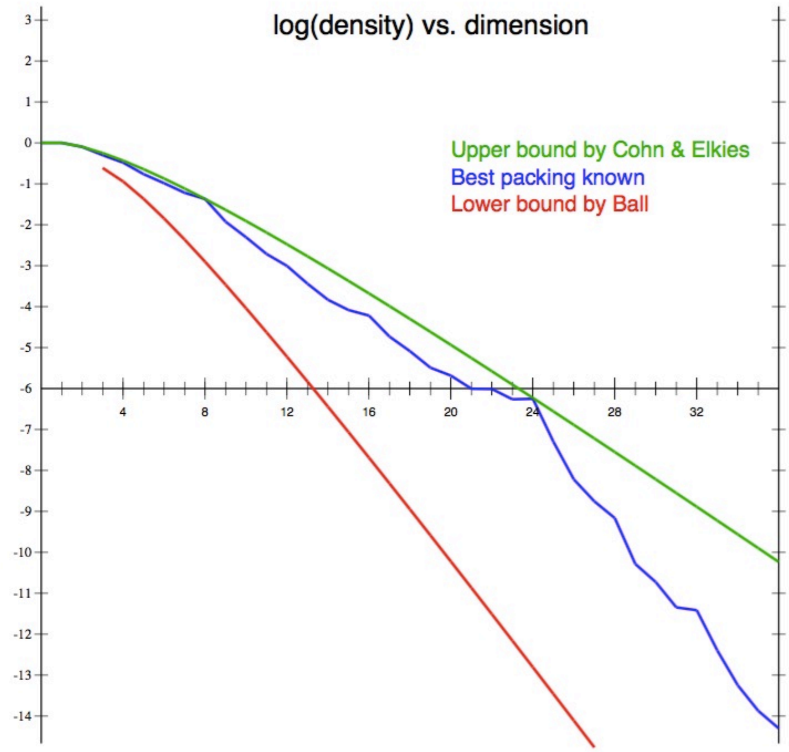

Figure 1.3: Lower and Upper Bounds for Different Dimensions [5]

Figure 1.3 is a plot of log of densities over dimensions. Values of log densities are always nonpositive because densities are less or equal to 1. As the dimension increases, the maximum possible density decreases. Moreover, although the sphere packing problem has more than 400 years history, there is still a huge amount of work needs to be done. Beyond dimension 3, our best known packings are far away from their corresponding upper bounds except dimension 8 and 24. As the dimensions increase, the discrepancy between our best known packings and their corresponding upper bounds grows. Besides the long history of sphere packing problems, since we still do not know too much about this classic question, mathematicians are very interested in this topic. In addition to theoretical challenges, higher dimensional packings are actually practical objects, too.

Sphere packings relate to the error-correcting codes used by cell phones, space probes and the Internet to send signals through noisy channels.

There are two different types of sphere packings, lattice packings and non-lattice packings. It's easy to tell from the name that lattice packings are given by lattices while non-lattice packings do not involve lattice structures.

A lattice in $\mathbb{R}^m$ refers to a set of all linear combinations of n linearly independent vectors in $\mathbb{R}^m$ with integer coefficients. Intuitively, it is a repetitive pattern of an arrangement of points. Each point in the lattice corresponds to a center of a sphere. Since no overlapping is allowed, the radii for these spheres should be a half of the shortest vector in the lattice. This kind of symmetric structure of lattice can reduce the complexity of our problem.

On the other hand, non-lattice packings do not have such lattice-like structures, therefore, studying non-lattice packing requires more work. Fortunately, in many dimensions, especially lower dimensions such as dimension 2, lattice packings are better than non-lattice packings. Therefore, we will focus on lattice packings in $\mathbb{R}^2$ in this paper.

After the brief introduction of sphere packing problem, we will discuss the main goal of this paper. In this paper, we will focus on lattice sphere packings in dimension 2. As we've mentioned before, the maximum possible density in dimension 2 is $\frac{\pi}{\sqrt{18}}$ which is achieved by so called hexagonal lattice. In the following sections, we will prove the fact that no lattice packings in dimension 2 can exceed this upper bound. This fact is a theorem due to Thue.

1.2 Solution for the $\mathbb{R}^2$ Lattice Packing Problem

In the previous section, we introduced the history of sphere packing problem and the goal of this paper. Now we will state the main theorem and will prove it in the following sections.

Theorem 1. *(Thue's Theorem) Let Λ be a full rank lattice in $\mathbb{R}^2$, then*

$$\Delta(\Lambda) \leq \Delta(\Lambda_h) = \frac{\pi}{\sqrt{12}}$$

Equality holds if and only if Λ is similar to Λ_h, where Δ is the density function [8].

As we've discussed at the beginning, density Δ refers to the ratio of area occupied by a specific arrangement over the entire space and the definition will be given in Section 2.1.1. Then, in the following chapters, we will prove this therorem. In order to do this, we will introduce some definitions. Then we will break down the proof into three steps and each step consists of several lemmas. We will prove each of those lemmas. Then we will give the proof to the theorem based on these lemmas.

Chapter 2 *Background and Strategy*

2.1 Definition

Before proceeding to the proof of the main theorem, we require some basic preliminaries in this chapter.

Definition 2.1.1. A *lattice* is a free $\mathbb{Z}$ module with finite rank.

To be more specific, a lattice Λ in $\mathbb{R}^2$ is a free $\mathbb{Z}$ module of rank two. Lattices are very important objects which appear in a wide variety of disciplines including group theory, number theory, finance, arts, cryptography and more. The definition we provided above (Definition 2.1.1) requires knowledge in abstract algebra.

Here is a simple mathematical description of an n dimensional lattice: Given n linearly independent vectors b_1, b_2, ..., $b_n \in \mathbb{R}^m$, the *lattice* generated by them is defined as

$$\mathcal{L}(b_1, b_2, ..., b_n) = \left\{ \sum_{i=1}^{n} x_i b_i : x_i \in \mathbb{Z} \right\}.$$

We refer to b_1, b_2, ..., b_n as a basis of the lattice. Equivalently, if we define B as $m \times n$ matrix whose columns are b_1, b_2, ..., b_n, then the lattice generated by B is

$$\mathcal{L}(B) = \mathcal{L}(b_1, b_2, ..., b_n) = \{Bx : x \in \mathbb{Z}\}.$$

We say that the *rank* of the lattice is n and the *dimension* is m. If $n = m$, the lattice is called a *full rank lattice*. Thus, for any basis in $\mathbb{R}^n$, the subgroup of all linear combinations with integer coefficients of the basis vectors forms a lattice [10]. Then, let's see some examples of lattices in $\mathbb{R}^2$.

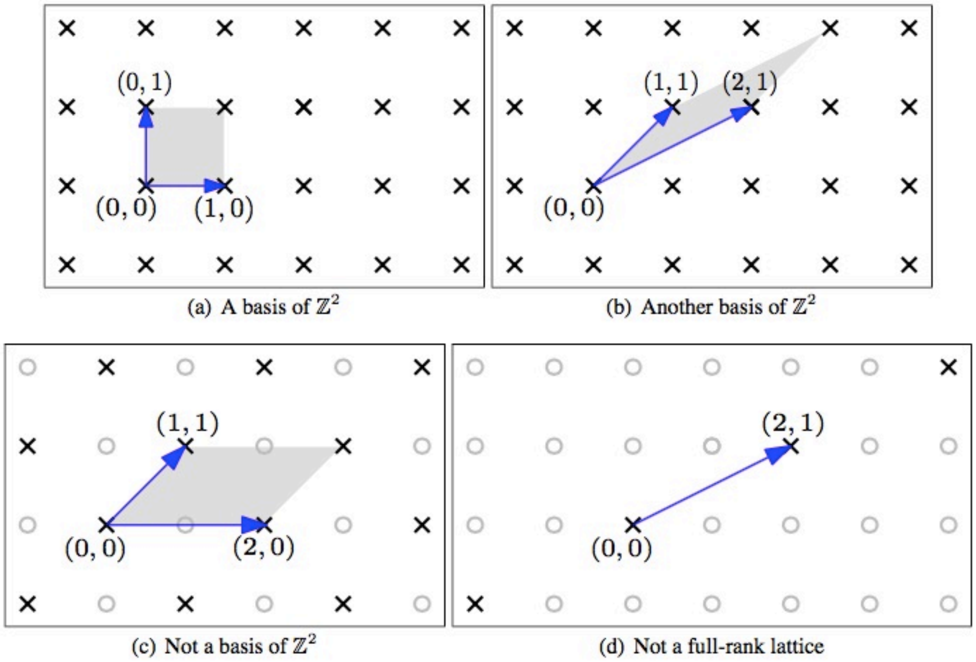

Figure 2.1: Lattice in $\mathbb{Z}^2$ [10]

