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Topics in Ramsey Theory

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M.S. in Computer Science, University of São Paulo, 2007

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Abstract of
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Abstract

In this dissertation we discuss two results in Ramsey Theory.

Result I: the size-Ramsey number of a graph H is the smallest number of edges a graph G must have in order to force a monochromatic copy of H in every 2-coloring of the edges of G . In 1990, Beck studied the size-Ramsey number of trees: he introduced a tree invariant $\beta(\cdot)$, and proved that the size-Ramsey number of a tree T is at least $\beta(T)/4$. Moreover, Beck showed an upper bound for this number involving $\beta(T)$, and further conjectured that the size-Ramsey number of any tree T is of order $\beta(T)$. We answer his conjecture affirmatively. Our proof uses the expansion properties of random bipartite graphs.

Result II: We prove the following metric Ramsey theorem. For any connected graph G endowed with a linear order on its vertex set, there exists a graph R such that in every coloring of the t -sets of vertices of R it is possible to find a copy G' of G inside R satisfying the following two properties:

- the distance between any two vertices $x, y \in V(G')$ in the graph R is the same as their distance within G' ;
- the color of each t -set in G' depends only on the graph-distance metric induced in G' by the ordered t -set.

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Chapter 1

Introduction

Ramsey theory, which is named after Frank P. Ramsey, author of the very influential article [34], concerns the study of partitions of discrete structures, such as graphs, hypergraphs, integers, vector spaces, partially ordered sets, points in the Euclidean space, etc. Statements in Ramsey theory are usually of the form “if a certain structure is large enough, no matter how its basic components are partitioned into r parts, it is possible to find a given configuration completely contained in one of the partition classes”. More refined statements provide some estimate for how large the structure needs to be in term of the configuration and number of parts. These abstract notions will be illustrated in the example that follows.

Example 1.1. *In a group of six people, you will always find either three people that know each other or three people that are strangers to each other.*

In the example above, the structure is the network of people and their acquaintanceships. The basic components are pairs of individuals, with each pair being classified according to whether the two individuals know each other or not. The sought-after configuration is a group of three people (such that all the three pairs of people that you can form among the three are contained in the same class).

The example above can be generalized in several directions. First, let us introduce some notation. For a positive integer n , denote by $[n]$ the

set $\{1, \dots, n\}$. For any integer $k \geq 0$ and set X , denote by $\binom{X}{k}$ the family of all subsets of X having exactly k elements.

Theorem 1.2 (Ramsey 1930). *For all integers $r, k \geq 1$, and $\ell \geq k$ there exists n_0 such that for all $n \geq n_0$ the following holds. For any partition of $\binom{[n]}{k}$ as $C_1 \cup \dots \cup C_r$, there exists $1 \leq j \leq r$ and an ℓ -element subset $Y \subset [n]$ such that $\binom{Y}{k} \subset C_j$.*

Theorem 1.2 was later rediscovered by Erdős and Szekeres [11] in a paper that popularized Ramsey theory in the mathematics community.¹ In this dissertation, we will present two Ramsey theorems on graphs. These two results and the corresponding Chapters 2 and 3 are completely independent and self-contained.

Some existence results in Ramsey theory follow from Theorem 1.2. For instance, it follows that for any graph H there exists $n_0 = n_0(H)$ such that, for any $n \geq n_0$, the complete graph K_n is such that in any 2-coloring of its edges, one can find a monochromatic copy of H . In such cases, it is desirable to obtain good bounds for the smallest n_0 for which the above holds. The result in Chapter 2 is a case where existence is a simple corollary of Theorem 1.2 but the real problem is to obtain good numerical bounds. Here we settle a conjecture of Beck [3] and obtain bounds which are optimal up to a constant multiple factor. Such a precise bound is rather the exception in Ramsey theory, which is a field notorious for having loose bounds that stand for decades without substantial improvement. In contrast, our result of Chapter 3 is an existence result and we made no effort to obtain numerical bounds (which certainly would be extremely large).

In Chapter 2 we discuss a *size-Ramsey* result, namely a result in which we obtain a Ramsey graph having the smallest possible number of edges (in contrast, Ramsey graphs are typically measured in terms of number of vertices).

¹Erdős was certainly one of the main contributors to Ramsey theory. Unfortunately, F. P. Ramsey died prematurely and did not see the large impact of his results.

Here we prove a conjecture of J. Beck [3] which states that for any tree T , its edge-Ramsey number is of order $\beta(T)$, where $\beta(\cdot)$ is a tree-invariant. Beck proved that any Ramsey graph for T must have at least $\beta(T)/4$ edges and conjectured that the edge-Ramsey number is of the same order as $\beta(T)$. What we prove here is in fact a stronger result: we use probabilistic methods to show the existence of sparse graphs with the property that every subgraph containing, say, 1% of the total number of edges must contain all trees having the same invariant parameters.

In Chapter 3 we prove a *metric* Ramsey result on graphs. In this Ramsey theorem the structures are ordered graphs endowed with metric embeddings. We call G a *metric subgraph* of H if $G \subset H$ and there is no shortcut path in the larger graph H for connecting two vertices of G (the shortest distance is attained by a path completely inside G). In particular, a metric subgraph is also an induced subgraph (but the converse is not always true). The components being colored in this Ramsey theorem are t -subsets of the vertex set that induce a given fixed metric ρ . The configuration being sought is a metric copy of a fixed graph G where every t -subset of $V(G)$ inducing the metric ρ has the same color.

An interesting particular case of our results is the following: for every graph G there exists a graph R such that for every two-coloring of the pairs of vertices of R one can find a metric subgraph $G' \subset R$, isomorphic to G , such that the color of a pair of vertices of G' is a function of the distance between the vertices. In other words, all pairs which are edges of G' have the same color, all pairs at distance two have the same color, etc.

Much of the notation used here is quite standard, including “big O ” notation such as $O(\cdot)$, $\Omega(\cdot)$, $o(\cdot)$. We also use **a.a.s.** (asymptotically almost surely) to denote that a sequence of random events has probability converging to 1. Special notation pertaining to each chapter will be introduced as needed.

Chapter 2

The size-Ramsey number of trees

2.1 Introduction

For graphs G and H , the size-Ramsey number $\hat{r}(G, H)$, introduced by Erdős et al. [9], is the smallest number m such that there exists a graph F on m edges with the property that, in any red-blue coloring of the edges of F , there exists either a red copy of G or a blue copy of H .

For a real number $\alpha \in [0, 1]$ and graphs F, G we shall write $F \rightarrow_\alpha G$ if any subgraph $F' \subseteq F$ with $e(F') \geq \alpha e(F)$ contains a copy of G as a subgraph. Notice that if $F \rightarrow_{1/2} G$ then $\hat{r}(G) = \hat{r}(G, G) \leq e(F)$.

It is well known that $\hat{r}(K_n)$ grows exponentially with n . In contrast, Beck [2], answering a question of Erdős, showed that for P_t , the path on t vertices, we have

$$\hat{r}(P_t) = \hat{r}(P_t, P_t) \leq 900t.$$

In fact, Beck proved that for any $\alpha \in (0, 1]$ there is $c = c(\alpha)$ such that **a.a.s.**

⁰The contents of this chapter were published in [4].

(asymptotically almost surely) the random graph $G = G_{n,c/n}$ satisfies $G \rightarrow_\alpha P_{\lfloor n/c \rfloor}$.

Friedman and Pippenger [13] improved this result by showing that any tree with maximum degree Δ and t vertices has size-Ramsey number $c(\Delta)t$, where $c(\Delta) = O(\Delta^4)$. This was later improved to $c(\Delta) = O(\Delta^2)$ by Ke [20] and to $c(\Delta) = O(\Delta)$ by Haxell and Kohayakawa [19].

Although certain trees \mathcal{T} have size-Ramsey number of order $\Delta(\mathcal{T})|\mathcal{T}|$, it is clear that the size-Ramsey number of the star $K_{1,t}$ is not of order t^2 . Indeed, $K_{1,\alpha^{-1}t} \rightarrow_\alpha K_{1,t}$ for any $\alpha \in (0, 1]$. Hence, the bound $\Delta(\mathcal{T})|\mathcal{T}|$ may be far from optimal in many cases.

In [3], Beck introduced the tree invariant $\beta(\mathcal{T})$ which is defined as follows. Let $V(\mathcal{T}) = V_0(\mathcal{T}) \cup V_1(\mathcal{T})$ be the partition determined by the unique proper two-coloring of the vertex set of \mathcal{T} . Set $\Delta_i = \Delta_i(\mathcal{T}) = \max\{d_{\mathcal{T}}(v) : v \in V_i(\mathcal{T})\}$ and $n_i = n_i(\mathcal{T}) = |V_i(\mathcal{T})|$ for $i = 0, 1$ and let $\beta(\mathcal{T}) = n_0\Delta_0 + n_1\Delta_1$. Improving his previous result, Beck [3] proved that for any tree \mathcal{T} ,

$$\beta(\mathcal{T})/4 < \hat{r}(\mathcal{T}) \leq O(\beta(\mathcal{T})(\log |\mathcal{T}|)^{12})$$

and conjectured that $\hat{r}(\mathcal{T}) = O(\beta(\mathcal{T}))$. For completeness, we include Beck's proof of the lower bound in Section 2.1.2. Haxell and Kohayakawa [19] significantly improved the upper bound to $\hat{r}(\mathcal{T}) = O(\beta(\mathcal{T}) \log \Delta(\mathcal{T}))$.

We settle this conjecture by showing that for any $(n_0, \Delta_0, n_1, \Delta_1)$ and $\alpha \in (0, 1]$ there exist $N_0, N_1, C(\alpha)$, and $p \in [0, 1]$ with $pN_0N_1 = C(\alpha)(n_0\Delta_0 + n_1\Delta_1)$ such that **a.a.s.** the random bipartite graph $G = G_{N_0, N_1; p}$ satisfies $G \rightarrow_\alpha \mathcal{T}$ for any tree \mathcal{T} with $\Delta_i(\mathcal{T}) \leq \Delta_i$ and $n_i(\mathcal{T}) \leq n_i$, for $i = 0, 1$. Since **a.a.s.** G has $O(pN_0N_1)$ edges, the size-Ramsey number of any tree \mathcal{T} is of the order of $\beta(\mathcal{T})$.

The embedding of \mathcal{T} into G is done algorithmically. We believe that this algorithmic method is interesting in its own right and that it could be useful in other similar contexts. In fact, we have used analogous techniques in an

algorithm that embeds graphs of bounded degree into sparse random graphs (see [5]).

2.1.1 Organization of the chapter

In order to prove Beck's conjecture we establish several properties that hold **a.a.s.** for random graphs. Any graph satisfying these properties may be used as an upper bound for the size-Ramsey number of trees. However, there is no known graph construction satisfying all these properties. Thus we have resorted to the probabilistic method in order to prove the existence of such graphs. The results on random graphs are stated in Theorem 2.6 of Section 2.4.

In Section 2.6 we exhibit an embedding scheme for trees, Algorithm 1, that finds an isomorphic copy of any tree with prescribed parameters in a graph satisfying the properties listed in Theorem 2.6.

We shall give an outline of a simpler (somewhat unrealistic) case for the sake of introducing, in an easier context, some of the techniques employed in the general case. This informal outline is given in Section 2.3.

2.1.2 The lower bound

Here we give Beck's proof that for any tree \mathcal{T} , $\hat{r}(\mathcal{T}) > \beta(\mathcal{T})/4$.

Let $n_i = n_i(\mathcal{T})$ and $\Delta_i = \Delta_i(\mathcal{T})$, $i = 0, 1$, be as above. Without loss of generality we assume that $n_0\Delta_0 \geq n_1\Delta_1$. Let G be any graph having fewer than $\beta(\mathcal{T})/4$ edges. We will now describe an explicit coloring of the edges of G which does not admit any monochromatic copy of \mathcal{T} .

First partition $V(G)$ into $V^+ = \{v \in V(G) : \deg(v) \geq \Delta_0\}$ and $V^- = V(G) \setminus V^+$. We now color all edges of G completely inside V^+ or completely inside V^- using the color blue. All the edges having endpoints in distinct parts V^-, V^+ are colored red.

If G contains a blue copy of \mathcal{T} , then this copy must be contained in V^+ since no vertex of V^- can be used to embed a vertex of degree Δ_0 and V^- is not connected to V^+ by any blue edge. Consequently, $|V^+| \geq n_0 + n_1$. However, by definition, the degrees of vertices in V^+ are all at least Δ_0 and thus we obtain a contradiction:

$$e(G) \geq \frac{1}{2} \sum_{v \in V^+} \deg(v) \geq \frac{\Delta_0 |V^+|}{2} \geq \frac{\Delta_0(n_0 + n_1)}{2} > \frac{\beta(\mathcal{T})}{4}.$$

It follows that there cannot be a blue copy of \mathcal{T} .

Let us now assume we may find a red copy of \mathcal{T} in G . Since the red subgraph of G is bipartite (with classes V^-, V^+), the class $V_0(\mathcal{T})$ must be embedded into V^+ (since we must be able to embed a vertex of degree Δ_0). Hence $|V^+| \geq n_0$ and thus $e(G) \geq \frac{1}{2}|V^+|\Delta_0 \geq \frac{n_0\Delta_0}{2} \geq \beta(\mathcal{T})/4$, again a contradiction.

2.2 Preliminaries

Given a graph $G = (V, E)$ and disjoint sets $S, T \subset V$, we denote by $E_G(S, T)$ the set of all edges with one endpoint in S and the other endpoint in T and let $e_G(S, T) = |E_G(S, T)|$. The neighborhood of a vertex $v \in V$ is denoted by $\Gamma_G(v)$ and the neighborhood of a set $S \subseteq V$ is denoted by $\Gamma_G(S) = \bigcup_{v \in S} \Gamma_G(v)$.

Definition 2.1. Given a graph $G = (V, E)$, for any set $S \subseteq V$, we define

$$\Gamma_G^*(S) = \{v \in V : e_G(\{v\}, S) = 1\}$$

as the set of *unique* neighbors of S . Let $d_G^*(S) = |\Gamma_G^*(S)|$.

We may omit the subscript if the graph is clear from the context.

If $x, t \in \mathbb{R}$, $\varepsilon > 0$ are such that $x \in [(1 - \varepsilon)t, (1 + \varepsilon)t]$ then we write $x \sim_\varepsilon t$. We shall also use the standard notations $\Omega_\gamma(f(n))$, $O_\gamma(f(n))$ for the classes of

all functions lower/upper bounded by $c(\gamma)f(n)$, where $c = c(\gamma)$ is a constant that only depends on γ . In many computations we implicitly use well-known inequalities such as

$$1 + x \leq e^x \quad \text{and} \quad \left(\frac{a}{b}\right)^b \leq \binom{a}{b} \leq \left(\frac{ea}{b}\right)^b. \quad (2.1)$$

The Chernoff inequality is also used extensively: for any $\varepsilon > 0$ and any Binomial random variable X with parameters n and p we have

$$\mathbf{P}[|X - np| \geq \varepsilon np] \leq \exp\{-\Omega_\varepsilon(np)\}. \quad (2.2)$$

Definition 2.2 (LE sets). We say that a set of vertices S in a graph G is ε -lossless expanding if $|\Gamma(S) \setminus S| \sim_\varepsilon e(S, V(G) \setminus S)$, that is, almost every edge in the S -cut corresponds to a unique neighbor of S . We may refer to S as an *LE set* for short.

A useful feature of LE sets is their resilience: even if a large fraction of the edges incident to an LE set is removed, the LE property persists. This is stated formally in the following simple lemma.

Lemma 2.3. *Let G be a graph and $S \subseteq V = V(G)$. For any $G' \subseteq G$ we have*

$$|\Gamma_{G'}(S) \setminus S| \geq e_{G'}(S, V \setminus S) + 2(|\Gamma_G(S) \setminus S| - e_G(S, V \setminus S)).$$

Proof. Let N denote the number of edges $e = uv$ in $E_G(S, V \setminus S)$ such that the end-vertex $v \in V \setminus S$ satisfies $e_G(v, S) \geq 2$. Note that $|\Gamma_G(S) \setminus S| \leq (e_G(S, V \setminus S) - N) + N/2$, since each edge not counted by N corresponds to exactly one unique neighbor of S and all the edges counted by N may contribute with at most $N/2$ neighbors. We obtain $-N \geq 2(|\Gamma_G(S) \setminus S| - e_G(S, V \setminus S))$. The claim follows as $|\Gamma_{G'}(S) \setminus S| \geq e_{G'}(S, V \setminus S) - N$. \square

Definition 2.4. Let \mathcal{T} be a tree and $V(\mathcal{T}) = V_0(\mathcal{T}) \cup V_1(\mathcal{T})$ be the canonical bipartition of \mathcal{T} . Set $n_i = |V_i(\mathcal{T})|$ and $\Delta_i = \max\{d_{\mathcal{T}}(v) : v \in V_i(\mathcal{T})\}$, for $i = 0, 1$. The parameter $\beta(\mathcal{T})$ is defined as

$$\beta(\mathcal{T}) = n_0\Delta_0 + n_1\Delta_1.$$

A tree with these parameters is called an $(n_0, \Delta_0, n_1, \Delta_1)$ -tree.

2.3 Outline of a simpler case

In this section we consider a simpler, specific case, where we can apply easier versions of the techniques used in the proof of our result. Let us assume that the n_i 's and Δ_i 's are fixed and satisfy $n_0\Delta_0 = n_1\Delta_1$. Our unrealistic¹ assumption is the existence of a bipartite graph G having classes V_0, V_1 with $100n_i \leq |V_i| = N_i = O(n_i)$, $i = 0, 1$, such that all vertices in V_i have degree $D_i = O(\Delta_i)$, $D_i > 32\Delta_i$, and such that for any i and any set $S \subseteq V_i$, with $|S| \leq |V_{i-1}|/D_i$, we have $|\Gamma_G(S)| \geq (1 - \varepsilon)D_i |S|$ for some small $\varepsilon \geq 0$. In particular, G is a bipartite graph for which we have lossless expansion for essentially all sets (obviously, if S is too large, it cannot expand losslessly).

Next we outline how one could find a copy of an $(n_0, \Delta_0, n_1, \Delta_1)$ -tree \mathcal{T} in any sufficiently dense subgraph of G . Suppose that $G' \subseteq G$ is such that $e(G') \geq e(G)/2$. By sequentially removing vertices of low degree, we may ensure that for $i = 0, 1$, every $v \in V'_i = V_i \cap V(G')$ has degree at least $D_i/8$ and that $e(G') \geq e(G)/4$.

Suppose that f is a partial embedding of \mathcal{T} into G' . A vertex $v \in V' = V(G')$ is *inactive* with respect to f if there is a vertex $u \in V(\mathcal{T})$ such that $v = f(u)$ and, moreover, all neighbors of u are already embedded by f (namely, $f^{-1}(V') \supset \Gamma_{\mathcal{T}}(u)$).

¹Such graphs do not exist for all range of parameters, for instance, if $N_0 = 2^{N_1}$ such a strong expander needs $D_0 \geq c(\varepsilon)N_1$, which means that the graph needs to be very dense (see [33, Theorem 1.5(a)]).

A vertex is called *free* with respect to some partial embedding f if it is neither *reserved* nor in the image of f . We shall describe how a vertex becomes reserved in what follows.

Critical vertices. The main ingredient in the embedding scheme is how to deal with active vertices in G' which have few free neighbors. These vertices will be called *critical*. We associate to every critical vertex v a subset R_v of its free neighborhood which shall be reserved exclusively to embed neighbors of $f^{-1}(v)$ (if v ever gets used in the embedding, otherwise they shall remain unused). In particular, those vertices in R_v will no longer be free.

Let $c \in (0, 1/8)$ be a fixed constant to be defined later. A vertex from class V'_i ($i = 0, 1$) is classified as *critical* if it has less than cD_i free neighbors.

There are basically two difficulties in dealing with critical vertices: since the reserved subsets must be exclusive, they must be disjoint from each other. Moreover, after reserving vertices, one may produce more critical vertices, as those reserved vertices are no longer free. It is therefore essential to make sure that the number of critical vertices is bounded at all times.

To ensure that there are not too many critical vertices, the set of reserved vertices for each critical vertex is relatively small—it has size Δ_0 or Δ_1 , depending on the class to which the critical vertex belongs. Therefore, for each new critical vertex, we reserve a small number of vertices (making them non-free). On the other hand, every critical vertex must send a considerable fraction of its edges into the set of non-free vertices. By the LE property and Lemma 2.3, the set of critical vertices must be small, otherwise the expansion of the LE set of critical vertices would contradict the fact that the set of non-free vertices is not large.

More formally, let C_i be the set of critical vertices in the class V_i at a certain moment in the embedding procedure. The number of non-free vertices in V_{1-i} is at most $n_{1-i} + |C_i| \Delta_i$. However, every vertex $v \in C_i$ sends at least $d_{G'}(v) - cD_i \geq (1/8 - c)D_i > D_i/16$ edges into the set of non-free

vertices of V_{1-i} . If $|C_i| \geq 32n_{1-i}/D_i$, one can establish a contradiction with the LE property by way of Lemma 2.3. Indeed, the set of non-free vertices would have to be larger than

$$\begin{aligned} |C_i| D_i/16 &\geq (16n_{1-i}/D_i + |C_i|/2) \cdot (D_i/16) = n_{1-i} + |C_i| D_i/32 \\ &> n_{1-i} + |C_i| \Delta_i, \end{aligned}$$

which contradicts the trivial upper bound on the number of non-free vertices.

Embedding scheme. Fix an arbitrary root $v_1 \in V_1(\mathcal{T})$ and map it to an arbitrary vertex in V'_1 . At each step we take an already embedded vertex and embed all of its children at once. Suppose that we have a partial embedding f of \mathcal{T} into G' . Let \mathcal{C} be the collection of critical vertices and $\mathcal{R} = \{R_v\}_{v \in \mathcal{C}}$ be the family of reserved sets. Let $u \in V(\mathcal{T})$ be an embedded vertex and $w = f(u)$.

If w is critical then $R_w \in \mathcal{R}$ contains enough vertices to embed every child of u . No other critical vertex can be created after this embedding occurs (since no free vertex is used).

If $w \in V'_i$ is not critical, then the number of free neighbors of w is at least $cD_i \gg \Delta_i$, which is more than enough to embed every child of u . After embedding the children of u (arbitrarily choosing vertices among the free neighbors of w), we might have created new critical vertices.

A new critical vertex had cD_i free neighbors before the above embedding extension. Since the extension can only make Δ_i vertices non-free and $cD_i \gg \Delta_i$, this new critical vertex still has many free neighbors immediately after the extension.

Pick one of the (possibly many) new critical vertices and choose an arbitrary Δ_i -subset of its free neighborhood. We construct reserved sets for the new critical vertices using the following iterative procedure.

Suppose that $\mathcal{C}^j \subset V'_i$ is the collection of the first j critical vertices in V'_i created by the embedding extension and which were already processed. (Ini-

tially, $j = 0$ and $\mathcal{C}^0 = \emptyset$.) Let $\{R_v^j\}_{v \in \mathcal{C}^j}$ be a family of disjoint Δ_i -subsets such that each R_v^j may only contain free neighbors of v . Set $X^j = \bigcup_{v \in \mathcal{C}^j} R_v^j$.

If there is a (non-critical) vertex w having less than cD_i free neighbors outside of X^j we set $\mathcal{C}^{j+1} = \mathcal{C}^j \cup \{w\}$ and obtain a new family of disjoint Δ_i -sets $\{R_v^{j+1}\}_{v \in \mathcal{C}^{j+1}}$ as above (we describe this process in more detail at the end of this Section). We also impose an extra restriction on this family: $X^j \subset X^{j+1}$, namely, once a vertex is chosen to be reserved to any critical vertex, it will be reserved to some critical vertex (but not necessarily to the one it was originally assigned to). This restriction is important since we use the fact that the set of non-free vertices is monotonically increasing. In particular, once a vertex is classified as critical, it always has less than cD_i free neighbors.

Suppose that the above procedure finishes when \mathcal{C}^k and $\{R_v^k\}_{v \in \mathcal{C}^k}$ were constructed. We set $R_v = R_v^k$ for all $v \in \mathcal{C}^k$ and thus consolidate the reserved set of every new critical vertex. At that point, every non-critical vertex of V_i' ($i = 0, 1$) has at least cD_i free neighbors and every critical vertex has an exclusive set of reserved vertices. Therefore, it is possible to continue the embedding until the whole tree is embedded.

Obtaining a family of reserved sets. The j th new critical vertex created after an extension must have at least $cD_i - \Delta_i$ neighbors that are either free or contained in X_{j-1} . Indeed, before that vertex became critical, it had cD_i free neighbors; after the extension, at most Δ_i vertices were used in the extension and became non-free.

Using the LE property of the graph and a Hall-type argument, it is simple to obtain a new family of reserved sets as long as $j = |\mathcal{C}^j|$ is not too large. However, since we have a global upper bound on the number of critical vertices, this strategy always works. (See Lemma 2.12 for a formal version of this argument.)

2.4 Properties of random bipartite graphs

In this section we state a technical theorem describing several properties of random bipartite graphs that we use when embedding trees. We remark that, in contrast with the assumptions of Section 2.3, having lossless expansion on both classes of a sparse bipartite graph is not always possible (see [33, Theorem 1.5(a)]). To overcome this shortcoming we show that there are plenty of LE sets in “useful places”, namely, most neighborhoods of vertices are rich in LE sets.

Definition 2.5. Let $\varepsilon > 0$, $p \in (0, \varepsilon/8)$, $N_0, N_1, D_0 = pN_1, D_1 = pN_0 \in \mathbb{N}$. A bipartite graph $G = (V_0, V_1; E)$ with $|V_0| = N_0$, $|V_1| = N_1$ satisfies *Property* (\ddagger) if there exists $V'_1 \subseteq V_1$ with $|V'_1| \geq (1 - 2\varepsilon)N_1$ such that the following conditions hold:

(i) $\deg(w) \sim_\varepsilon D_1$ for all $w \in V'_1$ and, moreover,

$$\#\{u \in \Gamma(w) : \deg(u) \not\sim_\varepsilon D_0\} < \varepsilon D_1;$$

(ii) for every $S \subseteq V'_1$ with $|S| \leq \varepsilon N_1 / (8D_0)$, we have $d^*(S) \sim_\varepsilon D_1 |S|$;

(iii) for every $S \subseteq V'_1$ with $|S| \leq \varepsilon N_1 / (D_0 D_1)$ and for every $T \subseteq \Gamma(S)$ with $\sqrt{\varepsilon} D_1 |S| \leq |T|$, we have $d^*(T) \geq (1 - 5\sqrt{\varepsilon}) D_0 |T|$;

(iv) if $\varepsilon N_1 < D_0 D_1$ then for every $w \in V'_1$ and every subset $T \subseteq \Gamma(w)$ with $|T| \geq \varepsilon D_1$ we have disjoint sets $T_1, \dots, T_r \subset T$, each of cardinality

$$\min\{\varepsilon D_1 / 8, \varepsilon N_1 / (4D_0)\},$$

such that

$$\left| \bigcup_{i=1}^r T_i \right| \geq \frac{3}{4} |T| \quad \text{and} \quad d^*(T'_i) \sim_\varepsilon D_0 |T'_i|$$

for every $T'_i \subseteq T_i$, $i = 1, \dots, r$;

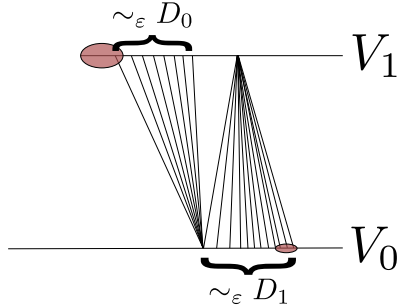


Figure 2.1: The graph obtained from Theorem 2.6. All vertices in V_1 which are not in the small shaded subset have $\sim_\varepsilon D_1$ neighbors and all but at most εD_1 such neighbors have degree $\sim_\varepsilon D_0$.

