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On Saturation Spectrum

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On Saturation Spectrum

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An abstract of
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Abstract

On Saturation Spectrum

By Jessica M. Fuller

Given a graph H, we say a graph G is *H-saturated* if G does not contain H as a subgraph and the addition of any edge $e \notin E(G)$ results in H as a subgraph. The question of the minimum number of edges of an H-saturated graph on n vertices, known as the *saturation number*, and the question of the maximum number of edges possible of an H-saturated graph, known as the Turán number, have been addressed for many different types of graphs. We are interested in the existence of H-saturated graphs for each edge count between the saturation number and the Turán number.

In this thesis, we determine the saturation spectrum of $(K_t - e)$-saturated graphs and F_t-saturated graphs. Let $K_t - e$ be the complete graph minus one edge. We prove that $(K_4 - e)$-saturated graphs do not exist for small values of $|E(G)|$ and construct $(K_4 - e)$-saturated graphs with $|E(G)|$ in the interval $\left[2n-4, \left\lfloor \frac{n}{2} \right\rfloor \left\lceil \frac{n}{2} \right\rceil - \left\lfloor \frac{n}{2} \right\rfloor + 1\right]$. We then extend the constructed $(K_4 - e)$-saturated graphs to create $(K_t - e)$-saturated graphs.

Let F_t be the graph consisting of t edge-disjoint triangles that intersect at a single vertex v. We prove that F_2-saturated graphs do not exist for small edge counts and construct a collection of F_2-saturated graphs with edge counts in the interval $[2n-4, \frac{n^2}{4} - \lfloor \frac{n}{2} \rfloor + 2]$ or the size of a complete bipartite graph with an additional edge. We also establish two general constructions that yield F_t-saturated graphs with edge counts in $\left[\left(\frac{3t}{2} - 1\right) n - \left\lfloor \frac{3t}{2} \right\rfloor \left\lceil \frac{3t}{2} \right\rceil + t^2 - 4, \left\lceil \frac{n}{2} \right\rceil \left\lfloor \frac{n}{2} \right\rfloor - \left\lfloor \frac{n}{2} \right\rfloor + 2 + (t-1)^2\right]$.

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Chapter 1

Introduction

1.1 Definitions

For this thesis we assume a basic knowledge of Graph Theory; for terms and concepts not defined see [3]. Also, we only consider simple graphs. For a graph G the graph H is a *subgraph*, denoted $H \subseteq G$, if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. The set of neighbors in G of a vertex v is called the *neighborhood of* v and is denoted $N(v)$. The degree of v is $|N(v)|$, denoted $\deg(v)$, and the *minimum degree* of a graph G, denoted $\delta(G)$, is the smallest degree among the vertices of G. For two vertices u and v in a graph G, a $u-v$ *path* P is a sequence of vertices in G beginning with u and ending at v such that consecutive vertices in P are adjacent in G and no vertex is repeated. A graph G is *connected* if there is a $u-v$ path for every pair of vertices $u, v \in V(G)$. The distance from a vertex u to a vertex v in a connected graph G is the minimum of the lengths of all $u-v$ paths in G. The *diameter* of G, denoted $\mathrm{diam}(G)$, is the greatest distance between any pair of vertices. If $G - v$ is not connected for some vertex v of G, then v is a *cut vertex* of G. The *union* of two graphs G and H, denoted $G \cup H$ is the graph having vertex set $V(G) \cup V(H)$ and edge set $E(G) \cup E(H)$. The *join* of two graphs G and H, denoted $G + H$, is the graph having vertex set $V(G) \cup V(H)$

and edge set $E(G) \cup E(H) \cup \{uw : u \in V(G), w \in V(H)\}$.

The graph in which every two distinct vertices are adjacent is the *complete graph* of order n, denoted K_n, having $\binom{n}{2}$ edges. The path P_n is a graph of order n and size $n-1$, for integer $n \geq 1$, whose vertices can be labeled by $v_1, v_2, ..., v_n$ and whose edges are $v_i v_{i+1}$ for $i = 1, 2, ..., n-1$. The cycle C_n is a graph of order n and size n, for integer $n \geq 3$, whose vertices can be labeled by $v_1, v_2, ..., v_n$ and whose edges are $v_1 v_n$ and $v_i v_{i+1}$ for $i = 1, 2, ..., n-1$. The cycle C_3 is also called a *triangle*. A graph G is *bipartite* if $V(G)$ can be partitioned into two sets U and W (called *partite sets*) so that every edge of G joins a vertex of U and a vertex of W. If $V(G)$ can be partitioned into two sets U and W and $uw \in E(G)$ if and only if $u \in U$ and $w \in W$, then G is called a *complete bipartite graph*. For $|U| = s$ and $|W| = t$, this complete bipartite graph has order $s+t$, size st and is denoted by $K_{s,t}$. The complete bipartite graph $K_{1,t}$ is called a *star*. A graph G is *k-partite*, for integer $k \geq 1$, if $V(G)$ can be partitioned into k subsets $V_1, V_2, ..., V_k$ such that every edge joins vertices in two different partite sets. A *complete k-partite graph* is a k-partite graph such that two vertices are adjacent in G if and only if the vertices belong to two different partite sets. If $|V_i| = n_i$, for $1 \leq i \leq k$, then G is denoted $K_{n_1, n_2, ..., n_k}$.

1.2 Extremal Numbers

A graph G is *H-saturated* if, given a graph H, G does not contain a copy of H but the addition of any edge $e \notin E(G)$ creates at least one copy of H within G. The maximum number of edges possible in a graph G on n vertices that is H-saturated is known as the *Turán number* and is denoted $ex(n, H)$. For a family of graphs $\mathscr{F}$, $ex(n, \mathscr{F})$ is the maximum number of edges in an H-saturated graph of order n for any $H \in \mathscr{F}$. The set of all H-saturated graphs of order n having size $ex(n, H)$ is denoted $EX(n, H)$.

Let $\mathscr{F}$ be a family of graphs. Then $ex(n, \mathscr{F})$ satisfies the following monotonicity properties:

1. $ex(n, \mathscr{F}) \leq ex(n+1, \mathscr{F})$,
2. If $\mathscr{F}_1 \subset \mathscr{F}$, then $ex(n, \mathscr{F}_1) \geq ex(n, \mathscr{F})$,
3. If $H \subseteq G$, then $ex(n, H) \leq ex(n, G)$.

In 1907, Mantel proved one of the first results on extremal numbers, in particular he proved that $ex(n, K_3) = \left\lfloor \frac{n^2}{4} \right\rfloor$ [10]. Mantel's result was generalized to all complete graphs in 1941 by Pál Turán [11].

Theorem 1.2.1. *There is a unique graph on n vertices with the maximum possible number of edges that is K_{p+1}-saturated, namely $T_{n,p}$.*

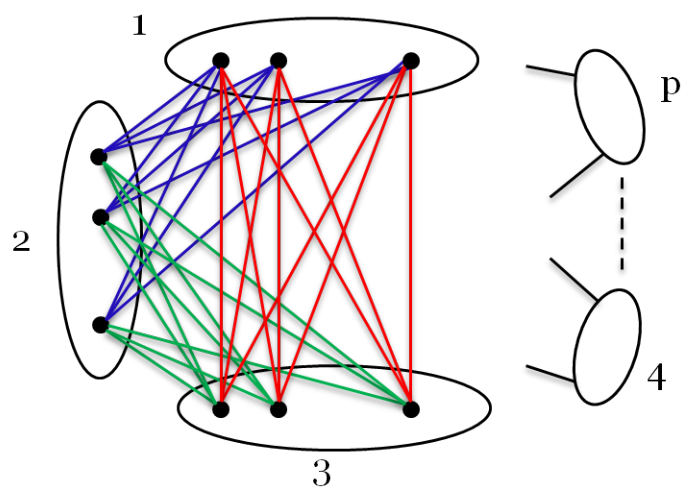

Figure 1.1: The Turán Graph $T_{n,p}$

The graph $T_{n,p}$ is a complete p-partite graph with each partite set having almost equal order. For example, the K_4-saturated graph on 10 vertices is $T_{10,3} = K_{4,3,3}$.

1.3 Saturation Numbers

The minimum number of edges of an H-saturated graph on n vertices is known as the *saturation number* and is denoted $sat(n, H)$. For a family of graphs $\mathscr{F}$, $sat(n, \mathscr{F})$

is the minimum number of edges in an $\mathscr{F}$-saturated graph G of order n, that is G is H-saturated for every $H \in \mathscr{F}$. The set of all H-saturated graphs of order n having size $sat(n, H)$ is denoted $SAT(n, H)$. In 1964, Erdös, Hajnal and Moon determined the saturation number for the complete graph [6]:

Theorem 1.3.1. *For $t \geq 3$, $sat(n, K_t) = (t-2)(n-1) - \binom{t-2}{2}$.*

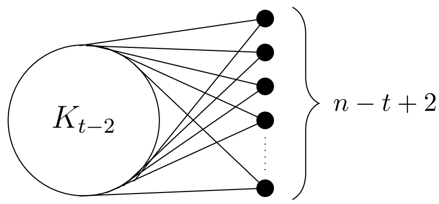

Figure 1.2: $K_{t-2} + \overline{K}_{n-t+2}$

The unique K_t-saturated graph with $sat(n, K_t)$ edges is $K_{t-2} + \overline{K}_{n-t+2}$ shown in Figure 2. In fact, $SAT(n, K_t)$ contains only $K_{t-2} + \overline{K}_{n-t+2}$ [6]. In particular, $sat(n, K_3) = n - 1$.

In 1986, Kàszonyi and Tuza found the saturation numbers for paths, stars and matchings [9].

Theorem 1.3.2.

1. *For $n \geq 3, sat(n, P_3) = \left\lfloor \frac{n}{2} \right\rfloor$.*

2. *For $n \geq 4$,*

$$sat(n, P_4) = \begin{cases} \frac{n}{2} & if \quad n \text{ is even} \\ \frac{n+3}{2} & if \quad n \text{ is odd} \end{cases}.$$

3. *For $n \geq 5, sat(n, P_5) = \left\lceil \frac{5n-4}{6} \right\rceil$.*

4. *Let*

$$a_k = \begin{cases} 3 \cdot 2^{t-1} - 2 & if \quad k = 2t \\ 4 \cdot 2^{t-1} - 2 & if \quad k = 2t+1 \end{cases}.$$

If $n \geq a_k$ and $k \geq 6$, then $sat(n, P_k) = n - \left\lfloor \frac{n}{a_k} \right\rfloor$.

5. *let* $S_t = K_{1,t-1}$ *denote a star on* t *vertices. Then,*

$$sat(n, S_t) = \begin{cases} \binom{t-1}{2} + \binom{n-t+1}{2} & if \quad t \leq n \leq \frac{3t-3}{2} \\ \left\lceil \frac{(t-2)n}{2} - \frac{(t-1)^2}{8} \right\rceil & if \quad \frac{3t-3}{2} \leq n \end{cases}.$$

6. *For* $n \geq 3t-3$, $sat(n, tK_2) = 3t-3$.

In general, finding the saturation number of a graph is a difficult problem because it does not satisfy the nice monotonicity properties satisfied by the Turán number, so inductive arguments generally do not work. For example, $sat(n, H)$ is not monotone in general for n as Theorem 1.3.2(4) shows that when considering the graph P_4, $sat(2k-1, P_4) = k+1 > k = sat(2k, P_4)$. We can also see that, in general, the saturation number is not monotone for subgraphs by considering the graphs $K_{1,t}$ and $K_{1,t}+e$. Since $K_{1,n-1}$ is $(K_{1,t}+e)$-saturated, $sat(n, K_{1,t}+e) \leq n-1$ whereas $sat(n, K_{1,t})$ is strictly greater when n is large enough as determined in Theorem 1.3.2(5). This example also shows how the saturation number is not monotone in general for subfamilies if we consider $\mathscr{F} = \{K_{1,t}, K_{1,t}+e\}$ and $\mathscr{F}' = \{K_{1,t}+e\}$ as $sat(n, \mathscr{F}) = sat(n, K_{1,t}) > n-1$ whereas $sat(n, \mathscr{F}') \leq n-1$ as before.

Kàszonyi and Tuza also provided a general bound on the saturation number for every graph [9].

Theorem 1.3.3. *For every graph* H*, there exists a constant* c *such that*

$$sat(n, H) < cn.$$

More recently in 2008, G. Chen, R. Faudree and R. Gould determined the saturation numbers for the book graph B_p, the union of p triangles sharing one edge, and the generalized book $B_{b,p}$, the union of p copies of K_{b+1} sharing a common K_b [4].

Theorem 1.3.4. *For $n \geq p^3 + p$,*

$$sat(n, B_p) = \frac{1}{2}\left((p+1)(n-1) - \left\lceil \frac{p}{2} \right\rceil \left\lfloor \frac{p}{2} \right\rfloor + \theta(n,p)\right),$$

$$where\ \theta(n,p) = \begin{cases} 1 & if \quad p \equiv n - \frac{p}{2} \equiv 0 \pmod 2 \\ \\ 0 & if \quad otherwise \end{cases}.$$

Theorem 1.3.5. *For $n \geq 4(p+2b)^b$,*

$$sat(n, B_{b,p}) = \frac{1}{2}\left((p+2b-3)(n-b+1) - \left\lceil \frac{p}{2} \right\rceil \left\lfloor \frac{p}{2} \right\rfloor + \theta(n,p,b) + (b-1)(b-2)\right),$$

$$where\ \theta(n,p,b) = \begin{cases} 1 & if \quad p \equiv n - \frac{p}{2} - b \equiv 0 \pmod 2 \\ \\ 0 & if \quad otherwise \end{cases}.$$

In particular, we have that the saturation number for two triangles sharing an edge is $sat(n, B_2) = \frac{3(n-1)}{2}$ if $n \geq 10$ and the saturation number of two copies of K_{t-1} sharing a common K_{t-2} is $sat(n, B_{t-2,2}) = \frac{(2t-5)(n-t+3)}{2} + (t-3)(t-4)$ when n is large enough.

1.4 Known Results on Saturation Spectrum

For a graph H, the *saturation spectrum* of the family of H-saturated graphs on n vertices is the set of all possible sizes of an H-saturated graph. Since determining the saturation number and extremal number for many graphs is difficult, the saturation spectrum has thus far only been explored for a few interesting graphs with known saturation numbers, including $K_3, K_t,$ P_t and S_t. In 1995 Barefoot, Casey, Fisher, Fraghnaugh, and Harary were the first to prove a result when they determined the saturation spectrum for K_3-saturated graphs [2].

Theorem 1.4.1. *For $n \geq 5$, there exists a K_3-saturated graph of order n with m*

edges if and only if it is complete bipartite or

$$2n-5 \le m \le \left\lfloor \frac{(n-1)^2}{4} \right\rfloor + 1.$$

The result in Theorem 1.4.1 was generalized for all complete graphs in 2010 by Amin, Faudree, Gould and Sidorowicz [1].

Theorem 1.4.2. *For $n \ge 3t+5$, there is a K_t-saturated graph G of order n with m edges if and only if G is complete $(t-1)$-partite or*

$$(t-1)\left(n-\frac{t}{2}\right)-2 \le m \le \left\lfloor \frac{(t-2)n^2-2n+(t-2)}{2(t-1)} \right\rfloor + 1.$$

In 2012, Gould, Tang, Wei, and Zhang addressed the saturation spectrum of paths of lengths five and six [8].

