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Some Cases of Erdős-Lovász Tihany Conjecture

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An abstract of
A dissertation submitted to the Faculty of the
James T. Laney School of Graduate Studies of Emory University
in partial fulfillment of the requirements for the degree of
Doctor of Philosophy
in Mathematics
2024

Dedication

To all Gazan students who did not get to graduate.

To the beautiful three souls, I will never be able to talk to again...

Some Cases of Erdős-Lovász Tihany Conjecture

By Juvaria Tariq

The Erdős-Lovász Tihany conjecture states that any G with chromatic number $\chi(G) = s+t-1 > \omega(G)$, with $s, t \geq 2$ can be split into two vertex-disjoint subgraphs of chromatic number s, t respectively. We prove this conjecture for pairs (s, t) if $t \leq s + 2$, whenever G has a K_s, and for pairs (s, t) if $t \leq 4s - 3$, whenever G contains a K_s and is claw-free. We also prove the Erdős Lovász Tihany Conjecture for the pair $(3, 10)$ for claw-free graphs.

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Acknowledgments

ٱلْحَمْدُ لِلَّٰهِ

All Praise to Allah

I want to start by thanking my parents, Muhammad Tariq Sabir and Samina Naz, the two people who made me the person I am today. I am grateful that they never stopped me from taking opportunities in life, whether those were to simply enjoy life or come so far away from home to do a PhD. Also, I can confidently say that my brother is THE BEST BROTHER out there, my biggest support. Also, Annum, Horrain, Yusuf, and Zainab, thank you for always making me smile and being there for me.

I would like to thank Dr. Hao Huang, my advisor, for being supportive throughout this journey. He encouraged me to not give up even when I was full of self-doubt. I have learned so much from Dr. Huang in these last 6 years.

I am also very grateful for all the support I got from the mathematics department. From the last-minute help from Terry Ingram, the best hugs and teaching advice from Dr. Bree Ettinger, the unconditional support from Dr. David Borthwick to the teaching knowledge shared by Dr. Juan Villeta-Garcia, everyone has helped me get to the finish line in their ways.

I also want to thank Dr. Vojtech Rodl and Dr. Dwight Duffus for working with me as teaching mentors and also as my dissertation committee members.

I want to thank Sean Longbrake for being a great team partner. For understanding my struggles and helping me through some of the most difficult times without any question. I mean it when I say that I would not have reached the end without him.

I can not talk about people who stayed there with me and not mention some of the best friends I made at Emory; Dr. Alexander Clifton, Dr. Dylanger Pittman, and Dr. Jayanth Guhan. I miss you three dearly.

Also, Merlene Peter, you are the best!! Sabrina Tariq, Shikufa Mousavi, Maliha Azizi, and Sadia Zahoor, I am so glad to have known you all and thankful that we were able to make some lovely memories in a short amount of time.

Lastly, I want to thank Muffin, Queenie, HSN, Gwenn, and Stony for being you! You guys mean a lot to me.

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Chapter 1

Introduction

In this thesis, we study some cases of Erdős-Lovász Tihany conjecture and explore the structure of graphs that can be possible counterexamples to the conjecture. In particular, we show the classifying graphs, in which the removal of any graph of chromatic number s reduces the chromatic number of the original graph by exactly s, for many small s, can be in some way viewed as a simple problem of asking whether the graph contains a complete graph or not. We exploit structural properties like minimum degree, independence number, absence of certain cliques, and many more to establish some interesting results.

1.1 Notation and Definitions

In this thesis, we consider G as a simple graph with no loops or multiple edges. We let K_ℓ denote the complete graph on ℓ vertices. $K_{s,t}$ denotes the complete bipartite graph with one part of size s and one of size t. We define *chromatic number* of G, denoted by $\chi(G)$, to be the least number of colors needed to color the vertices of a graph G such that no edge is monochromatic. We let $\omega(G)$ be the largest ℓ such that $K_\ell \subset G$, also known as *clique number* of G. The *independence number* of a graph, also known as $\alpha(G)$, is the size of the largest set of vertices in the graph that are not adjacent to each other. For a set $U \subseteq V(G)$, we let $G[U]$ be the subgraph induced by U. We say G is *claw-free* if there is no set W such that $G[W] \cong K_{1,3}$.

Furthermore, we define for a set $S \subset V(G)$, $N(S) = \bigcap_{v \in S} N(v)$ where $N(V)$ is the set of vertices that are adjacent to v. For any subgraph $H \subset G$, we define $N(H) = N(V(H))$. We call this set the *common neighborhood* of H. We define $N[S] = N(S) \cup S$. For a subgraph H, let the *degree* of H, $d(H) = |N(H)|$. Furthermore, for any subgraph $F \subseteq G$, $N_F(H)$ is defined to be $N(H) \cap V(F)$, consequently $d_F(H) = |N_F(H)|$. For other definitions, see the standard reference [Wes01].

Following [BKPS09,Tof95,NL82], given a graph G with k-coloring $\phi : V(G) \to [k]$ and a permutation $\pi : [k] \to [k]$ and a vertex $x \in V(G)$, we let N_1 to be the set of vertices adjacent to x with color $\pi(\phi(x))$, N_2 the set of vertices adjacent to some vertex in N_1 with color $\pi^2(\phi(x))$, N_3 the set of vertices adjacent to some vertex in N_2 with color $\pi^3(\phi(x))$, and so on. We call $N(x, \phi, \pi) = \{x\} \cup N_1 \cup N_2 \cup \ldots$ a *generalized Kempe chain* from x with respect to ϕ and π. Note that changing the color $\phi(y)$ for every $y \in N(x, \phi, \pi)$ to $\pi(\phi(y))$ defines a new k-coloring of G.

1.2 Background

We offer now a brief history of the Erdős-Lovász Tihany conjecture, with some particularly relevant results highlighted. We direct the reader to [Son22] for more details.

The Erdős-Lovász Tihany Conjecture states:

Conjecture 1.1 (Erdős-Lovász Tihany [Erd68])**.** *For $t \geq s \geq 2$, for any graph G with chromatic number $\chi(G) = s + t - 1 > \omega(G)$ there exists a vertex partition $S \sqcup T = V(G)$ such that $\chi(G[S]) \geq s$ and $\chi(G[T]) \geq t$.*

While this conjecture is quite old and has received much attention over the last fifty years, the exact result is known only for the following pairs: $(2, 2), (2, 3), (2, 4), (3, 3), (3, 4), (3, 5)$ $[\mathrm{BJ69}, \mathrm{Moz87}, \mathrm{Sti87a}, \mathrm{Sti87b}]$.

A particularly interesting case is for claw-free graphs. The most general result is the following by Chudnovsky, Fradkin, and Plumettaz [CFP13].