(v) for every $X \subseteq V_0$ and $Y \subseteq V_1$ with $|X| \geq \varepsilon^3 N_0$, $|Y| \geq \varepsilon^3 N_1$ we have $e_G(X, Y) \sim_{\varepsilon^2} p |X| |Y|$; in particular, $e(V_0, V_1) \geq (1 - 4\varepsilon) e(G)$.

Using the probabilistic method we show that there are graphs satisfying Property (\ddagger) .

Theorem 2.6. *Suppose that $n_0 \geq n_1$ and $n_0 \Delta_0 = n_1 \Delta_1$. Let $0 < \varepsilon < 1/100$ be given. There exists $C = C(\varepsilon)$ such that, with probability at least $1 - \varepsilon$, the bipartite random graph $G_{N_0, N_1; p} = (V_0, V_1; E)$, with $N_0 = Cn_0$, $N_1 = Cn_1$, and $p = \Delta_0/n_1 = \Delta_1/n_0 < \varepsilon/8$ satisfies **Property** (\ddagger) .*

Before proving the above theorem, we observe that the condition $p < \varepsilon/8$ is not very restrictive. In the case $p \geq \varepsilon/8$, we may use a complete bipartite graph.

Lemma 2.7. *Let $\alpha \in (0, 1]$ and \mathcal{T} be a tree with (bipartite) classes having cardinalities n_0 and n_1 . We have $G = K_{4n_0/\alpha, 4n_1/\alpha} \rightarrow_\alpha \mathcal{T}$.*

Proof. Suppose that the vertex classes of G are V_0 and V_1 ($|V_0| = n_0$ and $|V_1| = n_1$). First observe that G has $16n_0n_1/\alpha^2$ edges. When $p \geq \varepsilon/8$, we must have $\beta(\mathcal{T}) \geq pn_0n_1 \geq \varepsilon n_0n_1/8$ and hence $e(G) = O(\beta(\mathcal{T}))$.

Let $G' \subseteq G$ be any subgraph with $e(G') \geq \alpha e(G)$. While there is a vertex $v \in V_0$ (or a vertex $w \in V_1$) with $\deg_{G'}(v) < n_1$ (or $\deg_{G'}(w) < n_0$) remove v (or w) from G' together with all of the edges incident to the removed vertex. The total number of edges removed is at most $(4n_0/\alpha)n_1 + (4n_1/\alpha)n_0 = \frac{\alpha}{2}e(G)$. Therefore, the remaining graph G' is non-empty and has minimum degree on V_0 at least n_1 and minimum degree on V_1 at least n_0 .

Now we can inductively embed any tree \mathcal{T} with classes having cardinalities n_0 and n_1 . Fix an arbitrary root $v_0 \in V_0(\mathcal{T})$ and set $f: v_0 \mapsto w_0$ where w_0 is an arbitrary vertex on V_0 .

Suppose that we have a partial embedding f of \mathcal{T} into G . Pick some vertex $v \in V_i(\mathcal{T})$, $i = 0, 1$, that was already embedded together with some $w \in \Gamma_{\mathcal{T}}(v)$ which was not yet embedded. Since the degree of $f(v)$ in G' is at least as large as $|V_{1-i}(\mathcal{T})|$, there must be some $w' \in \Gamma_G(f(v))$ such that $w' \notin f(V_{1-i}(\mathcal{T}))$. Extend f by mapping w to w' . \square

To simplify the proof of Theorem 2.6 we shall avoid floors and ceilings by making every parameter—such as $\varepsilon, p, C, n_0, n_1, \Delta_0, \Delta_1$ —a power of 2. This is not a problem given our final goal since this shall affect the parameter $\beta(\mathcal{T})$ by only a multiplicative constant.

Proof. The proof of Theorem 2.6 is divided into several claims.

Claim 2.8. *Let $G = G_{N_0, N_1; p} = (V_0, V_1; E)$ be a random bipartite graph and $S \subseteq V_i$ ($i = 0, 1$) be a set with s vertices. Then $d^*(S)$ is a binomial variable with parameters N_{1-i} and $sp(1-p)^{s-1}$. Moreover, if $sp \leq \varepsilon$ then $\mathbf{E}[d^*(S)] \geq (1-2\varepsilon)spN_{1-i}$.*

Proof. We may represent $d^*(S)$ as a sum of indicator variables

$$I_v = \mathbb{I}[e_G(v, S) = 1], \quad v \in V_{1-i}.$$

Since the I_v 's are independent and each has probability $sp(1-p)^{s-1}$, the first part of the claim is proved. For the second part, notice that

$$\mathbf{E}[d^*(S)] = N_{1-i}sp(1-p)^{s-1} \geq spN_{1-i}e^{-2sp} \geq (1-2\varepsilon)spN_{1-i}, \quad (2.3)$$

since $(1-p) \geq e^{-p-p^2} \geq e^{-2p}$ (as $p \leq \varepsilon/s \leq 1/2$). \square

Claim 2.9. *With probability at least $1-3\varepsilon/4$ there exists $V'_1 \subseteq V_1$ with $|V'_1| \geq (1-\varepsilon)N_1$ for which (i) and (ii) from Property (\ddagger) hold.*

Proof. Notice that for any vertex $v \in V_i$, $i = 0, 1$, we have $\mathbf{E}[\deg(v)] = D_i$. By the Chernoff inequality, for any fixed vertex v ,

$$\mathbf{P}[|\deg(v) - D_i| \geq \varepsilon D_i] \leq \exp\{-\Omega_\varepsilon(D_i)\} \leq \varepsilon^2/8 \quad (2.4)$$

for sufficiently large C .

Note that the degrees in V_1 are independent random variables (since the graph is bipartite). Given a fixed vertex $w \in V_1$, let us estimate the probability that more than εD_1 of its neighbors have degree $\not\sim_\varepsilon D_0$ conditioned on $\deg(w) \sim_\varepsilon D_1$. For each $u \in \Gamma(w)$, the degree of u is one more than the number of its neighbors in $V_1 - w$, which is a binomial variable independent of other vertices in $\Gamma(w)$ and of w itself. Hence, the probability of having εD_1 neighbors failing to have the “correct” degree is bounded by

$$\binom{(1+\varepsilon)D_1}{\varepsilon D_1} \exp\{-\Omega_\varepsilon(D_0) \cdot \varepsilon D_1\} = \exp\{-\Omega_\varepsilon(D_0 D_1)\} < \varepsilon^2/8, \quad (2.5)$$

for sufficiently large C .

Let \mathcal{E}_0 denote the event in which the set of vertices having exceptional degree or having many neighbors of exceptional degree has at most $\varepsilon N_1/2$ elements. By (2.4) and (2.5), the expected number of such vertices is less than $\varepsilon^2 N_1/4$, by Markov's inequality, we obtain $\mathbf{P}[\mathcal{E}_0] \geq 1 - \varepsilon/2$.

Next, we prove that the event

$$\mathcal{E}_1 = \left\{ \text{for all } S \subseteq V_1 \text{ with } s = |S| \in \left[\frac{\varepsilon}{8p}, \frac{\varepsilon}{4p} \right], d^*(S) \geq (1-\varepsilon)D_1 |S| \right\}$$

holds with probability at least $1 - \varepsilon/4$. By Claim 2.8, we have $\mathbf{E}[d^*(S)] \geq (1 - \varepsilon/2)sD_1$ for all sets considered in \mathcal{E}_1 .

By the Chernoff inequality, the probability that one fixed set S in \mathcal{E}_1 has $d^*(S) < (1 - \varepsilon)sD_1$ is at most $\exp\{-\Omega_\varepsilon(sD_1)\}$. A simple union bound gives an upper bound on the probability that some set S has small $d^*(S)$, that is,

$$\begin{aligned} \sum_{s=\varepsilon/(8p)}^{\varepsilon/(4p)} \binom{N_1}{s} \exp\{-\Omega_\varepsilon(sD_1)\} &\leq \sum_s \left\{ \frac{eN_1 e^{-\Omega_\varepsilon(D_1)}}{s} \right\}^s \\ &\leq \sum_s \left\{ \frac{8eD_0 e^{-\Omega_\varepsilon(D_1)}}{\varepsilon} \right\}^s. \end{aligned}$$

Note that $D_1 \geq D_0$ (since by assumption $n_0 \geq n_1$), which means that we may take C sufficiently large in order to have $e^{1-\Omega_\varepsilon(D_1)} D_0/\varepsilon < \varepsilon/64$. In particular, the last sum is at most $\sum_{s=1}^{\infty} (\varepsilon/8)^s < \varepsilon/4$.

To prove (ii) let us assume that \mathcal{E}_1 holds. Suppose that there are disjoint sets S_1, S_2, \dots, S_k such that $|S_i| \leq \varepsilon/(8p) - 1$ and $d^*(S_i) < (1 - \varepsilon)D_1 |S_i|$. We call such sets S_i *non-expanding*. Suppose that $S = \bigcup_{i=1}^{k'} S_i$ ($k' \leq k$), is such that $\varepsilon/(8p) \leq |S| \leq 2(\varepsilon/(8p) - 1) \leq \varepsilon/(4p)$. Then $d^*(S) \leq \sum_{i=1}^{k'} d^*(S_i) < (1 - \varepsilon)D_1 |S|$, which contradicts \mathcal{E}_1 . It follows that by removing non-expanding sets from V_1 sequentially we eventually get rid of all of them while removing at most $\varepsilon/(4p) = \varepsilon N_1/(4D_0)$ vertices.

In total, if both \mathcal{E}_0 and \mathcal{E}_1 hold, we need to remove less than εN_1 vertices from V_1 to get (i) and (ii). Since $\mathbf{P}[\mathcal{E}_0 \wedge \mathcal{E}_1] \geq 1 - 3\varepsilon/4$ the claim is proved. \square

Set $s_0 = \varepsilon N_1/(D_0 D_1)$. We assume that $s_0 \geq 1$ as otherwise (iii) is trivial. Let us estimate the probability that a fixed $S \subseteq V_1$ with $s = |S| \in [s_0, 3s_0]$ and $|\Gamma(S)| \sim_\varepsilon D_1 |S|$ is such that there exists $T \subseteq \Gamma(S)$ with $\varepsilon s D_1 \leq |T|$ having $d^*(T) < (1 - 10\varepsilon)D_0 |T|$. Such (S, T) will be called a *bad pair*. Apply Claim 2.8 to the random subgraph $G[V_0, V_1 \setminus S]$ and the set T (observe that we have exposed the edges incident to S but no other edge of G , hence $G[V_0, V_1 \setminus$

$S]$ is a random graph independent of what was already exposed). Note that $p|T| \leq (1+\varepsilon)D_1sp \leq 3(1+\varepsilon)\frac{\varepsilon N_1}{D_0}p = 3\varepsilon(1+\varepsilon)$ and $|V_1 \setminus S| \geq (1-\varepsilon)N_1$. From Claim 2.8 we get that $\mathbf{E}[d^*(T)] \geq (1-8\varepsilon)D_0|T|$.

Applying the Chernoff inequality, we get that the probability that a fixed choice of (S, T) becomes a bad pair is at most $\exp\{-\Omega_\varepsilon(D_0|T|)\}$. The union bound over all choices of S and all choices of T gives the following upper bound for the probability of any bad pair occurring in G :

$$\begin{aligned} [*] &= \sum_{s=s_0}^{3s_0} \sum_{t=\varepsilon s D_1}^{2s D_1} \binom{N_1}{s} \binom{2s D_1}{t} \exp\{-\Omega_\varepsilon(t D_0)\} \\ &\leq \sum_{s=s_0}^{3s_0} \sum_{t=\varepsilon s D_1}^{2s D_1} \left(\frac{e N_1}{s}\right)^s \left(\frac{2es D_1}{t}\right)^t \exp\{-\Omega_\varepsilon(t D_0)\}. \end{aligned}$$

Replacing the occurrences of s and t in the denominators by lower bounds (s_0 and $\varepsilon s_0 D_1$, respectively) and their occurrences in the numerators or exponents by upper bounds ($3s_0$ and $6s_0 D_1$, respectively) we obtain

$$\begin{aligned} [*] &\leq \sum_{t=\varepsilon s_0 D_1}^{6s_0 D_1} \sum_s (e D_0 D_1 / \varepsilon)^{3s_0} (6e/\varepsilon)^t \exp\{-\Omega_\varepsilon(t D_0)\} \\ &\leq \sum_t 2s_0 \cdot \exp\{3s_0 \log(e D_0 D_1 / \varepsilon) + t \log(6e/\varepsilon) - \Omega_\varepsilon(t D_0)\} \\ &\leq 12s_0^2 D_1 \cdot \exp\{3s_0 \log(e D_0 D_1 / \varepsilon) + 6s_0 D_1 \log(6e/\varepsilon) - \Omega_\varepsilon(s_0 D_0 D_1)\} \\ &\leq \exp\{-\Omega_\varepsilon(N_1)\}, \end{aligned} \tag{2.6}$$

for a sufficiently large C .

Let \mathcal{E}_2 be the event

$$\begin{aligned} \mathcal{E}_2 = \left\{ \text{for all } S \subseteq V_1, \text{ with } s = |S| \in [s_0, 3s_0] \text{ and } |\Gamma(S)| \sim_\varepsilon s D_1, \right. \\ \left. \text{if } T \subseteq \Gamma(S), \varepsilon s D_1 \leq |T|, \text{ then } d^*(T) \geq (1-10\varepsilon)D_0|T| \right\}. \end{aligned} \tag{2.7}$$

By inequality (2.6), \mathcal{E}_2 holds with probability at least $1 - \varepsilon/16$.

Claim 2.10. *Conditioning on \mathcal{E}_0 , \mathcal{E}_1 and \mathcal{E}_2 , there is $V'_1 \subseteq V_1$ satisfying (i), (ii), (iii).*

Proof. Initially, let V'_1 be the set obtained by Claim 2.9 (here we use \mathcal{E}_0 and \mathcal{E}_1). Suppose that there exists $S_1 \subseteq V'_1$ with $|S_1| \leq s_0 - 1$ and $\Gamma_G(S_1) \sim_\varepsilon D_1 |S_1|$ such that there is $T_1 \subseteq \Gamma_G(S_1)$ with $\sqrt{\varepsilon} D_1 |S_1| \leq |T_1|$ and $d^*(T_1) < (1 - 5\sqrt{\varepsilon}) D_0 |T_1|$. Remove S_1 from V'_1 . Repeat this procedure until there are no more bad sets or until the union $S = \bigcup_i S_i$ has at least s_0 elements. We claim that $|S| \leq 2s_0$. Indeed, this follows since each S_i has at most s_0 elements. Next we show that S cannot have more than s_0 elements, namely, the union of all bad sets contains less than s_0 elements.

Suppose that $s_0 \leq |S| \leq 2s_0$ and let $T = \bigcup_{i=1}^k T_i \subseteq \Gamma_G(S)$. Exploiting the LE property of V'_1 we shall show that $|T|$ is close to $\sum_{i=1}^k |T_i|$ and, since $T \subseteq \Gamma(S)$, this contradicts \mathcal{E}_2 . Note that $e_G(S, T) \geq \sum_{i=1}^k e_G(S_i, T_i)$, since the S_i 's are disjoint. However, we know that $e_G(S_i, T_i) \geq |T_i| \geq \sqrt{\varepsilon} D_1 |S_i|$. Take $G' \subseteq G$ with $E(G') = \bigcup_{i=1}^k E_G(S_i, T_i)$. Clearly, $e(G') \geq \sum_{i=1}^k |T_i| \geq \sqrt{\varepsilon} D_1 |S|$. On the other hand, since V'_1 was initially obtained from Claim 2.9, every vertex of V'_1 has degree at most $(1 + \varepsilon) D_1$ and $|\Gamma_G(S)| \geq d_G^*(S) \geq (1 - \varepsilon) D_1 |S|$. Hence, by Lemma 2.3, it follows that

$$\begin{aligned} |T| &= |\Gamma_{G'}(S)| \geq e_{G'}(S, T) - 2\{e_G(S, T) - |\Gamma_G(S)|\} \\ &\geq \sum_{i=1}^k |T_i| - 4\varepsilon D_1 |S| \geq (1 - 4\sqrt{\varepsilon}) \sum_{i=1}^k |T_i|, \end{aligned} \tag{2.8}$$

where we have used that $\sqrt{\varepsilon} D_1 |S| \leq \sum_{i=1}^k |T_i|$. Therefore

$$|T| \geq \frac{1}{2} \sum_{i=1}^k |T_i| \geq \frac{1}{2} \sqrt{\varepsilon} D_1 |S| > \varepsilon D_1 |S|,$$

which means that \mathcal{E}_2 implies that $T \subset \Gamma_G(S)$ satisfies

$$d_G^*(T) \geq (1 - 10\varepsilon) D_0 |T|. \tag{2.9}$$

Since, $\Gamma_G^*(T) \subseteq \bigcup_{i=1}^k \Gamma_G^*(T_i)$ and $\varepsilon < 1/100$, we have from (2.8),

$$\begin{aligned} d_G^*(T) &\leq \sum_{i=1}^k d_G^*(T_i) < (1 - 5\sqrt{\varepsilon})D_0 \sum_{i=1}^k |T_i| \\ &\leq \frac{1 - 5\sqrt{\varepsilon}}{1 - 4\sqrt{\varepsilon}} D_0 |T| \\ &< (1 - 10\varepsilon)D_0 |T|, \end{aligned}$$

a contradiction with (2.9). Hence, by removing less than s_0 elements from V_1' we may ensure that (iii) holds together with (i) and (ii). \square

Claim 2.11. *If $\varepsilon N_1 < D_0 D_1$ then **a.a.s.** every $w \in V_1$ for which $\deg(w) \sim_\varepsilon D_1$ and every $T \subseteq \Gamma(w)$ with $|T| \geq \varepsilon D_1$ satisfy the conditions of Property (\ddagger) .(iv).*

Proof. Suppose that $\varepsilon N_1 < D_0 D_1$. Let $w \in V_1$ be fixed and assume that $\deg(w) \sim_\varepsilon D_1$ (as otherwise $w \notin V_1'$). Let $T = \{t_1, t_2, \dots, t_m\} \subseteq \Gamma(w)$ be an arbitrary set with $m \geq \varepsilon D_1$. Let $k = \min\{\varepsilon D_1/8, \varepsilon N_1/(4D_0)\}$ and $r = \lceil \frac{3m}{4k} \rceil$.

In the random graph $G[V_0, V_1 \setminus \{w\}]$, the vertex t_1 has expected degree

$$p(N_1 - 1) \sim_{\varepsilon/100} D_0.$$

Hence, by the Chernoff inequality,

$$\mathbf{P}[\deg(t_1) \sim_{\varepsilon/3} D_0] \geq 1 - \exp\{\Omega_\varepsilon(D_0)\}.$$

We shall (attempt to) construct a set T_1 with k elements satisfying condition (iv). Let $X = \{w\}$. We say that t_i *succeeds* if $|\Gamma(t_i) \setminus X| \sim_{\varepsilon/3} D_0$ and $\deg(t_i) \sim_{\varepsilon/3} D_0$, otherwise it *fails*. If t_i succeeds, we add t_i to T_1 and $\Gamma(t_i)$ to X . If it fails, both X and T_1 remain unchanged. If T_1 contains k elements then we have obtained our final T_1 . By construction, every $T_1' \subseteq T_1$ is such that $|\Gamma(T_1')| \sim_{\varepsilon/3} D_0 |T_1'|$. To estimate $d^*(T_1')$ for $T_1' \subseteq T_1$ we observe that $e_G(T_1', \Gamma(T_1')) \leq (1 + \varepsilon/3)D_0 |T_1'|$ while, on the other hand,

$e_G(T'_1, \Gamma(T'_1)) \geq 2|\Gamma(T'_1)| - d^*(T'_1)$. From these two inequalities we conclude that

$$d^*(T'_1) \geq \{2(1 - \varepsilon/3) - (1 + \varepsilon/3)\}D_0 |T'_1| \geq (1 - \varepsilon)D_0 |T'_1|.$$

Suppose that t_ℓ was the k th element added to T_1 . Then we start building $T_2 \subset \{t_{\ell+1}, \dots, t_m\}$ in the same way we constructed T_1 : set $X = \{w\}$ and sequentially add vertices t_i that succeed to T_2 and their neighborhoods $\Gamma(t_i)$ to X . Repeat the procedure for other T_j 's until we have finished constructing T_r or until the vertex t_m was reached. Note that we always have

$$|X| \leq (1 + \varepsilon)D_0 k + 1 \leq \frac{\varepsilon N_1}{4D_0}(1 + \varepsilon)D_0 + 1 \leq \frac{\varepsilon N_1}{3.9}.$$

In particular,

$$\mathbf{E}[|\Gamma(t_i) \setminus X|] = p(N_1 - |X|) \geq \left(1 - \frac{\varepsilon}{3.9}\right)D_0.$$

Therefore, from the Chernoff inequality it follows that for any $X \subset V_1$ with $|X| \leq \varepsilon N_1/(3.9)$ and fixed $t_i \in T$, we have

$$\begin{aligned} \mathbf{P}[t_i \text{ fails} \mid X] &= \mathbf{P}[\deg(t_i) \not\sim_{\varepsilon/3} D_0 \text{ or } |\Gamma(t_i) \setminus X| \not\sim_{\varepsilon/3} D_0] \\ &= \mathbf{P}[\deg(t_i) > (1 + \varepsilon/3)D_0 \text{ or } |\Gamma(t_i) \setminus X| < (1 - \varepsilon/3)D_0] \\ &\leq 2 \exp\{-\Omega_\varepsilon(D_0)\}. \end{aligned} \tag{2.10}$$

If we were unable to construct the desired collection T_1, \dots, T_r then at least $m/8$ elements from T have failed. Indeed, we need rk elements to succeed, where $3m/4 \leq rk < 3m/4 + k \leq 3m/4 + \varepsilon D_1/8 \leq 7m/8$. The sequence of indicator variables $\mathbb{I}[t_i \text{ fails}]$ is not independent, however, the i th event only depends on the size of the set X constructed after the first $i - 1$ events. Therefore, by (2.10), the probability that a fixed sequence of $(m/8) \geq \varepsilon D_1/8$ vertices fails is at most $\exp\{-\Omega_\varepsilon(mD_0/8)\} = \exp\{-\Omega_\varepsilon(D_0 D_1)\}$.

Consider the union bound over (1) all choices $w \in V_1$ having $\deg(w) \sim_\varepsilon D_1$; (2) all subsets $T \subseteq \Gamma(w)$ with $|T| \geq \varepsilon D_1$; (3) all possible $(m/8)$ -subsets of

failing vertices of T . The probability that we fail to construct the desired collection for some vertex is at most

$$N_1 \cdot 2^{2D_1} \cdot 2^{2D_1} \cdot \exp\{-\Omega_\varepsilon(D_0 D_1)\} \leq \exp\{\log N_1 + 4D_1 - c_\varepsilon D_0 D_1\}. \quad (2.11)$$

We choose a constant $C = C(\varepsilon)$ that is large enough to ensure that $D_0 = C\Delta_0$ satisfies $D_0 \geq 16/c_\varepsilon$ and that $N_1 = Cn_1$ satisfies $N_1 \geq \frac{4}{\varepsilon c_\varepsilon} \log N_1$. Since by this choice of C we have $D_0 D_1 \geq \varepsilon N_1 \geq \frac{4}{c_\varepsilon} \log N_1$, it follows that (2.11) is at most

$$\exp\left\{\left(\log N_1 - \frac{c_\varepsilon D_0 D_1}{2}\right) + \left(4D_1 - \frac{c_\varepsilon D_0 D_1}{2}\right)\right\} \leq \exp\{-\log N_1 - 4D_1\} = o(1).$$

Therefore the claim is proved by the union bound. \square

It is a well-known fact that the number of edges among linear-sized sets in a random graph is **a.a.s.** very close to the expected value. Indeed, let \mathcal{E}_3 be the event corresponding to $(\ddagger).(v)$ and let \mathcal{E}_4 denote the event described by Claim 2.11. Note that the events $\mathcal{E}_0, \dots, \mathcal{E}_4$ hold together with probability at least $1 - \varepsilon$. Conditioning on all those events, (v) is satisfied (by \mathcal{E}_3), Claim 2.10 ensures (i) – (iii) and \mathcal{E}_4 together with (i) imply (iv) . \square

2.5 Auxiliary results

In this section we prove lemmas that will be used to ensure that certain steps in our tree embedding scheme can be performed.

Lemma 2.12. *Let S_1, \dots, S_m be a collection of sets and $b \in \mathbb{N}^m$ be such that, for every $I \subseteq [m]$, we have $|\bigcup_{i \in I} S_i| \geq \sum_{i \in I} b_i$.*

Then, there exists a family $\mathcal{S} = \{S'_i \subseteq S_i\}_{i=1}^m$ of disjoint sets with $|S'_i| = b_i$ for all i . Moreover, if $\{S''_i \subseteq S_i\}_{i=1}^k$, $k \leq m$, is any family of disjoint sets with $|S''_i| = b_i$, we may require that \mathcal{S} satisfies $\bigcup_{i=1}^k S''_i \subseteq \bigcup_{i=1}^m S'_i$.

Proof. We reduce this problem to a matching problem. Consider a bipartite graph H with vertex classes $A = \bigcup_{i=1}^m \{i\} \times [b_i]$ and $B = \bigcup_{i=1}^m S_i$ and edges given by $\{(i, j), u\}$ for all $i \in [m]$, $j \in [b_i]$ and $u \in S_i$. Observe that we are adding b_i copies of a vertex i that has neighborhood S_i for all i .

Given a set $A' \subseteq A$, let $I = I(A')$ be the projection of A' onto the first coordinate. We have $|A'| \leq \sum_{i \in I} b_i$ and, on the other hand, $|\Gamma_H(A')| = |\bigcup_{i \in I} S_i| \geq \sum_{i \in I} b_i \geq |A'|$. Hence, Hall's condition is satisfied for H and there is a matching M covering A . From M we get sets $S'_i \subseteq S_i$ by letting S'_i be the set of elements matched to $(i, 1), \dots, (i, b_i)$.