The lattice generated by $(1, 0)^T$ and $(0, 1)^T$ is shown in Figure 2.1(a). Since the coefficients for linear combinations in this lattice are all integers. This lattice generates $\mathbb{Z}^2$. The basis is not unique. For example, in Figure 2.1(b), basis consists of $\{(1, 1), (2, 1)\}$ also generates $\mathbb{Z}^2$ and another basis for $\mathbb{Z}^2$ is $\{(2017, 1), (2018, 1)\}$. On the other hand, not all randomly picked two vectors can be basis for $\mathbb{Z}^2$. For example, Figure 2.1(c) shows that $\{(1, 1), (2, 0)\}$ is not a basis for $\mathbb{Z}^2$ because its linear combinations cannot generate all integer points. For instance, it cannot generate $(1, 0)$. In Figure 2.1(d), it is an example of lattice which is not full rank. Since in $\mathbb{Z}^2$, the dimension is 2 but the lattice has only one linearly independent vector. Thus, the lattice generated by $\{(2, 1)\}$ is not full rank in $\mathbb{Z}^2$. It is of dimension 2 and of rank 1.

Definition 2.1.2. A *Voronoi cell* of Λ is defined as

$$\mathcal{V}(\Lambda) = \{\mathbf{y} \in \mathbb{R}^2 : ||\mathbf{y}|| \leq ||\mathbf{y} - \mathbf{x}|| \, \forall \, \mathbf{x} \in \Lambda\}.$$

The notion of a Voronoi cell is a very important idea. Intuitively, a Voronoi cell $\mathcal{V}(\Lambda)$ in $\mathbb{R}^2$ is the closure of the set of all vectors in the real plane that are closer to $\mathbf{0}$ than to any other vectors in lattice Λ. Since we focus on lattice packings in this paper, Voronoi cells enable us to reduce the complexity of our problem. Since lattices are repetitive patterns generated by sets of linearly independent vectors, we can study a small proportion of these lattices instead of these lattices as a whole. In other words, we can reduce our problem of maximizing global densities over the entire lattices to maximizing the local densities over their corresponding Voronoi cells since the entire lattice consists of shifting and translating the Voronoi cell.

Now, recall the example of filling empty warehouse with basketballs at the beginning. Our goal is to find an arrangement of spheres to fill a space with non-overlapping spheres which maximizes the density of spheres in this space. Although this problem is easy to imagine, there are lots of questions we have to deal with. For example, what is an "arrangement" of spheres? What space are we filling? How should we measure the "density"? We will answer all of these questions in the rest of this section.

Definition 2.1.3. Let $\Lambda \subset \mathbb{R}^2$ be a discrete set of points such that $||x - y|| \geq 2r$ for any distinct points x, $y \in \Lambda$ and $r \in \mathbb{R}^+$. Then the union

$$\mathcal{P} = \bigcup_{x \in \Lambda} B(x, r)$$

is a *sphere packing.* $B(x, r)$ denotes a ball centered at x with radius r. If $\Lambda \subset \mathbb{R}^2$ is a lattice, then $\mathcal{P}$ is a *lattice packing.*

Definition 2.1.4. Let Λ be a full rank lattice in $\mathbb{R}^2$, so that

$$\Lambda = M\mathbb{Z}^2$$

where $M = (\mathbf{u_1}\ \mathbf{u_2}) \in GL_2(\mathbb{R})$, and the column vectors $\mathbf{u_1}$ and $\mathbf{u_2}$ of M form a basis for Λ. Then, M refers to the *basis matrix* and the *determinant* of Λ, denoted by $\det(\Lambda)$, is defined to be $|\det(M)|$.

Remark. Let us look at the basis martrix for hexagonal lattice Λ_h in $\mathbb{R}^2$

$$\Lambda_h := \begin{bmatrix} 1 & \frac{1}{2} \\ 0 & \frac{\sqrt{3}}{2} \end{bmatrix} \mathbb{Z}^2.$$

It's easy to conclude that hexagonal lattice Λ_h in $\mathbb{R}^2$ is generated by vectors $\{(1\ 0), (\frac{1}{2}\ \frac{\sqrt{3}}{2})\}$. They have the same norm and there is a $\frac{\pi}{3}$ angle between them. Moreover, the determinant of Λ, $\det(\Lambda) = \frac{\sqrt{3}}{2}$. The importance of determinant of lattice is that $\det(\Lambda)$ is the area of the Voronoi cell $\mathcal{V}(\Lambda)$ corresponding to the lattice Λ.

Here we give a short proof of this fact that

$$S_{\mathcal{V}(\Lambda)} = \det(\Lambda)$$

where $S_{\mathcal{V}(\Lambda)}$ denotes the area of Voronoi cell $\mathcal{V}(\Lambda)$.

Proof. Let Λ be a lattice in $\mathbb{R}^2$ that generated by $\{\mathbf{u}\,,\,\mathbf{w}\}$. So the Voronoi cell $\mathcal{V}(\Lambda)$ is the parallelogram generated by $\mathbf{u}$ and $\mathbf{w}$. Since vectors $\mathbf{u}$, $\mathbf{w}$ are in $\mathbb{R}^2$, let

$$\mathbf{u} = (u_1\,,\,u_2),$$

$$\mathbf{w} = (w_1\,,\,w_2).$$

Then, since cross products only make sense in $\mathbb{R}^3$, we can extend $\mathbf{u}$ and $\mathbf{w}$ to

$$\mathbf{u} = (u_1\,,\,u_2\,,\,0),$$

$$\mathbf{w} = (w_1\,,\,w_2\,,\,0).$$

Then by definition of cross product, we have that

$$S_{\mathcal{V}(\Lambda)} = \mathbf{u} \times \mathbf{w} = \begin{vmatrix} i & j & k \\ u_1 & u_2 & 0 \\ w_1 & w_2 & 0 \end{vmatrix} = k \det \begin{bmatrix} u_1 & u_2 \\ w_1 & w_2 \end{bmatrix}.$$

Then by taking norm to both sides we have that

$$S_{\mathcal{V}(\Lambda)} = ||\mathbf{u} \times \mathbf{w}|| = \det \begin{bmatrix} u_1 & u_2 \\ w_1 & w_2 \end{bmatrix} = \det \begin{bmatrix} u_1 & w_1 \\ u_2 & w_2 \end{bmatrix} = \det(\Lambda).$$

$\square$

Definition 2.1.5. Let Λ be a lattice in $\mathbb{R}^2$. There exists a vector $\mathbf{u_{min}}$ such that $|\mathbf{u_{min}}|$ is the minimum in Λ. Then, the *packing radius* $r(\Lambda)$ is defined as

$$r(\Lambda) = \frac{1}{2}|\mathbf{u_{min}}|.$$

One thing should be noticed is that the packing radius $r(\Lambda)$ is not necessaily equal to 1. A natural question is how will the length of packing radius affect the density function? In the following chapters, we will show that the packing radius will affect the density function but the maximum value of density function will not be affected by it.

