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On Problems in Extremal Graph Theory and Ramsey Theory

By

Steven J. La Fleur
Doctor of Philosophy

Mathematics

Vojtěch Rödl, Ph.D.
Advisor

Dwight Duffus, Ph.D.
Committee Member

Ronald Gould, Ph.D.
Committee Member

Andrzej Ruciński, Ph.D.
Committee Member

Accepted:

Lisa A. Tedesco, Ph.D.
Dean of the James T. Laney School of Graduate Studies

Date

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Steven J. La Fleur
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Abstract

On Problems in Extremal Graph Theory and Ramsey Theory

By Steven J. La Fleur

Extremal graph theory and Ramsey theory are two topics in graph theory with many problems which are being actively investigated. Both subjects involve finding substructures within graphs, or general graph-like structures, under certain conditions.

We consider an extremal problem regarding multigraphs with edge multiplicity bounded by a positive integer q. Given a family $\mathcal{F}$ of q-multigraphs, define $\mathrm{ex}(n, \mathcal{F})$ to be the maximum number of edges (counting multiplicities) that a q-multigraph on n vertices can have without containing a copy of any $F \in \mathcal{F}$ (not necessarily induced). It is well known that $\tau(\mathcal{F}) = \lim_{n\to\infty} \mathrm{ex}(n, \mathcal{F})/\binom{n}{2}$ exists for every family $\mathcal{F}$ (finite or infinite). Let $\mathcal{T} = \{\tau(\mathcal{F}) : \mathcal{F}$ is a family of q-multigraphs$\}$. We say the number α, $0 \le \alpha < q$ is a jump for q if there exists a constant $c = c(\alpha, q)$ such that if $\alpha' \in \mathcal{T}$ such that $\alpha' > \alpha$ then $\alpha' \ge \alpha + c$. We show that, for 3-multigraphs, every number in the interval $[0, 2)$ is a jump.

Given two (hyper)graphs T and S, the induced Ramsey number, $r_{\mathrm{ind}}(T, S)$, is defined to be the smallest integer N such that there exists a (hyper)graph R with the following property: In any two-coloring of the edges of R with red and blue, we can always find a red *induced* copy of T or a blue *induced* copy of S. In this dissertation we will discuss bounds for $r(K^{(k)}_{t,\ldots,t}, K^{(k)}_s)$ where $K^{(k)}_{t,\ldots,t}$ is the complete k-partite k-graph with partition classes of size t. We also present new upper bounds for $r_{\mathrm{ind}}(S, T)$, where $T \subseteq K^{(k)}_{t,\ldots,t}$ and $S \subseteq K^{(k)}_s$.

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To my wife, Heather.

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Chapter 1

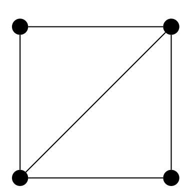
Introduction

1.1 Basic concepts

In this section, we will introduce the reader to some of the fundamental concepts in extremal graph theory and Ramsey theory.

Extremal graph theory and Ramsey theory both study graphs and generalizations of graphs, so understanding these objects will be crucial to reading this dissertation.

Definition 1.1. *A* graph $G = (V, E)$ *is a pair of sets, where* V *is a set of vertices and* $E \subseteq \binom{V}{2}$ *is a set of pairs of elements from* V*.*



$$\begin{pmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix}$$

Figure 1.1: A graph and its adjacency matrix

Given a graph G, we will use $V(G)$ and $E(G)$ to denote the vertex set and edge set of G, respectively. A *loop* in a graph is an edge that begins and ends at the same vertex (i.e. the pair (v, v) for a vertex $v \in V(G)$

is a loop). A graph without loops and without repeated edges is called a *simple graph*.

Throughout this dissertation, we will assume that the vertex set $V(G)$ is finite. In general, this is not a requirement, and there have been many papers that study graphs when the vertex set is infinite (even uncountably so).

Given a graph G on n vertices, its adjacency matrix is an $n \times n$ matrix, denoted A_G. Each row in A_G corresponds to a vertex of G, as does each column. This means that each entry of the matrix corresponds to a pair of vertices. For such a pair $v_i, v_j \in V(G)$, the matrix entry corresponding to this pair is 1 if there is an edge between them, and 0 otherwise. The entry v_i, v_i is 0 for a simple graph, by convention. Many properties of a graph G have been discovered by analyzing its corresponding adjacency matrix.

Definition 1.2. *A* multigraph *is a graph $G = (V, E)$ where we allow any edge to appear more than once.*

A multigraph with multiplicity bounded by q (or q-multigraph), is a multigraph where each edge is only allowed to appear at most q times. In this sense, a simple graph is a 1-multigraph, and hence the concept of multigraphs extend the notion of graphs.

Definition 1.3. *A k-uniform hypergraph (or k-graph) $\mathcal{G} = (V, E)$ is a pair of sets, where V is a set of vertices, and $E \subseteq \binom{V}{k}$ is a set of k-element subsets of V.*

1.2 Extremal Graph Theory

Extremal graph theory began with a simple question:

Question 1.4. *How many edges must a graph G of order n contain to guarantee the existence of a triangle as a subgraph of G?*

The triangle is K_3, the complete graph on 3 vertices.

We will use the notation $\mathrm{ex}(n, K_3)$ to denote the *extremal number* (i.e. the maximum number of edges in an order n graph which which is triangle-free). With this notation, the above question asks us to determine $\mathrm{ex}(n, K_3)$.

Of course, for all $n \geq 3$, the complete graph K_n contains a triangle. Therefore, the question must have an answer, and the answer was found in 1907 by Mantel, who proved the following.

Theorem 1.5. *For all $n \geq 3$,*

$$ex(n, K_3) = \left\lfloor \frac{n^2}{4} \right\rfloor.$$

Naturally, the same question may be asked with the triangle replaced by a more general graph. Paul Turán, in 1941, gave the answer for all complete graphs. More precisely, he proved the following theorem.

Theorem 1.6.

$$ex(n, K_{t+1}) = \left(1 - \frac{1}{t}\right)\frac{n^2}{2} + o(n^2).$$

Note that the term $o(n^2)$ in the above theorem is precisely known. In fact, Turán gave, for each n and t, the unique n vertex extremal graph for K_{t+1}, now called the *Turán graph* $T_{n,t}$. To construct this graph, partition the n vertices of $T_{n,t}$ into t parts as evenly as possible (i.e. each partition contains either $\lceil n/t \rceil$ or $\lfloor n/t \rfloor$ vertices). Now join every two vertices not in the same part by an edge.

Next, one might hope to determine the values of $\mathrm{ex}(n, H)$ for a fixed graph H which is not complete. However, this question is very difficult to answer precisely for all n. Aside from a few small examples, not many precise results exist. However, there are results in the case when n is "large."

If we fix a graph H, then $\mathrm{ex}(n, H)$ becomes a function of n.

Fact 1.7. *For every fixed graph H, $\frac{ex(n,H)}{\binom{n}{2}}$ is an increasing function of n.*

The proof of Fact 1.7 is straight forward and can be found, for example, in [10]. Thus the *extremal density* $\pi(H)$, defined by

$$\pi(H) = \lim_{n\to\infty} \frac{\mathrm{ex}(n, H)}{\binom{n}{2}},$$

exists for every graph H.

Erdős and Stone determine the value of $\pi(H)$ for every graph H.

Theorem 1.8. *Let H be a graph with chromatic number $\chi(H)$. Then*

$$\pi(H) = \Big(1 - \frac{1}{\chi(H) - 1}\Big).$$

This result implies that for any graph H with chromatic number $\chi(H)$,

$$\mathrm{ex}(n, H) = \Big(1 - \frac{1}{\chi(H) - 1}\Big)\binom{n}{2} + o(n^2).$$

Thus to determine the extremal number, we must determine the precise value of the $o(n^2)$ term above.

The Erdős-Stone theorem is interesting from another perspective as well. The theorem states that the set of extremal densities is precisely the set

$$\Big\{1 - \frac{1}{t} : t \geq 1\Big\}.$$

This immediately answers the following question.

Question 1.9. *What is the set of all extremal densities which can be obtained?*

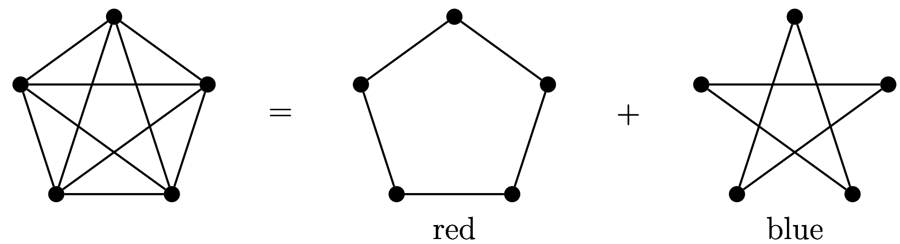


Figure 1.2: Triangle-free coloring of K_5

As mentioned, for graphs Question 1.9 is completely answered (and therefore somewhat uninteresting). However, one might ask both Question 1.4 and Question 1.9 with graphs replaced by more general structures, such as r-uniform hypergraphs with $r \geq 3$ or q-multigraphs with $q \geq 2$. It turns out that these questions are much harder in these settings. In Chapter 2 we investigate these questions in the setting of q-multigraphs. The answers given here are based on joint work with Paul Horn and Vojtěch Rödl [19].

1.3 Ramsey Theory

To understand the principles of Ramsey theory, consider the following simple problem. Is there a way to color the edges of K_6 with two colors, red and blue, in such a way that there is not a triangle with all red edges or all blue edges? As it turns out, the answer is no. No matter how you color the edges, you will always find a triangle which is monochromatic. However, there is a coloring of the edges of K_5 with red and blue which avoids a monochromatic triangle, as seen in Figure 1.2 below.

In 1928, Ramsey [25] proved the following theorem, which we state in the scope of graphs. (The original statement of Ramsey is more general and can be found in [10], for example.)

Theorem 1.10. *Given positive integers r and m, there exists an integer*

$n_0 = n_0(m, r)$ such that the following is true. For any $n \geq n_0$, any coloring of the edges of K_n with r colors yields a monochromatic copy of K_m in one of the r colors.

Notice that Theorem 1.10 guarantees the existence of the number n_0, but doesn't hint at what it is. Much of the work in Ramsey theory is determining the values of these so-called *Ramsey numbers*, denoted by $r(m)$. For the remainder of the dissertation, we will focus on the case when there are two colors (i.e. $r = 2$). As above, we will use red and blue as these two colors.

Instead of finding a red or blue copy of K_n, suppose that we wanted to find either a red copy of K_s or a blue copy of K_t for some, possibly different, values s and t. It can be easily argued using only Theorem 1.10, that there exists a number $n_0 = n_0(s, t)$ such that any coloring of the edges of K_n, $n \geq n_0$ with red and blue yields either a red copy of K_s or a blue copy of K_t as desired. Indeed, set $m = \max\{s, t\}$ and Theorem 1.10 guarantees that for n large enough, any coloring of K_n yields either a blue K_m or a red K_m. Since $K_s, K_t \subset K_m$, we immediately find the desired monochromatic subgraph. Note that the case when $s \neq t$ is often referred to as the *off-diagonal* case, for obvious reasons. In this case, the ramsey number will be denoted $r(s, t)$, and it follows that $r(m, m) = r(m)$.

There are not many values of these Ramsey number known precisely. In fact, the summary of all of the known Ramsey numbers for complete graphs is given in Table 1.1.

Since it is difficult to determine the Ramsey numbers precisely, the next best thing is to find bounds. Remarkably, in 1947 Erdős, using a probabilistic method, was able to determine the following lower bound for the Ramsey number $r(t)$:

$$r(t) \geq (1 + o(1)) \frac{t}{e\sqrt{2}} 2^{t/2}.$$

s,t	1	2	3	4	5	6	7	8	9	10
1	1	1	1	1	1	1	1	1	1	1
2	1	2	3	4	5	6	7	8	9	10
3	1	3	6	9	14	18	23	28	36	
4	1	4	9	18	25					
5	1	5	14	25						
6	1	6	18							
7	1	7	23							
8	1	8	28							
9	1	9	36							
10	1	10								

Table 1.1: Known ramsey numbers $r(s,t)$

In 1975 Spencer [31] was able to improve this bound by a factor of two, and this is still the best lower bound. On the other hand, the best upper bound known currently is due to Conlon [5]:

$$r(t) \le t^{-c\log t/\log\log t}4^t.$$

Therefore, the Ramsey number $r(t)$ grows exponentially in t, and the exponential growth factor is between $\sqrt{2}$ and 4.

1.3.1 Induced Ramsey Numbers

Several generalizations of Ramsey's original theorem have been studied in the last century. Here we discuss one such generalization. Given graphs S and T, we want to find a graph R with the following property. Any 2-coloring of the edges of R yield a red induced copy of T or a blue induced copy of S. Any such graph R is called the *induced Ramsey graph* with respect to S and T. It was shown in [9] [14] and [26] that there exists an induced Ramsey graph for every pair S and T. Therefore we denote by $r_{\text{ind}}(S,T)$ the minimum order of a Ramsey graph for S and T, called its *induced Ramsey number*.

1.3.2 Ramsey theory for hypergraphs

Not surprisingly, the questions posed for graphs can be extended to hypergraphs as well. In fact, Ramsey's original statement includes the existence of Ramsey numbers for k-uniform hypergraphs for all $k \geq 3$. The existence of induced Ramsey numbers for all pairs of k-graphs $\mathcal{S}$ and $\mathcal{T}$ is due to [1] and [23]. In Chapter 3, we give new bounds for the induced Ramsey numbers of a certain class of k-graphs. These results are based on joint work with Domingos Dellamonica and Vojtěch Rödl [8].

Chapter 2

Jumps and non-jumps in multigraphs

2.1 Introduction

For a positive integer q, we consider multigraphs with edge multiplicities bounded above by q, which we call q-multigraphs for convenience. Given a q-multigraph G, the *density* of G will be defined as $d(G) = |E(G)|\binom{n}{2}^{-1}$, where $|E(G)|$ counts the multiplicity of each edge. In particular, the density of a q-multigraph is on the interval $[0, q)$.

Given a family of q-multigraphs $\mathcal{F}$, we define the set $\mathrm{Forb}(\mathcal{F})$ to be the family of all graphs which do not contain a member of $\mathcal{F}$ as a subgraph (not necessarily induced). Let $\mathrm{ex}(n, \mathcal{F})$ be the maximum number of edges (counting multiplicity) of any q-multigraph $G \in \mathrm{Forb}(\mathcal{F})$ with $|V(G)| = n$. Finally we define the *extremal density* of $\mathcal{F}$ as

$$\tau(\mathcal{F}) = \lim_{n\to\infty} \frac{\mathrm{ex}(n, \mathcal{F})}{\binom{n}{2}}.$$

The limit in the above expression exists, a fact which follows from the averaging argument of Katona, Nemetz and Simonovits [21].

We want to examine the structure of the set

$$\mathcal{T}_q = \{\tau(\mathcal{F}) : \mathcal{F} \text{ is a (possibly infinite) family of } q\text{-multigraphs}\}.$$

Definition 2.1. *We say that the number* $\alpha \in [0, q)$ *is a* jump for q *if there exists a constant* $c = c(\alpha, q)$ *such that given any* $\alpha' \in \mathcal{T}_q$ *with* $\alpha' > \alpha$*, it follows that* $\alpha' \geq \alpha + c$*.*

For $q = 1$, a simple corollary of the Erdős-Stone theorem, [16], is that every $\alpha \in [0, 1)$ is a jump. Indeed, the set $\mathcal{T}_1$ is precisely

$$\mathcal{T}_1 = \left\{1 - \frac{1}{k}\right\}_{k=1}^{\infty}.$$

For $q \geq 2$, obtaining an explicit description of the set $\mathcal{T}_q$ of extremal densities is much harder. It might be easier, if we aim only to understand the structure of the set itself. To that end, we give the following definition.

Definition 2.2. *A set* $\mathcal{T}$ *is called* well-ordered *if it does not contain an infinite decreasing sequence.*

Since we know $\mathcal{T}_1$ explicitly, it is simple to observe that it is well-ordered. A question which arises naturally for q-multigraphs with $q \geq 2$ is the following.

Question 2.3. *For* $q \geq 2$*, is* $\mathcal{T}_q$ *well-ordered?*

Erdős, Brown and Simonovits, in [2, 3], resolved Question 2.3 for $q = 2$ in the affirmative, showing $\mathcal{T}_2$ is well-ordered. Sidorenko [30] gave an alternate proof of this fact which gives a somewhat more explicit description of the set $\mathcal{T}_2$. On the other hand, Rödl and Sidorenko showed in [28] that the answer to Question 2.3 is no for q-multigraphs where $q \geq 4$, by constructing a family of sequences of graphs with decreasing extremal densities.

Two interesting questions remain. What can one say about $\mathcal{T}_q$ when $q = 3$; is $\mathcal{T}_3$ well-ordered? Second, for $q \geq 4$, only some $\alpha \in (0, q)$ are known to be non-jumps. Can the jumps and non-jumps be characterized? The main results of this chapter give partial answers to each of these questions.

Regarding the first question, an easy argument, which we give in Section 2.2, shows that $\mathcal{T}_q \cap (0, \frac{q}{2})$ is well-ordered if and only if $\mathcal{T}_{q-1} \cap (0, \frac{q}{2})$ is well-ordered for all $q \geq 3$. It follows from this that, since $\mathcal{T}_2$ is well-ordered, then $\mathcal{T}_3 \cap (0, \frac{3}{2})$ is also well-ordered. Consequently every $\alpha \in (0, \frac{3}{2})$ is a jump for $q = 3$. However, in order to determine whether $\mathcal{T}_3$ outside of the interval $[0, \frac{3}{2})$ requires a non-trivial argument, and our first result proves that $\mathcal{T} \cap [3/2, 2)$ is well-ordered for $q = 3$, i.e. we show:

Theorem 2.4. *Every number $\alpha \in [0, 2)$ is a jump for $q = 3$.*

In order to better understand the struture of $\mathcal{T}_3 \cap [0, 2)$, we determine the order type of this set. Order type is a measure of the structural complexity of a well-ordered set which we define precisely in Section 2.4.

For $q \geq 4$, the argument of Rödl and Sidorenko shows that $\alpha = q - 1$, among some other values of α, is not a jump. We give an alternate proof of this fact, using spectral graph theory. Our proof allows us to show the following which suggests that the set of non-jumps gets richer as q increases.

Theorem 2.5. *Suppose $r \in \mathbb{Q}$ with $0 < r \leq 1$. Then there exists an integer $Q = Q(r)$ such that for any $q > Q$, $q - r$ is not a jump for q.*

The remainder of the chapter is organized as follows. In Section 2.2, we make some preliminary definitions, and state some results established by Sidorenko in [30] which we use in the proof of Theorem 2.4. In Section 2.3 we complete the proof of Theorem 2.4, which extends ideas of Sidorenko in the case $q = 2$. In Section 2.4, we determine the order type of $\mathcal{T}_3 \cap [0,2)$. We outline the definitions and facts from spectral graph theory which will be necessary for the proof of Theorem 2.5 in Section 2.5 and prove this theorem in Section 2.6.