Suppose that there is a family of disjoint sets $\{S''_i \subseteq S_i\}_{i=1}^k$, $k \leq m$, with $|S''_i| = b_i$. By performing small local changes to the family $\{S'_i \subseteq S_i\}_{i=1}^m$ we may ensure that $\bigcup_{i=1}^k S''_i \subseteq \bigcup_{i=1}^m S'_i$. If there exists $x \in \bigcup_{i=1}^k S''_i \setminus \bigcup_{i=1}^m S'_i$ then let $j \in [k]$ be such that $x \in S''_j$. Since $b_j = |S'_j| = |S''_j|$, there exists some $y \in S'_j \setminus S''_j$. Set $S'_j \leftarrow S'_j - y + x$. Note that this strictly decreases

$$\sum_{i=1}^k |S'_i \Delta S''_i|.$$

In particular, since this number is always non-negative, in at most

$$\sum_{i=1}^k |S'_i \Delta S''_i|$$

steps we can obtain the desired family. \square

Lemma 2.13. *Let $G = (V_0, V_1; E)$ be a graph with $V'_1 \subseteq V_1$ satisfying Property (\ddagger) . Let $\alpha \geq \alpha_0(\varepsilon) = 13\sqrt{\varepsilon}$.*

Suppose that $S \subseteq V'_1$, with $|S| \leq \varepsilon N_1 / (D_0 D_1)$, is such that there is a family of disjoint sets $\{A_v \subset \Gamma(v)\}_{v \in S}$ and a family (of not necessarily disjoint sets) $\{B_x \subset \Gamma(x)\}_{x \in \bigcup_{v \in S} A_v}$ with $|A_v| = \alpha D_1$ for every $v \in S$ and $|B_x| = \alpha D_0$ for every $x \in \bigcup_{v \in S} A_v$.

Then there is a family of disjoint Δ_1 -sets $\{X_v \subseteq A_v\}_{v \in S}$ and a family of disjoint Δ_0 -sets $\{Y_{v,x} \subseteq B_x\}_{v \in S, x \in X_v}$.

Proof. We shall assume that $D_0 D_1 \leq \varepsilon N_1$ as otherwise $S = \emptyset$ and there is nothing to prove. The desired families will be obtained in three steps. Let $m = (\alpha - \varepsilon) D_1$ and $\alpha_0(\varepsilon) = 13\sqrt{\varepsilon}$.

In step one we obtain a family of disjoint sets $\{X'_v \subseteq A_v\}_{v \in S}$ such that, for every $v \in S$, we have that $|X'_v| = m$ and every $u \in X'_v$ has $\deg(u) \sim_\varepsilon D_0$. This is possible because of Property $(\dagger).(i)$.

In step two we obtain a family of disjoint sets

$$\left\{ Y'_v \subseteq Y_v \stackrel{\text{def.}}{=} \bigcup_{x \in X'_v} B_x \right\}_{v \in S}$$

with $|Y'_v| = (\alpha - 12\sqrt{\varepsilon}) D_0 |X'_v|$.

To obtain this family we will use Lemma 2.12. For $S' \subseteq S$, denote by $X_{S'}$ the union $X_{S'} = \bigcup_{v \in S'} X'_v$. Note that we have $X_{S'} \subseteq \Gamma(S')$ with $|X_{S'}| = m |S'| \geq \sqrt{\varepsilon} D_1 |S'|$. Hence, from Property $(\dagger).(iii)$ we get that $|\Gamma(X_{S'})| \geq d^*(X_{S'}) \geq (1 - 5\sqrt{\varepsilon}) D_0 |X_{S'}|$. Using the degree hypothesis on the elements of the set X'_v and applying Lemma 2.3 we conclude that

$$\begin{aligned} \left| \bigcup_{v \in S'} Y'_v \right| &\geq \#\{\{x, y\} \in E(G) : x \in X_{S'}, y \in B_x\} + 2\{\Gamma(X_{S'}) - e(X_{S'}, V_1)\} \\ &\geq \sum_{x \in X_{S'}} |B_x| + 2\{(1 - 5\sqrt{\varepsilon}) D_0 |X_{S'}| - (1 + \varepsilon) D_0 |X_{S'}|\} \\ &\geq (\alpha - 12\sqrt{\varepsilon}) D_0 |X_{S'}| = (\alpha - 12\sqrt{\varepsilon}) D_0 m |S'|. \end{aligned}$$

Using Lemma 2.12 we may obtain the desired family of disjoint sets $Y'_v \subseteq Y_v$ with $|Y'_v| = (\alpha - 12\sqrt{\varepsilon}) D_0 m$ for all $v \in S$.

In step three we obtain the families described in the statement of this lemma.

Consider the pair (X'_v, Y'_v) constructed above for some $v \in S$. Let $X_v \subset X'_v$, with $|X_v| \leq \Delta_1$, be a maximal set such that there exists a family of disjoint Δ_0 -sets $\{Y_{v,x} \subset B_x \subset Y'_v\}_{x \in X_v}$. We claim that $|X_v| = \Delta_1$. Suppose for the sake of a contradiction that $|X_v| < \Delta_1$.

Let

$$Y_v'' = Y_v' \setminus \bigcup_{x \in X_v} B_x \quad (2.12)$$

Notice that we have

$$|Y_v''| \geq (\alpha - 12\sqrt{\varepsilon})D_0m - |X_v|\alpha D_0 > \Delta_0m.$$

Moreover,

$$Y_v'' \subset Y_v \setminus \bigcup_{x \in X_v} B_x = \bigcup_{x \in X_v'} B_x \setminus \bigcup_{x \in X_v} B_x \subset \bigcup_{x \in X_v' \setminus X_v} B_x.$$

Consequently,

$$\sum_{x \in X_v' \setminus X_v} |B_x \cap Y_v''| \geq |Y_v''| > \Delta_0m = \Delta_0 |X_v'|.$$

By averaging, there exists $x^* \in X_v' \setminus X_v$ such that $|B_{x^*} \cap Y_v''| > \Delta_0$. On the other hand, $\bigcup_{x \in X_v} Y_{v,x} \subset \bigcup_{x \in X_v} B_x$ which means that Y_v'' is pairwise disjoint with every set in the family $\{Y_{v,x}\}_{x \in X_v}$. It follows that there is $Y_{v,x^*} \subset B_{x^*} \cap Y_v''$, with $|Y_{v,x^*}| = \Delta_0$, which is pairwise disjoint with every member of the family. This contradicts the maximality of X_v .

Since the families $\{X_v'\}_{v \in S}$ and $\{Y_v'\}_{v \in S}$ are disjoint, it is clear that $\{X_v\}_{v \in S}$ and $\{Y_{v,x}\}_{v \in S, x \in X_v}$ satisfy the conclusions of the lemma. \square

2.6 An embedding scheme for trees

In this section we present Algorithm 1, which embeds trees in suitable graphs. This algorithm takes advantage of the lossless expansion property of the host graph when constructing the embedding. Although many of the techniques and ideas involved in this algorithm were already discussed at a superficial level in Section 2.3, there are many new details and subtleties that are addressed solely in this section.

A formal analysis of the algorithm is done through several invariants that must hold at the beginning of every iteration. Once the invariants are known to hold at the beginning of every iteration, we must prove that the algorithm does not abort. If the algorithm does not abort then it succeeds in embedding the tree, which is our goal.

In what follows, $\alpha \in (0, 1)$ will be a fixed number. For $n_0, \Delta_0, n_1, \Delta_1$ and $\varepsilon > 0$ given, let $G' = (V_0, V_1; E)$ be a graph and $V'_1 \subset V_1$ be a set for which the conclusion of Theorem 2.6 holds (namely, G' satisfies Property (\ddagger)). The algorithm takes as input an $(n_0, \Delta_0, n_1, \Delta_1)$ -tree \mathcal{T} and a subgraph $G \subset G'$ with $d_G(v) \geq \alpha D_i$ for all $v \in V(G) \cap V_i$, $i = 0, 1$, and such that $V(G) \cap V_1 \subset V'_1$. The output of the algorithm is an embedding of \mathcal{T} into $G \subset G'$.

2.6.1 Description of the algorithm

For convenience we will think of \mathcal{T} as a rooted tree (with root in $V_1(\mathcal{T})$). The algorithm constructs a sequence of partial embeddings in steps until the whole tree is embedded. Initially the root of \mathcal{T} is mapped to an arbitrary vertex in V_1 . In each step a vertex $p \in V_1$ which is already the image of some tree vertex is chosen and every child and grandchild of the pre-image of p is embedded during the step.

The vertices of \mathcal{T} and G will be marked according to their current status with respect to the partial embedding. A vertex may receive multiple markings.

Active: a vertex $v \in V_1$ which is the image of a tree vertex whose children and grandchildren are not yet embedded. The set of these vertices will be represented by the queue Q .

Used: a vertex of \mathcal{T} or G that is already in the partial embedding.

Reserved: if a vertex $w \in V(G)$ is reserved to $v \in \Gamma_G(w)$ then w can only

be used to embed a child of the pre-image of v . More formally, there is a family of disjoint vertex sets where each set is exclusively reserved for the children of a given vertex.

Dangerous: a vertex $v \in V_0$ with at least $\alpha D_0/2$ neighbors which are used or reserved. The set of dangerous vertices is denoted by \mathcal{D} .

Free: a vertex is free if it is not used nor reserved nor dangerous. The set of non-free neighbors will be contained in a set denoted by Z .

Critical: a vertex $v \in V_1$ with at least $\alpha D_1/2$ neighbors which are not free. Each critical vertex v will have an associate set S_v of reserved vertices to ensure that it is always possible to embed the children of v in the future. The set of critical vertices is denoted by \mathcal{C} . The family of disjoint reserved sets for critical vertices is denoted by $\mathcal{S} = \{S_v : v \in \mathcal{C}\}$.

Ultra-critical: a critical vertex $v \in V_1$ that, in addition, had at least half of its reserved vertices ($S_v \in \mathcal{S}$) become dangerous. An ultra-critical vertex v has an associate subset $S'_v \subset S_v$ of reserved vertices where each $w \in S'_v$ has a set of reserved neighbors $Z_{v,w} \subset \Gamma_G(w)$ which will ensure that it is possible to embed the grandchildren of v that are children of w . The set of ultra-critical vertices is denoted by \mathcal{U} . The family of disjoint reserved sets for children of ultra-critical vertices is denoted by $\mathcal{W} = \{Z_{v,w} : v \in \mathcal{U}, w \in S'_v\}$.

The challenge of this algorithm is to balance the reserved, dangerous and the critical/ultra-critical vertices. Indeed, a new critical vertex requires other free vertices to become reserved, which means that there will be fewer free vertices and possibly newer critical vertices. Similarly, new ultra-critical vertices cause other free vertices to become reserved, which might create new dangerous vertices which in turn can produce newer ultra-critical vertices.

We will show that the choices made by the algorithm ensure that it is always possible to find enough free vertices to be reserved.

More precisely, given the expanding nature of the graph G , if the set of, say, dangerous vertices, is large, then the set of non-free vertices of V_1 must be at least $\Omega(\alpha D_0)$ times as large. On the other hand, the number of non-free vertices of V_1 is easily expressed in terms of n_1 (to account for the embedded vertices) and the number of ultra-critical vertices (to account for the reserved vertices of V_1).

An important feature of the sets/markings described above is that they are all monotone. Namely, new elements are added to Z , \mathcal{C} , \mathcal{D} , \mathcal{U} while the old elements are maintained. This ensures, for instance, that if a vertex was marked critical at some step then in all future steps it will have a small number of free neighbors (since Z is monotone).

After the embedding is extended on a given step, some new dangerous, critical or ultra-critical vertices may have been created. An iterative procedure, **restoreinvariants**, defines the reserved sets for each new critical/ultra-critical vertex until there are no more such vertices. This procedure uses the auxiliary procedure **find-critical-vertices** which is responsible for listing the new critical vertices and reserving neighborhoods for them. The partial embedding is represented by a matching $M \subset V(\mathcal{T}) \times V(G)$ and f_M is the corresponding function.

Algorithm 1: Embedding trees

Input : A tree \mathcal{T} with root $r \in V_1(\mathcal{T})$;
A graph $G = (V_0, V_1; E)$.

Output: An embedding of \mathcal{T} into G represented by a matching M .

```
1.1  $M \leftarrow \{(r, v)\}$ ; // initialize embedding
1.2  $Q \leftarrow \{v\}$ ; // queue of active vertices
1.3  $\mathcal{C} \leftarrow \emptyset$ ; // critical vertices
1.4  $\mathcal{D} \leftarrow \emptyset$ ; // dangerous vertices
1.5  $\mathcal{S} \leftarrow \emptyset$ ; // reserved neighborhoods (family of subsets of  $V_0$ ,
     $\mathcal{S} = \{S_v\}_{v \in \mathcal{C}}$ )
1.6  $\mathcal{U} \leftarrow \emptyset$ ; // ultra-critical vertices
1.7  $\mathcal{S}_U \leftarrow \emptyset$ ; // reserved neighborhoods for children of
    ultra-critical vertices
1.8  $\mathcal{W} \leftarrow \emptyset$ ; // reserved neighborhoods for grandchildren of
    ultra-critical vertices
1.9  $Z \leftarrow \{v\}$ ; // contains the set of non-free vertices
1.10 while  $Q \neq \emptyset$  do
1.11      $p \leftarrow \text{pop}(Q)$ ; // obtain an active vertex
1.12     if  $p \in \mathcal{U}$  then
1.13         //  $S'_p \in \mathcal{S}_U$  and  $Z_{p,u} \in \mathcal{W}$  for  $u \in S'_p$ 
          $(M, S''_p, \{Z'_{p,u}\}_{u \in S''_p}) \leftarrow \text{embed-descendants}$ 
          $(M, p, S'_p, \{Z_{p,u}\}_{u \in S'_p})$ ;
1.14         enqueue  $(Q, \bigcup_{u \in S''_p} Z'_{p,u})$  go-to 1.10; // skip to the next
         iteration
1.15      $C_p \leftarrow \{v_1, \dots, v_\ell : v_i \text{ is a child of } f_M^{-1}(p)\}$ ;
1.16     if  $p \notin \mathcal{C}$  then
1.17          $S_p \leftarrow \Gamma_G(p) \setminus Z$ ; // if  $p \in \mathcal{C}$  then  $S_p \in \mathcal{S}$  is already
         defined
1.18     find a subset  $S'_p = \{u_1, \dots, u_\ell\} \subseteq S_p$  and a family of disjoint sets
          $\{Z_i \subseteq \Gamma_G(u_i) \setminus Z\}_{u_i \in S'_p}$ , with  $|Z_i| = \#\{\text{children of } v_i\}$ ; if not
         possible, abort;
1.19     extend  $M$ : match  $v_i$  to  $u_i$  and  $\{\text{children of } v_i\}$  to  $Z_i$  arbitrarily for
         all  $i$ ;
1.20     enqueue  $(Q, \bigcup_{i=1}^l Z_i)$ ;
1.21      $Z \leftarrow Z \cup S'_p \cup \bigcup_{i=1}^l Z_i$ ;
1.22     restore-invariants;
```

Procedure 2: embed-descendants($M, p, S_p, \{Z_{p,u}\}_{u \in S_p}$)

Input : M – current embedding, $f = f_M$ is the corresponding function;

p – a vertex in the host graph already used in the embedding;

S_p – children of p should be mapped into this set;

$\{Z_{p,u}\}_{u \in S_p}$ – if a child v of $f^{-1}(p)$ is mapped to $u \in S_p$, the children of v will be mapped into $Z_{p,u}$.

Output: M – updated embedding;

$S'_p \subseteq S_p$ – vertices used for children of $f^{-1}(p)$;

$\{Z'_{p,u} \subseteq Z_{p,u}\}_{u \in S'_p}$ – vertices used for grandchildren of $f^{-1}(p)$.

2.1 choose $S'_p \subseteq S_p$ arbitrarily with $|S'_p| = \deg_{\mathcal{T}}(f^{-1}(p))$;

2.2 match each $v \in \Gamma_{\mathcal{T}}(f^{-1}(p))$ to some vertex in S'_p and update M ;

2.3 for each $u \in S'_p$, take some arbitrary $Z'_{p,u} \subseteq Z_{p,u}$
with $|Z'_{p,u}| = \deg_{\mathcal{T}}(f^{-1}(u))$;

2.4 for each $u \in S'_p$ and each $w \in \Gamma_{\mathcal{T}}(f^{-1}(u))$, match w to a vertex in $Z'_{p,u}$
and update M ;

Procedure 3: restore-invariants

```

3.1  $R \leftarrow \emptyset$  ;
3.2  $D \leftarrow \{x \in V_0 : \deg_G(x, V_1 \setminus Z) < \alpha D_0/2\}$  ;
3.3  $Z' \leftarrow \emptyset$  ;
3.4 repeat
3.5    $(\mathcal{C}', \mathcal{S}') \leftarrow \mathbf{find-critical-vertices}(Z, \mathcal{C}, D)$  ;
      // consolidate critical vertices
3.6    $\mathcal{C} \leftarrow \mathcal{C} \cup \mathcal{C}'$ ,  $\mathcal{S} \leftarrow \mathcal{S} \cup \mathcal{S}'$ ,  $Z \leftarrow Z \cup \bigcup_{S \in \mathcal{S}'} S$  ;
      // promotion to ultra-critical
3.7    $\mathcal{U}' \leftarrow \{w \in \mathcal{C} \setminus \mathcal{U} : |S_w \setminus D| < |S_w|/2 = \alpha 2^{-1-rc} D_1\}$  ;
3.8    $\mathcal{U} \leftarrow \mathcal{U} \cup \mathcal{U}'$  ;
3.9   if  $|\mathcal{U}| > \varepsilon N_1 / (D_0 D_1)$  then
3.10     └ abort Algorithm 1 ; // comply with Invariant V
3.11     find sets  $S''_w \subseteq S_w$  with  $|S''_w| = |S_w|/4$ , for  $w \in \mathcal{U}'$ , and a family of
      (not necessarily disjoint)  $\alpha 2^{-ru} D_0$ -sets
       $\{Y_{w,u} \subseteq \Gamma_G(u) \setminus Z\}_{w \in \mathcal{U}', u \in S''_w}$ ; if not possible, abort Algorithm 1 ;
3.12      $Z' \leftarrow Z' \cup \bigcup_{w \in \mathcal{U}', u \in S''_w} Y_{w,u}$  ;
3.13      $R \leftarrow R \cup \mathcal{U}'$ 
3.14      $D \leftarrow \{x \in V_0 : \deg_G(x, V_1 \setminus (Z \cup Z')) < \alpha D_0/2\}$ 
3.15 until  $\mathcal{U}' = \emptyset$ ;
3.16 find sets  $S'_w \subseteq S''_w$  with  $|S'_w| = \Delta_1$ , for  $w \in R$ , and a family of disjoint
       $\Delta_0$ -sets  $\{Z_{w,u} \subseteq Y_{w,u}\}_{w \in R, u \in S'_w}$ ; if not possible, abort Algorithm 1 ;
3.17  $\mathcal{D} \leftarrow D$ ,  $Z \leftarrow Z \cup Z' \cup \mathcal{D}$  ;
3.18  $\mathcal{S}_U \leftarrow \mathcal{S}_U \cup \{S'_w\}_{w \in R}$  ;
3.19  $\mathcal{W} \leftarrow \mathcal{W} \cup \{Z_{w,u}\}_{w \in R, u \in S'_w}$  ;

```

Procedure 4: find-critical-vertices(Z, \mathcal{C}, D)

Input : Z – set of used/reserved/dangerous vertices;
 \mathcal{C} – current collection of critical vertices;
 D – set of vertices that will be marked dangerous.

Output: \mathcal{C}' – the set of critical vertices found;
 $\{S_w \subseteq \Gamma_G(w) \setminus Z\}_{w \in \mathcal{C}'}$ – a family of disjoint $\alpha 2^{-rc} D_1$ -sets.

4.1 $\mathcal{C}' \leftarrow \emptyset$
4.2 $X \leftarrow \emptyset$
4.3 **while** *there exists* $v \in V_1 \setminus (\mathcal{C} \cup \mathcal{C}')$
with $\deg_G(v, V_0 \setminus (Z \cup X \cup D)) < \alpha D_1/2$ **do**
4.4 $\mathcal{C}' \leftarrow \mathcal{C}' \cup \{v\}$
4.5 **if** $|\mathcal{C}'| + |\mathcal{C}| > \varepsilon N_0/(8D_1)$ **then**
4.6 **abort** Algorithm 1 ; // comply with Invariant IV
4.7 find family of disjoint $\alpha 2^{-rc} D_1$ -sets $\{S_w \subseteq \Gamma_G(w) \setminus Z\}_{w \in \mathcal{C}'}$
 covering X ; if not possible, **abort** Algorithm 1 ;
4.8 $X \leftarrow \bigcup_{w \in \mathcal{C}'} S_w$;
4.9 **return** $(\mathcal{C}', \{S_w\}_{w \in \mathcal{C}'})$

In what follows, $r_C, r_U \in \mathbb{N}$ will be sufficiently large absolute constants.

Invariants. At the beginning of every iteration of Algorithm 1 (line 1.10), the following holds:

I. (**cardinality of $|Z|$**), we have

$$|Z \cap V_0| \leq |f_M(\mathcal{T}) \cap V_0| + |\mathcal{D}| + |\mathcal{C}|(\alpha 2^{-rc} D_1)$$

and

$$|Z \cap V_1| \leq |f_M(\mathcal{T}) \cap V_1| + |\mathcal{U}|(\alpha^2 2^{-rc-r_U} D_0 D_1);$$

II. (**non-critical/non-dangerous vertices**) for every $u \in V_0 \setminus \mathcal{D}$, $w \in V_1 \setminus \mathcal{C}$, we have

$$\deg_G(u, V_1 \setminus Z) \geq \frac{\alpha D_0}{2} \quad \text{and} \quad \deg_G(w, V_0 \setminus Z) \geq \frac{\alpha D_1}{2};$$

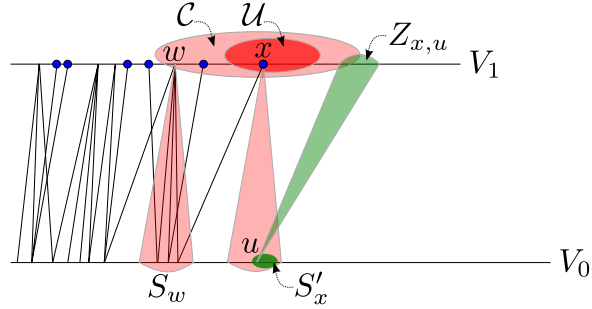


Figure 2.2: A partial embedding of \mathcal{T} into G . The edges represent the embedded part of \mathcal{T} . The active vertices in V_1 are represented by circles and some of the reserved sets (such as S_w , $Z_{x,u}$) are represented by shaded areas.

III. (**dangerous vertices**) we have $|\mathcal{D}| < \varepsilon^3 N_0$ and, for every $u \in \mathcal{D} \subseteq V_0 \cap Z$,

$$\deg_G(u, Z \cap V_1) \geq \frac{\alpha D_0}{2}.$$

IV. (**critical vertices**) we have $|\mathcal{C}| \leq \frac{\varepsilon N_0}{8D_1}$ and, for every $w \in \mathcal{C} \subseteq V_1$,

$$\deg_G(w, Z \cap V_0) \geq \frac{\alpha D_1}{2},$$

and the set $S_w \in \mathcal{S}$ has $\alpha 2^{-rc} D_1$ elements exclusively reserved for embedding the children of w ; moreover, if $w \notin \mathcal{U}$, then

$$\begin{aligned} |S_w \setminus \mathcal{D}| &= \#\{u \in S_w : \deg_G(u, Z \cap V_1) < \alpha D_0/2\} \\ &\geq |S_w|/2 = \alpha 2^{-1-rc} D_1; \end{aligned} \quad (2.13)$$

V. (**ultra-critical vertices**) we have $|\mathcal{U}| \leq \frac{\varepsilon N_1}{D_0 D_1}$ and, for every $w \in \mathcal{U} \subseteq \mathcal{C}$,

$$\begin{aligned} |S_w \cap \mathcal{D}| &= \#\{u \in S_w : \deg_G(u, Z \cap V_1) \geq \alpha D_0/2\} \\ &> |S_w|/2 = \alpha 2^{-1-rc} D_1; \end{aligned} \quad (2.14)$$

moreover, we also have $S'_w \in \mathcal{S}_U$ with $|S'_w| = \Delta_1$, $S'_w \subset S_w$, and a family of Δ_0 -sets $\{Z_{w,u}\}_{u \in S'_w} \subseteq \mathcal{W}$, where S'_w is reserved for children of w and $Z_{w,u}$ is reserved for children of u (grandchildren of w).

Theorem 2.14. *Let $n_0, n_1, \Delta_0, \Delta_1$ be given. Suppose that $G' = (V_0, V_1; E)$ is a graph satisfying Property (\ddagger) for some $\varepsilon > 0$, $C = C(\varepsilon)$ is a sufficiently large constant, $N_0 = Cn_0$, $N_1 = Cn_1$ and $p = \max\{\Delta_0/n_1, \Delta_1/n_0\} < \varepsilon/8$. Let $V'_1 \subset V_1$ be determined by Property (\ddagger) , $D_0 = pN_1$ and $D_1 = pN_0$.*

There exists an absolute constant $c > 0$ such that the following holds. Let $\alpha = c\sqrt{\varepsilon}$ and $G \subseteq G'[V_0, V'_1]$ be such that $d_G(u) \geq \alpha D_0$ for all $u \in V_0 \cap V(G)$ and $d_G(w) \geq \alpha D_1$ for all $w \in V'_1 \cap V(G)$. Then, Algorithm 1 embeds any $(n_0, \Delta_0, n_1, \Delta_1)$ -tree \mathcal{T} into G .

Proof. We shall abuse the notation and set $V_0 \leftarrow V_0 \cap V(G)$ and $V_1 \leftarrow V'_1 \cap V(G)$. Hence G has classes V_0 and V_1 . The proof is divided into three parts:

- The Invariants I-V hold at the beginning of every iteration;
- Algorithm 1 does not abort when the input is a graph G as above together with an $(n_0, \Delta_0, n_1, \Delta_1)$ -tree \mathcal{T} (with arbitrary root $r \in V_1(\mathcal{T})$);
- If the algorithm does not abort then it obtains an embedding of the tree into the host graph.

Clearly, this establishes that for any $(n_0, \Delta_0, n_1, \Delta_1)$ -tree \mathcal{T} the algorithm provides an embedding of \mathcal{T} into G .

For the base case, we have $|Z| = 1$ and there are no critical or dangerous vertices. It is then immediate that all the invariants hold.

Next, observe that when $p \in \mathcal{U}$, the sets $Z, \mathcal{C}, \mathcal{U}, \mathcal{S}, \mathcal{W}$ remain unchanged since the elements of $S''_p \subseteq S'_p \in \mathcal{S}_U$, $S'_p \subset Z$, are used for the children of p and the elements of $\{Z_{p,u} \subset Z\}_{u \in S''_p} \subseteq \mathcal{W}$ are used for the grandchildren of p .

Hence, in this case, the invariants are maintained and the algorithm does not abort.

Suppose that all the invariants hold at the beginning of some iteration and that $p \notin \mathcal{U}$. Let us prove that all the invariants hold at the beginning of the next iteration (if the algorithm does not abort).