Theorem 1.4.3. *Let $n \ge 5$ and $sat(n, P_5) \le m \le ex(n, P_5)$ be integers. There exists a P_5-saturated graph on n vertices and m edges if and only if $n \equiv 1, 2 \pmod 4$ or*

$$m \notin \begin{cases} \left\{\frac{3n-5}{2}\right\} & if \quad n \equiv 3 \pmod 4 \\ \left\{\frac{3n-6}{2}, \frac{3n-4}{2}, \frac{3n-2}{2}\right\} & if \quad n \equiv 0 \pmod 4 \end{cases}.$$

Theorem 1.4.4. *Let $n \ge 10$ and $sat(n, P_6) \le m \le ex(n, P_6)$ be integers. There exists a P_6-saturated graph on n vertices and m edges if and only if*

1. $(n, m) \notin \{(10, 10), (11, 11), (12, 12), (13, 13), (14, 14), (11, 14)\}$, *and*

2. $m \notin \begin{cases} \{2n-4, 2n-3, 2n-1\} & if \quad n \equiv 0 \pmod 5 \\ \{2n-4\} & if \quad n \equiv 2 \pmod 5 \\ \{2n-4\} & if \quad n \equiv 4 \pmod 5 \end{cases}.$

Finally, using [8] as a starting point, Faudree, Faudree, Gould, Jacobson and Thomas have recently submitted the following results on the saturation spectrum of stars and paths [7].

Theorem 1.4.5. *If $t \geq 3$ and $n \geq t+1$ there exists a $K_{1,t}$-saturated graph on n vertices and m edges for $m \in [sat(n, K_{1,t}), ex(n, K_{1,t})]$.*

Also, for a_k is defined as in Theorem 1.3.2, we have the following result for paths:

Theorem 1.4.6. *If $n = r(k-1) + a_k\left[\binom{k-1}{2} - (k-1)\right] + \beta$, where $0 \leq \beta < k-1$, then there exists a P_k-saturated graph on n vertices and m edges for*

$$m \in \left[sat(n, P_k), r\binom{k-1}{2} + a_k\left[\binom{k-1}{2} - (k-1)\right] + \beta - 1\right].$$

It is interesting to explore that saturation spectrum for any graph H, either constructing H-saturated graphs of size m with $m \in [sat(n, H), ex(n, H)]$ or determining if such a graph does not exist for some value of m. Complete graphs were interesting because they are so fundamental to many areas of Graph Theory, as are paths and stars. However, just as relevant are graphs comprised of subgraphs that share some element such as the nearly complete graphs $K_t - e$ and the graph comprised of multiple triangles sharing one vertex. As such, it is important to study the saturation spectrum of these graphs.

Chapter 2

On the Saturation Spectrum of $(K_t - e)$-saturated graphs

To determine the saturation spectrum of $K_t - e$ it is necessary to first determine the saturation spectrum of $(K_4 - e)$-saturated graphs, where $K_4 - e$ is the complete graph on four vertices with one edge removed. The graph $K_4 - e$ is isomorphic to the graph comprised of two triangles that share an edge, sometimes called a book. Further, $sat(n, K_4 - e) = \left\lfloor \frac{3(n-1)}{2} \right\rfloor$ [4]. The saturation number, $sat(n, K_4 - e)$, can be realized as the edge count of the graph on n vertices formed by $\frac{n-1}{2}$ triangles joined at a single vertex v when n is odd (Figure 1(a)) and $\frac{n-2}{2}$ triangles joined at the vertex v with an edge from v to the remaining vertex when n is even (Figure 2.1(b)). These graphs are $(K_4 - e)$-saturated as each vertex, except perhaps one, is a vertex of a triangle and an additional edge creates a second triangle with v, forming a copy of $(K_4 - e)$.

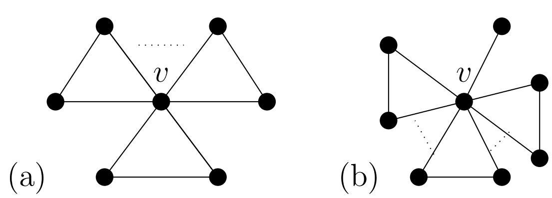

Figure 2.1: Saturation graphs for $K_4 - e$

The Turán number for an n vertex (K_4-e)-free graph G is $ex(n, K_4-e) = \lfloor \frac{n}{2} \rfloor \lceil \frac{n}{2} \rceil$ and can be realized by the complete bipartite graph $K_{\lfloor \frac{n}{2} \rfloor, \lceil \frac{n}{2} \rceil}$. The goal now is to construct graphs of size m that are $(K_4 - e)$-saturated with $\left\lfloor \frac{3(n-1)}{2} \right\rfloor \leq m \leq \left\lfloor \frac{n}{2} \right\rfloor \left\lceil \frac{n}{2} \right\rceil$ and to determine if such a graph exists for every possible value of m.

2.1 $(K_4 - e)$-saturated graphs

We begin begin our exploration of the saturation spectrum for $(K_4 - e)$-saturated graphs with a few useful lemmas. First, we see that a connected $(K_4 - e)$-saturated graph with a cut vertex is the saturation graph and then we determine that there is a gap in the saturation spectrum between the saturation number for $K_4 - e$ and the next possible edge count for a $(K_4 - e)$-saturated graph. We then construct $(K_4 - e)$-saturated graph for every possible edge value up to $\left\lfloor \frac{n}{2} \right\rfloor \left\lceil \frac{n}{2} \right\rceil - \left\lfloor \frac{n}{2} \right\rfloor + 1$, which is a difference of $\left\lfloor \frac{n}{2} \right\rfloor$ from the extremal number. We believe that there will be $(K_4 - e)$-saturated graphs with edge counts in $\left[\left\lceil \frac{n}{2} \right\rceil \left\lfloor \frac{n}{2} \right\rfloor - \left\lfloor \frac{n}{2} \right\rfloor + 2, \left\lceil \frac{n}{2} \right\rceil \left\lfloor \frac{n}{2} \right\rfloor\right]$, but only for sporadic values, which will be explained later.

Lemma 2.1.1. *If G is a connected $(K_4 - e)$-saturated graph, then $diam(G) = 2$.*

Proof. Suppose that G is a connected $(K_4 - e)$-saturated graph. Let x and y be vertices in G with $xy \notin E(G)$. Then $G + xy$ contains a $K_4 - e$ so there is a vertex $w \in V(G)$, distinct from x and y such that x, w, y is a path in G. Since this must be true for any pair of vertices in G, $\text{diam}(G) = 2$. □

Lemma 2.1.2. *If G is a $(K_4 - e)$-saturated graph on n vertices with a cut vertex, then $|E(G)| = \left\lfloor \frac{3(n-1)}{2} \right\rfloor$.*

Proof. Let G be a $(K_4 - e)$-saturated graph with a cut vertex, say x. By Lemma 2.1.1, $\text{diam}(G) = 2$ so every such path from u to v is of length 2, that is x is adjacent to every vertex $y \in V(G - x)$. Since it is possible to add an edge between two vertices of degree one without creating a copy of $(K_4 - e)$ and G is $(K_4 - e)$-saturated, there is a maximal matching in $V(G - x)$ that covers all except possibly one vertex. This creates $\left\lfloor \frac{3(n-1)}{2} \right\rfloor$ edge disjoint triangles, with one additional edge incident to x if n is even. This is precisely the graph that realizes the saturation number with an edge count of $|E(G)| = \left\lfloor \frac{3(n-1)}{2} \right\rfloor$. $\square$

Aside from the saturation number, small edge counts are not realizable by graphs that are $(K_4 - e)$-saturated. The following lemmas show the lower bound on the saturation spectrum of $(K_4 - e)$-saturated graphs.

Lemma 2.1.3. *Let G be a connected $(K_4 - e)$-saturated graph with $\delta(G) \geq 3$ on $n \geq 10$ vertices. Then $|E(G)| \geq 2n - 4$.*

Proof. Let G be a connected $(K_4 - e)$-saturated graph with $\delta(G) \geq 3$. If $\delta(G) \geq 4$, then $|E(G)| \geq 2n > 2n - 4$. Therefore there exists a vertex of degree exactly 3, say u. Note that $\operatorname{diam}(G) = 2$ by Lemma 2.1.1. Let u be adjacent to exactly three other vertices of G, say x, y and z. Let $X = \{x, y, z\}$ and let $A = V(G) - \{u, x, y, z\}$. Since $\operatorname{diam}(G) = 2$, every vertex in A is adjacent to at least one of the vertices in X. Let A_1 be the set of vertices in A that are adjacent to exactly one vertex of X, let A_2 be the vertices in A adjacent to exactly two vertices of X and let A_3 be the vertices in A adjacent to all vertices of X. The minimum degree condition implies that each $v \in A_1$ must be adjacent to at least two other vertices in A and each $w \in A_2$ must be adjacent to at least one other vertex in A. So we have a minimum edge count as follows:

$$\begin{aligned}
|E(G)| &\geq 3 + |A_1| + 2|A_2| + 3|A_3| + \left\lceil \frac{2|A_1| + |A_2|}{2} \right\rceil \\
&\geq 3 + 2|A_1| + 2|A_2| + \left\lceil \frac{|A_2|}{2} \right\rceil + 3|A_3| \\
&= 3 + 2(n - |A_3| - 4) + \left\lceil \frac{|A_2|}{2} \right\rceil + 3|A_3| \\
&= 2n - 5 + \left\lceil \frac{|A_2|}{2} \right\rceil + |A_3|.
\end{aligned}$$

If either A_2 or A_3 is non-empty, we are done. Thus, assume that $|A_2| = |A_3| = 0$. Then $|E(G)| \geq 2n - 5$ and it remains to show that there is at least one additional edge in G.

If at least one of the edges xy, yz, xz is in $E(G)$, we are done. Assume that xy, yz, and xz are not edges of G. Since $\delta(G) = 3$, there must be at least two vertices of A_1 adjacent to x, two vertices of A_1 adjacent to y and two vertices of A_1 adjacent to z. Then each vertex adjacent to x must be adjacent to at least one vertex adjacent to y and at least one vertex adjacent to z. Each vertex adjacent to y must be adjacent to at least one vertex adjacent to x and one vertex adjacent to z. Each vertex adjacent to z must be adjacent to at least one vertex adjacent to x and one vertex adjacent to y. This requirement allows the minimum possible edge count to remain at $|E(G)| \geq 2n - 5$ as it requires at least $|A_1|$ edges amongst the vertices of A_1. However, this graph is not $(K_4 - e)$-saturated, as adding xy does not create a copy of $K_4 - e$, so there must be at least one additional edge. This completes the proof of the lemma. □

Lemma 2.1.4. *Let G be a 2-connected $(K_4 - e)$-saturated graph on m edges and $n \geq 10$ vertices. Then $m \geq 2n - 4$.*

Proof. Let G be a $(K_4 - e)$-saturated, 2-connected graph on m edges. Since G is $(K_4 - e)$-saturated, $\text{diam}(G) = 2$ by Lemma 2.1.1 and it follows from Lemma 2.1.3, that $m \geq 2n - 4$ if $\delta(G) \geq 3$. Suppose $\delta(G) = 2$ with $\deg(z) = 2$ for some $z \in V(G)$. Then z is adjacent to some $x, y \in V(G)$ and we can partition the remaining vertices of G into three sets A, B, C with $A \subset N(x)$, $B \subseteq N(x) \cap N(y)$ and $C \subseteq N(y)$, (see Figure 2.2). Since G is 2-connected, A and B cannot both be empty, as this would make y a cut vertex. Similarly, C and B cannot both be empty. Note that if $B \neq \emptyset$ the edge from x to y is not in $E(G)$ as it would create a copy of $K_4 - e$ and for similar reasons, B must be an independent set.

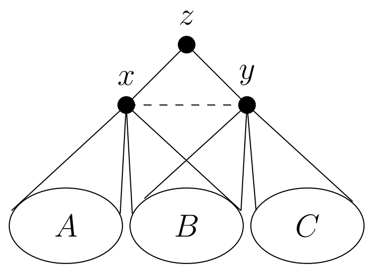

Figure 2.2: Tree structure of 2-connected $(K_4 - e)$-saturated graphs with $\delta(G) = 2$

Case 1: Suppose both A and C are empty and B is not empty.

In this case, each vertex in B is adjacent to both x and y, which creates a copy of C_4 for each vertex of B with the vertices x, y and z. Adding the edge xy, an edge between any two vertices of B, or the edge vz for some $v \in B$ will create a copy of $K_4 - e$. So the graph G is $(K_4 - e)$-saturated and it has as an edge count $m = 2 + 2(n - 3) = 2n - 4$.

Case 2: Suppose that A is empty and B, C are non-empty.

Since $\text{diam}(G) = 2$, there must be a path of length two from x to each $w \in C$ hence, there must be an edge from at least one $v \in B$ to each $w \in C$. Since G cannot contain a copy of $K_4 - e$, each $w \in C$ must be adjacent to a distinct vertex in B and hence $|B| \geq |C|$. Then each $w \in C$ is in a distinct triangle and is not adjacent to another vertex in C or a copy of $K_4 - e$ would exist in G. Additional edges will increase the edge count so $|E(G)|$ must be at least:

$$
\begin{aligned}
m &\geq 2 + 2|B| + 2|C| \\
&= 2 + 2(n - |C| - 3) + 2|C| \\
&= 2n - 2|C| + 2|C| + 2 - 6 \\
&= 2n - 4.
\end{aligned}
$$

Note that, by symmetry, a similar argument holds when C is empty and A is nonempty.

Case 3: Suppose that A and C are both non-empty with $1 \leq |C| \leq |A|$ and B is empty.

Since G is $(K_4 - e)$-saturated, xy must be an edge of G and there can be no path of length 2 or more between any two vertices in A or between any two vertices in C. Also, $\mathrm{diam}(G) = 2$ implies that there is a $u - w$ path of length 1 or 2 for each $u \in A$ and each $w \in C$. So each $u \in A$ must be adjacent to at least $\left\lceil \frac{|C|}{2} \right\rceil$ vertices of C. There must also be at least $\left\lfloor \frac{|C|}{2} \right\rfloor$ additional edges, either within C in the form of a matching, or between A and C if there is a vertex of A that is not in an edge in A.

If $|C| = 1$, $\mathrm{diam}(G) = 2$ requires that either $w \in C$ is adjacent to all vertices in A or $w \in C$ is adjacent to $\left\lceil \frac{|A|}{2} \right\rceil$ vertices in A and there are $\left\lfloor \frac{|A|}{2} \right\rfloor$ edges within A. In either case, $|E(G)| \geq 4 + |A| + |A| = 2(n-3) + 4 = 2n - 2 > 2n - 4$. Otherwise, we have the following edge count for G:

$$\begin{aligned}
m &\geq 3 + |A| + |C| + |A| \left\lceil \frac{|C|}{2} \right\rceil + \left\lfloor \frac{|C|}{2} \right\rfloor \\
&= n + \left\lceil \frac{|C|}{2} \right\rceil \left(n - \left\lceil \frac{|C|}{2} \right\rceil - \left\lfloor \frac{|C|}{2} \right\rfloor - 3 \right) + \left\lfloor \frac{|C|}{2} \right\rfloor \\
&= 2n - 4 + \left(\left\lceil \frac{|C|}{2} \right\rceil - 1 \right) n - \left\lceil \frac{|C|}{2} \right\rceil^2 - 3 \left\lceil \frac{|C|}{2} \right\rceil + 4 - \left\lfloor \frac{|C|}{2} \right\rfloor \left(\left\lceil \frac{|C|}{2} \right\rceil - 1 \right) \\
&= 2n - 4 + \left(\left\lceil \frac{|C|}{2} \right\rceil - 1 \right) - \left(\left\lceil \frac{|C|}{2} \right\rceil - 1 \right) \left(\left\lceil \frac{|C|}{2} \right\rceil + 4 \right) - \left\lfloor \frac{|C|}{2} \right\rfloor \left(\left\lceil \frac{|C|}{2} \right\rceil - 1 \right) \\
&= 2n - 4 + \left(\left\lceil \frac{|C|}{2} \right\rceil - 1 \right) (n - |C| - 4).
\end{aligned}$$

Since $\left\lceil \frac{|C|}{2} \right\rceil \geq 1$ and $|A| = n - |C| - 3 \geq 1$ both clearly hold, it follows that $|E(G)| \geq 2n - 4$ is always true.

Case 4: Suppose that A, B and C are non-empty with $1 \leq |C| \leq |A|$.