2.2 Preliminaries

The basic idea of the proof of Theorem 2.4 is, first, to observe that we may restrict our attention to the extremal densities obtained by a special class of 'globally dense' graphs. Second, we show that these dense graphs may be constructed in an appropriate manner from a *bounded* number of graphs. The fact that the number of these graphs is bounded is the essential reason why we may derive the fact that $\mathcal{T}_3 \cap [0,2)$ is well-ordered.

Let us begin by setting out some notation. Throughout, G denotes a q-multigraph. For a vertex $v \in V(G)$ and set $S \subseteq V$, the *neighborhood* of v in S, denoted $N_S(v)$ is a multiset consisting of all neighbors of v in S with multiplicity. The t-neighborhood of v in S, denoted $N_S^t(v)$ is the *set* (not multiset!) of neighbors of G which occur in $N(v)$ with multiplicity exactly t. In the case, $S = V(G)$, we simply refer to $N(v)$ and $N^t(v)$. As a slight abuse of notation, for any two vertices u and v, we say that $N(u) = N(v)$ if $N_{V(G)\setminus\{u,v\}}(u) = N_{V(G)\setminus\{u,v\}}(v)$ as multisets. That is, the two vertices u and v are called *symmetric* if they have the same neighborhood (with respect to multiplicities) in $V(G) \setminus \{u,v\}$.

Given two q-multigraphs, H and G with vertex sets $\{u_1, \ldots, u_m\}$ and $\{v_1, \ldots, v_n\}$ respectively with $m \leq n$, we say that H is a subgraph of G, denoted $H \subseteq G$ if there is an injective map $\varphi : V(H) \to V(G)$ such that if there are r edges between u_i and u_j in H, then there are at least r edges between $\varphi(u_i)$ and $\varphi(u_j)$ in G.

We denote by $K_k^{(t)}$, for $t \leq q$, the complete graph where every pair of vertices is joined by t edges.

2.2.1 Globally dense graphs

We begin this section with a definition which we use to express the densities of a q-multigraphs constructed from G.

Definition 2.6. *The* Lagrangian *of a q-multigraph G on n vertices is defined to be*

$$\lambda(G) = \max\{\mathbf{u}^* A_G \mathbf{u} : \sum_{i=1}^{n} u_i = 1,\ u_i \geq 0\ \forall i \leq n\}$$

where $\mathbf{u}^*$ denotes the transpose of the vector $\mathbf{u}$, and A_G denotes the adjacency matrix of G.

Notice that if G and H are q-multigraphs with $H \subset G$ then it follows that $\lambda(H) \leq \lambda(G)$. To prove this, let $\mathbf{x} = (x_1, x_2, \ldots, x_m)^*$ be such that $\mathbf{x}^* A_H \mathbf{x} = \lambda(H)$. For the injective mapping $\varphi : V(H) \to V(G)$, define a vector $\mathbf{y} = (y_1, y_2, \ldots, y_n)^*$ with length $|V(G)|$ as follows. Let the component y_j of $\mathbf{y}$ correspond to the vertex $u_j \in V(G)$. Then

$$y_j = \begin{cases} x_i & \text{if } \varphi(v_i) = u_j \\ 0 & \text{else.} \end{cases}$$

A computation, then yields

$$\lambda(G) \geq \mathbf{y}^* A_G \mathbf{y} \geq \mathbf{x}^* A_H \mathbf{x} = \lambda(H),$$

as claimed.

Definition 2.7. *Let G be a q-multigraph, on $\{v_1, \ldots, v_n\}$. The* blowup *of G by a vector $\mathbf{x} \in \mathbb{Z}^n_{\geq 0}$, denoted by $G(\mathbf{x})$, is defined as the graph constructed by the following procedure:*

(*i*) *Replace each vertex $v_i \in G$ with a set of vertices V_i of size x_i*

(*ii*) *If there are p edges between v_i and v_j in G, then adjoin every vertex of V_i with every vertex of V_j by p edges.*

(*iii*) *Each of the vertex sets V_i is independent. If vertex v_i has p' loops in G, then every pair of vertices in V_i will be joined p' times, and each vertex in V_i will have p' loops.*

A *modified blowup* of G is the same, but replacing condition (*iii*) by

(*iii'*) Each of the vertex sets V_i is a $K_{x_i}^{(1)}$.

Throughout the chapter, we will use $G(\mathbf{x})$ interchangeably for both a blowup and a modified blowup. However, we will always make it clear which we are referring to, so that there is never any confusion.

Given a q-multigraph G, and an integer vector $\mathbf{x}$, the quantity $\mathbf{x}^* A_G \mathbf{x}$ counts the number of edges in the blowup $G(\mathbf{x})$ (see Defintion 2.7 below). For each k, we may find at least one vector $\mathbf{x}_k$ with component sum equal to k which maximizes $\mathbf{x} A_G \mathbf{x}$ over all such vectors. In this way, $\mathbf{x_k}$ also gives the maximum density

$$d(G(\mathbf{x}_k) = \frac{\mathbf{x}_k^* A_G \mathbf{x}_k}{\binom{k}{2}}$$

of the blowup $G(\mathbf{x}_k)$ over all blowups of G with k vertices. As $k \to \infty$, the theory of Lagrange functions implies $d(G(\mathbf{x_k})) \to \lambda(G)$, where $\lambda(G)$ is defined as above. On the other hand, this limit measures the densest blowup of G as the number of vertices in the blowup tends to infinity.

Observe that, for any q-multigraph G and any vector $\mathbf{x} \in \mathbb{N}^n$, $G \subset G(\mathbf{x})$.

Definition 2.8. *A q-multigraph G is* globally dense *if, for any induced subgraph G' of G such that $G' \neq G$, it follows that $\lambda(G') < \lambda(G)$.*

Recall that we defined $\mathcal{T}_q$ to be the set of extremal densities of famlilies of q-multigraphs. We now define several related sets which are useful in the proof of Theorem 2.4. First we define the following sets:

$$\begin{aligned} \mathcal{M}_q &= \{G : G \text{ is a globally dense } q\text{-multigraph}\} \\ \mathcal{L}_q &= \{\lambda(G) : G \in \mathcal{M}_q\}. \end{aligned} \tag{2.1}$$

A simple observation, which follows from the definition of globally dense graphs and the monotonicity of $\lambda(\cdot)$ is the following. Given a q-multigraph G, either G is globally dense or it contains a globally dense subgraph G' with $\lambda(G') = \lambda(G)$. Therefore, we may have just as easily defined $\mathcal{L}_q$ to be $\{\lambda(G)\colon G \text{ is a } q\text{-multigraph}\}$. While these definitions are equivalent, the fact that we may consider only globally dense multigraphs is helpful to our proof.

We also define truncated versions of $\mathcal{T}_q$ and $\mathcal{L}_q$ as follows. For $\alpha \geq 0$, we define $\mathcal{T}_q^\alpha$ to be $\mathcal{T}_q^\alpha = \mathcal{T}_q \cap [0, \alpha)$, and similarly define $\mathcal{L}_q^\alpha$ to be $\mathcal{L}_q^\alpha = \mathcal{L}_q \cap [0, \alpha)$. In this notation, the fact mentioned in the paragraph preceding the statement of Theorem 2.4 is that $\mathcal{T}_q^{q/2}$ is well-ordered if and only if $\mathcal{T}_{q-1}^{q/2}$ is well-ordered.

The next proposition states a key observation made by Brown and Simonovits [4].

Proposition 2.9. *For any $\alpha \geq 0$ and $q \geq 1$, we have the following:*

(i) $\mathcal{T}_q^\alpha$ is well-ordered if and only if $\mathcal{L}_q^\alpha$ is well-ordered.

(ii) If we denote by $\overline{\mathcal{L}_q^\alpha}$ the closure of the set $\mathcal{L}_q^\alpha$ with regard to its limit points then $\mathcal{L}_q^\alpha \subset \mathcal{T}_q^\alpha \subseteq \overline{\mathcal{L}_q^\alpha}$.

Fact 2.10.

$$\mathcal{L}_q^{q/2} = \mathcal{L}_{q-1}^{q/2} \textit{ for any } q \geq 1.$$

Fact 2.10 follows from the fact that, if a globally dense q-multigraph G contains an edge of multiplicity q, then $\lambda(G) \geq \lambda(K_2^q) = q/2$ where K_2^q is the graph consisting of two vertices joined by an edge of multiplicity

q. Together with this, Proposition 2.9 (i) immediately implies that $\mathcal{T}_q^{q/2}$ is well-ordered if and only if $\mathcal{T}_{q+1}^{q/2}$ is well-ordered.

In order to better understand the Lagrangian of globally dense multigraphs, we recall some results of Sidorenko and make a few additional observations which will be useful in our proof. We will denote by $\mathbf{1}$ the vector $(1, 1, \ldots, 1)^*$. The length of this vector will be apparent from the context in which the vector is used.

In [30], Sidorenko gave the following useful characterization of globally dense q-multigraphs:

Theorem 2.11 ([30, Theorem 1]). *A q-multigraph is globally dense if and only if its adjacency matrix A_G satisfies*

(a) A_G is non-singular, and all components of the vector $\mathbf{1}A_G^{-1}$ are positive; and

(b) A_G is of negative type, i.e. $\mathbf{x}^* A_G \mathbf{x} < 0$ holds for every vector $\mathbf{x}$ such that $\mathbf{x}^*\mathbf{1} = 0$.

He proved the sufficiency of the conditions by exhibiting a vector $\mathbf{y}$ and argued that, when A_G satisfies (a) and (b), $\mathbf{y}^* A_G \mathbf{y} = \lambda(G)$.

For our purposes, the most useful aspect of Theorem 2.11 is that it allows us to show a multigraph G is not globally dense by showing that its adjacency matrix A_G is not of negative type. Note that if a principle submatrix of A_G is not of negative type, neither is A_G. As a slight abuse of terminology, we shall say that G is of negative type if its adjacency matrix A_G is. To summarize, we observed that if G is globally dense then every induced subgraph of G is of negative type. Now we list a few examples of graphs that are not of negative type which will be relevant in the next section.

Note that in each of the following examples, we give a 3-multigraph as well as an integer vector $\mathbf{x}$ of *vertex weights* which show that $\mathbf{x}^* A_G \mathbf{x} \geq 0$ and hence G is not of negative type. Therefore, when we refer the weight of a vertex, we are actually referring to the component of the vector $\mathbf{x}$ which corresponds to the vertex.

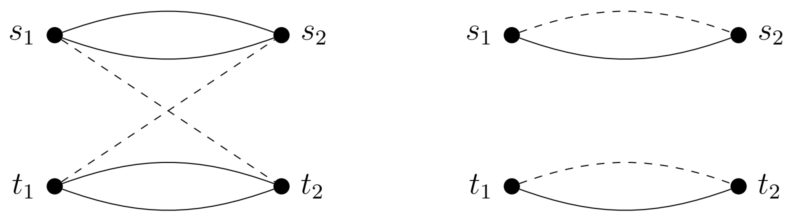
Example 2.12. *The following 3-multigraphs are not of negative type:*

(1) *The 3-multigraph consisting of two independent vertices is not of negative type, as the adjacency matrix is the zero matrix. Thus in a globally dense graph every pair of vertices will be joined by at least one edge.*

(2) *Sidorenko [30] observed the following family is not of negative type. Let $E_{a,b,c}$ (with $c(ab-1) \geq (2ab+a+b)$) be the 3-multigraph with three sets of vertices A, B and C of sizes a, b and c respectively. Every vertex of A is connected to every vertex in B by at least two edges, and every other pair of vertices of $A \cup B \cup C$ is connected by only a single edge. To observe these graphs are not of negative type, we take $\mathbf{x}$ so that the weight for each vertex in A is $c(b+1)$, the weight of each vertex in B is $c(a+1)$ and the weight of each vertex in C is $-(a(b+1)+b(a+1))$. A short calculation shows that, if E is the adjacency matrix of $E_{a,b,c}$, then $\mathbf{x}^* E \mathbf{x} \geq 0$ as long as $c(ab-1) \geq (2ab+a+b)$. If we further require $a = 1$ (which we will when we use $E_{a,b,c}$ later) then the previous inequality reduces to $c(b-1) \geq 3b+1$ which is satisfied when $b \geq 2,\ c \geq 4,$ and $b+c \geq 9$.*

(3) *If G is a q-multigraph on vertex set $S \cup T$, where $|S| = |T|$, such that the total number of edges completely contained in one of S or T is at least the number of edges in between them then G is not of negative type. Indeed, this follows by setting the weights of the vertices in S and T to 1 and -1 respectively. Of particular use to us are the 3-multigraphs of type D_4 and E_4 in the figure below. The top two vertices are in S and the bottom two are in T for both D_4 and E_4. For simplicity, each pair of vertices of D_4*

and E_4 are connected by one more edge than pictured (e.g. s_1 and t_1 is joined by a single edge in both). A dashed line indicates that both the multigraph with and without this edge are in D_4 and E_4 respectively. All of these are of this type where $|S| = |T| = 2$.



D_4 E_4

2.2.2 Irreducible Graphs

In this subsection we characterize irreducible q-multigraphs. To begin with, we give the following definition.

Definition 2.13. *For a q-multigraph, a pair of distinct vertices $u, v \in V(G)$ are called* equivalent *if,*

(1) $N(u) = N(v)$ (recall this is in $G \setminus \{u, v\}$).

(2) u and v are joined by a single edge, $v \in N^1(u)$.

Further we define any vertex to be equivalent to itself, in order to ensure that this is an equivalence relation.

Let us call a q-multigraph G *irreducible* if no pair of distinct vertices in G are equivalent, otherwise we call G *reducible*.

We call the unique maximal irreducible subgraph of G the *core* of G and denote it by $G/\sim$. The uniqueness of the core of a q-multigraph G comes from the fact that the partition of the vertices by the equivalence relation is completely determined, based on the distribution of the edges of G. Note that $G/{\sim}$ is a subgraph of G induced by one vertex from each equivalence class.

Any reducible q-multigraph G is a modified blowup of $G/{\sim}$. To prove this, consider an arbitrary modified blowup of $G/{\sim}$ (or any q-multigraph). Suppose that u is obtained by blowing up the vertex $v \in V(G/\sim)$. By the definition of a modified blowup, the vertices u and v are symmetric and joined by a single edge. However this directly implies that u and v are equivalent. Since $G/\sim$ contains precisely one vertex from each equivalence class of G, we can obtain G by performing a modified blowup of $G/\sim$, where each vertex of $G/\sim$ is blown up to the size of its equivalence class in G.

Example 2.14.

(1) *Recall,* $K_k^{(t)}$ *denotes the complete graph of multiplicity* t*. For any* $2 \leq t \leq q$*, observe* $K_k^{(t)}$ *is irreducible. On the other hand,* $K_k^{(1)}/{\sim}$ *is a single vertex.*

(2) *Given a* q*-multigraph* G *and a vector* $\mathbf{x} > \mathbf{1}$*, then the modified blowup* $G(\mathbf{x})$ *is reducible. If, additionally* G *is irreducible itself, then* $G(\mathbf{x})/\sim = G$*.*

2.3 Proof of Theorem 2.4

The strategy of the proof of Theorem 2.4 is to show that for globally dense 3-multigraphs G with $\lambda(G) < 2$, the size of $G/{\sim}$ can be bounded in terms of $2 - \lambda(G)$.

Throughout this section a few particular classes of graphs will be important in addition to those in Example 2.12 of the previous section. For a positive integer a, let $K_{a,a}^{(1,1,3)}$ be the bipartite 3-multigraph with a vertices in each partite set, three edges between each pair of vertices from opposite partite sets and a single edge between any two vertices of the same partite set. Then the following can be shown by a direct calculation:

Proposition 2.15. $\lim_{a\to\infty} \lambda(K_{a,a}^{(1,1,3)}) = 2$

Proof. In fact, $\lambda(K_{a,a}^{(1,1,3)}) = 2 - 1/n$. To prove this, notice that $\lambda(\cdot)$ is linear over graph edge decomposition. A simple decomposition of $K_{a,a}^{(1,1,3)}$ is into a complete simple graph K_{2a} and two complete bipartite graphs $K_{a,a}$. Now, we simply compute

$$\lambda(K_{2a}) = 1 - \frac{1}{2a},$$
$$\lambda(K_{a,a}) = \frac{1}{2},$$

using the fact that, asymptotically, the densest blowup of a $2a$-partite (or bipartite) graph is the Turán graph $T_{2a,k}$ (or $T_{2,k}$). Thus $\lambda(K_{a,a}^{(1,1,3)}) = (1 - 1/(2a)) + 2 \cdot 1/2 = 2 - 1/(2a)$, as claimed. □

Also consider a complete 3-multigraph on k vertices such that there is an edge of multiplicity two or three between any two vertices, which we will call a *graph of type* $K_k^{(2,3)}$. We can find a lower bound on the density of any 3-multigraph G of type $K_k^{(2,3)}$. Since $K_k^{(2)} \subseteq G$, it follows that $\lambda(G) \geq \lambda(K_k^{(2)}) = 2 - \frac{2}{k}$. (The latter equality here is obtained by decomposing $K_k^{(2)}$ into two copies of K_k, and calculating $\lambda(K_k) = 1 - 1/k$). Therefore, given any $\alpha < 2$, we may choose $k = k(\alpha)$ large enough so that $\lambda(G) > \alpha$.

To summarize, given any $\alpha \in [0, 2)$, there exist integers $a = a(\alpha)$ and $k = k(\alpha)$ so that any globally dense 3-multigraph G with $\lambda(G) \leq \alpha$ does not contain $K_{a,a}^{(1,1,3)}$ or $K_k^{(2,3)}$ as a subgraph. We now state the following lemma and show how it implies Theorem 2.4.

Lemma 2.16. *Let G be a globally dense 3-multigraph with $\lambda(G) <$ 2. Let $a = a(\lambda(G))$ and $k = k(\lambda(G))$ be defined as in the previous* *paragraph (and $G/\sim$ as in Sec. 2.2.2). Then*

$$|V(G/\sim)| \leq r(k, (3k+6)2^{r(2a,k)}),$$

where $r(a,b)$ denotes the usual Ramsey number (see Sec. 1.3).

Proof of Theorem 2.4. First note that since the set $\mathbb{N}_\infty = \{1, 2, \ldots, \infty\}$ is well-ordered, then the set $\mathbb{N}_\infty^r = \mathbb{N}_\infty \times \cdots \times \mathbb{N}_\infty$ has the descending chain condition (i.e. all decreasing sequences of elements of $\mathbb{N}_\infty^r$ are finite) under the ordering $\mathbf{x} \leq \mathbf{y}$ where we say $\mathbf{x} \leq \mathbf{y}$ if and only if $x_i \leq y_i$ for all $i \leq r$. This is an important observation to keep in mind as we proceed.

