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April 9, 2025

Higher-order topological knots in non-Hermitian lattices

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Abstract

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In two dimensions, Hermitian lattices with non-zero Chern numbers and non-Hermitian lattices with a higher-order skin effect (HOSE) bypass the constraints of the Nielsen–Ninomiya “no-go” theorem at their one-dimensional boundaries. This allows the realization of topologically-protected one-dimensional edges with nonreciprocal dynamics. However, unlike the edge states of Chern insulators, the nonreciprocal edges of HOSE phases exist only at certain edges of the two-dimensional lattice, not all, leading to corner localized states. In this work, we investigate the topological connections between these two systems and uncover novel non-Hermitian topological phases possessing “higher-order topological knots” (HOTKs). These phases arise from multiband topology protected by crystalline symmetries and host point-gap-protected nonreciprocal edge states that circulate the entire boundary of the two-dimensional lattice. We show that phase transitions typically separate HOTK phases from “Complex Chern insulator” phases –non-Hermitian lattices with nonzero Chern numbers protected by imaginary line gaps in the presence of time-reversal symmetry.

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Chapter 1

Introduction

Some contents of this thesis have been submitted for publication, available as Yifan Wang and Wladimir Benalcazar “Higher-order topological knots in non-Hermitian lattices”, arXiv:2412.05809, 2024 (Ref.[85]).

1.1 Symmetry, topology, and condensed matter physics

One of the ultimate goals of condensed matter physics is to understand and classify different quantum systems. The discovery of integer quantum Hall effect (IQHE), for example, for the first time linked topology to quantum systems. Through years of development, the physical intuition behind IQHE results in a whole new class of quantum systems, called topological insulators and topological superconductors. In this section, we aim to provide a general review of the Hermitian topological insulator and topological superconductors to provide a better context for the main results of the paper.

In 1980, Klaus von Klitzing discovered that, under a strong magnetic field, the bulk of MOSFET (metal-oxide-semiconductor field-effect transistor) remains an insulator,

while its edge becomes a conductor. Surprisingly, the conductance is quantized to the value $\sigma_{x,y} = \frac{e^2}{h}C$, where C is an integer $\mathbb{Z}$ [37]. This phenomenon is later referred to as the integer quantum hall effect.

In 1982, Thouless, Kohmoto, Nightingale, and den Nijs identified C as the Chern number, linking the integer quantum Hall effect to topology [81]. Notably, the existence of non-trivial Chern number in the system requires the breakdown of time-reversal symmetry. In the case of IQHE, this was realized by adding a magnetic field. Another relevant discovery in the 80s is the quantum anomalous Hall effect (QAHE) proposed by Duncan Haldane [19]. In contrast to IQHE which requires a strong magnetic field, QAHE does not require a magnetic field. The breakdown of the time-reversal symmetry originated from the complex next-nearest-neighbor hopping in Haldane's model. Toy models that realize QAHE are generally referred to as Chern insulators. In 2001, Alexei Kitaev proposed a 1D p-wave superconducting chain, which hosts unpaired Majorana zero modes at the ends of the chain [35]. The Kitaev chain contains an $\mathbb{Z}_2$ invariant that identifies the existence of Majorana zero modes. The existence of such an invariant depends on the particle-hole symmetry of the system. This results in the so-called topological superconductors. Around 2005, building on top of IQHE, the discovery of the quantum spin Hall effect (QSHE) once again linked topology and condensed matter physics [30, 31, 38]. Similarly to IQHE, QSHE also exhibits edge-conducting states. But instead of the Chern number, the existence of QSHE edge states is identified through a $\mathbb{Z}_2$ invariant, which is sometimes referred to as Kane-Mele $\mathbb{Z}_2$ invariant [31]. The existence of Kane-Mele $\mathbb{Z}_2$ invariant also depends on the *existence* of TRS.

After these discoveries, a surge of linking symmetry, topology, and condensed matter physics occurs, forming the so-called topological insulators (TI) and topological superconductors (TSC) [18, 49, 17, 25]. In the following, to limit our scope, we will only mention TIs.