Proof of Invariant I. Examining the steps where Z is updated (see lines 1.9, 1.21, 3.17), it is clear that, by the end of the iteration, $Z \cap V_0$ consists of vertices used by the embedding $(f_M(\mathcal{T}) \cap V_0)$, dangerous vertices (namely, \mathcal{D}) and reserved vertices $(\bigcup_{w \in \mathcal{C}} S_w)$ which account for $|\mathcal{C}|(\alpha 2^{-rc} D_1)$ vertices. It is also clear that $Z \cap V_1$ contains vertices used in the embedding $(f_M(\mathcal{T}) \cap V_1)$ and the vertices added to Z' by line 3.12, which are at most $\sum_{w \in \mathcal{U}} |S_w| \alpha 2^{-ru} D_0$. Note that Z' contains the reserved vertices of V_1 . The invariant follows. \square

Proof of Invariant II. Let us analyze the Procedure `restore-invariants`. By construction (see line 4.3), immediately after Procedure `find-critical-vertices` returns (line 3.5) and the critical vertices are consolidated (in particular, Z now contains the newly reserved neighborhoods), no vertex $w \in V_1 \setminus \mathcal{C}$ satisfies $\deg_G(w, V_0 \setminus (Z \cup D)) < \alpha D_1/2$.

If \mathcal{U}' is empty on some iteration of the inner loop, the loop will be complete at that iteration without changing D or Z any further. In particular, the degree condition for non-critical vertices ($V_1 \setminus \mathcal{C}$) is ensured at the end of the iteration and this part of Invariant II holds at the next iteration.

The case of $u \in V_0 \setminus \mathcal{D}$ is simpler: any vertex that does not satisfy the degree condition by the end of the iteration is either already dangerous or becomes dangerous (see lines 3.2 and 3.14). \square

Proof of Invariant III. The degree part of Invariant III follows immediately from the updates made to \mathcal{D} (lines 3.2 and 3.14) and the fact that Z never loses any element. The fact that $\mathcal{D} \subset Z$ easily follows from line 3.17.

It remains to upper bound the number of dangerous vertices. Observe that $|Z \cap V_1|$ is determined by Invariant I (which is already proven to hold at the next iteration). Also note that a dangerous vertex v must have, by the end of the iteration, $\deg_G(v, Z \cap V_1) \geq \alpha D_0/2$. From the bound $|\mathcal{U}| \leq \frac{\varepsilon N_1}{D_0 D_1}$ (which is enforced by line 3.10) we get

$$|Z \cap V_1| \leq n_1 + \frac{\varepsilon N_1}{D_0 D_1} \alpha^2 2^{-r_C - r_U} D_0 D_1 \leq \varepsilon N_1.$$

If $|\mathcal{D}| \geq \varepsilon^3 N_0$ then let $A \supset Z \cap V_1$ be an arbitrary set with $|A| = \varepsilon N_1$, $A \subset V_1$, and observe that Property(\dagger).(v) implies that

$$e_{G'}(\mathcal{D}, Z \cap V_1) \leq e_{G'}(\mathcal{D}, A) \leq (1 + \varepsilon^2) p |\mathcal{D}| \cdot \varepsilon N_1.$$

On the other hand,

$$e_{G'}(\mathcal{D}, Z \cap V_1) \geq |\mathcal{D}| (\alpha D_0/2) = p |\mathcal{D}| \left(\frac{\alpha}{2} N_1 \right).$$

These inequalities imply that $\alpha < 3\varepsilon$, a contradiction. Therefore $|\mathcal{D}| < \varepsilon^3 N_0$. \square

Proof of Invariant IV. There is only one place in the algorithm where the set of critical vertices grows—just after a call to Procedure `find-critical-vertices` (l. 3.5) these critical vertices are consolidated. A subtle, but very important detail of `find-critical-vertices` consists in requiring that the family obtained in line 4.7 covers the set X (which is the union of the reserved sets of the previous iteration). Hence, a vertex that had a small number of free neighbors could only have less free neighbors in following iterations.

We also note that once an element is added to D , it remains in D (and is subsequently added to \mathcal{D}). It is immediate that the number of edges a critical vertex sends into Z cannot become smaller than $\alpha D_1/2$.

The reserved neighborhoods are defined to have $\alpha 2^{-r_C} D_1$ elements each and, once a reserved neighborhood is finally determined (after Procedure

`find-critical-vertices` returns), it is consolidated by being merged into Z . Since the reserved neighborhoods are disjoint, and new reserved neighborhoods must be chosen outside of Z , no other vertex can have its children embedded in a reserved neighborhood.

Moreover, if a vertex $w \in \mathcal{C}$ fails equation (2.13) at the end of the iteration, the line 3.7, together with the condition of the inner loop (that $\mathcal{U}' = \emptyset$), ensures that $w \in \mathcal{U}$ will hold when the iteration ends.

The cardinality of \mathcal{C} is enforced by line 4.6. □

Proof of Invariant V. If no new ultra-critical vertex was found at the iteration, the invariant is preserved. Hence, let us assume that some ultra-critical vertex was found.

Following the construction of \mathcal{U}' (see line 3.7), at the moment a vertex w becomes ultra-critical, equation (2.14) holds. Since the set Z is monotonically increasing, this equation must continue to hold subsequently.

The family of reserved sets of Invariant V is obtained at line 3.16. Those reserved vertices will not be used to embed the children/grandchildren of any other vertex because the reserved sets are merged into Z and no other vertex can reserve or use vertices in Z to embed their children/grandchildren.

The cardinality of \mathcal{U} is enforced by line 3.10. □

In the following analysis we shall denote by \tilde{Z} the set Z at the beginning of an iteration of the Algorithm 1. We also let \hat{Z} denote the set Z just after line 1.21.

The algorithm does not abort at find-critical-vertices. Let us suppose for the sake of a contradiction that the algorithm aborts at line 4.6. This means that there is a set $\mathcal{C} \cup \mathcal{C}'$, with $|\mathcal{C} \cup \mathcal{C}'| = \varepsilon N_0 / (8D_1)$, such that each vertex $v \in \mathcal{C} \cup \mathcal{C}'$ sends at least $\alpha D_1 / 2$ edges into $(Z \cup X \cup D) \cap V_0$. On the other hand, $|D| \leq \varepsilon^3 N_0$ (see the proof of Invariant III), which together with

Invariant I imply that $|(Z \cup X \cup D) \cap V_0|$ is at most

$$|f_M(\mathcal{T}) \cap V_0| + |D| + (|\mathcal{C}| + |\mathcal{C}'|)\alpha 2^{-r_C} D_1 \leq n_0 + \varepsilon^3 N_0 + \frac{\varepsilon N_0}{8D_1} \alpha 2^{-r_C} D_1 < \frac{\alpha \varepsilon N_0}{32},$$

if we set $r_C \geq 3$.

In G' the Property $(\ddagger).(ii)$ ensures that every subset of V_1 having at most $\varepsilon N_1 / (8D_0) = \varepsilon N_0 / (8D_1)$ elements expands by at least $(1 - \varepsilon)D_1$. Hence, using Lemma 2.3 we obtain

$$|(Z \cup X \cup D) \cap V_0| \geq (\alpha/2 - 2\varepsilon)D_1 \frac{\varepsilon N_0}{8D_1} > \frac{\alpha \varepsilon N_0}{32},$$

a contradiction.

Now we show that the procedure does not abort at line 4.7. By the above argument, the set \mathcal{C}' has cardinality at most $\varepsilon N_0 / (8D_1)$. Moreover, $(\hat{Z} \setminus \tilde{Z}) \cap V_0$ contains at most Δ_1 elements.

Observe that every call to `find-critical-vertices` is made with $Z = \hat{Z}$. Since vertices in \mathcal{C}' were not critical, Invariant II states that every $w \in \mathcal{C}'$ has degree at least $\alpha D_1 / 2 - \Delta_1$ on \hat{Z} .

Note that, although we consider the degree of vertices $w \in V_1 \setminus \mathcal{C}$ on the set $Z \cup X \cup D$ in `find-critical-vertices` to classify a vertex as critical, the reserved neighborhood of new critical vertices may include recent dangerous vertices (those in $D \setminus Z$). The reason is that vertices which were just classified as dangerous still have reasonably large degree outside Z .

To prove that the desired family of disjoint sets can be found, we invoke Properties $(\ddagger).(i)$, $(\ddagger).(ii)$ and Lemma 2.3 to establish that any subset $\mathcal{C}'' \subset \mathcal{C}' \subset V_1'$ must have at least $(\frac{\alpha D_1}{2} - 4\varepsilon)|\mathcal{C}''|$ neighbors outside of Z (in G). Hence we may apply Lemma 2.12 to obtain a family of disjoint sets, each having cardinality $\alpha 2^{-r_C} D_1$, such that the union of these sets covers X . \square

The algorithm does not abort at line 1.18. If $p \notin \mathcal{C}$, then

$$|\Gamma_G(p) \setminus Z| \geq \alpha D_1 / 2$$

because of Invariant II. Otherwise, $p \in \mathcal{C} \setminus \mathcal{U}$ and because of Invariant IV, we have that $|S_p \setminus \mathcal{D}| \geq |S_p|/2 = \alpha 2^{-1-rc} D_1$. Since $\mathcal{D} \subset Z$, in both cases, p has at least $\alpha 2^{-1-rc} D_1$ neighbors u (either free or reserved to p) satisfying $|\Gamma_G(u) \setminus Z| \geq \alpha D_0/2$.

If $\varepsilon N_1 \geq D_0 D_1$, apply Lemma 2.13 to $S \leftarrow \{p\}$, with $\alpha_{2.13} \leftarrow \alpha 2^{-1-rc}$, $A_p \subseteq S_p \setminus \mathcal{D}$ with $|A_p| = \alpha_{2.13} D_1$ and $B_x \subseteq \Gamma_G(x) \setminus Z$ with $|B_x| = \alpha_{2.13} D_0$ for all $x \in A_p$. Refine the families obtained from Lemma 2.13 in such a way that the corresponding cardinalities match the degrees in the tree. This will produce the set S'_p and the family of disjoint sets $\{Z_u\}_{u \in S'_p}$ of line 1.18.

Now we deal with the case $\varepsilon N_1 < D_0 D_1$. We may require every vertex in S_p to have degree at most $(1 + \varepsilon)D_0$ by possibly deleting at most εD_1 vertices from S_p (see Property $(\ddagger).(i)$). Use Property $(\ddagger).(iv)$ applied to $S_p \setminus \mathcal{D} \subset \Gamma_G(p)$ to obtain disjoint sets $T_1, \dots, T_r \subset S_p \setminus \mathcal{D}$. We need to find $S'_p = \{u_1, \dots, u_\ell\} \subset S_p$ and a family of disjoint sets $\{Z_i \subseteq \Gamma(u_i) \setminus Z\}_{i=1}^\ell$.

Given an arbitrary set $J \subset V_1$ such that $|J| \leq \min\{n_1, \Delta_0 \Delta_1\}$, we shall prove that the number of vertices $u \in T_i$ ($i = 1, \dots, r$) having $|\Gamma_G(u) \setminus (Z \cup J)| < \alpha D_0/4$ is at most $|T_i|/2$. Indeed, since $T_i \subseteq S_p \setminus \mathcal{D}$, we have $|\Gamma_G(u) \setminus Z| \geq \alpha D_0/2$ for all $u \in T_i$. Let $T'_i = \{u \in T_i : |\Gamma_G(u) \setminus (Z \cup J)| < \alpha D_0/4\}$. Note that for every vertex $u \in T'_i$,

$$|\Gamma_G(u) \cap J| \geq |\Gamma_G(u) \setminus Z| - |\Gamma_G(u) \setminus (Z \cup J)| \geq \alpha D_0/4.$$

Since T'_i is an LE set (by Property $(\ddagger).(iv)$), we can apply Lemma 2.3 to show that $\min\{\Delta_0 \Delta_1, n_1\} \geq |J| \geq \alpha D_0 |T'_i|/8$. For C sufficiently large, it follows that

$$|T'_i| \leq \frac{8}{\alpha D_0} n_1 = \frac{8}{\alpha C D_0} N_1 \leq \frac{\varepsilon N_1}{8 D_0}$$

and

$$|T'_i| \leq \frac{8}{\alpha D_0} \Delta_0 \Delta_1 \leq \frac{8}{\alpha C^2} D_1 \leq \frac{\varepsilon D_1}{16},$$

thus $|T'_i| \leq \frac{1}{2} \min\{\varepsilon D_1/8, \varepsilon N_1/(4D_0)\} = \frac{1}{2}|T_i|$.

We now construct S'_p and its corresponding family sequentially. Suppose that u_1, \dots, u_k have been selected from $\bigcup_{i=1}^r T_i$ together with a family of disjoint sets $\{Z_i\}_{i=1}^k$. Set $J = \bigcup_{i=1}^k Z_i$ (initially $J = \emptyset$) and note that $|J| \leq \Delta_0 \Delta_1$ since $k \leq \ell \leq \Delta_1$ and $|Z_i| \leq \Delta_0$ for all i . It is also clear that $|J| \leq n_1$ since to each vertex in J there corresponds a vertex in $V_1(\mathcal{T})$.

By the above argument, at least half of the elements in $\bigcup_{i=1}^r T_i$ have large degree outside of $Z \cup J$. Pick an arbitrary u_{k+1} (distinct from u_1, \dots, u_k) having at least $\alpha D_0/4$ neighbors outside of $Z \cup J$. Set Z_{k+1} to be an arbitrary subset of $\Gamma_G(u_{k+1}) \setminus (Z \cup J)$ having the same number of elements as the number of children of u_k (which is at most $\Delta_0 < \alpha D_0/4$). Since $\ell \leq \Delta_1 < \frac{1}{8}|S_p| < \frac{1}{2}|\bigcup_{i=1}^r T_i|$, it is always possible to extend the selection and the corresponding family. \square

The algorithm does not abort at line 3.10. Suppose for the sake of a contradiction that the algorithm aborts because \mathcal{U} grew larger than $\varepsilon N_1/(D_0 D_1)$. Let us start with the case $\varepsilon N_1 \geq D_0 D_1$. This means that we can find a set S of $\varepsilon N_1/(D_0 D_1)$ elements together with a family of disjoint $\alpha 2^{-1-rc} D_1$ -sets $\{X_w \subseteq \Gamma_G(w)\}_{w \in S}$ where each $u \in X_w$ sends at least $\alpha D_0/2$ edges into $Z \cap V_1$. We may also require that every vertex in X_w should have degree at most $(1 + \varepsilon)D_0$ in G' by possibly deleting at most $\varepsilon D_1 < \alpha 2^{-2-rc} D_1$ vertices from X_w (see Property $(\ddagger).(i)$).

From Invariant I we know that

$$|Z \cap V_1| \leq |f_M(\mathcal{T}) \cap V_1| + \frac{\varepsilon N_1}{D_0 D_1} (\alpha^2 2^{-rc-rc} D_0 D_1) \leq N_1 \left(\frac{1}{C} + \varepsilon \alpha^2 2^{-rc-rc} \right).$$

On the other hand, if we take the set $T = \bigcup_{w \in S} X_w$, then

$$|T| = \alpha 2^{-2-rc} D_1 |S| > \sqrt{\varepsilon} D_1 |S|$$

and by Property $(\ddagger).(iii)$,

$$d_{G'}^*(T) \geq (1 - 5\sqrt{\varepsilon}) \alpha 2^{-2-rc} (|S| D_0 D_1) = (1 - 5\sqrt{\varepsilon}) \varepsilon \alpha 2^{-2-rc} N_1.$$

Since the degrees of the vertices of T (in G') are at most $(1+\varepsilon)D_0$, applying Lemma 2.3 over the graph $G[T, Z \cap V_1] \subset G'$, we obtain

$$\begin{aligned} |Z \cap V_1| &\geq |T| \frac{\alpha D_0}{2} + 2\{(1 - 5\sqrt{\varepsilon})D_0 |T| - (1 + \varepsilon)D_0 |T|\} \\ &\geq \left(\frac{\alpha}{2} - 12\sqrt{\varepsilon}\right) D_0 |T| \\ &\geq \alpha^2 2^{-4-r_C} N_1, \end{aligned}$$

when $c = 48$ (that is, $\alpha = 48\sqrt{\varepsilon}$). However, for $r_U > 4$ and $C = C(\varepsilon)$ sufficiently large this is a contradiction with the upper bound on $|Z \cap V_1|$.

For the case $\varepsilon N_1 < D_0 D_1$ we do not allow even a single ultra-critical vertex. Let us suppose for the sake of contradiction that some vertex w became an ultra-critical vertex. Apply Property $(\ddagger).(iv)$ to S_w . Let $T_1, \dots, T_r \subset S_w$ be the disjoint sets obtained from the property. By assumption, there is a set $B \subseteq S_w$ with at least $|S_w|/2$ elements $u \in S_w$ with $\deg_G(u, Z \cap V_1) \geq \alpha D_0/2$. Since $\sum_{i=1}^r |T_i| \geq \frac{3}{4}|S_w|$, there exists some T_i with $|T_i \cap B| \geq |T_i|/4$. Because $T_i \cap B$ is an LE set, from Lemma 2.3 we obtain

$$|Z \cap V_1| \geq \frac{\alpha D_0}{4} |T_i \cap B| \geq \frac{\alpha D_0}{16} |T_i|.$$

Notice that

$$|T_i| = \min\{\varepsilon D_1/8, \varepsilon N_1/(4D_0)\} \geq \frac{\varepsilon N_1}{8D_0},$$

which implies that $|Z \cap V_1| \geq \alpha \varepsilon N_1/128 = (C\alpha\varepsilon/128)n_1$. On the other hand, Invariant I and the fact that $\mathcal{U} = \emptyset$ imply that $|Z \cap V_1| \leq n_1$. For our choice of large C , this is a contradiction. \square

Claim 2.15. *For any $w \in V_1$, the number of vertices $u \in \Gamma_G(w)$ having $\deg_{G'}(u) \sim_\varepsilon D_0$ and $\deg_G(u, (\hat{Z} \setminus \tilde{Z}) \cap V_1) \geq \alpha D_0/4$ is at most $\sqrt{\varepsilon} D_1$.*

Proof of Claim 2.15. Given any $w \in V_1$, let us bound the number N_w of vertices $u \in \Gamma_G(w)$ such that $d_{G'}(u) \sim_\varepsilon D_0$ and $\deg_G(u, (\hat{Z} \setminus \tilde{Z}) \cap V_1) \geq \alpha D_0/4$. Since $(\hat{Z} \setminus \tilde{Z}) \cap V_1 = \bigcup_{i=1}^l Z_i$ (see line 1.21), it is clear that $|(\hat{Z} \setminus$

$\tilde{Z}) \cap V_1| \leq \Delta_0 \Delta_1$. If $N_w \geq \sqrt{\varepsilon} D_1$, by Property (\ddagger) .(iii) and Lemma 2.3, we should have

$$|(\hat{Z} \setminus \tilde{Z}) \cap V_1| \geq (\alpha/4 - 10\sqrt{\varepsilon} - 2\varepsilon)\sqrt{\varepsilon} D_0 D_1,$$

a contradiction for sufficiently large C . \square

The algorithm does not abort at line 3.11. Note that $Z \cap V_1 = \hat{Z} \cap V_1$ holds throughout the inner loop. Let $w \in \mathcal{U}'$. Since w was not ultra-critical before, by Invariant IV and equation (2.13), at least half of the elements $u \in S_w \in \mathcal{C}$ are such that $\deg_G(u, V_1 \setminus \tilde{Z}) \geq \alpha D_0/2$. (It is possible that a vertex becomes critical and is promoted to ultra-critical during the execution of **restore-invariants**; the claim above is still true in that case since the reserved neighborhood for such a vertex would only contain vertices outside $\mathcal{D} \subseteq \tilde{Z}$.)

Since at most εD_1 vertices $u \in S_w$ fail to satisfy $\deg_{G'}(u) \sim_\varepsilon D_0$, by Claim 2.15, less than $2\sqrt{\varepsilon} D_1$ neighbors of $w \in \mathcal{U}'$ may have more than $\alpha D_0/4$ edges going into $\hat{Z} \setminus \tilde{Z}$. Therefore, the number of $u \in S_w$ such that

$$\deg_G(u, V_1 \setminus Z) = \deg_G(u, V_1 \setminus \tilde{Z}) - \deg_G(u, \hat{Z} \setminus \tilde{Z}) \geq \alpha D_0/4 > \alpha 2^{-r\nu} D_0$$

is greater than $|S_w|/4$. Since the family $\{Y_{w,u} \subseteq \Gamma(u) \setminus Z\}_{w \in \mathcal{U}', u \in S'_w}$ does not need to be disjoint, we are done. \square

The algorithm does not abort at line 3.16. We shall apply Lemma 2.13 with $S \leftarrow R$, $\alpha_{2.13} \leftarrow \alpha 2^{-r\nu}$, $A_w \subset S''_w$ with $|A_w| = \alpha_{2.13} D_1$ for all $w \in S$ and $B_x = Y_{w,x}$ with $|B_x| = \alpha_{2.13} D_0$ for all $w \in S$, $x \in A_w$. The families obtained through Lemma 2.13 are precisely the ones required at line 3.16. \square

We have covered all invariants and all places where the algorithm could have aborted. It remains to show that the algorithm provides an embedding of \mathcal{T} into G . Notice that the root of \mathcal{T} is mapped to an active vertex in V_1 . Every

embedding step consists in embedding the children and grandchildren of an active vertex from Q and then adding all of the embedded grandchildren to the queue Q . It follows by induction that every vertex of the tree is eventually embedded. \square

It is possible to apply Theorem 2.14 to every sufficiently dense subgraph of a graph satisfying Property (\ddagger) by pre-processing the graph in a simple way.

Theorem 2.16. *Let $n_0, n_1, \Delta_0, \Delta_1$ be given. Suppose that $G' = (V_0, V_1; E)$ is a graph satisfying Property (\ddagger) for some $\varepsilon > 0$, $C = C(\varepsilon)$ is a sufficiently large constant, $N_0 = Cn_0$, $N_1 = Cn_1$ and $p = \max\{\Delta_0/n_1, \Delta_1/n_0\} < \varepsilon/8$.*

There exists an absolute constant $c > 0$ such that any subgraph $G \subseteq G'$ with $e(G) \geq c\sqrt{\varepsilon}e(G')$ contains every $(n_0, \Delta_0, n_1, \Delta_1)$ -tree.

Proof. Let $D_0 = pN_1$ and $D_1 = pN_0$. Notice that, by assumption, $e(G') \sim_{\varepsilon^2} pN_0N_1 = D_0N_0 = D_1N_1$.

Let $V'_1 \subseteq V_1$ be the set described by Property (\ddagger) and let $\alpha = 8\alpha_{2.14}$, where $\alpha_{2.14}$ is defined on Theorem 2.14. Suppose that $e(G) \geq 2\alpha e(G')$. By $(\ddagger).(v)$, we may assume that G does not contain any edge incident to $V_1 \setminus V'_1$ by removing edges from G while having $e(G) \geq \frac{3}{2}\alpha e(G')$ (the number of edges removed is at most $(1 + \varepsilon^2)pN_0(2\varepsilon N_1) < 3\varepsilon e(G')$).

While there exists a vertex in $V(G) \cap V_0$ having degree less than $\alpha D_0/8$ or a vertex in $V(G) \cap V'_1$ having degree less than $\alpha D_1/8$, remove this vertex from G together with all the edges incident to the removed vertex. The number of edges incident to removed vertices is at most $N_0(\alpha D_0/8) + N_1(\alpha D_1/8) \leq \frac{1}{2}\alpha e(G')$. Hence, the remaining graph G is non-empty and the graphs $G \subset G'$ satisfy the conditions of Theorem 2.14. Consequently, G contains every $(n_0, \Delta_0, n_1, \Delta_1)$ -tree. \square

From Theorem 2.16 we may prove Beck's conjecture:

Corollary 2.17. *The size-Ramsey number of a tree \mathcal{T} is $\Theta(\beta(\mathcal{T}))$.*

Proof. Given the constant c of Theorem 2.16, let $\varepsilon > 0$ be such that $c\sqrt{\varepsilon} < 1/2$. Without loss of generality, assume that $\varepsilon = 2^{-a}$ for some $a > 0$. Let $n_0, \Delta_0, n_1, \Delta_1$ be the parameters of \mathcal{T} . By possibly enlarging these values, we may assume that each of them is a power of 2. Since for every integer b there is an n such that $2^n \leq b < 2^{n+1}$, in the worst case, we may have to double each parameter. We may also assume that $n_0\Delta_0 = n_1\Delta_1$ by possibly increasing some Δ_i . These changes may only affect $n_0\Delta_0 + n_1\Delta_1$ by a multiplicative constant. The embedding algorithm is not affected since the parameters are only used as upper bounds on the cardinalities of the classes and their respective degrees.

Let $p = \Delta_1/n_0 = \Delta_0/n_1$. If $p \geq \varepsilon/8$ then we use the complete bipartite graph K_{N_0, N_1} as our Ramsey graph (see Lemma 2.7). The complete bipartite graph has $O(n_0n_1)$ edges while $\beta(\mathcal{T}) = 2pn_0n_1 = \Omega(n_0n_1)$.

If $p < \varepsilon/8$, we let $C = C(\varepsilon)$ be a sufficiently large constant and use Theorem 2.6 to obtain a graph G' satisfying Property (\ddagger) for ε , $N_0 = Cn_0$, $N_1 = Cn_1$ and p . By our choice of ε , from Theorem 2.16 we get that any subgraph $G \subseteq G'$ with at least $\frac{1}{2}e(G')$ edges contains \mathcal{T} .

Since in any two-coloring of the edges of G' there will be one color containing at least half of its edges, the graph induced by the most frequent color contains \mathcal{T} . Moreover, we have

$$\begin{aligned} e(G') &\leq 2pN_0N_1 = 2C^2pn_0n_1 = C^2(pn_0)n_1 + C^2(pn_1)n_0 \\ &= C^2(\Delta_0n_0 + \Delta_1n_1) = O(\beta(\mathcal{T})). \end{aligned}$$

This shows that $\hat{r}(\mathcal{T}) = O(\beta(\mathcal{T}))$. Together with the lower bound proved by Beck, the conjecture is proved. \square

Chapter 3

Distance preserving Ramsey graphs

3.1 Introduction

In [7], [10] and [35, 37] the following extension of the Ramsey Theorem was proved.

Theorem 3.1. *For any graph G there exists a graph R with the property that in any 2-coloring of the edges of R there exists an induced copy $\mathbf{G} \subset R$ which is monochromatic.[†]*

In other words, Theorem 3.1 states that the class of all graphs and induced embeddings has the *edge-Ramsey property*. This theorem, proved in 1973, together with some generalizations and other related results that soon followed gave rise to the study of restricted/induced/sparse families of Ramsey Theorems (for a survey on these topics see [14, 28]).

⁰The contents of this chapter will appear in [6].

[†]For a graph G , we will use \mathbf{G} (typeset in a sans-serif font) to denote an isomorphic copy of G .

Remark 3.2. For simplicity we state Ramsey theorems only for 2-colorings when in fact it is straightforward to extend them to an arbitrary number of colors by applying the 2-color version inductively.

