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Bar Diagramming in School Mathematics:
A Review of the Literature

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An abstract of
A thesis submitted to the Faculty of the
James T. Laney School of Graduate Studies of Emory University
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Abstract

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This thesis is a review of the extant empirical literature about the school mathematics practices of bar diagramming, strip diagramming, tape diagramming, the Model Method, and bar modeling. For a few decades, policy makers, curriculum designers, and educators in several of the nations that have repeatedly ranked as top performers in mathematics have prioritized the introduction and featuring of these diagrammatic classroom practices. As incorporation of these diagrammatic practices has increased, so has the need to provide stakeholders with additional information about the effective and ineffective handling of diagrams in the classroom. In response to this need, I conducted a review of the empirical literature surrounding these diagrammatic practices. A total of 16 studies across 15 research articles met the literature search criteria. Major findings are as follows: (1) Bar diagrams can represent mathematical concepts in ways that show their connection to other mathematical concepts. (2) Bar diagrams can simultaneously show the informal, qualitative aspects of a concept and the formal, quantitative aspects of the concept. (3) Bar diagrams allow students to become comfortable with operating upon unknowns, undoing operations, and maintaining balance across the equal sign. (4) Habituation to the visual nature of bar diagrams and circumvention of algebra via arithmetic can cause some students problems. Further research about how stakeholders can better employ bar diagramming to develop and deepen students' understanding of problem structure and students' appropriation of algebraic concepts and principles is suggested.

Keywords: bar diagramming, bar modeling, diagrammatic reasoning, instructional technique, mathematics instruction, school mathematics, the Singapore Model Method, strip diagramming, tape diagramming

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Statement of the Problem

At its essence, mathematical thinking consists largely of reasoning about structure and relationship, or diagrammatic reasoning (Bakker & Hoffmann, 2005; Dörfler, 2001, 2005; Hoffmann, 2005, 2007; Hoopes, 1991; Otte, 2011, pp. 276–278). Thus it is perhaps unsurprising that diagrams and diagramming are central features of the mathematics curricula of countries whose students repeatedly rank highest on international mathematics assessments, as the table in Appendix A shows. Singapore Math, for example, is the national curriculum of top-ranking Singapore, and throughout the curriculum arithmetic, prealgebra, and early algebra are structured to be taught, learned and performed via diagrams and diagramming.

Due to Singapore's repeated international ranking as first or second in mathematics, Singapore Math has garnered much attention, with one U.S. writer even referring to it as "Miracle Math" (Garelick, 2006). As appealing as the notion of a miracle math might be, it would be a "technocratic" and "somewhat simplistic" view to deem didactic materials as "somehow miraculously [able to] develop mathematical knowledge" (Gellert, 2004, pp. 163-164). Didactical materials cannot work miraculously: They are "a means for setting ideas and intentions into teaching practice" (p. 163) and students do not appropriate ideas and intentions automatically (Biesta, 2007). Rather, student appropriation of ideas and intentions requires acts of mutual interpretation on the parts of students and teachers (Biesta, 2007).

For this reason, it is important for teachers to know the ideas and intentions behind didactic material. My initial survey of the literature reveals that some teachers have an incomplete understanding of the ideas and intentions behind bar diagramming and have incomplete models of or ways of interpreting bar diagramming. For instance, a research report detailing the adoption of Math in Focus® in Georgia's Hall County school district documents such mixed experiences with and mixed sentiments about bar diagramming as the following (Badger, Spence, & Velatini, 2010):

- Many teachers stated that bar modeling, number bonds, and manipulatives were important approaches to foster students' conceptual learning of mathematics rather than simply memorizing them. Some reported difficulties, however, to effectively teach word problems and bar modeling strategies thus presenting further challenges for students' learning. (p. 71)
- While a few teachers noted they were confident and comfortable demonstrating bar modeling to solve word problems—with others stating that they found it difficult to teach bar modeling because they did not understand why a bar model would be drawn in such a manner—they noted that students expressed confusion with bar modeling. (p. 91)
- One Grade Three teacher wrote in a journal entry in the second year, "I would LOVE to have more training on bar modeling!" (p. 135)

The authors of the report viewed these and other findings as indicative of issues with fidelity of implementation and a need for "specific areas of training and ongoing support from the county and school administration" (p. 95). Because some teachers experience challenges when working with bar diagrams, and because knowledge of the ideas underlying bar diagrams affects the utility of bar diagrams, a synthesis of research knowledge about helpful and unhelpful handling of bar diagrams would be useful.

Purpose of the Literature Review

My purpose in conducting this review is to examine the relevant empirical research surrounding the practice of bar diagramming in classroom mathematics. Specifically, the review addresses the following questions:

1. What is beneficial, detrimental, and challenging about bar diagrams in terms of their capacity for mathematical representation?

2. What is beneficial, detrimental, and challenging about bar diagrams in terms of supporting student problem solving?
3. What is beneficial, detrimental, and challenging about bar diagrams in terms of their capacity to support students' growth in mathematical understanding?

Significance of the Review

How students learn mathematics matters at least as much as what mathematics students learn: As Boaler found, a three-way interaction occurs among a learner’s way of practicing mathematics, his or her mathematical knowledge, and his or her identity as a mathematics learner (see Appendix B). It is therefore important to not only understand what the learning outcomes of an educational intervention are, but how and why an intervention supports or hampers learning and understanding. By including context-specific recommendations and details about bar diagramming, the findings of the proposed review have the potential to provide decision makers and practitioners with further insight about how to handle bar diagrams in the classroom.

The social significance of the proposed study lies in its potential to contribute practical knowledge about how to guide students along the arithmetic-to-algebra trajectory, preparing them for algebra instruction in a timely fashion. Students who complete algebra early enough can complete more advanced mathematics courses and other STEM courses in high school to position themselves for greater access to a wider variety of college majors and for greater success in college. Currently, scalable ways to equitably foster timely algebra preparedness are not clearly articulated in the United States (Stein, Kaufman, Sherman, & Hillen, 2011). Bar diagramming is a practice reportedly employed successfully in top-performing countries to teach students Early

Algebra and prealgebra in preparation for algebra. It is a low-tech intervention thus accessible independent of a school's technological resources. Greater understanding of the intended ideas and optimal handling of bar diagramming might benefit U.S. students and teachers in this area.

Literature Review Search Strategy

I completed a background familiarization search as part of an earlier, informal literature review about bar diagramming. During that informal literature review, I began collecting articles containing theory to plausibly explain the mechanisms underlying bar diagrams and bar diagramming. I also began collecting empirical studies about bar diagrams and bar diagramming.

As Table 1 shows, searched databases included Dissertations and Theses, Educational Resources Information Center (ERIC), PsycInfo, and Web of Science. Search strings included the following: bar diagram* & math*, bar model* & math*, model method, Singapore Math, strip diagram* & math*, and tape diagram* & math*. Search results were screened to include only peer-reviewed reports in either English or Spanish. Additionally, I contacted researcher Swee Fong Ng and research team Julie Booth and Kenneth Koedinger to inquire about additional bar diagramming studies.

Table 1

Preliminary Search Hits for Empirical Studies about Bar Diagramming

	Dissertations and Theses			ERIC			PsycInfo			Web of Science		
	hits	after screen by title/abstract	**after screen by full-text**	hits	after screen by title/abstract	**after screen by full-text**	hits	after screen by title/abstract	**after screen by full-text**	hits	after screen by title/abstract	**after screen by full-text**
bar diagram* & math*	185	1	**1**	2	1	**1**	4	1	**1**	8	1	**1**
bar model* & math*	266	15	**2**	4	2	**2**	2	0	**N/A**	80	1	**1**
“Model Method”	2408	63	**1**	31	3	**2**	144	6	**5**	1276	5	**3**
Singapore Math	86	17	**3**	6	1	**1**	2	1	**1**	4	0	**N/A**
strip diagram* & math*	10	4	**0**	0	N/A	**N/A**	0	N/A	**N/A**	0	N/A	**N/A**
tape diagram* & math*	5	2	**0**	1	1	**1**	0	N/A	**N/A**	0	N/A	**N/A**
Totals			**7**			**7**			**7**			**5**

Definition of Key Terms

Two key terms are central to this study.

Bar diagram

A bar diagram is a rectangular figure (or group of rectangular figures) used in school mathematics to depict mathematics objects, concepts, and word problems. Synonyms include strip diagram, tape diagram, and bar model (Booth & Koedinger, 2012, p. 494). Appendices D and E contain images, illustrations, explanations, and applications of bar diagrams and bar diagramming. As one can see from Appendices D and E, a bar diagram can be singular in nature or can consist of several related bar diagrams. Many times, two or three bar diagrams are drawn pointedly adjacent for the purposes of comparison, equalizing, etc. One important note is that the physical length of a bar diagram, but never its height, is what indicates relative magnitude of the quantities that adjacent bar diagrams represent.

Bar diagrams can depict a wide variety of mathematical objects, such as the particular fraction three-fourths, a known or unknown quantity, or a mathematics word problem in its entirety. A bar diagram that depicts the fraction 3/4, for instance, can be converted to a bar diagram depicting the percentages 25% and 75%, the ratio 3:4, or the decimals 0.25 and 0.75. Notably, this conversion is possible simply by converting only the labels on the bar diagram and not the bar diagram itself. This can demonstrate for students the fact that fractions, percents, ratios, and decimals can be equivalent and are all different ways of doing the same thing—namely, of "talking about" parts and wholes.

Booth and Koedinger (2012, p. 494) provide the following useful discussion that defines bar diagrams and explains their use:

> Diagrams are often used in math instruction in Singapore (Beckmann, 2004) and Japan (Murata, 2008), two countries in which mathematics achievement is consistently outstanding by world standards (National Center for Education Statistics, 2003). The style of diagrams used in these countries, sometimes called tape diagrams (Murata, 2008), strip diagrams (Beckmann, 2004), or bar models (Hoven & Garelick, 2007), uses strings of objects and/or strips of different lengths to represent the magnitude of and relationships between the quantities in the problem. Like algebraic equations (but unlike symbolic arithmetic problems), these diagrams are not meant to help users carry out operations, but to help them decide what operations to use and to understand why those operations are conceptually sound (Beckmann, 2004). Tape diagrams are also found in textbooks and some educational software (e.g., Carnegie Learning, 2009) in the United States, but their use is less frequent and inconsistent (Murata, 2008).

For the purposes of the proposed study, I will use one term—the term bar diagram—to refer to each and any of the above-mentioned constructions and any like constructions I may encounter in the literature that are largely indistinguishable from these.

Bar diagramming

Bar diagramming (creating bar diagrams) is likely to function differently—that is, is likely to present different opportunities for learning—than viewing bar diagrams, thus the emphasis upon distinguishing the two. For the scope of this study, I define bar diagramming as the school mathematics representational and problem solving technique that consists of two practices: the practice of creating one or more rectangles to depict mathematical objects or situations and the practice of acting mathematically (either mentally or physically) upon said rectangle(s). Bar diagrams and the rectangular subcomponents comprising bar diagrams can be named, partitioned, lengthened, shortened, labeled, and assigned numerical value. Thus for the scope of this study, I include these acts in the definition of bar diagramming.