The connection between non-trivial topological invariants in the bulk and the existence of edge states formed the bulk-boundary correspondence. Importantly, the bulk-boundary correspondence of TIs and TSCs also linked mathematical concepts (topological invariants) to physical phenomena (edge states). The topological edge states for a 2D crystal lattice, such as those of IQHE and QSHE mentioned previously, are of particular importance because they circumvent *Nielsen and Ninomiya (NN) theorem* [56, 54, 55]. NN theorem states that in a translational invariant, local, Hermitian lattice system, the system cannot have a single chiral fermion in the Brillouin zone. Therefore, by the constraints of the NN theorem, all chiral fermions must come in pairs of opposite chirality, a phenomenon called fermion doubling. IQHE forms a chiral fermion at one edge by compensating the chiral fermion with *opposite* chirality at the opposite edge. In this way, a circulation current can be formed at the boundary of the systems carried by chiral fermions with the same chirality. The edge states of QSHE consist of two counter-propagating edge modes of opposite spin and chirality. Time-reversal symmetry prevents back-scattering between the two edge states. Thus, the edge of QSH systems can carry two currents traveling in the opposite direction. The promising physical phenomena of TIs drive the physicist to understand them in a systematic way, that is, to classify them.

Trying to understand TIs and TSCs systematically, the condensed matter community begins the classification of Hamiltonians around 2010. Since the classification methods relate symmetries and topological invariants, which is essential to our results in this paper, in the following, we briefly recount the classification methods.

The starting points for analyzing most condensed matter systems begin by taking two considerations: (1) periodic boundary conditions (PBC) of the crystal, reflecting the translational symmetry of the crystal lattice, and (2) the tight-binding approximation, which models electrons as being localized to atoms with hopping between neighboring sites. The periodic boundary conditions allow us to define the concept

of unit cell: the smallest repeating unit of the lattice i.e. by repetitively translating unit cell, we can reproduce the whole lattice. Thus, the Hamiltonian that describes the crystal lattice should have the same periodicity as the unit cell. Translational invariance also makes the lattice momentum $\mathbf{k}$ a good quantum number, allowing us to use $\mathbf{k}$ to label eigenstates. The tight-binding approximation discretizes the Hamiltonian, which greatly benefits numerical calculations.

Due to the above two conditions, the effective Hamiltonian in real space can be exactly Fourier transformed into k space. The resulting Hamiltonian is called the Bloch Hamiltonian $h(\mathbf{k})$. The translational symmetry in the real space implies translational symmetry in the k space. We call the lattice in the k space the reciprocal lattice. The unit cell will also transform into the k space, forming the Brillouin zone (BZ): the smallest repeating unit of the reciprocal lattice. In 1D, the k space periodicity implies $h(k + 2\pi/a) = h(k)$, where we set the lattice constant $a = 1$ throughout the paper. The BZ is given by $k \in [0, 2\pi)$. The topological invariants mentioned previously, such as the Chern number or the Kane-Mele $\mathbb{Z}_2$ invariant, are all defined in the BZ i.e. under PBC. More concretely, non-trivial topological invariants in the BZ tell us the existence of edge states when the boundary is open, which is the meaning of bulk-boundary correspondence. Thinking in a more abstract sense, we consider BZ as a parameter space given by a circle S^1 . More generally, in n -dimension, BZ is given by the n -torus T^n . The Bloch Hamiltonian can thus be considered as a mapping of the T^n torus to some matrix space. The classification problems of condensed matter system Hamiltonians thus become a problem of classification of matrices and mappings in mathematics.