Then $\mathrm{diam}(G) = 2$ implies that there must be a path of length at most 2 from each $u \in A$ to each $w \in C$. The vertices x and y cannot be adjacent as B is non-empty, which results in at least one C_4 with $v \in B$, x, y and z. Also, the vertices in B must be independent as any edge between two vertices in B will result in a $K_4 - e$ with x

and y. If some $u \in A$ is not adjacent to any vertex in A or B, then the edge uz does not create a copy of $K_4 - e$, hence the graph is not $(K_4 - e)$-saturated. So if a vertex $u \in A$ is independent within A, then $uv \in E(G)$ for some $v \in B$ and either uw or vw is an edge of G for every $w \in C$. If a vertex $u \in A$ is not independent within A, then $uw \in E(G)$ for some $w \in C$ as an edge from A to B gives a $K_4 - e$. By symmetry, the same is true for vertices in C.

If $|C| = 1$, $\operatorname{diam}(G) = 2$ requires that there is a path of length 1 or 2 between $w \in C$ and each vertex in A. Since G is $(K_4 - e)$-saturated, wv is an edge in G for some $v \in B$ and either vu or wu is also an edge of G for some $u \in A$. Then $w \in C$ is adjacent to at least $\left\lceil \frac{|A|-1}{2} \right\rceil$ vertices in A and there are at most $\left\lfloor \frac{|A|-1}{2} \right\rfloor$ edges within A or from A to B. In any case, $|E(G)| \geq 3 + |A| + 2|B| + 1 + |A| = 2(n-4) + 4 = 2n - 4$.

Otherwise, each $u \in A$ must be adjacent to at least $\left\lceil \frac{|C|}{2} \right\rceil$ vertices of C. Then there must also be at least $\left\lfloor \frac{|C|}{2} \right\rfloor$ additional edges, either within C in the form of a matching, or between A and C if there is a vertex of A that is not in an edge in A. This yields the following edge count of G:

$$
\begin{aligned}
m &\geq 2 + |A| + 2|B| + |C| + |A| \left\lceil \tfrac{|C|}{2} \right\rceil + \left\lfloor \tfrac{|C|}{2} \right\rfloor \\
&= n - 1 + |B| + (n - |B| - |C| - 3) \left\lceil \tfrac{|C|}{2} \right\rceil + \left\lfloor \tfrac{|C|}{2} \right\rfloor \\
&= 2n - 1 + \left(\left\lceil \tfrac{|C|}{2} \right\rceil - 1 \right) (n - |B|) - \left\lceil \tfrac{|C|}{2} \right\rceil^2 - 3 \left\lceil \tfrac{|C|}{2} \right\rceil - \left(\left\lceil \tfrac{|C|}{2} \right\rceil - 1 \right) \left\lfloor \tfrac{|C|}{2} \right\rfloor \\
&= 2n - 5 + \left(\left\lceil \tfrac{|C|}{2} \right\rceil - 1 \right) \left(n - |B| - \left\lfloor \tfrac{|C|}{2} \right\rfloor \right) - \left(\left\lceil \tfrac{|C|}{2} \right\rceil - 1 \right) \left(\left\lceil \tfrac{|C|}{2} \right\rceil + 4 \right) \\
&= 2n - 5 + \left(\left\lceil \tfrac{|C|}{2} \right\rceil - 1 \right) (n - |B| - |C| - 4). \\
&= 2n - 5 + \left(\left\lceil \tfrac{|C|}{2} \right\rceil - 1 \right) (|A| - 1).
\end{aligned}
$$

So $|E(G)| \geq 2n - 4$ if $|C| \geq 3$ and $|A| \geq 2$. However, if $|A| = 1$ then we have $|E(G)| \geq 2n - 4$, similar to the case when $|C| = 1$, so it remains to determine the edge count of G when $|A| = |C| = 2$.

Suppose that $C = \{w, w'\}$. If $ww' \in E(G)$, wv and $w'v$ are not edges of G for any $v \in B$. Then $\mathrm{diam}(G) = 2$ implies that each $u \in A$ is adjacent to w or w'. Also, since G is $(K_4 - e)$-saturated, either there is an edge in A or there is an edge from A to B as adding one of those edges does not create a copy of $K_4 - e$. This yields $|E(G)| \geq 6 + 2|B| + 2 + 1 + 1 = 10 + 2(n-7) = 2n - 4$. On the other hand, if $ww' \notin E(G)$, both wv and $w'v'$ are edges of G for distinct $v \in B$ and $v' \in B$ without creating a copy of $K_4 - e$ and each vertex of C must be adjacent to at least one vertex in A such that $|E(G)| \geq 6 + 2|B| + 2 + 2 = 2n - 4$.

This completes the proof of the lemma. □

We will now show that there is a $(\mathrm{K}_4 - e)$-saturated graph for every integer value of m in the interval $[2n-4, \lceil \frac{n}{2} \rceil \lfloor \frac{n}{2} \rfloor - \lfloor \frac{n}{2} \rfloor + 1]$ by combining three different constructions.

Theorem 2.1.1. *There exists a* $(K_4 - e)$*-saturated graph on* $n \geq 6$ *vertices and* m *edges where* $2n - 4 \leq m \leq \lceil \frac{n}{2} \rceil \lfloor \frac{n}{2} \rfloor - \lfloor \frac{n}{2} \rfloor + 2$.

Proof. Case 1: Suppose $2n - 4 \leq m \leq 3n - 9$.

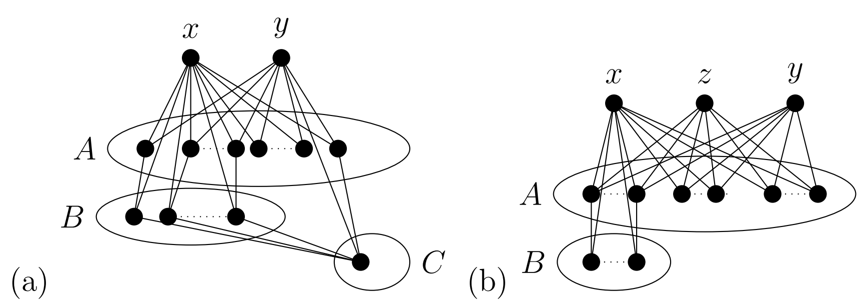

Figure 2.3: Construction of $(K_4 - e)$-saturated graphs with $m \in [2n-4, 3n-9]$

To construct $(K_4 - e)$-saturated graphs we modify $K_{2,n-2}$ and $K_{3,n-3}$. For the graph in Figure 2.3(a), we form a set B by removing at most $\lfloor \frac{n-3}{2} \rfloor - 1$ vertices from A so that the vertices are adjacent to x and to distinct vertices in A and we form a set C with a single vertex from A so that it is adjacent to y and a vertex in A that is not adjacent to any vertex in B. We then join each vertex of B to the vertex in C.

We now show the resulting graphs are $(K_4 - e)$-saturated. First, the edge xy will create at least two triangles on that edge if $|A| \geq 2$. Any edge added within A will create a $K_4 - e$ with x and y and any edge added within B (or within C) will create a triangle with x (or y, respectively), which creates a copy of $K_4 - e$ as every pair of vertices in B (or C) is a part of two triangles. Each edge from A to B is an edge of a triangle with x so any additional edge between $u \in A$ and $v \in B$ will create another triangle with x and the edge xv, resulting in a copy of $(K_4 - e)$. Similarly, an edge between any vertex in A and $w \in C$ with create a $K_4 - e$. Finally, any additional edge from x to $w \in C$ or y to $v \in B$ will create a $(K_4 - e)$ with the triangle constructed between x or y and B or C, sharing the edge from A to w or v, respectively.

For the graph in Figure 2.3(a), if $|A| = n - b - 3$ where $|B| = b$ and $|C| = 1$, the edge count is:

$$\begin{aligned} m &= 2|A| + 3|B| + 2 \\ &= 2(n - b - 3) + 3b + 2 \\ &= 2n - 4 + b. \end{aligned}$$

So we have an edge count of $m = 2n - 4 + b$, which increases by one as the size of B increases by one. Since $0 \leq |B| \leq \left\lfloor \frac{n-3}{2} \right\rfloor - 1$, we have the range of edge counts $2n - 4 \leq m \leq 2n - 4 + \left\lfloor \frac{n-2}{2} \right\rfloor - 1 = \left\lfloor \frac{5n}{2} \right\rfloor - 6$.

For the graph in Figure 2.3(b), we form a set B with vertices from A so that the vertices are adjacent to x and to distinct vertices in A. Similar to the graphs in Figure 2.3(a), such graphs are $(K_4 - e)$-saturated and, if $|A| = n - b - 3$ where $|B| = b$, they have edge count:

$$\begin{aligned} m &= 3|A| + 2|B| \\ &= 3(n - b - 3) + 2b \\ &= 3n - 9 - b. \end{aligned}$$

So the edge count decreases by one as the size of B is increased by one. Since

$0 \le |B| \le \lfloor \frac{n-3}{2} \rfloor$, we have $3n-9 \ge m \ge 3n-9-\lfloor \frac{n-3}{2} \rfloor = \lfloor \frac{5n}{2} \rfloor - 7$, which clearly intersects the interval constructed with the graphs of Figure 2.3(a). Thus, we have constructed saturated graphs of size $2n-4$ to $3n-9$ for $n \ge 6$ and this case completes the proof of Theorem 1 for $n \le 11$.

Case 2: Suppose $3n-9 \le m \le 4n-18$.

We can similarly modify the complete bipartite graphs $K_{3,n-3}$ and $K_{4,n-4}$ by adding triangles to the vertices in the smaller vertex set in the same way as before, to obtain a (K_4-e)-saturated graph for $n \ge 11$.

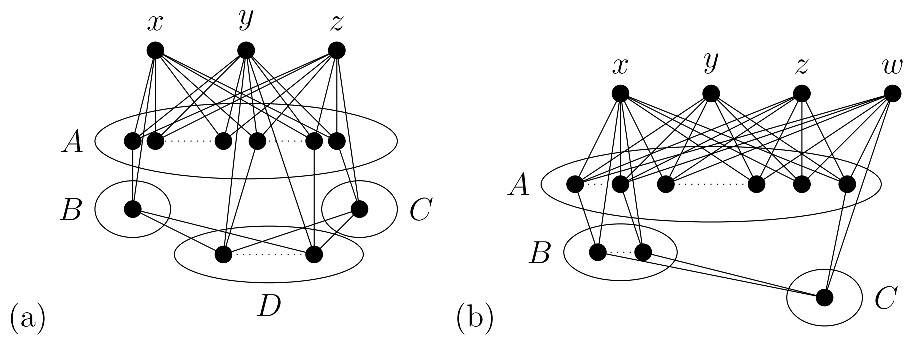

Figure 2.4: Construction of (K_4-e)-saturated graphs with $m \in [3n-9, 4n-18]$

For the graph in Figure 2.4(a), we form a set B with vertices from A so that the vertices are adjacent to x and to distinct vertices in A. We form a set C with the vertices from A that are adjacent to z and distinct vertices in A and we a form set D with vertices from A that are adjacent to y and distinct vertices in A. In forming the sets B, C and D, it is necessary that their neighbors in A do not overlap. We then join each vertex of B and C to all of D. If $|A| = n-b-c-d-3$ where $|B| = b$, $|C| = c$ and $|D| = d \ge 2$, then the edge count is:

$$\begin{aligned} m &= 3|A| + 2|B| + 2|C| + 2|D| + |B||D| + |C||D| \\ &= 3(n-b-c-d-3) + 2b + 2c + (2+b+c)d \\ &= 3n - b - c + (b+c-1)d - 9. \end{aligned}$$

If we fix $|B| = |C| = 1$, we have an edge count of $m = 3n - 11 + d$, which increases by one as the size of D increases by one. Since the construction requires $2 \leq |B| + |C| + |D| \leq \left\lfloor \frac{n-3}{2} \right\rfloor$, we have $2 \leq |D| \leq \left\lfloor \frac{n-3}{2} \right\rfloor - 2$, which yields the range of edge counts $3n - 9 \leq m \leq 3n - 11 + \left\lfloor \frac{n-3}{2} \right\rfloor - 2 = \left\lfloor \frac{7n}{2} \right\rfloor - 13$.

For the graph in Figure 2.4(b), we form a set B with vertices from A so that the vertices are adjacent to x and to distinct vertices in A, we form a set C with the vertices from A that are adjacent to w and distinct vertices in A. Then we join each vertex in C to all vertices in B. If $|A| = n - b - c - 4$ where $|B| = b$ and $|C| = c$, then the edge count is:

$$\begin{aligned} m &= 4|A| + 2|B| + 2|C| + |B||C| \\ &= 4(n - b - c - 4) + 2b + 2c + bc \\ &= 4n - 2b - 2c + bc - 16. \end{aligned}$$

Thus, if we fix $|C| = c = 1$, we have an edge count of $m = 4n - 18 - b$, which decreases by one as the size of B increases by one. Then $0 \leq |B| \leq \left\lfloor \frac{n-4}{2} \right\rfloor - 1$ implies $4n - 18 \geq m \geq 4n - 18 - \left(\left\lfloor \frac{n-4}{2} \right\rfloor - 1 \right) = \left\lfloor \frac{7n}{2} \right\rfloor - 15$, which intersects the interval for the graphs of Figure 2.4(a). Thus, we have constructed graphs of size $3n - 9$ to $4n - 18$ for $n \geq 11$.

Case 3: Suppose $4n - 18 \leq m \leq \left\lfloor \frac{n}{2} \right\rfloor \left\lceil \frac{n}{2} \right\rceil - n + 5$.

We blow up the graph C_5 such that each vertex becomes a set of independent vertices with adjacencies according to the original C_5, where an edge $xy \in E(C_5)$ becomes a $K_{s,t}$, when $x \in V(C_5)$ is blown-up to be a set of s independent vertices and $y \in V(C_5)$ is blown-up to be a set of t independent vertices.