Definition 2.1.6. The *density function* of a sphere packing is defined as

$$\Delta(\Lambda) = \frac{\text{area of one circle}}{\text{area of a Voronoi cell}} = \frac{\pi r(\Lambda)^2}{\det(\Lambda)}.$$

Remark. Definition 2.1.6 is extremely important, therefore it deserves further discussion. Let us have a look at the graph of hexagonal lattice packing in dimension 2.

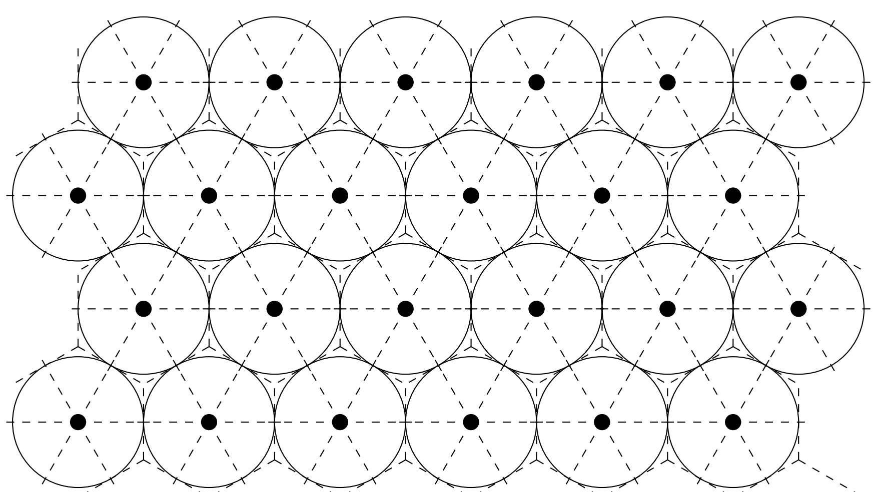

Figure 1. *Hexagonal Packing*

As we can see from the graph above, hexagonal packing is a very symmetric structure. We know that hexagonal lattice, denoted by Λ_h is generated by two linearly independent vectors in $\mathbb{R}^2$ with the same norm and a $\frac{\pi}{3}$ angle between them. Moreover, from the definition of Voronoi cell, a Voronoi cell $\mathcal{V}(\Lambda)$ is the closure of the set of all vectors in

$\mathbb{R}^2$ that are closer to $\mathbf{0}$ than to any other vector in Λ. Therefore, the real plane $\mathbb{R}^2$ is tiled with the translations of $\mathcal{V}(\Lambda)$. In other words, we can fill the entire real plane by shifting the Voronoi cell. The following graph is a Voronoi cell.

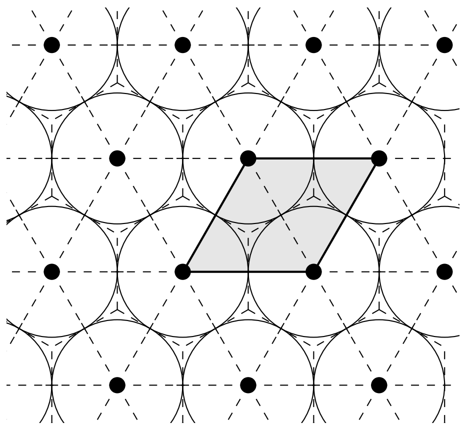

Figure 2. *Voronoi Cell*

We call the density over a Voronoi cell the local density. Since the real plane can be obtained by shifting Voronoi cell and the global density of a packing is simiply a weighted average of the local densities, the universal upper bound of the local density is automatically an upper bound of the global density. Therefore, our problem is reduced to the problem of maximizing the density function over a Voronoi cell.

Moreover, if we look at the Voronoi cell more closely, we can see that the density over a Voronoi cell is the ratio of the area of one circle in the packing over the area of Voronoi cell. Let us call the area of one circle in the packing as S_{circle} and number the sectors located at the four corners in the Voronoi cell as part 1,2,3 and 4 clockwisely starting from the left bottom corner. From Figure 2 we can see that, since the angle between the two linearly independent vectors is $\frac{\pi}{3}$ and the norms of them are equal, areas of part 1 and 3 are equal and both equal to $\frac{1}{6}$ of the area of one circle. Therefore, the sum of areas of part 1 and 3 is $\frac{1}{3}S_{\text{circle}}$. Then, since the Voronoi cell is a parallelogram, it is easy to conclude that both angles for sector 2 and 4 are $\frac{2\pi}{3}$. Therefore, $S_2 = S_4 = \frac{1}{3}S_{\text{circle}}$. Thus, the sum of sector 1,2,3 and 4 is $S_{\text{total}} = S_1 + S_2 + S_3 + S_4 = (\frac{1}{6} + \frac{1}{3} + \frac{1}{6} + \frac{1}{3})S_{\text{circle}} = S_{\text{circle}}$. Then, since the area of a parallelogram is the cross product of the two vectors that share the same starting point, which is the $\det(\Lambda)$, the density function is defined as

$$\Delta(\Lambda) = \frac{\text{area of one circle}}{\text{area of a Voronoi cell}} = \frac{\pi r(\Lambda)^2}{\det(\Lambda)}.$$

Definition 2.1.7. Given a lattice Λ in $\mathbb{R}^2$, let $\lambda_1 \leq \lambda_2$, and

$$\lambda_i(B) = \min\{\lambda \in \mathbb{R}_{>0} : \Lambda \cap \lambda B \text{ contains i linearly independent nonzero vectors}\},$$

then λ_1, λ_2 are *Minkowski successive minima* of Λ, where B is a unit circle centered at the origin. We say that vectors $\mathbf{u_1}$, $\mathbf{u_2}$ correspond to successive minimas λ_1, λ_2 if they are linearly independent and

$$||\mathbf{u_1}|| = \lambda_1,\ ||\mathbf{u_2}|| = \lambda_2.$$

The definition of Successive minima is used repeatedly in the following proofs of lemmas and theorem. Intuitively, values of successive minimas correspond to the lengths of two linearly independent vectors in $\mathbb{R}^2$.

Definition 2.1.8. A lattice $\Lambda \subset \mathbb{R}^2$ is called *well rounded* if its successive minimas λ_1 and λ_2 are equal.

Well rounded lattices are very important and will be used repeatedly in the following chapters. We will prove that the maximum density in $\mathbb{R}^2$ can only be achieved by well rounded lattices.

Definition 2.1.9. Let Λ and Ω be two different lattices in $\mathbb{R}^2$. Then Λ is said to be *similar* to Ω if there exists a real constant α and a 2×2 orthogonal real matrix M such that

$$\Omega = \alpha M \Lambda.$$

Remark. Definition 2.1.9 tells us that if Ω is similar to Λ, then Ω can be obtained from Λ by rotation and dilation. Moreover, in the following sections, we will prove that similarity is an equivalence relation and the density function for each similarity class is a constant.

Definition 2.1.10. Let Λ be a lattice in $\mathbb{R}^2$ with successive minima $\lambda_1 \leq \lambda_2$ and let $\mathbf{u_1}$, $\mathbf{u_2}$ be vectors in Λ corresponding to λ_1, λ_2, respectively. Then the basis $\{\mathbf{u_1}, \mathbf{u_2}\}$ is called the *minimum basis.*

In the next chapter, we will show that there exists a minimum basis for each lattice in $\mathbb{R}^2$.

2.2 Some Sample Packings in $\mathbb{R}^2$

Since we have given many definitions, let us get familiar with some of them. Now, we will discuss some sample sphere packings in $\mathbb{R}^2$ and calculate their densities.

First, let us consider the following example. Let $\Lambda \subset \mathbb{R}^2$ be a lattice with basis $\{\mathbf{u_1},\, \mathbf{u_2}\}$ where $\mathbf{u_1} = (2\,,\, 0)$ and $\mathbf{u_2} = (0\,,\, 4)$.