Theorem 3.1 was generalized in [8] and [30] where it was proved that the same statement remains true if the coloring of edges (K_2) is replaced by the coloring of cliques (K_k) or induced independent sets (\overline{K}_k). Moreover, Theorem 3.1 fails to be true if one colors copies of an arbitrary non-homogeneous graph F . More formally, for any graph $F \neq K_k, \overline{K}_k$ there exists G such that for every graph H there is a 2-coloring of the set of all induced copies of F in H such that no induced copy G in H is monochromatic (that is, there must be induced copies of F in G of both colors).

With terminology used in [22] this can be rephrased as follows.

Proposition 3.3. *The class of graphs and induced embeddings has the F -Ramsey property if and only if F is a complete graph or an independent set.*

Let us show by means of a simple example, that for non-homogeneous unordered graphs F , the class of (unordered) graphs and induced embeddings does not have the F -Ramsey property. Consider the graph $F = P_2$, the path with two edges. Let $G = C_4$ be the cycle of length four and R be an arbitrary graph. We will now introduce a 2-coloring of the (unordered) induced copies of P_2 in R . First, label the vertices of $V(R)$ with integers $1, 2, \dots, |V(R)|$. For a path ijk of length two in R , color ijk red if the middle vertex j is the smallest of the three ($j < i$ and $j < k$); otherwise, color it blue. Under this coloring, any induced copy of $G = C_4$ in R must contain P_2 's of both colors. Indeed, among the four vertices of the C_4 , the smallest vertex is the middle vertex of a P_2 colored red and the largest vertex is the middle vertex of a P_2 colored blue.

However, it was shown in [30] that if one considers graphs with linearly ordered vertex sets and induced monotone embeddings then the theorem becomes true for all graphs $(F, <)$. This is stated in Theorem 3.6 below.

Remark 3.4 (Ordered graphs). Since our result deals with an extension of Theorem 3.6, in this chapter we typically assume (as in [1] and [29]) that each graph has a linear order on its vertex set. The example we described above (coloring P_2 's) shows that this assumption is crucial. All maps between ordered vertex sets are considered to be *monotone*, that is $\phi(u) < \phi(v)$ whenever $u < v$. In particular, all isomorphisms between ordered graphs are unique.

Definition 3.5 (Subgraphs). We say that the graph G is an *induced subgraph* of the graph H (we write $G \subset H$) if $V(G) \subset V(H)$, $E(G) = \{e \in E(H) : e \subset V(G)\}$ and the order $<_G$ in $V(G)$ respects the order $<_H$ in $V(H)$, that is, for every $u, v \in V(G)$ we have $u <_G v$ if and only if $u <_H v$.

To avoid cumbersome notation, we will omit the linear orders $<_H, <_G$ and denote by $\binom{H}{G}_{\text{ind}}$ the set of all induced subgraphs of H which are (monotone) isomorphic to G .

With this definition we may now state Ramsey's theorem for graphs with monotone induced embeddings.

Theorem 3.6 ([1, 29]). *For any ordered graphs F and G there exists an ordered graph R such that for any partition*

$$\binom{R}{F}_{\text{ind}} = \mathcal{A}_1 \cup \mathcal{A}_2$$

there exists some $G \in \binom{R}{G}_{\text{ind}}$ such that $G \in \mathcal{A}_i$ for some $i \in \{1, 2\}$.

In other words, Theorem 3.6 states that the class of ordered graphs and induced monotone embeddings has the $(F, <)$ -Ramsey property for any ordered graph $(F, <)$.

Remark 3.7. If a class \mathcal{K} endowed with a set of embeddings has the K -Ramsey property for all $K \in \mathcal{K}$ it is called a *Ramsey class* (see, for instance, [15]). Theorem 3.6 shows that the class of ordered graphs with induced monotone embeddings is a Ramsey class. See [14, 15, 17, 21, 24, 25] for other examples of Ramsey classes such as

- finite partially ordered sets (with a fixed linear extension);
- finite vector spaces (over a fixed field F);
- finite labeled partitions;
- finite linearly ordered metric spaces.

Another way to refine Theorem 3.1 is to consider distance preserving embeddings rather than induced ones. (Distance preserving embeddings have been considered in other contexts, for instance, in [16, 38].) For ordered graphs R and G , let $\mathbf{G} \in \binom{R}{G}_{\text{ind}}$ be fixed. If for all $x, y \in V(\mathbf{G}) \subset V(R)$

$$\text{dist}_{\mathbf{G}}(x, y) = \text{dist}_R(x, y) \tag{3.1}$$

then \mathbf{G} is called a *metric copy* of G in R and the (unique) monotone isomorphism $\phi: V(G) \rightarrow V(\mathbf{G}) \subset V(R)$ is called a *distance preserving embedding* of G into R . Denote by $\binom{R}{G}_{\text{metric}}$ the set of all metric copies of G in R . Notice that $\binom{R}{G}_{\text{metric}} \subset \binom{R}{G}_{\text{ind}}$.

The following theorem is a consequence of our main result, Theorem 3.11.

Theorem 3.8. *For any ordered connected graphs F and H there exists an ordered graph R such that for any partition*

$$\binom{R}{F}_{\text{metric}} = \mathcal{A}_1 \cup \mathcal{A}_2$$

there exists some $\mathbf{H} \in \binom{R}{H}_{\text{metric}}$ such that $\binom{H}{F}_{\text{metric}} \subset \mathcal{A}_i$ for some $i \in \{1, 2\}$.

In effect, Theorem 3.8 shows that the class of ordered connected graphs with metric embeddings is also a Ramsey class. Our proof of Theorem 3.8 will use a slightly more general setting.

A discrete *metric* ρ on the set $[t] = \{1, 2, \dots, t\}$ is a symmetric function $\rho: [t]^2 \rightarrow \mathbb{N} \cup \{\infty\}$ satisfying $\rho(i, j) = 0$ if and only if $i = j$ and the triangle inequality:

$$\rho(i, j) + \rho(j, k) \geq \rho(i, k).$$

In this chapter, the metrics considered correspond to the distance given by shortest paths in a graph. For instance, the metric of a clique would satisfy $\rho(i, j) = 1$ for all $i \neq j$ and the metric of an empty graph would satisfy $\rho(i, j) = \infty$ for all $i \neq j$.

Definition 3.9 (Metric induced on a set; (ρ, G) -tuples). Let G be an ordered graph and $S = \{v_1, \dots, v_t\} \subset V(G)$, $v_1 < v_2 < \dots < v_t$, be an arbitrary set. The *metric* ρ induced by G on S is given by $\rho(i, j) = \text{dist}_G(v_i, v_j)$.

Let ρ be a fixed metric. A set S which induces the metric ρ in G is called a (ρ, G) -*tuple*. The set of all (ρ, G) -tuples of G is denoted $\binom{G}{\rho}$.

We prove a slightly stronger statement from which Theorem 3.8 is derived as a corollary:

Lemma 3.10. *Let $t \in \mathbb{N}$, ρ be a metric on $[t]$ and H be an ordered connected graph.*

Then there exists an ordered graph R such that for every 2-coloring of $\binom{R}{\rho}$ there exists $\mathbf{H} \in \binom{R}{H}_{\text{metric}}$ such that $\binom{\mathbf{H}}{\rho}$ is monochromatic.

We now derive Theorem 3.8 from Lemma 3.10 as follows. Let F and H be given ordered graphs. Take $t = |V(F)|$ and without loss of generality assume that $V(F) = [t]$ (with the usual order $<$). Let ρ be the metric corresponding to dist_F , namely, $\rho(i, j) = \text{dist}_F(i, j)$.

We first obtain an ordered graph R from Lemma 3.10 applied to H and ρ . We claim that the graph R has the Ramsey property of Theorem 3.8.

Notice that $\binom{R}{F}_{\text{metric}} \cong \binom{R}{\rho}$ since the vertex set of a metric copy of F is necessarily a (ρ, R) -tuple. Consequently, we can view any coloring χ of $\binom{R}{F}_{\text{metric}}$ as a coloring of $\binom{R}{\rho}$. By the hypothesis on R , there exists a graph $\mathbf{H} \in \binom{R}{H}_{\text{metric}}$ such that every (ρ, \mathbf{H}) -tuple has the same color c under χ . For every $\mathbf{F} \in \binom{H}{F}_{\text{metric}}$ the set $V(\mathbf{F})$ is a (ρ, \mathbf{H}) -tuple and therefore $\chi(\mathbf{F}) = c$. It follows that $\binom{H}{F}_{\text{metric}}$ is monochromatic.

In Section 3.4 we prove Lemma 3.10 and use it to establish our main result, Theorem 3.11.

Theorem 3.11. *Let $t \in \mathbb{N}$ and H be a connected ordered graph.*

There exists an ordered graph R with the following property. For every 2-coloring of $\binom{V(R)}{t}$ there exists $\mathbf{H} \in \binom{R}{H}_{\text{metric}}$ such that $\binom{H}{\rho}$ is monochromatic for every metric ρ on $[t]$.

After fixing connected graphs H and F note that Theorem 3.8 asserts that coloring all metric copies of F in R yields a monochromatic $\binom{H}{F}_{\text{metric}}$. On the other hand, Theorem 3.11 applies to all subgraphs of H on t vertices (even those which are not connected). It guarantees that there exists a copy of H in which the color of a t element subgraph depends only on its metric within H .

Note that Theorem 3.11 extends Theorem 3.8

Remark 3.12. The particular case $t = 2$ of Theorem 3.11 implies that for any connected graph H it is possible to find some graph R such that every coloring of the pairs in $\binom{V(R)}{2}$ yields a metric copy $\mathbf{H} \in \binom{R}{H}_{\text{metric}}$ in which the color of $\{x, y\} \in \binom{V(\mathbf{H})}{2}$ is a function of $\text{dist}_{\mathbf{H}}(x, y)$. (In particular, the edges of \mathbf{H} are monochromatic.) This special case $t = 2$ was stated in the survey [28].

Remark 3.13. Notice that for $t = 2$ the linear order on the vertices is irrelevant. In Section 3.4.2 we show a version of Lemma 3.10 that can be applied to unordered graphs (provided that the metric is “homogeneous”).

Definition 3.14 (ρ_ℓ -metric sets and (ρ_ℓ, G) -tuples). Let $\ell, t \in \mathbb{N}$ be fixed and ρ be a metric on $[t]$. Let $H = (H, <)$ be a graph and $S = \{v_1, v_2, \dots, v_t\}$ be a subset of $V(H)$ with $v_1 < v_2 < \dots < v_t$. We say that S is ρ_ℓ -metric with respect to H if for all $1 \leq i < j \leq t$

- $\text{dist}_H(v_i, v_j) = \rho(i, j)$ whenever $\rho(i, j) \leq \ell$;
- $\text{dist}_H(v_i, v_j) \geq \ell$ whenever $\rho(i, j) > \ell$.

A set S as above is called a (ρ_ℓ, H) -tuple. We denote by $\binom{H}{\rho_\ell}$ the family of all (ρ_ℓ, H) -tuples of H .

A graph G naturally induces a metric $\rho(G)$ over its vertices by defining the distance between pairs of vertices as the length of a shortest path connecting them (when the pair is not connected, their distance is ∞).

Definition 3.15 (ℓ -metric (sub)graph). For a graphs $G \subset R$, the graph G is said to be ℓ -metric in R if $V(G)$ is $\rho(G)_\ell$ -metric with respect to R . A connected graph G is *metric in R* if it is ℓ -metric in R for all ℓ —namely, $\text{dist}_G(x, y) = \text{dist}_R(x, y)$ for every $x, y \in V(G)$.

Notice that G is ℓ -metric in R if no pair of vertices in G admits a shortcut path in R of length smaller than ℓ . For instance, G is 2-metric in R if and only if it is an induced subgraph of R .

Recalling that all vertex sets are linearly ordered, for $A, B \subset V(G)$ we will write $A \prec B$ if $\max(A) < \min(B)$.

Definition 3.16 (q -partite graphs). For $q \geq 2$, the graph G together with the linear order $<$ on $V(G)$ and a partition $V(G) = V_1^q(G) \cup \dots \cup V_q^q(G)$ is called q -partite if

- every edge $e \in G$ is *crossing*, that is, $|e \cap V_i^q(G)| \leq 1$ for all $i = 1, \dots, q$;
- the partition satisfies $V_1^q(G) \prec V_2^q \prec \dots \prec V_q^q(G)$.

Definition 3.17 (Partite embedding/isomorphism). If G and H are ordered q -partite graphs, a *partite embedding* is an injective monotone map $\phi: V(G) \rightarrow V(H)$ which is edge-preserving ($\phi(e) \in E(H)$ for all $e \in E(G)$) and satisfies $\phi(V_j^q(G)) \subset V_j^q(H)$ for all $j = 1, \dots, q$. If, in addition, ϕ is an isomorphism then we call it a *partite isomorphism*.

Definition 3.18 (Notation). We will use the following notation.

- For a (hyper)graph G we abuse the notation and write $e \in G$ to denote $e \in E(G)$.
- For a (hyper)graph G and a one-to-one map $\phi: V(G) \rightarrow X$, set

$$\phi(G) = (\phi(V(G)), \{\phi(e) : e \in G\}).$$

- For q -partite graphs G and H we denote by $\binom{H}{G}_{\text{Part}(q)}$ the set of all subgraphs $\phi(G)$ of H where $\phi: V(G) \rightarrow V(H)$ is a partite embedding.
- If \mathbb{G} is an isomorphic copy of G with (unique) monotone isomorphism $\sigma: V(G) \rightarrow V(\mathbb{G})$ and \mathcal{I} is a hypergraph with $V(\mathcal{I}) \subset V(G)$ then we denote by $\mathcal{I}_{\mathbb{G}}$ the hypergraph $\sigma(\mathcal{I})$.

Lemma 3.19 below is a technical result which will be used in the proof of our main result, Theorem 3.11.

Lemma 3.19 (Partite Lemma). *Let $\ell, t, q \in \mathbb{N}$, $t \leq q$. Suppose that*

- ρ is a fixed metric on $[t]$;
- G is a q -partite (ordered) graph with partition $V(G) = V_1^q(G) \cup \dots \cup V_q^q(G)$;

- for some $1 \leq j_1 < j_2 < \dots < j_t \leq q$, $\mathcal{I} \subset \binom{G}{\rho_\ell}$ is a t -partite t -uniform hypergraph with classes $\{V_{j_i}^q(G)\}_{i=1}^t$ consisting of selected (ρ_ℓ, G) -tuples.

Then there exists a q -partite ordered graph R and $\mathcal{G} \subset \binom{R}{G}_{\text{Part}(q)}$ satisfying the following properties.

(L1) For any 2-coloring of the (ρ_ℓ, R) -tuples in $\bigcup_{G \in \mathcal{G}} \mathcal{I}_G$ there exists $G \in \mathcal{G}$ such that $\mathcal{I}_G \subset \binom{G}{\rho_\ell} \subset \binom{R}{\rho_\ell}$ is monochromatic.

(L2) Every $G \in \mathcal{G}$ is ℓ -metric in R .

Remark 3.20. Note that $\bigcup_{G \in \mathcal{G}} \mathcal{I}_G$ is a t -partite t -uniform hypergraph with classes $\{V_{j_i}^q(R)\}_{i=1}^t$. This is because by the definition of $\binom{R}{G}_{\text{Part}(q)}$ every $G \in \mathcal{G} \subset \binom{R}{G}_{\text{Part}(q)}$ is the image of G under a partite embedding into R (and thus $V_j^q(G) \subset V_j^q(R)$ for all $j = 1, \dots, q$).

Moreover, it will follow from our proof that for any pair of distinct $G, G' \in \mathcal{G}$ we have $V(G) \cap V(G') \subset \bigcup_{i=1}^t V_{j_i}^q(R)$.

The proof of Lemma 3.19 uses the *partite construction* method, which was introduced in [23] and has been a successful tool for proving the existence of several Ramsey structures such as metric spaces [21], systems of sets [32], Steiner systems [31] etc. Perhaps, a novelty here is that the Partite Lemma, which was usually proved using the Hales–Jewett theorem [18] directly, is proved here by induction using the partite construction as well.

3.2 Proof of Lemma 3.19

We will prove a slightly stronger statement by double induction. The main induction is over ℓ . The base case ($\ell = 2$) is presented in Section 3.3 (Lemma 3.35). In this section we will prove the induction step from ℓ to $\ell + 1$.

Remark 3.21. The somewhat complicated intersection conditions (A) and (B) serve the purpose of imposing useful constraints on how the copies in the family may intersect while at the same time being weak enough to be carried by the induction. The condition (B) is later used to guarantee that when two vertices are shared by two copies then the distances with respect to each copy are “compatible”. More precisely, if we wish to obtain a family of ℓ -metric subgraphs then it is obvious that any pair of vertices at distance $\ell' < \ell$ in some copy should not have distance different than ℓ' in another copy.

Induction over ℓ – Hypothesis for R_ℓ and \mathcal{G}_ℓ

For a q -partite graph G , a metric ρ on $[t]$ and a t -partite t -uniform hypergraph $\mathcal{I} \subset \binom{G}{\rho_\ell}$ there is a graph $R_\ell = R_\ell(q, G, \rho, \mathcal{I})$ and $\mathcal{G}_\ell = \mathcal{G}_\ell(q, G, \rho, \mathcal{I}) \subset \binom{R_\ell}{G}_{\text{Part}(q)}$ satisfying conditions (L1) and (L2) of Lemma 3.19 and

$$(L3) \quad E(R_\ell) = \bigcup_{G \in \mathcal{G}_\ell} E(G).$$

Moreover, \mathcal{G}_ℓ satisfies the conditions (A) and (B) below.

Intersection conditions for a family \mathcal{G} of copies of G

- (A) If $G_1, G_2 \in \mathcal{G}$ and $u \in V(G_1) \cap V(G_2)$ then there are (ρ_ℓ, G_j) -tuples $I^j \in \mathcal{I}_{G_j}$, $j = 1, 2$, such that $u \in I^1 \cap I^2$.
- (B) If $G_1, G_2 \in \mathcal{G}$ and $u, v \in V(G_1) \cap V(G_2)$ then either
 - (B1) there exist (ρ_ℓ, G_j) -tuples $I^j \in \mathcal{I}_{G_j}$, $j = 1, 2$, such that $\{u, v\} \subset I^1 \cap I^2$ or
 - (B2) the (unique) isomorphisms $\sigma_j: V(G_j) \rightarrow V(G)$, $j = 1, 2$, satisfy $\sigma_1(u) = \sigma_2(u)$ and $\sigma_1(v) = \sigma_2(v)$.

Given q, G, ρ and $\mathcal{I} \subset \binom{G}{\rho_{\ell+1}} \subset \binom{G}{\rho_\ell}$ as in the statement of the lemma, we obtain $R_\ell = R_\ell(q, G, \rho, \mathcal{I})$ and $\mathcal{G}_\ell = \mathcal{G}_\ell(q, G, \rho, \mathcal{I})$ from the induction hypothesis over ℓ . Our goal is to construct $R_{\ell+1}$ and $\mathcal{G}_{\ell+1}$ satisfying the hypothesis for $\ell + 1$.

Consider the family

$$\bigcup_{G \in \mathcal{G}_\ell} \mathcal{I}_G = \{I_1, I_2, \dots, I_m\} \subset \binom{R_\ell}{\rho_\ell}. \quad (3.2)$$

This family is a t -partite t -uniform hypergraph with partition $\{V_{j_i}^q(R_\ell)\}_{i=1}^t$ (see Figure 3.1).

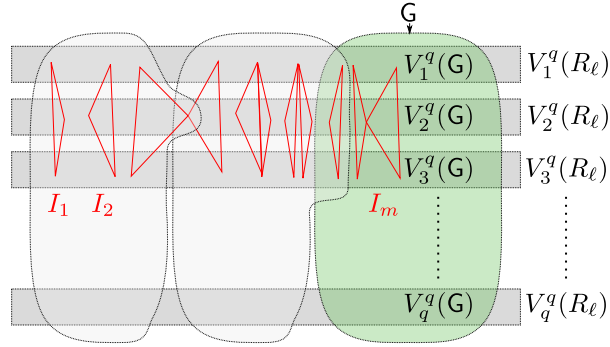
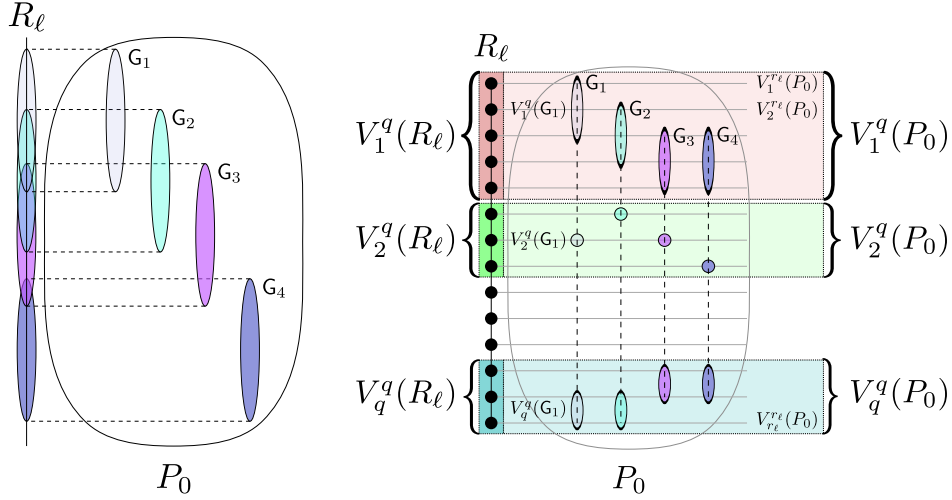


Figure 3.1: An illustration of R_ℓ and $G \in \mathcal{G}_\ell$. Here we assume $t = 3$, $j_1 = 1$, $j_2 = 2$ and $j_3 = 3$. The triples of (3.2) are represented by the crossing triangles.

We now construct a sequence of $|V(R_\ell)|$ -partite graphs P_0, P_1, \dots, P_m , which we will call *pictures*¹, and families $\mathcal{G}(P_k) \subset \binom{P_k}{G}_{\text{Part}(q)}$, $k = 0, 1, \dots, m$. We will then show that $R_{\ell+1} = P_m$ and $\mathcal{G}_{\ell+1} = \mathcal{G}(P_m)$ satisfy conditions (L1), (L2), (L3), (A), and (B). This will establish the induction step and conclude the proof of Lemma 3.19.

¹The name ‘pictures’ was used before, e.g. in [27].



(a) P_0 is a disjoint union of copies of G where each copy is projected by π_0 into a copy of G in \mathcal{G}_ℓ .

(b) P_0 with its coarse q -partition and the refined r_ℓ -partition (see (3.3) and (3.4)). Notice that the copies of G are partite embedded in the q -partite graph P_0 (see Definition 3.17).

Figure 3.2:

Let us start by constructing P_0 (see Figure 3.2). For convenience, let $r_\ell = |V(R_\ell)|$. For each $u \in V(R_\ell)$, let

$$V_u^{r_\ell}(P_0) = \{(u, G) : G \in \mathcal{G}_\ell, V(G) \ni u\}. \quad (3.3)$$

Recalling the total order on $V(R_\ell)$ we may assume in fact that $V(R_\ell) = \{1, 2, \dots, r_\ell\}$. We then impose a total order in $V(P_0)$ that satisfies $V_j^{r_\ell}(P_0) \prec V_{j+1}^{r_\ell}(P_0)$ for all $j = 1, \dots, r_\ell - 1$.

The edges of P_0 are of the form $\{(u, G), (w, G)\}$, where $uw \in E(G)$, $G \in \mathcal{G}_\ell$. Notice that the r_ℓ -partition of P_0 given by (3.3) is indeed such that every edge of P_0 is crossing. We set $\mathcal{G}(P_0)$ to be the set of copies of G in correspondence with \mathcal{G}_ℓ . In particular, $|\mathcal{G}(P_0)| = |\mathcal{G}_\ell|$. Moreover, the projection $\pi_0(u, G) = u$ defines a monotone homomorphism from P_0 to R_ℓ .

Assuming that the hypothesis holds for some $\ell \geq 2$ we will now describe the induction over k .

Induction over k – Hypothesis on P_k and $\mathcal{G}(P_k)$

- (K1) The picture P_k is r_ℓ -partite with classes $V_j^{r_\ell}(P_k)$, $j = 1, \dots, r_\ell$. The projection map $\pi_k: V(P_k) \rightarrow V(R_\ell) = [r_\ell]$ given by $\pi_k(x) = j$ if and only if $x \in V_j^{r_\ell}(P_k)$ is a homomorphism of P_k into R_ℓ . Moreover, $\pi_k(\mathbf{G}) \in \mathcal{G}_\ell$ for every $\mathbf{G} \in \mathcal{G}(P_k)$.
- (K2) The family $\mathcal{G}(P_k)$ is contained in $\binom{P_k}{G}_{\text{Part}(q)}$.
- (K3) The family $\mathcal{G}(P_k)$ satisfies conditions (A) and (B).
- (K4) Every $\mathbf{G} \in \mathcal{G}(P_k)$ is $(\ell + 1)$ -metric in P_k .

Claim 3.22. *The graph P_0 satisfies the induction hypothesis for $k = 0$.*

Since the copies of G in P_0 are vertex-disjoint (and thus metric) and are projected by π_0 into copies of G in R_ℓ it is clear that (K1), (K3) and (K4) hold for P_0 and $\mathcal{G}(P_0)$. It remains to check (K2), namely, that $\mathcal{G}(P_0)$ is contained in $\binom{P_0}{G}_{\text{Part}(q)}$.

We now observe that the q -partition of $V(P_0)$ may be expressed in terms of π_0 as

$$V_j^q(P_0) = \pi_0^{-1}(V_j^q(R_\ell)) = \bigcup_{u \in V_j^q(R_\ell)} V_u^{r_\ell}(P_0) \quad (3.4)$$

for $j = 1, \dots, q$ (see Figure 3.2). For every $\mathbf{G} \in \mathcal{G}(P_0)$, we have $\mathbf{G}' = \pi_0(\mathbf{G}) \in \mathcal{G}_\ell$. From the induction hypothesis over ℓ we have $\mathbf{G} \in \mathcal{G}(P_0) \subset \binom{R_\ell}{G}_{\text{Part}(q)}$ and hence the isomorphism $\sigma: V(G) \rightarrow V(\mathbf{G}')$ must be a partite isomorphism. Then $\pi_0^{-1} \circ \sigma: V(G) \rightarrow V(\mathbf{G})$ is a partite isomorphism of G into \mathbf{G} by our choice of $V_j^q(P_0)$, $j = 1, \dots, q$.

Hence P_0 satisfies the induction hypothesis for $k = 0$ and Claim 3.22 is proved.

Suppose that P_k , $\mathcal{G}(P_k)$, and π_k , $k \geq 0$, are constructed and satisfy the induction hypothesis. Since every $G \in \mathcal{G}(P_k)$ is $(\ell + 1)$ -metric in P_k , it follows that $\mathcal{I}_G \subset \binom{G}{\rho_{\ell+1}} \subset \binom{P_k}{\rho_{\ell+1}}$ for every $G \in \mathcal{G}(P_k)$. Define

$$\mathcal{I}^{(k)} = \left\{ I \in \bigcup_{G \in \mathcal{G}(P_k)} \mathcal{I}_G : \pi_k(I) = I_{k+1} \right\} \subset \binom{P_k}{\rho_{\ell+1}}, \quad (3.5)$$

where the (ρ_ℓ, R_ℓ) -tuple $I_{k+1} = \{w_1, w_2, \dots, w_t\}$ is the $(k + 1)$ th tuple from equation (3.2).