Delimitations

I delimited the empirical studies I include in the review to peer reviewed empirical articles published after January 1, 1970 that report in English or Spanish on bar diagrams or bar diagramming of school mathematics

Limitations of the Review

Although bar diagramming has been an official component of Singapore's mathematics curriculum since the 1980s, researchers have published relatively little empirical research about bar diagramming. The review thus addresses a relatively small number of empirical studies.

Review of the Literature

To familiarize the reader with the literature and to lay the groundwork for addressing my research questions, I begin the review by introducing each text and identifying the texts' main claims. I have arranged this introductory discussion of the texts according to the main type of data used as the basis of the texts' claims. Data examined in this collection of studies includes (1) curricular materials, (2) participants' assessment scores, (3) participants' mathematics work, (4) interview data, (5) transcripts of classroom conversation, (6) videotaped teaching sessions, and (7) neuroimaging data captured as participants performed mathematical tasks. Appendix F contains a table of major findings relevant to the review questions, along with details about each study's sample.

Researchers' Examinations of Curricular Materials

Ng (2004) and Murata (2008) both primarily examined curricular materials, however their research aims differed: Ng looked at Singapore's primary mathematics curriculum to determine how it develops algebraic thinking. Murata looked at Japanese curricular materials to determine how tape diagrams might be mediating the teaching and learning of mathematics from Grade 1 to Grade 6. An additional aim of Murata's was to ascertain any differences in US and Japanese instructional approaches as suggested by differences in the mathematical representations in US and Japanese textbooks.

Ng's (2004) study. Ng (2004) made four main claims after examining four types of curricular materials (the syllabi, teachers' notes, textbooks, and workbooks) produced by Singapore's Ministry of Education for primary grade levels one to six. Ng's first main claim is that algebraic thinking in general develops via various approaches, including a problem solving approach, a generalizing approach, and a functional approach. Ng's second main claim is that when Singapore's mathematics curriculum emphasizes thinking about parts and wholes, the curriculum is employing the problem solving approach to developing algebraic thinking. Similarly, Ng's third main claim is that when Singapore's curriculum emphasizes the acts of generalizing and specializing, the curriculum is employing the generalizing approach to developing algebraic thinking. Ng's fourth and final main claim is that when the curriculum emphasizes doing and undoing, the curriculum is employing a functional approach to developing algebraic thinking.

Ng presented strong evidence to support her claims that Singapore's primary mathematics curriculum emphasizes thinking about part-and-whole relationships, generalizing and specializing, and doing and undoing. For instance, to back her third claim she provided excerpts of curriculum pages containing number pattern and geometrical pattern activities that clearly constituted generalizing and specializing. Ng, however, did not provide a definition of algebra or a discussion of what constitutes algebraic thinking. Her third claim would have been strengthened had Ng clarified why completing a number pattern or geometrical pattern develops, entails, or constitutes algebraic thinking. The same critique applies to Ng's second and fourth claims, though to a lesser extent as the curriculum excerpts Ng provided to back those claims more clearly connected to algebra.

Murata's (2008) study. Unlike Ng's singular aim to determine how Singapore's primary mathematics curriculum supports development of algebraic thinking, Murata's (2008) aim was two-fold: Murata sought to understand and describe how tape diagrams support mathematics teaching and learning from grades one to six, and she sought to compare and contrast US and Japanese instructional approaches as suggested by representations in textbooks.

Murata's (2008) investigation is significant because it gets at the question of whether it is powerful to maintain use of one representation (among others) to consistently undergird the learning and teaching of a large variety of mathematics for one's entire elementary mathematics education. Does it matter that tape diagrams travel with learners, so to speak, from Grade 1 to Grade to 2, all the way to Grade 6 and beyond? What is gained from use of tape diagrams year after year, and are those gains in any way dependent on long term use of tape diagrams? These questions are important because such gains might be forfeited in curricula that do not emphasize a consistent mathematical representation.

In her analysis, Murata paid special attention to units on addition and subtraction, as addition and subtraction are foundational skills and honing in on them allowed for greater depth of analysis. Also, when examining US and Japanese representations to compare instructional approaches, Murata focused on the top five most prevalent representations and sought to report on findings different than those previously reported in the literature.

Before presenting her findings, Murata led with an extensive discussion of theory that explicated how representations function as tools, how working with tools forms the mind, and how representations "become a part of the learners' cognition" (p. 376). This initial theoretical discussion lay a strong foundation for Murata's subsequent presentation of a model of mathematics learning that she and a colleague had developed, namely the Zone of Proximal Development Math Learning Model. In this four-phase model of learning, visual representations "work as a tool to support the development of understanding and to carry potential meanings from one context to the next" (p. 378). Visual representations "bring . . . the new concept close to students at the initial stage" and later "provide 'self-assistance,'" connect various concepts, and enable deep understanding (p. 378).

Murata made three main claims regarding differences in representations between the US and Japanese textbooks. First, representations in the Japanese textbooks more often accompany word problems, while representations in the US textbooks more often accompany computational exercises. Second, representations in the Japanese textbooks were presented to help students analyze the mathematical information in word problems, whereas representations in the US textbooks were presented to guide the solving of the word problem. Third, to represent the concept of addition (i.e., that a part plus a part equals a whole), the US textbooks use three rectangles: one rectangle for each part and a third rectangle as the new sum, or whole. Japanese textbooks only present the two parts, and the sum of those two parts is considered to be the length of the two parts taken together. No third rectangle is presented to represent the sum, which Murata deemed a more accurate representation.

Murata’s claims regarding the mediating power of tape diagrams were numerous and detailed. She identified both the mathematics that tape diagrams represent in grades one through six along with the meaning supports possible with tape diagrams. Regarding the mathematics that tape diagrams represent in the Japanese curriculum, Murata concluded that tape diagrams represent concepts and relationships such as whole numbers, rational numbers, the four basic operations for both whole number and rational numbers, how operations relate, how rational numbers relate, how ratio works, and how proportion works. Tape diagrams are simple enough to evoke students’ earlier addition and subtraction work on strings of concrete objects and complex enough to depict multiplication, division, ratios, and proportions—all of these with rational numbers. Finally, tape diagrams work flexibly in conjunction with single and double number lines.

Regarding meaning supports possible with tape diagrams, Murata determined that tape diagrams relate different quantities, allow understanding of word problems before abstract numbers are involved, relate different mathematical operations, freeze mathematical problems in time, provide continuity across mathematics topics, render mathematics as systematic, and support (classroom) mathematical discussion. Overall Murata's claims were well-founded. She included sample images and pages from the textbooks. She presented a mathematics learning model with explanatory power for justifying her claims. She exercised caution and acknowledged that representations do not have inherent meaning and that her findings were simply what could occur.

Researchers' Analyses of Participants' Assessment Scores

Student assessment scores served as the main source of data in two of the studies under review: Mahoney's (2012) dissertation study and the first of Booth and Koedinger's (2012) two studies. Mahoney aimed to determine whether learning the Model Method increases student performance on multiplicative comparison word problems and fraction word problems (see Appendix D). Alternatively, Booth and Koedinger had three research purposes. Their first aim was to determine whether middle school students enjoy a textual advantage, that is, whether these students perform better at solving the textual (story) version of algebraic word problems versus the equation version. Their second aim was to determine whether middle school students enjoy a diagrammatic advantage, that is, whether these students benefit from the presence of a bar diagram with the story version of an algebraic word problem. Lastly, their third aim was to determine whether the textual advantage or diagrammatic advantage varies by students' mathematical ability levels.

Mahoney's (2012) study. Mahoney (2012) conducted a single case, experimental research design four times individually with one female and three male fourth grade students from a rural, mostly lower middle-class area in New Hampshire. Pre-intervention, each participant completed an assessment probe several times to establish a baseline score on solving multiplicative comparison word problems and fraction word problems. Mahoney then delivered separately to each student a researcher-designed teaching intervention. For each of 8 teaching sessions, participants received instruction on using bar diagrams to solve multiplicative comparison and fraction word problems and then completed an assessment probe. Participants later completed two post-intervention assessment probes, the first one week and the second three to four weeks after the teaching intervention had ended.

The assessment probes were researcher-designed. They contained five multiplicative comparison word problems and five fraction word problems. Participants completed 13 parallel-designed assessment probes in all—three assessment probes at the beginning in order to establish a baseline performance score, one assessment probe to close each of eight intervention teaching sessions, and two post-intervention assessment probes.

The eight-session teaching intervention that participants received was scripted, and the scripted instruction followed a version of the “I do, we do, you do” instructional model. For the “I do” component the instructor demonstrated solving a problem while the student watched and asked questions. For the “we do” component the instructor and student solved the same problem separately and then compared their work. Subsequently, the student solved a word problem independently, explaining his or her steps to the instructor. Lastly, for the “you do” component of the teaching intervention, the student solved a word problem independently, and the instructor asked the student questions as desired.

Mahoney’s methodology had many strengths: He took efforts to reduce researcher bias, to screen participants, to administer a social validity survey, and to establish inter-rater reliability for the participants’ assessment probe scores. To form his conclusions, he generated and analyzed line graphs depicting the trajectory of each participant’s assessment probe scores.

Mahoney concluded that instruction in bar diagramming increases students’ performance on multiplicative comparison and fraction word problems. He noted that a “correct schema choice led to a correct problem solution among the four participants 83.75% of the time” and surmised that “the fact that 95% of correctly drawn models led to correct solutions is further convincing evidence of the effect of model drawing on problem-solving performance” (p. 107).

One weakness of Mahoney's study was that all of the word problems on the assessment probes were either multiplicative word problems or fraction problems. No distractor word problems such as fraction addition, fraction subtraction, or fraction division problems were included. As such, the mere presence or absence of a fraction in the word problem may have automatically cued participants to configure their bar diagrams in a certain way. If so, the assessment probe would not have necessarily assessed students' mathematical understanding of problem situations.

Booth and Koedinger’s (2012) first study. Noting the inconclusive nature of prior research on the benefits of diagrams, one of Booth and Koedinger’s (2012) aims was to determine whether it is advantageous for middle school students if a bar diagram accompanies the text of difficult algebraic word problems. They also sought to determine whether a textual advantage is in effect for middle schoolers, that is, whether middle school students perform better at solving the textual version as opposed to the equation version of difficult algebraic word problems. Booth and Koedinger’s third and final aim was to determine whether any such diagrammatic advantage or textual advantage depended upon the middle school student's level of mathematical ability.

Three hundred seventy-three students (121 sixth, 117 seventh, and 135 eighth graders) from an ethnically diverse middle school in the Midwestern United States participated in their study. Booth and Koedinger employed what amounts to a cross-sectional correlation methodology, treating each grade level as a cohort.

The assessment items that Booth and Koedinger scored consisted of three algebra problems embedded in a larger algebra assessment. They used three different assessment forms constructed such that each form contained the three different problems and each form contained three different formats (equation, story, or story + diagram). Participants randomly received one of the three forms.