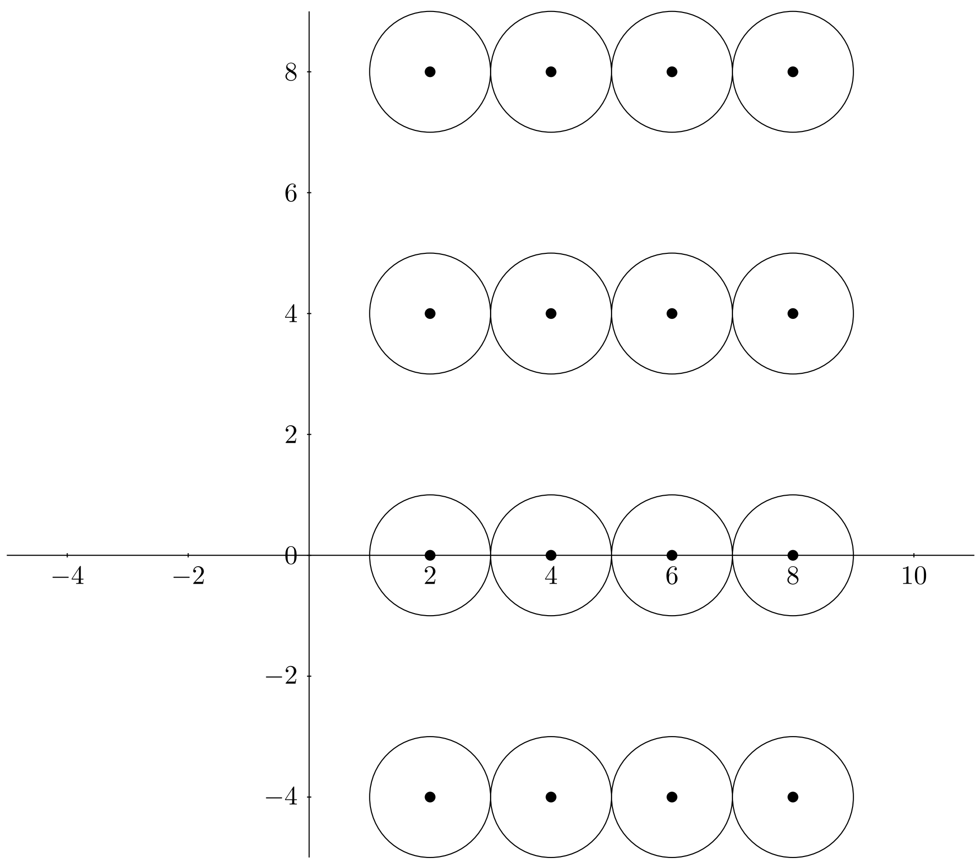

Figure 3. *Sphere Packing Given by* Λ

It is obvious that Λ is **not** a well rounded lattice because from the definition of well roundedness and successive minima, $\lambda_1 = 2 \neq 4 = \lambda_2$. Then let us calculate the packing density of Λ. From the definition of density function we have,

$$\Delta(\Lambda) = \frac{\pi r(\Lambda)^2}{\det(\Lambda)}.$$

So let us get the packing radius $r(\Lambda)$. Since the minimum norm in lattice Λ is 2, by Definition 2.1.5 we have

$$r(\Lambda) = 1.$$

Then, by Definition 2.1.4, we can write out the basis matrix of Λ,

$$M = \begin{bmatrix} 2 & 0 \\ 0 & 4 \end{bmatrix}.$$

Again by Definition 2.1.4, we have,

$$\det(\Lambda) = |\det(M)| = 8.$$

Therefore, we can calculate the density of packing given by lattice Λ

$$\Delta(\Lambda) = \frac{\pi r(\Lambda)^2}{\det(\Lambda)} = \frac{\pi}{8} \approx 0.3926991...$$

The maximum density we can get in $\mathbb{R}^2$ is about 0.907 which means the packing density given by lattice Λ is way under the maximum possible value. So, now let us try to improve the density of this packing by doing some proper adjustments to lattice Λ.

As we've mentioned before, well rounded lattices give better densities. So let us rescale the basis of Λ and make it to be well rounded. We can shrink vector $\mathbf{u_2}$ to $(0\,,\,2)$. Then, the basis for Λ becomes $\{(2\,,\,0), (0\,,\,2)\}$. By definition of well roundedness and successive minima, we now find that $\lambda_1 = 2 = \lambda_2$. So Λ is well rounded lattice now. See graph below.

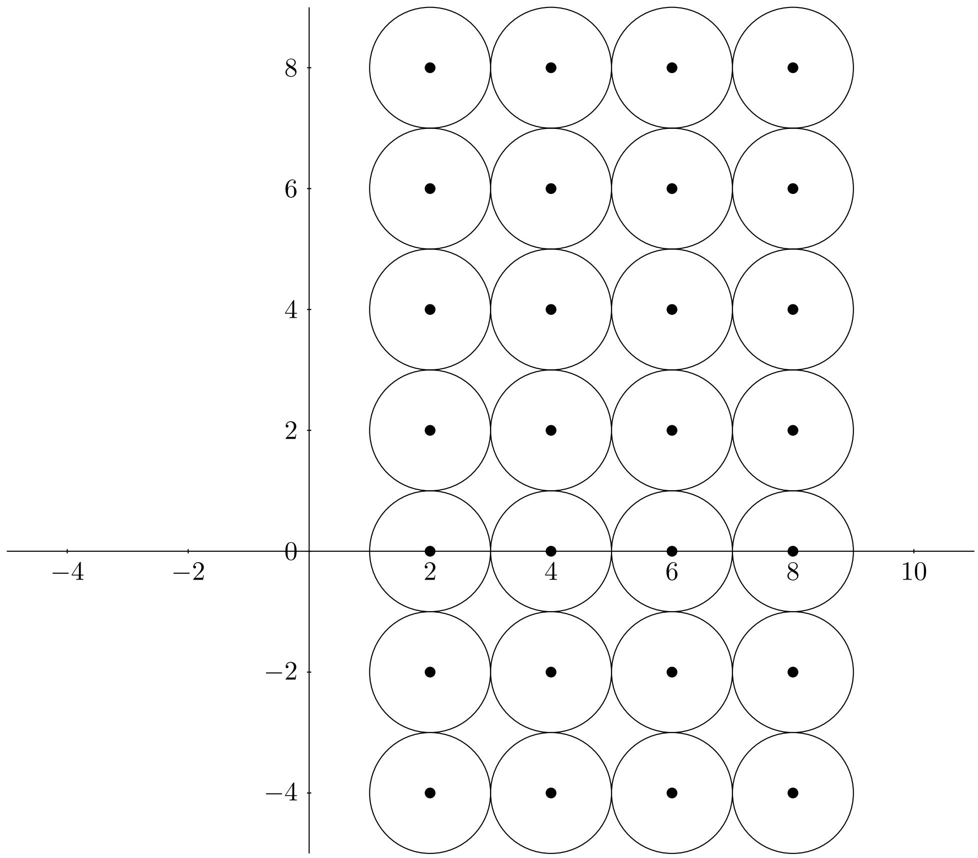

Figure 4. *After Rescaling*

Then, we calculate its packing radius and determinant. We have

$$r(\Lambda) = \frac{1}{2} \times 2 = 1,$$

$$M = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}.$$

So we have that

$$\det(\Lambda) = |\det(M)| = 4.$$

Therefore, we can get the new density

$$\Delta(\Lambda) = \frac{\pi}{4} \approx 0.7853982...$$

As we can see, rescaling the basis for Λ doubled the original density. However, there is still some room for the density to improve. Unfortunately, further rescaling cannot make progress any more. That means, although rescaling the original lattice to a well rounded one improves the density, the density is not maximized. Thus, we have to do something other than rescaling. That is to change the **angle** between the linearly independent vectors in the basis.