Observe that by construction, $\mathcal{I}^{(k)}$ is a t -partite t -uniform hypergraph. Indeed, every tuple in $\mathcal{I}^{(k)}$ is crossing with respect to the sets $\{\pi_k^{-1}(u) = V_u^{r_\ell}(P_k)\}_{u \in I_{k+1}}$. To construct P_{k+1} we invoke our induction assumption over ℓ with

- r_ℓ in place of q ;
- P_k in place of G ;
- $\mathcal{I}^{(k)} \subset \binom{P_k}{\rho_{\ell+1}} \subset \binom{P_k}{\rho_\ell}$ in place of \mathcal{I} .

We then obtain the graph $P_{k+1} = R_\ell(r_\ell, P_k, \rho, \mathcal{I}^{(k)})$ and a family $\mathcal{P}_{k+1} = \mathcal{G}_\ell(r_\ell, P_k, \rho, \mathcal{I}^{(k)}) \subset \binom{P_{k+1}}{P_k}_{\text{Part}(r_\ell)}$ satisfying conditions (L1), (L2), (L3), (A) and (B). More specifically, the following holds:

- (1) _{$k+1$} For every 2-coloring of the (ρ_ℓ, P_{k+1}) -tuples in $\bigcup_{P \in \mathcal{P}_{k+1}} \mathcal{I}_P^{(k)}$ there exists $P \in \mathcal{P}_{k+1}$ such that $\mathcal{I}_P^{(k)} \subset \binom{P}{\rho_{\ell+1}} \subset \binom{P_{k+1}}{\rho_\ell}$ is monochromatic (recall that the hypergraph $\mathcal{I}_P^{(k)}$ is an isomorphic copy of $\mathcal{I}^{(k)}$ in P).
- (2) _{$k+1$} Every $P \in \mathcal{P}_{k+1}$ is ℓ -metric in P_{k+1} .
- (3) _{$k+1$} $E(P_{k+1}) = \bigcup_{P \in \mathcal{P}_{k+1}} E(P)$.

- (A)_{k+1} If $P^1, P^2 \in \mathcal{P}_{k+1}$ are distinct and $u \in V(P^1) \cap V(P^2)$ then there are $(\rho_{\ell+1}, P^j)$ -tuples $I_*^j \in \mathcal{I}_{P^j}^{(k)}$, $j = 1, 2$, such that $u \in I_*^1 \cap I_*^2$.
- (B)_{k+1} If $P^1, P^2 \in \mathcal{P}_{k+1}$ are distinct and $u, v \in V(P^1) \cap V(P^2)$ then either
- (B1)_{k+1} there exist $(\rho_{\ell+1}, P^j)$ -tuples $I_*^j \in \mathcal{I}_{P^j}^{(k)}$, $j = 1, 2$, such that $\{u, v\} \subset I_*^1 \cap I_*^2$ or
- (B2)_{k+1} the isomorphisms $\phi_j: V(P^j) \rightarrow V(P_k)$, $j = 1, 2$, satisfy $\phi_1(u) = \phi_2(u)$ and $\phi_1(v) = \phi_2(v)$.

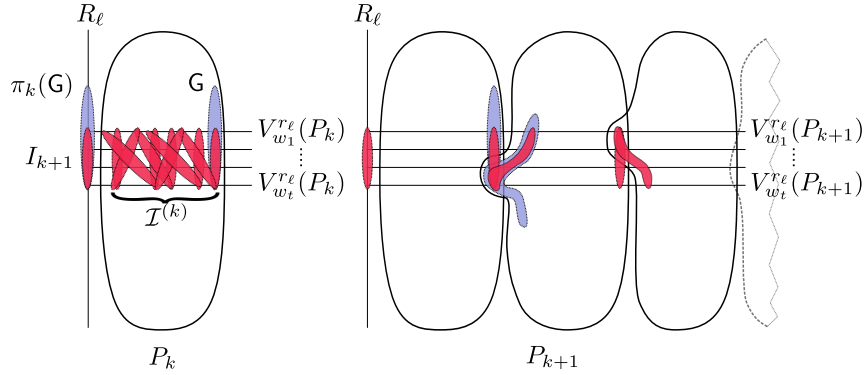
Remark 3.23. The graph P_{k+1} is obtained by *amalgamating* copies of P_k in a particular way determined by the induction over ℓ . For instance, due to (A)_{k+1}, only vertices in $V_j^{r_\ell}(P_{k+1})$, with $j \in I_{k+1}$, may be shared by distinct copies of P_k in P_{k+1} .

See Figure 3.3 for an illustration of the amalgamation.

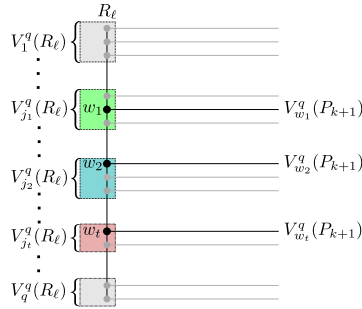
The projection $\pi_{k+1}: V(P_{k+1}) \rightarrow V(R_\ell)$ is defined in terms of the partition $\{V_j^{r_\ell}(P_{k+1})\}_{j=1}^{r_\ell}$ given by the induction hypothesis over ℓ . More concretely, $\pi_{k+1}(u) = j$ if and only if $u \in V_j^{r_\ell}(P_{k+1})$. For any $P \in \mathcal{P}_{k+1}$, with isomorphism $\phi: V(P_k) \rightarrow V(P)$, we claim that the following diagram commutes:

$$\begin{array}{ccccc}
 P_k & \xleftarrow{\phi} & P & \xrightarrow{i} & P_{k+1} \\
 \downarrow \pi_k & & & \searrow \pi_{k+1} & \\
 R_\ell & & & &
 \end{array} \tag{3.6}$$

Indeed, because ϕ is a partite embedding, we have $\phi(V_j^{r_\ell}(P_k)) \subset V_j^{r_\ell}(P_{k+1})$ for all $j = 1, \dots, r_\ell$. Hence, for $u \in V(P_k)$, $\pi_k(u) = j$ if and only if $u \in V_j^{r_\ell}(P_k)$ if and only if $\phi(u) \in V_j^{r_\ell}(P_{k+1})$ if and only if $\pi_{k+1} \circ \phi(u) = j$. This shows that $\pi_k = \pi_{k+1} \circ \phi$ and thus the diagram (3.6) commutes.



(a) The picture P_{k+1} is obtained from picture P_k by applying the induction hypothesis over ℓ . To simplify the figure, the vertical order of the vertices in the illustration does not coincide with the order of $V(R_\ell) = \{1, \dots, r_\ell\}$.



(b) The tuple $I_{k+1} = \{w_1, \dots, w_t\}$ and the corresponding classes $V_{w_i}^q(P_{k+1})$ are drawn according to the order of $V(R_\ell)$.

Figure 3.3:

NB: It is rather cumbersome to draw the elements of (ρ_ℓ, R_ℓ) -tuples in their correct order. For this reason we will refrain from having $V(R_\ell)$ vertically ordered in the next figures.

Constructing the q -partition of P_{k+1} . The graph P_{k+1} is q -partite with partition given by the classes

$$V_j^q(P_{k+1}) = \pi_{k+1}^{-1}(V_j^q(R_\ell)) = \bigcup_{u \in V_j^q(R_\ell)} V_u^{r_\ell}(P_{k+1}), \quad j = 1, \dots, q. \quad (3.7)$$

Notice that because $V_1^q(R_\ell) \prec V_2^q(R_\ell) \prec \dots \prec V_q^q(R_\ell)$ and $V_1^{r_\ell}(P_{k+1}) \prec \dots \prec V_{r_\ell}^{r_\ell}(P_{k+1})$ we also have $V_1^q(P_{k+1}) \prec \dots \prec V_q^q(P_{k+1})$ —see Figure 3.4.

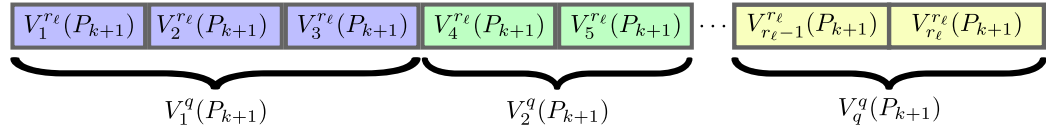


Figure 3.4: The linearly ordered vertices of P_{k+1} (from left to right) and both q - and r_ℓ -partitions. Note that the r_ℓ -partition of P_{k+1} is a refinement of its q -partition.

Constructing the family $\mathcal{G}(P_{k+1}) \subset \binom{P_{k+1}}{G}_{\text{Part}(q)}$. For any $\mathbf{P} \in \mathcal{P}_{k+1} \subset \binom{P_{k+1}}{P_k}_{\text{Part}(r_\ell)}$, given the (unique monotone) isomorphism $\phi: V(P_k) \rightarrow V(\mathbf{P})$, set

$$\mathcal{G}(\mathbf{P}) = \{\phi(\mathbf{G}) : \mathbf{G} \in \mathcal{G}(P_k)\}.$$

Define

$$\mathcal{G}(P_{k+1}) = \bigcup_{\mathbf{P} \in \mathcal{P}_{k+1}} \mathcal{G}(\mathbf{P}). \quad (3.8)$$

Observe that there is a rich structure of copies of G in P_{k+1} which is inherited by the many overlapping copies of P_k in P_{k+1} .

We will now start the proof of the induction step over k . The proof is divided in several claims, one for each of the conditions (K1)–(K4) of the induction over k (see the box above).

Claim 3.24. *Condition (K1) holds for P_{k+1} , namely, the projection map π_{k+1} is a homomorphism of P_{k+1} into R_ℓ satisfying $\pi_{k+1}(\mathbf{G}) \in \mathcal{G}_\ell$ for every $\mathbf{G} \in \mathcal{G}(P_{k+1})$.*

We will start by showing that the projection map π_{k+1} is a homomorphism of P_{k+1} into R_ℓ . To this end we must prove that $\pi_{k+1}(E(P_{k+1})) \subset E(R_\ell)$.

By the induction hypothesis over P_k , the projection $\pi_k: V(P_k) \rightarrow V(R_\ell)$ is a homomorphism. Consequently, the diagram (3.6) shows that for every $\mathbf{P} \in \mathcal{P}_{k+1}$, the map $\pi_{k+1}|_{V(\mathbf{P})}$ is a homomorphism of \mathbf{P} into R_ℓ and thus $\pi_{k+1}(E(\mathbf{P})) \subset E(R_\ell)$. Since by (3)_{k+1} we have $E(P_{k+1}) = \bigcup_{\mathbf{P} \in \mathcal{P}_{k+1}} E(\mathbf{P})$ it follows that $\pi_{k+1}(E(P_{k+1})) \subset E(R_\ell)$.

It remains to show that $\pi_{k+1}(\mathbf{G}) \in \mathcal{G}_\ell$ for every

$$\mathbf{G} \in \mathcal{G}(P_{k+1}) = \bigcup_{\mathbf{P} \in \mathcal{P}_{k+1}} \mathcal{G}(\mathbf{P}).$$

For any $\mathbf{G} \in \mathcal{G}(\mathbf{P})$, $\mathbf{P} \in \mathcal{P}_{k+1}$, we have $\phi^{-1}(\mathbf{G}) \in \mathcal{G}(P_k)$, where $\phi: V(P_k) \rightarrow V(\mathbf{P})$ is the unique isomorphism. By the induction hypothesis (K1) over P_k it follows that $\pi_k(\phi^{-1}(\mathbf{G})) \in \mathcal{G}_\ell$. Since the diagram (3.6) commutes,

$$\pi_k(\phi^{-1}(\mathbf{G})) = \pi_{k+1} \circ \phi(\phi^{-1}(\mathbf{G})) = \pi_{k+1}(\mathbf{G})$$

and thus $\pi_{k+1}(\mathbf{G}) \in \mathcal{G}_\ell$. This concludes the proof that (K1) holds for P_{k+1} .

Claim 3.25. *Condition (K2) holds for P_{k+1} , namely, $\mathcal{G}(P_{k+1}) \subset \binom{P_{k+1}}{G}_{\text{Part}(q)}$.*

First observe that for every $\mathbf{P} \in \mathcal{P}_{k+1} \subset \binom{P_{k+1}}{P_k}_{\text{Part}(r_\ell)}$ we have $V_j^{r_\ell}(\mathbf{P}) \subset V_j^{r_\ell}(P_{k+1})$ for all $j = 1, \dots, r_\ell$. Consequently,

$$V_j^q(\mathbf{P}) \stackrel{(3.4),(3.7)}{=} \bigcup_{u \in V_j^q(R_\ell)} V_u^{r_\ell}(\mathbf{P}) \subset \bigcup_{u \in V_j^q(R_\ell)} V_u^{r_\ell}(P_{k+1}) \stackrel{(3.7)}{=} V_j^q(P_{k+1}) \quad (3.9)$$

for all $j = 1, \dots, q$.

For all $\mathbf{G} \in \mathcal{G}(\mathbf{P}) \subset \binom{\mathbf{P}}{G}_{\text{Part}(q)}$ we have $V_j^q(\mathbf{G}) \subset V_j^q(\mathbf{P}) \subset V_j^q(R_\ell)$. It follows that

$$\mathcal{G}(P_{k+1}) = \bigcup_{\mathbf{P} \in \mathcal{G}(P_{k+1})} \mathcal{G}(\mathbf{P}) \subset \binom{P_{k+1}}{G}_{\text{Part}(q)}.$$

Therefore the claim is proved.

Claim 3.26 (Auxiliary). *If $\mathbf{P}^1, \mathbf{P}^2 \in \mathcal{P}_{k+1}$ are distinct and $u \in V(\mathbf{P}^1) \cap V(\mathbf{P}^2)$ then $\pi_{k+1}(u) \in I_{k+1}$. Consequently, for each $\mathbf{G} \in \mathcal{G}(P_{k+1})$ there is a unique $\mathbf{P} \in \mathcal{P}_{k+1}$ such that $\mathbf{G} \subset \mathbf{P}$.*

From condition (A)_{k+1} there exist $I_*^j \in \mathcal{I}_{\mathbf{P}^j}^{(k)}$, $j = 1, 2$, such that $u \in I_*^1 \cap I_*^2$. From diagram (3.6) we conclude that the isomorphism $\phi_1: V(P_k) \rightarrow V(\mathbf{P}^1)$ satisfies $\pi_k = \pi_{k+1} \circ \phi_1$. Because $I^1 = \phi_1^{-1}(I_*^1) \in \mathcal{I}^{(k)}$, we have

$$\pi_{k+1}(I_*^1) = \pi_{k+1} \circ \phi_1(I^1) = \pi_k(I^1) \stackrel{(3.5)}{=} I_{k+1}.$$

Consequently, $\pi_{k+1}(u) \in I_{k+1}$.

Since each $\mathbf{G} \in \mathcal{G}(P_{k+1})$ is mapped by π_{k+1} onto a member of \mathcal{G}_ℓ , the projection must be one-to-one over $V(\mathbf{G})$. Therefore $|\pi_{k+1}(V(\mathbf{G}))| = |V(\mathbf{G})| > t$ and thus $\pi_{k+1}(V(\mathbf{G})) \not\subset I_{k+1}$. It follows that $V(\mathbf{G}) \not\subset V(\mathbf{P}^1) \cap V(\mathbf{P}^2)$.

Claim 3.27. *Condition (K3) holds for P_{k+1} , namely, $\mathcal{G}(P_{k+1})$ satisfies the intersection conditions (A) and (B).*

Let $\mathbf{G}_1, \mathbf{G}_2 \in \mathcal{G}(P_{k+1})$ be distinct and arbitrary. By Claim 3.26 there are unique $\mathbf{P}^1, \mathbf{P}^2 \in \mathcal{P}_{k+1}$ such that $\mathbf{G}_j \subset \mathbf{P}^j$, $j = 1, 2$. If $\mathbf{P}^1 = \mathbf{P}^2$ then the induction hypothesis over $\mathbf{P}^1 = \mathbf{P}^2 \cong P_k$ implies that both conditions (A) and (B) hold for \mathbf{G}_1 and \mathbf{G}_2 . Hence let us suppose that $\mathbf{P}^1 \neq \mathbf{P}^2$.

Proof of (A). Since \mathcal{P}_{k+1} satisfies (A)_{k+1}, it follows that for any $u \in V(\mathbf{G}_1) \cap V(\mathbf{G}_2) \subset V(\mathbf{P}^1) \cap V(\mathbf{P}^2)$ there exist $(\rho_{\ell+1}, \mathbf{P}^j)$ -tuples $I_*^j \in \mathcal{I}_{\mathbf{P}^j}^{(k)}$, $j = 1, 2$, such that $u \in I_*^1 \cap I_*^2$. Let $\mathbf{G}_j^* \in \mathcal{G}(\mathbf{P}^j)$ be such that $I_*^j \in \mathcal{I}_{\mathbf{G}_j^*}$. For each $j = 1, 2$ we are going to obtain $I^j \in \mathcal{I}_{\mathbf{G}_j}$ with $u \in I^1 \cap I^2$.

First we show that there exists $I^1 \in \mathcal{I}_{\mathbf{G}_1}$ such that $u \in I^1$. If $\mathbf{G}_1 = \mathbf{G}'_1$, we are done by taking $I^1 = I_*^1$ so let us assume that $\mathbf{G}_1 \neq \mathbf{G}'_1$. The induction hypothesis (K3) applied to $\mathbf{P}^1 \cong P_k$ implies that $\mathcal{G}(\mathbf{P}^1)$ satisfies condition (A): since $u \in V(\mathbf{G}_1) \cap V(\mathbf{G}'_1)$ there exists $I^1 \in \mathcal{I}_{\mathbf{G}_1}$ such that $u \in I^1 \cap I_*^1$. Similarly we find $I^2 \in \mathcal{I}_{\mathbf{G}_2}$ such that $u \in I^2$ and hence $u \in I^1 \cap I^2$ thus proving that condition (A) holds for $\mathcal{G}(P_{k+1})$.

Proof of (B). Suppose that there are two distinct $u, v \in V(\mathbf{G}_1) \cap V(\mathbf{G}_2) \subset V(\mathbf{P}^1) \cap V(\mathbf{P}^2)$. Condition (B) $_{k+1}$ applies to \mathcal{P}_{k+1} which means that either (B1) $_{k+1}$ or (B2) $_{k+1}$ holds for $u, v, \mathbf{P}^1, \mathbf{P}^2$.

In case (B1) $_{k+1}$ holds for $u, v, \mathbf{P}^1, \mathbf{P}^2$ we will show that (B1) holds for $u, v, \mathbf{G}^1, \mathbf{G}^2$. Consider the $(\rho_{\ell+1}, \mathbf{P}^j)$ -tuples $I_*^j \in \mathcal{I}_{\mathbf{P}^j}^{(k)}$, $j = 1, 2$, such that $u, v \in I_*^1 \cap I_*^2$. Let $\mathbf{G}_j^* \in \mathcal{G}(\mathbf{P}^j)$ be such that $I_*^j \in \mathcal{I}_{\mathbf{G}_j^*}$, $j = 1, 2$.

First we will show that there exists $I^1 \in \mathcal{I}_{\mathbf{G}_1}$ such that $u, v \in I^1$. If $\mathbf{G}_1^* = \mathbf{G}_1$, set $I^1 = I_*^1$. Otherwise, observe that $u, v \in V(\mathbf{G}_1) \cap V(\mathbf{G}_1^*)$ and $\mathbf{G}_1, \mathbf{G}_1^* \in \mathcal{G}(\mathbf{P}^1)$. We may now use the induction hypothesis (K β) on $\mathbf{P}^1 \cong P_k$ which states that condition (B) holds for $\mathcal{G}(\mathbf{P}^1)$. In particular, either (B1) applies and we immediately obtain $I^1 \in \mathcal{I}_{\mathbf{G}_1}$ satisfying $u, v \in I^1 \cap I_*^1$ or (B2) applies and the isomorphisms σ_1, σ_1^* from $\mathbf{G}_1, \mathbf{G}_1^*$ to G are such that $\sigma_1(u) = \sigma_1^*(u)$ and $\sigma_1(v) = \sigma_1^*(v)$. However, in the latter case, set $I^1 = \sigma_1^{-1} \circ \sigma_1^*(I_*^1) \in \mathcal{I}_{\mathbf{G}_1}$ and observe that $u, v \in I^1$.

In the same way we obtain $I^2 \in \mathcal{I}_{\mathbf{G}_2}$ such that $u, v \in I^2$ and thus establish that (B1) holds for $u, v, \mathbf{G}^1, \mathbf{G}^2$.

Consider now the case that (B2) $_{k+1}$ holds for $u, v, \mathbf{P}^1, \mathbf{P}^2$. In other words, for the (unique) isomorphisms $\phi_j: V(\mathbf{P}^j) \rightarrow V(P_k)$, $j = 1, 2$, we have $\phi_1(u) = \phi_2(u)$ and $\phi_1(v) = \phi_2(v)$. Let $\mathbf{G}_j^* = \phi_j(\mathbf{G}_j) \in \mathcal{G}(P_k)$, $j = 1, 2$ and set $x = \phi_1(u)$, $y = \phi_1(v)$. Since $x, y \in V(\mathbf{G}_1^*) \cap V(\mathbf{G}_2^*)$ and $\mathcal{G}(P_k)$ satisfies condition (B), one of the following must hold:

- *There exist $I_*^j \in \mathcal{I}_{\mathbf{G}_j^*}$, $j = 1, 2$, such that $x, y \in I_*^1 \cap I_*^2$.*
Letting $I^j = \phi_j^{-1}(I_*^j) \in \mathcal{I}_{\mathbf{G}_j}$ for $j = 1, 2$, we have $u, v \in I^1 \cap I^2$. Hence condition (B1) holds for $u, v, \mathbf{G}^1, \mathbf{G}^2$.
- *The isomorphisms $\sigma_j^*: V(\mathbf{G}_j^*) \rightarrow V(G)$ satisfy $\sigma_1^*(x) = \sigma_2^*(x)$, $\sigma_1^*(y) = \sigma_2^*(y)$.*

Since the (unique) isomorphisms $\sigma_j: V(\mathbf{G}_j) \rightarrow V(G)$ satisfy

$$\sigma_j = \sigma_j^* \circ \phi_j,$$

we have

$$\sigma_1(u) = \sigma_1^*(\phi_1(u)) = \sigma_1^*(x) = \sigma_2^*(x) = \sigma_2^*(\phi_2(u)) = \sigma_2(u)$$

and, similarly, $\sigma_1(v) = \sigma_2(v)$. Consequently, condition (B2) holds for $u, v, \mathbf{G}^1, \mathbf{G}^2$.

This concludes the proof that $\mathcal{G}(P_{k+1})$ satisfies condition (B).

Before showing that condition (K4) holds we will prove two auxiliary claims.

Claim 3.28 (Auxiliary). *Suppose that $\mathbf{P}^1, \mathbf{P}^2 \in \mathcal{P}_{k+1}$, $u, v \in V(\mathbf{P}^1) \cap V(\mathbf{P}^2)$, $d_1 = \text{dist}_{\mathbf{P}^1}(u, v)$ and $d_2 = \text{dist}_{\mathbf{P}^2}(u, v)$. Then either $\min\{d_1, d_2\} \geq \ell + 1$ or $d_1 = d_2$.*

Without loss of generality assume that $\mathbf{P}^1 \neq \mathbf{P}^2$, $d_1 = \min\{d_1, d_2\} \leq \ell$, and $u \neq v$. Since \mathcal{P}_{k+1} satisfies condition (B) $_{k+1}$, either condition (B1) $_{k+1}$ or condition (B2) $_{k+1}$ applies to $u, v \in V(\mathbf{P}^1) \cap V(\mathbf{P}^2)$.

Suppose first that (B2) $_{k+1}$ holds for $u, v, \mathbf{P}^1, \mathbf{P}^2$, namely, the isomorphisms $\phi_j: V(\mathbf{P}^j) \rightarrow V(P_k)$ are such that $\phi_1(u) = \phi_2(u)$ and $\phi_1(v) = \phi_2(v)$. In this case, $\phi = \phi_2^{-1} \circ \phi_1: V(\mathbf{P}^1) \rightarrow V(\mathbf{P}^2)$ is the isomorphism from \mathbf{P}^1 to \mathbf{P}^2 . Moreover, ϕ satisfies $\phi(u) = u$ and $\phi(v) = v$. It follows that

$$\text{dist}_{\mathbf{P}^1}(u, v) = \text{dist}_{\mathbf{P}^2}(\phi(u), \phi(v)) = \text{dist}_{\mathbf{P}^2}(u, v).$$

The equality in this case holds even for arbitrary distances d_1, d_2 .

Suppose now that condition (B1) $_{k+1}$ holds for $u, v, \mathbf{P}^1, \mathbf{P}^2$, namely, there exist $(\rho_{\ell+1}, \mathbf{P}^j)$ -tuples $I^j \in \mathcal{I}_{\mathbf{P}^j}^{(k)} \subset \binom{\mathbf{P}^j}{\rho_{\ell+1}}$, $j = 1, 2$, such that $u, v \in I^1 \cap I^2$.

Let $\mathbf{G}_j \in \mathcal{G}(\mathbf{P}^j)$ be such that $I^j \in \mathcal{I}_{\mathbf{G}_j}$ for $j = 1, 2$. By the induction hypothesis over $\mathbf{P}^j \cong P_k$, the graph \mathbf{G}_j is $(\ell + 1)$ -metric in \mathbf{P}^j . In particular, $\text{dist}_{\mathbf{P}^1}(u, v) = d_1 \leq \ell$ implies that $\text{dist}_{\mathbf{G}_1}(u, v) = d_1$.

Recall that

$$\pi_{k+1}(I^1) = \pi_{k+1}(I^2) = I_{k+1} = \{w_1 < w_2 < \cdots < w_t\} \subset V(R_\ell).$$

In particular, $\pi_{k+1}(u) = w_a$ and $\pi_{k+1}(v) = w_b$, for some $1 \leq a, b \leq t$. Consequently, u is the a th element of I^j ($j = 1, 2$) and v is the b th element of I^j ($j = 1, 2$). Because $\text{dist}_{\mathbf{G}_1}(u, v) = d_1 \leq \ell$,

$$d_1 = \underbrace{\text{dist}_{\mathbf{G}_1}(u, v)}_{I^1 \text{ is a } (\rho_{\ell+1}, \mathbf{G}_1)\text{-tuple}} = \overbrace{\rho(a, b)}^{I^2 \text{ is a } (\rho_{\ell+1}, \mathbf{G}_2)\text{-tuple}} = \text{dist}_{\mathbf{G}_2}(u, v) \geq \text{dist}_{\mathbf{P}^2}(u, v) = d_2 = \max\{d_1, d_2\}$$

and thus $d_1 = d_2$. Hence, Claim 3.28 follows.