Analyses of variance, follow-up repeated measures ANOVAs, and follow-up t-tests yielded the following results: When comparing the story format (i.e., text format) to only the equation format, a textual advantage manifested for students in all grade levels for both single-reference and double-reference problems (problems in which a single variable is referenced once or twice, respectively). The story + diagram format was beneficial for double-reference problems and for seventh and eighth graders in general, but not for sixth graders in general. Regarding low ability students, low ability sixth graders performed worse when a bar diagram accompanied a word problem. On the contrary, low ability eighth graders performed much better when a bar diagram accompanied a word problem.

Booth and Koedinger's conclusions were well-supported and convincing to the extent that they applied to the participants in the study and not necessarily to middle school students generally. The participants in this study learned mathematics using the reform-based Connect Mathematics curriculum. This would not be the case for many other middle school students. Also, unlike the students in Mahoney's dissertation study, these participants had little to no experience with bar diagramming upon taking the assessment as they received no formal instruction in bar diagramming prior to or during the study. Another important note is that overall, study participants did not typically perform well on any of the problems of interest, regardless of format.

Researchers' Examinations of Participants' Mathematical Work

Participants' work served as the main data source in four of the studies under review: Koedinger and Terao's (2002) task analysis study, Van den Heuvel-Panhuizen's (2003) longitudinal study, Ng and Lee's (2005) task analysis study, and Booth and Koedinger's (2012) longitudinal study. Two of these studies share the common purpose of determining how students use bar diagrams. In a third one of these studies, the researchers sought to ascertain whether bar diagrams support sixth grade students in solving algebraic word problems and to determine which types of algebraic problems are difficult for sixth grade students to solve via bar diagramming. In the remaining study, the researchers sought to determine whether students who initially did not benefit from bar diagrams benefitted over time.

Koedinger and Terao's (2002) study. Koedinger and Terao's (2002) participants had received no formal instruction in algebra upon participating in their study. However, prior to completing the study's assessment instrument, participants did receive bar diagramming instruction in solving algebraic word problems. Findings had implications for improving their instructional approach.

Koedinger and Terao (2002) examined student work in order to ascertain which type of algebraic word problem is more difficult for students: word problems with three unknown quantities, word problems with “less-than” relationships between variables/quantities, or word problems containing both of these dimensions of difficulty. They also sought to determine whether the representation phase or the solution phase of bar diagramming is more difficult for students.

Their sample consisted of 35 sixth graders. Though demographic information was not specified, these students were likely US students, given that Koedinger and Terao referred to bar diagramming as “Picture Algebra” and described it as “part of a new middle school mathematics curriculum” (Koedinger & Terao, 2002, p. 542).

The student work that Koedinger and Terao analyzed consisted of students’ solutions to three word problems referred to in the study as the Cans problem, the Beanie problem, and the CD problem. Half of the students were provided a bar diagram for the Cans problem, half of the students were provided a bar diagram for the Beanie problem, and no students were provided a bar diagram for the CD problem.

Sixty-eight percent of the participants solved the Beanie problem correctly, 68% solved the Cans problem correctly, and 32% solved the CD problem correctly. Participants were much more successful on the CD problem when they drew a correct diagram than otherwise (68% correct with a diagram versus 17% with no diagram or an incorrect one). This reinforced Koedinger and Terao’s conclusion that students can use bar diagrams successfully. Overall these students performed better on these problems than much older students had performed on similar problems, according to Koedinger and Terao.

Koedinger and Terao employed task analysis methods. They also created and analyzed production rule traces that documented common problem solving moves that students made. From this data Koedinger and Terao made two main claims. Typically when students draw horizontal, rectangular bars in the bar diagramming process, drawn bars stand for existing amounts. When students attempted to draw abstract bar diagrams with rectangular parts to indicate missing amounts, students encountered problems.

Booth and Koedinger also found that under certain conditions students struggled to draw bar diagrams that convert “y is d less than x” to “x is d more than y.” The challenge occurred for three-variable word problems wherein two of the variables were both described as less than the third variable.

The evidence Koedinger and Terao presented consisted of images and descriptions of student work and was thus convincing. A note of interest is that instead of attributing student failures to students’ inadequacies or any inadequacies of bar diagrams, they interpreted their findings to indicate that their own bar diagramming instruction could be improved. Their study could have been strengthened had they provided demographic information about their participants and had they detailed any scoring and reliability procedures.

Van den Heuvel-Panhuizen’s (2003) study. Though Van den Heuvel-Panhuizen (2003) and Ng and Lee (2005) commonly examined how students use bar diagrams, their approaches, ultimate purposes, mathematical topics, and likely their participants’ demographics differed vastly. Van den Heuvel-Panhuizen conducted a longitudinal study (participant demographics unspecified) as part of a Realistic Mathematics Education curriculum design project, focusing on students' developing understanding of percent as supported by bar diagrams. Ng and Lee conducted a task analysis study with 151 Grade 5 Singaporean students to determine how they used the Model Method to solve one arithmetic and four algebraic word problems.

Researcher Van den Heuvel-Panhuizen sought to determine how bar diagrams were used to support student growth in understanding of percent over an extended learning trajectory. Her study is of particular interest due to its emphasis on the Realistic Mathematics Education (RME) approach to student handling of mathematical models. In RME, models for general use emerge from situation-specific representational activities so that models-of become models-for. That is, educators guide students' natural, problem-specific representational activities such that students eventually view models of a particular problem situation as suitable to model categories of problem situations.

Van den Heuvel-Panhuizen examined student work and curriculum pages from units being developed for the US textbook series Mathematics in Context. While participant demographics were unspecified, the pertinent curriculum units were fifth and sixth grade units. Van den Heuvel-Panhuizen made several claims. She asserted that students drew bar diagrams spontaneously when prompted to draw models showing percent. She noted that students did not receive direct instruction in bar diagramming. Instead, bar diagrams emerged naturally in the process of guided problem solving. She also concluded that bar diagrams functioned in several ways for students: as an estimation model, a calculation model, and a thought model.

Van den Heuvel-Panhuizen provided clear evidence for most of her claims: She provided images depicting students' spontaneously drawn, informal bar diagrams beneath a prompt to represent a certain percentage. She provided a sample assignment evidencing how students—as opposed to receiving direct instruction in bar diagramming—shaded in a rectangular image of theater which developed into a bar diagram on which students independently equated percents with fractions.

According to Van den Heuvel-Panhuizen, when students began viewing the rectangle as useful for modeling other situations, this was the first shift toward context-free (more generic) usage of the bar diagram. She asserted that such a shift from a model-of (i.e., a model of a specific situation or word problem) to a model-for (i.e., a model for generally showing x amount out of y amount) constitutes a rise in students' level of mathematical understanding.

Overall Van den Heuvel-Panhuizen provided strong evidence that students used the bar diagram as an estimation model, as a calculation model, and as a thought model. Bar diagrams appeared particularly suited for aiding estimation. For example, images of students' bar diagrams showed how students used benchmark fractions of one-half and three-fourths used to estimate that about 60% of the bar was shaded.

Using the term "calculation model" to mean that the bar diagram indicated for students which operations they should perform, she shared a student's bar diagram with 1% of 1,603 labeled as 16 that supported the student in choosing division to estimate that 91 is roughly 5% of 1,603. Though she asserted that the 100-segment bar helped students to connect percents to decimals and to understand the role of decimals in computing percents, that was unclear from the included images of student work.

Van den Heuvel-Panhuizen included several images of bar diagrams used as thought models. A particularly illustrative image was a vertical stack of bar diagrams placed on a graph to help students visualize how and why 6% growth gets larger over time and how to compute exponential growth. Another use of the bar diagram as a thought model depicted $96 as a sales price at 25% off of the original price. The student labeled the three-fourths marker on the bar diagram with $96, independently determined the original price, and encountered further instruction that connected the bar diagram to the solution strategy of dividing $96 by 0.75.

Van den Heuvel-Panhuizen noted that it was not just one form of the bar diagram but a chain of slightly changing forms of the bar diagram that over time led to growth in students' understanding of mathematics. In her estimation, quite powerful for supporting student learning is the ability of the bar diagram to continue to hold initial lower-level meanings of a mathematical topic while simultaneously displaying higher-level, formal meanings. In this way, bar diagrams can at once accommodate students with various levels of understanding of a mathematical topic.

Ng and Lee's (2005) study. Ng and Lee examined the work of 151 Primary 5 students in order to determine how they use the model method. These students were from lower-middle class to middle-class areas. In terms of achievement level, top stream students comprised 21% of the sample, and the rest were middle stream students.

All participants received a problem set containing one arithmetic problem and four algebra problems. The percent of students who solved each problem correctly was 63% for the arithmetic word problem and 44%, 37%, 20%, and 15% for the four algebra word problems.

Ng and Lee made several assertions about students' successes and difficulties with the model method. Students with a correct bar model of the arithmetic problem used one of two ways to solve it. Most students used only arithmetic, but some performed an elegant combination of multiplication and arithmetic that the bar diagram demonstrated was possible. Unsuccessful students included those who made computational errors despite having generated a correct bar diagram, as well as students who possibly lacked understanding of terms such as "more than."

Ng and Lee made observations and drew several conclusions after analyzing student work on the four algebra problems. Successful students structured their bar models as pictorial equations with the left sides of their equations consisting of bars representing the relationship between known and unknown numbers, a right brace serving as an equal sign, and the right sides of their equations consisting of known values (written as a numeral) or unknown values (written as a question mark). Successful students' work revealed understanding of the efficacy of using inverse operations to solve problems. Having yet learned algebra, successful students bypassed using algebra and used arithmetic instead.

When one value is defined in a word problem in terms of another, for example if a word problem indicates that Brian has 7 more than Sheila has, students have a choice: Students can first draw a rectangle for Brian's amount and then base the size of the rectangle for Sheila's amount off of the size of Brian's rectangle. Likewise, students can first draw a rectangle for Sheila's amount and base the rectangle for Brian's amount off of the size of Sheila's rectangle. In the model method this is referred to as selecting the generator. Ng and Lee noted that a handful of students chose generators strategically.

Strategic choice of generator was important. Echoing Koedinger and Terao, Ng and Lee noted that some students encountered problems after attempting to draw rectangles to indicate missing or negative amounts. Later in the problem solving process these students treated such rectangles as positive amounts to be subtracted instead of negative amounts to be added back. Had these students chosen a different generator, drawing missing or negative amounts would have been unnecessary.

The algebra problem that contained a fraction posed a significant challenge for students. Generating a correct bar model required prior knowledge of how to find common denominators and create equivalent fractions. The small percentage of students who modeled and solved the problem correctly typically solved the problem by dividing and multiplying whole numbers instead of dividing by a fraction—a skill yet learned.

Ng and Lee noted that the two-variable systems of linear equations problem was less amenable to bar diagramming than were the other three algebra problems. Few students attempted the problem. Students who persevered in modeling the problem with bar diagrams used the individual rectangles in their models in unconventional ways.

Ng and Lee concluded that, overall, successful students had the prerequisite conceptual knowledge of the mathematics involved in the problem. They understood language such as “more than,” “less than,” “as times as much,” and “as times as many.” Unsuccessful students—aside from those who only made computational errors—made conceptual errors such as interpreting phrases the phrase “as many times” to mean “as many times more.” When stalled they erroneously converted algebraic relationships into simple arithmetic relationships, deleting pertinent unknowns from the problem.