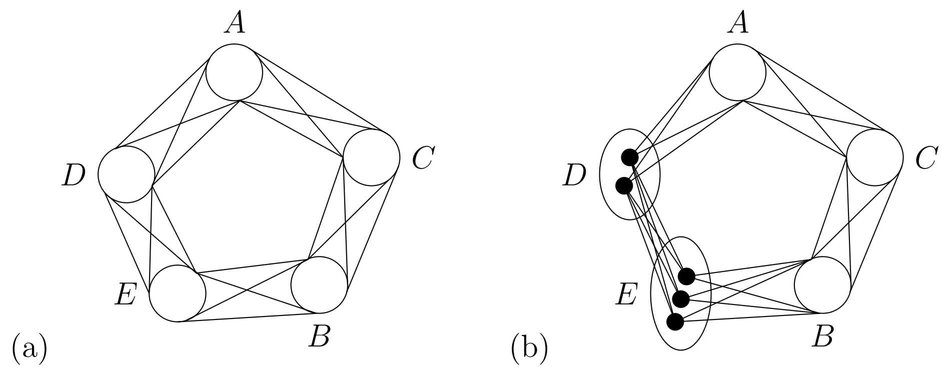

Figure 2.5: (a) Blowup of C_5; (b) Construction of $(K_4 - e)$-saturated graphs

Any edge added within a set of independent vertices will create at least two triangles on that edge with vertices of two adjacent sets. Also, any edge added between vertices in two different vertex sets will create at least two triangles on that edge with vertices of the common adjacent set, if the common adjacent set has order at least 2. As such, a blown-up C_5 in Figure 2.5(a), with at least two vertices in each vertex set, is $(K_4 - e)$-saturated. So the blown-up C_5 in Figure 2.5(b), which we denote as $G = C_5[A, D, E, B, C]$, is $(\mathrm{K}_4 - e)$-saturated with $|A| = n - b - c - 5$ provided $|B| = b \geq 2$, $|C| = c \geq 2$, $|D| = 2$ and $|E| = 3$ with $|E(G)| = m$ given by the products of the orders of consecutive vertex sets, hence:

$$\begin{aligned} m &= |A||D| + |D||E| + |E||B| + |B||C| + |C||A| \\ &= 2(n - b - c - 5) + 2(3) + 3b + bc + c(n - b - c - 5) \\ &= cn + 2n - c^2 - 7c + b - 4 \\ &= (n - c)(c + 2) - 5c + b - 4. \end{aligned}$$

Then for fixed values of c, when b increases by 1, that is, as vertices are moved from A to B, the edge count increases by 1. To maintain at least two vertices in each set of the blown-up C_5, we must have $b \in [2, n - c - 7]$. If we let $b = n - c - 7$ for fixed n, then we have $m = cn + 3n - c^2 - 8c - 11$, which is maximized when

$c = \lceil \frac{n}{2} \rceil - 4$ such that $c \in [2, \lceil \frac{n}{2} \rceil - 4]$. Then the smallest edge count for G is when $a = n-9, b = 2, c = 2$ and is $m = (n-2)(4) - 10 + 2 - 4 = 4n - 20 \leq 4n - 18$ and the largest possible edge count is given when $a = 2, b = \lfloor \frac{n}{2} \rfloor - 3, c = \lceil \frac{n}{2} \rceil - 4$ and is:

$$\begin{aligned}
m &= \left(n - \lceil \tfrac{n}{2} \rceil + 4\right)\left(\lceil \tfrac{n}{2} \rceil - 4 + 2\right) - 5\left(\lceil \tfrac{n}{2} \rceil - 4\right) + \left(\lfloor \tfrac{n}{2} \rfloor - 3\right) - 4 \\
&= \left(\lfloor \tfrac{n}{2} \rfloor + 4\right)\left(\lceil \tfrac{n}{2} \rceil - 2\right) - 5\lceil \tfrac{n}{2} \rceil + 20 + \lfloor \tfrac{n}{2} \rfloor - 3 - 4 \\
&= \lceil \tfrac{n}{2} \rceil \lfloor \tfrac{n}{2} \rfloor - 2\lfloor \tfrac{n}{2} \rfloor + 4\lceil \tfrac{n}{2} \rceil - 8 - 5\lceil \tfrac{n}{2} \rceil + 13 + \lfloor \tfrac{n}{2} \rfloor \\
&= \lceil \tfrac{n}{2} \rceil \lfloor \tfrac{n}{2} \rfloor - \lfloor \tfrac{n}{2} \rfloor - \lceil \tfrac{n}{2} \rceil + 5 \\
&= \lceil \tfrac{n}{2} \rceil \lfloor \tfrac{n}{2} \rfloor - n + 5.
\end{aligned}$$

Next, we check that the construction accounts for each edge count in the interval $\left[4n - 18, \lceil \frac{n}{2} \rceil \lfloor \frac{n}{2} \rfloor - n + 5\right]$ of length $\lceil \frac{n}{2} \rceil \lfloor \frac{n}{2} \rfloor - 5n + 24$. For each fixed c, we will have an interval of values S_c determined by the range of values for b, namely, each interval has a left endpoint given when $b = 2$ such that the interval starts at $m = (n-c)(c+2) - 5c - 2$. So we have a $(K_4 - e)$-saturated graph on n vertices and m edges for an interval of length $(n - c - 7) - 2 + 1 = n - c - 8$ and we have $\left(\lceil \frac{n}{2} \rceil - 4\right) - 2 + 1 = \lceil \frac{n}{2} \rceil - 5$ such intervals. The next consecutive interval starts at:

$$\begin{aligned}
m &= (n - (c+1))(c + 1 + 2) - 5(c+1) - 2 \\
&= (n-c)(c+2) + (n-c) - (c+2) - 1 - 5c - 7 \\
&= (n-c)(c+2) - 5c - 2 + (n - 2c - 8).
\end{aligned}$$

Thus, the end of each interval S_c will overlap with the next interval S_{c+1} in the first $(n-c-8) - (n-2c-8) = c$ numbers. There are $\lceil \frac{n}{2} \rceil - 5$ intervals, each with $(n-c-8) - (c) + 1 = n - 2c - 7$ distinct elements. As the lowest interval starts at $4n - 18$ and the largest interval ends at $\lceil \frac{n}{2} \rceil \lfloor \frac{n}{2} \rfloor - n + 5$ with each interval having a nonempty overlap with the next highest interval, all values are covered.

Case 4: Suppose $m \in [\lceil \frac{n}{2} \rceil \lfloor \frac{n}{2} \rfloor - n + 6, \lceil \frac{n}{2} \rceil \lfloor \frac{n}{2} \rfloor - \lfloor \frac{n}{2} \rfloor + 2]$.

The blown-up C_5 in Figure 2.6 is another construction that yields additional values in the saturation spectrum of $(K_4 - e)$-saturated graphs. The blown-up C_5 is $(\mathrm{K}_4 - e)$-saturated when $|A| = n - b - c - 3$, $|B| = b \geq 2$, $|C| = c \geq 2$, $|D| = 1$ and $|E| = 2$ and two edges added from E to A between distinct vertices.

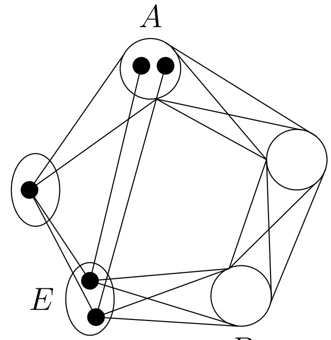

Figure 2.6: Another Construction of $(K_4 - e)$-saturated graphs

If an edge is added within A, B, C or E, then a copy of $K_4 - e$ is created with that edge and two vertices from adjacent sets. If an edge uv is added from A to B, from C to E, from B to D or from D to C, a $K_4 - e$ is created on that edge with two vertices in the shared neighborhood of u and v. Finally, if an additional edge is added between A and E, a $K_4 - e$ will be created on an edge from D to E with two vertices from A.

The general edge count of the graph m is given by the products of the orders of consecutive vertex sets, hence:

$$\begin{aligned} m &= |A||D| + |D||E| + |E||B| + |B||C| + |C||A| + 2 \\ &= (n - b - c - 3) + 1(2) + 2b + bc + c(n - b - c - 3) + 2 \\ &= (n - c) - b - 3 + 3 + 2b - 3c + c(n - c) + 1 \\ &= (n - c)(c + 1) - 3c + b + 1. \end{aligned}$$

Then for fixed values of c, when b increases by 1, that is, as vertices are moved from A to B, the edge count increases by 1. To maintain the minimum required number of vertices in each set we must have $b \in [2, n-c-5]$. If we let $b = n-c-5$ for fixed n, then we have $m = cn + 2n - c^2 - 5c - 4$, which is maximized when $c = \left\lceil \frac{n}{2} \right\rceil - 3$ such that $c \in [2, \left\lceil \frac{n}{2} \right\rceil - 3]$. Then the smallest edge count for G is when $a = n-7, b = 2, c = 2$ and is $m = (n-2)(3) - 6 + 2 + 1 = 3n - 9$ and the largest possible edge count is given when $a = 2, b = \left\lfloor \frac{n}{2} \right\rfloor - 2, c = \left\lceil \frac{n}{2} \right\rceil - 3$ and is:

$$\begin{aligned} m &= \left(n - \left\lceil \tfrac{n}{2} \right\rceil + 3\right)\left(\left\lceil \tfrac{n}{2} \right\rceil - 3 + 1\right) - 3\left(\left\lceil \tfrac{n}{2} \right\rceil - 3\right) + \left(\left\lfloor \tfrac{n}{2} \right\rfloor - 2\right) + 1 \\ &= \left(\left\lfloor \tfrac{n}{2} \right\rfloor + 3\right)\left(\left\lceil \tfrac{n}{2} \right\rceil - 2\right) - 3\left\lceil \tfrac{n}{2} \right\rceil + 9 + \left\lfloor \tfrac{n}{2} \right\rfloor - 2 + 1 \\ &= \left\lceil \tfrac{n}{2} \right\rceil \left\lfloor \tfrac{n}{2} \right\rfloor - 2\left\lfloor \tfrac{n}{2} \right\rfloor + 3\left\lceil \tfrac{n}{2} \right\rceil - 6 - 3\left\lceil \tfrac{n}{2} \right\rceil + 8 + \left\lfloor \tfrac{n}{2} \right\rfloor \\ &= \left\lceil \tfrac{n}{2} \right\rceil \left\lfloor \tfrac{n}{2} \right\rfloor - \left\lfloor \tfrac{n}{2} \right\rfloor + 2. \end{aligned}$$

In particular, when $c = \left\lceil \frac{n}{2} \right\rceil - 3$ and $b \in [2, n-c-5]$, the construction yields graphs that are $(K_4 - e)$-saturated with edge counts in the remaining interval of values $\left[\left\lceil \frac{n}{2} \right\rceil \left\lfloor \frac{n}{2} \right\rfloor - n + 6, \left\lceil \frac{n}{2} \right\rceil \left\lfloor \frac{n}{2} \right\rfloor - \left\lfloor \frac{n}{2} \right\rfloor + 2\right]$.

This completes the proof of the theorem. □

We conjecture that graphs with sizes in the interval $\left[\left\lceil \frac{n}{2} \right\rceil \left\lfloor \frac{n}{2} \right\rfloor - \left\lfloor \frac{n}{2} \right\rfloor + 2, \left\lceil \frac{n}{2} \right\rceil \left\lfloor \frac{n}{2} \right\rfloor\right]$ are of two types: complete bipartite graphs with partite sets of nearly equal size, and 3-partite graphs with two partite sets of nearly equal size and one partite set of order one. The complete bipartite graph is $(K_4 - e)$-saturated as adding an edge between any two nonadjacent vertices will create a $K_4 - e$. In the 3-partite graph, we let the two larger partite sets induce a complete bipartite graph and the single vertex set be adjacent to exactly one vertex in each of the other partite sets. This 3-partite graph, then, contains a complete bipartite graph on $n-1$ vertices as well as a single triangle and is $(K_4 - e)$-saturated. If an edge is added within either of the independent sets a copy of $K_4 - e$ is created and if an edge is added between a vertex of the triangle and any other vertex of the graph, a $K_4 - e$ is created. Such graphs would have the highest possible edge counts when the larger partite sets are almost the same order.

Let the graph G be a complete bipartite graph with one partite set of order $\lfloor \frac{n}{2} \rfloor + k$. Then the size of G is $m = \left(\lfloor \frac{n}{2} \rfloor + k\right)\left(\lceil \frac{n}{2} \rceil - k\right)$. This gives a few additional values of m in the interval $\left[\lceil \frac{n}{2} \rceil \lfloor \frac{n}{2} \rfloor - \lfloor \frac{n}{2} \rfloor + 2, \lceil \frac{n}{2} \rceil \lfloor \frac{n}{2} \rfloor\right]$. Let the graph H be a 3-partite graph described above with partite sets of order $\lceil \frac{n}{2} \rceil + k$, $\lfloor \frac{n}{2} \rfloor - k - 1$, and 1. Then the size of H is:

$$\begin{aligned} m &= \left(\lceil \tfrac{n}{2} \rceil + k\right)\left(\lfloor \tfrac{n}{2} \rfloor - k - 1\right) + 2 \\ &= \lceil \tfrac{n}{2} \rceil \lfloor \tfrac{n}{2} \rfloor - k \lceil \tfrac{n}{2} \rceil - \lceil \tfrac{n}{2} \rceil + k \lfloor \tfrac{n}{2} \rfloor - k^2 - k + 2. \end{aligned}$$

This is $\lceil \frac{n}{2} \rceil \lfloor \frac{n}{2} \rfloor - \lceil \frac{n}{2} \rceil - k^2 - k + 2$ when n is even and $\lceil \frac{n}{2} \rceil \lfloor \frac{n}{2} \rfloor - \lceil \frac{n}{2} \rceil - k^2 - 2k + 2$ when n is odd. We believe that this completes the saturation spectrum of $(K_4 - e)$-saturated graphs since the graphs G and H are completely saturated with four vertex complete graphs missing two edges.

Thus, the saturation spectrum of $(K_4 - e)$-saturated graphs has a jump from the saturation number to the next possible edge count, is continuous in the interval $\left[2n - 4, \lceil \frac{n}{2} \rceil \lfloor \frac{n}{2} \rfloor - \lfloor \frac{n}{2} \rfloor + 2\right]$ and then,we believe, has sporadic values in the interval $\left[\lceil \frac{n}{2} \rceil \lfloor \frac{n}{2} \rfloor - \lfloor \frac{n}{2} \rfloor + 2, \lceil \frac{n}{2} \rceil \lfloor \frac{n}{2} \rfloor\right]$.

2.2 Constructing $(K_t - e)$-saturated Graphs

Given a graph G that is $(K_4 - e)$-saturated, it is possible to construct a graph that is $(K_5 - e)$-saturated by adding a vertex v and all edges from v to each vertex in G. Then, by joining a vertex to each of the $(K_4 - e)$-saturated graphs we have constructed, there is a $(K_5 - e)$-saturated graph on n vertices for each edge count in

$$\begin{aligned} & \left[2(n-1) - 4 + (n-1), \left\lfloor \tfrac{n-1}{2} \right\rfloor \left\lceil \tfrac{n-1}{2} \right\rceil - \left\lfloor \tfrac{n-1}{2} \right\rfloor + 2 + (n-1)\right] \\ = \quad & \left[3n - 7, \left\lfloor \tfrac{n-1}{2} \right\rfloor \left\lceil \tfrac{n-1}{2} \right\rceil + \left\lceil \tfrac{n-1}{2} + 2 \right\rceil\right]. \end{aligned}$$

Similarly, given a graph H that is $(K_{t-1} - e)$-saturated, it is possible to construct a graph $H' = H + v$ that is $(K_t - e)$-saturated where $H + v$ is constructed by adding a vertex v and all edges from v to each vertex in H.

Note that if we add an edge to a $(K_t - e)$-saturated graph, at least one copy of $K_4 - e$ is created. As such, we believe that there is a gap between the saturation number of $K_t - e$ and the next possible edge count of a $(K_t - e)$-saturated graph. Since a $(K_t - e)$-saturated graph must have diameter 2, similar to a $(K_4 - e)$-saturated graph, we conjecture the following:

Conjecture 2.2.1. *For integers n and $t \geq 4$, there does not exist a $(K_t - e)$-saturated graph on n vertices and m edges where $m \neq sat(n, K_t - e)$ and $m \geq f(n,t)$ for some function $f(n,t) \geq 2n - 4$.*

Joining $t - 4$ vertices, one at a time, to a $(K_4 - e)$-saturated graph on $n - t + 4$ vertices will result in a graph that is $(K_t - e)$-saturated. As such, there is a $(K_t - e)$-saturated graph on n vertices and m edges for each value in the following interval:

$$\begin{aligned} & \left[2n - 2t + 4 + \sum_{i=1}^{t-4}(n - t + 3 + i), \left\lfloor \tfrac{n-t+4}{2} \right\rfloor \left\lceil \tfrac{n-t+4}{2} \right\rceil - \left\lfloor \tfrac{n-t+4}{2} \right\rfloor + 2 + \sum_{i=1}^{t-4}(n - t + 3 + i)\right] \\ = & \left[(t-2)n - \tfrac{t^2}{2} + \tfrac{3}{2}t - 2, \left(\left\lfloor \tfrac{n-t}{2} \right\rfloor + 2\right)\left(\left\lceil \tfrac{n-t}{2} \right\rceil + 2 - 1\right) + 2 + n(t-4) - \tfrac{t^2}{2} + \tfrac{7}{2}t - 6\right] \\ = & \left[(t-2)n - \binom{t-1}{2} - 1, \left\lfloor \tfrac{n-t}{2} \right\rfloor \left\lceil \tfrac{n-t}{2} \right\rceil + \left\lceil \tfrac{n-t}{2} \right\rceil + 1 + (t-3)n - \binom{t-2}{2}\right]. \end{aligned}$$

Also, there are $(K_t - e)$-saturated graphs for sporadic values of m in

$\left[\left\lfloor \tfrac{n-t}{2} \right\rfloor \left\lceil \tfrac{n-t}{2} \right\rceil + (t-3)n - \binom{t-2}{2}, \left\lfloor \tfrac{n-t}{2} \right\rfloor \left\lceil \tfrac{n-t}{2} \right\rceil + (t-2)n - \binom{t-1}{2}\right]$.