The heart of the classification relies on three symmetries: particle-hole symmetry (PHS)

$$\mathcal{P}h(\mathbf{k})\mathcal{P}^{-1} = -h(-\mathbf{k}), \mathcal{P}^2 = \pm 1, \quad (1.1)$$

which comes from adopting the mean-field approximation of TSC [65]; time-reversal

symmetry (TRS)

$$\mathcal{T}h(\mathbf{k})\mathcal{T}^{-1} = h(-\mathbf{k}), \mathcal{T}^2 = \pm 1, \quad (1.2)$$

the existence of which depends on the existence of magnetic field or complex hoppings;

chiral symmetry (CS)

$$\mathcal{C}h(\mathbf{k})\mathcal{C}^{-1} = -h(\mathbf{k}), \mathcal{C}^2 = 1, \quad (1.3)$$

which comes from the combination of PHS and TRS. As mentioned previously, we can consider the classification of Hamiltonians as equivalent mappings from T^n torus to matrix space. The matrix space is constrained by the above three symmetries. The classification of equivalent mappings belongs to homotopy theory, which is well studied in mathematics [24]. This results in the ten-fold classification of TIs and TSCs [1, 12, 36, 80, 64, 67, 78], which accounts for all possible combinations and exhaust “strong” topological invariants in all possible dimensions. Notably, none of these three symmetries originates from the geometry of the lattice. Thus, PHS, TRS, and CS are called internal symmetries: Lattice deformation and distortion will not break these symmetries. The topological invariants based on the classification of internal symmetries are called strong invariants: their existence is robust against disorder. The robustness of topological invariants also means that edge states, and thus the current carried by edge states, is robust. Indeed, it is known that lattice defects will not hinder current transportation at the boundary of TIs.

Having been convinced by great promises of TIs, physicists began to search for more exotic topological phenomenon, even by breaking down one of the main assumptions of quantum mechanics: Hermiticity. We review the physical motivations and consequences of breaking Hermiticity of lattice systems in the next section.

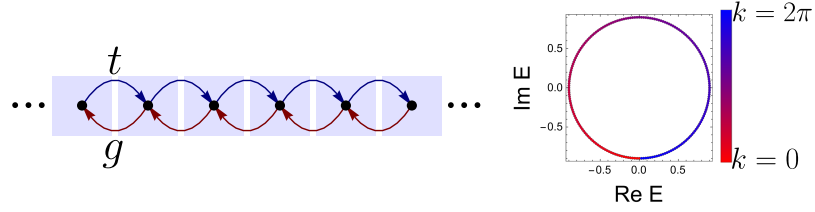


Figure 1.1: Lattice of the Hatano-Nelsen model 1.5 and its spectrum (plotted at $t = 1$, $g = 0.5$) on the complex plane. The spectrum is colored according to the value of k in BZ.

1.2 Rise of non-Hermiticity and topology

One of the most fundamental assumptions of quantum mechanics is that observables must be described by Hermitian operators. Hamiltonian, for example, is imposed to be Hermitian such that its eigenvalues, which is energy, must be real. However, complex energy also has a physical meaning. The imaginary part of the eigen-energy reflects the exponential decay or growth of the corresponding eigenstates. Indeed, the non-Hermitian (NH) Hamiltonian was known in the 1960s to describe the effective edge Hamiltonian of an open quantum system [13]. Due to the exchange of energy between the system and the environment, the complex energy from the effective Hamiltonian is useful for understanding finite-life-time particles on a phenomenological level. However, a systematical understanding of non-Hermitian Hamiltonian and their possible application was still lacking until recently.

It is worth noticing that, as mentioned in the previous section, one of the assumptions of NN theorem is the Hermiticity of Hamiltonian. Non-Hermitian Hamiltonian is a natural platform for bypassing the NN theorem and realizing transportation properties that are otherwise impossible in Hermitian systems. It is known that robust and nonreciprocal transportation similar to that of the edge of Chern insulator is impossible to realize in 1D Hermitian systems. However, this limitation can be easily bypassed in non-Hermitian systems. A famous example is the Hatano-Nelson (HN) model [22, 23, 21]. HN model considered a 1D tight-binding Hamiltonian with

in-equivalent hopping strength in left and right direction

$$H = \sum_n t c_{n+1}^\dagger c_n + g c_n^\dagger c_{n+1}, \quad (1.4)$$

which has the form in k space

$$h_{HN}(k) = t e^{ik} + g e^{-ik}. \quad (1.5)$$

The imbalance in hopping strength will create a preferable hopping direction for the electrons, that is, towards the direction with stronger hopping. This can be reflected in the complex spectrum of $h_{HN}(k)$ in Fig. 1.1. The right mover (located at $k = \pi$) has the largest imaginary energy, while the left mover (located at $k = 0$) has the smallest imaginary energy. Under time evolution, this leads to the exponential growth of the right mover, which will eventually dominate the movement of wave packet. On the other hand, the left mover will evanescent exponentially. This is how HN model explicitly realizes unidirectional propagation, and thus, bypassing NN theorem in 1D. But the properties HN model hold are not fully understood until it meets with the insight from TIs.