Recall that $\mathcal{T}_3^2$ is the set of extremal densities in the interval $[0, 2)$ and $\mathcal{L}_3^2$ is

$$\begin{aligned}\mathcal{L}_3^2 &= \mathcal{L}_3 \cap [0,2)\\ &= \{\lambda(G) : G \text{ is a globally dense 3-multigraph, } \lambda(G) < 2\}.\end{aligned}$$

By Proposition 2.9, if $\mathcal{L}_3^2$ is well-ordered, then so is $\mathcal{T}_3^2$. Hence it suffices to show that $\mathcal{L}_3^2$ is well-ordered. Actually, we prove an equivalent condition, namely that the sets $\mathcal{L}_3^\alpha$ are well-ordered for every $\alpha < 2$.

Fix $\alpha < 2$ and let $k = k(\alpha)$ and $a = a(\alpha)$ be constants such that any globally dense 3-multigraph G with $\lambda(G) \leq \alpha$ does not contain $K_{a,a}^{(1,1,3)}$ or a subgraph of type $K_k^{(2,3)}$. Therefore Lemma 2.16 implies that for any globally dense graph G with $\lambda(G) \in \mathcal{L}_3^\alpha$ the number of equivalence classes of G is bounded by $r(k, (3k+6)2^{r(k,2a)})$ where k and a depend only on α. Hence the set of irreducible 3-multigraphs G with $\lambda(G) \in \mathcal{L}_3^\alpha$ is finite. Call this set $\mathcal{I}_\alpha$ so that

$$\mathcal{I}_\alpha = \{G : \lambda(G) \in \mathcal{L}_3^\alpha \text{ and } G \text{ is irreducible}\}.$$

Since any 3-multigraph G with $\lambda(G) \in \mathcal{L}_3^\alpha$ is globally dense, every pair of vertices must be joined by at least one edge. Otherwise, G would contain a pair of independent vertices as an induced subgraph, which is not of negative type, as demonstrated in Example 2.12 (1). This immediately implies that G itself is not of negative type, contradicting the fact that G is globally dense and therefore must be of negative type by Theorem 2.11.

As a consequence of this fact, G is a modified blowup of its irreducible part $G/{\sim}$. Moreover, $\lambda(G/{\sim}) \in \mathcal{L}_3^\alpha$ since $\lambda(G/{\sim}) \leq \lambda(G)$. Thus we can partition the set $\mathcal{L}_3^\alpha$ into a finite number of sets $\bigcup_{G\in\mathcal{I}_\alpha} \mathcal{L}_G^\alpha$ where

$$\mathcal{L}_G^\alpha = \{\lambda(G(\mathbf{x})) < \alpha : \mathbf{x} \in \mathbb{N}_\infty^{|V(G)|}\}.$$

For a fixed 3-multigraph $G \in \mathcal{I}_\alpha$ with $|V(G)| = r$, there is an obvious mapping from the set $\mathbb{N}_\infty^r$ to the set of modified blowups of G, (e.g. map $\mathbf{x}$ to $G(\mathbf{x})$). Note that $\mathbf{x} \leq \mathbf{y}$ implies $\lambda(G(\mathbf{x})) \leq \lambda(G(\mathbf{y}))$ (since $G(\mathbf{x}) \subseteq G(\mathbf{y})$, in this case). Using this fact and the fact that $\mathbb{N}_\infty^r$ has the descending chain condition, it follows that $\mathcal{L}_G^\alpha$ is well-ordered (see Proposition 2.21 below). Indeed, if there was an infinite decreasing sequence $\{\lambda(G(\mathbf{x}_i))\}_{i=1}^\infty$ then the sequence $\{\mathbf{x}_i\}_{i=1}^\infty$ must also be decreasing, which contradicts the descending chain condition.

Since $\mathcal{L}_3^\alpha$ is the union of finitely many well-ordered sets, $\mathcal{L}_G^\alpha$, it follows that $\mathcal{L}_3^\alpha$ is well-ordered. Therefore $\mathcal{T}_3^2$ is well-ordered, completing the proof of Theorem 2.4. □

Lemma 2.16 is a consequence of the following:

Lemma 2.17. *Let G be a 3-multigraph with $\lambda(G) < 2$. Further assume that:*

(α) Every pair of symmetric vertices (i.e. $u, v \in V(G)$ such that $N(u) = N(v)$ in $G - \{u, v\}$) is connected by at least two edges, and

(β) G is of negative type.

Let $k = k(\lambda(G))$ and $a = a(\lambda(G))$ be as defined above. Then $|V(G)| < r(k, (3k+6)2^{r(2a,k)})$, where $r(a,b)$ denotes the usual Ramsey number.

Proof of Lemma 2.16. Let G be a globally dense 3-multigraph with $\lambda(G) < 2$. By (b) of Theorem 2.11, G is of negative type, and hence so is $G/\sim$. Since $G/\sim$ is irreducible, it also satisfies condition (α) of Lemma 2.17. Thus, applying Lemma 2.17 to $G/\sim$, Lemma 2.16 follows. $\square$

We now give the proof of Lemma 2.17, which is the crux of the argument.

Proof of Lemma 2.17 . First note that since G is a 3-multigraph of negative type, then there are no induced subgraphs isomorphic to our classes from Example 2.12, namely E_4, D_4, or $E_{a,b,c}$ (with $a = 1, b \geq 4, c \geq 2, b + c \geq 9$) and moreover any pair of vertices is joined by at least one edge.

Let $S \subset V(G)$ be a maximal clique on edges of multiplicity one. Since G contains no subgraph of type $K_k^{(2,3)}$ showing $|S| \leq s$ would imply that $|G|$ is less than the Ramsey number $r(s,k)$. The rest of the proof shows that such a bound exists. The proof follows in two steps: First we find a subset $T \subset S$, with the property that $N^3(u)$ is the same for every $u \in T$ and moreover $|T| > c(a,k)|S|$ where $c(a,k)$ is a constant depending only on a and k. The second stage is to bound $|T|$. We begin by finding our subset T.

For $v \in V \setminus S$ we define S_v^i to be neighborhood of v in S in edge multiplicity i. For simplicity of notation, we set $R = V \setminus S$ and for $v \in R$ we let $S_v = N_S^3(v)$ denote the 3-neighborhood of v into S. For each subset $\widetilde{S} \subseteq S$ define $R_{\widetilde{S}} = \{v \in R : S_v = \widetilde{S}\}$. Note that each vertex in R lies in exactly one $R_{\widetilde{S}}$, namely R_{S_v}. Define X by taking precisely one vertex from each nonempty $R_{\tilde{S}}$. Thus for each vertex

$w \in R \setminus X$ there is a vertex $v \in X$ such that $S_w = S_v$. Moreover, for any pair of vertices $u, v \in X$, $S_u \neq S_v$. We show that $|X| < r(2a, k)$.

Since G contains no induced copy of D_4 it follows that if $u, v \in X$ are joined by a single edge then either $S_u \subset S_v$ or $S_v \subset S_u$. Since X contains no $K_k^{(2,3)}$, the inequality $|X| < r(2a, k)$ will follow if we prove that X contains no $K_{2a}^{(1)}$ as well. Suppose instead, X does contain an induced $K_{2a}^{(1)}$. Denote the vertices of this clique $\{v_1, \ldots, v_{2a}\}$. As for any pair v_i, v_j, either $S_{v_i} \subset S_{v_j}$ or vice-versa, we may order the v_i so that $S_{v_i} \subset S_{v_{i+1}}$ for $1 \leq i \leq 2a-1$. Since these inclusions are strict, we have that $|S^3_{v_i}| \geq i-1$. But then G contains an induced $K_{a,a}^{(1,1,3)}$ on vertex set $N^3_S(v_{a+1}) \cup \{v_{a+1}, \ldots, v_{2a}\}$. This contradicts our assumptions and hence X contains no $K_{2a}^{(1)}$. Thus $|X| < r(2a, k)$ as claimed.

Observe that for any $u \in S$, $N^3_X(u)$ completely determines $N^3(u)$. Therefore there must exist a subset $T \subset S$ of size $|S|/2^{|X|}$ with the property that $N^3(u)$ is the same for every $u \in T$.

Now that we have defined T, we move to the second part of the proof; bounding $|T|$ and hence $|S|$. We hence assume that $|T| \geq 9$, as otherwise we have the simple bound that $|S| \leq 9 \cdot 2^{|X|} \leq 9 \cdot 2^{r(2a,k)}$. For each $v \in R$ and $i = 1, 2, 3$ we set $T^i_v = N^i_T(v)$. For any $\widetilde{T} \subseteq T$ we define the set $R^2_{\widetilde{T}} = \{v \in R : T^2_v = \widetilde{T}\}$. Similarly as before, $\{R^2_{\widetilde{T}} \colon \widetilde{T} \in T\}$ partitions R. Next, we define Y by taking precisely one vertex from each non-empty $R^2_{\widetilde{T}}$ as $\widetilde{T}$ ranges over all subsets of T. Note that for any pair of distinct vertices $u, v \in Y$, it follows that $T^2_u \neq T^2_v$, and for any vertex $w \in R - Y$ there is a vertex $v \in Y$ such that $T^2_w = T^2_v$.

Set

$$Y_i = \{v \in Y \mid |T^2_v| = i\}.$$

We observe that $|Y_i| = 0$ for $i = 2, 3, \ldots, |T|-4$. Indeed, if $v \in Y_i$ with $i \in \{2, 3, \ldots, |T|-4\}$ then the sets $A = \{v\}$, $B = T^2_v$ and $C = T - T^2_v$ induce an $E_{a,b,c}$ of the type forbidden (i.e. with $a = 1, b \geq 2, c \geq 4, b + c \geq 9$) as we shall check. Note that in our case $|A| = 1$, $|B| \geq 2$,

$|C| \geq 4$, and $|B| + |C| = |T| \geq 9$, so we only must verify that the proper edges are present. Only single edges are induced on $B \cup C$ as $B \cup C = T \subseteq S$. Since $T_v^2 \neq \emptyset$, it follows that $T_v^3 = \emptyset$. This is because either $T_v^3 = T$ or $T_v^3 = \emptyset$ for all $v \in R$ by the definition of T. In particular this implies that $C \subseteq T_v^1$. This induced $E_{a,b,c}$ would contradict assumption (β) of the lemma, thus $|Y_i| \neq 0$ is possible only for $i \leq 1$ or $i \geq |T| - 3$.

It is clear from the definition of Y that $|Y_0| \leq 1$. We further claim that $|Y_1| \leq 1$. Indeed, if there are two distinct vertices $v, u \in Y_1$ then one of two things must occur:

(Case 1) $u \in N^1(v)$: The set of vertices $\{u, v\} \cup T_v^2 \cup T_u^2$ induces a copy of E_4, which is forbidden by assumption.

(Case 2) $u \in N^2(v) \cup N^3(v)$: In this case set $A = \{v\}$, $B = \{u\} \cup T_v^2$ and $C = T \setminus (T_v^2 \cup T_u^2)$. Since $|T| \geq 9$, the set $A \cup B \cup C$ induces a copy of $E_{1,2,7}$, which is likewise forbidden.

Let $t = |T|$, and consider the collection $\{T_v^2 : v \in Y_{t-3} \cup Y_{t-2} \cup Y_{t-1}\}$. We define $Y' \subseteq Y$ to be the set of v' so that $T_{v'}^2$ is a minimal element of this collection under inclusion. More formally define sets

$$
\begin{aligned}
Y'_{t-2} &= \left\{v \in Y \mid \forall v' \in Y_{t-3}, \quad T_{v'}^2 \not\subset T_v^2\right\} \\
Y'_{t-1} &= \left\{v \in Y \mid \forall v' \in Y_{t-3} \cup Y'_{t-2}, \quad T_{v'}^2 \not\subset T_v^2\right\} \\
Y' &= Y_{t-3} \cup Y'_{t-2} \cup Y'_{t-1}.
\end{aligned}
$$

For the remainder of the proof, we will focus on the subgraph induced by T and Y'. In particular, we will use the fact that each vertex in Y' has a distinct neighborhood in T of size at least $|T| - 3$ to bound $|T|$.

By definition of Y', for any two vertices $v, v' \in Y'$, $T_v^2 \not\subseteq T_{v'}^2$ and $T_{v'}^2 \not\subseteq T_v^2$. Therefore, for each $v, v' \in Y'$, there exists vertices $u \in T_v^2$ and $u' \in T_{v'}^2$ such that $v \in N^1(u')$ and $v' \in N^1(u)$. In order to prevent

$\{u, u', v, v'\}$ from inducing a copy of E_4, v and v' must be joined by at least two edges. Thus the vertices of Y' form a graph of type $K^{(2,3)}_{|Y'|}$, so by assumption $|Y'| \leq k-1$.

If we take, as slight abuse of notation, $\overline{T^2_v} = T \backslash T^2_v$, we have $\bigcap_{v\in Y'} T^2_v = T \setminus \bigcup_{v\in Y'} \overline{T^2_v}$. On the other hand, by definition of Y', we have $|\overline{T^2_v}| \leq 3$, and thus

$$\left| T \setminus \bigcup_{v\in Y'} \overline{T^2_v} \right| \geq |T| - 3|Y'| \geq |T| - 3(k-1).$$

Therefore if we can bound $\left|\bigcap_{v\in Y'} T^2_v\right|$ then we obtain a bound on $|T|$.

We claim that in fact $\left|\bigcap_{v\in Y'} T^2_v\right| \leq 2$. Indeed, assume that $\widetilde{T} = \bigcap_{v\in Y'} T^2_v$ contains three or more vertices. Then we claim that at least two of these three vertices have identical neighborhoods (the third vertex may be the lone neighbor of the vertex in Y_1).

To prove this claim, suppose that $x, y \in \widetilde{T}$ have neighborhoods $N^2(x)$, $N^2(y) \subset V(G) \setminus Y_1$. Consider a vertex $v \in Y$ and without loss of generality, suppose $v \in N^2(x)$. Either $v \in Y'$ or else $v \in Y \setminus Y'$. If $v \in Y'$ then it follow immediately from the intersection in the definition of $\widetilde{T}$ that $v \in N^2(y)$ as well. Suppose instead that $v \in Y \setminus Y'$. In this case there must be some $u \in Y'$ such that $T^2_u \subset T^2_v$ by the construction of Y'. However, $y \in T^2_u$ and hence $y \in T^2_v$. In either case $v \in N^2(y)$ from which it follows that $N^2_Y(x) = N^2_Y(y)$. Finally, it follows directly from the construction of Y that $N^2(x) = N^2(y)$. Recall that we chose the set $T \subset S$ so that every vertex in T has the same neighborhood in multiplicity three, and thus in particular $N^3(x) = N^3(y)$. Finally, since we have shown that x and y have identical neighborhoods in multiplicity two and three, it immediately follows that they have identical neighborhoods in multiplicity one as well. Thus we conclude that x and y are indeed symmetric. By assumption (α) of this lemma, two symmetric vertices must be connected by at least two edges. This contradicts the fact that the vertices are in S and hence connected by only a single edge. Thus

$$\begin{aligned} 2 \geq \left|\bigcap_{v\in Y'} T_v^2\right| = \left|T\setminus \bigcup_{v\in Y'} \overline{T_v^2}\right| \\ \geq |T| - 3|Y'| \\ \geq \frac{|S|}{2^{|X|}} - 3(k-1) \end{aligned}$$

This yields the inequality $|S| \leq (3k-1)2^{|X|}$. We previously showed that $|X| < r(2a, k)$ hence it follows that $|S| \leq (3k-1)2^{r(2a,k)}$. On the other hand, we only have this bound under the assumption that $|T| \geq 9$ and hence it is only guaranteed if $|S| \geq 9 \cdot 2^{r(2a,k)}$. Combining,

$$|S| \leq \max\{(3k-1), 9\}2^{r(2a,k)} \leq (3k+6)2^{r(2a,k)}.$$

completing the result. □

2.4 Order type of $\mathcal{L}_3^2$ and $\mathcal{T}_3^2$

In the previous section, we showed that the sets $\mathcal{L}_3^2$ and $\mathcal{T}_3^2$ are well-ordered. This immediately raises the question, what are their order types, where by order type we mean the following.

Definition 2.18. *Given a well-ordered set S, the* order type *of S, denoted ord(S), is the class of well-ordered sets, of which S is a member, such that there is an order preserving isomorphism between any two elements of the class.*

We will now briefly describe some of the notions central to the concept of order type. For anything not mentioned here, the reader should see [29], a very accessible text on the subject. Every well-ordered set is order-equivalent to exactly one ordinal number. The ordinal numbers are taken to be the canonical representatives of their classes, and so the order type of a well-ordered set is usually identified with the corresponding ordinal. For example, the order type of the natural numbers is ω, the smallest countably infinite ordianl. The list of countably infinite ordinals then continues, $\omega+1, \omega+2, \ldots, \omega \cdot 2, \omega \cdot 2 + 1, \ldots \omega^2, \ldots, \omega^3, \ldots, \omega^\omega, \ldots, \omega^{\omega^\omega}, \ldots$. Here addition and multiplication are not commutative. In particular $1+\omega$ is ω, rather than $\omega+1$ which is the smallest ordinal larger than ω. Likewise, $2 \cdot \omega$ is ω while $\omega \cdot 2$ is the ordinal type of two infinite increasing sequences in which the limit point of one is the initial point of the other.

Notice that $\sup(\mathcal{L}_3^2) = 2$ which is not in the closure $\overline{\mathcal{L}_3^2}$ by the definition of the set (specifically the fact that $\mathcal{L}_3^2 \subseteq [0, 2)$). This along with Proposition 2.9 (ii) immediately yields the following.

Fact 2.19. *$ord(\mathcal{T}_3^2) = ord(\mathcal{L}_3^2)$*

Therefore we simply need to compute $\mathrm{ord}(\mathcal{L}_3^2)$.

Proposition 2.20.

$$ord(\mathcal{L}_3^2) = \omega^\omega.$$

Proof. We prove Proposition 2.20 by bounding $\mathrm{ord}(\mathcal{L}_3^2)$ from both sides.

We first give a proof of the lower bound. Observe that, for the complete 2-multigraph on n vertices with all edges of multiplicity two K_n^2, the set

$$\mathcal{L}_{K_n^2} = \{\lambda(K_n^2(\mathbf{x})) : \mathbf{x} \in \mathbb{Z}_{\geq 0}^n\}$$

is contained in $\mathcal{L}_3^2$. In Claim 2.22 below, we show that $\mathrm{ord}(\mathcal{L}_{K_n^2}) = \omega^n$. Since $\mathcal{L}_{K_n^2} \subseteq \mathcal{L}_3^2$ for all $n \geq 1$, this implies that $\mathrm{ord}(\mathcal{L}_3^2) \geq \omega^n$ for all $n \geq 1$, whence $\mathrm{ord}(\mathcal{L}_3^2) \geq \omega^\omega$. Thus the upper bound will be established once we prove Claim 2.22.

On the other hand, give a proof of the upper bound. This proof is based on the fact that, for a fixed globally dense q-multigraph G on n

vertices the order on $\{\lambda(G(\mathbf{x})) : \mathbf{x} \in \mathbb{N}^n\}$ is a linear extension of $\mathbb{N}^n$ where we use the usual partial ordering of $\mathbb{N}^n$. This follows from a more general result which is due to deJongh and Parikh [7].