Claim 3.29 (Auxiliary). *Suppose that $\mathbf{G}_1, \mathbf{G}_2 \in \mathcal{G}_\ell$ and there exist distinct $u, v \in V(\mathbf{G}_1) \cap V(\mathbf{G}_2)$. Moreover, assume that there exists $I^1 \in \mathcal{I}_{\mathbf{G}_1}$ such that $u, v \in I^1$. Then there exists $I^2 \in \mathcal{I}_{\mathbf{G}_2}$ such that $u, v \in I^2$.*

If $\mathbf{G}_1 = \mathbf{G}_2$ then the claim is trivial so let us assume the graphs are distinct. By assumption, \mathcal{G}_ℓ satisfies condition (B). If (B1) holds then the existence of I^2 is immediate.

If, on the other hand, (B2) holds, then the isomorphisms $\sigma_j: V(\mathbf{G}_j) \rightarrow V(G)$ satisfy $\sigma_1(u) = \sigma_2(u)$ and $\sigma_1(v) = \sigma_2(v)$. The map $\sigma = \sigma_2^{-1} \circ \sigma_1: V(\mathbf{G}_1) \rightarrow V(\mathbf{G}_2)$ is clearly the isomorphism from \mathbf{G}_1 to \mathbf{G}_2 . Since $\sigma(u) = u$ and $\sigma(v) = v$, it follows that $I^2 = \sigma(I^1) \in \mathcal{I}_{\mathbf{G}_2}$ satisfies the conditions of the claim.

Claim 3.30. *Condition (K4) holds for P_{k+1} , namely, every $\mathbf{G} \in \mathcal{G}(P_{k+1})$ is $(\ell + 1)$ -metric.*

For an arbitrary $\mathbf{G} \in \mathcal{G}(P_{k+1})$ and $u, v \in V(\mathbf{G})$ we will show the following:

- (i) If $\text{dist}_{\mathbf{G}}(u, v) \leq \ell$ then $\text{dist}_{P_{k+1}}(u, v) = \text{dist}_{\mathbf{G}}(u, v)$.
- (ii) If $\text{dist}_{\mathbf{G}}(u, v) \geq \ell + 1$ then $\text{dist}_{P_{k+1}}(u, v) \geq \ell + 1$.

The two conditions above imply that \mathbf{G} is $(\ell + 1)$ -metric in P_{k+1} . Indeed, when $\text{dist}_{\mathbf{G}}(u, v) = \ell + 1$ we have

$$\ell + 1 \stackrel{(ii)}{\leq} \text{dist}_{P_{k+1}}(u, v) \leq \text{dist}_{\mathbf{G}}(u, v) = \ell + 1$$

and equality holds. Consequently, for all $u, v \in V(\mathbf{G})$ we have $\text{dist}_{P_{k+1}}(u, v) = \text{dist}_{\mathbf{G}}(u, v)$ whenever $\text{dist}_{\mathbf{G}}(u, v) \leq \ell + 1$ and $\text{dist}_{P_{k+1}}(u, v) \geq \ell + 1$ whenever $\text{dist}_{\mathbf{G}}(u, v) > \ell + 1$.

We start by proving (i). Assume that $\text{dist}_{\mathbf{G}}(u, v) \leq \ell$. If $\text{dist}_{P_{k+1}}(u, v) < \text{dist}_{\mathbf{G}}(u, v)$, consider a shortest path $P(u, v)$ in P_{k+1} . The projection of this path, $\pi_{k+1}(P(u, v))$, is a trail in R_{ℓ} starting at $x = \pi_{k+1}(u)$ and ending at $y = \pi_{k+1}(v)$. Since $\mathbf{G}' = \pi_{k+1}(\mathbf{G}) \in \mathcal{G}_{\ell}$ and π_{k+1} is an isomorphism between \mathbf{G} and \mathbf{G}' , it follows that $\text{dist}_{\mathbf{G}'}(x, y) = \text{dist}_{\mathbf{G}}(u, v) \leq \ell$. On the other hand, the trail $\pi_{k+1}(P(u, v))$ shows that

$$\begin{aligned} \text{dist}_{R_{\ell}}(x, y) &\leq |\pi_{k+1}(P(u, v))| \leq |P(u, v)| \\ &= \text{dist}_{P_{k+1}}(u, v) < \text{dist}_{\mathbf{G}}(u, v) = \text{dist}_{\mathbf{G}'}(x, y). \end{aligned} \tag{3.10}$$

However, this contradicts the fact that \mathbf{G}' is ℓ -metric in R_{ℓ} .

Now let us prove (ii). Suppose for the sake of contradiction that there exists a path $P(u, v)$ in P_{k+1} with

$$|P(u, v)| \leq \ell \quad \text{and} \quad \text{dist}_{\mathbf{G}}(u, v) \geq \ell + 1. \tag{3.11}$$

By Claim 3.26, there exists a unique $\mathbf{P}^1 \in \mathcal{P}_{k+1} \subset \binom{P_{k+1}}{P_k}_{\text{Part}(r_{\ell})}$ such that $\mathbf{G} \subset \mathbf{P}^1$.

Fact 3.31. The path $P(u, v)$ satisfies the following:

- (a) $P(u, v) \not\subset \mathbf{P}^1$;
- (b) there is no internal vertex of $P(u, v)$ in $V(\mathbf{P}^1)$, hence $E(P(u, v)) \cap E(\mathbf{P}^1) = \emptyset$;

(c) $\pi_{k+1}(u), \pi_{k+1}(v) \in I_{k+1}$;

(d) $P(u, v) \not\subset P^2$ for every $P^2 \in \mathcal{P}_{k+1}$;

By the induction hypothesis over the picture $P^1 \cong P_k$ the graph G must be $(\ell + 1)$ -metric in P^1 and thus

$$\text{dist}_{P^1}(u, v) \geq \ell + 1. \quad (3.12)$$

In particular, (a) holds, that is, the path $P(u, v)$ cannot be entirely contained in P^1 .

Suppose that the path $P(u, v)$ contains an internal vertex $w \in V(P^1)$. Then the (non-trivial) induced sub-paths $P(u, w)$ and $P(w, v)$ have length strictly shorter than ℓ . Our assumption that P^1 is ℓ -metric in P_{k+1} implies that $|P(u, w)| \geq \text{dist}_{P^1}(u, w)$ and $|P(w, v)| \geq \text{dist}_{P^1}(w, v)$. Therefore

$$\begin{aligned} |P(u, v)| &= |P(u, w)| + |P(w, v)| \geq \text{dist}_{P^1}(u, w) + \text{dist}_{P^1}(w, v) \\ &\geq \text{dist}_{P^1}(u, v) \stackrel{(3.12)}{\geq} \ell + 1, \end{aligned} \quad (3.13)$$

which contradicts the fact that $|P(u, v)| \leq \ell$. Therefore (b) holds.

Because of (b), the edge of the path incident to u , say $e = \{u, w\}$, must be contained in some $P^2 \in \mathcal{P}_{k+1}$, $P^2 \neq P^1$, otherwise w would be an internal vertex of $P(u, v)$. In particular, $u \in V(P^1) \cap V(P^2)$. From Claim 3.26 we conclude that $\pi_{k+1}(u) \in I_{k+1}$. For the same reason we conclude that $\pi_{k+1}(v) \in I_{k+1}$ and therefore (c) holds.

To show that (d) is satisfied, suppose that $P(u, v) \subset P^2$ for some $P^2 \in \mathcal{P}_{k+1}$, $P^2 \neq P^1$. Then $d_2 = \text{dist}_{P^2}(u, v) \leq \ell$. From Claim 3.28 we conclude that

$$\text{dist}_{P^1}(u, v) = d_1 = d_2 = \ell,$$

which contradicts (3.12). Therefore (d) holds.

We now return to the proof of Claim 3.30(ii). From (a)–(d) we conclude that the path $P(u, v)$ can be decomposed into sub-paths contained in at

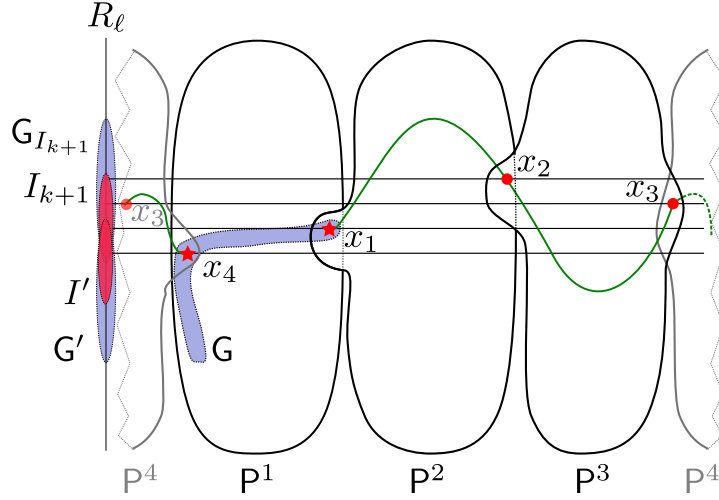


Figure 3.5: An illustration of a path $P(u, v)$ and its sub-paths from case (ii) of Claim 3.30 with $u = x_1$ and $v = x_4$. We also have $t = 4$, $a_1 = 3$, $a_2 = 1$, $a_3 = 2$ and $a_4 = 4$. The vertex x_3 is repeated because P^4 is wrapped around and effectively intersects both P^3 and P^1 . Note that $G' = \pi_{k+1}(G)$ and that $G_{I_{k+1}}$ contains I_{k+1} .

least two distinct copies of P_k in \mathcal{P}_{k+1} . Therefore we may find vertices $u = x_1, x_2, \dots, x_r = v$, $r \geq 3$, belonging to $P(u, v)$ such that each (non-trivial) sub-path $P(x_j, x_{j+1})$, $j = 1, \dots, r - 1$, is entirely contained in some $P^{j+1} \in \mathcal{P}_{k+1}$, and $P^{j+1} \neq P^{j+2}$ for $j = 1, \dots, r - 2$ (see the illustration in Figure 3.5).

Note that each $P(x_j, x_{j+1})$ has length at most $\ell - 1$ since the sum of the lengths of each sub-path equals $|P(u, v)| \leq \ell$. From Claim 3.26 and (c) we infer that $\pi_{k+1}(x_j) \in I_{k+1} = \{w_1 < w_2 < \dots < w_t\}$ for $j = 1, \dots, r$. For each $j = 1, \dots, r$, let $a_j \in [t]$ be such that $\pi_{k+1}(x_j) = w_{a_j}$.

For every $j = 1, \dots, r - 1$, the projection $\pi_{k+1}(P(x_j, x_{j+1}))$ is a trail connecting w_{a_j} and $w_{a_{j+1}}$ of length $|P(x_j, x_{j+1})| \leq \ell - 1$. Consequently, $\text{dist}_{R_\ell}(w_{a_j}, w_{a_{j+1}}) \leq \ell - 1$. Let $G_{I_{k+1}} \in \mathcal{G}_\ell \subset \binom{R_\ell}{G}_{\text{Part}(q)}$ be such that $I_{k+1} \in$

$\mathcal{I}_{\mathbf{G}_{I_{k+1}}} \subset \binom{\mathbf{G}_{I_{k+1}}}{\rho_{\ell+1}}$. Since $\mathbf{G}_{I_{k+1}}$ is ℓ -metric in R_ℓ it follows that

$$\text{dist}_{\mathbf{G}_{I_{k+1}}}(w_{a_j}, w_{a_{j+1}}) = \text{dist}_{R_\ell}(w_{a_j}, w_{a_{j+1}}) \leq |\mathbf{P}(x_j, x_{j+1})| \leq \ell - 1.$$

Because $I_{k+1} \in \binom{\mathbf{G}_{I_{k+1}}}{\rho_{\ell+1}}$ we must have $\text{dist}_{\mathbf{G}_{I_{k+1}}}(w_{a_j}, w_{a_{j+1}}) = \rho(a_j, a_{j+1})$ and thus

$$\begin{aligned} |\mathbf{P}(u, v)| &= \sum_{j=1}^{r-1} |\mathbf{P}(x_j, x_{j+1})| \geq \sum_{j=1}^{r-1} \text{dist}_{\mathbf{G}_{I_{k+1}}}(w_{a_j}, w_{a_{j+1}}) \\ &= \sum_{j=1}^{r-1} \rho(a_j, a_{j+1}) \geq \rho(a_1, a_r), \end{aligned} \tag{3.14}$$

where in the last part we used the triangle inequality.

Let $\mathbf{G}' = \pi_{k+1}(\mathbf{G}) \in \mathcal{G}_\ell$. Notice that $w_{a_1} = \pi_{k+1}(u), w_{a_r} = \pi_{k+1}(v) \in V(\mathbf{G}') \cap V(\mathbf{G}_{I_{k+1}})$. From Claim 3.29 applied to \mathbf{G}' and $\mathbf{G}_{I_{k+1}}$ we conclude that there exists $I' \in \mathcal{I}_{\mathbf{G}'}$ such that $w_{a_1}, w_{a_r} \in I' \cap I_{k+1}$. Moreover, by the induction hypothesis (over ℓ) every graph in \mathcal{G}_ℓ is partite embedded into R_ℓ , that is $\mathcal{G}_\ell \subset \binom{R_\ell}{G}_{\text{Part}(q)}$. In particular, $V_j^q(\mathbf{G}'), V_j^q(\mathbf{G}_{I_{k+1}}) \subset V_j^q(R_\ell)$ for all $j = 1, \dots, q$. Because $\mathcal{I} \subset \binom{G}{\rho_{\ell+1}}$ is a t -partite hypergraph with classes $\{V_{j_i}^q(G)\}_{i=1}^t$, it follows that $\mathcal{I}_{\mathbf{G}'}$ is t -partite with classes $\{V_{j_i}^q(\mathbf{G}') \subset V_{j_i}^q(R_\ell)\}_{i=1}^t$ and $\mathcal{I}_{\mathbf{G}_{I_{k+1}}}$ is t -partite with classes $\{V_{j_i}^q(\mathbf{G}_{I_{k+1}}) \subset V_{j_i}^q(R_\ell)\}_{i=1}^t$. This ensures that both $I' \in \mathcal{I}_{\mathbf{G}'}$ and $I_{k+1} \in \mathcal{I}_{\mathbf{G}_{I_{k+1}}}$ are crossing with respect to $\{V_{j_i}^q(R_\ell)\}_{i=1}^t$. Therefore, the a_1 th element in I' is w_{a_1} and the a_r th element in I' is w_{a_r} . Because $I' \in \binom{\mathbf{G}'}{\rho_{\ell+1}}$ and $\rho(a_1, a_r) \leq \ell$, we have $\text{dist}_{\mathbf{G}'}(w_{a_1}, w_{a_r}) = \rho(a_1, a_r) \leq \ell$

Since π_{k+1} is the isomorphism of \mathbf{G} into \mathbf{G}' we have

$$\text{dist}_{\mathbf{G}}(u, v) = \text{dist}_{\mathbf{G}'}(w_{a_1}, w_{a_r}) = \rho(a_1, a_r) \leq \ell,$$

which is a contradiction with the original assumption (3.11) that $\text{dist}_{\mathbf{G}}(u, v) \geq \ell + 1$. This finishes the proof of Claim 3.30.

Remark 3.32. A subtle point in the proof Claim 3.30(ii) is that while the copies of G in \mathcal{G}_ℓ are only guaranteed to be ℓ -metric in R_ℓ , for $\mathbf{G}^1, \mathbf{G}^2 \in$

\mathcal{G}_ℓ and $u, v \in V(\mathbf{G}^1) \cap V(\mathbf{G}^2)$ —similarly as in Claim 3.28—we have either $\text{dist}_{\mathbf{G}^1}(u, v) = \text{dist}_{\mathbf{G}^2}(u, v)$ or $\min\{\text{dist}_{\mathbf{G}^1}(u, v), \text{dist}_{\mathbf{G}^2}(u, v)\} \geq \ell + 1$. In other words, if $\text{dist}_{\mathbf{G}^1}(u, v) = \ell + 1$ there may exist a path $P(u, v)$ in R_ℓ of length ℓ but this path cannot be entirely contained in any $\mathbf{G}^2 \in \mathcal{G}_\ell$.

We have proved the induction step over k by establishing Claims 3.24, 3.25, 3.27 and 3.30. In order to prove that

$$R_{\ell+1} = P_m \quad \text{and} \quad \mathcal{G}_{\ell+1} = \mathcal{G}(P_m) \quad (3.15)$$

satisfy the induction hypothesis for $\ell + 1$, it remains to show that (L1) and (L3) hold.

The property (L3) follows from $(3)_m, (3)_{m-1}, \dots, (3)_1$ since every edge $e \in E(P_m)$ must belong to some copy \mathbf{P}^0 of P_0 and thus $e \in E(\mathbf{G})$ for some $\mathbf{G} \in \mathcal{G}(\mathbf{P}^0) \subset \mathcal{G}(P_m) = \mathcal{G}_{\ell+1}$. More formally,

$$\begin{aligned} E(R_{\ell+1}) &= E(P_m) = \bigcup_{\mathbf{P}^{m-1} \in \mathcal{P}_m} E(\mathbf{P}^{m-1}) \\ &= \bigcup_{\mathbf{P}^{m-1} \in \mathcal{P}_m} \bigcup_{\mathbf{P}^{m-2} \in \mathcal{P}_{m-1}(\mathbf{P}^{m-1})} \dots \bigcup_{\mathbf{P}^0 \in \mathcal{P}_1(\mathbf{P}^1)} E(\mathbf{P}^0) \\ &= \bigcup_{\mathbf{P}^{m-1}, \dots, \mathbf{P}^0} \bigcup_{\mathbf{G} \in \mathcal{G}(\mathbf{P}^0)} E(\mathbf{G}) \\ &\stackrel{(3.8)}{=} \bigcup_{\mathbf{G} \in \mathcal{G}(P_m)} E(\mathbf{G}). \end{aligned} \quad (3.16)$$

To prove² that the condition (L1) is satisfied by $R_{\ell+1}$ and $\mathcal{G}_{\ell+1}$ we first show that under certain assumptions on a coloring of P_0 one can obtain $\mathbf{G} \in \mathcal{G}(P_0)$ with $\mathcal{I}_{\mathbf{G}}$ monochromatic. Our goal is then reduced to finding some $\mathbf{P}^0 \subset R_{\ell+1}$, $\mathbf{P}^0 \cong P_0$, which is colored in such a way.

Claim 3.33 (Auxiliary). *Suppose that the tuples in $\bigcup_{\mathbf{G} \in \mathcal{G}(P_0)} \mathcal{I}_{\mathbf{G}}$ are colored in such a way that the color of any $I \in \bigcup_{\mathbf{G} \in \mathcal{G}(P_0)}$ depends only on the projection $\pi_0(I) \in \bigcup_{\mathbf{G} \in \mathcal{G}_\ell} \mathcal{I}_{\mathbf{G}}$.*

²This proof closely follows [31].

Then there exists $G \in \mathcal{G}(P_0)$ with \mathcal{I}_G monochromatic.

Under the assumptions of the claim there is an induced coloring of the tuples in $\bigcup_{G \in \mathcal{G}_\ell} \mathcal{I}_G$ given by assigning to each $\mathcal{I}' \in \bigcup_{G \in \mathcal{G}_\ell} \mathcal{I}_G$ the same color of the tuples $I \in \bigcup_{G \in \mathcal{G}(P_0)}$ satisfying $\pi_0(I) = \mathcal{I}'$.

By the induction hypothesis (L1) over R_ℓ and \mathcal{G}_ℓ , there must be some $G^* \in \mathcal{G}_\ell$ such that \mathcal{I}_{G^*} is monochromatic under this induced coloring. By construction, $G = \pi_0^{-1}(G^*)$ is contained in $\mathcal{G}(P_0)$ (see Figure 3.2). Since the color of any tuple $I \in \mathcal{I}_G$ is given by the color of $\pi_0(I) \in \mathcal{I}_{G^*}$, it is clear that \mathcal{I}_G is monochromatic.

Claim 3.34 below establishes (L1).

Claim 3.34. *For every 2-coloring of $\bigcup_{G \in \mathcal{G}_{\ell+1}} \mathcal{I}_G \subset \binom{R_{\ell+1}}{\rho_{\ell+1}}$ there exists some $G \in \mathcal{G}_{\ell+1}$ such that \mathcal{I}_G is monochromatic.*

Let a 2-coloring of $\bigcup_{G \in \mathcal{G}_{\ell+1}} \mathcal{I}_G$ be given. In view of Claim 3.33 we now look for a copy $P^0 \subset R_{\ell+1}$ such that the coloring of $\bigcup_{G \in \mathcal{G}(P^0)} \mathcal{I}_G$ satisfies the conditions of the claim.

Notice that because of (3.5) and (3.8), we have

$$\bigcup_{P \in \mathcal{P}_m} \mathcal{I}_P^{(m-1)} \subset \bigcup_{P \in \mathcal{P}_m} \bigcup_{G \in \mathcal{G}(P)} \mathcal{I}_G = \bigcup_{G \in \mathcal{G}(P_m)} \mathcal{I}_G.$$

Hence there is an induced 2-coloring of $\bigcup_{P \in \mathcal{P}_m} \mathcal{I}_P^{(m-1)}$. By Property (1)_m, there exist some $P^{m-1} \in \mathcal{P}_m$ such that $\mathcal{I}_{P^{m-1}}^{(m-1)}$ is monochromatic. Denote by $\pi^{m-1}: V(P^{m-1}) \rightarrow V(R_\ell)$ be the natural projection/homomorphism of P^{m-1} onto R_ℓ . Notice that because $P^{m-1} \cong P_{m-1}$, $\mathcal{I}_{P^{m-1}}^{(m-1)} \cong \mathcal{I}^{(m-1)}$ and π^{m-1} is the map induced by π_{m-1} , the definition in (3.5) translates to

$$\mathcal{I}_{P^{m-1}}^{(m-1)} = \left\{ I \in \bigcup_{G \in \mathcal{G}(P^{m-1})} \mathcal{I}_G : \pi^{m-1}(I) = I_m \right\}. \quad (3.17)$$

Hence, the color of all the tuples in $\bigcup_{G \in \mathcal{G}(\mathbf{P}^{m-1})} \mathcal{I}_G$ projecting onto I_m is the same.

Applying Property $(1)_{m-1}$ to $\mathbf{P}^{m-1} \cong P_{m-1}$ we obtain some graph $\mathbf{P}^{m-2} \in \mathcal{P}_{m-1}(\mathbf{P}^{m-1}) \subset \binom{\mathbf{P}^{m-1}}{P_{m-2}}_{\text{Part}(r_\ell)}$ such that $\mathcal{I}_{\mathbf{P}^{m-2}}^{(m-2)}$ is monochromatic. Similarly as before, the projection π^{m-2} of \mathbf{P}^{m-2} onto R_ℓ is such that

$$\mathcal{I}_{\mathbf{P}^{m-2}}^{(m-2)} = \left\{ I \in \bigcup_{G \in \mathcal{G}(\mathbf{P}^{m-2})} \mathcal{I}_G : \pi^{m-2}(I) = I_{m-1} \right\}.$$

Moreover, because $\mathbf{P}^{m-2} \in \binom{\mathbf{P}^{m-1}}{P_{m-2}}_{\text{Part}(r_\ell)}$ we have $\pi^{m-2} = \pi^{m-1}|_{V(\mathbf{P}^{m-2})}$. Since $\mathcal{G}(\mathbf{P}^{m-2}) \subset \mathcal{G}(\mathbf{P}^{m-1})$, from (3.17) we have

$$\left\{ I \in \bigcup_{G \in \mathcal{G}(\mathbf{P}^{m-2})} \mathcal{I}_G : \pi^{m-2}(I) = I_m \right\} \subset \mathcal{I}_{\mathbf{P}^{m-1}}^{(m-1)}.$$

By repeating this argument sequentially (invoking $(1)_{m-2}, \dots, (1)_1$) we obtain $\mathbf{P}^{m-1} \supset \mathbf{P}^{m-2} \supset \dots \supset \mathbf{P}^0$ satisfying the following. For all $k = 0, \dots, m-1$, the family $\mathcal{I}_{\mathbf{P}^k}^{(k)}$ is monochromatic and

$$\left\{ I \in \bigcup_{G \in \mathcal{G}(\mathbf{P}^0)} \mathcal{I}_G : \pi^0(I) = I_{k+1} \right\} \subset \mathcal{I}_{\mathbf{P}^k}^{(k)},$$

where $\pi^0 = \pi^1|_{V(\mathbf{P}^0)} = \dots = \pi^{m-1}|_{V(\mathbf{P}^0)}$ is the projection/homomorphism of \mathbf{P}^0 onto R_ℓ .

Consequently, the color of a tuple $I \in \bigcup_{G \in \mathcal{G}(\mathbf{P}^0)} \mathcal{I}_G$ depends only on its projection $\pi^0(I)$. This means that the assumptions of Claim 3.33 are satisfied by \mathbf{P}^0 . The claim then yields $\mathbf{G} \in \mathcal{G}(\mathbf{P}^0) \subset \mathcal{G}_{\ell+1}$ such that $\mathcal{I}_{\mathbf{G}}$ is monochromatic, thus proving that $(L1)$ holds for $R_{\ell+1}$ and $\mathcal{G}_{\ell+1}$.

The conditions $(K1)$ – $(K4)$, which hold for $R_{\ell+1} = P_m$ and $\mathcal{G}_{\ell+1} = \mathcal{G}(P_m)$, together with (3.16) and Claim 3.34 establish that the induction hypothesis holds for $\ell + 1$. Lemma 3.19 then follows by induction.

3.3 The base of the induction

Here we state Lemma 3.35, the induction base of the proof of Lemma 3.19. The proof of this lemma is based on an application of the Hales–Jewett theorem and will be given in Section 3.5.

Lemma 3.35. *Let $t, q \in \mathbb{N}$, $t \leq q$. Suppose that*

- ρ is a fixed metric on $[t]$;
- G is a q -partite (ordered) graph with partition $V(G) = V_1^q(G) \cup \dots \cup V_q^q(G)$;
- for some $1 \leq j_1 < j_2 < \dots < j_t \leq q$, $\mathcal{I} \subset \binom{G}{\rho_2}$ is a t -partite t -uniform hypergraph with classes $\{V_{j_i}^q(G)\}_{i=1}^t$ consisting of selected (ρ_2, G) -tuples.

Then there exists a q -partite graph R and $\mathcal{G} \subset \binom{R}{G}_{\text{Part}(q)}$ satisfying the following properties.

(L1) *For any 2-coloring of the (ρ_2, R) -tuples in $\bigcup_{G \in \mathcal{G}} \mathcal{I}_G$ there exists $G \in \mathcal{G}$ such that every $\mathcal{I}_G \subset \binom{G}{\rho_2} \subset \binom{R}{\rho_2}$ is monochromatic.*

(L2) *Every $G \in \mathcal{G}$ is 2-metric in R .*

(L3) $E(R) = \bigcup_{G \in \mathcal{G}} E(G)$.