The HN model was revisited after 20 years of discovery in the context of topological phases. Ref [92] discovered the non-Hermitian skin effect within the HN model¹: upon the boundary condition changes from periodic boundary condition to open boundary conditions, the eigenstates of the models are exponentially localized at the boundary. This phenomenon is protected by an integer topological invariant called the spectral winding number (see Eq. 2.8 in Chapter 2). Ref [92] also provides a new insight into the definition of “gaps,” which is referred to as the point gap for the HN model spectrum in Fig. 1.1 (See Chapter 2 for more discussion regarding

¹Strictly speaking, Ref [92] considered the non-Hermitian Su-Schreiffer-Heeger model instead of the HN model. But two models have essentially the same properties.

point gaps). Notably, non-trivial spectral winding number also necessarily leads to an unequal number of left and right movers [11]. Therefore, the two phenomena can be understood uniformly through a single topological invariant.

Even for an extremely simple non-Hermitian lattice such as the HN model 1.5, it already exhibits both nonreciprocal transportation and localization properties that are impossible to realize for Hermitian lattices. Following these discoveries, classification effort for non-Hermitian Hamiltonian is called for. The topological classification of NH Hamiltonians is largely expanded from the 10-fold classification of Hermitian Hamiltonians [1, 12, 36, 80, 64, 67, 78]. This expansion arises because the three basic symmetries – time-reversal ($\mathcal{T}$), particle-hole ($\mathcal{P}$), and chiral ($\mathcal{C}$) – split into two distinct versions. Since the majority of models appearing in this paper obey time-reversal symmetry, we focus on the classification of Hamiltonians that obey TRS only in the following. Under non-Hermiticity, TRS bifurcates into the conventional TRS, under which a Bloch Hamiltonian $h(\mathbf{k})$ satisfies

$$\mathcal{T}h(\mathbf{k})\mathcal{T}^{-1} = h(-\mathbf{k}), \quad (1.6)$$

and an additional “pseudo-TRS”, denoted here as $\text{TRS}^\dagger$,

$$\mathcal{T}h(\mathbf{k})\mathcal{T}^{-1} = h(-\mathbf{k})^\dagger, \quad (1.7)$$

where $\mathbf{k}$ is the crystal momentum, $\mathcal{T} = U\mathcal{K}$ is the time reversal operator, U is a unitary matrix and $\mathcal{K}$ denotes complex conjugation. Taking into account the ramification of TRS, particle-hole symmetry, and chiral symmetry, as well as the additional property of pseudo-Hermiticity, $\eta h(\mathbf{k})\eta^{-1} = h(\mathbf{k})^\dagger$, where η is unitary, the 10-fold classification of Hermitian Hamiltonians expands into a 38-fold classification for NH Hamiltonians [32, 100, 7].

In the absence of pseudo-Hermiticity, a NH Hamiltonian necessarily breaks either

Gap	1D	2D
Point gap	$\mathbb{Z}$	0
Real line	0	0
Imaginary line	$\mathbb{Z}_2$	$\mathbb{Z}$

Table 1.1: Topological classification in class AI of the 38-fold classification of NH Hamiltonians in 1D and 2D lattices.

TRS or TRS[†]. Systems exhibiting a skin effect can satisfy (1.6), but they must break pseudo-TRS (1.7) (Appendix A.1). Here, we focus on 2D lattices belonging to class AI in the 38-fold classification. Class AI encompasses Hamiltonians that obey TRS (1.6) with $\mathcal{T}^2 = 1$.