Proposition 2.21 ([7])**.** *Let $\phi : \mathbb{N}^n \to \mathbb{R}$ be a function for which $\phi(\mathbf{x}) \leq \phi(\mathbf{y})$ whenever $\mathbf{x} < \mathbf{y}$. Then*

(1) $\{\phi(\mathbf{x}) : \mathbf{x} \in \mathbb{N}^n\}$ *is a well-ordered set, and*

(2) *ord*$\{\phi(\mathbf{x}) : \mathbf{x} \in \mathbb{N}^n\} \leq \omega^n$.

We show that Proposition 2.21 implies, in particular, that $\mathrm{ord}(\mathcal{L}_G) \leq \omega^n$ for all globally dense G on n vertices. Lemma 2.16 implies, for any $\alpha < 2$, that the total number of globally dense, irreducible 3-multigraphs G with $\lambda(G) < \alpha$ is finite. However, for an arbitrary 3-multigraph H with $\lambda(H) < \alpha$, there is a globally dense induced subgraph H' with $\lambda(H') = \lambda(H)$. Further H' is a modified blowup of an irreducible graph G and $\lambda(G) \leq \lambda(H') < \alpha$. Since H' is a modified blowup of G, the ordinal type of the set

$$\{\lambda(G(\mathbf{x}))\colon \mathbf{x} \in \mathbb{N}^{|V(G)|}\}$$

is at least the ordinal type of

$$\{\lambda(H'(\mathbf{x}))\colon \mathbf{x} \in \mathbb{N}^{|V(H')|}\}.$$

Together with the fact that, by Lemma 2.16, there exists an $r = r(\alpha)$ such that $|V(G)| < r$, this yields

$$\mathrm{ord}(\mathcal{L}_3^\alpha) \leq \sum_{G \in \mathcal{J}_\alpha} \{\omega^r : |V(G)| \leq r\} < \omega^\omega$$

Finally since, $\mathcal{L}_3^2 = \bigcup_{\alpha<2} \mathcal{L}_3^\alpha$, we arrive at $\mathrm{ord}(\mathcal{L}_3^2) \leq \omega^\omega$. □

It should be noted that, $\{K_n^2\colon n \geq 2\}$ is a family of 2-multigraphs, and as such, it can be quickly observed that the set $\mathcal{L}_2$ has ordinal number ω^ω as well.

The proof of Proposition 2.20 hinges on the truth of Claim 2.22. Before we prove this claim, we will establish some notation. Given a vector $\mathbf{x} = (x_1, \ldots, x_n)$ with possibly some infinite coordinates, we define a sequence $\{\mathbf{y_m}\}_m$ as

$$(\mathbf{y_m})_i = \begin{cases} x_i & \text{if } x_i < \infty, \\ m & \text{if } x_i = \infty. \end{cases}$$

and define

$$\lambda(G(\mathbf{x})) = \lim_{n\to\infty} \lambda(G(\mathbf{y_n}))$$

It is easy to observe that whenever $\mathbf{x}$ has all finite components and $\mathbf{x} < \mathbf{y}$ then $\lambda(G(\mathbf{x})) \leq \lambda(G(\mathbf{y}))$ because $G(\mathbf{x})$ is an induced subgraph of $G(\mathbf{y})$.

With this in mind, notice that since, for some integer $a > 0$ the graph $K_1^2(a)$ is a simple clique, then

$$\mathcal{L}_{K_1^2} = \left\{1 - \frac{1}{k} : k \geq 0\right\}$$

which clearly has order type ω. Using this as our base case, we will show by induction the following.

Claim 2.22. *For every integer $n \geq 1$,*

$$ord(\mathcal{L}_{K_n^2}) = \omega^n.$$

Proof. We show this by induction on n. For $n = 1$, the claim is easy and we demonstrated the set $\mathcal{L}_{K_1^2}$ above. Assume then that

$$\text{ord}(\mathcal{L}_{K_{n-1}^2}) = \omega^{n-1}$$

and consider the case for K_n^2. We may consider $\lambda(\cdot)$ as a function of $\mathbb{N}^n$ which maps each $\mathbf{x} \in \mathbb{N}^n$ to $\lambda(K_n^2(\mathbf{x}))$. Then it follows from Proposition 2.21 that $\text{ord}\mathcal{L}_{K_n^2} \leq \omega^n$. Thus we only need to show $\text{ord}\mathcal{L}_{K_n^2} \geq \omega^n$ to finish the proof of the claim.

For each fixed integer $b \geq 0$,

$$\text{ord}\{\lambda(K_n^2(\mathbf{y}, b)) : \mathbf{y} \in \mathbb{Z}_{\geq 0}^{n-1}\} \geq \omega^{n-1}.$$

This follows from the fact that the map $f : \lambda(K_n^2(\mathbf{y}, b)) \mapsto \lambda(K_{n-1}^2(\mathbf{y}))$ is surjective and $\text{ord}(\mathcal{L}_{K_{n-1}^2}) \geq \omega^{n-1}$ by our inductive hypothesis. Further $\lambda(K_n^2(\infty, \ldots, \infty, b))$ is the limit point of type ω^{n-1} of this set. Therefore, for any $\varepsilon > 0$ the set

$$\{\lambda(K_n^2(\mathbf{y}, b)) : \mathbf{y} \in \mathbb{N}^{n-1}\} \cap (\lambda(K_n^2(\infty, \ldots, \infty, b)) - \varepsilon, \lambda(K_n^2(\infty, \ldots, \infty, b)))$$

has ordinal type ω^{n-1}. Since this holds for every $b \geq 0$, then we generate a sequence of limit points $\{\lambda(K_n^2(\infty, \ldots, \infty, b))\}_b$. Finally observe that

$$\begin{aligned}
\lambda(K_n^2(\infty, \ldots, \infty, b)) &= \lim_{k\to\infty} \lambda(K_n^2(k, \ldots, k, b)) \\
&= \lim_{k\to\infty} \lambda(K_{(n-1)k+b} \cup K_{k,\ldots,k,b}) \\
&< \lim_{k\to\infty} \lambda(K_{(n-1)k+b}) + \lim_{k\to\infty} \lambda(K_{k,\ldots,k,b}) \\
&= 1 + (1 - \frac{1}{n}) = 2 - \frac{1}{n} = \lambda(K_n^2(\infty, \ldots, \infty))
\end{aligned}$$

where the strict inequality above holds because $\lambda(K_{(n-1)k+b})$ and $\lambda(K_{k,\ldots,k,b})$ are acheived at different vectors.

The above implies that the set $\{\lambda(K_n^2(\infty, \ldots, \infty, b))\}_b$ contains a monotone increasing sequence whose limit is $\lambda(K_n^2(\infty, \ldots, \infty))$. Thus, since each of the points $\lambda(K_n^2(\infty, \ldots, \infty, b))$ is a limit point of type

ω^{n-1} then it follows immediately that

$$\mathrm{ord}(\mathcal{L}_{K_n^2}) \geq \omega^n,$$

and taken with the lower bound, we get $\mathrm{ord}(\mathcal{L}_{K_n^2}) = \omega^n$ as desired. $\square$

2.5 Spectral Prerequisites

Let A denote adjacency matrix of a d-regular simple graph G of order n. Then A has eigenvalues

$$d = \lambda_0 \geq \lambda_1 \geq \cdots \geq \lambda_{n-1} \geq -d$$

Note that if G is connected then $\lambda_1 < \lambda_0$. For simplicity we will say that λ is an eigenvalue of the graph G when λ is an eigenvalue of its adjacency matrix.

A has orthonormal eigenvectors $\phi_0, \ldots, \phi_{n-1}$ associated with $\lambda_0, \ldots, \lambda_{n-1}$, where $\phi_0 = \frac{1}{\sqrt{n}}\mathbf{1}$ with $\mathbf{1} = (1, 1, \ldots, 1)$.

For a graph G, let $|\lambda|$ denote the second largest eigenvalue of G. The following result of Friedman [17] implies that, for a random d-regular graph on n vertices, chosen uniformly at random, λ has the following bound with high probability.

Proposition 2.23. *For any fixed $\epsilon > 0$ and $d \geq 3$, a random d-regular graph on n vertices has $|\lambda| < 2\sqrt{d-1} + \epsilon$ with probability $1 - o(1)$.*

For $d \geq 5$, this implies that, so long as n is sufficiently large, there exist d-regular graphs with $|\lambda| \leq d - 1$ since $2\sqrt{d-1} \leq d - 1$.

We also need the following well known facts, see eg. [18]:

Example 2.24. *The adjacency matrix of the following special classes of graphs have eigenvalues as follows:*

1. *The complete graph* K_n *has eigenvalues* $\lambda_0 = n - 1$ *and* $\lambda_i = -1$ *for* $1 \leq i \leq n - 1$.

2. *For* $a \geq 1$ *and* n *with* $a|n$, *the complete multipartite graph* $K_a(n/a)$ *with* a *parts of size* n/a *has* $(a-1)\frac{n}{a}$ *as an eigenvalue with multiplicity* 1, $-\frac{n}{a}$ *as an eigenvalue with multiplicity* $a - 1$ *and zero as an eigenvalue with multiplicity* $a(\frac{n}{a} - 1)$.

2.6 Proof of Theorem 2.5

From Definition 2.1, it follows that if a number α is not a jump for q, then for any given constant $c > 0$ there must exist at least one graph G_c with $\alpha < \lambda(G_c) < \alpha + c$. By choosing appropriate values of c, we can construct a sequence of q-multigraphs $\{G_n\}$ with

$$\alpha < \lambda(G_n) = \alpha + o(1).$$

Throughout this section, we use the following notation. Let G_1 and G_2 be two multigraphs on the same vertex set. $G = G_1 \cup G_2$ means that the multigraph G is the edge-disjoint union on G_1 and G_2. That is, the multiplicity of the edge xy in G is the sum of the the multiplicity of xy in G_1 and xy in G_2. For example, under this notation $K_k^{(2)} = K_k \cup K_k$. A property of particular interest is the following: If G, G_1 and G_2 are multigraphs so that $G = G_1 \cup G_2$ and A, A_1 and A_2 are their corresponding adjacency matrices then for any $\mathbf{x} \in \mathbb{R}^n$,

$$\mathbf{x}^* A\mathbf{x} = \mathbf{x}^* A_1\mathbf{x} + \mathbf{x}^* A_2\mathbf{x}. \tag{2.2}$$

We will use this in the proof of Theorem 2.5 by constructing a q-multigraph as a union of simple graphs and using (2.2) to analyze $\lambda(G)$.

Proof of Theorem 2.5. Let $0 < r \leq 1$ be a rational number. We want to show that there exists an integer Q such that $q-r$ is not a jump for any $q \geq Q$. Let m to be the smallest integer such that r can be written as the sum of m unit fractions. Furthermore, let

$$r = \sum_{j=1}^{m} \frac{1}{a_j}, \tag{2.3}$$

where $a_j \in \mathbb{N}$ and $a_1 \leq a_2 \leq \cdots \leq a_m$. Fix $Q = m+5$. We will show that $q-r$ is not a jump for any $q \geq Q$.

Let n be an integer such that $a_i | n$ for all $1 \leq i \leq m$, and for which a $d = (q-m+1)$-regular graph on n/a_1 vertices with second largest eigenvalue less than $(d-1)$ exists. Note that there are infinitely many values of n for which such graphs exist. Since $q-m \geq 5$, sufficiently large random regular graphs satisfy this property with high probability, by Proposition 2.23. We will define a q-multigraph G_n in terms of auxiliary graphs H_i, for $i = 1, \ldots, m$ and R which we describe below.

We begin by defining $H_i = K_{a_i}(n/a_i)$, the complete a_i-partite graph where all parts are of size n/a_i. We also let R denote the n-vertex graph consisting of a_1 vertex-disjoint copies of the $(q-m+1)$ regular graph on n/a_1 vertices whose existence we asserted above.

We then write $G_n = K_n^{(q-m)} \cup (\bigcup_{i=1}^{m} H_i) \cup R$. We require that the disjoint graphs in R align with the empty partite sets in H_1, but the placement of H_i for $i = 2, \ldots, m$ is arbitrary. Alternately, if A_i, K and B denotes the adjacency matrix of H_i, K_n and R respectively, then the adjacency matrix of G_n is $A = (q-m)K + \sum A_i + B$. Note that any edge in G_n has multiplicity at most q.

We now wish to compute $\lambda(G)$. Note that for any vector $\mathbf{x}$ where $\sum x_i = 1$,

$$\begin{aligned}\mathbf{x}^*A\mathbf{x} &= \mathbf{x}^*((q-m)K + \sum_{i=1}^{m} A_i + B)\mathbf{x} \\ &= (q-m)\mathbf{x}^*K\mathbf{x} + \sum_{i=1}^{m} \mathbf{x}^*A_i\mathbf{x} + \mathbf{x}^*B\mathbf{x}. \qquad (2.4)\end{aligned}$$

We have

$$(q-m)\mathbf{x}^*K\mathbf{x} = (q-m)\big((\sum_{i=1}^{n} x_i)^2 - \sum_{i=1}^{n} x_i^2\big) = (q-m) - (q-m)\sum_{i=1}^{n} x_i^2. \qquad (2.5)$$

Since H_i is $\frac{(a_i-1)}{a_i}n$ regular, we have that $\mathbf{1}/\sqrt{n}$ is the principal eigenvector of A_i. Since $\mathbf{x}^*\mathbf{1} = \sum x_i = 1$ and all other eigenvalues of A_i are non-positive (c.f. Example 3, item 2), we have that

$$\mathbf{x}^*A_i\mathbf{x} \leq \frac{1}{n}\mathbf{1}^*A_i\mathbf{1} = 1 - \frac{1}{a_i}. \qquad (2.6)$$

Finally, note that B has eigenvalue $(q-m+1)$ with multiplicity a_1, and all other eigenvalues are at most $q-m$ in absolute value. We take a set of orthonormal eigenvectors of B, $\phi_1, \ldots, \phi_n$ such that $\phi_1, \ldots, \phi_{a_1}$ are normalized indicator vectors for the a_1 disjoint copies of the graph inside R. In other words, if X_i is the vertex set one of the a_1 copies of the graph within R, we have that $\phi_i = \frac{1}{\sqrt{n/a_i}}\mathbf{1}_{X_i}$. We write

$$\mathbf{x} = \sum_{i=1}^{n} \alpha_i\phi_i,$$

where we note that $\alpha_1, \ldots, \alpha_{a_1}$ are bounded by $\sqrt{a_1/n}$.

Further note that

$$\sum_{i=1}^{n} \alpha_i^2 = \sum_{i=1}^{n} x_i^2, \qquad (2.7)$$

and

$$\mathbf{x}^* B \mathbf{x} = \sum_{i=1}^{n} \alpha_i^2 \lambda_i. \tag{2.8}$$

Due to the fact that $\lambda_1 = \lambda_2 = \cdots = \lambda_{a_1 - 1} = q - m + 1$ with corresponding $\alpha_i' s$ bounded by $\sqrt{a_1/n}$, and also recalling $q - m \geq \lambda_{a_1} \geq \cdots \geq \lambda_{m-1}$, we infer that

$$\sum_{i=1}^{n} \alpha_i^2 (\lambda_i - (q - m)) \leq \frac{a_1^2}{n} \tag{2.9}$$

Combining (2.3) – (2.9), we have that

$$\begin{aligned}
\mathbf{x}^* A \mathbf{x} \quad & \overset{(2.5),(2.6),(2.8)}{\leq} \quad (q-m) - (q-m)\sum_{i=1}^{n} x_i^2 + \sum_{j=1}^{m}\Big(1 - \frac{1}{a_j}\Big) + \sum_{i=1}^{n} \alpha_i^2 \lambda_i \\
& \overset{(2.3),(2.7)}{=} \quad (q-m) + (m-r) + \sum_{i=1}^{n} \alpha_i^2(\lambda_i - (q-m)) \\
& \overset{(2.9)}{\leq} \quad q - r + \frac{a_1^2}{n}.
\end{aligned}$$

Thus $\lambda(G_n) \leq q - r + \frac{a_1^2}{n}$. On the other hand, taking $\mathbf{x} = \mathbf{1}/n$ shows that $\lambda(G_n) \geq q - r + \frac{1}{n}$. Thus, $\lambda(G_n) = q - r + o(1)$, which shows that $q - r$ is not a jump. $\square$

Remark: Theorem 2.5 shows that for any rational $r \in (0, 1]$, eventually $q - r$ will become a non-jump. An interesting open question is to find the dependence (if any!) of q on r. It is known that if $r = \frac{p}{s}$ in lowest terms, that there is a unit fraction decomposition in $O(\sqrt{\log s})$ terms, see [32]. Of course, it is possible that there are no jumps on the interval $[q - 1, q)$, even for $q \geq 4$.

Chapter 3

Ramsey and induced Ramsey results for k-graphs

3.1 Introduction

For simplicity we refer to k-uniform hypergraphs as k-graphs throughout the chapter. We denote the uniformity of a k-graph with a superscript (e.g. $\mathcal{H}^{(k)}$) when it is not immediately obvious from the context. For a k-graph $\mathcal{H}$, we use $\mathcal{H}$ to denote both the k-graph and its set of edges interchangeably. We denote the vertex set of $\mathcal{H}$ by $V(\mathcal{H})$. Throughout the chapter, we will denote a k-tuple by lower case letters, e.g. $e \in \binom{V(\mathcal{H})}{k}$.

Given two k-graphs, $\mathcal{T}$ and $\mathcal{S}$, the Ramsey number of this pair, denoted $r(\mathcal{T}, \mathcal{S})$ is the minimum integer n such that any two coloring of the k-tuples of the complete k-graph on n vertices, $\mathcal{K}_n^{(k)}$, by red and blue yields either a red $\mathcal{T}$ or a blue $\mathcal{S}$. We will denote by $\mathcal{K}_{t,\dots,t}^{(k)}$ the complete k-partite, k-graph with partite sets of size t. In the first part of the chapter we will show that

$$s^{c_1 t^{k-1}} \leq r(\mathcal{K}_{t,\dots,t}^{(k)}, \mathcal{K}_s^{(k)}) \leq s^{c_2 t^{k-1}} \tag{3.1}$$

for $c_1 = 1 - (1+o(1))/k$, $c_2 = k$ and, in the case of the lower bound, $t = s^{o(1)}$, where $o(1) \to 0$ as $s \to \infty$.

The second, and more substantial part of the chapter, is devoted to induced Ramsey numbers. For k-graphs $\mathcal{R}, \mathcal{S}$ and $\mathcal{T}$ we write $\mathcal{R} \xrightarrow{\text{ind}} (\mathcal{T}, \mathcal{S})$ if, for every two-coloring of the k-tuples of $\mathcal{R}$ with red and blue, one can find either a red induced copy of $\mathcal{T}$ or a blue induced copy of $\mathcal{S}$. Let $r_{\text{ind}}(\mathcal{T}, \mathcal{S})$ denote the smallest number n such that there exists a k-graph $\mathcal{R}$ on n vertices with $\mathcal{R} \xrightarrow{\text{ind}} (\mathcal{T}, \mathcal{S})$. We will adopt the abbreviated notation $r_{\text{ind}}(\mathcal{T})$ to mean $r_{\text{ind}}(\mathcal{T}, \mathcal{T})$. Clearly $r_{\text{ind}}(\mathcal{T}, \mathcal{S}) \geq r(\mathcal{T}, \mathcal{S})$ for all pairs $\mathcal{T}, \mathcal{S}$.