Theorem 1.2. *Let G be a claw-free graph with $\chi(G) > w(G)$. Then, there exists a clique K with $|V(K)| \leq 5$ such that $\chi(G - K) > \chi(G) - |V(K)|$.*

Kostochka and Stiebitz proved the conjecture under the condition G is a line graph [KS08]. A graph is a *quasi-line* graph if, for every vertex v, the set of neighbors of v is expressible as the union of two cliques. The previous result was extended to the following by Balogh, Kostochka, Prince, and Stiebitz [BKPS09]:

Theorem 1.3. *Any quasi-line graph G with chromatic number $\chi(G) = s + t - 1 > \omega(G)$ can be split into two disjoint subgraphs of chromatic number s, t respectively.*

Furthermore, if $\alpha(G) = 2$ *and* $\chi(G) = s + t - 1 > \omega(G)$*,* G *can be split into two vertex-disjoint subgraphs of chromatic number* s, t *respectively.*

This work was extended by Song [Son19] to the following. Recall that a *hole* is a cycle such that no two vertices of the cycle are connected by an edge that does not itself belong to the cycle.

Theorem 1.4. *If* $\alpha(G) \geq 3$ *and* G *has no hole of length between* 4 *and* $2\alpha(G) - 1$ *and* $\chi(G) = s + t - 1 > \omega(G)$*,* G *can be split into two vertex-disjoint subgraphs of chromatic number* s, t *respectively.*

As noted by Erdős and Lovasz if $s = 2$, the Erdős-Lovász Tihany conjecture is equivalent to the following:

Conjecture 1.5 (Double-Critical Graph Conjecture [Erd68])**.** *If* G *is a graph such that removing every edge reduces the chromatic number by two, then* G *is a complete graph.*

This variant has received much attention over the years. In particular, Huang and Yu [HY16] proved:

Theorem 1.6. *If* G *is a claw-free graph of chromatic number six, such that removing every edge reduces the chromatic number by two, then* G *is a complete graph*

Building on this work and work by Kawarabayashi, Pedersen, and Toft [KPT10], Rolek and Song [RS17] were able to prove the following:

Theorem 1.7. *If* G *is a claw-free graph of chromatic number less than or equal to eight, such that removing every edge reduces the chromatic number by two, then* G *is a complete graph.*

1.3 K_ℓ-Critical Graphs

We generalize the idea of double-critical graphs as follows;

Definition 1.8. *For $\ell \geq 2$, we say a graph G is K_ℓ-critical if it satisfies the following three conditions:*

(i) G has a K_ℓ as a subgraph.

(ii) G is critical, i.e. removing any vertex reduces the chromatic number of G by one.

(iii) Removing the vertex set of any K_ℓ reduces the chromatic number of G by ℓ.

The first two conditions are to remove some trivial examples from the family, such as taking the disjoint union of a K_ℓ-critical graph with chromatic number k with a K_ℓ-free graph of chromatic number $k - \ell$, or taking a K_ℓ-free graph. Note that K_2-critical graphs are double-critical graphs.

Note that if G is a counterexample to the Erdős-Lovász Tihany Conjecture for a pair (s, t) and contains a K_s as a subgraph, then G contains a K_s-critical subgraph of the same chromatic number as G. Indeed, if $\chi(G - S) > s + t - 1 - s = t - 1$ for any copy S of K_s, we have found a partition satisfying the Erdős-Lovász Tihany Conjecture. As $\chi(G - S) \geq \chi(G) - |V(S)|$, for all subgraphs S, we see that if G is a counterexample, then $\chi(G - S) = \chi(G) - s$ for all copies S of K_s. In particular, we will prove that any graph with the property that removing any K_s reduces the chromatic number by s contains a K_s-critical graph as an induced subgraph.

In light of this, we make the following conjecture.

Conjecture 1.9. *If G is K_ℓ-critical for some $\ell \geq 2$, then G is a complete graph.*

Note that while a proof of this conjecture would imply Erdős-Lovász Tihany for graphs containing a K_s as a subgraph, the other direction does not hold.

In [Ped08], Pedersen offered a similar definition that requires edges to lie on a K_ℓ. In this setting, he proved Conjecture 1.9 for $\chi(G) \leq 6$ and $\ell = 3$. In our work, we drop this requirement that edges lie on a K_ℓ and are able to reprove this result, as seen in Corollary 2.4.

1.4 Results

In the language of K_ℓ-critical graphs, our main results are the following:

Theorem 1.10. *If G is a K_ℓ-critical graph with $\chi(G) \leq 2\ell+1$, then G is a complete graph.*

Theorem 1.11. *If G is a K_ℓ-critical claw-free graph with $\chi(G) \leq 5\ell - 4$, then G is a complete graph.*

Throughout this work, we call a graph G *triangle-critical* if it is K_3-critical. In this case, we can extend the result one step further.

Theorem 1.12. *If G is a triangle-critical claw-free graph with $\chi(G) = 12$, then G is a complete graph.*

1.5 Organization

In chapter 2, we will study the structure of K_ℓ-critical graphs in detail. Later in chapter 3 we will see the proof of Theorem 1.10. Afterward, we will prove Theorem 1.11 and 1.12 in chapter 4. We will end with a brief discussion of a further research question in chapter 5.

Chapter 2

Preliminary Lemmas

In this chapter, we will establish structural properties of K_ℓ-critical graphs.

Lemma 2.1. *Every graph G containing a K_ℓ that has the property such that $\chi(G - L) = \chi(G) - |L|$ for every copy L of K_ℓ contains a K_ℓ-critical subgraph G' of the same chromatic number of G.*

Proof. Let $G_0 = G$. Given G_i, let G_{i+1} be formed from G_i by removing some vertex $x \in V(G_i)$ such that $\chi(G_i - x) = \chi(G_i)$. The process stops if no such x remains in G_i, and set G' to that graph.

We claim that at every stage of the process, every copy L of K_ℓ has the property that $\chi(G_i - L) = \chi(G_i) - \ell$. In particular, this says that at no stage do we remove a vertex x that lies on a K_ℓ. Note that the following holds hold for all copies L of K_ℓ in G_i:

$$\chi(G - L) \geq \chi(G_i - L) \geq \chi(G_i) - |L|$$

$$\chi(G) - \ell \geq \chi(G_i - L) \geq \chi(G_i) - \ell$$

$$\chi(G_i) - \ell \geq \chi(G_i - L) \geq \chi(G_i) - \ell$$

Thus, in particular, every G_i still has the property that removing a K_ℓ reduces the chromatic number by ℓ. Note that by definition, G' is a critical graph. By our earlier arguments, it still has a K_ℓ and in particular, is thus K_ℓ-critical. □

The following two lemmas are equivalent to Lemma 3.1 and Lemma 3.7 of Stiebitz [Sti87b]. We include proofs for the sake of completeness.

Lemma 2.2. *Let G be a K_ℓ-critical graph with $\chi(G) = k$. Then, $d(v) \geq k - 1$ for all $v \in V(G)$ and for any $L \subseteq G$, with L a copy of K_ℓ, $d(L) \geq k - \ell$. In particular in any $(k - \ell)$-coloring ϕ of $G - L$, for all $i \in [k - \ell]$, $\phi^{-1}(i) \cap N(L) \neq \emptyset$.*

Proof. Since G is critical, for all v in $V(G)$, $\chi(G - v) = k - 1$. Fix a coloring of $G - v$ in $k - 1$ colors. If v does not have a neighbor in every color class, then we can color v with the color not used in $N(v)$. This would give a $(k - 1)$-coloring of G, contradicting that $\chi(G) = k$. Thus, v sees a neighbor in every color class, and so has degree at least $k - 1$.