Booth and Koedinger's (2012) second study. Having determined in a first study that many sixth graders benefitted less from the presence of bar diagrams than did seventh and eighth graders, in their second study Booth and Koedinger set out to determine whether and how bar diagrams became beneficial to sixth graders later in the seventh or eighth grades. In the process, they analyzed student errors each year in the attempt to gain a deeper understanding of students' struggles and successes.

Participants were 84 of the sixth graders who had participated in Booth and Koedinger's first study. The procedures in this longitudinal study were a repeat of the procedures followed in the initial study. Participants completed the same assessment instrument in their seventh and eighth grade years as they had in their sixth grade year.

Booth and Koedinger came to several conclusions based on their ANOVA of the participants' scores. The textual advantage remained intact: Over time these participants still performed better on the story version of the algebraic problems verses the equation version. While students improved over time on the story + diagram version of problems, there was no improvement over time on the story version of problems. Analysis revealed that this applied not only for the group overall but for low ability students when viewed as a subgroup. Specifically, Booth and Koedinger found that "low-ability students are much more avoidant of diagram problems than story problems in sixth grade (65% vs. 39% no response errors)" (p 504).

The study followed a cross-sectional correlational design. Given the research question, the study may have benefitted from inclusion of within-subject comparison. However, given the design of the studies, Booth and Koedinger's claims were convincing. Assessment forms were randomly assigned to participants and initial ANOVAs were followed up with additional analyses to clarify findings.

This study's findings are significant in that they call into question the notion that a more concrete representation such as bar diagrams will necessarily aid lower ability students. Likewise, these findings call into question assumptions that more accomplished students do not benefit from such concrete representations. This study revealed the opposite effect, for reasons which might be clarified with further research.

Researchers' Analyses of Interview Data

In four of the reviewed research articles, the authors interviewed participants in addition to analyzing participants' assessment scores or mathematical work. These studies include Ng's (2003) task-based interviews, Ng and Lee's (2008) task-based and semi-structured interviews, Looi and Lim's (2009) quasi-experimental study and clinical interviews that were part of a larger curriculum design research project, and Ng and Lee's (2009) task analysis and interviews. Besides sharing in common the use of interviewing, one common thread among the studies is an interest in the quality and nature of the algebraic knowledge of students who have spent years in school learning and performing mathematics with bar diagrams.

Ng's (2003) study. Ng (2003) conducted task-based interviews with the purpose of answering four questions: (1) Which method would secondary two express stream students use to solve a given problem? (2) Were secondary two express stream students able to recall the model method? (3) Were secondary two express stream students able to identify how unknowns are represented in the model method? and (4) What were secondary two express stream students' perceptions of the model method? Ng's participants were 145, express stream, secondary two students (87 females and 58 males), ages 14+, from four classes. Sample A (77 of the 151 students) was differentiated from Sample B (n = 68) in that Sample A students completed the topic of simple linear equations and simultaneous equations one week earlier than Sample B.

Given a maximum time allotment of one hour, participants completed a four-item assessment instrument. Ng audio recorded 15-minute interviews with ten randomly selected students (five from each sample) directly after the written assessment. The written assessment contained four items. Item 1 was a prompt to solve a mathematics word problem using any method of the participant's own choosing. Only 4 of the 145 students opted to use bar diagrams alone to solve the problem in Item 1, while 9 of the students used a combination of bar diagrams and algebra. The vast majority of the students solved the problem using algebra (others used a listing method or a grouping method or were incorrect).

Interview data revealed that though a few students appreciated how the visual nature of bar diagrams increased their grasp of the problem, students deemed algebra to be quicker, less exacting, and less tedious than bar diagramming. Students had recently learned algebra and one student had forgotten how to use bar diagrams. One student noted that algebra was easier to check for correctness. Another student mentioned that bar diagramming required extra thought and was actually harder than algebra. Some students mentioned that their secondary teacher discouraged use of bar diagrams.

Item 2 asked students to assess a bar diagram solution and use bar diagrams to solve a similar problem if they agreed with their usage in the provided, solved problem. Over 94% of the participants correctly assessed the provided solution as correct and used bar diagrams to solve a similar provided problem. Item 3 prompted students to identify the unknowns in a solved model method problem. Sixty-seven students (less than half) did so successfully. Item 4 asked students for their perceptions of bar diagrams. The vast majority spoke favorably of bar diagrams for their ability to help students visualize mathematics problems. However, most students noted drawbacks about bar diagramming: Bar diagrams were too exacting (one mistake during the representation phase usually renders the final solution incorrect), bar diagrams for large numbers or more complex equations are not as useful, bar diagrams are for younger or less accomplished students, and algebra is simpler and more powerful.

Ng (2003) interpreted the study data to indicate that although students were competent in the Model Method, a link between the Model Method and algebra was not strong. The inability of students to identify the unknowns in a bar diagram suggested to Ng (2003) that perhaps requiring students to translate bar diagrams into algebraic equations would access the full benefits of bar diagrams.

Ng and Lee's (2008) study. While one of Ng's (2003) research questions centered on students' perception of the Model Method as a technique, Ng and Lee (2008) were interested in students' perception specifically of the Model Method's rectangle itself. Ng and Lee's study participants were 10 Primary 5 students (3 females and 7 males) from neighborhood schools. Using a clinical interview approach, Ng and Lee presented each participant with an assessment instrument that contained a written production task and a written validation task. Immediately after the written tasks, which took approximately 30 minutes, Ng and Lee interviewed participants. Each participant's semi-structured interview lasted between 30 and 60 minutes. Analysis consisted of comparing across-participant and within-participant analysis of written and verbal data.

Ng and Lee's first major claim was that participants were abstract thinkers if their bar diagram representing numbers in the tens were relatively the same size as their bar diagram representing numbers in the hundreds and thousands. Ng and Lee's second claim was that most of the students were abstract thinkers. Ng and Lee's third major claim was that interview data confirmed that participants agreed that for word problems having the same structure, the same bar diagram can be used regardless of the order of magnitude of the input numbers.

Ng and Lee's fourth and final claim was strong: They interpreted student written and verbal data to indicate that students were viewing the rectangles in bar diagrams as variables and not simply as unknowns. Because Ng and Lee did not provide definitions for their usage of the terms unknown and variable, the reader was left unclear as to the meaning and significance of this fourth claim.

Looi and Lim's (2009) study. Concerned about students' challenges with transitioning from use of bar diagrams to use of traditional symbolic algebra, Looi and Lim (2009) designed pedagogy and software (ALGEBAR) to help students bridge bar diagrams with algebraic expressions and equations. They conducted a quasi-experimental study and clinical interviews to assess the effectiveness of the pedagogy and software (ALGEBAR).

Looi and Lim's participants were 68 secondary one students in two class of 34 students each. The intervention lasted for four weeks. The control class and experimental class matched on pertinent variables. The two teachers used the same lesson plans.

The experimental class replaced one hour a week of class instruction with one hour in the computer lab using the ALGEBAR software. Also, instruction in the experimental class weaned students off of bar diagrams and prioritized use of algebraic symbolization. The instruction in the control class emphasized bridging, that is, creating bar diagrams and then translating them into equivalent algebraic equations.

Both classes took post-intervention tests that were analyzed and scored for use of algebraic symbolization. Looi and Lim conducted pre-intervention and post-intervention clinical interviews with three students from the experimental class (one weak, one average, and one high achieving student). Finally, Looi and Lim interviewed both teachers.

Looi and Lim conducted t-tests comparing the control and experimental classes' scores on the test instrument and found that the experimental class used more algebra. They conducted an ANCOVA to control for mathematics ability and the finding was the same. Looi and Lim found that pre-intervention, the three interviewed students were successful at solving routine problems with bar diagrams. However, when a complex problem required a more sophisticated bar diagram, this posed a problem for students.

Looi and Lim's major claim was that "after 4 weeks of intervention, the stronger student is able to define variables and formulate equations, the average student is able to define variables and formulate equations for routine questions, but the weaker student is only able to define variables correctly" (p. 370). Teachers' interview data indicated that students had difficulty transitioning from years of using the highly visual bar diagramming technique to use of algebraic symbols. Teachers felt that students could perform arithmetic and algebra separately from a word problem, but modeling a word problem with algebraic symbolization was challenging for students. Looi and Lim noted that bar diagramming acclimates students to writing equations using a forward calculations process that allows students to bypass using algebraic methods. Looi and Lim observed that students seemed to be confusing forward calculation equations with algebra.

The experimental teacher felt that ALGEBAR helped participants by making bar diagramming easier. Once students had created diagrams of word problems, students were then able to translate bar diagrams into algebraic equations, which ALGEBAR allowed students to check for consistency. Teachers noted that algebraic ratios were not amenable to representation in ALGEBAR and that this had the potential to help students value the representational power and flexibility of algebra.

Ng and Lee's (2009) study. Ng and Lee (2009) presented students with five mathematics word problems of increasing difficulty to determine how children would use the Model Method to solve them. They analyzed students' resultant work and then interviewed teachers and department heads to determine when and how teachers introduce children to the Model Method. Ng and Lee's (2009) participants were 151 middle-class to lower-class Primary 5 students (74 females and 77 males). Regarding their mathematics achievement level, 21% were top stream (high ability) students and 79% were middle stream (average ability) students.

The assessment instrument used was researcher-designed (based on problems from curricular materials), piloted, and approved by nonparticipating teachers. Student participants had one hour to complete the assessment. Problems were coded first by accuracy and then by type of error. Interviews with department heads and teachers lasted about one hour. Teachers completed the assessment instrument during the interview which was audio recorded.

Ng and Lee drew their first main claim from the interview data. Teachers reported that the model method is helpful because its visual format supports student understanding. The model method was also valuable in that it revealed to teachers students' processes and thinking. Teachers noted that students should draw rectangle lengths to reflect their appropriate proportions in magnitude. Teachers noted that in bar diagramming, a detail as small as using a dotted line versus a solid line to partition a bar diagram matters. Finally, teachers explained how students were able to use the model method to solve algebra problems prior to learning algebra: Using what is known as the unitary method, students represented the relationships in the bar diagram via a series of arithmetic equations that allowed students to undo operations in order to solve for unknowns. This allowed students to circumvent writing and manipulating algebraic equations.

From the student work, Ng and Lee drew several major claims: Complete models included every detail. Partly correct models—even those with a single missing detail—could yield incorrect solutions. One error that students made was to change generators midway into solving the problem. A generator is the variable or rectangle from which other quantities in the word problem are defined. In rare instances students drew a correct model yet failed to answer the question asked in the word problem. This error accounted for less than 3% of all errors.

Students with conceptual knowledge of fractions were able to model and solve an algebra problem containing fractions but bypassed computing with fractions. Instead students used division and multiplication. Lastly, about one-fifth of the students correctly solved a two-variable system of equations problem. Another third of the students had successfully modeled the problem though ultimately they did not answer it correctly.

Researchers' Analyses of Classroom Conversation

Shreyar, Zolkower, and Pérez's (2010) study is the one study under review wherein researchers primarily analyzed classroom conversation. Shreyar, Zolkower, and Pérez based their claims upon their analysis of a sixth grade classroom conversation about percents. Their aim was to answer the following two research questions: (1) What meanings were realized in the whole-class conversation by the teacher and students and how were these meanings realized? and (2) How did the teacher's lexico-grammatical choices guide the students' choices?