Let us recap a little bit. We have a lattice $\Lambda \subset \mathbb{R}^2$, and the basis is $\{\mathbf{u_1}, \mathbf{u_2}\}$ where $\mathbf{u_1} = (2\,,\,0)$ and $\mathbf{u_2} = (0\,,\,2)$. Thus, the angle $\theta = \frac{\pi}{2}$. Now if we change the angle θ from $\frac{\pi}{2}$ to $\frac{\pi}{3}$ but keep the norms of $\mathbf{u_1}$, $\mathbf{u_2}$ untouched, we will end up with

$$\mathbf{u_1} = (2\,,\,0),$$

$$\mathbf{u_2} = (1\,,\,\sqrt{3}).$$

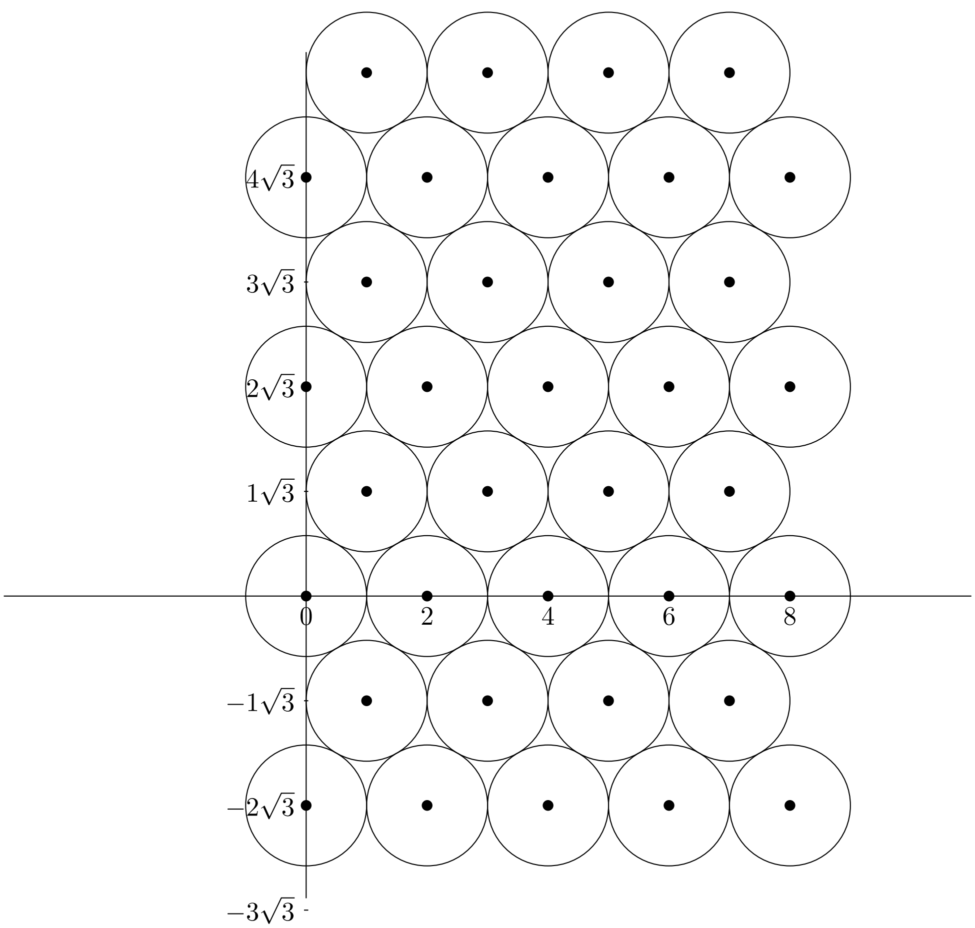

Figure 5. *After Changing the Angle*

Then, we calculate the packing radius and determinant. Obviously,

$$r(\Lambda) = 1,$$

$$M = \begin{bmatrix} 2 & 1 \\ 0 & \sqrt{3} \end{bmatrix}$$

$$\det(\Lambda) = |\det(M)| = 2\sqrt{3}.$$

Therefore, we have that

$$\Delta(\Lambda) = \frac{\pi}{2\sqrt{3}} \approx 0.906899...$$

We've reached the maximum possible density for all lattice packings in $\mathbb{R}^2$! We will prove this fact which is a theorem due to Thue. As you can see, the packing density depends on not only the norms of vectors in the basis, but the angles between the basis vectors as well. The final version of Λ we ended up with is actually the well known hexagonal lattice, denoted by Λ_h. In the following chapters, we will prove that a lattice packing yields the maximum possible density in $\mathbb{R}^2$ if and only if it is similar to hexagonal lattice Λ_h.

2.3 Three-Step Strategy for Proving Theorem 1

In this section, we will briefly introduce the strategy we will use in the proof of the main theorem. There are three key steps:

Step 1. We will show that each lattice in $\mathbb{R}^2$ has a minimal basis.

Step 2. We will show that the maximum density can be attained if and only if the lattice is well rounded.

Step 3. We will show that if a lattice attains the maximum density, it must similar to hexagonal lattice Λ_h.

Chapter 3 *Confirming the Three Steps*

In this chapter we will prove the main theorem. Before that, we have to prove some lemmas. Remember in the last chapter, we will follow the strategy we have mentioned.

3.1 Every Lattice Has a Minimal Basis

In Section 3.1, we will prove that each lattice in $\mathbb{R}^2$ with successive minima $\lambda_1 \leq \lambda_2$ has a minimum basis.

Lemma 3.1. *Let* $\mathbf{u_1}$ *and* $\mathbf{u_2}$ *be nonzero vectors in* $\mathbb{R}^2$ *and let the angle between* u_1, u_2 *be* θ. *Then if* $0 < \theta < \frac{\pi}{3}$, *we have that*

$$||\mathbf{u_1} - \mathbf{u_2}|| < \max\{||\mathbf{u_1}||, ||\mathbf{u_2}||\}.$$

Proof. Since $\theta < \frac{\pi}{3}$ and the cosine function is decreasing in the interval $[0, \frac{\pi}{3}]$, we have that

$$\cos\theta = \frac{\mathbf{u_1} \cdot \mathbf{u_2}}{||\mathbf{u_1}||||\mathbf{u_2}||} > \frac{1}{2}.$$

So, it follows that

$$2\mathbf{u_1} \cdot \mathbf{u_2} > ||\mathbf{u_1}||||\mathbf{u_2}||.$$

Then we have the identity,

$$||\mathbf{u_1}-\mathbf{u_2}||^2 = (\mathbf{u_1}-\mathbf{u_2})(\mathbf{u_1}-\mathbf{u_2}) = ||\mathbf{u_1}||^2+||\mathbf{u_2}||^2-2\mathbf{u_1}\cdot\mathbf{u_2} < ||\mathbf{u_1}||^2+||\mathbf{u_2}||^2-||\mathbf{u_1}||||\mathbf{u_2}||.$$

Without loss of generality, we may assume $||\mathbf{u_2}|| > ||\mathbf{u_1}||$. Then, it follows that

$$||\mathbf{u_1}||^2 < ||\mathbf{u_1}||||\mathbf{u_2}||,$$

$$||\mathbf{u_1}||^2 + ||\mathbf{u_2}||^2 - ||\mathbf{u_1}||||\mathbf{u_2}|| < ||\mathbf{u_1}||^2 + ||\mathbf{u_2}||^2 - ||\mathbf{u_1}||^2 = ||\mathbf{u_2}||^2 = \max\{||\mathbf{u_1}||, ||\mathbf{u_2}||\}^2.$$