(L4) *The family \mathcal{G} satisfies conditions (A) and (B).*

Remark 3.36. For the fixed (discrete³) metric ρ on $[t]$, consider a graph F_ρ with vertex set $[t]$ such that $ij \in F_\rho$ if and only if $\rho(x, y) = 1$. With this definition we have $\binom{G}{\rho_2} \cong \binom{G}{F_\rho}$, i.e., $\binom{G}{\rho_2}$ coincides with the set of all induced copies of F_ρ in G .

³Recall that all metrics in this dissertation are discrete.

Notice also that the fact that every $G \in \mathcal{G}$ is 2-metric in R implies that G is an induced subgraph of R . Indeed, by the definition, for all $x, y \in V(G)$, when $\text{dist}_R(x, y) \leq 2$ we must have $\text{dist}_G(x, y) = \text{dist}_R(x, y)$ and when $\text{dist}_R(x, y) > 2$ we must have $\text{dist}_G(x, y) \geq 2$. In particular, $xy \in R$ if and only if $xy \in G$.

Lemma 3.35 appears in [31] without explicitly stating condition (L_4) , which is needed here for technical reasons to carry on the induction. For completeness we include the proof of [31] modified to explicitly establish (L_4) in Section 3.5.

3.4 Proof of Theorem 3.11

In this section we give a sketch of the proof of Lemma 3.10 and later use it to prove Theorem 3.11 in §3.4.1. Since this proof is very similar to the proof of the induction step in Lemma 3.19 (albeit simpler), we avoid repeating some details and instead refer the reader to parts of the proof of Lemma 3.19 that present similar arguments. The main difference between this proof and that of Lemma 3.19 is that here the “metric” part of the result follows rather trivially from our use of the Partite Lemma 3.19. On the other hand, we are now able to partition (color) all of $\binom{R}{\rho}$ and not just a t -partite system.

Let H be a given connected graph on n vertices and ρ be a metric on t elements. Set $N = R_t(n)$, where $R_t(n)$ is the smallest number such that for every 2-coloring of the complete t -uniform hypergraph $\binom{[N]}{t}$ there exists a monochromatic $\binom{S}{t}$ with $|S| = n$.

Similarly as in the proof of Lemma 3.19 we construct an N -partite graph P_0 consisting of disjoint copies of H (see Figure 3.2). Set $V(P_0) = [N] \times \binom{[N]}{n}$. For a set $S \in \binom{[N]}{n}$, let $\phi_S: V(H) \rightarrow S$ be the unique monotone map and set H_S to be a graph with vertex set $S \times \{S\}$ and edges given by

$$\{ \{ (\phi_S(x), S), (\phi_S(y), S) \} : xy \in H \}.$$

Let

$$E(P_0) = \bigcup_{S \in \binom{[N]}{n}} E(H_S).$$

Notice that P_0 indeed is the disjoint union of the copies of H in the family $\mathcal{H}(P_0) = \{H_S : S \in \binom{[N]}{n}\}$. Set $\pi_0: V(P_0) \rightarrow [N]$ be the projection onto the first coordinate.

Define

$$\mathcal{H}_0 = \left\{ \pi_0(H_S) : S \in \binom{[N]}{n} \right\}.$$

Consider the hypergraph

$$\bigcup_{H \in \mathcal{H}_0} \binom{H}{\rho} = \{I_1, \dots, I_m\} \subset \binom{[N]}{t},$$

and set

$$\mathcal{I}^{(0)} = \left\{ I \in \bigcup_{H \in \mathcal{H}(P_0)} \binom{H}{\rho} : \pi_0(I) = I_1 \right\} \subset \binom{P_0}{\rho}.$$

(Note that $\mathcal{I}^{(0)}$ is defined in a similar way as the hypergraph in (3.5).) Observe that the t -uniform hypergraph $\mathcal{I}^{(0)}$ is t -partite with respect to the vertex partition $\{V_j^N(P_0) = \pi_0^{-1}(j)\}_{j \in I_1}$.

Set $\ell = \max\{\text{dist}_H(x, y) : x, y \in V(H)\} < \infty$ and apply Lemma 3.19 to the N -partite graph P_0 (instead of a q -partite G) and the family $\mathcal{I}^{(0)} \subset \binom{P_0}{\rho_\ell}$. We then obtain the Ramsey N -partite graph P_1 and $\mathcal{P}_1 \subset \binom{P_1}{P_0}_{\text{Part}(N)}$ for which (L1) and (L2) hold. In particular, (L2) ensures that every $\mathbf{P} \in \mathcal{P}_1$ is ℓ -metric in P_1 . By our choice of ℓ , this implies that every $H \in \mathcal{H}(\mathbf{P})$ is metric in P_1 .

In general, we obtain P_{k+1} from P_k , $k = 0, \dots, m-1$, by applying Lemma 3.19 to the N -partite graph P_k and the t -partite t -uniform hypergraph

$$\mathcal{I}^{(k)} = \left\{ I \in \bigcup_{H \in \mathcal{H}(P_k)} \binom{H}{\rho} : \pi_k(I) = I_{k+1} \right\} \subset \binom{P_k}{\rho_\ell}.$$

The graph P_{k+1} and the family $\mathcal{P}_{k+1} \subset \binom{P_{k+1}}{P_k}_{\text{Part}(N)}$ we obtain are such that every $\mathbf{H} \in \mathcal{H}(P_{k+1}) = \bigcup_{\mathbf{P} \in \mathcal{P}_{k+1}} \mathcal{H}(\mathbf{P})$ is metric in P_{k+1} and $\pi_{k+1}(\mathbf{H}) \in \mathcal{H}_0$ (where $\pi_{k+1}: V(P_{k+1}) \rightarrow [N]$ is defined as the projection that maps every $v \in V_j^N(P_{k+1})$ to j for all $j = 1, \dots, N$).

Take $R = P_m$ and $\mathcal{H} = \mathcal{H}(P_m) \subset \binom{R}{H}$. Just as in Claim 3.34 one may show that in any 2-coloring of $\bigcup_{\mathbf{H} \in \mathcal{H}(P_m)} \binom{\mathbf{H}}{\rho} \subset \binom{R}{\rho}$ there exists a copy of P_0 in R , say $\mathbf{P}^0 \subset R$, such that the color of a tuple $I \in \binom{\mathbf{H}}{\rho} \subset \binom{R}{\rho}$, $\mathbf{H} \in \mathcal{H}(\mathbf{P}^0)$, depends only on the projection $\pi^0(I) \in \{I_1, \dots, I_m\}$, where $\pi^0: V(\mathbf{P}^0) \rightarrow [N]$ is the natural projection of \mathbf{P}^0 onto $[N]$. In particular, there is an induced 2-coloring of the tuples $I_1, I_2, \dots, I_m \in \binom{[N]}{t}$. Extend this induced 2-coloring to all of $\binom{[N]}{t}$ arbitrarily.

By the definition of N , there must be a monochromatic $\binom{S}{t}$ with $|S| = n$. Let $\mathbf{H} \in \mathcal{H}(\mathbf{P}^0)$ be the (unique) graph such that $\pi^0(V(\mathbf{H})) = S$. Since the color of every $I \in \binom{\mathbf{H}}{\rho}$ is the same as the color of $\pi^0(I) \in \binom{S}{t}$, it follows that $\binom{\mathbf{H}}{\rho}$ is monochromatic. Moreover \mathbf{H} is metric in $R = P_m$ since it belongs to $\mathcal{H}(P_m)$. \square

3.4.1 Proof of Theorem 3.11

By repeated applications of Lemma 3.10, we will obtain Theorem 3.11.

Let $\mathcal{M} = \{\rho^1, \dots, \rho^m\}$ be the set of all metrics induced by t vertices of H . Apply Lemma 3.10 to $R_0 = H$ and ρ^1 to obtain a graph R_1 . After R_i is constructed, $1 \leq i \leq m-1$, obtain R_{i+1} by applying Lemma 3.10 to R_i and ρ^{i+1} .

We claim that $R = R_m$ satisfies the conditions of Theorem 3.11. Indeed, given any 2-coloring of $\binom{V(R)}{t}$, we can find a metric copy \mathbf{R}^{m-1} of R_{m-1} in which every (ρ^m, t) -tuple in $\binom{\mathbf{R}^{m-1}}{\rho^m}$ is colored by c_m . Iterating this argument yields a sequence $\mathbf{R}^0 \subset \mathbf{R}^1 \subset \dots \subset \mathbf{R}^{m-1} \subset R$ such that $\mathbf{R}^i \cong R_i$ is metric in \mathbf{R}^{i+1} and every (ρ^{i+1}, t) -tuple in $\binom{\mathbf{R}^i}{\rho^{i+1}}$ has the same color c_{i+1} .

The graph $\mathbf{H} = \mathbf{R}^0 \cong H$ is metric in R and is such that $\binom{\mathbf{H}}{\rho^i} \subset \binom{\mathbf{R}^{i-1}}{\rho^i}$ is monochromatic (with color c_i) for $i = 1, \dots, m$.

3.4.2 An unordered version of Lemma 3.10

We now address the question of what could be an “unordered version” of Lemma 3.10. Let (M, ρ) be a finite unordered metric space with $|M| = t$ and integer distances. For any connected graph H , let $\binom{H}{\rho}$ be the set of all t -sets $T \subset V(H)$ such that the metric spaces (T, dist_H) and (M, ρ) are isometric.

Analogously to Proposition 3.3 one can show the following characterization of the metric spaces (M, ρ) for which the class of unordered graphs with metric embeddings has the (M, ρ) -Ramsey property.

Proposition 3.37. *Let (M, ρ) be a finite metric space with integer distances. The following statements are equivalent:*

- (a) *For any unordered connected graph H there exists an unordered graph R such that for any partition*

$$\binom{R}{\rho} = \mathcal{A}_1 \cup \mathcal{A}_2$$

there exists $i \in \{1, 2\}$ and $\mathbf{H} \in \binom{R}{H}_{\text{metric}}$ satisfying

$$\binom{\mathbf{H}}{\rho} \subset \mathcal{A}_i.$$

- (b) *ρ is homogeneous, that is, there exists a positive integer c such that for any pair of distinct elements $m, m' \in M$ we have $\rho(m, m') = c$.*

The proof of (b) \implies (a) is a direct consequence of Theorem 3.11. Indeed, due to the symmetry of homogeneous metrics the ordering is irrelevant.

The proof of (a) \implies (b) follows closely the arguments from [30] and [26] and therefore we omit it.

3.5 Proof of Lemma 3.35

Before proving the lemma, we recall some definitions relevant to the Hales–Jewett theorem.

Suppose that $\mathcal{I} \subset \binom{G}{\rho_2}$ is a t -partite t -uniform hypergraph with vertex set V and classes $V_1 = V_{j_1}^q(G), \dots, V_t = V_{j_t}^q(G)$. Let \mathcal{I}^n be the set of n -tuples of elements of \mathcal{I} . A *combinatorial line* L in \mathcal{I}^n associated with a partition $[n] = M_L \cup F_L$, $M_L \neq \emptyset$, and an $|F_L|$ -tuple $(I_k^L)_{k \in F_L} \in \mathcal{I}^{F_L}$ is given by

$$L = \{(I_1, I_2, \dots, I_n) \in \mathcal{I}^n : I_r = I_s \text{ for } r, s \in M_L \text{ and } I_k = I_k^L \text{ for } k \in F_L\}.$$

The set M_L is called the set of *moving* coordinates, while F_L is called the set of *fixed* coordinates. Notice that every combinatorial line has precisely $|\mathcal{I}|$ elements.

The Hales–Jewett theorem is stated as follows. For a proof, see for instance [17].

Theorem 3.38 ([18]). *For any integer $r \geq 2$ and finite set \mathcal{I} there exists n such that in every r -coloring of \mathcal{I}^n there exists a monochromatic line.*

For our purposes it will be useful to interpret an element $I \in \mathcal{I}$ as a vector with t coordinates where the j th coordinate is simply the unique vertex in $I \cap V_j$. In this way, an element in \mathcal{I}^n may be viewed as a $t \times n$ matrix. Consequently, a line L of \mathcal{I}^n may be described as a collection of size $|\mathcal{I}|$ consisting of $t \times n$ matrices Q_I^L , $I \in \mathcal{I}$, where the columns of Q_I^L indexed by F_L are fixed and independent of I while every column indexed by M_L is precisely I . For example, for $n = 4$, $M_L = \{1, 2\}$, $F_L = [4] \setminus M_L = \{3, 4\}$ and $L = \{(I, I, I_3^L, I_4^L) : I \in \mathcal{I}\}$, the elements of L are the matrices

$$Q_I^L = \begin{bmatrix} | & | & | & | \\ I & I & I_3^L & I_4^L \\ | & | & | & | \end{bmatrix} \quad (3.18)$$

for all $I \in \mathcal{I}$.

Proof of Lemma 3.35. Suppose that G and \mathcal{I} are given as in the statement of the lemma. Let $J = \{j_1, \dots, j_t\}$ be the set of indices with the property of the assumption, namely, \mathcal{I} is a t -partite t -uniform hypergraph with classes $\{V_j^q(G)\}_{j \in J}$. Let n be given by Theorem 3.38 (with $r = 2$) applied to \mathcal{I} . Let $\{L_1, \dots, L_N\}$ denote the set of all lines in \mathcal{I}^n

Let $W = \bigcup_{I \in \mathcal{I}} I$ and $W_j = V_j^q(G) \cap W$. (Notice that $W_j = \emptyset$ when $j \notin J$.) The vertex set of R is given by

$$V(R) = ([N] \times (V(G) \setminus W)) \cup \bigcup_{j \in J} W_j^n.$$

The edge set of R will be defined later (see (3.20) below).

In our construction, the family \mathcal{G} will be in direct correspondence with the set of lines in \mathcal{I}^n , namely, to each line L_j there will be a corresponding $\mathbf{G}_j \in \mathcal{G}$. In order to guarantee that \mathcal{G} satisfies (A) we will have $V(\mathbf{G}_j) \setminus \bigcup_{I \in \mathcal{I}_{\mathbf{G}_j}} I = \{j\} \times (V(G) \setminus W)$ for $j = 1, \dots, N$.

For a line L_a determined by the values $(I_k^a)_{k \in F_a}$ of its fixed coordinates F_a , we represent $I_k^a = \{I_{k,j}^a \in W_j\}_{j \in J}$ as a column-vector $[I_{k,j}^a]_{j \in J}$. Let us define the map $\psi_a: V(G) \rightarrow V(R)$ as follows:

$$\psi_a(v) = \begin{cases} (a, v) & \text{for } v \in V(G) \setminus W; \\ (v_1, v_2, \dots, v_n) & \text{for } v \in W_j, j \in J, \text{ where} \\ v_k = v \text{ for } k \in M_a \text{ and } v_k = I_{k,j}^a \text{ for } k \in F_a. \end{cases} \quad (3.19)$$

Fix some $I = \{u_1 < u_2 < \dots < u_t\} \in \mathcal{I}$. Because \mathcal{I} is t -partite with classes $\{V_{j_i}^q(G)\}_{i=1}^t$, we have $u_i \in W_{j_i}$ and thus $\psi_a(u_i)$ is an n -tuple for all $i = 1, \dots, t$. Therefore, in view of (3.18) and (3.19),

$$Q_I^{L_a} = \psi_a(I) = \begin{bmatrix} \psi_a(u_1) \\ \psi_a(u_2) \\ \vdots \\ \psi_a(u_t) \end{bmatrix}.$$

Indeed, the equality above is true because

- For $k \in M_a$ we have $\psi_a(u_i)_k = u_i$ for all i and hence the k th column of the matrix on the right is simply I ;
- For $k \in F_a$, we have $\psi_a(u_i)_k = I_{k,j_i}^a$ for all i and hence the k th column of the matrix on the right is simply I_k^a .

Observe that the rows of the matrices $Q_I^{L_a}$ correspond to vertices of R .

Claim 3.39. *The map $\psi_a: V(G) \rightarrow V(R)$ is one-to-one.*

Suppose for the sake of contradiction that two distinct $u, v \in V_j^q(G)$, $1 \leq j \leq q$, are such that $\psi_a(u) = \psi_a(v)$. We cannot have $\psi_a(u) = (a, u)$ since that would imply $u = v$. Consequently, $u, v \in W_j$ with $j \in J$. Hence both $\psi_a(u)$ and $\psi_a(v)$ must be n -tuples such that $\psi_a(u)_k = u \neq v = \psi_a(v)_k$ for all $k \in M_a$. Therefore u cannot be distinct from v and hence Claim 3.39 holds.

Set

$$E(R) = \bigcup_{a=1}^N E(\psi_a(G)) \quad (3.20)$$

and let $\mathcal{G} = \{\mathbf{G}_a = \psi_a(G) : a = 1, \dots, N\}$. Observe that by our definition of \mathcal{G} , (L3) follows directly from (3.20).

We now must prove that the conclusions of the lemma hold for R and \mathcal{G} . This will be accomplished by the following steps.

Step I Define a total order on $V(R)$ and a q -partition $V(R) = V_1^q(R) \cup V_2^q(R) \cup \dots \cup V_q^q(R)$ such that every ψ_a is a monotone map satisfying $\psi_a(V_j^q(G)) \subset V_j^q(R)$ for every j .

Step II Show that \mathcal{G} satisfies the intersection conditions (A) and (B) and thus prove (L4).

Step III Use Step II to show that every $G_a \in \mathcal{G}$ is an induced subgraph of R and thus prove (L2).

Step IV Show that the family \mathcal{G} is Ramsey in R , namely, prove (L1).

Proof of Step I: For all j , define

$$V_j^q(R) = ([N] \times (V_j^q(G) \setminus W)) \cup W_j^n. \quad (3.21)$$

Observe that $V(R) = V_1^q(R) \cup V_2^q(R) \cup \dots \cup V_q^q(R)$. Moreover, it is simple to check that $\psi_a(V_j^q(G)) \subset V_j^q(R)$ for all j . Let us now define a total order on $V(R)$ for which every map ψ_a is monotone. It is enough to define the order for each $V_j^q(R)$ since we require $V_1^q(R) \prec V_2^q(R) \prec \dots \prec V_q^q(R)$.

For $j \notin J$, we have $W_j = \emptyset$ and thus $V_j^q(R) = [N] \times V_j^q(G)$. Order the vertices lexicographically and observe that for every $a \in [N]$, $\psi_a(v) < \psi_a(w)$ if and only if $v < w$.

Since for $j \in J$ the class $V_j^q(R)$ may contain both pairs and n -tuples as elements, our ordering is somewhat more complicated than a simple lexicographical order on tuples.

Let $f: V_j^q(R) \rightarrow V_j^q(G)^n \times \{0, 1, \dots, N\}$ be defined as follows. For a tuple $(v_1, \dots, v_n) \in W_j^n$, set $f(v_1, \dots, v_n) = (v_1, \dots, v_n, 0)$; for $(a, v) \in [N] \times (V_j^q(G) \setminus W)$ set $f(a, v) = (v_1, \dots, v_n, a)$, where $v_k = v$ for all $k \in M_a$ and $v_k = I_{k,j}^a$ for all $k \in F_a$. The ordering on $V_j^q(R)$ is induced by f and the lexicographic order on the image of f , namely, we set $x < y$ if and only if $f(x) < f(y)$.

Let $v, w \in V_j^q(G)$ be such that $v < w$. By definition, for every $a \in [N]$, $\psi_a(v) < \psi_a(w)$ if and only if $f(\psi_a(v)) < f(\psi_a(w))$. Since $f(\psi_a(v))_k = f(\psi_a(w))_k = I_{k,j}^a$ for every $k \in F_a$, the first coordinate where the elements $f(\psi_a(v))$ and $f(\psi_a(w))$ differ is in M_a . On the other hand, for $k \in M_a$ we have

$$f(\psi_a(v))_k = v < w = f(\psi_a(w))_k.$$

We conclude that $f(\psi_a(v)) < f(\psi_a(w))$ if and only if $v < w$. Hence $\psi_a(v) < \psi_a(w)$ if and only if $v < w$.

Proof of Step II: Suppose that $x \in V(\mathbf{G}_a) \cap V(\mathbf{G}_b)$ with $a \neq b$. We must have $x \in W_j^n$ for some $j \in J$ since otherwise for some $v \in V(G) \setminus W$, we have $x = (a, v) = (b, v)$ which contradicts $a \neq b$. It follows therefore that $\psi_a^{-1}(x), \psi_b^{-1}(x) \in W_j$. Since $W_j \subset W = \bigcup_{I \in \mathcal{I}} I$, there exists $I'_a, I'_b \in \mathcal{I}$ such that $\psi_a^{-1}(x) \in I'_a$ and $\psi_b^{-1}(x) \in I'_b$. Consequently, $x \in I_a = \psi_a(I'_a) \in \mathcal{I}_{\mathbf{G}_a}$ and $x \in I_b = \psi_b(I'_b) \in \mathcal{I}_{\mathbf{G}_b}$. This establishes the intersection condition (A) for members of \mathcal{G} .

Now let us prove condition (B). Suppose that there are distinct elements $x = (x_1, \dots, x_n), y = (y_1, \dots, y_n) \in V(\mathbf{G}_a) \cap V(\mathbf{G}_b), a \neq b$.

We distinguish between two cases.

- (i) $M_a \cap M_b \neq \emptyset$.
- (ii) $M_a \cap M_b = \emptyset$ (then $M_a \subset F_b$ and $M_b \subset F_a$).

Suppose first that (i) holds and fix $k \in M_a \cap M_b$. We have $\psi_a^{-1}(x) = x_k = \psi_b^{-1}(x)$, and similarly $\psi_a^{-1}(y) = \psi_b^{-1}(y)$. Consequently, in this case condition (B2) holds as the isomorphisms $\sigma_a = \psi_a^{-1}: V(\mathbf{G}_a) \rightarrow V(G)$ and $\sigma_b = \psi_b^{-1}: V(\mathbf{G}_b) \rightarrow V(G)$ satisfy $\sigma_a(x) = \sigma_b(x)$ and $\sigma_a(y) = \sigma_b(y)$.

Now suppose that (ii) holds; in particular, we must have $M_a \subset F_b$ and $M_b \subset F_a$. Let $(I_k^a = [I_{k,j}^a]_{j \in J})_{k \in F_a}$ and $(I_k^b = [I_{k,j}^b]_{j \in J})_{k \in F_b}$ be the tuples of fixed elements that define the lines L_a and L_b respectively. Let $j, j' \in J$ be such that $x \in W_j^n$ and $y \in W_{j'}^n$.

For $k \in M_a \subset F_b$, (3.19) implies that

$$\psi_a^{-1}(x) \stackrel{k \in M_a}{=} x_k \stackrel{k \in F_b}{=} I_{k,j}^b$$

and similarly $\psi_a^{-1}(y) = y_k = I_{k,j'}^b$. In particular,

$$\{\psi_a^{-1}(x), \psi_a^{-1}(y)\} = \{I_{k,j}^b, I_{k,j'}^b\} \subset I_k^b \in \mathcal{I}.$$

Let $I_a = \psi_a(I_k^b) \in \mathcal{I}_{\mathbf{G}_a}$ and notice that

$$\{x, y\} = \psi_a(\{\psi_a^{-1}(x), \psi_a^{-1}(y)\}) \subset \psi_a(I_k^b) = I_a.$$

A symmetric argument yields $I_b \in \mathcal{I}_{\mathbf{G}_b}$ such that $\{x, y\} \in I_b$. Hence, condition (B1) follows.

To summarize, case (i) implies condition (B2) and case (ii) implies condition (B1).

Proof of Step III: Let $\mathbf{G}_a \in \mathcal{G}$ be arbitrary. To prove that \mathbf{G}_a is an induced subgraph of R we must check that for every pair of distinct $x, y \in V(\mathbf{G}_a)$ if $x, y \in V(\mathbf{G}_b)$ for some $b \neq a$ then $\{x, y\} \in \mathbf{G}_a$ if and only if $\{x, y\} \in \mathbf{G}_b$. Since $x, y \in V(\mathbf{G}_a) \cap V(\mathbf{G}_b)$, we may invoke the intersection properties of \mathcal{G} proved in Step II.

In case condition (B2) holds, the unique isomorphisms σ_a, σ_b of $\mathbf{G}_a, \mathbf{G}_b$ into G satisfy $\sigma_a(x) = \sigma_b(x)$ and $\sigma_a(y) = \sigma_b(y)$. Since σ_a is an isomorphism, $\{x, y\} \in \mathbf{G}_a$ if and only if $e = \{\sigma_a(x), \sigma_a(y)\} \in G$. Similarly, $\{x, y\} \in \mathbf{G}_b$ if and only if $e' = \{\sigma_b(x), \sigma_b(y)\} \in G$. Because $e = e'$ we infer that $\{x, y\} \in \mathbf{G}_a$ if and only if $\{x, y\} \in \mathbf{G}_b$.

In case condition (B1) holds, let $I_a \in \mathcal{I}_{\mathbf{G}_a}$ and $I_b \in \mathcal{I}_{\mathbf{G}_b}$ be such that $x, y \in I_a \cap I_b$. Let $j_r, j_s \in J$ ($1 \leq r, s \leq t$) be such that $x \in V_{j_r}^q(R)$ and $y \in V_{j_s}^q(R)$. Because $I_a \in \binom{\mathbf{G}_a}{\rho_2}$ it follows that $\text{dist}_{\mathbf{G}_a}(x, y) = \rho(r, s)$ whenever $\rho(r, s) \leq 2$ and $\text{dist}_{\mathbf{G}_a}(x, y) \geq 2$ whenever $\rho(r, s) > 2$. In particular, $\{x, y\} \in \mathbf{G}_a$ if and only if $\rho(r, s) = 1$. Similarly, $\{x, y\} \in \mathbf{G}_b$ if and only if $\rho(r, s) = 1$. Therefore $\{x, y\} \in \mathbf{G}_a$ if and only if $\{x, y\} \in \mathbf{G}_b$.

Proof of Step IV: We will now show that for any 2-coloring of the (ρ_2, R) -tuples in $\bigcup_{\mathbf{G} \in \mathcal{G}} \mathcal{I}_{\mathbf{G}}$ there exists $\mathbf{G} \in \mathcal{G}$ such that every t -tuple in $\mathcal{I}_{\mathbf{G}} \subset \binom{\mathbf{G}}{\rho_2}$ is monochromatic. It will be convenient to assume that all t -tuples in $V_{j_1}^q(R) \times \cdots \times V_{j_t}^q(R)$ are colored.

Consider $Q = (I_1, \dots, I_n) \in \mathcal{I}^n$ as a $t \times n$ matrix with columns I_1, \dots, I_n .

The k th row of the matrix is in $V_{j_k}^q(R)$ (recall that $J = \{j_1, \dots, j_t\}$). In particular, Q is in correspondence with a t -tuple of $V_{j_1}^q(R) \times \dots \times V_{j_t}^q(R)$. Define the color of Q as the color of the corresponding t -tuple.

By the Hales–Jewett theorem, there is a monochromatic line L_a , $a \in [N]$, in such a coloring. It follows that $\mathbf{G} = \mathbf{G}_a$ is such that $\mathcal{I}_{\mathbf{G}}$ is monochromatic. Indeed, every t -tuple $\psi_a(I) \in \mathcal{I}_{\mathbf{G}_a}$, $I \in \mathcal{I}$, corresponds to the matrix $Q_I^{L_a}$ contained in the line L_a (see (3.19)). \square

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