The terms realized and lexo-grammatical in Shreyar, Zolkower, and Pérez's research questions are inspired by Halliday's theory of systemic functional linguistics (SFL). According to Shreyar and colleagues, in Halliday's SFL theory, meanings are realized when they become words, and words are realized when they become sound or writing. Lexo-grammatical choices are choices related to semantics and syntax (lexical sets and grammatical sets). Thus, Shreyar and colleagues' research questions indicate that they sought to determine the meanings that became spoken and written text in the classroom and that they sought to determine how the teacher's speech influenced students' choices.

The study sample consisted of one class of 21 sixth graders (of medium to high economic status) in a private bilingual (English/Spanish) school in Argentina. Shreyar, Zolkower, and Pérez audiotaped and generated a transcript of a whole-class conversation that focused on determining which was the better buy: 20% off of $68.99 or 25% off of $74.99. Four coders including the teacher coded the transcript according to SLF theory. This included identifying speakers, conversational turns, clause types, and clause speech function.

Most of the students had previously solved the problem, and thus the focus of the classroom conversation was to develop ideas about percent and about why various solution attempts were appropriate or not. Twenty of the 21 students spoke, and student contributions were balanced among genders and mathematical performance level. The conversation consisted of almost 600 speaking turns, which Shreyar, Zolkower, and Pérez divided into 12 phases, or topics.

Ms. P. guided students to compare and make connections among the various plans and solution attempts that she and the students wrote on the whiteboard or chart paper. She guided students to refer to written plans and solution attempts as ideas. Such ideas were written in the form of ratio tables, bar diagrams, and expressions, and Ms. P used questions and declarative statements to guide students to see if “something from there coincides with the ideas that are here” (Shreyar et al., 2010, p. 35).

Shreyar et al. documented several examples of the roles that bar diagrams had in supporting meaning making. When students wondered if 20% of off $69 could be computed by dividing $69 by 20, Ms. P. asked students to make sense of $69 divided by 20 as shown on a bar diagram divided into 20 pieces. When a student offered a way to repair this solution idea, Ms. P. again guided students to use the bar model as a sense-making tool to understand the student’s suggestions.

When a student offered a standard algorithm for computing 20% off of $69 and 25% off of $75, Ms. P. guided the students to see the connection between the standard algorithm and the 1% strategy. The 1% strategy consists of first calculating 1% of a number and then multiplying the result by 5 to find 5%, by 20 to find 20%, and so forth. Students offered the idea of replicating their ratio table work on a bar diagram, which eventually resulted in students understanding how the calculations in the standard algorithm accomplished the same process as the ratio table and bar diagramming work did.

Regarding the teachers’ lexico-grammatical choices, Ms. P. spoke overwhelmingly in the positive tense and spent a large amount of time using present simple modal clauses such as Could that work? Rather than prompting much metacognition, Shreyar, Zolkower, and Pérez felt that Ms. P used language intended to elicit the quick offering of as many ideas as possible. Her language left assessment of ideas up to the students.

The evidence that Shreyar, Zolkower, and Pérez provided to support their claims was strong. They included coded excerpts of the analyzed conversation, and they provided tables that presented data about the interpersonal grammar of the conversation, the mood and speech function of the teacher, the main grammatical subjects the teacher used, and the teache’'s lexico-grammatical choices.

Overall, it was clear that the multimodal nature of the conversation (use of gesture, words, and diagramming) was a key part of the progression of ideas in the classroom. Also key was the teacher’s use of tense, modulation, and questions to invite the students to offer, assess, and reconcile ideas. This study’s findings were significant in that it demonstrated the role of the teacher's language in guiding students to use representations to make their own sense of the mathematics involved. Such a use for bar diagrams differed vastly from the approach of presenting bar diagrams to students as assumed-to-be correct representations of problems and solutions.

Researchers' Examinations of Teaching Sessions and Students' Session Work

Green's (2009) study is the one study under review wherein the researcher primarily examined teaching sessions and students' session work**.** Green examined video recordings of teaching sessions in order to determine the nature of the algebraic understandings students developed as a result of solving algebra problems with bar diagrams. She sought to determine how students developed their algebraic understandings and to determine what aspects of the environment seemed to afford or constrain students' development of algebraic understandings.

Green's (2009) sample consisted of three, sixth-grade students from a predominately African American school in Philadelphia. Employing a teaching experiment methodology, Green worked with each student individually. For each learning goal, Green assessed the participant's understanding, modeled the participant's understanding, designed a learning trajectory plan for developing desired understandings, and videotaped subsequent instructional lessons conducted with the participant. This cycle was repeated throughout the experiment, after which Green conducted a retroactive analysis of the data.

Green (2009) noted the following findings: First, after working with known numbers (in bar diagram form) as examples, participants spontaneously understood to operate upon unknowns (in bar diagram form) as if they were knowns. Second, participants learned that an additive quantity is decomposable and recomposable. Third, they gained understanding of the relationship between quantities in equations of the form $ax + b = cx$ and $ax + b = cx + d$. Fourth, they gained understanding that the parts of a quantity are commutative. Fifth, participants were able to anticipate that creating equivalence is helpful in solving unknowns. Sixth, participants gained understanding that operating on one side of an equation must be balanced out with operating on the other side of the equation. Particularly noteworthy is that participants transitioned to use of symbolic algebra without the typical pitfalls and with seeming understanding. The one drawback to working with bar diagrams was that some of the participants inappropriately overrode quantitative information when visual cues were confusing or unhelpful.

Green (2009) provided robust, credible evidence to back her claims, including extensive amounts of transcript excerpts and many images of student work. Green's write-up of her findings was quite detailed and included awareness and acknowledgment of rare moments when her instruction may have led students to particular courses of action.

Researchers' Examinations of Neuroimaging

Neuroimaging scans served as the main data source in the final two of the studies under review. The purpose of both studies was to determine whether use of the model method engages different cognitive processes than use of traditional symbolic algebra. The first study focused on the representation phase of algebraic problem solving, and the second study focused on the solution phase of algebraic problem solving.

Lee, Lim, Yeong, Ng, Venkatraman, and Chee's (2007) study. Eighteen right-handed adults (10 males, 8 females) between the ages of 20 and 25 participated in Lee and colleague's (2007) study. The study followed a 2 (method: model vs. symbolic) x 2 (type: experimental vs. control) within-subject design. Each participant viewed all four conditions—the model experimental condition, the model control condition, the symbolic experimental condition, and the symbolic control condition.

In the model experimental condition, participants viewed a worded algebra problem and then were asked to "hold in mind" a bar diagram representation of the word problem. The goal was to select the correct bar diagramming image that represented the word problem, once the image later appeared. In the model control condition, participants viewed text that contained similar content as an algebra word problem but that had no mathematical meaning that would prompt computation.

In the symbolic experimental condition, participants viewed a worded algebra problem and then were asked to "hold in mind" an algebraic equation to represent the word problem. The goal was to select the image with an algebraic equation matching the word problem, once the image appeared. In the symbolic control condition, the text contained wording and numbers, but the text was not a solvable word problem. It was not clear from the report what the goal was for the symbolic control condition.

Findings indicate that the model and symbolic experimental conditions activate the brain in both similar and different ways. Use of both the model method and symbolic algebra activated the precuneus and the posterior superior parietal lobules (PSPL)—regions of the brain associated with attentional resources. However, the symbolic algebra method activated those areas more strongly, suggesting that use of symbolic algebra is more taxing. Both the model and symbolic experimental conditions activated an area of the brain associated with a mental number line, an area known as the horizontal segment of the intra-parietal sulcus, or HIPS.

Differences in brain activity between the two conditions suggest that use of symbolic algebra may involve procedural retrieval more than does the model method. This was suggested by greater activation of the caudate area of the brain. Another difference, higher activation of the occipital lobe in the symbolic experimental condition, suggested that perhaps participants spent more time viewing symbolic stimuli than viewing bar diagrams.

Lee and colleagues claims seem well-founded. A within-subject design ensured that individual brain differences did not factor into the results. Additionally, the researchers included control conditions so that results measured were due to mathematical cognition and not simply the stimuli's modality (diagram versus symbols). Their design and methodology were strong.

Lee, Yeong, Ng, Venkatraman, Graham, and Chee's (2010) study. In a second neuroimaging study, 17 right-handed adults (10 males, 7 females) between the ages of 20 and 29 participated in the study (Lee et al, 2010). The study followed a 2 (method: model vs. symbolic) x 2 (type: experimental vs. control) within-subject design. Each participant viewed all four conditions—the model experimental condition, the model control condition, the symbolic experimental condition, and the symbolic control condition—distributed over 144 trials.

In the model experimental condition, participants viewed two bar diagram rectangles juxtaposed to represent a system of algebraic equations in two variables, J and M. The task was to solve for J and select the stimulus representing the correct solution. In the model control condition, participants viewed two bar diagram rectangles configured to have no mathematically useful meaning. The task was to identify the number associated with J and select the corresponding image.

The symbolic experimental and symbolic control conditions were analogous to the model conditions. In the symbolic experimental condition, participants viewed a pair of simple, linear algebraic equations in two variables (J and M), configured as a system of equations. The task was to solve for J and select the stimulus representing the correct solution.

Both the model and symbolic experimental conditions activated the intraparietal sulcus, an area of the brain associated with the mental number line. The symbolic experimental condition activated this area more strongly, suggesting that traditional algebraic symbolization might rely more on numerical processing than does bar diagramming. Also, the symbolic experimental condition activated the brain's left hemisphere's portion of the intraparietal sulcus more than it did the right hemisphere's. This hemispheric asymmetry could indicate either that symbolic algebra was more reliant upon basic fact retrieval than bar diagramming or that it was more language-like than bar diagramming. Lee and colleagues felt the latter explanation was more likely.

Another difference between the two methods was that use of symbolic algebra more strongly activated a part of the precuneus. This suggests that symbolic algebra demands more attentional and executive resources. The symbolic method activated areas in the occipital lobe less strongly, suggesting that it is less visually taxing than the model method.

Lee and colleagues were tentative about their findings, acknowledging that one limitation of their study was that the tasks performed were more difficult than typically used in functional magnetic resonance imaging studies. Their design and methodology were strong: A within-subject design ensured that individual brain differences did not factor into the results. Additionally, the researchers included control conditions so that results measured were due to mathematical cognition and not simply the stimuli's modality (diagram versus symbols). Follow-up t-tests were conducted after the initial ANOVA.

Because the participants were adults screened to perform very well at solving algebra problems (both using bar diagrams and symbolic algebra), study findings do not necessarily apply to younger, less skilled learners. Also, the study's participants performed all of their tasks mentally, which is a restriction not likely to be placed upon student learners. It is unknown whether the ability to use paper and pencil would render the symbolic experimental condition less taxing.

Literature Review Findings

This aim of this review is to address the following questions:

1. What is beneficial, detrimental, and challenging about bar diagrams in terms of their capacity for mathematical representation?
2. What is beneficial, detrimental, and challenging about bar diagrams in terms of supporting student problem solving?
3. What is beneficial, detrimental, and challenging about bar diagrams in terms of their capacity to support students' growth in mathematical understanding?