□

Lemma 3.2. *Let $\Lambda \subset \mathbb{R}^2$ be a lattice of full rank with successive minima $\lambda_1 \leq \lambda_2$, and let u_1,u_2 be the vetors in Λ corresponding to λ_1,λ_2, respectively. Let $\theta \in [0, \frac{\pi}{2}]$ be angle between $\mathbf{u_1}$,$\mathbf{u_2}$. Then we have that*

$$\frac{\pi}{3} \leq \theta \leq \frac{\pi}{2}.$$

Proof. Since $\mathbf{u_1}$ and $\mathbf{u_2}$ corresponds to $\lambda_1 \leq \lambda_2$, we have

$$||\mathbf{u_1}|| = \lambda_1$$

$$||\mathbf{u_2}|| = \lambda_2.$$

Suppose $\theta < \frac{\pi}{3}$, then by Lemma 1, we have

$$||\mathbf{u_1} - \mathbf{u_2}|| < ||\mathbf{u_2}|| = \lambda_2.$$

By definition of successive minima, $\mathbf{u_1}$ and $\mathbf{u_2}$ are linearly independent. So $\mathbf{u_1} - \mathbf{u_2}$ is also linearly independent. This contradicts the definition of successive minima. □

Lemma 3.3. *let Λ be a lattice in $\mathbb{R}^2$ with successive minima $\lambda_1 \leq \lambda_2$ and let $\mathbf{u_1}$,$\mathbf{u_2}$ be vectors in Λ corresponding to λ_1,λ_2, respectively. Then $\mathbf{u_1}$, $\mathbf{u_2}$ form a basis for Λ.*

Proof. Let $\mathbf{v_1} \in \Lambda$ be a vector with shortest length extendable to a basis in Λ, and let $\mathbf{v_2} \in \Lambda$ be the shortest vector such that $\{\mathbf{v_1}, \mathbf{v_2}\}$ is a basis of Λ. By picking a proper choice of signs in $\pm\mathbf{v_1}$, $\pm\mathbf{v_2}$ we can ensure that the angle between them is no greater than $\frac{\pi}{2}$. Then, without loss of generality, we may assume that

$$0 < ||\mathbf{v_1}|| \leq ||\mathbf{v_2}||.$$

Then, for any vector $\mathbf{w} \in \Lambda$ with $||\mathbf{w}|| < \mathbf{v_2}$, $\{\mathbf{w}, \mathbf{v_1}\}$ will not be a basis for Λ. Since $\mathbf{u_1}$, $\mathbf{u_2} \in \Lambda$, there must exist a_1,a_2, b_1, $b_2 \in \mathbb{Z}$ such that

$$\begin{bmatrix}\mathbf{u_1} & \mathbf{u_2}\end{bmatrix} = \begin{bmatrix}\mathbf{v_1} & \mathbf{v_2}\end{bmatrix}\begin{bmatrix}a_1 & b_1 \\ a_2 & b_2\end{bmatrix} \tag{3.1}$$

where $\mathbf{u_1}$, $\mathbf{u_2}$, $\mathbf{v_1}$, $\mathbf{v_2}$ are vectors in $\mathbb{R}^2$. Then, let θ_u be the angle between $\mathbf{u_1}$, $\mathbf{u_2}$ and θ_v be the angle between $\mathbf{v_1}$, $\mathbf{v_2}$. Then, by Lemma 2, we know $\theta_u \in [\frac{\pi}{3}, \frac{\pi}{2}]$. Then suppose $\theta_v < \frac{\pi}{3}$, by Lemma 1 we have that

$$||v_1 - v_2|| < ||v_2||.$$

However, v_1, $v_1 - v_2$ are basis for Λ which contradicts to our choice of v_2. Thus, we conclude that $\theta_v \in [\frac{\pi}{3}, \frac{\pi}{2}]$ Then, let

$$D = \left|\det\begin{bmatrix}a_1 & b_1 \\ a_2 & b_2\end{bmatrix}\right|.$$

Then take determinant to both sides of (3.1), we have

$$||\mathbf{u_1}||||\mathbf{u_2}||\sin\theta_u = D\,||\mathbf{v_1}||||\mathbf{v_2}||\sin\theta_v.$$

By definition of successive minima, $||\mathbf{u_1}||||\mathbf{u_2}|| \leq ||\mathbf{v_1}||||\mathbf{v_2}||$. Moreover, since both θ_u and $\theta_v \in [\frac{\pi}{3}, \frac{\pi}{2}]$, in order to get the upper bound of D, we have to let $\theta_u = \frac{\pi}{2}$ and $\theta_v = \frac{\pi}{3}$ Then we have

$$D = \frac{||\mathbf{u_1}||||\mathbf{u_2}||\sin\theta_u}{||\mathbf{v_1}||||\mathbf{v_2}||\sin\theta_v} \leq \frac{2}{\sqrt{3}} \approx 1.1547...$$

Since $D \in \mathbb{Z}^+$, we have $D = 1$. From (3.1), we can easily conclude that

$$\begin{bmatrix} \mathbf{u_1} & \mathbf{u_2} \end{bmatrix} \begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix}^{-1} = \begin{bmatrix} \mathbf{v_1} & \mathbf{v_2} \end{bmatrix}.$$

Since $D = 1$, it follows that

$$\begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix}^{-1} = \begin{bmatrix} b_2 & -b_1 \\ -a_2 & a_1 \end{bmatrix}.$$

Thus, all entries in the matrix are integers. So vectors $\mathbf{v_1}$, $\mathbf{v_2}$ can be represented by vectors $\mathbf{u_1}$ and $\mathbf{u_2}$ and therefore $\{\mathbf{u_1}, \mathbf{u_2}\}$ is a basis for Λ.

□

3.2 Well Rounded Lattices

In Section 3.2, we will prove that the problem of maximizing density function for all lattices in $\mathbb{R}^2$ can be reduced to maximizing density function for the set of well rounded lattices. In other words, we will show that the density function is maximized if and only if the lattice is well rounded.

Lemma 3.4. *Let Λ and Ω be lattices of full rank in $\mathbb{R}^2$ with successive minima $\lambda_1(\Lambda)$,$\lambda_2(\Lambda)$ and $\lambda_1(\Omega)$,$\lambda_2(\Omega)$ respectively. Let $\mathbf{u_1}$, $\mathbf{u_2}$, and $\mathbf{v_1}$, $\mathbf{v_2}$ be vectors in Λ and Ω, respectively, corresponding to successive minima. Suppose that $\mathbf{u_1} = \mathbf{v_1}$, and angles between the vectors $\mathbf{u_1}, \mathbf{u_2}$, and $\mathbf{v_1}, \mathbf{v_2}$ are equal. Let the common angle be θ. If*

$$\lambda_1(\Lambda) = \lambda_2(\Lambda),$$

then we have that

$$\Delta(\Lambda) \geq \Delta(\Omega).$$

Proof. By Lemma 3, $\mathbf{u_1}$,$\mathbf{u_2}$ and $\mathbf{v_1}$,$\mathbf{v_2}$ are minimal basises for Λ and Ω, respective. We have

$$||\mathbf{u_1}|| = \lambda_1(\Lambda),$$

$$||\mathbf{u_2}|| = \lambda_2(\Lambda)$$

and it follows that

$$||\mathbf{v_1}|| = \lambda_1(\Omega),$$

$$||\mathbf{v_2}|| = \lambda_2(\Omega).$$

From what we are given and definition of successive minima, we know that

$$\mathbf{u_1} = \mathbf{v_1}$$

and so it follows that

$$\lambda_1(\Lambda) = \lambda_2(\Lambda),$$

$$\lambda_1(\Omega) \leq \lambda_2(\Omega).$$

Then we find that

$$\lambda_1(\Lambda) = \lambda_2(\Lambda) = ||\mathbf{u_1}|| = ||\mathbf{u_2}|| = ||\mathbf{v_1}|| = \lambda_1(\Omega) \leq \lambda_2(\Omega) = ||\mathbf{v_2}||.$$

By the definition of the density function, we have

$$\begin{aligned} \Delta(\Lambda) &= \frac{\pi\lambda_1(\Lambda)^2}{4\det(\Lambda)} = \frac{\pi\lambda_1(\Lambda)^2}{4\,||\mathbf{u_1}||||\mathbf{u_2}||\sin\theta} = \frac{\pi}{4\sin\theta} \\ &\geq \frac{\pi\lambda_1(\Omega)^2}{4\,||\mathbf{v_1}||||\mathbf{v_2}||\sin\theta} = \frac{\pi\lambda_1(\Omega)^2}{4\det(\Omega)} = \Delta(\Omega). \end{aligned} \tag{3.2}$$

□

Lemma 3.5. *Let $\Lambda \subset \mathbb{R}^2$ be a lattice of full rank, and let $\mathbf{u_1}$, $\mathbf{u_2}$ be a basis for Λ such that*

$$||\mathbf{u_1}|| = ||\mathbf{u_2}||$$

and the angle θ between these vectors lies in the interval $[\frac{\pi}{3}, \frac{\pi}{2}]$. $\{\mathbf{u_1}\,,\, \mathbf{u_2}\}$ is a minimal basis for Λ. In particular, this implies that Λ is well rounded.