Let L be a K_ℓ in G and suppose on the contrary that there is a $(k - \ell)$-coloring ϕ of $G - L$ where for some $i \in [k - \ell]$, $N(L) \cap \phi^{-1}(i) = \emptyset$. Fix this i.

Let $V(L) = \{v_{k-\ell+1}, v_{k-\ell+2}, \ldots v_k\}$. Let $\psi : V(L) \to [k-\ell+1, k] : \psi(v_j) = j$. For each vertex $w \in \phi^{-1}(i)$, there is at least one vertex v_{j_w} among $V(L)$ such that w is not adjacent to v_{j_w}. Let $f : \phi^{-1}(i) \to [k-\ell+1, k]$ such that $f(w) = j_w$.

Define $\phi' : V(G) \to [k] - \{i\}$, a coloring of G as follows

$$\phi'(v) = \begin{cases} \psi(v) & v \in V(L) \\ f(v) & v \in \phi^{-1}(i) \\ \phi(v) & \text{otherwise} \end{cases}$$

Note that ϕ' forms a $(k-1)$-coloring of G, a contradiction to G having chromatic number k.

□

Lemma 2.3. *Let G be a K_ℓ-critical graph with chromatic number k. If G contains $K_{k-\ell+1}$, then $G \cong K_k$.*

Proof. We will prove this by induction. Note that if G contains K_k the result follows by criticality. Suppose $1 \leq i \leq \ell - 1$ and G has a K_{k-i} as a subgraph. Then, if we can show that G contains K_{k-i+1}, the result would follow. Now, by definition, G has a K_ℓ, so we may assume $k - i \geq \ell$. Let $X = \{x_1, x_2, \ldots, x_{k-i}\}$ be the vertices of a K_{k-i}. Suppose G has no K_{k-i+1}. Note that $G[\{x_1, x_2, \ldots, x_\ell\}] \cong K_\ell$, and let $L = G[\{x_1, x_2, \ldots, x_\ell\}]$. Also, $\{x_{\ell+1}, \ldots x_{k-i}\} \subseteq N(L)$, but since $|N(L)| \geq k - \ell$ by Lemma 2.2, we have $|N(L) - X| \geq k - \ell - (k - i - \ell) \geq i$.

Since G has no K_{k-i+1}, for every vertex y in $N(L) - X$ there exists an $x \in$

$V(X) - V(L)$ such that xy is not an edge, so without loss of generality fix $y_1 \in N(L) - X$ such that y_1 is not adjacent to $x_{\ell+1}$. Then, $G[\{y_1, x_2, \dots x_\ell\}] \cong K_\ell = L_1$, so $|N(L_1)| \geq k - \ell$. Note that $x_{\ell+1}$ is not among the common neighbors of L_1, so

$$
\begin{aligned}
|N(L_1) - X| &\geq |N(L_1)| - |X - \{x_2, \dots x_\ell, x_{\ell+1}\}| \\
&\geq k - \ell - (k - i - \ell) \\
&\geq i.
\end{aligned}
$$

Thus, there is a y_2 in $N(L_1) - X$.

Continuing, if $j \leq i$, we have $L_j = G[\{y_1, y_2, \dots y_j, x_{j+1}, \dots x_\ell\}] = L_j$ is a copy of K_ℓ, and we note that $|N(L_j) - X| \geq i + 1 - j$. Then, for all $j \leq i$, there is a $y_{j+1} \in N(L_j) - X$. At the end, we have found a K_ℓ, $G[\{y_1, \dots y_{i+1}, x_{i+2}, \dots x_\ell\}] = L_{i+1}$. Let $X' = X - L_{i+1}$. Note that $|V(X')| = k - i - (\ell - i - 1) = k - \ell + 1$. Thus, we have found a K_ℓ, namely L_{i+1}, which is vertex-disjoint from a clique X' of size $k - \ell + 1$, a contradiction to G being K_ℓ-critical . Thus, G has a K_{k-i+1}. □

Note the following immediate corollary:

Corollary 2.4. *If G is a K_ℓ-critical graph with $\chi(G) \leq 2\ell$, G is a complete graph.*

Proof. Let G be a K_ℓ-critical graph with $\chi(G) \leq 2\ell$. Note that G has a K_ℓ. If $\chi(G) = \ell$, the result is clear. Assume then $\chi(G) > \ell$. In particular, by Lemma 2.2, we have that the K_ℓ is contained in a $K_{\ell+1}$. Thus, by Lemma 2.3, G is a complete graph. □

This result is a weaker version of Theorem 1.10. We will improve it in the next section.

Following [BKPS09,Tof95,NL82], given a graph G with k-coloring $\phi : V(G) \to [k]$ and a permutation $\pi : [k] \to [k]$ and a vertex $x \in V(G)$, we let N_1 to be the set of vertices adjacent to x with color $\pi(\phi(x))$, N_2 the set of vertices adjacent to some vertex in N_1 with color $\pi^2(\phi(x))$, N_3 the set of vertices adjacent to some vertex in N_2 with color $\pi^3(\phi(x))$, and so on. We call $N(x, \phi, \pi) = \{x\} \cup N_1 \cup N_2 \cup \dots$ a *generalized Kempe chain* from x with respect to ϕ and π. Note that changing the color $\phi(y)$ for every $y \in N(x, \phi, \pi)$ to $\pi(\phi(y))$ defines a new k-coloring of G.

Lemma 2.5. *Let G be a K_ℓ-critical graph and L be a copy of $K_\ell$in G. Let $\chi(G) = k$ and ϕ be a $(k - \ell)$-coloring of $G - L$. Then for any nonempty repeat-free sequence $j_1, j_2, \dots j_t$ in $[k - \ell]$, and $x, y \in V(L)$, there is a path on $t + 2$ vertices starting at x and ending at y with the $i + 1$th vertex v being in $G - L$ with $\phi(v) = j_i$.*

Proof. Let G' be the graph on $V(G)$ with edges $E(G) - \{xy\}$. Let ϕ' be a $(k - 1)$-coloring of G' extending ϕ and giving unique colors to every vertex of L besides x, y, with $\phi(x) = \phi(y) = k - 1$. Let π be the cyclic permutation defined by $(k - 1, j_1, j_2, \dots, j_t)$. If $N(x, \phi, \pi)$ does not contain y, then reassigning the colors by applying π to the chain (as described above) gives a coloring of G' where x, y have distinct colors. Thus, this would extend to a $k - 1$ coloring of G by adding back the edge xy, a contradiction. Therefore, y must be on this generalized Kempe chain. Since only y and x have color $k - 1$, it follows that $G[N(x, \phi, \pi)]$ must contain a path from x to y of order $t + 2$ satisfying our conditions. □

Lemma 2.6. *Let G be a K_ℓ-critical graph with chromatic number k which is not K_k. Then there exists a copy S of $K_{\ell+1}$, such that for every vertex $x \in V(S)$, there is copy L of K_ℓ, satisfying $L \nsubseteq N[x]$.*

Proof. Let L' be a K_ℓ. We will construct S by induction via the following claim.