Given the previous analysis of the reviewed studies, below is a discussion of the claims relevant to the review questions.

Benefits of Bar Diagrams for Representing Mathematics

Bar diagrams have the capacity to represent a wide variety of mathematics. Murata (2008) identified all of the mathematical concepts and relationships that tape diagrams displayed in the Japanese textbook series she examined, and her list included all core concepts, operations, and problem types from Grade 1 to Grade 6 for both whole numbers and rational numbers (p. 388-389).

One particularly noteworthy representational benefit of bar diagrams is their capacity to represent each of the different types of arithmetic and subtraction problems in a way that clarifies the distinction among problem types. Consider the problem situation "Austin has 12 cookies and gives 7 away" versus the problem situation "Seven of Austin's 12 cookies are sugar cookies and the rest are chocolate chip." For both of these problem situations, 12-7 is an appropriate numerical (abstract) representation. This means that subtraction is both an appropriate operator for situations involving loss or removal of items as well as for situations where no items were lost or removed. This can be confusing for some students. Bar diagrams can clarify and demystify this state of affairs by making it explicit that as a Separate (Result-Unknown) problem, the first situation differs from the second situation as a Part-Part-Whole (Part Unknown) problem.

Bar diagrams can be configured in a variety of ways. Multiple bar diagrams can be associated or used in conjunction (Murata, 2008; Van den Heuvel-Panhuizen, 2003). A bar diagram can emerge from a model of a rectangular garage (Van den Heuvel-Panhuizen, 2003). A stack of bar diagrams can visually and mathematically represent exponential growth (Van den Heuvel-Panhuizen, 2003). Bar diagrams can be used in conjunction with ratio tables and double number lines (Shreyar, Zolkower, & Pérez, 2010; Van den Heuvel-Panhuizen, 2003). Bar diagrams themselves can function as double number lines (Van den Heuvel-Panhuizen, 2003). A 100-segment bar diagram can show percentage and its relationship to decimals (Shreyar, Zolkower, & Pérez, 2010; Van den Heuvel-Panhuizen, 2003).

Bar diagrams add a visual support to the abstract symbolization of mathematics, and bar diagrams can display both a qualitative and quantitative sense of a math concept—simultaneously. Van den Heuvel-Panhuizen (2003), for example, noted that as a representation of percentage, bar diagrams show a qualitative sense of fullness while also functioning as a calculation model.

Detriments of Bar Diagrams for Representing Mathematics

Though the number of representational benefits discussed in the literature largely outweighed the detriments, a number of representational problems stemming from the concrete nature of bar diagrams surfaced. Green (2009) noted that two of her students tended to rely on the visual features of bar diagrams instead of their numerical or logical features (p. 338). Also, Koedinger and Terao (2002) found that when students creatively attempted to represent negative numbers or “less-than” amounts with physically-present marks such as rectangles containing negative numbers, this created a visual miscue: Students attempted to subtract off the physical presence of these rectangles and thus used the mathematical operation of subtraction, but subtraction of a negative number requires addition, an algebraic principle that was not visually cued (p. 546).

A detriment unique to bar diagrams is that it is possible to select a problematic, initial quantity to use as a generator of subsequent quantities in the problem (Ng & Lee, 2009, p. 310). With traditional, symbolic algebra it typically makes no difference whether the variables are defined, for example, as x = y + 5 or y = x - 5. On the contrary, depending on the relationship between quantities, drawing a bar diagram of x in terms of y can render the problem solving process very different from drawing a bar diagram of y in terms of x. This favors student problem solvers who are efficient and poses challenges for those who are not.

Lastly, although the breadth of elementary mathematics that bar diagrams can represent is extensive, bar diagrams are limited in the scope of mathematics they can represent. Looi and Lim (2009) noted that when algebra problems reached a certain level of challenge or when algebra problems contained algebraic ratios, these problems could not be represented with bar diagrams (p. 373). Students who had been trained to translate bar diagrams into algebra thus could not proceed (p. 370).

Challenges with Bar Diagrams for Representing Mathematics

The main challenge with bar diagrams, in terms of representing mathematics, is that bar diagrams are so exacting that a single, small error can lead to an incorrect solution (Ng & Lee, 2009, p. 310). The teachers in Ng and Lee's (2009) study confirmed that students needed to proceed carefully when bar diagramming: Teachers felt that it could matter whether dotted or full lines were used to partition rectangles and felt that rectangles should be drawn in proportion when representing known amounts (p. 296). The students in an earlier study by Ng and Lee (2008) relaxed the latter requirement, however, and were still successful problem solvers.

Green (2009) noted that two of her participants encountered a challenge with operating upon unknown rectangles (p. 340). However, after working with known values as examples, these students were able to successfully transition to operating upon unknown rectangles with understanding.

Ng and Lee (2005) noted that lack of conceptual knowledge of fractions prevented some students from solving a word problem containing fractions. This was noteworthy because often bar diagrams allow students to circumvent otherwise necessary prerequisite knowledge and use arithmetic to solve problems successfully. For example, even if a student cannot perform calculations with the fraction 3/4, the way a bar diagram depicts 3/4 usually empowers the student to understand that dividing by four and then multiplying by three will yield 3/4 of an amount. However, the fraction problem Ng and Lee's participants attempted necessitated some knowledge of fractions.

Benefits of Bar Diagrams for Supporting Students' Problem Solving

Bar diagrams clearly benefitted students in terms of providing students with access to solving algebra problems that otherwise would have been beyond their mathematical ability (Booth & Terao, 2002; Ng & Lee, 2005, 2008). For example, after receiving instruction in bar diagramming, Booth and Terao's (2002) sixth grade students outperformed much older students on basic algebra problems (p. 543). Even without instruction in bar diagramming, seventh and eighth graders benefitted from the presence of bar diagrams that modeled algebraic story problems (Booth & Koedinger, 2012).

Students' problem solving success in Mahoney's (2012) and Looi and Lim's (2009) studies may have been influenced by other factors, such as lack of diversity of problem type and increased focus due to a computer environment, respectively. However, teachers and researchers did share, anecdotally, that student usage of bar diagrams permits teachers and researchers to gain insight into student thinking and conceptual understanding or misunderstanding.

Detriments of Bar Diagrams for Supporting Students' Problem Solving

Regarding support of students' problem solving, one clear detriment of bar diagrams discussed in the reviewed literature is that sixth graders who lack prior instruction in bar diagramming are less likely to attempt algebraic word problems if a bar diagram accompanies the problem (Booth & Koedinger, 2012, p. 500, 505). In other words, sixth grade students attempted story problems that lacked accompanying bar diagrams, but in many cases bar diagrams they could have ignored seemed to distract them away from the story problems that they otherwise typically attempted.

A second detriment of bar diagrams is a side effect of a benefit: Bar diagrams permit students to use a series of forward calculation arithmetic equations to circumvent using algebra when solving algebra problems (Ng & Lee, 2009). Looi and Lim (2009) noted that this can become problematic once students begin learning algebra: Many students confuse forward calculation arithmetic equations with the algebraic equations they needed to be generating. Habituating to solving algebra problems without algebra may cause some students later problems, an idea supported by Ng's (2003) finding that less than half of the study's participants were able to identify the unknowns of a problem in bar diagram form.

Challenges with Bar Diagrams for Supporting Students' Problem Solving

Most challenges with bar diagramming for supporting students' problem solving overlap or coincide with the challenges and detriments for representing mathematics. Because bar diagrams are so exacting, small and lone errors in representing mathematical situations can lead to a final incorrect solution. Because it can matter which variable is selected as the generator for drawing the other quantities in a problem, the sequence in which variables are presented can pose extra challenge. Because representing negative values or amounts "less-than" requires adding a physical mark, bar diagrams can miscue an incorrect choice of mathematical operation.

One challenge not previously mentioned is the potential for inadvertently changing generators midway during the problem solving representation phase (Ng & Lee, 2009). This is the mistake, for example, of accidently using the wrong bar as the base for drawing another variable in the problem. The equivalent error in traditional symbolic algebra would be, for example, to define z as y + 5 when z should be defined as x + 5.

Benefits of Bar Diagrams for Supporting Students' Growth in Mathematical Understanding

Bar diagrams are particularly beneficial for supporting students' understanding that real-world mathematical problem situations have underlying structure. Though not the main focus of Ng and Lee's (2008) study about students' perceptions of unknowns, one clear finding from the study was that bar diagrams capture and display problem structure and bring problem structure to the forefront for students. Participants in the study noted that as long as a problem situation had the same structure as another, then the same configuration of bar diagrams could be used for both. The study also clearly revealed that students perceive empty bar diagrams as receptacles of yet-to-be-assigned values, and this successfully conveys to students the notion of an unknown value.

Both Van den Heuvel-Panhuizen's (2003) study and Shreyar, Zolkower, and Pérez's (2010) study revealed that the visual nature of bar diagrams allow students to access the mathematical concept of percent from their current, individual level of understanding. Participants' entry level understandings were on a spectrum from very informal and qualitative to formal and quantitative. Bar diagrams allowed students to use their own reasoning, for example, to see how determining 1% is helpful and inherent in a formal percent formula. The presence of bar diagrams permitted the teacher to step back and allow student participants to think, diagram, communicate, and demonstrate their way from sense-making visuals to formal and symbolic solutions (Shreyar, Zolkower, & Pérez, 2010).

Van den Heuvel-Panhuizen (2003) noted that general-use bar diagrams have the capacity to emerge from diagrams used to model single, specific situations. This was the mechanism for students' mathematical growth documented in Van den Heuvel-Panhuizen's study, and one of the most significant findings was that bar diagrams are of great benefit due to their ability to continue to display early, informal conceptions of a mathematics concept while also displaying later, high-level, formal conceptions of a mathematics concept.

One of the interesting side effects of the representational challenges and detriments of bar diagrams is that algebra students developed an appreciation for algebra (Ng, 2003) and an understanding of the value algebra's increased abstractness adds. Specifically, because bar diagrams are exacting and are not able to represent more complex problems, students saw value in the directness, efficiency, and representational power of algebra (Ng, 2003).

Detriments of Bar Diagrams for Supporting Students' Growth in Mathematical Understanding

The only detriment of bar diagramming for supporting students' growth in mathematical understanding overlaps with a detriment discussed above related to problem solving: That of student habituation to forward (arithmetic) calculations which later can interfere with some students' understanding of how to generate algebraic equations to represent algebraic word problems. As Ng and Lee (2009) note, teachers in their study explained how primary students solved algebra problems without using algebra:

> Because they used the unitary method strategy to solve for the unknown unit, they avoided the need to construct equivalent equations. The objects of the procedural phase were a series of logical arithmetic equations, not a set of equivalent algebraic equations. Such a strategy circumvented the need to teach transformation of equations, which the teachers explained was beyond the primary school syllabus. (Ng & Lee, 2009, p. 297)

As previously mentioned, resonating with this state of affairs is Ng's (2003) finding that less than half of her participants could identify the unknowns in a bar diagram representation of an algebraic word problem. Weak connection between bar diagramming and algebraic symbolization was profound enough of a problem that Looi and Lim (2009) designed software and pedagogy for helping students bridge the two.