While the existence of Ramsey numbers follows from Ramsey's theorem, the existence of induced Ramsey numbers was shown in [9],[14], and [26] for simple graphs and in [1] and [23] for k-graphs with $k \geq 3$.

When we refer to a k-graph being s-partite for some integer $s \geq k$, we mean precisely the following.

Definition 3.1 (Partite Hypergraph). *For $k \leq s$, a k-graph $\mathcal{H}^{(k)}$ is called* s-partite *if $V(\mathcal{H}^{(k)})$ admits a partition $V_1 \cup \cdots \cup V_s$ such that, for every $e \in \mathcal{H}^{(k)}$ and $i \in [s]$, we have $|e \cap V_i| \leq 1$.*

For simple graphs, due to a well-known result of Erdős, $r_{\text{ind}}(K_t) = r(K_t) \geq 2^{t/2}$, Erdős and Rödl asked whether there exists a constant $c > 0$ such that $r_{\text{ind}}(G) < c^t$ for all graphs G on t vertices. It was proved in [26], and in a strengthened density form in [12], that this indeed is true if G is bipartite. For G arbitrary, improving on an earlier bound from [22], Conlon, Fox and Zhao [6] recently proved that $r_{\text{ind}}(G) \leq 2^{ct \log t}$, which is currently the best general upper bound.

For k-graphs with $k \geq 3$, well-known open problems remain about determining the order of magnitude of the classical Ramsey numbers. Since the induced Ramsey numbers seem harder to compute in general, it will be difficult to make progress for the induced case. The questions seem to be more tractable if we restrict to k-partite k-graphs. For example, using an extremal result of Erdős [13], one can easily show that $r_{\text{ind}}(\mathcal{K}^{(k)}_{t,\dots,t}) \le 2^{ct^{k-1}}$, where c is a constant depending only on k, and this bound is best possible, up to the value of c. It was conjectured in [12] that the same upper bound holds for $r_{\text{ind}}(\mathcal{T})$ for all $\mathcal{T} \subseteq \mathcal{K}^{(k)}_{t,\dots,t}$.

Dudek [11] showed that for a general k-partite, k-graph $\mathcal{T} \subseteq \mathcal{K}^{(k)}_{t,\dots,t}$ there exists a constant $c = c(k)$ such that $r_{\text{ind}}(\mathcal{T}) < 2^{2^{ct^{k-1}}}$. Here we give a result which improves this bound for all $k \ge 3$. In particular, we show the following.

Theorem 3.2. *Let integers $k \ge 3$, s and t be given. For any k-graphs $\mathcal{T} \subseteq \mathcal{K}^{(k)}_{t,\dots,t}$ and any $\mathcal{S}$, with $|V(\mathcal{S})| = s$, there exists a constant $c_k > 0$ such that*

$$r_{\text{ind}}(\mathcal{T}, \mathcal{S}) \le \exp(c_k t^{2k} s^{k^2}),$$

and thus there is a c'_k such that $r_{\text{ind}}(\mathcal{T}) \le \exp(c'_k t^{k^2+2k})$.

Remark 3.3. *In the proof of Theorem 3.2, we show that for $s \ge k$ and $t \ge t_0$ we can take $c_k = (4k)^k$ and correspondingly $c'_k = 4^k k^{k^2+k}$.*

Remark 3.4. *Throughout the chapter, we consider the k-partite k-graph $\mathcal{T} \subseteq \mathcal{K}^{(k)}_{t,\dots,t}$ to be on the vertex set $\bigcup_{i=1}^k U_i$, $|U_1| = \cdots = |U_k| = t$.*

This chapter is organized as follows. We prove that the bounds given in (3.1) hold for the non-induced Ramsey number in Section 3.2. In Section 3.3 we state some auxiliary results, and in Section 3.4 we use these results to prove Theorem 3.2. Section 3.5 contains the proof of a technical lemma stated in Section 3.3. The proof of the embedding lemma that we use (Lemma 3.17) is in Section 3.6. Finally, Section 3.7 contains the proof of a theorem of Erdős with the precise bounds we require in Section 3.2.

3.2 Non-induced Ramsey Numbers

In this section, we will prove for $c_1 = 1 - (1 + o(1))/k$ and $c_2 = k$,

$$s^{c_1 t^{k-1}} \leq r(\mathcal{K}^{(k)}_{t,\ldots,t},\, \mathcal{K}^{(k)}_s) \leq s^{c_2 t^{k-1}}.$$

where the lower bound holds under the condition $t = s^{o(1)}$ as $s \to \infty$.

First, we establish the upper bound, the proof of which is fairly easy. We then show the lower bound using the probabilistic method.

Proposition 3.5. *Given integers $k \geq 3$, and $s, t \geq k$,*

$$r(\mathcal{K}^{(k)}_{t,\ldots,t},\ \mathcal{K}^{(k)}_s) \leq s^{kt^{k-1}}.$$

In the proof of Proposition 3.5, we will use the following result of Erdős, (see Theorem 1 of [13]).

Theorem 3.6. *Let $t > 1$, $c = k \log k$ and $n \geq 2^{ct^{k-1}}$. Then any k-graph $\mathcal{R}$ on n vertices with $|\mathcal{R}| \geq n^{k - \frac{1}{t^{k-1}}}$ contains a copy of $\mathcal{K}^{(k)}_{t,\ldots,t}$.*

In [13], Erdős does not give an explicit lower bound for the required number of vertices in the statement of the theorem. In Section 3.7 of the current chapter, we restate the original proof of Erdős with the explicit bound $n \geq 2^{ct^{k-1}}$ on the number of vertices.

We will use this theorem in the following way. Fix a coloring of the k-tuples of $\mathcal{K}^{(k)}_n$ by red and blue. If there are at least $n^{k - \frac{1}{t^{k-1}}}$ red k-tuples, then by Theorem 3.6 there is a $\mathcal{K}^{(k)}_{t,\ldots,t}$ in red. If this is not the case, we find a blue copy of $\mathcal{K}^{(k)}_s$ using Lemma 3.7 below.

Lemma 3.7. *Let $s \geq k$ be given, and let$\mathcal{R}$ be a k-graph on $n \geq s^{kt^{k-1}}$ vertices. If there are no copies of $\mathcal{K}^{(k)}_{t,\ldots,t} \subset \mathcal{R}$ then $\mathcal{R}$ must contain an independent set of size s.*

Proof. Arbitrarily partition the vertex set of $\mathcal{R}$ into s classes V_i, $1 \leq i \leq s$ of size as equal as possible. Let us call a set $S \subset V(\mathcal{R})$ *crossing* with respect to this partition if $\left|S \cap V_i\right| = 1$ for each $1 \le i \le s$. We will count the number of crossing sets S, with respect to the vertex partition $V_1 \cup \cdots \cup V_s$ of $\mathcal{R}$, which contain at least one edge of $\mathcal{R}$ and show that this number is less than $(n/s)^s$, and thus there exists an independent set as desired.

We first choose a k-tuple $e \in \mathcal{R}$ for which $|e \cap V_i| \le 1$ for all $1 \le i \le s$, and include its vertices in S. Since $\mathcal{R}$ does not contain any copies of $\mathcal{K}^{(k)}_{t,\ldots,t}$, there must be fewer than $n^{k-\frac{1}{t^{k-1}}}$ k-tuples in $\mathcal{R}$ by Theorem 3.6. Now, choose the other $s-k$ vertices arbitrarily to complete S; there are at most $(n/s)^{s-k}$ choices for these vertices. Thus, if we denote by m the total number of crossing sets S which contain at least one edge from $\mathcal{R}$, we see

$$\begin{aligned} m &< (n^{k-\frac{1}{t^{k-1}}})\left(\frac{n}{s}\right)^{s-k} \\ &\le \left(\frac{n}{s}\right)^{s} \end{aligned}$$

where the second inequality holds because $n \ge s^{kt^{k-1}}$ by assumption. In conclusion, there must be at least one independent crossing set in $\mathcal{R}$. □

Note that, in a red-blue coloring of $\mathcal{K}^{(k)}_n$, the independent set of Lemma 3.7 corresponds to a blue clique. Therefore, we now have the upper bound for $r(\mathcal{K}^{(k)}_{t,\ldots,t}, \mathcal{K}^{(k)}_s)$ which we claimed. To obtain the lower bound, we use the standard probabilistic argument to show the following.

Proposition 3.8. *Given $k, s \ge 3$ and $t \ge 2$ such that $t = s^{o(1)}$ as $s \to \infty$,*

$$r(\mathcal{K}^{(k)}_{t,\ldots,t}, \mathcal{K}^{(k)}_s) \ge s^{t^{k-1}\left(\frac{k-1-o(1)}{k}\right)},$$

where $o(1) \to 0$ as $s \to \infty$.

Proof. Let $\varepsilon > 0$ be given. We will show that there exists an s_0 such that for all $s \geq s_0$,

$$r(\mathcal{K}^{(k)}_{t,\dots,t}, \mathcal{K}^{(k)}_s) \geq s^{t^{k-1}\left(\frac{k-1-\varepsilon}{k}\right)}.$$

Set $N = s^{t^{k-1}\left(\frac{k-1-\varepsilon}{k}\right)}$, $p = N^{\frac{-k}{t^{k-1}}}$, and consider the random two-coloring of $\binom{[N]}{k}$ with each k-tuple colored red independently with probability p and blue with probability $1-p$. We will verify that, given our choices of p and N, the expected number of red copies of $\mathcal{K}^{(k)}_{t,\dots,t}$ and blue copies of $\mathcal{K}^{(k)}_s$ are, in total, fewer than one, i.e. we show

$$\binom{N}{t}^k p^{t^k} + \binom{N}{s}(1-p)^{\binom{s}{k}} < 1.$$

Therefore, there exists a k-graph on N vertices which has neither a red copy of $\mathcal{K}^{(k)}_{t,\dots,t}$ nor a blue copy of $\mathcal{K}^{(k)}_s$.

First observe that, due to our choice of p and $t \geq 2$,

$$\binom{N}{t}^k p^{t^k} < \frac{N^{tk}}{(t!)^k} N^{-kt} < \frac{1}{2}.$$

Next we will verify that

$$\binom{N}{s}(1-p)^{\binom{s}{k}} \leq \left(\frac{Ne}{s}\right)^s e^{-p\binom{s}{k}} < \frac{1}{2},$$

by showing

$$s \log N < p\binom{s}{k}.$$

In view of the fact that $p = N^{\frac{-k}{t^{k-1}}} = s^{1-k+\varepsilon}$ and $\log N \leq t^{k-1}\log s$, and using our assumption that $t = s^{o(1)}$, we conclude that for $s \geq 3$,

$$s \log N \leq s t^{k-1} \log s \leq s^{1+o(1)} \leq s^{1+\varepsilon-k}\binom{s}{k} = p\binom{s}{k},$$

finishing the argument. □

3.3 Some Preliminaries

In this section, we set up some definitions and state a couple of technical lemmas which will be used to prove Theorem 3.2. Throughout the section, we will let $\mathcal{H}$ be a k-partite k-graph and $\mathcal{G}$ a k-partite $(k-1)$-graph. Both $\mathcal{H}$ and $\mathcal{G}$ will have the same vertex set, $V = \bigcup_{i=1}^{k} |V_i|$, with $|V_1| = |V_2| = \cdots = |V_k| = n$.

Definition 3.9 (Clique hypergraph)**.** *For $\mathcal{G}$ as above, let $\mathcal{K}_k(\mathcal{G})$ denote the k-uniform* clique hypergraph *of $\mathcal{G}$ on the same vertex set. The edges of $\mathcal{K}_k(\mathcal{G})$ are formed by the vertex sets of the cliques of $\mathcal{G}$, i.e.,*

$$\mathcal{K}_k(\mathcal{G}) = \left\{ e \in \binom{V(\mathcal{G})}{k} \ :\ \binom{e}{k-1} \subset \mathcal{G} \right\}.$$

Note that, since $\mathcal{G}$ is a k-partite $(k-1)$-graph, $\mathcal{K}_k(\mathcal{G})$ is a k-partite k-graph. The next definition is crucial.

Definition 3.10 ((ε,ρ)-dense)**.** *Given constants $0 < \rho, \varepsilon < 1$, a k-graph $\mathcal{H}$ as described above is* (ε,ρ)-dense *if, for any $(k-1)$-graph $\mathcal{G}$ as above and with $|\mathcal{K}_k(\mathcal{G})| \geq \varepsilon n^k$, it follows that*

$$|\mathcal{H} \cap \mathcal{K}_k(\mathcal{G})| \geq \rho |\mathcal{K}_k(\mathcal{G})|.$$

Given k-graphs $\mathcal{T} \subseteq \mathcal{K}^{(k)}_{t,\ldots,t}$ and $\mathcal{S} \subseteq \mathcal{K}^{(k)}_{s}$, our goal is to construct a k-graph $\mathcal{R}$ such that $\mathcal{R} \xrightarrow{\text{ind}} (\mathcal{T}, \mathcal{S})$. To construct such a k-graph, we will first build an auxiliary k-partite k-graph $\mathcal{H}$ with the following property. In any red-blue coloring of the edges of $\mathcal{H}$, either there is a red induced copy of $\mathcal{T}$ or the set of blue edges of $\mathcal{H}$ is (ε,ρ)-dense. Later, we will show how to use this condition on $\mathcal{R}$ in order to embed S.

Fix an arbitrary $(k-1)$-graph $\mathcal{G}$ as above with $|\mathcal{K}_k(\mathcal{G})| \geq \varepsilon n^k$. In order to show that the blue edge set (which we label Blue) of $\mathcal{H}$ is (ε, ρ)-dense for some ε and ρ, we must show that $|\mathrm{Blue} \cap \mathcal{K}_k(\mathcal{G})| \geq \rho |\mathcal{K}_k(\mathcal{G})|$. Before introducing the technical lemma of this section, which guarantees the existence of such a k-graph $\mathcal{H}$, we need additional definitions.

Definition 3.11 (Underlying hypergraph, $\mathcal{G}$-complete)**.** *Let $\mathcal{H}$ and $\mathcal{G}$ be given as above, and let $\mathcal{T}_0$ be an induced subgraph of $\mathcal{H}$.*

(i) The subgraph of $\mathcal{K}_k(\mathcal{G})$ induced on $V(\mathcal{T}_0)$, denoted $U_{\mathcal{T}_0}$, is called the underlying hypergraph *of $\mathcal{T}_0$.*

*(ii) An induced subgraph $\mathcal{T}_0$ of $\mathcal{H}$ is called $\mathcal{G}$-*complete *if its underlying hypergraph $U_{\mathcal{T}_0}$ is a complete k-partite k-graph.*

We will denote by $\binom{\mathcal{H}}{\mathcal{T}}_{\mathcal{G}}$ the family of all complete k-partite k-graphs which are underlying hypergraphs of induced copies of $\mathcal{T}$ in $\mathcal{H}$.

Definition 3.12 (Transversal)**.** *Let $\mathcal{H}$ and $\mathcal{G}$ be given as above, and also let $\mathcal{T} \subseteq \mathcal{K}^{(k)}_{t,\ldots,t}$ be given.*

(i) A set $\mathcal{L} \subset \mathcal{K}_k(\mathcal{G})$ is transversal *to the set $\binom{\mathcal{H}}{\mathcal{T}}_{\mathcal{G}}$ if the intersection of $\mathcal{L}$ with the edge set of each member of $\binom{\mathcal{H}}{\mathcal{T}}_{\mathcal{G}}$ is nonempty.*

(ii) Denote by $\mathrm{tr}(\mathcal{T}, \mathcal{G}, \mathcal{H})$ the minimum size of a such a transversal set $\mathcal{L}$.

We now continue the monologue which preceded Definition 3.11. Fix a red-blue edge coloring of $\mathcal{H}$ which contains no red induced copy of $\mathcal{T}$. Let Blue denote the set of blue edges of $\mathcal{H}$. We want to show that Blue is (ε, ρ)-dense for some constants $0 < \varepsilon, \rho \leq 1$. Therefore, for any $(k-1)$-graph $\mathcal{G}$ as above with $|\mathcal{K}_k(\mathcal{G})| \geq \varepsilon n^k$, we must show that $|\mathrm{Blue} \cap \mathcal{K}_k(\mathcal{G})| \geq \rho |\mathcal{K}_k(\mathcal{G})|$.

Since, by definition, any $\mathcal{G}$-complete copy of $\mathcal{T}$, say $\mathcal{T}_0 \subset \mathcal{H}$, is induced in $\mathcal{H}$, it follows that $\mathcal{T}_0$ must contain at least one blue edge, and so

$U_{\mathcal{T}} \supset \mathcal{T}$ must also contain a blue edge. Let $\mathcal{B} \subseteq$ Blue be the set of blue edges of $\mathcal{G}$-complete copies of $\mathcal{T}$ in $\mathcal{H}$. Since $\mathcal{H}$ contains no induced red copy of $\mathcal{T}$, it follows that $\mathcal{B}$ is transversal to $\binom{\mathcal{H}}{\mathcal{T}}_{\mathcal{G}}$. Therefore, by the definition of $\mathrm{tr}(\mathcal{T}, \mathcal{G}, \mathcal{H})$, it follows that

$$|\mathrm{Blue} \cap \mathcal{K}_k(\mathcal{G})| \geq |\mathcal{B}| \geq \mathrm{tr}(\mathcal{T}, \mathcal{G}, \mathcal{H}).$$

Thus in order to show that Blue is (ε, ρ)-dense, it suffices to show that $\mathrm{tr}(\mathcal{T}, \mathcal{G}, \mathcal{H}) \geq \rho|\mathcal{K}_k(\mathcal{G})|$.

Remark 3.13. *Note that in our definition of the transversal set $\mathcal{L}$, we have not required that $\mathcal{L} \subset \mathcal{H}$. In fact, in Definition 3.12 we allow $\mathcal{L}$ to contain edges of both $\mathcal{H}$ and $\mathcal{K}^{(k)}_{n,\ldots,n} \setminus \mathcal{H}$ as long as each edge belongs to a complete k-partite k-graph which is underlying some induced copy of $\mathcal{T}$ in $\mathcal{H}$.*

While this is somewhat less straightforward, it turns out to be easier to bound $\mathrm{tr}(\mathcal{T}, \mathcal{G}, \mathcal{H})$ than bounding $|\mathrm{Blue}|$ directly.

Lemma 3.14. *Let $\varepsilon > 0$, integers $k \geq 3$ and t sufficiently large as well as $\mathcal{T} \subseteq \mathcal{K}^{(k)}_{t,\ldots,t}$ be given. Set*

$$n = e^{t^k \log^k (2^{k+1}/\varepsilon)} \quad \text{and} \quad \rho = e^{-3kt} \tag{3.2}$$

Then there exists $\mathcal{H} \subseteq \mathcal{K}^{(k)}_{n,\ldots,n}$ such that, for all $\mathcal{G}$ as above and with $|\mathcal{K}_k(\mathcal{G})| \geq \varepsilon n^k$,

$$\mathrm{tr}(\mathcal{T}, \mathcal{G}, \mathcal{H}) \geq \rho|\mathcal{K}_k(\mathcal{G})|. \tag{3.3}$$

The proof of Lemma 3.14 will be given in Section 3.5. The next corollary follows directly from Lemma 3.14 and the discussion preceding it.