Chapter 3

On the Saturation Spectrum of F_t-saturated graphs

The t-fan, F_t, is the graph consisting of t edge-disjoint triangles that intersect at a single vertex v. To determine the saturation spectrum of F_t it is necessary to first determine the saturation spectrum of F_2-saturated graphs. The saturation number for F_2 is given by $sat(n, F_2) = n + 2$ and can be realized as the edge count of the graph G consisting of a K_4 with $n - 4$ pendant edges on one vertex u of the K_4 (see Figure 3.1(a)). The K_4 provides two triangles, T_1 and T_2, containing u that are not edge-disjoint, however adding any other edge will create a triangle with u that is edge-disjoint from T_1 or T_2. At the other end of the spectrum, the extremal number for F_2 is given by $ex(n, F_2) = \lfloor \frac{n}{2} \rfloor \lceil \frac{n}{2} \rceil + 1$ for $n \geq 5$ [5] and can be realized as the edge count of the complete bipartite graph $K_{\lfloor \frac{n}{2} \rfloor, \lceil \frac{n}{2} \rceil}$ with any additional edge $e = uv$ for $u, v \in V(G)$ (see Figure 3.1(b)). The graph $K_{p,n-p} + e$ ($p \geq 2$) is F_2-saturated as vertices u and v are contained in at least two triangles that intersect only at e so adding any other edge creates an additional triangle intersecting with one of the triangles containing u or v at exactly u or v, respectively, forming a copy of F_2.

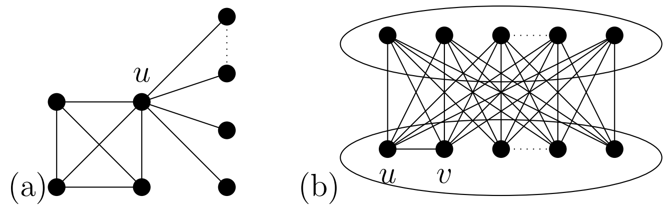

Figure 3.1: (a) $K_4 + \overline{K}_{n-4}$; (b) $K_{\lceil \frac{n}{2} \rceil \lfloor \frac{n}{2} \rfloor}$

We prove the following theorem for the saturation spectrum of F_2:

Theorem 3.0.1. *There exists an F_2-saturated graph on $n \geq 6$ vertices and m edges where $2n - 4 \leq m \leq \lceil \frac{n}{2} \rceil \lfloor \frac{n}{2} \rfloor - \lfloor \frac{n}{2} \rfloor + 2$.*

3.1 F_2-saturated graphs

We begin begin our exploration of the saturation spectrum for F_2-saturated graphs with a few useful lemmas. First, we see that a connected F_2-saturated graph with a cut vertex is the saturation graph and then we determine that there is a gap in the saturation spectrum between the saturation number for F_2 and the next possible edge count for a F_2-saturated graph. We then construct F_2-saturated graph for every possible edge value up to $\lceil \frac{n}{2} \rceil \lfloor \frac{n}{2} \rfloor - \lfloor \frac{n}{2} \rfloor + 2$, which is $\lfloor \frac{n}{2} \rfloor$ less than the extremal number.

Lemma 3.1.1. *If G is a connected F_2-saturated graph with $n \geq 5$ vertices, then $diam(G) = 2$.*

Proof. Suppose that G is a connected F_2-saturated graph with $\text{diam}(G) \geq 3$. Then for some $u, v \in V(G)$, there is no path from u to v of length two. Since G is F_2-saturated, the edge uv must create a copy of F_2, so it creates the triangle $\{u, v, w\}$ for some $w \in V(G)$. Then $uw \in E(G)$ and $vw \in E(G)$ and there is a path of length two from u to v through w, which is a contradiction. Thus, $\text{diam}(G) = 2$ if G is a connected F_2-saturated graph. $\square$

Lemma 3.1.2. *If G is a connected F_2-saturated graph on n vertices with a cut vertex, then* $|E(G)| = n + 2$.

Proof. Let G be a connected F_2-saturated graph with cut vertex x. By Lemma 3.1.1, $\text{diam}(G) = 2$, so for any pair u, v from different components of $V(G - x)$, there is a path through x of length 2 joining u and v. Hence, x must be adjacent to all vertices $u \in V(G-x)$. Let $\{u_1, u_2, ..., u_{n-1}\}$ be the vertices of G adjacent to x. An edge u_iu_j, for $i \neq j$ and $1 \leq i, j \leq n - 1$, will create a triangle $\{x, u_i, u_j\}$ but not a copy of F_2 and, since G is F_2-saturated, we must have that edge in G. The edge u_iu_k, for some k distinct from i, j and $1 \leq k \leq n - 1$, will create a triangle $\{x, u_i, u_k\}$, which shares the edge xu_i with the triangle $\{x, u_i, u_j\}$ but does not create a copy of F_2, so $u_iu_k \in E(G)$.

Without loss of generality, say $i = 1, j = 2$ and $k = 3$. Then any edge u_su_t for $s \neq t$ and $3 \leq s, t \leq n - 1$ will create the triangle $\{x, u_s, u_t\}$ in a copy of F_2 with $\{x, u_1, u_2\}$ so $\{u_3, u_4, ..., u_{n-1}\}$ form an independent set in G. Adding the edge u_2u_3 creates a triangle $\{x, u_2, u_3\}$ that shares an edge with the existing triangle $\{x, u_1, u_2\}$ but not a single vertex so no copy of F_2 is created. If $u_2u_3 \in E(G)$, then $u_1u_3 \in E(G)$, as this creates no copy of F_2 but $u_tu_1, u_tu_2 \notin E(G)$ as each edge would create a copy of F_2 with the triangles $\{u_1, x, u_t\}$ (or $\{u_2, x, u_t\}$) and $\{u_1, u_2, u_3\}$. This is precisely the graph that realizes the saturation number with an edge count of $|E(G)| = n + 2$.

If $u_2u_3 \notin E(G)$, one of $\{u_1, u_2\}$, say u_1, can be adjacent to each u_t for each $3 \leq t \leq n - 1$ without creating a copy of F_2, as each triangle created shares the edge xu_1 but no single vertex. However, such a graph would be 2-connected as both x and u_1 are adjacent to every other vertex in the graph. Thus, there is only one F_2-saturated graph with a cut vertex and it has an edge count of $|E(G)| = n + 2$. □

Aside from the saturation number, small edge counts are not realizable by graphs that are F_2-saturated. The following lemmas establish the lower bound of the saturation spectrum of F_2.

Lemma 3.1.3. *Let G be a connected F_2-saturated graph with $\delta(G) \geq 3$ on $n \geq 10$ vertices. Then $|E(G)| \geq 2n-4$.*

Proof. Let G be a connected F_2-saturated graph with $\delta(G) \geq 3$. Then $\mathrm{diam}(G) = 2$ by Lemma 3.1.1. Note that if $\delta(G) \geq 4$, then $|E(G)| \geq 2n > 2n-4$ and we are done. So there is a vertex u in G adjacent to exactly three other vertices of G, say x, y and z. Let $X = \{x, y, z\}$ and let $A = V(G) - \{u, x, y, z\}$. Since $\mathrm{diam}(G) = 2$, every vertex in A is adjacent to at least one of the vertices in X. Let A_1 be the set of vertices in A that are adjacent to exactly one vertex of X, let A_2 be the vertices in A adjacent to exactly two vertices of X and let A_3 be the vertices in A adjacent to all vertices of X. The minimum degree condition implies that each $v \in A_1$ must be adjacent to at least two other vertices in A and each $w \in A_2$ must be adjacent to at least one other vertex in A. So we have at most a minimum edge count as follows:

$$\begin{aligned}
|E(G)| &\geq 3 + |A_1| + 2|A_2| + 3|A_3| + \left\lceil \tfrac{2|A_1|+|A_2|}{2} \right\rceil \\
&\geq 3 + 2|A_1| + 2|A_2| + \left\lceil \tfrac{|A_2|}{2} \right\rceil + 3|A_3| \\
&= 3 + 2(n - |A_3| - 4) + \left\lceil \tfrac{|A_2|}{2} \right\rceil + 3|A_3| \\
&= 2n - 5 + \left\lceil \tfrac{|A_2|}{2} \right\rceil + |A_3|.
\end{aligned}$$

If either A_2 or A_3 is non-empty, we are done. Thus, assume that $|A_2| = |A_3| = 0$. Then $|E(G)| \geq 2n-5$ and it remains to show that there is at least one additional edge in G.

If at least one of the edges xy, yz, xz is in $E(G)$, we are done. Assume that xy, yz, and xz are not edges of G. Since $\delta(G) = 3$, there must be at least two vertices of A_1 adjacent to x, two vertices of A_1 adjacent to y and two vertices of A_1 adjacent to z. Also, to maintain $\mathrm{diam}(G) = 2$, each vertex adjacent to x must be adjacent to at least one vertex adjacent to y and at least one vertex adjacent to z. Similarly, each vertex adjacent to y must be adjacent to at least one vertex adjacent to x and one vertex adjacent to z and each vertex adjacent to z must be adjacent to at least one vertex adjacent to x and one vertex adjacent to y. This requirement allows the minimum possible edge count to remain at $|E(G)| \geq 2n - 5$ as it requires at least $|A_1|$ edges amongst the vertices of A_1. However, this graph is not F_2-saturated, as it is possible to add xy without creating a copy of F_2, so there must be at least one additional edge. This completes the proof of the lemma. □

Lemma 3.1.4. *Let G be a 2-connected F_2-saturated graph on m edges and $n \geq 5$ vertices. Then $m \geq 2n - 4$.*

Proof. Let G be a 2-connected F_2-saturated graph on m edges and $n \geq 5$ vertices.

Since G is F_2-saturated, $\text{diam}(G) = 2$ by Lemma 3.1.1. Then it follows from Lemma 3.1.3, that $m \geq 2n - 4$ if $\delta(G) \geq 3$. Suppose $\delta(G) = 2$ and let $\deg(z) = 2$ for some $z \in V(G)$. Let z be adjacent to $x, y \in V(G)$ and partition the remaining vertices of G into three sets A, B, C with every $u \in A$ adjacent only to x, every $v \in B$ adjacent to x and y, and every $w \in C$ adjacent only to y as in Figure 3.2. Since G is 2-connected, A and B cannot both be empty, as this would make y a cut vertex. Similarly, both C and B cannot be empty.

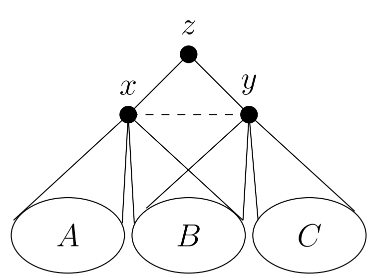

Figure 3.2: Tree structure of 2-connected F_2-saturated graphs with $\delta(G) = 2$

Case 1: Suppose both A and C are empty and B is not empty. The resulting graph is isomorphic to $K_{2,n-2}$, which is F_2-saturated only if we add one edge within one of the independent sets as described above. In this case, $m = 1 + 2(n-2) = 2n-3 > 2n-4$.

Case 2: Suppose that A is empty and B, C are non-empty. Since $\text{diam}(G) = 2$, there must be a path of length 2 from x to all $w \in C$. If $xy \in E(G)$, then B and C

must be independent sets, or a copy of F_2 is created, so $deg(w) = 1$ for every $w \in C$, which violates the minimum degree condition. This implies w must have adjacencies in B, however this creates a copy of F_2. Then C must also be empty in this case, which is a contradiction. Now suppose that $xy \notin E(G)$. To maintain $\text{diam}(G) = 2$, there must be an edge from at least one $v \in B$ to a vertex $w \in C$ for all $w \in C$. Since G cannot contain a copy of F_2, each $w \in C$ must be adjacent to the same vertex in B, otherwise a copy of F_2 is created with the two edge disjoint triangles sharing the vertex y. Additional edges will increase the edge count so G must have an edge count of at least:

$$\begin{aligned} m &\geq 2 + 2|B| + 2|C| \\ &= 2 + 2(n - |C| - 3) + 2|C| \\ &= 2n - 2|C| + 2|C| + 2 - 6 \\ &= 2n - 4. \end{aligned}$$

Note that a similar argument holds when C is empty and A is nonempty.

Case 3: Suppose that A and C are both non-empty with $1 \leq |A| \leq |C|$ and B is empty. By Lemma 3.1.1, $\text{diam}(G) = 2$. If $xy \in E(G)$, there can be no edges within A or C. However, all edges between A and C can be added without creating an F_2. As such, G must have an edge count of at least:

$$\begin{aligned} m &\geq 3 + |A| + |C| + |A||C| \\ &= 3 + (n - |C| - 3) + |C| + (n - |C| - 3)|C| \\ &= n + n|C| - |C|^2 - 3|C| \\ &= 2n - 4 + n(|C| - 1) - (|C|^2 + 3|C| - 4) \\ &\geq 2n - 4 + (n - |C| - 4)(|C| - 1) \\ &\geq 2n - 4. \end{aligned}$$

Now suppose that $xy \notin E(G)$. Then there are several possibilities. If there are no edges within A or C, then all edges exist between A and C without creating an F_2. Hence, this graph is not yet F_2-saturated. Any additional edge gives an edge count as large as the previous count.

Now suppose that there are edges within A or C or both. Any edges in A cannot be independent, as this creates a copy of F_2, and similarly any edges in C cannot be independent, thus A and C contain at most a star (which can span the set). So A and C contain at least $|A|-1$ and $|C|-1$ independent vertices, respectively. Let A have t vertices not in a star and let C have s vertices not in a star. The edge between the center of the star in A and the center of the star in C can exist without creating an F_2. All edges between the independent vertices of A and C are possible without creating an F_2. Further, the t independent vertices in A can be adjacent to the center of the star in C without creating a copy of F_2 and similarly the s independent vertices of C can be adjacent to the center of the star in A. As such, G must have an edge count as follows:

$$\begin{aligned} m &\geq 2+|A|+|C|+1+(|A|-1)(|C|-1)+(|A|-1-t)+(|C|-1-s)+t+s \\ &= 1+2|A|+2|C|+(|A|-1)(|C|-1) \\ &= 2(|A|+|C|+3)-5+(|A|-1)(|C|-1) \\ &= 2n-5+(|A|-1)(|C|-1) \\ &\geq 2n-4 \leftrightarrow |A|>1. \end{aligned}$$

If $|A|=1$, then $u \in A$ is adjacent to every vertex in C. The resulting graph is not F_2-saturated, however, since adding the edge xy does not create a copy of F_2, hence we have $m \geq 3+2(n-4)+1 = 2n-4$ edges in this case and similarly for $|C|=1$. This concludes the case and all cases for $n=5$.

Case 4: Suppose that A, B and C are non-empty with $1 \leq |A| \leq |C|$. Since diam$(G) = 2$, there is a path or length one or length two between vertices in A and vertices in C. A path of length two between vertex in A and a vertex in C that includes any vertex in B will create a copy of F_2, so each vertex in B can at most have edges to A or C but not both. As such, there must be a path of length at most two from each vertex in A to each vertex in C that are contained in $A \cup C$. If $xy \in E(G)$, the vertices of A, B and C must be independent. However, it is possible to add all edges between A and C without creating an F_2. Then G must have the minimum edge count:

$$\begin{aligned} m &\geq 3 + |A| + 2|B| + |C| + |A||C| \\ &= 3 + 2(n - |A| - |C| - 3) + |A| + |C| + |A||C| \\ &= 2n + |A||C| - |A| - |C| - 3 \\ &= 2n - 4 + (|A| - 1)(|C| - 1) \\ &\geq 2n - 4. \end{aligned}$$

Now suppose $xy \notin E(G)$. If there A and C contain no edges, then all edges exist between A and C without creating an F_2. Hence, this graph is not yet F_2-saturated. Any additional edge gives an edge count as large as the previous count and we are done.