Challenges with Bar Diagrams for Supporting Students' Growth in Mathematical Understanding

None of the major claims in the reviewed studies identified challenges with bar diagrams for supporting students' growth in mathematical understanding.

Conclusion

Bar diagrams have the capacity to represent a wide variety of primary mathematics in profound, empowering ways. Bar diagrams are uniquely adept at representing primary mathematical concepts in ways that show their connection to other mathematical concepts. Bar diagrams can represent mathematics in ways that simultaneously show informal, qualitative aspects of a concept and formal, quantitative aspects of the concept. However, the scope of mathematics that bar diagrams can represent is not unlimited: At some point, more complex algebraic equations and more abstract mathematics are not amenable to representation via bar diagrams.

Bar diagrams provide prealgebraic students with the ability to solve algebra problems they otherwise would not be able to solve. When instructed in the use of bar diagrams, middle schoolers benefit from the presence of bar diagrams and from generating bar diagrams. On the other hand, without previous instruction with bar diagrams, only older and more accomplished middle schoolers benefit from the inclusion of bar diagrams with algebraic word problems.

Bar diagrams bring to the fore the structure of mathematical concepts and of problem situations in a sense-making way accessible to students at various levels of understanding. Bar diagrams allow students to become comfortable with operating upon unknowns, undoing operations, and maintaining balance across the equal sign. However, habituation to the visual nature of bar diagrams and circumvention of algebra via forward calculation arithmetic equations can cause some students problems, particularly once these students begin learning algebra.

Bar diagrams help students think, choose operations, estimate, understand formal procedures and formulas, communicate, and holistically represent mathematical problem situations. Bar diagrams grow with students from Grade 1 to Grade 6, from situation-specific modeling to general-use modeling, and from informal understanding to formal understanding. Bar diagrams support students in using their own reasoning in sense-making and problem solving and function as a common base to support classroom conversation and shared meanings.

Suggestions for Further Research

One of the developers' main goals for bar diagramming is that it facilitate mathematics word problem solving. From the literature under review, it is clear that bar diagramming aids some students' problem solving of some types of problems. However, overall students missed many of the word problems included in the various studies, and many word problems were omitted from analysis because so few students attempted those word problems.

Consequently, it would be valuable to more deeply explore the effect of bar diagramming upon word problem solving. In their work Children's mathematics: Cognitively guided instruction, Carpenter et al. (1999) have provided an analysis that categorizes mathematical word problems. Their categorization of word problems into types such as joining, separating, and part-part-whole problems with either start unknown, change unknown, or result unknown components reveals why some word problems are more challenging than others. It also lays the foundation for a schematic approach to mathematical word problem solving which is compatible with bar diagramming. A bar diagramming study incorporating Carpenter and colleague's categorizations might address research questions such as, (1) Which problem types are more amenable to bar diagramming, if any? (2) Does bar diagramming facilitate students' understanding of the similarities and differences among word problem types, and if so, how? (3) Can bar diagramming be used to help students understand why one operation can be appropriate for various problem types?

Another of the developers' goals for bar diagramming is that it facilitate the learning of algebra. From the literature under review, it is clear that while bar diagramming supports understanding of the concept of an unknown and permits students to work with algebraic problems earlier than otherwise, researchers and teachers believe that the link between algebra and bar diagramming can be improved. One problem revealed in the literature is the tendency of students to confuse forward calculations (bar diagramming inspired expressions that look like algebraic expressions but only demand arithmetic) with algebra.

Final Word

There is clearly potential for bar diagramming in the United States if it is implemented effectively, but as educators have learned in curriculum reform, no all-encompassing "magic bullets" exist. The effectiveness of any approach lies in its implementation and educators must recognize that the power of the approach will necessarily vary across students and settings. This is further influenced by the attitudes and knowledge base of the teachers asked to implement any innovation. Bar diagramming can provide mathematics educators with a potentially powerful tool for supporting student learning, and research to date does provide hints of the benefits, detriments, and challenges of implementing such an approach. The next question is: How do we best proceed from here?

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Appendix A

Countries Ranking Fifth or Higher on the Mathematics TIMSS and Whether Their Curriculum Features Bar Diagramming

1999 Grade 4

No Grade 4 TIMSS Mathematics assessment this year

1999 Grade 8

Rank	Country	Features Bar Diagramming?
1	Singapore	✓
2	Korea	
3	Chinese Taipei	✓
4	Hong Kong	✓
5	Japan	✓

2003 Grade 4

Rank	Country	Features Bar Diagramming?
1	Singapore	✓
2	Hong Kong	✓
3	Japan	✓
4	Chinese Taipei	✓
5	Belgium Flemish	

2003 Grade 8

Rank	Country	Features Bar Diagramming?
1	Singapore	✓
2	Korea	
3	Chinese Taipei	✓
4	Japan	✓
5	Belgium Flemish	

2007 Grade 4

Rank	Country	Curriculum Features Bar Diagramming?
1	Hong Kong	✓
2	Singapore	✓
3	Chinese Taipei	✓
4	Japan	✓
5	Kazakhstan	

2007 Grade 8

Rank	Country	Features Bar Diagramming?
1	Chinese Taipei	✓
2	Korea	
3	Singapore	✓
4	Hong Kong	✓
5	Japan	✓

2011 Grade 4

Rank	Country	Features Bar Diagramming?
1	Singapore	✓
2	Korea	
3	Hong-Kong	✓
4	Japan	✓
5	North Ireland	

2011 Grade 8

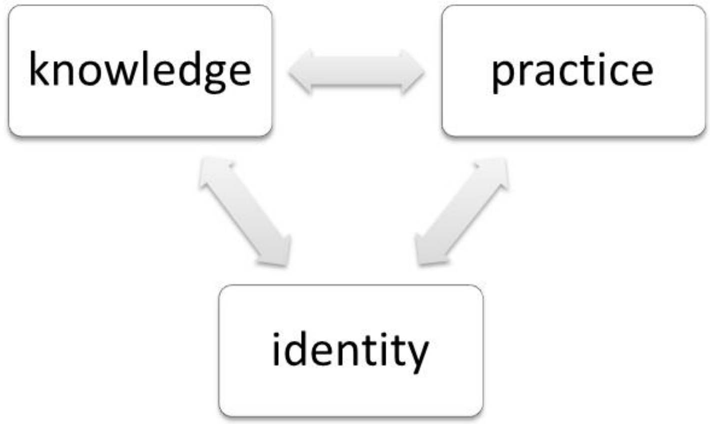
Rank	Country	Features Bar Diagramming?
1	Korea	
2	Singapore	✓
3	Chinese-Taipei	✓
4	Japan	✓
5	Russian Federation	

Appendix B

Boaler's Model Depicting the Interrelationship Among Knowledge, Practice, and Identity

Disciplinary

relationship



Note**.** Boaler's model depicting the interrelationship among knowledge, practice, and identity. It is not solely what a student learns but how the student does mathematics and learns mathematics that determines the student's relationship with mathematics and self-identity as a mathematics learner (or not). Thus Boaler (2002) discovered, for example, that learners' classroom practice of engaging in a "dance of agency"—that is, of "working at the interplay of their own agencies and disciplinary agencies"—is the practice that enabled observed learners to have an adaptive disciplinary relationship with mathematics and persist when they "hit a wall" during mathematical tasks (p. 46).

When students learn mathematics with bar diagrams, are there various ways for students to handle the bar diagrams? If so, do differing ways of handling bar diagrams make a difference in how students see mathematics and themselves as learners? Because each pair in Boaler's disciplinary relationship model has a relationship, understanding the how of a mathematics intervention such as bar diagramming is important. Adapted from "The development of disciplinary relationships: Knowledge, practice and identity in mathematics classrooms" by J. Boaler, 2002, For the Learning of Mathematics, 22(1), p. 46

Appendix C

List of Empirical Studies under Review

Booth, J. L., & Koedinger, K. R. (2012). Are diagrams always helpful tools? Developmental and individual differences in the effect of presentation format on student problem solving. British Journal of Educational Psychology, 82(3), 492–511.

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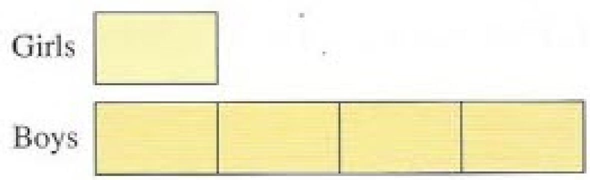
van Den Heuvel-Panhuizen, M. (2003). The didactical use of models in realistic mathematics education: An example from a longitudinal trajectory on percentage. Educational Studies in Mathematics, 54(1), 9–35.

Appendix D

Bar Diagrams Can Depict How the Same Situation Can Be Discussed with Different Mathematical Language Corresponding to Different but Related Concepts

Two quantities can be compared in many ways. For example, the following comparison model shows that there are 4 times as many boys as girls:



Difference	The difference between the number of girls and the number of boys is 3 units.
Multiple	The number of boys is 4 times the number of girls.
Fraction	The number of girls is $\frac{1}{4}$ times the number of boys.
Ratio	The ratio of the number of boys to the number of girls is 4 : 1.
	The ratio of the number of girls to the number of boys is 1 : 4.

Note. What is the conceptual relationship among fractions, ratios, and multiples? How bar diagrams show the relatedness of various ways of comparing numbers. Reprinted from The Singapore model method for learning mathematics p. 40, by T.H. Kho, S.M. Yeo, & J. Lim, 2009, EPB Pan Pacific: Publisher.

Appendix E

Bar Diagrams Can Represent Algebra Equations in Addition to Arithmetic Equations

Integration of the Model Method and Algebra

The following examples illustrate how the Model Method can be integrated with the algebraic method in solving algebra word problems.

Example 1

There are 50 children in a dance group. If there are 10 more boys than girls, how many girls are there?

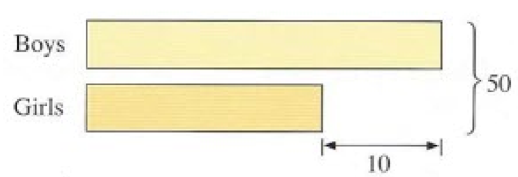
The algebraic method involves using a letter symbol such as x to represent an unknown quantity. For example, let x be the number of girls. As there are 10 more boys than girls, the number of boys is $x + 10$. The total number of boys and girls is $x + (x + 10)$, which is equal to 50. Thus students obtain the following equation to solve the problem:

$$x + (x + 10) = 50$$

The solution of the equation is $x = 20$.

There are 20 girls.

Students can draw the comparison model to represent the problem situation, and use it to solve the problem using the unitary method or the algebraic method.

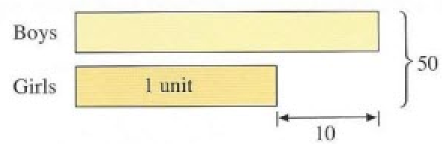


This model shows that the total number of boys and girls is 50, and that the difference between the number of boys and the number of girls is 10.