Proof. Let $\mathbf{w} \in \Lambda$, then $\exists\, a, b \in \mathbb{Z}$ such that $\mathbf{w} = a\,\mathbf{u_1} + b\,\mathbf{u_2}$. Then we have that

$$||\mathbf{w}||^2 = a^2\,||\mathbf{u_1}||^2 + b^2\,||\mathbf{u_2}||^2 + 2ab\,\mathbf{u_1}\cdot\mathbf{u_2} = (a^2 + b^2 + 2ab\,\cos\theta)||\mathbf{u_1}||^2.$$

If $ab \geq 0$, then $||\mathbf{w}||^2 \geq ||\mathbf{u_1}||^2$.
If $ab < 0$, then since $\theta \in [\frac{\pi}{3}, \frac{\pi}{2}]$, $\cos\theta \in [0, \frac{1}{2}]$.
And since $ab < 0$, we have

$$a^2 + b^2 + 2ab\,\cos\theta \geq a^2 + b^2 - 2 \times \frac{1}{2}|ab| = a^2 + b^2 - |ab|.$$

So we find that

$$||\mathbf{w}||^2 \geq (a^2 + b^2 - |ab|)||\mathbf{u_1}||^2 \geq ||\mathbf{u_1}||^2.$$

Therefore, for all $\mathbf{w} \in \Lambda$, $||\mathbf{w}|| \geq ||\mathbf{u_1}|| = ||\mathbf{u_2}||$. We can conclude that $\mathbf{u_1}$ and $\mathbf{u_2}$ are two nonzero vectors with shortest norms in Λ. Therefore, they correspond to the successive minima for Λ and by Lemma 3, $\{\mathbf{u_1}, \mathbf{u_1}\}$ form a minimal basis for Λ. Moreover, since $\lambda_1 = ||\mathbf{u_1}|| = ||\mathbf{u_1}|| = \lambda_2$, Λ is well rounded. □

Lemma 3.6. *Let Λ be a lattice in $\mathbb{R}^2$ with successive minima λ_1,λ_2 and corresponding basis vectors, $\mathbf{u_1}$,$\mathbf{u_2}$, respectively. Then the lattice*

$$\Lambda_{WR} = \begin{bmatrix} \mathbf{u_1} & \frac{\lambda_1}{\lambda_2}\mathbf{u_2} \end{bmatrix} \mathbb{Z}^2$$

is well rounded with successive minima equal to λ_1.

Proof. Since $\mathbf{u_1}$, $\mathbf{u_2}$ correspond to λ_1 and λ_2 we have that

$$||\mathbf{u_1}|| = \lambda_1,$$

$$||\mathbf{u_2}|| = \lambda_2.$$

Thus it follows that

$$||\frac{\lambda_1}{\lambda_2}\mathbf{u_2}|| = \frac{\lambda_1}{\lambda_2}||\mathbf{u_2}|| = \frac{\lambda_1}{\lambda_2} \times \lambda_2 = \lambda_1 = ||\mathbf{u_1}||.$$

Moreover, since the angle between $\mathbf{u_1}$, $\mathbf{u_2}$ is the same as the angle between $\mathbf{u_1}$, $\frac{\lambda_1}{\lambda_2}\mathbf{u_2}$. By Lemma 2, the angle between $\mathbf{u_1}$, $\frac{\lambda_1}{\lambda_2}\mathbf{u_2}$ is in $[\frac{\pi}{3}, \frac{\pi}{2}]$. Then by lemma 5, Λ_{WR} is well rounded and the successive minima is λ_1. □

Remark. Lemma 4,5,6 implies that

$$\Delta(\Lambda_{WR}) \geq \Delta(\Lambda). \tag{3.3}$$

for any lattice $\Lambda \subset \mathbb{R}^2$. Moreover, from definition of density function, we know that the equality of (3.2) holds if and only if $\Lambda = \Lambda_{WR}$. Therefore, the maximum possible packing density among all lattices in $\mathbb{R}^2$ can be achieved if and only if the lattice is well rounded. Then, our problem can be reduced to maximize density for all well rounded lattices.

3.3 Well Rounded Similarity Class

In Section 3.3, we will prove the sufficient condition for two well rounded lattices to be similar in $\mathbb{R}^2$ and show the density function for a similarity class is a constant.

Lemma 3.7. *Let Λ be a well rounded lattice in $\mathbb{R}^2$. A lattice $\Omega \subset \mathbb{R}^2$ is similar to Λ if and only if Ω is also well rounded and $\theta(\Lambda) = \theta(\Omega)$.*

Proof. First, we will prove the statement in $\Rightarrow$ direction. Let $\Omega \subset \mathbb{R}^2$ be a lattice. Suppose that Ω is similar to Λ. Then, by what we have proven in Section 3.2, there exists a minimal basis $\{\mathbf{u_1}, \mathbf{u_2}\}$ for Λ. Since Λ is similar to Ω, there exists a constant $\alpha \in \mathbb{R}$ and a real orthogonal 2×2 matrix M such that

$$\Omega = \alpha M \Lambda.$$

Then, let $\{\mathbf{v_1}, \mathbf{v_2}\}$ be the basis for Ω. So we have that

$$\begin{bmatrix} \mathbf{v_1} & \mathbf{v_2} \end{bmatrix} = \alpha M \begin{bmatrix} \mathbf{u_1} & \mathbf{u_2} \end{bmatrix}.$$

Since Λ is well rounded, $||\mathbf{u_1}|| = ||\mathbf{u_2}||$. Since the orthogonal matrix M preserves the lengths of vectors $\mathbf{u_1}$, $\mathbf{u_2}$, and since the rescaling factor α acts on both of $\mathbf{u_1}$ and $\mathbf{u_2}$ at the same time, we can conclude that

$$||\mathbf{v_1}|| = ||\mathbf{v_2}||.$$

From (3.2), for any well-rounded lattice $\Lambda \subset \mathbb{R}^2$, we have that

$$\Delta(\Lambda) = \frac{\pi}{4\sin\theta}.$$

Then, we have

$$\sin\theta = \frac{\pi}{4\Delta(\Lambda)}.$$

Since there is only one maximum value of $\Delta(\Lambda)$ and by Lemma 3.2, $\theta \in [\frac{\pi}{3}, \frac{\pi}{2}]$, we conclude that $\theta(\Lambda)$ is independent of the choice of lattices. Moreover, since orthogonal matrices preserve angles, we have

$$\theta(\Lambda) = \theta(\Omega).$$

Therefore, we have that

$$||\mathbf{v_1}|| = ||\mathbf{v_2}||.$$

and

$$\theta(\Lambda) = \theta(\Omega) \in [\frac{\pi}{3}, \frac{\pi}{2}].$$

By Lemma 3.5, we conlucde that $\{\mathbf{v_1}, \mathbf{v_2}\}$ is a minimal basis for Ω and Ω is a well-rounded lattice and $\theta(\Lambda) = \theta(\Omega)$.