Now suppose that there are edges within A or C or both. Edges in A cannot be independent, as this creates a copy of F_2, and similarly in C, thus A and C contain at most a star (which can span the set). So A and C contain at least $|A| - 1$ and $|C| - 1$ independent vertices, respectively. Let A have t vertices not in a star and let C have s vertices not in a star. The edge between the center of the star in A and the center of the star in C can exist without creating an F_2. All edges between the independent vertices of A and C can exist without creating a copy of F_2. Further, the t independent vertices in A can be adjacent to the center of the star in C without creating a copy of F_2 and similarly the s independent vertices of C can be adjacent to the center of the star in A. Then the graph G has an edge count of:

$$\begin{aligned} m &\geq 3+|A|+2|B|+|C|+(|A|-1)(|C|-1)+|A|-1-t+|C|-1-s+t+s \\ &= 1+2(|A|+|B|+|C|+3)-6+(|A|-1)(|C|-1) \\ &= 2n-5+(|A|-1)(|C|-1) \\ &\geq 2n-4 \leftrightarrow |A|>1. \end{aligned}$$

If $|A|=1$, there are several possibilities. The vertices in C can form a star in C. This gives an edge count of $m \geq 3+2|B|+|C|+|C|-1=2n-1$. The vertices in C can be independent, however the edge xy and all edges from A to C can be added without creating a copy of F_2, which yields an edge count of $m \geq 4+2|B|+2|C| = 4+2(n-4)=2n-4$. There can be edges from vertices in C to one vertex (the same one) in B, however there must also be an edge between A and C since $\mathrm{diam}(G)=2$. Such a graph would have an edge count of $m \geq 3+2|B|+2|C|+1=2n-4$. Or there can be a combination of some vertices in C in a star and some independent, which gives the same edge count. This completes the proof of the lemma. □

Note that $ext(n,F_2)=\lceil \frac{n}{2} \rceil \lfloor \frac{n}{2} \rfloor +1$ with the extremal graph, a balanced bipartite graph with one additional edge, and $sat(n,F_2)=n+2$.

Theorem 3.1.1. *There exists an F_2-saturated graph on $n \geq 6$ vertices and m edges where $2n-4 \leq m \leq \lceil \frac{n}{2} \rceil \lfloor \frac{n}{2} \rfloor - \lfloor \frac{n}{2} \rfloor +2$.*

Proof. We expand the graph C_5 such that each vertex of C_5 becomes a set of independent vertices with adjacencies according to the original C_5, that is, where an edge xy becomes a $K_{s,t}$, when $x \in V(C_5)$ expands to a set of s vertices and $y \in V(C_5)$ expands to a set of t vertices. Then we say that graph $C_5[A,C,B,D,E]$ is an expanded C_5

with each vertex set A, B, C, D, E an independent set.

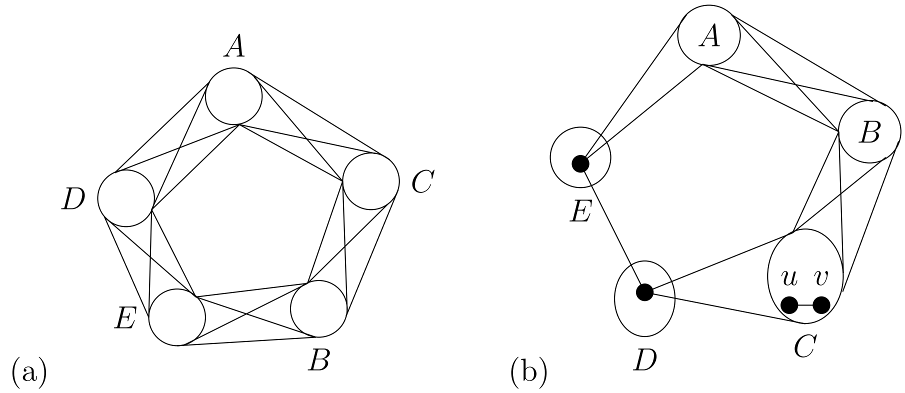

Figure 3.3: (a) Blowup of C_5; (b) Construction of F_2-saturated graphs

The expanded C_5 graph in Figure 3.3(b), which we denote $G = C_5[A, B, C, D, E]$, with $|E| = 1$ and exactly one edge $e = uv$ for some $u, v \in V(C)$, is F_2-saturated with $|A| = n - b - c - d - 1$ provided $|B| = b \geq 1$, $|C| = c \geq 2$, $|D| = d \geq 1$ and $a + c \geq 4$. To see that this is true, we note that since one edge $e = uv$ is added in C, each vertex $w \in B$ is in a triangle $\{u, v, w\}$, each triangle sharing the edge e. Then an additional edge a_1a_2 within A would create a copy of F_2 with the triangle $\{a_1, a_2, w\}$ and $\{u, v, w\}$ for some $w \in B$. An additional edge b_1b_2 in B would create a copy of F_2 with the triangle $\{b_1, b_2, u\}$ and $\{u, v, d_1\}$ for $d_1 \in D$. Also, adding an edge d_1d_2 in D would create a copy of F_2 with the triangle $\{d_1, d_2, u\}$ and $\{u, v, w\}$ for $w \in B$. Finally, adding an edge between any two non-adjacent vertex sets is easily seen to create an F_2. Thus, G is F_2-saturated with edge count $|E(G)| = m$ given by the products of the orders of consecutive vertex sets such that:

$$\begin{aligned} m &= (n - b - c - d - 1)b + bc + cd + 1 + d + (n - b - c - d - 1) \\ &= bn - b^2 - bd - 2b + cd + n - c \\ &= (n - b)(b + 1) - b(d + 1) + c(d - 1). \end{aligned}$$

Then for $d = 2$, $m = (n - b)(b + 1) - 3b + c$. Hence, for fixed values of b, when c increases by 1, as vertices are moved from A to C, the edge count increases by 1.

To maintain the required number of vertices in each set of the expanded C_5, we must have $c \in [2, n-b-5]$. So for a fixed value of b, we can create an F_2-saturated graph having an edge count in $[(n-b)(b+1)-3b+2, (n-b)(b+2)-3b-5]$. If we let $c = n-b-5$ for fixed n, then we have $m = bn+2n-b^2-5b-5$, which is maximized when $b = \lfloor \frac{n-5}{2} \rfloor$. Since the function calculating the edge count increases until $|B|$ and $|C|$ are approximately the same before decreasing, the construction only produces unique edge values for $b \in [1, \lfloor \frac{n-5}{2} \rfloor]$.

Then the smallest edge count for an F_2-saturated $C_5[A, B, C, D, E]$ is given when $b = 1, c = 3, d = 1$ and is $m = (n-1)(2) - 1(2) + 3(0) = 2n-4$. The largest count is given when $a = 1, b = \lfloor \frac{n}{2} \rfloor - 2, c = \lceil \frac{n}{2} \rceil - 2, d = 2$ and is:

$$\begin{aligned} m &= \left(n - \lfloor \tfrac{n}{2} \rfloor + 2\right)\left(\lfloor \tfrac{n}{2} \rfloor - 2 + 1\right) - 3\left(\lfloor \tfrac{n}{2} \rfloor - 2\right) + \left(\lceil \tfrac{n}{2} \rceil - 2\right) \\ &= \left(\lceil \tfrac{n}{2} \rceil + 2\right)\left(\lfloor \tfrac{n}{2} \rfloor - 1\right) - 3\lfloor \tfrac{n}{2} \rfloor + \lceil \tfrac{n}{2} \rceil + 4 \\ &= \lceil \tfrac{n}{2} \rceil \lfloor \tfrac{n}{2} \rfloor - \lceil \tfrac{n}{2} \rceil + 2\lfloor \tfrac{n}{2} \rfloor - 2 - 3\lfloor \tfrac{n}{2} \rfloor + \lceil \tfrac{n}{2} \rceil + 4 \\ &= \lceil \tfrac{n}{2} \rceil \lfloor \tfrac{n}{2} \rfloor - \lfloor \tfrac{n}{2} \rfloor + 2. \end{aligned}$$

When $d = 2$, as c increases by 1, we have F_2-saturated graphs with an edge count in the intervals

$$[2n-3, 3n-10], [3n-10, 4n-18], [4n-19, 5n-28], [5n-30, 6n-40], \cdots,$$

$$\left[\left\lceil \frac{n}{2} \right\rceil \left\lfloor \frac{n}{2} \right\rfloor - n + 5, \left\lceil \frac{n}{2} \right\rceil \left\lfloor \frac{n}{2} \right\rfloor - \left\lfloor \frac{n}{2} \right\rfloor + 2\right], \left[\left\lceil \frac{n}{2} \right\rceil \left\lfloor \frac{n}{2} \right\rfloor - n + 6, \left\lceil \frac{n}{2} \right\rceil \left\lfloor \frac{n}{2} \right\rfloor - \left\lfloor \frac{n}{2} \right\rfloor + 2\right]$$

It is clear that the $\lfloor \frac{n}{2} \rfloor - 2$ intervals overlap however, since the difference between the minimum values of any two sequential intervals is $[(n-b-1)(b+1+1) - 3(b+1)+2] - [(n-b)(b+1) - 3b+2] = n-2b-5$ and the length of each interval is $[(n-b)(b+2) - 3b-4] - [(n-b)(b+1) - 3b+2] + 1 = n-b-5$, any interval will overlap with the previous interval in the first $b-1$ numbers. As there are $\lfloor \frac{n}{2} \rfloor - 2$

intervals, each with $(n-b-5)-(b-1)=n-2b-4$ distinct elements, the entire interval is covered as b and c vary since:

$$\begin{aligned}
\textstyle\sum_{b=1}^{\lfloor \frac{n}{2}\rfloor -2}(n-2b-4) &= (n-4)\left(\lfloor \tfrac{n}{2}\rfloor -2\right)-\left(\lfloor \tfrac{n}{2}\rfloor -2+1\right)\left(\lfloor \tfrac{n}{2}\rfloor -3+1\right)\\
&= n\lfloor \tfrac{n}{2}\rfloor -2n-4\lfloor \tfrac{n}{2}\rfloor +8-\lfloor \tfrac{n}{2}\rfloor\lfloor \tfrac{n}{2}\rfloor +2\lfloor \tfrac{n}{2}\rfloor -2\\
&= \left(n-\lfloor \tfrac{n}{2}\rfloor\right)\lfloor \tfrac{n}{2}\rfloor -2n-\lfloor \tfrac{n}{2}\rfloor +6\\
&= \lceil \tfrac{n}{2}\rceil\lfloor \tfrac{n}{2}\rfloor -\lfloor \tfrac{n}{2}\rfloor +2-(2n-3)+1.
\end{aligned}$$

This completes the proof of the theorem. □

3.2 Constructing F_3-saturated Graphs

We can construct F_3-saturated graphs similarly to how we construct F_2-saturated graphs with an expanded $C_5[A,B,C,2,1]$. However, in place of the edge $uv \in C$ from the F_2-construction, we need a C_4 as it has two vertex disjoint edges inducing a copy of F_2. This graph is F_3-saturated when $|A|\geq 1, |B|\geq 1$ and $|C|\geq 4$.

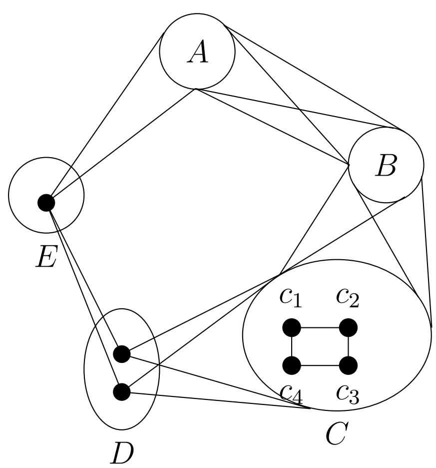

Figure 3.4: Construction of F_3-saturated graphs

Note that each vertex $b\in B$ is in two edge disjoint triangles as the C_4 has two vertex disjoint edges as an induced subgraph, say $\{c_1,c_2,b\}$ and $\{c_3,c_4,b\}$. Then an additional edge a_1a_2 within A would create a copy of F_3 with the triangles $\{a_1, a_2, b\}$, $\{c_1, c_2, b\}$ and $\{c_3, c_4, b\}$ for some $b \in B$. Since each vertex in the C_4 is the shared vertex of an induced copy of F_2, an additional edge in B, say b_1b_2, would create a copy of F_3 with triangles $\{b_1, b_2, c_1\}$, $\{c_1, c_2, d_1\}$ and $\{c_1, c_4, d_2\}$ for $d_1, d_2 \in D$. If $|B| \geq 2$, adding an edge in D similarly creates a copy of F_3. If $|B| = 1$, however, adding an edge d_1d_2 in D would still create a copy of F_3 with the triangles $\{d_1, d_2, e\}$ $\{c_1, c_2, d_1\}$ and $\{c_3, c_4, d_1\}$ for $e \in E$. Adding an edge from C to A creates a triangle $\{a, b, c\}$ for $a \in A, b \in B$ and $c \in C$, and if $c \notin \{c_1, c_2, c_3, c_4\}$ we have a copy of F_3 with three edge disjoint triangles sharing b while $c \in \{c_1, c_2, c_3, c_4\}$ creates a copy of F_3 with triangles sharing c. Similarly for adding edges from C to E or from B to D. Finally, adding an edge between B and E or A and D is easily seen to create a copy of F_3. Thus, the graph is F_3-saturated with edge count m given by the products of the orders of consecutive vertex sets as follows:

$$\begin{aligned} m &= (n-b-c-3)b + bc + 2c + 2 + (n-b-c-3) + 4 \\ &= bn - b^2 - 4b + n + c + 3 \\ &= (n-b)(b+1) - 3b + c + 3. \end{aligned}$$

Hence, for fixed values of b, when c increases by 1, as vertices are moved from A to C, the edge count increases by 1. To maintain the required number of vertices in each set of the expanded C_5, we must have $c \in [4, n-b-4]$. So for a fixed value of b, we can create an F_3-saturated graph having an edge count in $[(n-b)(b+1) - 3b + 7, (n-b)(b+2) - 3b - 1]$. If we let $c = n - b - 4$ for fixed n, then we have $m = bn + 2n - b^2 - 5b - 2$, which is maximized when $b = \lfloor \frac{n-5}{2} \rfloor$ such that $b \in [1, \lfloor \frac{n-5}{2} \rfloor]$ similarly to the previous construction.