Corollary 3.15. *Let $\varepsilon > 0$, integers $k \geq 3$ and t sufficiently large as well as $\mathcal{T} \subseteq \mathcal{K}^{(k)}_{t,\ldots,t}$ be given. There exists a k-partite k-graph $\mathcal{H} \subset$*

$\mathcal{K}^{(k)}_{n,\ldots,n}$, *with* $n = e^{t^k \log^k(2^{k+1}/\varepsilon)}$*, such that the following holds. For every red-blue coloring of the edges of* $\mathcal{H}$*, there is either a red induced copy of* $\mathcal{T}$ *or the blue edges of* $\mathcal{H}$ *form an* (ε, ρ)*-dense* k*-graph with* $\rho = e^{-3kt}$. □

Construction 3.16. *(*of Ramsey k-graph $\mathcal{R}$*) Now we will use the* k*-partite* k*-graph* $\mathcal{H}$ *obtained in Corollary 3.15 to construct a* k*-graph* $\mathcal{R}$*. For a given integer* $s > 0$*, let a* k*-graph* $\mathcal{S} \subseteq \mathcal{K}_s$*, be given on the vertex set* $[s]$*. We construct an* s*-partite* k*-graph* $\mathcal{R}$ *with vertex partition* $\bigcup_{i=1}^{s} W_s$*, with* $|W_1| = \cdots = |W_s| = n$ *as follows. For each* $(i_1, \ldots, i_k) \in \mathcal{S}$*, let* $\mathcal{R}[W_{i_1} \cup \cdots \cup W_{i_k}]$ *be an isomorphic copy of* $\mathcal{H}$ *(with an arbitrary isomorphism* $V_j \to W_{i_j}$*), which is obtained from Corollary 3.15. If* $(j_1, \ldots, j_k) \notin \mathcal{S}$ *then let* $\mathcal{R}[W_{j_1} \cup \cdots \cup W_{j_k}]$ *be empty. In summary,* $\mathcal{R}$ *has*

$$V(\mathcal{R}) = \bigcup_{i=1}^{s} W_i$$
$$E(\mathcal{R}) = \bigcup_{(j_1,\ldots,j_k)\in\mathcal{S}} \mathcal{H}(W_{j_1} \cup \cdots \cup W_{j_k}),$$

where $\mathcal{H}(W_{j_1} \cup \cdots \cup W_{j_k})$ *is a copy of* $\mathcal{H}$ *on the vertex set* $W_{j_1} \cup \cdots \cup W_{j_k}$*.*

If $\mathcal{R}$ contains no induced red copy of $\mathcal{T}$, we will find an embedding of a given k-graph $\mathcal{S}$ into $\mathcal{R}$ such that every edge in the image is blue. To accomplish this, we will use the following embedding lemma:

Lemma 3.17. *For a fixed positive integer* s*, let* $0 < \rho < 1/2$*,* $\varepsilon < \rho^{s^k}$ *and* $2 \leq k \leq s$ *be given. Suppose that* $\mathcal{R}$ *is a* k*-graph which satisfies:*

- $V(\mathcal{R}) = V_1 \cup \cdots \cup V_s$*,* $|V_1| = \cdots = |V_s| = n$*, and*
- *for each* $(i_1, \ldots, i_k) \subset [s]$*, the subgraph* $\mathcal{R}[V_{i_1} \cup \cdots \cup V_{i_k}]$ *is* (ε, ρ)*-dense.*

Then, for every k-graph $\mathcal{S}$ on the vertex set $V(\mathcal{S}) = [s]$, there are

$$(\rho/2)^{|\mathcal{S}^{(k)}|}n^s$$

partite copies of $\mathcal{S}$ in $\mathcal{R}$, i.e. copies of $\mathcal{S}$ in $\mathcal{R}$ whose embedding maps each $i \in [s]$ to a vertex of V_i.

Note that the idea for the proof of Lemma 3.17 is based on the proof of Theorem 2.2 in [27]. In particular, we modify the proof to accommodate for (ε, ρ)-dense k-graphs.

3.4 Proof of Theorem 3.2

The reader may observe that the Construction 3.16 and Lemma 3.17 together form the basis of the proof of Theorem 3.2. Here we will formalize the proof by combining these two elements as well as give the argument bounding the order of $\mathcal{R}$. In the proof we will assume that s and t given in the theorem are sufficiently large to satisfy Corollary 3.15 and Lemma 3.17. Note that this can be assumed without loss of generality since the finitely many cases not covered by our approach can be dealt with by the choice of c_k large enough.

Proof of Theorem 3.2. Let $t \geq t_0$ where t_0 is the minimum value of t which satisfies the conditions of Lemma 3.14. Set $\rho = e^{-3kt}$ and let $\mathcal{H}$ be the k-partite k-graph obtained from Corollary 3.15 with $\varepsilon = \rho^{s^k}$. Let n be the size of each class of $\mathcal{H}$ given by Corollary 3.15, namely

$$\begin{aligned} n &= \exp(t^k \log^k(2^{k+1}/\varepsilon)) = \exp\left(t^k \log^k\left(\frac{2^{k+1}}{\rho^{s^k}}\right)\right) \\ &= \exp\left(t^k \log^k\left(2^{k+1}(e^{3kt})^{s^k}\right)\right) \end{aligned}$$

$$
\begin{aligned}
&< \exp\Big(t^k \log^k\big(e^{4kts^k}\big)\Big) \\
&= e^{(4k)^k t^{2k} s^{k^2}}.
\end{aligned}
$$

Let $\mathcal{R}$ be the k-graph given in Construction 3.16. We need to show that any 2-coloring of $\mathcal{R}$ necessarily yields either a red induced copy of $\mathcal{T}$ or a blue induced copy of $\mathcal{S}$.

Fix a 2-coloring of $\mathcal{R}$ with no red induced copy of $\mathcal{T}$, and let Blue be the subgraph of $\mathcal{R}$ consisting of blue edges. If $(i_1, \dots, i_k) \in \mathcal{S}$ then, $\mathcal{R}[V_{i_1}, \cup \cdots \cup V_k]$ is isomorphic to $\mathcal{H}$. Therefore, by Corollary 3.15, Blue$[V_{i_1}, \cup \cdots \cup V_{i_k}]$ is (ε, ρ)-dense. Further, by the construction of $\mathcal{R}$, it follows that for $(i_1, \dots, i_k) \notin \mathcal{S}$ the subgraph $\mathcal{R}[V_{i_1} \cup \cdots \cup V_{i_k}]$ is empty.

Notice that the k-graph Blue $\subset \mathcal{R}$ satisfies the conditions of Lemma 3.17. It follows that there are at least

$$
(\rho/2)^{|\mathcal{S}|} n^s \geq (\rho/2)^{s^k} (1/\varepsilon)^s \geq (\rho/2)^{s^k} (1/\rho^{s^{k+1}}) \geq 1
$$

partite copies of $\mathcal{S}$ in $\mathcal{R}$. Consequently, there is an isomorphism ϕ of $\mathcal{S}$ into Blue such that $\phi(i) \in V_i$ for all $i \in [s]$. In conclusion, in view of Construction 3.16,

$$
\mathcal{R} \xrightarrow{\text{ind}} (\mathcal{T}, \mathcal{S}),
$$

so for $t \geq t_0$, $r_{\text{ind}}(\mathcal{T}, \mathcal{S}) \leq sn \leq s \cdot e^{(4k)^k t^{2k} s^{k^2}}$. Thus there exists a constant c_k such that for all t and s, $r_{\text{ind}}(\mathcal{T}, \mathcal{S}) \leq \exp(c_k t^{2k} s^{k^2})$ as claimed. □

3.5 Proof of Lemma 3.14

We begin this section by providing an outline of the proof of Lemma 3.14. To show the existence of a k-graph $\mathcal{H} \subseteq \mathcal{K}^{(k)}_{n,\dots,n}$, we will generate a random k-partite k-graph by sampling each of the n^k crossing k-tuples independently with probability 1/2.

We need to show that, for all choices of a k-partite $(k-1)$-graph $\mathcal{G}$ on the same vertex set as $\mathcal{H}$ and with $|\mathcal{K}_k(\mathcal{G})| \geq \varepsilon n^k$, inequality (3.3) holds. To this end, for a fixed $\mathcal{G}$ denote by $A(\mathcal{G})$ the event that any set $\mathcal{L} \subset \mathcal{K}_k(\mathcal{G})$ transversal to $\binom{\mathcal{H}}{\mathcal{T}}_{\mathcal{G}}$ must satisfy $|\mathcal{L}| \geq \rho|\mathcal{K}_k(\mathcal{G})|$. We need to show that

$$\mathbf{P}\Big(\bigcap_{\mathcal{G}} A(\mathcal{G})\Big) > 0 \tag{3.4}$$

where $\mathcal{G}$ runs over all k-partite $(k-1)$-graphs with $|\mathcal{K}_k(\mathcal{G})| > \varepsilon n^k$. This will follow by showing

$$\mathbf{P}(\overline{A}(\mathcal{G})) < 2^{-kn^{k-1}} \tag{3.5}$$

for each $\mathcal{G}$ since there are only at most $2^{kn^{k-1}}$ possible choices of $\mathcal{G}$.

For a fixed $\mathcal{L} \subset \mathcal{K}_k(\mathcal{G})$, $|\mathcal{L}| < \rho|\mathcal{K}_k(\mathcal{G})|$ let $A(\mathcal{G}, \mathcal{L})$ denote the event that $\mathcal{L}$ is transversal to $\binom{\mathcal{H}}{\mathcal{T}}_{\mathcal{G}}$.

Since $\overline{A}(\mathcal{G}) = \bigcup_{\mathcal{L}} A(\mathcal{G}, \mathcal{L})$ where the union is taken over all $\mathcal{L}$ with $|\mathcal{L}| < \rho|\mathcal{K}_k(\mathcal{G})|$, (3.5) will follow from:

Lemma 3.18. *Let $\varepsilon > 0$, integers $k \geq 3$ and t sufficiently large be given as well as a k-graph $\mathcal{T} \subseteq \mathcal{K}^{(k)}_{t,\dots,t}$. Set n and ρ as in (3.2). Fix a k-partite $(k-1)$-graph $\mathcal{G}$ on vertex set $V_1 \cup \dots \cup V_k$, $|V_1| = \dots = |V_k| = n$ with $|\mathcal{G}| \geq \varepsilon n^k$ and $\mathcal{L} \subset \mathcal{G}$ with $|\mathcal{L}| < \rho|\mathcal{K}_k(\mathcal{G})|$. Let the random k-graph $\mathcal{H}$ and the event $\overline{A}(\mathcal{G}, \mathcal{L})$ be as above. Then*

$$\mathbf{P}(\overline{A}(\mathcal{G}, \mathcal{L})) < 2^{-kn^{k-1}} \binom{|\mathcal{K}_k(\mathcal{G})|}{\rho|\mathcal{K}_k(\mathcal{G})|}^{-1}$$

Indeed (3.5) follows since there are at most $\binom{|\mathcal{K}_k(\mathcal{G})|}{\rho|\mathcal{K}_k(\mathcal{G})|}$ choices of $\mathcal{L}$ satisfying $|\mathcal{L}| < \rho|\mathcal{K}_k(\mathcal{G})|$. Consequently, in order to prove Lemma 3.14, it will be sufficient to verify Lemma 3.18.

An equivalent phrasing of the event $\overline{A}(\mathcal{G}, \mathcal{L})$ is that there is no $\mathcal{G}$-

complete copy of $\mathcal{T}$ in $\mathcal{H}$ with underlying clique contained in $\mathcal{K}_k(\mathcal{G})\setminus\mathcal{L}$. Therefore, in order to prove Lemma 3.18, consider the set of copies of $\mathcal{K}^{(k)}_{t,\ldots,t}$ in $\mathcal{K}_k(\mathcal{G})\setminus\mathcal{L}$. Fix one such copy of $\mathcal{K}^{(k)}_{t,\ldots,t}$ and consider the event that this $\mathcal{K}^{(k)}_{t,\ldots,t}$ underlies an induced copy of $\mathcal{T}$. The event $\overline{A}(\mathcal{G},\mathcal{L})$ occurs if and only if this happens for at least one $\mathcal{K}^{(k)}_{t,\ldots,t}\subset\mathcal{K}_k(\mathcal{G})\setminus\mathcal{L}$. Therefore we want to compute the probability that none of these events occur. To achieve this we use the following proposition from [20].

Proposition 3.19. *Let A,Q be finite sets, $\{E_q\}_{q\in Q}$ be a collection of independent random indicator variables and $\{Q(\alpha)\}_{\alpha\in A}$ be a family of subsets of Q. Define $I_\alpha=\prod_{q\in Q(\alpha)}E_q$ and $X=\sum_{\alpha\in A}I_\alpha$. Moreover, define*

$$\mu=\mathbf{E}X=\sum_{\alpha\in A}\mathbf{E}(I_\alpha)\quad \textit{and}\quad \Delta=\sum_{\alpha\sim\beta}\mathbf{E}(I_\alpha I_\beta),$$

where $\alpha\sim\beta$ if $Q(\alpha)\cap Q(\beta)\neq\emptyset$ but $\alpha\neq\beta$.

Then,

$$\mathbf{P}(X=0)\leq\exp\left(-\frac{1}{2}\frac{\mu^2}{\mu+\Delta}\right).$$

□

In Lemma 3.18, the objects of interest are $\mathcal{G}$-complete induced copies of $\mathcal{T}$ in $\mathcal{H}$. Proposition 3.19 cannot be directly used to count induced subgraphs. However, by more carefully selecting a family of copies of $\mathcal{K}^{(k)}_{t,\ldots,t}$ on which to look for induced copies of $\mathcal{T}$, we will be able to apply the proposition. In particular we will do the following (see i.e Claim 3.21). Subpartition each class of vertices $V_i\subset V(\mathcal{H})$ into t parts, $V_i=V_{i,1}\cup\cdots\cup V_{i,t}$. Find a "large" family of copies of $\mathcal{K}^{(k)}_{t,\ldots,t}$ in $\mathcal{K}_k(\mathcal{G})\setminus\mathcal{L}$ with precisely one vertex in each class $V_{i,j}$. In order to bound Δ in Proposition 3.19 it will also be necessary that the number of copies of $\mathcal{K}^{(k)}_{t,\ldots,t}$ which contain any given edge is bounded. Finally, we fix an ordering of the vertices of $\mathcal{T}$, and require that the embedding of $\mathcal{T}$ into $\mathcal{H}$ be partite induced, which is defined in Definition 3.20 below.

Definition 3.20 (Partite induced embedding/copy)**.** *For a given integer $k \geq 3$, let $\mathcal{H}$ be a k-partite k-graph with vertex classes $V_1, \ldots, V_k$. Further, for a given integer $t \geq 1$ and $i = 1, \ldots, k$, let $\{V_{i,j}\}_{j=1}^t$ be a fixed partition of V_i into t subsets. Given a k-graph $\mathcal{T} \subseteq \mathcal{K}^{(k)}_{t,\ldots,t}$ on labelled vertex set $V(\mathcal{T}) = \bigcup_{i=1}^k U_i$, $U_i = \{u_{i,1}, \ldots, u_{i,t}\}$. we call a map $\varphi\colon V(\mathcal{T}) \to V(\mathcal{H})$ a* partite induced embedding *of $\mathcal{T}$ if*

- $\varphi(u_{i,j}) \in V_{i,j}$ *for all* $u_{i,j} \in V(\mathcal{T})$
- $\varphi(e) \in \mathcal{H}$ *for all* $e \in \mathcal{T}$,
- $\varphi(f) \notin \mathcal{H}$ *for all* $f \in \binom{V(\mathcal{T})}{k} \setminus \mathcal{T}$.

We say that $\mathcal{T}' \subseteq \mathcal{H}$ is a partite induced copy *of $\mathcal{T}$ if it is the image of a partite induced embedding of $\mathcal{T}$ into $\mathcal{H}$.*

Recall we want to show that, for fixed $\mathcal{G}$ and $\mathcal{L}$, the probability of avoiding a $\mathcal{G}$-complete copy of $\mathcal{T}$ in $\mathcal{H}$ whose underlying graph is contained in $\mathcal{K}_k(\mathcal{G}) \setminus \mathcal{L}$ is less than $2^{-kn^{k-1}}\binom{|\mathcal{K}_k(\mathcal{G})|}{\rho|\mathcal{K}_k(\mathcal{G})|}^{-1}$. We will use Proposition 3.19 with the family $\{Q_\alpha\} = \mathcal{F}$, which will be defined in the following claim.

Claim 3.21. *Let $\varepsilon > 0$ and n be as in Lemma 3.18. Further, fix a k-partite $(k-1)$-graph $\mathcal{G}$ on vertex set $\bigcup_{i=1}^k V_i$, $|V_1| = \cdots = |V_k| = n$ with at least εn^k edges. Also let $\mathcal{L} \subset \mathcal{G}$, $|\mathcal{L}| < \rho|\mathcal{K}_k(\mathcal{G})|$ be given. There exist partitions $V_i = \bigcup_{j\in[t]} V_{i,j}$, $1 \leq i \leq k$, and a family $\mathcal{F} = \mathcal{F}(\mathcal{K}_k(\mathcal{G}) \setminus \mathcal{L})$ of partite induced copies of $\mathcal{K}^{(k)}_{t,\ldots,t}$ in $\mathcal{K}_k(\mathcal{G}) \setminus \mathcal{L}$ satisfying:*

(i) $|\mathcal{F}| \geq e^{-kt}\left(\dfrac{\varepsilon}{2^{k+1}}\right)^{kt^k}\dbinom{n}{t}^k.$

(ii) For every k-tuple $e \in \mathcal{K}_k(\mathcal{G}) \setminus \mathcal{L}$, the number of copies of $\mathcal{K}^{(k)}_{t,\ldots,t}$ in $\mathcal{F}$ that contain the k-tuple e is at most $e^{kt}\dfrac{6t^k\,|\mathcal{F}|}{|\mathcal{K}_k(\mathcal{G})|}$.

We postpone the proof of Claim 3.21 until Subsection 3.5.1 and are now ready to start the formal proof of Lemma 3.18.

Proof of Lemma 3.18. Fix the vertex set $V = \bigcup_{i=1}^{k} V_i$, $V_i = \bigcup_{j=1}^{t} V_{i,j}$ as well as a k-partite $(k-1)$-graph $\mathcal{G}$ and family $\mathcal{F}$ of copies of $\mathcal{K}_{t,\ldots,t}^{(k)}$ in $\mathcal{G}$ as in Claim 3.21. Let $\mathcal{H}$ be a random k-graph on $\bigcup_{i=1}^{k} V_i$ generated by sampling each k-tuple in $V_1 \times \cdots \times V_k$ independently and with probability 1/2. Finally, fix a labeling of the vertices of $\mathcal{T}$ by $u_{i,j}$, $i = 1, \ldots, k$, $j = 1, \ldots, t$.