Claim. *For any subgraph S contained in a copy L of K_ℓ, there exists x such that $x \in N(S)$ and there is a copy T of K_ℓ, with $T \nsubseteq N[x]$. Moreover, if $|S| \leq \ell - 1$, we can pick x such that $S \cup \{x\}$ is contained in a K_ℓ.*

Note by Lemma 2.2, that $|N(L)| \geq k - \ell$. Since $G \ncong K_k$, we have the existence of a pair $x, y \in N(L) \subseteq N(S)$ such that $x \nsim y$, as otherwise $G[L \cup N(L)] \cong K_k$. Now, y along with $\ell - 1$ vertices of L forms a K_ℓ not in $N[x]$ as xy is not an edge. If $|S| < \ell$, then we have that $\{x\} \cup S$ with some vertices from $L - S$ forms a K_ℓ. Thus, x is the desired vertex to fulfill the claim.

For our base case, note that $\emptyset \subseteq L'$ satisfies the conditions of the claim. Suppose we have an S satisfying the conditions of claim with $|S| \leq \ell$. Then, by repeatedly applying the above claim, we have that there is an x such that there is a T a copy of K_ℓ, with $T \nsubseteq N[x]$, and $S \cup \{x\}$ satisfies the claim if $|S| \leq \ell - 1$ and proves Lemma 2.6 if $|S| = \ell$. □

Lemma 2.7. *Let G be a K_ℓ-critical graph with $\chi(G) = k$ and x a vertex in G such that there is a copy L of K_ℓ with $L \nsubseteq N[x]$. Then, $N(L) \nsubseteq N(x)$. In particular, as $x \notin N(L)$, this implies $N(L) \nsubseteq N[x]$.*

Proof. Suppose otherwise and remove L from G. Observe that if $x \in V(L)$, then

$L \subseteq N[x]$, so $x \in V(G-L)$. Furthermore, $G-L$ is $(k-\ell)$-colorable. Fix a $(k-\ell)$-coloring ϕ, and note that by Lemma 2.2, there is a vertex $y \in N(L)$ such that $\phi(y) = \phi(x)$. As $N(L) \subseteq N(x)$, we have a monochromatic edge, contradicting ϕ being a coloring. So $N(L) \nsubseteq N(x)$. □

Lemma 2.8. *Let G be a K_ℓ-critical graph and x a vertex in G such that there is a copy L_0 of K_ℓ with $L_0 \nsubseteq N[x]$. Then, $\chi(G[N(x)]) \leq k-\ell-1$.*

Proof. We will need the following claim.

Claim. *Let L_i be a K_ℓ intersecting $N(x)$ in $1 \leq s < \ell$ vertices with $x \notin V(L_i)$. Then there exists a copy L_{i+1} of K_ℓ that intersects $N(x)$ in $s-1$ places with $x \notin V(L_{i+1})$.*

By Lemma 2.7, there exists a $z \in N(L_i) - N(x)$. Since $V(L_i) \cap N(x) \neq \emptyset$, there is some vertex $w \in V(L_i) \cap N(x)$. Let $L_{i+1} = G[V(L_i) \cup \{z\} - \{w\}]$.

With this claim, we see there is some copy L_j of K_ℓ not containing x that intersects $N(x)$ in zero places. Remove L_j from the graph. We have that the remainder is $(k-\ell)$-colorable, so $N(x) \cup \{x\}$ is $(k-\ell)$-colorable. Thus, $N(x)$ is $(k-\ell-1)$-colorable, as x is adjacent to every vertex within. □

Lemma 2.9. *Let G be a K_ℓ-critical graph with chromatic number k which is not K_k, with $\ell \geq 2$. Then, every vertex which lies on a K_ℓ has degree at least $k+2\ell-3$. In particular, for all $1 \leq i \leq \ell$, if H is a copy of K_i is contained in some K_ℓ in G, then $d(H) \geq k-\ell+3(\ell-i)$.*

Proof. Let x_1 be a vertex in G that lies on a K_ℓ. Take the L copy of K_ℓ containing x_1 such that over all copies S of K_ℓ containing x_1, $|N(L)| \leq |N(S)|$. Let $V(L) = \{x_1, \ldots, x_\ell\}$. By Lemma 2.2, the number of common neighbors of $V(L)$ is at least $k - \ell$. Since G is not a K_k, there is at least one nonedge between two vertices u, v in $N(L)$.

Let L_i denote the K_ℓ formed by taking $G[\{x_1, \ldots, x_{i-1}, u, x_{i+1}, \ldots, x_\ell\}]$, for $i \in [2, \ell]$. Each such L_i has at least as many neighbors as L but does not have v as a neighbor. Thus, $N(L_i) - (\{x_i\} \cup N(L)) \neq \emptyset$. Let z_i be in $N(L_i) - (\{x_i\} \cup N(L))$. Since $z_i \notin N(L)$, but $z_i \in N(\{x_1, \ldots x_{i-1}, x_{i+1}, \ldots x_\ell\})$, we know that $z_i \notin N(x_i)$. Thus by Lemma 2.7, we have that $L_i' = G[\{x_1, \ldots, x_{i-1}, z_i, x_{i+1}, \ldots, x_\ell\}]$ satisfies $N(L_i') \not\subseteq N[x_i]$. Let z_i' be a vertex in $N(L_i') - N[x_i]$. Note in particular, $z_i' \notin N(L)$.

Note that $z_i \neq z_j$ for $i \neq j$, as then z_i would be in $N(L)$. Furthermore, we have that $z_i' \neq z_j$, as z_i' is not adjacent to x_i, yet z_j is adjacent to x_i. Similarly, $z_i' \neq z_j'$ for $i \neq j$.

Thus, $d(x_1) \geq |V(L) - \{x_1\}| + d(L) + |\{z_2, z_2', \ldots z_\ell, z_\ell'\}| \geq \ell - 1 + k - \ell + 2(\ell - 1) \geq k + 2\ell - 3$. Via the previous argument, any K_i contained in a K_ℓ has at least $k - \ell + 3(\ell - i)$ many common neighbors. □

Lemma 2.10. *Let G be a K_ℓ-critical graph. Let x be any vertex of G and v be a vertex lying on a copy L_0 of K_ℓ which contains a vertex outside $N[x]$. Then v has at least ℓ neighbors outside of $N[x]$.*

Proof. We will use the following claim.

Claim. *Let L_i be a K_ℓ containing a vertex v but not x such that $|(L_i - \{v\}) \cap N(x)| = s$*

with $1 \leq s < \ell - 1$*. Then there exists* L_{i+1} *such that* $|(L_{i+1} - \{v\}) \cap N(x)| = s - 1$*,* $x \notin V(L_{i+1})$*,* $L_{i+1} \cong K_\ell$*, and* v *still lies on* L_{i+1}*.*

By Lemma 2.7, there exists a $z \in N(L_i) - N[x]$. Since $V(L_i) \cap N(x) - \{v\} \neq \emptyset$, there is some vertex $w \in V(L_i) \cap N(x) - \{v\}$. Let $L_{i+1} = G[V(L_i) \cup \{z\} - \{w\}]$.