Note. Bar diagramming that permits use of arithmetic to solve algebra problems. This bar diagram enables students to use arithmetic to solve this word problem, which is otherwise typically solved with algebra or via the guess-and-check problem solving heuristic. Reprinted from The Singapore model method for learning mathematics p. 53, by T.H. Kho, S.M. Yeo, & J. Lim, 2009, EPB Pan Pacific: Publisher.

Unitary method

Take the number of girls as 1 unit. Students find the value of 1 unit and solve the problem as follows:

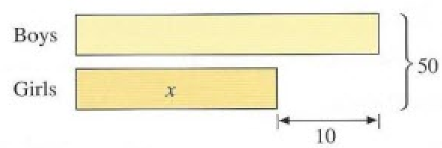


2 units = 50 – 10 = 40
1 unit = 40 ÷ 2 = 20

There are 20 girls.

Algebraic method

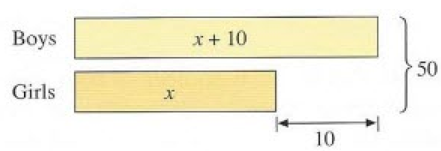
Let x be the number of girls.



The model enables students to express the number of boys in terms of x as shown in the following two variations.

Variation 1

The number of boys is 10 more than the number of girls. It is expressed as $x + 10$.



From the model, students see that the sum of $(x + 10)$ and x is 50, so they obtain the equation:

$$(x + 10) + x = 50$$

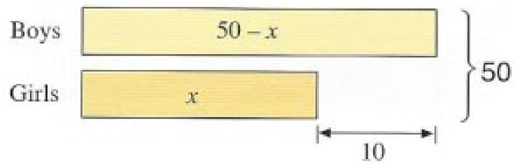
The solution of the equation is $x = 20$.

There are 20 girls.

Note. Two ways: Example of an algebra problem solved with arithmetic then algebra. Students can bypass the guess-and-check heuristic—and bypass algebra as well—and still determine the correct solution. Reprinted from The Singapore model method for learning mathematics p. 54, by T.H. Kho, S.M. Yeo, & J. Lim, 2009, EPB Pan Pacific: Publisher.

Variation 2

The total number of boys and girls is 50. The number of boys can be expressed as $50 - x$.



From the model, students see that the difference between $(50 - x)$ and x is 10, so they obtain the equation:

$$(50 - x) - x = 10$$

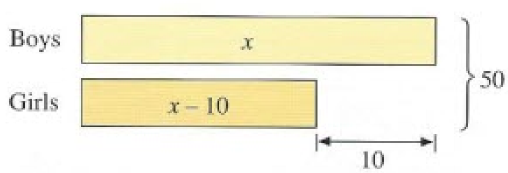
The solution of the equation is $x = 20$.

There are 20 girls.

As the number of boys and the number of girls are both unknown quantities, students may instead let x be the number of boys, and express the number of girls in terms of x. They will obtain a different equation to solve the problem. The following are alternative solutions to the problem.

Variation 3

The number of girls is $x - 10$.



From the model, students obtain the equation:

$$x + (x - 10) = 50$$

The solution of the equation is $x = 30$.

$$x - 10 = 20$$

There are 20 girls.

Note. Bar diagrams that show multiple ways of assigning variables in the same algebra problem. Reprinted from The Singapore model method for learning mathematics p. 55, by T.H. Kho, S.M. Yeo, & J. Lim, 2009, EPB Pan Pacific: Publisher.

Appendix F

Table A1
Studies' Locations, Sample Characteristics, and Relevant Research Findings

Year, author(s)	Sample location	Sample characteristics	Relevant research findings
(2002) Koedinger & Terao	not specified	35 6th grade students	Diagramming helps students solve algebra problems typically inaccessible to them and frequently missed by older students. Students' devising of abstract diagrams (e.g., diagrams representing negative quantities) typically led to error. The representation phase is difficult for these students when not one but multiple quantities are constructed in terms of a "less-than" relationship with a base quantity.
(2003) Ng	Singapore	Secondary two express stream students, ages 14+, females = 87, males = 58	Only 6.2% of the students opted to use the model method to solve the given word problem. About 95% of the students successfully recalled (verified and used) the model method when prompted. About 46% of the students articulated relatively fully how unknowns are represented in the model method. Some students viewed the model method as good for supporting understanding while others viewed the model method as too exacting and/or unsophisticated.
(2003) van Den Heuvel-Panhuizen	not specified	not specified	The bar model: (1) functions as on occupation meter to show fullness, (2) supports students' conception of percentage; (3) shows the relationship between percentage and fractions; (4) allows students to work flexibly at once with both low level aspects of percentage and increasingly advanced aspects; (5) easily shifts from a context-laden representation to a context-free representation; (6) supported student understanding of the 1% calculation strategy for understanding percents; (7) supported student understanding of the 1% estimation strategy for understanding percents; and (8) supported student understanding of complex percentage problems requiring backwards reasoning.
(2004) Ng	not applicable	not applicable	The model method helps develop algebraic thinking by supporting students in analyzing parts and wholes, working with unknowns, working with proportional reasoning, and doing and undoing.

Continued

Year, author(s)	Sample location	Sample characteristics	Relevant research findings
(2005) Ng & Lee	Singapore	151 students: male = 77, female = 74, average age = 10.7, grade = Primary 5, middle-class to lower middle-class	Students' success rate for the first of five assessment problems was 63%, 44%, 37%, 20%, and 15% respectively. Students set up diagram models to function as equations. On one problem, students used the bar model to bypass fractional computation that was beyond their present skill level. The last problem was less amenable to bar diagramming than typical. Errors revealed computational mistakes, lack of understanding of less-than and more-than relationships between quantities, and selection of a "generator" variable that increased the likelihood that students would make mistakes in defining remaining variables in the word problem.
(2007) Lee, Lim, Yeong, Ng, Venkatraman, & Chee	Singapore	18 adults: right-handed, 20-25 years old, male = 10, female = 8	During the problem representation phase, compared to the model method, symbolic algebra: (1) seemed more demanding on attentional resources; (2) activated the caudate region of the brain more, suggesting that symbolic algebra relies more on procedural retrieval; and (3) activated a part of the precuneus thought to be involved in attention orientation and information retrieval.
(2008) Murata	Not applicable	Not applicable	Tape diagrams: (1) assign related meanings to quantities in the word problem; (2) allow for understanding of the problem before use of abstract (numerical) representations; (3) allow for the relating of different mathematical operations; (4) allow the problem to be frozen in time at various times; (5) provide continuity in mathematics from grades 1 to 6; (6) allow students to experience mathematics as a subject of systematic relationships; and (7) support classroom discussion. Japanese tape diagrams were presented in such a way as to help students (1) analyze the givens in word problems and (2) ask and solve multiple questions about the same data. With Japanese tape diagrams, the sum of two addend bar diagrams is represented by the total lengths of the two bars (no third bar is drawn to function as the sum). On the contrary, US tape diagrams were presented in such a way as to model the solution of the word problem. A third, extra bar diagram is drawn to represent the sum of two addend bar diagrams.
(2008) Ng & Lee	Singapore	10 students: Primary 5, males = 7, females = 3, neighborhood school students	Each student perceived the rectangles as variables and not simply as unknowns.

Continued

Year, author(s)	Sample location	Sample characteristics	Relevant research findings
(2009) Green	Philadelphia	6th grade, from a predominately African American school: 3 students for Phase 1, 3 students for Phase 2	These students spontaneously understood to operate upon unknowns as if they were knowns (contradicting prior research findings.) They learned that an additive quantity is decomposable and recomposable. They gained understanding of the relationship between quantities in equations of the form $ax + b = cx$ and $ax + b = cx + d$. They gained understanding that the parts of a quantity are commutative. They were able to anticipate that creating equivalence is helpful in solving unknowns. They gained understanding that operating on one side of an equation must be balanced out with operating on the other side of the equation. They transitioned to use of symbolic algebra without the typical pitfalls and with seeming understanding. Some of these students inappropriately overrode quantitative information because visual cues were confusing or unhelpful.
(2009) Looi & Lim	Singapore	2 classes, each containing 34 secondary 1 students	After four weeks of the treatment (i.e., utilizing the ALGEBAR software and receiving algebra instruction lacking bar diagrams), the experimental class performed symbolic algebra better than did the control group. Pre-intervention, students had difficulty diagramming complex word problems. Post-intervention, a strong student correctly defined variables and set up equations, an average student did the same for routine word problems, and a weak student defined variables correctly. ALGEBAR seemed to improve students' facility with symbolic algebra in a round-about way—by making diagramming easier which led to increased problem comprehension that then made equation production easier for students. Inability to diagram word problems often coincided with inability to algebraically define variables. The ability to check for consistency between diagrams and algebraic representation gave students feedback and some independence from teachers.
(2009) Ng & Lee	Singapore	4 department heads, 14 teachers, 151 students Students: male = 77, female = 74, average age = 10.7, grade = Primary 5, middle-class to lower middle-class, 21% from the top stream, 79% from the middle stream	Students used bar models to solve a system of equations problem bypassing the traditional solution method. Students used bar models to solve fraction-containing word problems without use of fractions. Bar diagramming was exacting, and students were unsuccessful when they made the slightest errors in the representation phase. Some students inadvertently changed generators midway, resulting in incorrect solutions. Some students created correct bar diagrams but failed to answer the question asked in the word problem.

Continued

Year, author(s)	Sample location	Sample characteristics	Relevant research findings
(2010) Lee, Yeong, Ng, Venkatraman, Graham, & Chee	Singapore	17 adults: right-handed, 20-29 years old, male = 10, female = 7	During the problem solution phase, compared to the model method, symbolic algebra: (1) activated the intraparietal sulcus more strongly suggesting that symbolic algebra instigates more numerical processing; (2) yielded hemispheric asymmetry with higher left-side activation possibly indicative that symbolic algebra is more language-like or more dependent upon fact retrieval; (3) more strongly activated a part of the precuneus, suggesting that symbolic algebra demands more attentional and executive resources; (4) prompted stronger activation in the basal ganglia (no conclusive inferences made); and (5) activated areas in the occipital lobe less strongly, suggesting that symbolic algebra is less visually taxing than the model method.
(2010) Shreyar, Zolkower, & Pérez	Argentina	1 class of 21 students at a private bilingual (English/Spanish) school: 6th grade, medium to high economic status	Class participants used the bar diagram: (1) to connect a standard algorithm for calculating percents with a 1% strategy for calculating percents; (2) to demonstrate and record ideas; (3) to compare ideas with possible solutions; (4) to show the commutative property; and (5) to show the practicality of doing particular mathematical operations. Bar diagrams supported the teacher's use of questions to guide student thought, reflection, understanding, and learning.
(2012) Booth & Koedinger	American Middle West	373 6th-8th grade students, ethnically diverse	Typically the presence of diagrams did not aid these sixth graders' problem solving and may have even harmed it. However, these sixth grade students, including the low ones, improved with age on story + diagram problems, though not on story problems. These seventh graders and eighth graders benefitted from diagrams: Diagrams particularly aided these seventh and eighth grade students' problem solving of double reference problems.
(2012) Mahoney	Rural New Hampshire	4 students: 9-10 years, male = 3, female = 1	Model drawing instruction improved students' performance on multiplicative comparison and fraction word problems.