Then the smallest edge count for an F_3-saturated $C_5[A, B, C, 2, 1]$ is given when $b = 1$ and $c = 4$ and is $m = (n-1)(2) - 3(1) + 4 + 3 = 2n + 2$. The largest count is given when $a = 1, b = \lfloor \frac{n}{2} \rfloor - 2, c = \lceil \frac{n}{2} \rceil - 2$ and is:

$$\begin{aligned} m &= \left(n - \lfloor \tfrac{n}{2} \rfloor + 2\right)\left(\lfloor \tfrac{n}{2} \rfloor - 2 + 1\right) - 3\left(\lfloor \tfrac{n}{2} \rfloor - 2\right) + \left(\lceil \tfrac{n}{2} \rceil - 2\right) + 3 \\ &= \left(\lceil \tfrac{n}{2} \rceil + 2\right)\left(\lfloor \tfrac{n}{2} \rfloor - 1\right) - 3\lfloor \tfrac{n}{2} \rfloor + \lceil \tfrac{n}{2} \rceil + 7 \\ &= \lceil \tfrac{n}{2} \rceil \lfloor \tfrac{n}{2} \rfloor - \lceil \tfrac{n}{2} \rceil + 2\lfloor \tfrac{n}{2} \rfloor - 2 - 3\lfloor \tfrac{n}{2} \rfloor + \lceil \tfrac{n}{2} \rceil + 7 \\ &= \lceil \tfrac{n}{2} \rceil \lfloor \tfrac{n}{2} \rfloor - \lfloor \tfrac{n}{2} \rfloor + 5. \end{aligned}$$

If we let $|A| = n - 8$ and move vertices from A to C such that $|C|$ increases by 1, we have F_3-saturated graphs with an edge count in the intervals

$$[2n + 2, 3n - 7], [3n - 5, 4n - 15], [4n - 14, 5n - 25], [5n - 25, 6n - 37],$$

$$[6n - 38, 7n - 51], [7n - 53, 8n - 67], [8n - 70, 9n - 85], \cdots$$
$$\cdots, \left[\left\lceil \frac{n}{2} \right\rceil \left\lfloor \frac{n}{2} \right\rfloor - n + 11, \left\lceil \frac{n}{2} \right\rceil \left\lfloor \frac{n}{2} \right\rfloor - \left\lfloor \frac{n}{2} \right\rfloor + 5\right].$$

So we have constructed F_3-saturated graphs with an edge count in $[2n+2, \lceil \frac{n}{2} \rceil \lfloor \frac{n}{2} \rfloor - \lfloor \frac{n}{2} \rfloor + 5] - \{3n - 6\}$. The following graph in Figure 3.3.5 has edge count $3n - 6$ and is clearly F_3-saturated for $n \geq 7$, as adding any edge in A will create a triangle that is edge disjoint from the two edge disjoint triangles sharing v.

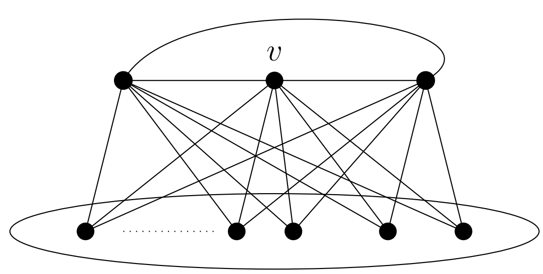

Figure 3.5: An F_3-saturated graph with $m = 3n - 6$

3.3 Constructing F_4-saturated Graphs

We can similarly construct F_4-saturated graphs with a modified $C_5[A, B, C, 2, 1]$. However, in place of the edge $uv \in C_5[A, B, C, 2, 1]$ from the F_2-construction, we have a chorded C_6 with chords such that the degree of each vertex within the cycle is three. This chorded cycle has three independent edges inducing a copy of F_3 with sets B and D and each vertex of the cycle is the shared vertex of an induced copy of F_3. This graph is F_4-saturated when $|A| \geq 1, |B| \geq 2$ and $|C| \geq 6$.

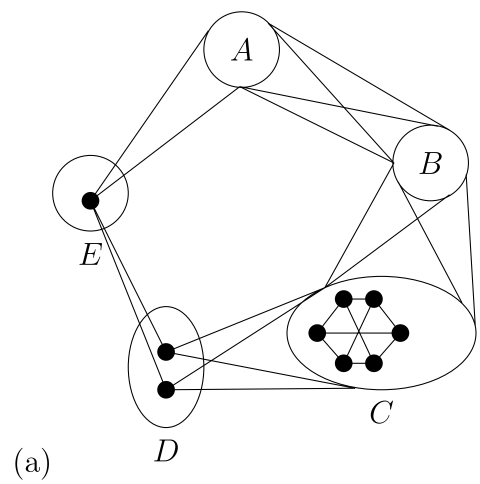

Figure 3.6: Construction of F_4-saturated graphs

Note that the graph is F_4-saturated similarly to the graph in the previous section with edge count m given by the products of the orders of consecutive vertex sets as follows:

$$\begin{aligned} m &= (n - b - c - 3)b + bc + 2c + 2 + (n - b - c - 3) + 9 \\ &= bn - b^2 - 4b + n + c + 8 \\ &= (n - b)(b + 1) - 3b + c + 8. \end{aligned}$$

Hence, for fixed values of b, when c increases by 1, as vertices are moved from A to C, the edge count increases by 1. To maintain the required number of vertices in each set of the expanded C_5, we must have $c \in [6, n - b - 4]$. So for a fixed value of b, we can construct an F_4-saturated graph having an edge count in the interval $[(n-b)(b+1)-3b+14,(n-b)(b+2)-3b+4]$. If we let $c=n-b-4$ for fixed n, then we have $m=bn+2n-b^2-5b+4$, which is maximized when $b=\lfloor\frac{n-5}{2}\rfloor$ such that $b\in[1,\lfloor\frac{n-5}{2}\rfloor]$.

Then the smallest edge count for an F_4-saturated graph is given when $b=2$ and $c=6$ and is $m=(n-2)(3)-3(2)+6+8=3n+2$. The largest count is given when $a=1, b=\lfloor\frac{n}{2}\rfloor-2, c=\lceil\frac{n}{2}\rceil-2$ and is:

$$\begin{aligned}
m &= \left(n-\lfloor\tfrac{n}{2}\rfloor+2\right)\left(\lfloor\tfrac{n}{2}\rfloor-2+1\right)-3\left(\lfloor\tfrac{n}{2}\rfloor-2\right)+\left(\lceil\tfrac{n}{2}\rceil-2\right)+8\\
&= \left(\lceil\tfrac{n}{2}\rceil+2\right)\left(\lfloor\tfrac{n}{2}\rfloor-1\right)-3\lfloor\tfrac{n}{2}\rfloor+\lceil\tfrac{n}{2}\rceil+14\\
&= \lceil\tfrac{n}{2}\rceil\lfloor\tfrac{n}{2}\rfloor-\lceil\tfrac{n}{2}\rceil+2\lfloor\tfrac{n}{2}\rfloor-2-3\lfloor\tfrac{n}{2}\rfloor+\lceil\tfrac{n}{2}\rceil+14\\
&= \lceil\tfrac{n}{2}\rceil\lfloor\tfrac{n}{2}\rfloor-\lfloor\tfrac{n}{2}\rfloor+10.
\end{aligned}$$

If we let $|A|=n-8$ and move vertices from A to C such that $|C|$ increases by 1, we have F_3-saturated graphs with an edge count in the intervals

$$[3n+2,4n-10],[4n-7,5n-20],[5n-18,6n-32],[6n-31,7n-46],$$

$$[7n-46,8n-62],[8n-63,9n-80],[9n-82,10n-100],\cdots$$
$$\cdots,\left[\left\lceil\frac{n}{2}\right\rceil\left\lfloor\frac{n}{2}\right\rfloor-n+18,\left\lceil\frac{n}{2}\right\rceil\left\lfloor\frac{n}{2}\right\rfloor-\left\lfloor\frac{n}{2}\right\rfloor+10\right].$$

So we have constructed F_4-saturated graphs with an edge count in the interval $[3n+2,\lceil\frac{n}{2}\rceil\lfloor\frac{n}{2}\rfloor-\lfloor\frac{n}{2}\rfloor+10]-\{4n-9,4n-8,5n-19\}$. We can partially fill in the missing edge counts using a similar construction for F_4-saturated graphs by altering the chorded cycle as seen below in Figure 3.3.7.

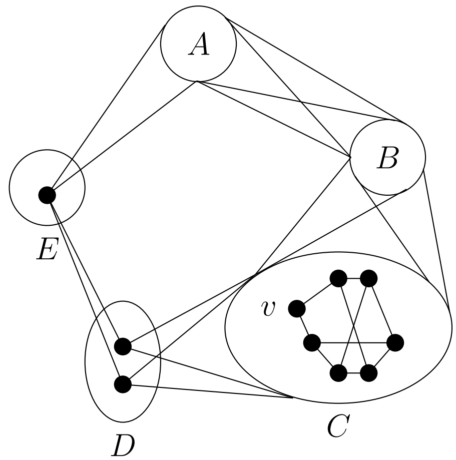

Figure 3.7: Another construction of F_4-saturated graphs

The construction for F_4-saturated graphs shown in Figure 3.3.7 is a modified expanded $C_5[A, B, C, 2, 1]$ similar to the previous construction. However, in place of the the chorded C_6, we have a chorded C_7 with chords such that the degree of a vertex within the cycle is three for each vertex in $V(C_7) - v$. This chorded cycle has three independent edges inducing a copy of F_3 with sets B and D and each vertex of the cycle except v is the shared vertex of an induced copy of F_3. This graph is F_4-saturated when $|A| \geq 1, |B| \geq 2$ and $|C| \geq 7$ with the following edge count:

$$\begin{aligned} m &= (n-b-c-3)b + bc + 2c + 2 + (n-b-c-3) + 10 \\ &= bn - b^2 - 4b + n + c + 9 \\ &= (n-b)(b+1) - 3b + c + 9. \end{aligned}$$

So when $a = 1, b = 2$ and $c = n - 6$, $m = 3(n-2) + n - 3 = 4n - 9$ and for $a = 1, b = 3, c = n - 7, m = 4(n-3) + n - 7 = 5n - 19$.

We cannot currently determine a construction for a F_4-saturated graph with an edge count of $4n - 8$. There has not yet been a saturation spectrum with a seemingly continuous interval having a gap at a single value. We pose the following interesting question:

Question 3.3.1. *Does there exist an F_4-saturated graph on n vertices with an edge count of $4n-8$?*

3.4 Constructing F_t-saturated Graphs

We can generalize the two constructions for F_4-saturated graphs to construct F_t-saturated graphs. First, however, we consider the extremal number and saturation number.

The extremal number for F_t-saturated graphs is given in the following theorem:

Theorem 3.4.1. *[4] For every $t \geq 1$, and for every $n \geq 50t^2$, if a graph G on n vertices has more than*

$$\left\lfloor \frac{n^2}{4} \right\rfloor + \begin{cases} t^2 - t & \text{if } t \text{ is odd} \\ t^2 - \frac{3}{2}t & \text{if } t \text{ is even} \end{cases}$$

edges, then G contains a copy of the t-fan. Furthermore, the number of edges is best possible.

The saturation number for F_t-saturated graphs is determined in the following theorem:

Theorem 3.4.2. *Let G be a graph on $n \geq 3t-2$ vertices, $sat(n, F_t) = n + t - 2$.*

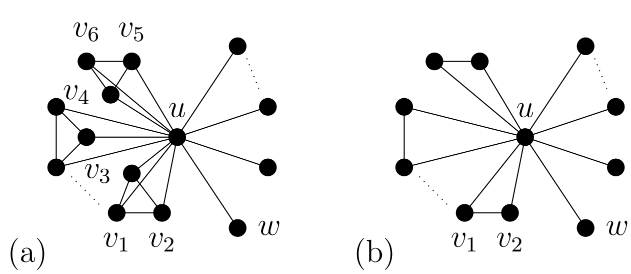

Figure 3.8: (a) Upper bound for $sat(n, F_t)$; (b) Lower bound for $sat(n, F_t)$

Proof. The graph in Figure 3.8(a) comprised of $t-1$ copies of K_4 and $n-3t+2$ edges intersecting at a single vertex u is F_t-saturated. The vertex u is shared by $t-1$ edge disjoint triangles and adding an edge between two vertices not in a K_4 creates an additional triangle sharing u that is edge disjoint from all others, so it yields a copy of F_t. Adding an edge between two copys of K_4, say v_1v_6, creates the three edge disjoint triangles $\{u, v_1, v_6\}, \{u, v_2, v_3\}$ and $\{u, v_4, v_5\}$ that, together with the remaining original $t-3$ edge disjoint triangles, creates a copy of F_t. Finally, adding an edge between a copy of K_4 and a vertex not in a K_4, say v_2w, creates the two edge disjoint triangles $\{u, v_2, w\}$ and $\{u, v_1, v_3\}$ that, together with the remaining original $t-2$ edge disjoint triangles, creates a copy of F_t. So we have that the minimum edge count for an F_t-saturated graph is at most $m = n-1+4(t-1)-3(t-1) = n+t-2$.

On the other hand, it is impossible to add one edge and create two edge disjoint triangles, so an F_t-saturated graph must already contain exactly $t-1$ edge disjoint triangles that share a single vertex. Also, as in Lemma 3.1.1, in order for an added edge to create a triangle, the graph must have diameter two. So we have a graph comprised of $t-1$ edge disjoint triangles and $n-t$ disjoint edges all sharing a single vertex u as in Figure 3.8(b). So the minimum edge count for an F_t-saturated graph on n vertices is at least $m = 3(t-1)+n-2(t-1)-1 = n+t-1$. □

Theorem 3.4.3. *If $t \geq 4$, there exists an F_t-saturated graph with $n \geq 3t$ vertices and edge count m where*

$$\left(\tfrac{3t}{2}-1\right)n - \left\lfloor \tfrac{3t}{2} \right\rfloor \left\lceil \tfrac{3t}{2} \right\rceil + t^2 - 4 \leq m \leq \left\lceil \tfrac{n}{2} \right\rceil \left\lfloor \tfrac{n}{2} \right\rfloor - \left\lfloor \tfrac{n}{2} \right\rfloor + (t-1)^2 + \left\lfloor \tfrac{t}{2} \right\rfloor .$$

Proof. Each expanded $C_5[A, B, C, 2, 1]$ that is F_2-saturated can be made F_t-saturated by replacing the edge $uv \in C$ with a chorded cycle $\hat{C}$ on $2t-2$ vertices. The chords of the cycle $\hat{C}$ must be distributed amongst the vertices such that each vertex in $\hat{C}$ is adjacent to exactly $t-1$ other vertices in $\hat{C}$.

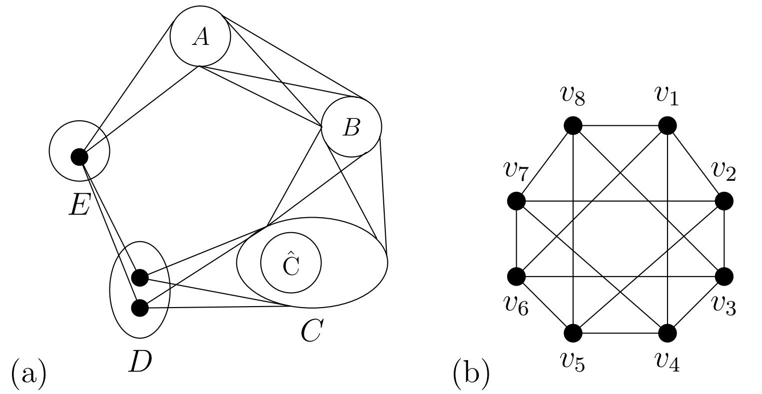

Figure 3.9: (a) Construction of F_t-saturated graphs;(b) Example of $\hat{C}$ for $n = 5$

One way to distribute the chords when t is is seen in Figure 3.7(b) for $t = 5$. In the cycle $\hat{C}$ we label the vertices clockwise $v_1, v_2, ..., v_{2t-2}$. Then we add the edge v_iv_j, if the distance between v_i and v_j is exactly k where $k = 3, 4, ..., \lceil \frac{t}{2} \rceil$ and, when t is even, $k = t - 1$. In this way, each vertex in $\hat{C}$ is adjacent to $t - 1$ other vertices of $\hat{C}$ so each $u \in \hat{C}$ is in exactly $t - 1$ edge disjoint triangles $\{uv, uw, vw\}$ where $v \in \hat{C}$ and w is a vertex in B or D.