We want to count the number of partite induced copies of $\mathcal{T}$ in $\mathcal{H} \cap (\mathcal{K}_k(\mathcal{G}) \setminus \mathcal{L})$ for which the vertex $u_{i,j}$ is mapped into the set $V_{i,j}$. Since $\mathcal{H}$ is a random k-graph, then this number is a random variable. Therefore, we will define a random variable $E_{x_1,\ldots,x_k}$ by

$$E_{x_1,\ldots,x_k} = \begin{cases} 1, & \text{if } (x_1, \ldots, x_k) \in \mathcal{H} \\ 0, & \text{otherwise.} \end{cases}$$

This random variable only counts the number of k-tuples in $\mathcal{H}$. If we let $j_i \in [t]$ be such that $x_i \in V_{i,j_i}$ for $i = 1, \ldots, k$, then we only wish to count the k-tuple $(x_1, \ldots, x_k)$ if one of the following holds:

- $(x_1, \ldots, x_k) \in \mathcal{H}$ and $(u_{1,j_1}, \ldots, u_{k,j_k}) \in \mathcal{T}$
- $(x_1, \ldots, x_k) \notin \mathcal{H}$ and $(u_{1,j_1}, \ldots, u_{k,j_k}) \notin \mathcal{T}$.

To this end we will define the random variable $E'_{x_1,\ldots,x_k}$ by

$$E'_{x_1,\ldots,x_k} = \begin{cases} E_{x_1,\ldots,x_k} & \text{if } (u_{1,j_i}, \ldots, u_{k,j_k}) \in \mathcal{T} \\ 1 - E_{x_1,\ldots,x_k} & \text{otherwise.} \end{cases} \tag{3.6}$$

Note that the indicator variables $\{E'_{x_1,\ldots,x_k} : (x_1, \ldots, x_k) \in V_1 \times \cdots \times V_k\}$ are mutually independent. Further, for any fixed $\mathcal{K} \in \mathcal{F}$, we can describe the event "$\mathcal{K} \cap \mathcal{H}$ is a partite induced copy of $\mathcal{T}$" as a product of the indicator variables defined in (3.6), namely

$$I_{\mathcal{K}} = \prod_{(x_1,\ldots,x_k)\in\mathcal{K}} E'_{x_1,\ldots,x_k}. \tag{3.7}$$

Indeed, if $X_1 \cup \cdots \cup X_k$ is the vertex set of $\mathcal{K}$, then $E_{x_1,\ldots,x_k} = 1$ for all $(x_1, \ldots, x_k) \in X_1 \times \cdots \times X_k$ if and only if $u_{i,j_i} \mapsto x_{i,j_i}$ is an isomorphism.

Let $X = \sum_{\mathcal{K}\in\mathcal{F}} I_{\mathcal{K}}$, i.e. X counts the number of partite induced copies of $\mathcal{T}$ are in $\mathcal{H}$ with an underlying k-graph from $\mathcal{F}$. Recall that if the event $\overline{A}(\mathcal{G}, \mathcal{L})$ occurs, then there are no $\mathcal{G}$-complete copies of $\mathcal{T}$ in $\mathcal{H}$. However, this immediately implies that no member of $\mathcal{F}$ can be the underlying k-graph of an induced copy of $\mathcal{T}$ in $\mathcal{H}$. Thus if $\overline{A}(\mathcal{G}, \mathcal{L})$ occurs, then $X = 0$ necessarily. In conclusion, we deduce that

$$\mathbf{P}(\overline{A}(\mathcal{G}, \mathcal{L})) \leq \mathbf{P}(X = 0). \tag{3.8}$$

Therefore to prove Lemma 3.18, it is sufficient to show that $\mathbf{P}(X = 0) \leq 2^{-kn^{k-1}} \binom{|\mathcal{K}_k(\mathcal{G})|}{\rho|\mathcal{K}_k(\mathcal{G})|}^{-1}$. To obtain this bound, we can apply Janson's inequality to the random variable X. For this, it will be necessary to calculate the expectation of X as well as Δ, where the sum for the latter is taken is over all pairs of distinct $\mathcal{K}, \mathcal{K}' \in \mathcal{F}$ that share at least one k-tuple. Since each indicator variable $E_{x_1,\ldots,x_k}$ is independent and has probability $1/2$ of being equal to 1, it is clear that

$$\mathbf{E}X = |\mathcal{F}|\, 2^{-t^k} \tag{3.9}$$

If $\mathcal{K}, \mathcal{K}' \in \mathcal{F}$ intersect at ℓ edges, then

$$\mathbf{E} I_{\mathcal{K}} I_{\mathcal{K}'} = 2^{\ell - 2t^k}.$$

The number of pairs $\mathcal{K}, \mathcal{K}' \in \mathcal{F}$ that intersect in more than one edge is at most

$$|\mathcal{F}| \cdot n^{kt-(k+1)},$$

since once $\mathcal{K}$ is fixed, any $\mathcal{K}'$ intersecting $\mathcal{K}$ at two or more k-tuples must be such that $|V(\mathcal{K}) \cap V(\mathcal{K}')| \geq k+1$. On the other hand, the number of pairs $\mathcal{K}, \mathcal{K}' \in \mathcal{F}$ that intersect in exactly one edge (and only the k vertices of that edge) can be bounded by Claim 3.21(ii) as

$$|\mathcal{F}| \cdot t^k \cdot e^{kt} \frac{6t^k\,|\mathcal{F}|}{|\mathcal{K}_k(\mathcal{G})|} = 6t^{2k}e^{kt}\frac{|\mathcal{F}|^2}{|\mathcal{K}_k(\mathcal{G})|}.$$

Consequently,

$$\begin{aligned}\Delta &= \sum_{\mathcal{K}\sim\mathcal{K}'} \mathbf{E} I_{\mathcal{K}} I_{\mathcal{K}'} \\ &\leq 2^{1-2t^k}\left(6t^{2k}e^{kt}\frac{|\mathcal{F}|^2}{|\mathcal{K}_k(\mathcal{G})|}\right) + 2^{-t^k}\left(|\mathcal{F}| \cdot n^{kt-(k+1)}\right).\end{aligned} \tag{3.10}$$

We will show that the RHS of (3.10) can be bounded above by

$$2^{1-2t^k}\left(7t^{2k}e^{kt}\frac{|\mathcal{F}|^2}{|\mathcal{K}_k(\mathcal{G})|}\right). \tag{3.11}$$

This is equivalent to showing,

$$2^{-t^k}|\mathcal{F}|n^{kt-(k+1)} \leq 2^{1-2t^k}t^{2k}e^{kt}\frac{|\mathcal{F}|^2}{|\mathcal{K}_k(\mathcal{G})|},$$

or rather

$$|\mathcal{F}| \geq 2^{t^k-1}n^{kt-(k+1)}\frac{1}{t^{2k}e^{kt}}|\mathcal{K}_k(\mathcal{G})|.$$

Since $|\mathcal{K}_k(\mathcal{G})| \leq n^k$, the previous inequality will hold if

$$|\mathcal{F}| \geq 2^{t^k-1}n^{kt-1}\frac{1}{t^{2k}e^{kt}}.$$

Consequently, in view of Claim 3.21 it will be sufficient to verify that

$$e^{-kt}\left(\frac{\varepsilon}{2^{k+1}}\right)^{kt^k}\left(\frac{n}{t}\right)^{kt} \geq 2^{t^k-1}n^{kt-1}\frac{1}{t^{2k}e^{kt}}, \tag{3.12}$$

or equivalently that

$$n \geq \left(\frac{2^{k+1}}{\varepsilon}\right)^{kt^k} t^{kt}2^{t^k-1}\frac{1}{t^{2k}}.$$

Further, it will suffice to show

$$n \geq \left(\frac{2^{k+1}}{\varepsilon}\right)^{kt^k} t^{kt}2^{t^k}.$$

This is true however, since

$$\begin{aligned} n^{1/t^k} = e^{\log^k(2^{k+1}/\varepsilon)} &= \left(\frac{2^{k+1}}{\varepsilon}\right)^{\log^{k-1}(2^{k+1}/\varepsilon)} \\ &\geq \left(\frac{2^{k+1}}{\varepsilon}\right)^{((k+1)\log 2)^{k-1}} > \left(\frac{2^{k+1}}{\varepsilon}\right)^{k+2} \\ &> \left(\frac{2^{k+1}}{\varepsilon}\right)^{k} \cdot 4 > \left(\frac{2^{k+1}}{\varepsilon}\right)^{k} \cdot 2 \cdot t^{k/t^{k-1}}. \end{aligned}$$

Therefore, in view of (3.11), we have verified

$$\Delta \leq 2^{1-2t^k}\left(7t^{2k}e^{kt}\frac{|\mathcal{F}|^2}{|\mathcal{K}_k(\mathcal{G})|}\right). \tag{3.13}$$

Now we will apply Janson's inequality (Proposition 3.19) to bound $\mathbf{P}(X = 0)$ by,

$$\mathbf{P}(X = 0) \leq \exp\left(-\frac{1}{2}\frac{(\mathbf{E}X)^2}{\mathbf{E}X + \sum_{\mathcal{K}\sim\mathcal{K}'}\mathbf{E}I_{\mathcal{K}}I_{\mathcal{K}'}}\right) \tag{3.14}$$

$$\leq \exp\left(-\frac{1}{2}\frac{(\mathbf{E}X)^2}{\mathbf{E}X+\Delta}\right).$$

We will show that the RHS of (3.14) can be bounded above by

$$\exp\left(\frac{-|\mathcal{K}_k(\mathcal{G})|}{e^{2kt}}\right). \tag{3.15}$$

If $\mathbf{E}X \geq \Delta$ then the last expression in (3.14) can be simplified to $\exp\left(-\frac{1}{4}\mathbf{E}X\right)$ which, in view of (3.9), is smaller than $\exp(-|\mathcal{F}|/(4\cdot 2^{t^k}))$. Therefore, it suffices to show that

$$\exp\left(-\frac{|\mathcal{F}|}{4\cdot 2^{t^k}}\right) \leq \exp(-|\mathcal{K}_k(\mathcal{G})|),$$

since $\exp(-|\mathcal{K}_k(\mathcal{G})|) < \exp(-|\mathcal{K}_k(\mathcal{G})|/e^{2kt})$. Next we show the equivalent statement,

$$|\mathcal{F}| \geq 4\cdot 2^{t^k}|\mathcal{K}_k(\mathcal{G})|. \tag{3.16}$$

Recall that the family $\mathcal{F}$ has the properties given in Claim 3.21 and hence,

$$|\mathcal{F}| \geq e^{-kt}\left(\frac{\varepsilon}{2^{k+1}}\right)^{kt^k}\left(\frac{n}{t}\right)^{kt} \overset{(3.12)}{\geq} 2^{t^k-1}n^{kt-1}\left(\frac{1}{t^{2k}e^{kt}}\right). \tag{3.17}$$

Next we will show that the right hand side of (3.17) is larger than

$$2^{t^k+2}n^k \geq 4\cdot 2^{t^k}|\mathcal{K}_k(\mathcal{G})|,$$

establishing (3.16). This is equivalent to showing

$$n^{k(t-1)-1} > n^{k(t-2)} \geq 8t^{2k}e^{kt},$$

which, in view of the fact that $n = e^{t^k \log^k(2^{k+1}/\varepsilon)}$ from the assumption of Lemma 3.18, clearly holds.

On the other hand, if $\mathbf{E}X < \Delta$ then the last expression in (3.14) can be simplified to $\exp\bigl(-\frac{1}{4}\frac{(\mathbf{E}X)^2}{\Delta}\bigr)$. Then it follows that

$$\begin{aligned}
\mathbf{P}(X=0) &\leq \exp\left(-\frac{(\mathbf{E}X)^2}{4\Delta}\right) \\
&\overset{(3.9),(3.13)}{\leq} \exp\left(-\frac{(|\mathcal{F}|2^{-t^k})^2}{4\cdot 2^{1-2t^k}(7t^{2k}e^{kt}|\mathcal{F}|^2/|\mathcal{K}_k(\mathcal{G})|)}\right) \\
&= \exp\left(-\frac{|\mathcal{K}_k(\mathcal{G})|}{56t^{2k}e^{kt}}\right) \\
&\leq \exp\left(-\frac{|\mathcal{K}_k(\mathcal{G})|}{e^{2kt}}\right),
\end{aligned}$$

where the last inequality hold for t sufficiently large.

We have now verified, in both cases, that

$$\mathbf{P}(X=0) \leq \exp(-|\mathcal{K}_k(\mathcal{G})|/e^{2kt})$$

as claimed. Finally, we will show that

$$\exp\left(-\frac{|\mathcal{K}_k(\mathcal{G})|}{e^{2kt}}\right) \leq 2^{-kn^{k-1}}\binom{|\mathcal{K}_k(\mathcal{G})|}{\rho|\mathcal{K}_k(\mathcal{G})|}^{-1},$$

which will complete the proof since, recalling (3.8), $\mathbf{P}(X=0) \geq \mathbf{P}(\overline{A}(\mathcal{G},\mathcal{L}))$. We will now use the value of ρ that we set in (3.2), namely $\rho = e^{-3kt}$. In what follows next, we will use the bound,

$$\begin{aligned}
\binom{|\mathcal{K}_k(\mathcal{G})|}{\rho|\mathcal{K}_k(\mathcal{G})|} &\leq \exp\bigl(\rho|\mathcal{K}_k(\mathcal{G})|\log(e/\rho)\bigr) \\
&\leq \exp\left(\frac{(3kt+1)|\mathcal{K}_k(\mathcal{G})|}{e^{3kt}}\right) \\
&\leq \exp\left(\frac{|\mathcal{K}_k(\mathcal{G})|}{2e^{2kt}}\right).
\end{aligned} \tag{3.18}$$

Using this we deduce,

$$\binom{|\mathcal{K}_k(\mathcal{G})|}{\rho|\mathcal{K}_k(\mathcal{G})|}\mathbf{P}(X=0) \overset{(3.15),(3.18)}{<} \exp\left(\frac{|\mathcal{K}_k(\mathcal{G})|}{2e^{2kt}}\right)\exp\left(-\frac{|\mathcal{K}_k(\mathcal{G})|}{e^{2kt}}\right)$$
$$= \exp\left(-\frac{|\mathcal{K}_k(\mathcal{G})|}{2e^{2kt}}\right)$$
$$\leq \exp\left(-\frac{\varepsilon n^k}{2e^{2kt}}\right)$$
$$\ll e^{-kn^{k-1}}, \tag{3.19}$$

where the last inequality holds because by (3.2), $n = e^{t^k \log^k(2^{k+1}/\varepsilon)} \gg 2e^{2kt}/\varepsilon$. □

3.5.1 Proof of Claim 3.21

For the proof of Claim 3.21, we use the following result of Nikiforov [24, Theorem 3] which also follows from a more general result of Erdős and Spencer [15, Theorem 12.2].

Lemma 3.22. *Let $\alpha \in \mathbb{R}$, and $t, n \in \mathbb{N}$ be such that*

$$2^k \exp\left(-\frac{1}{k}(\log n)^{1/k}\right) \leq \alpha \leq 1, \quad \textit{and} \quad t^k \leq \log n. \tag{3.20}$$

If $\mathcal{A} \subseteq \mathcal{K}^{(k)}_{n,\ldots,n}$ contains at least αn^k k-tuples, then there are at least

$$\left(\frac{\alpha}{2^k}\right)^{kt^k}\binom{n}{t}^k \tag{3.21}$$

copies of $\mathcal{K}^{(k)}_{t,\ldots,t}$ in $\mathcal{A}$. □

Proof of Claim 3.21. Let us start by defining a sequence of families $\emptyset = \mathcal{F}_0 \subset \mathcal{F}_1 \subset \cdots \subset \mathcal{F}_M$ of copies of $\mathcal{K}^{(k)}_{t,\ldots,t} \subseteq \mathcal{K}_k(\mathcal{G}) \setminus \mathcal{L}$, with $|\mathcal{F}_i| = i$, and

$$M = \left(\frac{\varepsilon}{2^{k+1}}\right)^{kt^k} \binom{n}{t}^k \tag{3.22}$$

as follows.

Let $0 \le i < M$ and suppose that $\mathcal{F}_i$ has been constructed already ($\mathcal{F}_0 = \emptyset$). For a k-tuple $e \in \mathcal{K}_k(\mathcal{G}) \setminus \mathcal{L}$, let $\deg_i(e)$ denote the number of $\mathcal{K} \in \mathcal{F}_i$ such that $e \in \mathcal{K}$.

Let $\mathcal{A}_i \subset \mathcal{K}_k(\mathcal{G}) \setminus \mathcal{L}$ be the k-graph consisting of the $|\mathcal{K}_k(\mathcal{G})|/2$ k-tuples with smallest $\deg_i(\cdot)$, breaking ties arbitrarily. By construction $|\mathcal{A}_i| = |\mathcal{K}_k(\mathcal{G})|/2 \ge \varepsilon n^k/2$. We will now apply Lemma 3.22 in order to show that there is a copy of $\mathcal{K}^{(k)}_{t,\dots,t}$ with edges in $\mathcal{A}_i$ which we did not yet include in $\mathcal{F}_i$. To this end, let $\alpha = \varepsilon/2$, and recall that $n = e^{t^k \log^k(2^{k+1}/\varepsilon)}$. Solving for ε yields

$$\alpha = \frac{\varepsilon}{2} = 2^k \exp\left(-\frac{1}{t}(\log n)^{1/k}\right) \ge 2^k \exp\left(-\frac{1}{k}(\log n)^{1/k}\right),$$

thus α, t, n, and $\mathcal{A}_i$ together satisfy the conditions of Lemma 3.22. Consequently, there are at least M copies of $\mathcal{K}^{(k)}_{t,\dots,t}$ in $\mathcal{A}_i$. Since $|\mathcal{F}_i| = i < M$, there is a copy $\mathcal{K}_{i+1}$ of $\mathcal{K}^{(k)}_{t,\dots,t}$ in $\mathcal{A}_i$ which does not belong to $\mathcal{F}_i$. Let $\mathcal{F}_{i+1} = \mathcal{F}_i \cup \{\mathcal{K}_{i+1}\}$.