Then we will prove the statement in $\Leftarrow$ direction. Suppose Ω is well rounded and $\theta(\Omega) = \theta(\Lambda)$. Then since both Λ and Ω are well rounded, there are unique successive minimas for both of these two lattices. So let $\lambda(\Lambda)$ and $\lambda(\Omega)$ be successive minimas of Λ, Ω respectively. Then by Lemma 3.3, both Λ, Ω have minimal basises. So let $\{\mathbf{u_1}, \mathbf{u_2}\}$ and $\{\mathbf{v_1}, \mathbf{v_2}\}$ be minimal basises for Λ and Ω, respectively. Since we have that

$$||\mathbf{u_1}|| = ||\mathbf{u_2}|| = \lambda(\Lambda)$$

$$||\mathbf{v_1}|| = ||\mathbf{v_2}|| = \lambda(\Omega).$$

It follows that

$$\mathbf{v_1} = \frac{\lambda(\Omega)}{\lambda(\Lambda)}\mathbf{u_1}$$

$$\mathbf{v_2} = \frac{\lambda(\Omega)}{\lambda(\Lambda)}\mathbf{u_2}.$$

Moreover, since orthogonal matrices preserve angles and from assumption, $\theta(\Omega) = \theta(\Lambda)$, there exists a 2×2 real orthogonal matrix M such that

$$\begin{bmatrix} \mathbf{v_1} & \mathbf{v_2} \end{bmatrix} = \frac{\lambda(\Omega)}{\lambda(\Lambda)} M \begin{bmatrix} \mathbf{u_1} & \mathbf{u_2} \end{bmatrix}.$$

Therefore, by definition of similarity, it follows that Λ is similar to Ω.

$\square$

Lemma 3.8. *Similarity is an equivalence relation.*

Proof. Let $\Lambda, \Omega, \Gamma \subset \mathbb{R}^2$ be three different lattices. To prove an equivalence relation, we must prove reflexivity, symmetry and transitivity. Here we establish these conditions one by one.
(1). Reflexivity: It's obvious that let I denotes 2×2 identity matrix. Then $\Lambda = 1 \times I \times \Lambda$. So reflexivity holds.
(2). Symmetry: Suppose Λ is similar to Ω. Then there exists a nonzero constant α and a 2×2 orthogonal real matrix M such that

$$\Lambda = \alpha M \Omega.$$

Since orthogonal matrices are invertible, we have

$$\Omega = \frac{1}{\alpha} M^{-1} \Lambda.$$

So Ω is similar to Λ and symmetry holds.
(3). Transitivity: Suppose Λ is similar to Ω and Ω is similar to Γ. Then there exist nonzero constants α, β and 2×2 orthogonal real matrices M and N such that

$$\Lambda = \alpha M \Omega$$

$$\Omega = \beta N \Gamma.$$

Then since $\beta \neq 0$ and $\det(N) \neq 0$, we have that

$$\Gamma = \frac{1}{\beta} N^{-1} \Omega.$$

So it follows that

$$\Lambda = \frac{\alpha}{\beta} M N^{-1} \Gamma.$$

So Λ is similar to Γ and transitivity holds. Therefore, similarity is an equivalence relation.

□

Lemma 3.9. *The density function Δ for a well rounded similarity class is a constant.*

Proof. Let Λ and $\Omega \subset \mathbb{R}^2$ be two different lattices but belong to the same well rounded similarity class. Namely, both of these lattices are well rounded and Λ is similar to Ω. Therefore, we have

$$||\mathbf{u_1}|| = ||\mathbf{u_2}|| = \lambda(\Lambda)$$

$$||\mathbf{v_1}|| = ||\mathbf{v_2}|| = \lambda(\Omega).$$

By definition of density function, we have that

$$\Delta(\Lambda) = \frac{\lambda(\Lambda)^2 \pi}{4 \det(\Lambda)} = \frac{\lambda(\Lambda)^2 \pi}{4||\mathbf{u_1}||||\mathbf{u_2}|| \sin \theta(\Lambda)} = \frac{\pi}{4 \sin \theta(\Lambda)}.$$

Similarly, we have that

$$\Delta(\Omega) = \frac{\lambda(\Omega)^2 \pi}{4 \det(\Omega)} = \frac{\lambda(\Omega)^2 \pi}{4||\mathbf{v_1}||||\mathbf{v_2}|| \sin \theta(\Omega)} = \frac{\pi}{4 \sin \theta(\Omega)}.$$

By Lemma 3.7, since Λ is similar to Ω, we know $\theta(\Lambda) = \theta(\Omega)$, we have $\Delta(\Lambda) = \Delta(\Omega)$

□

3.4 Proof of Theorem 1

Here we recall the main result of this thesis.

Theorem 1. *Let Λ be a full rank lattice in $\mathbb{R}^2$, then*

$$\Delta(\Lambda) \leq \Delta(\Lambda_h) = \frac{\pi}{\sqrt{12}}.$$

Equality holds if and only if Λ is similar to Λ_h.

Proof. From what we have discussed before, the maximum value of density is achieved by well rounded lattices. So let $\Lambda \subset \mathbb{R}^2$ be a well rounded lattice. Then by definition of density function,

$$\Delta(\Lambda) = \frac{\lambda(\Lambda)^2\pi}{4\det(\Lambda)} = \frac{\pi}{4\sin\theta(\Lambda)}.$$

So we need to minimize the value of $\sin\theta(\Lambda)$ in order to maximize $\Delta(\Lambda)$. By Lemma 3.2, we know $\theta \in [\frac{\pi}{3}, \frac{\pi}{2}]$. So $\sin\theta(\Lambda) \in [\frac{\sqrt{3}}{2}, 1]$. Obviously, $\sin\theta(\Lambda)$ is minimized when $\theta(\Lambda) = \frac{\pi}{3}$. So consider the hexagonal lattice Λ_h.

$$\Lambda_h = \begin{bmatrix} 1 & \frac{1}{2} \\ 0 & \frac{\sqrt{3}}{2} \end{bmatrix} \mathbb{Z}^2.$$

Let $\mathbf{u_1} = \begin{bmatrix} 1 & 0 \end{bmatrix}$ and $\mathbf{u_2} = \begin{bmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix}$. And by definition of successive minima, we have that

$$||\mathbf{u_1}|| = ||\mathbf{u_2}|| = \lambda(\Lambda) = 1$$

and

$$\theta(\Lambda_h) = \frac{\pi}{3}.$$

Therefore, hexagonal lattice Λ_h is well rounded and

$$\Delta(\Lambda_h) = \frac{\pi}{4\sin\theta(\Lambda_h)} = \frac{\pi}{4 \times \frac{\sqrt{3}}{2}} = \frac{\pi}{2\sqrt{3}}.$$

From the previous arugments, we know density function Δ is maximized if and only if $\theta(\Lambda) = \frac{\pi}{3}$. Thus, we have shown that

$$\Delta(\Lambda) \leq \Delta(\Lambda_h) = \frac{\pi}{2\sqrt{3}}.$$

Moreover, suppose $\Delta(\Lambda) = \Delta(\Lambda_h)$, by previous arguments, Λ must be well rounded and

$$\Delta(\Lambda) = \frac{\pi}{4\sin\theta(\Lambda)} = \frac{\pi}{4\sin\frac{\pi}{3}} = \Delta(\Lambda_h).$$

By Lemma 3.2, $\theta(\Lambda) \in [\frac{\pi}{3}, \frac{\pi}{2}]$, so $\theta(\Lambda) = \theta(\Lambda_h)$. Therefore, by Lemma 3.7, Λ is similar to Λ_h. Thus, if a lattice maximizes the density function, it is similar to the hexagonal lattice Λ_h.

$\square$

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