By this claim, there is some copy L_j of K_ℓ such that $(L_j - \{v\}) \cap N[x] = \emptyset$. By Lemma 2.7, there is a $z \in N(L_j) - N[x]$. Thus, there are at least ℓ vertices in $N(v) - N[x]$, namely $V(L_j) - \{v\}$ and z.

□

Chapter 3

K_ℓ-Critical Graphs with $\chi(G) \leq 2\ell + 1$

We first show the proof for triangle-critical graphs with chromatic number seven. Later we will generalize the same argument for $k = 2\ell + 1$.

Theorem 3.1. *The only triangle-critical graph with chromatic number seven is K_7.*

Proof. Assume otherwise, and let G be such a graph. Fix a triangle $X = \{x, y, z\}$ inside G, and fix a coloring of $G - X$, $\phi : V(G - X) \to [1, 2, 3, 4]$. Let a_1 be a common neighbor of $\{x, y, z\}$ among the four neighbors that exist by Lemma 2.2. Let a_2 be a common neighbor of $\{x, y, a_1\}$, which is not z, and let a_3 be a common neighbor of $\{a_1, a_2, x\}$ which is not y, z. Continue this sequence in the following way. Given $\{a_1, \ldots a_i\}$, let a_{i+1} be a common neighbor in $G - X$ of the triangle $\{a_i, a_{i-1}, a_{i-2}\}$ that has yet to appear on the sequence. We stop if no such vertex exists.

Note that this sequence is uniquely 4-colorable by construction. Since it is a subgraph of $V(G-X)$, it is 4-colorable, and as it is a sequence of K_4 intersecting in triangles, there is a unique way to do it: coloring a_i with $i \pmod 4$. Let $T = \{a_{p-2}, a_{p-1}, a_p\}$ be the last triangle on the sequence. Lemma 2.3 implies that G is K_5-free, hence $d_X(T) \leq 1$. Furthermore, by Lemma 2.2, T has at least 3 more neighbors, $\{a_{b_1}, a_{b_2}, a_{b_3}\}$ in G . Given that T is a triangle, in any 4-coloring of $G-X$, $N_{G-X}(T)$ is monochromatic. As T is the last triangle, $\{a_{b_1}, a_{b_2}, a_{b_3}\}$ lie on the sequence. Thus, $b_1, b_2, b_3 \equiv p+1 \pmod 4$. So, letting b_1 be smallest among $\{b_1, b_2, b_3\}$, we have that $p+1-12 \geq b_1 \geq 1$, therefore $p \geq 12$.

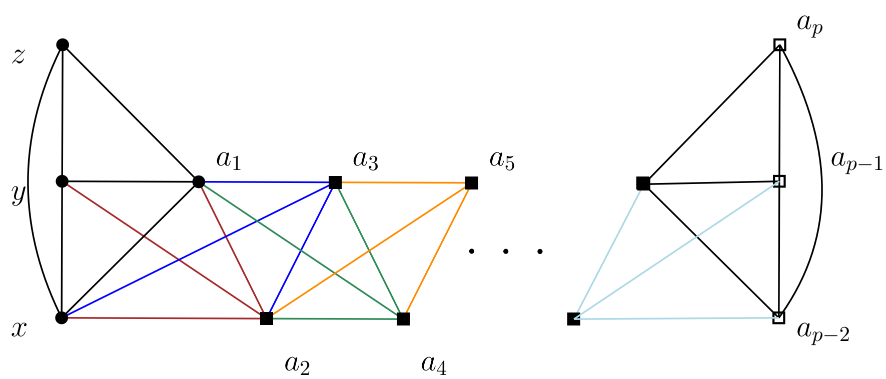

Figure 3.1: Uniquely 4-colorable sequence of k_4's

Let $\{A_i\}_{i=1}^4$ denote the color classes of the unique 4-coloring of $\{a_1, a_2, \ldots a_{p-3}\}$. Since $p \geq 12$, these classes are nonempty. Suppose without loss of generality, that the common neighbors of $\{a_{p-2}, a_{p-1}, a_p\}$ all lie inside A_1. By the triangle-critical condition, $G-T$ is four colorable, and in particular, T has a common neighbor in every color class. By construction, the only common neighbors of T lie inside A_1 or are among $\{x, y, z\}$. By Lemma 2.3, there can be at most one of $\{x, y, z\}$. But this means the neighborhood of $\{a_{p-2}, a_{p-1}, a_p\}$ see at most two colors by the unique colorability of $\{a_1, a_2, \ldots a_{p-3}\}$. This is a contradiction to Lemma 2.2, and so the only triangle-critical seven chromatic graph is K_7. $\square$

Note that Theorem 1.10 follows from below and Corollary 2.4.

Theorem 3.2. *Let G be a K_ℓ-critical graph with chromatic number $2\ell+1$ with $\ell \geq 2$. Then, $G \cong K_{2\ell+1}$.*

Proof. Assume otherwise, and let G be such a graph. Fix a K_ℓ, $X = \{x_1, x_2 \ldots x_\ell\}$ inside G, and fix a $(\ell+1)$-coloring of $G - X$, $\phi : V(G-X) \to [\ell+1]$. Let a_1 be one of the at least $\ell+1$ common neighbors of X. For $i < \ell$, having defined $a_1, \ldots a_i$, let a_{i+1} be a common neighbor of $\{a_1, \ldots a_i, x_{i+1} \ldots x_\ell\}$ which is not among $x_1, \ldots x_i$. As by Lemma 2.2, the common neighborhood has size $\ell+1$, we have that there is such a choice.

Now, for $i \geq \ell$, having defined $a_{i-\ell+1}, a_{i-\ell+2}, \ldots a_i$, we define a_{i+1} as any common neighbor of these vertices among $V(G-X)$ yet to appear on our sequence. We stop when no choices remain.

Note that this sequence $\{a_1, a_2 \ldots, a_p\}$ is uniquely $(\ell+1)$-colorable by construction. Since the sequence induces a subgraph of $V(G-X)$, it is $(\ell+1)$-colorable, and as it can be seen as a sequence of $K_{\ell+1}$'s intersecting in K_ℓ's, there is a unique up to relabeling way to do it: coloring a_i with $i \pmod{\ell+1}$, making it uniquely $(\ell+1)$-colorable. Let $L = \{a_{p-\ell+1}, a_{p-\ell+2}, \ldots, a_p\}$ be the last K_ℓ on the sequence. Lemma 2.3 implies that G is $K_{\ell+2}$-free, hence $d_X(L) \leq 1$. Furthermore, by Lemma 2.2, $d_{G-X}(L) \geq \ell$. Given that L is a K_ℓ, in any $(\ell+1)$-coloring of $G-X$, $N_{G-X}(L)$ is monochromatic. As L is the last K_ℓ, $N_{G-X}(L) \supseteq \{a_{b_1}, a_{b_2}, \ldots a_{b_\ell}\}$ lie in the sequence. Thus, $b_1, b_2, \ldots b_\ell \equiv$

$p+1 \pmod{\ell+1}$. So, letting b_1 be smallest among $\{b_1, b_2, \ldots b_\ell\}$, we have that $p+1-\ell(\ell+1) \geq b_1 \geq 1$, therefore $p \geq \ell(\ell+1)$.