We call the graph obtained by making this modification $\hat{C}_5[A, B, C, 2, 1]$ and it is F_t-saturated for $|A| = a \geq 1$, $|B| = b \geq t-2$ and $|C| = c \geq 2t-2$. In $\hat{C}_5[A, B, C, 2, 1]$, each vertex in B and D is a shared vertex of $t-1$ edge disjoint triangles with vertices from Ĉ. Also, each vertex of Ĉ is a shared vertex of $t - 1$ edge disjoint triangles. So adding an edge between A and D (or B and E) will create a triangle that is edge disjoint from all preexisting triangles sharing a vertex of D (or B) so a copy of F_t is created on that vertex. Adding an edge between A and C (or E and C) will create a triangle that is edge disjoint from $t-1$ preexisting triangles that either share a vertex in Ĉ or in B (or D), so a copy of F_t is created on that vertex. Similarly, adding an edge between B and D will create a triangle with a vertex in C, which either creates a copy of F_t sharing a vertex in B or a vertex in Ĉ. Also, if an edge is added between independent vertices of A, B, C or D, then we create a triangle that is edge disjoint from $t - 1$ triangles that share a vertex in B or D, so a copy of F_t is created on that vertex. Finally, since each vertex of Ĉ is a vertex of $t-1$ edge disjoint triangles with vertices in B and D, adding another edge within the cycle will create a copy of F_t is created on either of the vertices incident to that edge.

In general, $\hat{C}_5[A,B,C,2,1]$ will have $m=(n-b)(b+1)-3b+c+(t-1)^2-1$ edges. For fixed values of b, when c increases by 1, as vertices are moved from A to C, the edge count increases by 1. To maintain the required number of vertices in each set of the graph, we must have $c \in [2t-2, n-b-4]$. So for a fixed value of b and n large enough, we can create an $F_t$saturated graph having an edge count in $[(n-b)(b+1)-3b+t^2-2, (n-b)(b+1)+n-4b+(t-1)^2-5]$. If we let $c = n-b-4$ for fixed n, then we have $m = bn+2n-b^2-5b+t^2-2t-4$, which is maximized when $b=\lfloor\frac{n-5}{2}\rfloor$ such that the graphs from the construction have unique edge counts for $b \in [t-2, \lfloor\frac{n-5}{2}\rfloor]$.

Then the smallest edge count for an F_t-saturated $\hat{C}_5[A,C,B,2,1]$ on $n \geq 3t$ vertices is given when $b=t-2$ and $c=2t-2$ and is $m=(n-t+2)(t-1)-3(t-2)+2t-2+(t-1)^2=n(t-1)+3$. The largest count is given when $a=1, b=\lceil\frac{n}{2}\rceil-2$, and $c=\lfloor\frac{n}{2}\rfloor-2$ and is:

$$\begin{aligned} m &= \left(\left\lfloor\tfrac{n}{2}\right\rfloor+2\right)\left(\left\lceil\tfrac{n}{2}\right\rceil-1\right)-3\left(\left\lceil\tfrac{n}{2}\right\rceil-2\right)+\left\lfloor\tfrac{n}{2}\right\rfloor-2+(t-1)^2-1 \\ &= \left\lceil\tfrac{n}{2}\right\rceil\left\lfloor\tfrac{n}{2}\right\rfloor-\left\lceil\tfrac{n}{2}\right\rceil+(t-1)^2+1. \end{aligned}$$

If we let $|A|=n-t-1$ and move vertices from A to C such that $|C|$ increases by 1, we have F_t-saturated graphs with an edge count in the following intervals:

$$[nt-n+2, nt-3t+2], [nt-2t+1, nt-5t+n], [nt+n-4t-2, nt+2n-7t-4], \cdots$$

$$[2nt-3n-3t^2+8t-2, 2nt-2n-3t^2+4t], [2nt-2n-3t^2+4t+1, 2nt-n-3t^2+2],$$

$$[2nt-n-3t^2+2, 2nt-3t^2-4t+2], \cdots$$

$$\left[\left\lceil\frac{n}{2}\right\rceil\left\lfloor\frac{n}{2}\right\rfloor - 2n + 2\left\lceil\frac{n}{2}\right\rceil + t^2 + 1, \left\lceil\frac{n}{2}\right\rceil\left\lfloor\frac{n}{2}\right\rfloor - \left\lfloor\frac{n}{2}\right\rfloor + t^2 - 2t + 2\right],$$

$$\left[\left\lceil\frac{n}{2}\right\rceil\left\lfloor\frac{n}{2}\right\rfloor - n + t^2 + 2, \left\lceil\frac{n}{2}\right\rceil\left\lfloor\frac{n}{2}\right\rfloor - \left\lceil\frac{n}{2}\right\rceil + t^2 - 2t + 2\right].$$

Note that for the first $t-2$ intervals, there is a gap in the edge counts of length $2t-b-4$, which is the distance between the end of an interval and the beginning of the next consecutive interval. However, once $b \geq t-2$ the intervals begin to overlap. To partially fill this gap we use a modification of the previous construction.

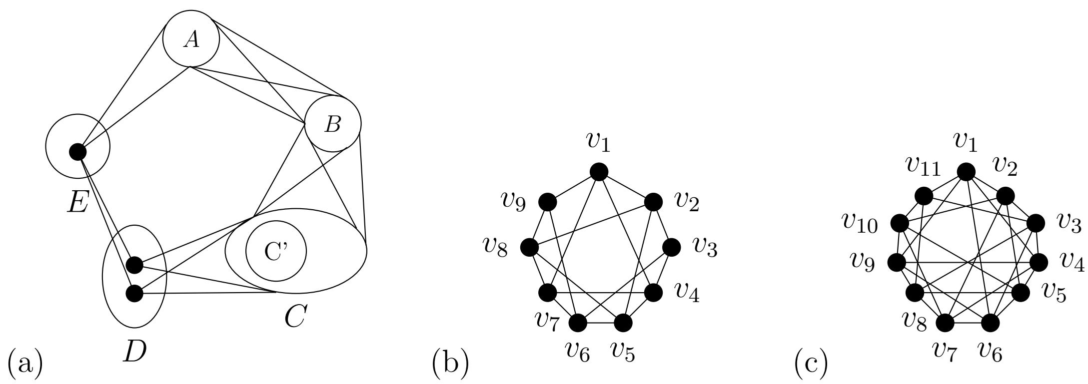

Figure 3.10: (a) Construction;(b) Example C' for $n = 5$; (c) Example C' for $n = 6$

We take each expanded $C_5[A, B, C, 2, 1]$ that is F_2-saturated and make it F_t-saturated by modifying the edge $uv \in C$ into a chorded cycle C' on $2t-1$ vertices. If t is even, we distribute the chords of the cycle C' amongst the vertices such that all but one vertex, say v, in C' is adjacent to exactly $t-1$ other vertices in C' and v is adjacent to exactly $t-2$ vertices in C'. If t is odd, we distribute the chords of the cycle C' amongst the vertices such that all but two vertices, say u and w in C', are adjacent to exactly $t-1$ other vertices in C' where u and w are distance exactly two apart on the cycle and adjacent to exactly $t-2$ vertices in C'.

We call this graph $C_5'[A, B, C, 2, 1]$ and it is F_t-saturated for $|A| = a \geq 1, |B| = b \geq t-2$ and $|C| = c \geq 2t-1$. The only difference between this new construction and the previous construction are the vertices v, u and w so we only need to consider how they impact the saturation of the graph. We need only consider the vertex v, as u and v behave similarly. If an edge is added from v to A, then a copy of F_t is created with a vertex of B as the shared vertex and the $t-1$ edge disjoint triangles from the $t-1$ disjoint edges in C'. Similarly, adding an edge from v to E will create a copy of F_t with a vertex from D as the shared vertex. If an edge is added between v and another vertex in C that is not in the cycle, then we have an edge that is disjoint from the $t-1$ disjoint edges in C' so a copy of F_t is created with a vertex from B or D as the shared vertex. Finally, if an edge is added between v and any other vertex in C', that vertex will now have degree t within C' and be the shared vertex of copy of F_t utilizing vertices from B and D to form the edge disjoint triangles.

In general $C_5'[A, B, C, 2, 1]$ will have an edge count given by the following:

$$m = (n-b)(b+1) - 3b + c - 1 + (t-1)^2 + \begin{cases} \frac{1}{2}(t-3) & \text{if } t \text{ is odd} \\ \\ \frac{1}{2}(t-2) & \text{if } t \text{ is even} \end{cases}.$$

Then $|E(C_5'[A, B, C, 2, 1])| = |E(\hat{C}_5[A, B, C, 2, 1])| + \left\lfloor \frac{t}{2} \right\rfloor - 1$ and we see that this modification yields intervals of edge counts that are shifts of the edge intervals given by the construction of $\hat{C}[A, B, C, 2, 1]$, shifting them up by $\frac{t}{2} - 1$. This means that once the gap between consecutive intervals given by $\hat{C}_5[A, B, C, 2, 1]$ is less than $\frac{t}{2}$, $C_5'[A, B, C, 2, 1]$ will yield edge counts to fully fill in the gaps. In particular, when $b \geq \left\lceil \frac{3t}{2} \right\rceil - 2$ the edge counts of the construction $C_5'[A, B, C, 2, 1]$ will fill in the gap in the edge counts of $\hat{C}_5[A, B, C, 2, 1]$.

This completes the proof. $\square$

Chapter 4

Future Work

Determining the saturation number for a specific graph is a difficult problem. The question remains unanswered for many interesting graphs, including cycles C_n for $n \geq 6$ where an approximate solution is known but exact values have yet to be determined. There are, however, several graphs for which the saturation number is known but the saturation spectrum has not yet been explored. It would be interesting, next, to explore the saturation spectrum for the book graph, small cycles, generalized book graphs and complete bipartite graphs.

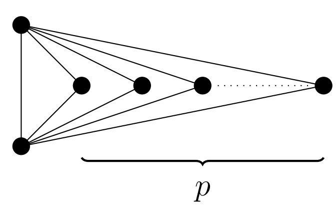

Figure 4.1: A book graph, B_p

A *book* B_p is a union of p triangles sharing one edge, as seen in Figure 4.1. The shared edge is called the *base* of the book. In 2008 [4], G. Chen, R. Faudree and R. Gould determined that the saturation number of the book graph in the following theorem:

Theorem 4.0.4. *Let n and p be two positive integers such that $n \geq p^3 + p$. Then*

$$sat(n, B_p) = \frac{1}{2}\left((p+1)(n-1) - \left\lceil \frac{p}{2} \right\rceil \left\lfloor \frac{p}{2} \right\rfloor + \theta(n,p)\right),$$

$$\text{where } \theta(n,p) = \begin{cases} 1 & if \quad p \equiv n - p/2 \equiv 0 \pmod 2 \\ \\ 0 & \quad otherwise \end{cases}.$$

In particular, $sat(n, B_3) = \frac{1}{2}(4n - 4 - 2) = 2n - 1$. We establish part of the saturation spectrum of B_3 in the following theorem:

Theorem 4.0.5. *For $n \geq 12$, there is a B_3-saturated graph G with an edge count*

$$|E(G)| \in \left[4n - 15, \left\lceil \frac{n}{2} \right\rceil \left\lfloor \frac{n}{2} \right\rfloor - \left\lfloor \frac{n}{2} \right\rfloor + 3\right].$$

Proof. The blown-up C_5 graph G in Figure 4.2 is a general construction that yields values in the saturation spectrum of B_3-saturated graphs. The graph G is B_3-saturated with $|A| = n - b - c - 3$ provided $|B| = b \geq 3$, $|C| = c \geq 3$, $|D| = 2$ and $|E| = 1$ if we add edges between the two vertices in D and two of the vertices in A.

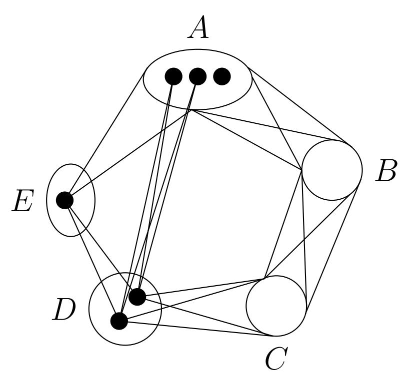

Figure 4.2: A Construction of B_3-saturated graphs

If an edge is added within A, B, C or D, then a copy of B_3 is created with that edge as the base and three vertices from adjacent sets. If an edge uv is added from A to C, from B to D or from B to E a B_3 is created on that edge as the base with three vertices in the shared neighborhood of u and v. If an edge is added from C to E, then a copy of B_3 is created using an edge from D to E as the base, the two vertices from A adjacent to vertices in D and the vertex from C in the new edge. Finally, if an additional edge is added between A and D, a B_3 will be created with an edge from D to E as the base and three vertices from A. Thus, G is B_3-saturated.

The general edge count of the graph m is given by the products of the orders of consecutive vertex sets, hence:

$$\begin{aligned} m &= |A| + |D||E| + |D||C| + |B||C| + |B||A| + 4 \\ &= (n-b-c-3) + 1(2) + 2c + bc + b(n-b-c-3) + 4 \\ &= (n-b) - c - 3 + 6 + 2c - 3b + b(n-b) \\ &= (n-b)(b+1) - 3b + c + 3. \end{aligned}$$

Then for fixed values of c, when b increases by 1, that is, as vertices are moved from A to C, the edge count increases by 1. To maintain the minimum required number of vertices in each set we must have $c \in [3, n-b-6]$. If we let $c = n-b-6$ for fixed n, then we have $m = nb + 2n - b^2 - 5b - 3$, which is maximized when $b = \left\lceil \frac{n}{2} \right\rceil - 2$ such that $b \in [3, \left\lceil \frac{n}{2} \right\rceil - 2]$. Then the smallest edge count for G is when $a = n-9, b = 3, c = 3$ and is $m = (n-3)(4) - 9 + 6 = 4n - 15$ and the largest possible edge count is given when $a = 3, b = \left\lceil \frac{n}{2} \right\rceil - 3, c = \left\lfloor \frac{n}{2} \right\rfloor - 3$ and is:

$$\begin{aligned} m &= \left(n - \left\lceil \tfrac{n}{2} \right\rceil + 3\right)\left(\left\lceil \tfrac{n}{2} \right\rceil - 3 + 1\right) - 3\left(\left\lceil \tfrac{n}{2} \right\rceil - 3\right) + \left(\left\lfloor \tfrac{n}{2} \right\rfloor - 3\right) + 3 \\ &= \left(\left\lfloor \tfrac{n}{2} \right\rfloor + 3\right)\left(\left\lceil \tfrac{n}{2} \right\rceil - 2\right) - 3\left\lceil \tfrac{n}{2} \right\rceil + 9 + \left\lfloor \tfrac{n}{2} \right\rfloor \\ &= \left\lceil \tfrac{n}{2} \right\rceil \left\lfloor \tfrac{n}{2} \right\rfloor - 2\left\lfloor \tfrac{n}{2} \right\rfloor + 3\left\lceil \tfrac{n}{2} \right\rceil - 6 - 3\left\lceil \tfrac{n}{2} \right\rceil + 9 + \left\lfloor \tfrac{n}{2} \right\rfloor \\ &= \left\lceil \tfrac{n}{2} \right\rceil \left\lfloor \tfrac{n}{2} \right\rfloor - \left\lfloor \tfrac{n}{2} \right\rfloor + 3. \end{aligned}$$

We can see that the individual intervals, produced when b is fixed and c varies, overlap such that they span the entire interval $\left[4n-15, \left\lceil \frac{n}{2} \right\rceil \left\lfloor \frac{n}{2} \right\rfloor - \left\lfloor \frac{n}{2} \right\rfloor + 3\right]$. The gap between two consecutive intervals has length $4-b$ and is at least two when $2 \geq b$, which is never true. So we have constructed B_3-saturated graphs with edge counts in $\left[4n-15, \left\lceil \frac{n}{2} \right\rceil \left\lfloor \frac{n}{2} \right\rfloor - \left\lfloor \frac{n}{2} \right\rfloor + 3\right]$. $\square$

It remains to be seen if there are additional values in the saturation spectrum of B_3 and if this construction can be extended to help determine the saturation spectrum of B_p for $p \geq 3$. Another interesting and related problem would be to determine how many copies of H are created with the addition of an edge in an H-saturated graph. Specifically, working to find constructions of H-saturated graphs where adding an edge creates exactly one copy of H, for any interesting graph H, would be a great possible project for the future.

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