Now that we have constructed $\mathcal{F}_M$, let us bound $\max_e \deg_M(e)$ as follows. Let $e^* \in \mathcal{K}_k(\mathcal{G}) \setminus \mathcal{L}$ be a k-tuple with maximum $\deg_M(e^*)$. Suppose that $i = i(e^*) \in \{0, 1, \dots, M-1\}$ is the largest index for which the element $\mathcal{K}_{i+1} \in \mathcal{F}_{i+1} \setminus \mathcal{F}_i$ satisfies $e^* \in \mathcal{K}_{i+1}$. By the definition of i, we have $\deg_M(e^*) = \deg_{i+1}(e^*) = \deg_i(e^*) + 1$. Since $e^* \in \mathcal{K}_{i+1} \subset \mathcal{A}_i$, by the definition of $\mathcal{A}_i$ every k-tuple $e \in \mathcal{K}_k(\mathcal{G}) \setminus (\mathcal{L} \cup \mathcal{A}_i)$ satisfies

$$\deg_i(e) \ge \deg_i(e^*) = \deg_M(e^*) - 1.$$

Since our goal is to show that an upper bound for $\deg_M(e^*)$ is given by Claim 3.21(ii), we may assume $\deg_M(e^*) > 1$. Also, recalling from (3.2) that $\rho = e^{-3kt}$, we will use the fact that $|\mathcal{L}| < \rho|\mathcal{K}_k(\mathcal{G})| < |\mathcal{K}_k(\mathcal{G})|/6$. From this we derive,

$$\sum_{e\in\mathcal{K}_k(\mathcal{G})\setminus\mathcal{L}} \deg_i(e) \geq \left(|\mathcal{K}_k(\mathcal{G})\setminus\mathcal{L}| - \frac{|\mathcal{K}_k(\mathcal{G})|}{2}\right)(\deg_M(e^*) - 1)$$
$$> \frac{|\mathcal{K}_k(\mathcal{G})|}{3}\frac{\deg_M(e^*)}{2}.$$

On the other hand,

$$\sum_{e\in\mathcal{K}_k(\mathcal{G})\setminus\mathcal{L}} \deg_M(e) = Mt^k,$$

since each element of $\mathcal{F}_M$ contributes to the degree of t^k k-tuples. It follows that

$$\max_{e\in\mathcal{K}_k(\mathcal{G})\setminus\mathcal{L}} \deg_M(e) = \deg_M(e^*) \leq 6t^k\frac{M}{|\mathcal{K}_k(\mathcal{G})|}. \tag{3.23}$$

We will use (3.23) to establish condition (ii) after we finish selecting a subfamily $\mathcal{F} \subset \mathcal{F}_M$.

Consider random partitions $V_i = \bigcup_{j\in[t]} V_{i,j}$ of the sets $V_1, \ldots, V_k$. More precisely, an element $v \in V_i$ is selected to be in part $V_{i,j}$, $j \in [t]$, independently and uniformly with probability $1/t$. Let $\mathcal{K} \in \mathcal{F}_M$ be fixed and consider the probability that $|V(\mathcal{K}) \cap V_{i,j}| = 1$ for all $1 \leq i \leq k$, $1 \leq j \leq t$. Such an event happens if and only if all of the t vertices of $V(\mathcal{K}) \cap V_i$ are selected to different parts for all $i = 1, \ldots, k$. The probability of the event is then

$$\left(\frac{t!}{t^t}\right)^k \geq \frac{1}{e^{kt}}.$$

It follows that the expected number of $\mathcal{K} \in \mathcal{F}_M$ satisfying $|V(\mathcal{K}) \cap V_{i,j}| = 1$ for all i, j, is $e^{-kt}M$. In particular, there exist partitions $V_i =$

$\bigcup_{j\in[t]} V_{i,j}$, $i = 1,\ldots,k$ for which at least $e^{-kt}M$ elements $\mathcal{K} \in \mathcal{F}_M$ satisfy $|V(\mathcal{K}) \cap V_{i,j}| = 1$ for all i, j. Let $\mathcal{F} \subset \mathcal{F}_M$ be the set of all such $\mathcal{K}$. Clearly, this choice of $\mathcal{F}$ implies that every $\mathcal{K} \in \mathcal{F}$ is partite induced with respect to the chosen partition. Since $|\mathcal{F}| \geq e^{-kt}M$ and in view of (3.22), $\mathcal{F}$ satisfies (i) of Claim 3.21. Moreover, by (3.23), $\mathcal{F} \subset \mathcal{F}_M$ also satisfies (ii). □

3.6 Proof of Embedding Lemma

Proof of Lemma 3.17. The proof uses induction on the number of edges of $\mathcal{S} = \mathcal{S}^{(k)}$. Without loss of generality, assume that $V(\mathcal{S}) = [s]$.

The result is trivial for an empty k-graph $\mathcal{S}$ so let us assume that the hypergraph has at least one edge (without loss of generality, let $[k] = \{1,\ldots,k\} \in \mathcal{S}$ be that edge). Let $\mathcal{S}^- = \mathcal{S} \setminus [k]$ be the k-graph which arises by removing the edge $[k]$ from $\mathcal{S}$, and let $\mathcal{S}^*$ be the subgraph of $\mathcal{S}$ (or $\mathcal{S}^-$) induced on the vertex set $\{k+1, k+2, \ldots, s\}$.

Let $d = \rho/2$. By the induction assumption, the number of partite copies of $\mathcal{S}^-$ in $\mathcal{R}$ is at least $d^{|\mathcal{S}|-1}n^s$. For a copy $\mathcal{S}^*_{\mathsf{copy}}$ of $\mathcal{S}^*$ in $\mathcal{R}$, define $\mathrm{ext}(\mathcal{S}^*_{\mathsf{copy}})$ by,

$$\mathrm{ext}\big(\mathcal{S}^*_{\mathsf{copy}}\big) = \Big\{e \in V_1 \times \cdots \times V_k : \mathcal{S}^- \subseteq \mathcal{R}[e \cup V(\mathcal{S}^*_{\mathsf{copy}})]\Big\}.$$

Consider the family

$$\Lambda = \big\{\mathcal{S}^*_{\mathsf{copy}} \;:\; |\mathrm{ext}(\mathcal{S}^*_{\mathsf{copy}})| \geq d^{|\mathcal{S}|-1}n^k/2\big\}.$$

Since the number of (partite) copies of $\mathcal{S}^*$ in $\mathcal{R}$ is at most n^{s-k}, the total number of copies of $\mathcal{S}^-$ in $\mathcal{R}$ which extend some $\mathcal{S}^*_{\mathsf{copy}}$ not in Λ is at most

$$n^{s-k} \cdot d^{|\mathcal{S}|-1}n^k/2 \leq d^{|\mathcal{S}|-1}n^s/2.$$

Consequently, at least $d^{|\mathcal{S}|-1}n^s/2$ copies of $\mathcal{S}^-$ extend a copy of $\mathcal{S}^*$ contained in Λ.

Fix an arbitrary $\mathcal{S}^*_{\mathsf{copy}} \in \Lambda$ and note that $\mathrm{ext}(\mathcal{S}^*_{\mathsf{copy}})$ is a k-partite k-graph with vertex classes $V_1, \dots, V_k$. Define the k-partite, $(k-1)$-graph $\mathcal{G}$ on the vertex set $V(\mathcal{G}) = V_1 \times \cdots \times V_k$ by

$$\mathcal{G}_{\mathcal{S}^*_{\mathsf{copy}}} = \left\{ \binom{e}{k-1} : e \in \mathrm{ext}(\mathcal{S}^*_{\mathsf{copy}}) \right\}.$$

Claim 3.23. *$\mathcal{K}_k(\mathcal{G}_{\mathcal{S}^*_{\mathsf{copy}}}) = ext(\mathcal{S}^*_{\mathsf{copy}})$.*

To prove Claim 3.23, notice that to say $\mathrm{ext}(\mathcal{S}^*_{\mathsf{copy}}) \subseteq \mathcal{K}_k(\mathcal{G}_{\mathcal{S}^*_{\mathsf{copy}}})$ means that every k-tuple $e \in \mathrm{ext}(\mathcal{S}^*_{\mathsf{copy}})$ induces a clique in $\mathcal{G}_{\mathcal{S}^*_{\mathsf{copy}}}$. This however, directly follows from the definition of $\mathcal{G}_{\mathcal{S}^*_{\mathsf{copy}}}$. Thus it suffices to prove that $\mathcal{K}_k(\mathcal{G}_{\mathcal{S}^*_{\mathsf{copy}}}) \subseteq \mathrm{ext}(\mathcal{S}^*_{\mathsf{copy}})$, i.e. there are no k-tuples $f \notin \mathrm{ext}(\mathcal{S}^*_{\mathsf{copy}})$ such that the clique $\binom{f}{k-1}$ is contained in $\mathcal{G}_{\mathcal{S}^*_{\mathsf{copy}}}$.

Suppose $f \in V_1 \times \cdots \times V_k$ is given such that $f \notin \mathrm{ext}(\mathcal{S}^*_{\mathsf{copy}})$. Then the induced subgraph of $\mathcal{R}$ on $f \cup V(\mathcal{S}^*_{\mathsf{copy}})$ does not contain a partite copy of $\mathcal{S}^-$. This implies that there is some edge, say f', missing from $\mathcal{R}[f \cup V(\mathcal{S}^*_{\mathsf{copy}})]$, which should be present to form a partite copy of $\mathcal{S}^-$. Further, $f \cap f' \neq \emptyset$, otherwise $f' \subset V(\mathcal{S}^*_{\mathsf{copy}})$ which is impossible since we know that all of the edges of $\mathcal{S}^*_{\mathsf{copy}}$ are present. Let $Z \in \binom{f}{k-1}$ be an arbitrary $(k-1)$-subset of f containing $f \cap f'$. Since Z contains $f \cap f'$, then for any $e' \in V_1 \times \cdots \times V_k$ containing Z, $\mathcal{R}[e' \cup V(S^*_{\mathsf{copy}})]$ is not a copy of S^-, i.e. there is no $e \in \mathrm{ext}(S^*_{\mathsf{copy}})$ containing Z. It follows from this and the definition of $\mathcal{G}_{\mathcal{S}^*_{\mathsf{copy}}}$ that $Z \notin \mathcal{G}_{\mathcal{S}^*_{\mathsf{copy}}}$, and hence $\binom{f}{k-1}$ is not contained in $\mathcal{G}_{\mathcal{S}^*_{\mathsf{copy}}}$. This finishes the proof of Claim 3.23.

Claim 3.23 implies that $\mathrm{ext}(\mathcal{S}^*_{\mathsf{copy}})$ is a clique hypergraph (see Definition 3.9. Since by construction $|\mathrm{ext}(\mathcal{S}^*_{\mathsf{copy}})| \geq d^{|\mathcal{S}|-1}n^k/2 \geq \varepsilon n^k$, the fact that $\mathcal{R}[V_{i_1} \cup \cdots \cup V_{i_k}]$ is (ε, ρ)-dense and Claim 3.23 together imply that

$$|\mathcal{R} \cap \mathrm{ext}(\mathcal{S}^*_{\mathsf{copy}})| \geq \rho\, |\mathrm{ext}(\mathcal{S}^*_{\mathsf{copy}})|.$$

Note that for every k-tuple $e \in \text{ext}(\mathcal{S}^*_{\mathsf{copy}})$ which is also an edge in $\mathcal{R}$, $\mathcal{R}[e \cup V(\mathcal{S}^*_{\mathsf{copy}})]$ contains a copy of $\mathcal{S}$. Therefore, the number of partite copies of $\mathcal{S}$ in $\mathcal{R}$ is at least

$$\sum_{\mathcal{S}^*_{\mathsf{copy}} \in \Lambda} \rho\, |\text{ext}(\mathcal{S}^*_{\mathsf{copy}})| \geq \rho d^{|\mathcal{S}|-1} n^s/2 = d^{|\mathcal{S}|} n^s.$$

Thus we have the desired result. □

3.7 A Theorem of Erdős

Here we will restate Erdős' proof of the following theorem using the notation in the current chapter.

Theorem 3.24. *Let $t > 1$, $c = k \log k$ and $n > 2^{ct^{k-1}}$. Then any k-graph $\mathcal{R}$ on n vertices with $|\mathcal{R}| \geq n^{k - \frac{1}{t^{k-1}}}$ contains a copy of $\mathcal{K}^{(k)}_{t,\ldots,t}$.*

In order to prove Theorem 3.24, Erdős uses the following lemma, which he states and proves in [13]. We will first give this result before moving on to the proof of Theorem 3.24.

Lemma 3.25. *Let S be a set of N elements $y_1, \ldots, y_N$ and let A_i, $1 \leq i \leq n$, be subsets of S satisfying,*

$$\sum_{i=1}^{n} |A_i| \geq \frac{nN}{w} \tag{3.24}$$

for some $w \geq 1$. Then for all $t > 0$ such that $n \geq 2t^2 w^t$, there are t distinct A's, $A_{i_1}, \ldots, A_{i_t}$, so that

$$\left| \bigcap_{j=1}^{t} A_{i_j} \right| \geq \frac{N}{2w^t}. \tag{3.25}$$

Proof. Let $f_i\colon S \to \{0,1\}$ be the characteristic function of the set A_i (i.e. $f_i(y_j) = 1$ if $y_j \in A_i$ and $f_i(y_j) = 0$ otherwise). Define another function $F\colon S \to \mathbb{N}$ as

$$F(y) = \sum_{i=1}^{n} f_i(y)$$

Clearly by (3.24),

$$\sum_{j=1}^{N} F(y_j) \geq \frac{nN}{w}. \tag{3.26}$$

Thus from (3.26) we obtain by an elementary inequality (Jensen's) that

$$\sum_{j=1}^{N} F(y_j)^t$$

is minimal if for all j, $F(y_j) = n/w$, or

$$\sum_{j=1}^{N} F(y_j)^t \geq N\left(\frac{n}{w}\right)^t. \tag{3.27}$$

On the other hand we obtain by a simple argument (i.e. expand $F(y_j)^t$),

$$\sum_{j=1}^{N} F(y_j)^t = \sum \left|A_{i_1} \cap \cdots \cap A_{i_t}\right| \tag{3.28}$$

where the summation in (3.28) is extended over all the choices of $i_1, \ldots, i_t$, $(1 \leq i_r \leq n)$. There are $\prod_{i=0}^{t-1}(n-i) \leq n^t$ choices of $i_1, \ldots, i_t$ where all of the indices are distinct, and (3.25) would be false if the contribution of these terms to the sum (3.28) would be less than

$$\frac{Nn^t}{2w^t}. \tag{3.29}$$

The number of summands in (3.28) where not all of the indices are distinct is easily seen to be less than $t^2 n^{t-1}$. The contribution of each of these terms to the right side of (3.28) is clearly at most N. Thus finally from (3.28) and (3.29)

$$\sum_{j=1}^{N} F(y_j)^t < \frac{Nn^t}{2w^t} + t^2 n^{t-1} N. \tag{3.30}$$

Now since $n \geq 2t^2 w^t$, (3.30) contradicts (3.27). Thus (3.25) must hold for at least one choice of distinct A_i's $1 \leq i \leq t$ which completes the proof of the lemma. $\square$

Proof of Theorem 3.24. The proof proceeds by induction with respect to k. The proof of the case $k = 2$ is given by Turán's well-known theorem. However we give Erdős' proof of this as it shed's some light on the methods used for the rest of the proof.

Consider now the case $k = 2$. Denote the vertices of the graph $\mathcal{R}^{(2)}$, $|\mathcal{R}^{(2)}| \geq n^{2-1/t}$ by $x_1, \ldots, x_n$ and by $v(x_i)$ we denote the degree of vertex x_i. Clearly

$$\sum_{i=1}^{n} v(x_i) \geq 2n^{2-1/t}. \tag{3.31}$$

We want to count the pairs (x_i, T) where $T \subseteq N(x_i)$ of cardinality t. Clearly the number of such pairs is

$$\sum_{i=1}^{n} \binom{v(x_i)}{t}. \tag{3.32}$$

Jensen's inequality states that the sum (3.32) is minimal if all of the $v(x_i)$ are equal. Thus by a simple computation,

$$\sum_{i=1}^{n} \binom{v(x_i)}{t} \geq n \binom{2n^{1-1/t}}{t} > t \binom{n}{t}.$$

Hence there are t vertices $y_1, \ldots, y_t$ which are joined to the same t vertices T which means that $\mathcal{R}^{(2)}$ contains a $\mathcal{K}^{(2)}(t,t)$ as stated.

Assume now that the theorem holds for $k-1$ when $n \geq e^{(k-1)\log(k-1)t^{k-2}}$. We shall prove it for k if $n \geq e^{k \log k t^{k-1}}$. Suppose then that we have a k-graph $\mathcal{R}^{(k)}$ with $|\mathcal{R}^{(k)}| \geq n^{k-1/t^{k-1}}$. Denote by $x_1, \ldots, x_n$ the vertices of $\mathcal{R}^{(k)}$ and by $y_1, \ldots, y_N$, $N = \binom{n}{k-1}$ the set of all $(k-1)$-tuples formed from the x_i, $1 \leq i \leq n$. $E_1^{(k)}, \ldots, E_m^{(k)}$ denotes the k-tuples of $\mathcal{R}^{(k)}$, $m \geq n^{k-1/t^{k-1}}$. To apply Lemma 3.25, denote by A_i the set of all $(k-1)$-tuples y_j such that $y_j \cup x_i = E_\ell^{(k)}$ for some $1 \leq \ell \leq m$. We evidently have

$$\sum_{i=1}^{n} |A_i| = km \geq kn^{k-1/t^{k-1}} > nN(k!n^{-1/t^{k-1}}).$$

Thus Lemma 3.25 applies with $N = \binom{n}{k-1}, w = n^{1/t^{k-1}}/k!$ as long as $w \geq 1$. Indeed since $n \geq e^{k \log k t^{k-1}}$, it follows that

$$\begin{aligned} w = \frac{n^{1/t^{k-1}}}{k!} &\geq \frac{\left(e^{k \log k t^{k-1}}\right)^{1/t^{k-1}}}{k!} \\ &= \frac{k^k}{k!} > 1. \end{aligned}$$

We thus obtain from Lemma 3.25 that there are t distinct A's $A_{i_1}, \ldots, A_{i_t}$ for which

$$\left| \bigcap_{j=1}^{t} A_{i_j} \right| \geq \frac{1}{2} \binom{n}{k-1} (k!n^{-1/t^{k-1}}) > n^{(k-1)-(1/t^{k-2})}. \tag{3.33}$$

By (3.33) there are more than $n^{(k-1)-1/t^{k-2}}$ $(k-1)$-tuples

$$E_1^{(k-1)}, \ldots, E_{m_1}^{(k-1)}, \quad m_1 > n^{(k-1)-1/t^{k-2}}, \tag{3.34}$$

so that all the k-tuples

$$\{x_{i_j} \cup E_s^{(k-1)} : 1 \leq j \leq t,\ 1 \leq s \leq m_1\} \tag{3.35}$$

are in $\mathcal{R}^{(k)}$.

These $(k-1)$-tuples define a $(k-1)$-graph $\mathcal{R}^{(k-1)}$ on $n-t$ vertices for which

$$|\mathcal{R}^{(k-1)}| = m_1 > n^{(k-1)-(1/t^{k-2})}.$$

By our induction hypothesis, $\mathcal{R}^{(k-1)}$ contains a $\mathcal{K}^{(k-1)}_{t,\ldots,t}$ since clearly $n \geq e^{k\log(k)t^{k-1}} > t + n^{(k-1)\log(k-1)t^{k-2}}$. By (3.35) this implies that our $\mathcal{R}^{(k)}$ contains a $\mathcal{K}^{(k)}_{t,\ldots,t}$ which proves the theorem. $\square$

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