Note that since $p \geq \ell(\ell+1)$, L is distinct from $a_1, a_2, \ldots, a_\ell$. By our earlier observation, there is a j such that $N_{G-X}(L) \subseteq \phi^{-1}(j)$. Let $A = \phi^{-1}(j) \cap \{a_1, a_2, \ldots a_{p-\ell}\}$. Now, by unique colorability, any $(\ell+1)$-coloring of $G-L$ colors A with one color. As L has at most one common neighbor among X, we have that the neighborhood of L sees at most two color classes of the coloring of $G-L$, and thus misses at least one. But this contradicts Lemma 2.2, thus $G \cong K_{2\ell+1}$. □

Chapter 4

Claw-Free Graphs

In this chapter, we focus our attention on the class of graphs that are claw-free and are K_ℓ-critical.

4.1 Claw-Free Graphs with $\chi(G) \leq 5l - 4$

We first show the proof of our theorem for chromatic number 8 case and then we will show the more general proof. We include this proof to highlight the alternative method using Ramsey numbers and their lower-bound constructions.

Theorem 4.1. *If G is triangle-critical, has chromatic number eight, and is claw-free, then $G \cong K_8$.*

Proof. Let G be a triangle-critical claw-free graph with chromatic number eight. Then, $d(T) \geq 5$ for all triangles T by Lemma 2.2.

Fix any triangle T. We will now show that $G[N(T)] \cong C_5$ for T. If G is not

K_8 then the neighborhood of T is K_3-free by Lemma 2.3. Furthermore, since G is claw-free, the independence number of $G[N(T)]$ is at most two. Suppose on the contrary that $d(T) \geq 6$. Then since $R(3,3) = 6$, $N(T)$ contains either a triangle or an independent set of size three, a contradiction. Thus, $d(T) = 5$. Consequently, from the uniqueness of the lower bound Ramsey construction, $G[N(T)] \cong C_5$.

Let T_1 be a triangle in G, with $V(T_1) = \{x, y, z\}$. Let $N(T_1) = \{a, b, c, d, e\}$, which forms a cycle (a, b, c, d, e). Now, take the triangle T_a induced by $\{x, y, a\}$, the common neighborhood of this triangle certainly contains the vertices z, b, and e. As before, $G[N(T_a)] \cong C_5$, so there must be vertices, which we will call suggestively a_1, e_1 lying in the common neighborhood such that (z, b, a_1, e_1, e) is a cycle. Note that $a \not\sim c, d$, therefore $a_1, e_1 \neq c, d$.

Now, let us examine the triangle T_b induced by $\{x, y, b\}$. $N(T_b)$ includes a, a_1, z, c. We already know that $c \sim z \sim a \sim a_1$, so there must be a $b_1 \in N(T_b)$ such that $c \sim b_1 \sim a_1$. As $b_1 \sim b$, we have that $b_1 \neq e_1$ since $e_1 \not\sim b$. Similarly, $b_1 \neq d, e$.

Let us now look at the triangle T_c induced by $\{x, y, c\}$. We note that $\{b, d, b_1, z\} \subseteq N(T_c)$. There must be a fifth vertex c_1 such that $c_1 \sim b_1, d$ but $c_1 \not\sim b, z$. Thus, $c_1 \neq a_1, b_1, a, e$, yet it may be true that $c_1 = e_1$. We will discount this possibility later.

Consider now the triangle T_d induced by $\{x, y, d\}$. We note that $\{c, e, c_1, z\} \subseteq N(T_d)$. There must be a fifth vertex d_1 such that $d_1 \sim c_1, e$ but $d_1 \not\sim z, c$. Note that for $G[N(T_d)] \cong C_5$, we must have that $c_1 \not\sim e$. Therefore, $c_1 \neq e_1$. Since $d_1 \not\sim c$, we have that $d_1 \neq b_1, a, b$.

Let us look at the triangle T_e induced by $\{x, y, e\}$. We note that $\{a, d, z, d_1, e_1\} \subseteq$

$N(T_e)$. If $e_1 = d_1$, then a subset of $N(T_e)$ would induce a C_4, so $e_1 \neq d_1$. Thus, in particular $d_1 \not\sim a$, so $d_1 \neq a_1$. Therefore all five a_1, b_1, c_1, d_1, e_1 are distinct and distinct from $\{a, b, c, d, e\}$.

Now, let us take the triangle T_{a_1} induced by $\{x, y, a_1\}$. We note that $N(T_{a_1})$ contains the vertices a, b, b_1, e_1 and a vertex w such that (e_1, a, b, b_1, w) is a C_5. Then, we look at the triangle T_{b_1} induced by $\{x, y, b_1\}$, $N(T_{b_1})$ contains b, c, a_1, c_1, w. If $w = c_1$, $G[N(T_{b_1})]$ would contain a C_4, so we have that $w \neq c_1$ and $w \sim c_1$. Via similar arguments examining $N(\{x, y, c_1\})$, $N(\{x, y, d_1\})$, $N(\{x, y, e_1\})$, we have that $N(\{x, y, w\}) = \{a_1, b_1, c_1, d_1, e_1\}$. Note in particular that w is distinct from all five of these vertices. By triangle-criticality, $G - \{a, b, a_1\}$ has chromatic number five. Fix a coloring. Note that x, y must have distinct colors from $\{w, z, c, d, e, b_1, c_1, d_1, e_1\}$, so we must color the rest with three colors. $\{b_1, c_1, c\}$ must all receive three distinct colors, say respectively 1, 2, 3. $\{b_1, c_1, w\}$ is a triangle, so w must see color 3. $\{c, c_1, d\}$ is a triangle so d sees color 1. $\{d, c_1, d_1\}$ is a triangle, so d_1 sees color 3. Yet $d_1 \sim w$, a contradiction.

Thus, $G \cong K_8$. □

Furthermore, we will prove the following statement:

Theorem 4.2 (Restatement of Theorem 1.11). *Let $\ell \geq 2$. Let G be K_ℓ-critical claw-free graph with chromatic number $k \leq 5\ell - 4$. Then $G \cong K_k$.*

Proof. Suppose on the contrary, $G \ncong K_k$.

Then, by Lemma 2.6, there is a $S \cong K_\ell$ such that for every $x \in V(S)$, there is some $L \cong K_\ell$ such that $L \nsubseteq N(x)$ and $x \notin V(L)$.

Moreover, for any $x \in V(S)$, Lemma 2.8 implies $\chi(G[N(x)]) \leq k - \ell - 1$, and hence by claw-freeness $d(x) \leq 2(k - \ell - 1)$.

Let u, v be a nonadjacent pair inside $N(S)$. Note that G is a K_k if no such pair exists. Then for every $x \in V(S)$, by Lemma 2.9, $d(u, x) \geq k-\ell+3(\ell-2) \geq k+2\ell-6$.

Take $y \in V(S)$, let $S' = G[V(S)-\{y\}\cup\{v\}]$. Then, S' is not in the neighborhood of u and does not contain u, but contains every $x \in S - \{y\}$. Thus, by Lemma 2.10, for every $x \in V(S) - \{y\}$ has at least ℓ neighbors outside $N[u]$. Thus, since x is adjacent to u, we have that

$$\begin{aligned} d(x) &\geq d(x, u) + |N(x) - N[u]| + |\{u\}| \\ &\geq k + 2\ell - 6 + \ell + 1 \\ &\geq k + 3\ell - 5. \end{aligned}$$

Combining this with the upper bound on $d(x)$, we have

$$\begin{aligned} k + 3\ell - 5 &\leq 2(k - \ell - 1) \\ k + 3\ell - 5 &\leq 2k - 2\ell - 2 \\ 5\ell - 3 &\leq k. \end{aligned}$$

Yet, by assumption, we have that $k \leq 5\ell - 4$, a contradiction. So $G \cong K_k$. $\square$

4.2 Claw-Free Graphs with $\chi(G) = 12$

We will now prove the following statement:

Theorem 4.3 (Restatement of Theorem 1.12)**.** *Let G be a triangle-critical, claw-free graph of chromatic number twelve. Then $G \cong K_{12}$*

Proof. Assume on the contrary that $G \ncong K_{12}$. Let a be a vertex of G that lies on a triangle L such that there is a triangle L' in $G - \{a\}$ not fully contained in $N(a)$. By Lemma 2.6, such a vertex exists. Let b, d be a nonedge in $N(L)$. Since $G \ncong K_{12}$, such a nonedge exists. Note that there is a triangle containing a, which does not lie inside $N(b)$ and does not contain b. Recall that by claw-freeness and Lemma 2.8, $d(a), d(b) \leq 2(12 - 3 - 1) \leq 16$.

If every triangle containing ab has degree at least ten, then, following the proof of Lemma 2.9, we have that $d(a, b) \geq 13$. As by Lemma 2.10, $|N(a) - N[b]| \geq 3$, we have that

$$d(a) = d(a, b) + |N(a) - N[b]| + |\{b\}| \geq 17,$$

a contradiction. Thus, by Lemma 2.2, there is a $c \in N(a, b)$ such that $d(a, b, c) = 9$. Let $T = G[\{a, b, c\}]$. Now, by Lemma 2.9, we have that $d(a, b), d(a, c), d(b, c) \geq 12$. Thus, there are six vertices, x, x', y, y', z, z' such that $x, x' \in N(a, b) - N[c]$, $y, y' \in N(a, c) - N[b]$, and $z, z' \in N(b, c) - N[a]$.

Now, we have that for each vertex among $\{a, b, c\}$ there is a triangle not containing it such that misses its neighborhood, so by Lemma 2.8 and claw-freeness, $d(a), d(b), d(c) \leq 16$.

Let us examine the triangle $S = G[\{a, b, x\}]$. Since $x \notin N[c]$, S has a neighbor outside $N[c]$ by Lemma 2.7; without loss of generality, we may assume it is x'. Also by Lemma 2.7, a, x, x' has a common neighbor outside the $N[c]$, let us call it a'. Suppose $a' \sim b$. Then, $d(a, b) \geq 13$. But, then by Lemma 2.10, $d(a) \geq 17$, a contradiction. So $a' \not\sim b$. Similar logic gives a vertex b' that is adjacent to x, x', b but not a, c.

Now, let us examine $S' = G[\{a, c, y\}]$. Since $y \notin N[b]$, S' has a neighbor outside $N[b]$ by Lemma 2.7; without loss of generality, we may assume it is y'. Furthermore, a, y, y' has a common neighbor outside $N[b]$, let us call it a''. As above $a'' \not\sim c$. If $a'' \neq a'$, then $d(a) \geq 17$, a contradiction. Thus, $a'' = a'$.

Following this logic to its natural conclusion, we have found that $x \sim x'$, $y \sim y'$, and $z \sim z'$, and the existence of three vertices a', b', c' such that $a' \sim a, x, x', y, y'$; $a' \not\sim b, c$; $b' \sim b, x, x', z, z'$; $b' \not\sim a, c$; $c' \sim c, y, y', z, z'$; and $c' \not\sim a, b$.

Note that the edge aa' lies on a triangle, so $d(a, a') \geq 12$. In particular, $N(a, a')$ contains x, x', y, y' and eight vertices among $N(T)$. Similar logic holds for bb' and cc'. In particular $N_{N(T)}(a', b', c') \geq 6$. Fix $w \in N_{N(T)}(a', b', c')$. If $a' \not\sim b'$, then $G[\{w, a', b', c\}]$, would be a claw, a contradiction. So $a' \sim b'$, and similar logic shows $T' := G[\{a', b', c'\}]$ satisfies $T' \cong K_3$, as shown in Figure 1.

Now, $G - T$ is 9-chromatic by triangle-criticality. Fix one such 9-coloring ϕ. Under ϕ, $N(T)$ receives all nine colors by Lemma 2.2. Since $d_{N(T)}(a') \geq 8$, we have that a' has exactly one non-neighbor in $N(T)$. Suppose inside $N(T)$, a', b' share a common non-neighbor. Let v be the non-neighbor of a' in $N(T)$, and so under ϕ, $\phi(v) = \phi(a')$. Under the assumption a', b' are both nonadjacent to v, we have that $\phi(v) = \phi(b')$. But then, $\phi(b') = \phi(a')$, contradicting that ϕ is proper coloring.

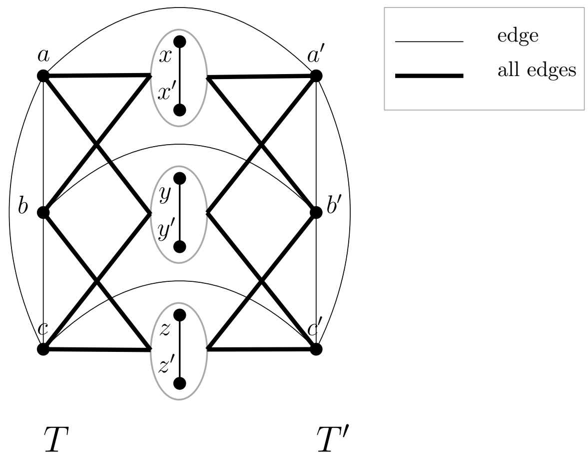

Figure 4.1: Some edges in $G[\{x, x', y, y', z, z'\} \cup T \cup T']$

This gives us a complete description of the connectivity between $N(T)$ and T', as shown in Figure 2.

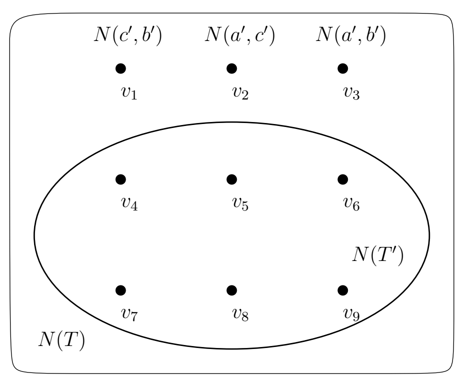

Figure 4.2: Structure of $N(T)$

Let us now examine the edge ax. As it lies on a triangle $d(ax) \geq 12$. Since ax has at most three neighbors among $\{x', y, y', z, z'\}$ and exactly two neighbors, a', b among $\{b, c, a', b', c'\}$, we have that ax have at least seven common neighbors among

$N(T)$. Similarly, $d_{N(T)}(x'), d_{N(T)}(y), d_{N(T)}(y'), d_{N(T)}(z), d_{N(T)}(z') \geq 7$.

Note that under ϕ, six colors appear among x, x', y, y', z, z'. Indeed, suppose that two of them received the same color, say without loss of generality $\phi(x) = \phi(y)$. Letting w be the vertex in $N(T)$ receiving color $\phi(x)$, then $G[\{a, x, y, w\}]$ would be a claw, a contradiction.

We seek to show that, under ϕ, every vertex in $\{x, x', y, y', z, z', a', b', c'\}$ receives a distinct color. Suppose on the contrary that one vertex among x, x', y, y', z, z' under the coloring ϕ shares a color with one of $\{a', b', c'\}$. Without loss of generality, we may assume it is z and a'. Let v_i be the vertex in $N(T)$ such that $\phi(v_i) = \phi(z) = \phi(a')$. Now, as $z, a' \not\sim v_i$, we have that $d_{N(T)}(z, a') \geq 7$. If $d_{N(T)}(v_i) \geq 2$, then $N_{N(T)}(v_i, a', z) \neq \emptyset$, and so G would a contain a claw, a contradiction. Thus, $d_{N(T)}(v_i) \leq 1$. Now, $d(a, b, v_i) \geq 9$, so $|N(a) - N(c)| \geq d(a, b, v_i) - d_{N(T)}(v_i) \geq 8$. But, then $d(a) = d(a, c) + |N(a) - N(c)| \geq 12 + 8 > 16$, a contradiction. Thus, under ϕ, every vertex in $\{x, x', y, y', z, z', a', b', c'\}$ receives a distinct color.

Without loss of generality, assume $\phi(a') = 1, \phi(b') = 2, \phi(c') = 3, \phi(x) = 4, \phi(x') = 5, \phi(y) = 6, \phi(y') = 7, \phi(z) = 8, \phi(z') = 9$. For every $i \in [9]$, let $v_i \in N(T)$ be the unique vertex colored i under ϕ.

Claim. *For all $i \in \{1, 2, 3\}$ and all $j \in \{4, 5, 6, 7, 8, 9\}$, $v_i v_j$ is an edge.*

For simplicity, let us first examine $i = 1, j = 4$. By Lemma 2.5, there is a path of order four from b to c where the second vertex receives color 1 and the third vertex receives color 4. Yet b is adjacent to exactly one vertex of color 1, v_1, and c is adjacent to exactly one vertex of color 4, v_4. So $v_1 v_4$ is an edge. Similar arguments complete this claim.

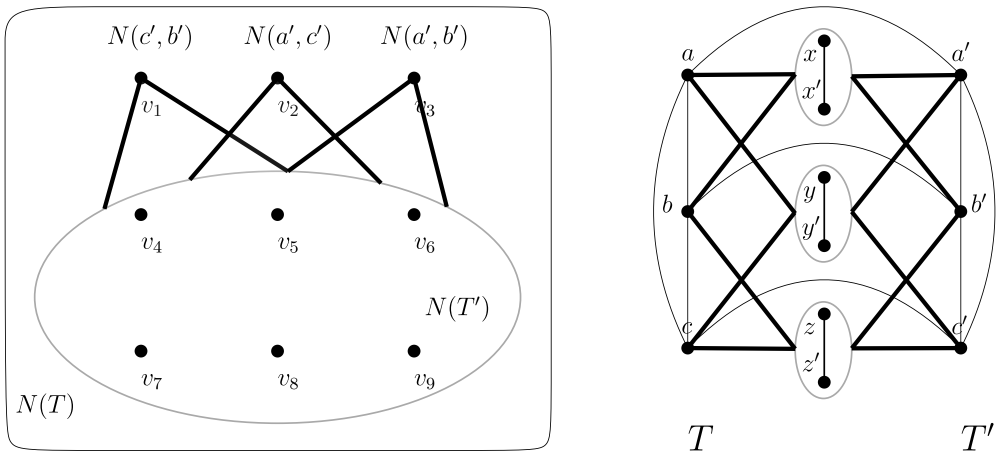

Figure 4.3: Key Structure of $N(a) \cup N(b) \cup N(c)$

Our final claim before our contradiction is that $d(T') = 9$. Now, as T is a triangle that does not lie completely in any of their neighborhoods, by Lemma 2.8 and claw-free, we have that $d(a'), d(b'), d(c') \leq 16$. Suppose on the contrary that $d(T') \geq 10$. Now, a' is adjacent to at least seven vertices that are not in $N(T')$, namely a, v_2, v_3, b', c', and then at least two of x, x', y, y', as c' cannot be adjacent to both of x, x' and b' cannot be adjacent to both of y, y', as then either $d(c') \geq 17$ or $d(b') \geq 17$ respectively. Indeed, if c' were adjacent to both x, x', then c' would be adjacent to eight vertices in $N(T)$, c, x, x', y, y', z, z' and a', b'. Yet then, $d(a') \geq 17$, so $d(T') = 9$.

We will now show that $\chi(G - T') \geq 10$, contradicting triangle-criticality. Suppose there is a 9-coloring of $G - T'$, call it ψ. Then, by Lemma 2.2, all nine colors must appear in the $N(T')$. As there are only nine vertices, every vertex must get a distinct color. Suppose without loss of generality that $\psi(v_4) = 4, \psi(v_5) = 5, \psi(v_6) =$

$6, \psi(v_7) = 7, \psi(v_8) = 8, \psi(v_9) = 9$. Then, as each of $\{a, b, c\}$ is adjacent to all six of these vertices, and form a triangle, we may assume $\psi(a) = 1, \psi(b) = 2$, and $\psi(c) = 3$. Thus, all nine colors appear in the neighborhood of v_1, and so ψ cannot be a proper coloring.

Therefore, $G \cong K_{12}$. □

Chapter 5

Summary and Future Research

We note that if G is a counterexample for the Erdős-Lovász Tihany conjecture for a pair (s,t) with $s = 3$, then it must have a K_3. In particular, Stiebitz [Lemma 3.6, [Sti87b]] showed it must have a K_4. Thus, our main results reprove Erdős-Lovász Tihany for $(3,3),(3,4),(3,5)$ and prove it for claw-free graphs for $(3,t)$ with $t \in \{6,7,8,9,10\}$.

In particular, this leads to the open question:

Question 5.1. *Does any counterexample G to the Erdős-Lovász Tihany Conjecture for a pair (s,t) with $s,t \geq 4$ require K_s to be a subgraph of G?*

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