The topological classification for class AI is indicated in Table 1.1. In 1D, NH Hamiltonians with a point gap are topologically classified by a $\mathbb{Z}$ invariant, which is the winding number mentioned earlier. Systems with a real line gap are always trivial, whereas those with an imaginary line gap possess a $\mathbb{Z}_2$ classification, where the topological invariant is the Berry phase, constrained to take values of 0 or π (Appendix A.2). In 2D, there are nontrivial classes only in the presence of an imaginary line gap. The corresponding invariant is the Chern number, which protects “complex Chern bands”, bands in the complex energy plane possess nonzero Chern numbers and support edge states that traverse either a real line gap [Fig. 2.2(a)] or an imaginary line gap [Fig. 2.2(b)].

The classification presented in Table 1.1 pertains only to first-order topological phases. In the presence of additional crystalline symmetries, the classification is further expanded to account for the protection of higher-order topological phases. This protection is weaker, however, as crystalline symmetries are not local, and disorder in the lattice can disrupt them. Nonetheless, the topological properties of the disordered system persist as long as the energy gap remains open.

Inspired by the similarity between the Chern insulator and the HN model in terms of nonreciprocal transportation, we aim to find novel phases that support such

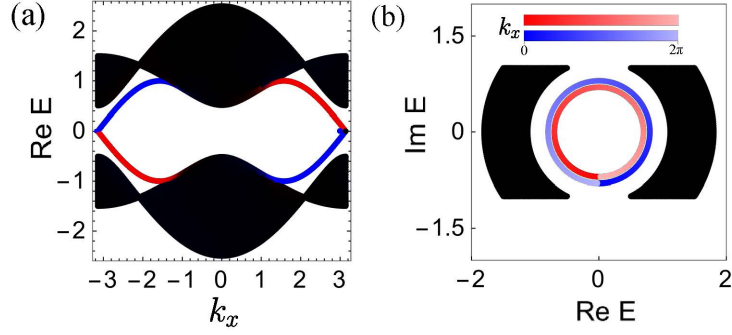


Figure 1.2: Energy spectra under PBC along x and OBC along y of (a) a Chern insulator and (b) a HOSE phase. Bulk bands are depicted in black, while red and blue represent states localized at opposite boundaries. In (b), the brightness of the red and blue colors indicates the value of the crystal momentum k_x . As indicated in the color bar, the colors are brightest at $k_x = 0$, and gradually fade as k_x approaches 2π . The red and blue spectra in (b) are degenerate; they have been slightly offset for clarity.

phenomena that are only realizable in non-Hermitian lattices. Therefore, we ask if nonreciprocal transportation of the HN model can be realized at the boundary of a 2D NH lattice. In fact, this question is already being partially answered. 2D NH lattices with a higher-order skin effect (HOSE) display edges with unidirectional propagation, albeit not along all edges [33, 58, 47, 45].

In both cases of Chern insulator and HOSE, a bulk-boundary correspondence connects the nontrivial topological properties of bulk states to unconventional spectral characteristics of edge states across the Brillouin zone.

As mentioned previously, for Chern insulators, the bulk invariant is the Chern number, which ensures that edge states connect with bulk states in certain regions of the spectrum [Fig. 1.2(a)]. For HOSE phases, the precise bulk invariant has not been completely elucidated [33, 69]. Furthermore, the edge states in HOSE phases are completely separated from the bulk states by a gap, and their topological nature can be determined by the winding number of the edge complex spectrum across the Brillouin zone alone [Fig. 1.2(b)].

The nonreciprocal nature of the edge states of Chern insulators and (some of) the edges of HOSE phases motivates us to look into their topological connections.

Specifically, what is the minimal sequence of phase transitions that separate these two phases, and how do the edge states of a Chern insulator spectrally separate from its bulk bands as it transitions into a HOSE phase? Moreover, can this process provide insights into the bulk-boundary correspondence in HOSE phases?

In this paper, we address these questions. We begin by demonstrating how the minimal model of a Chern insulator deforms into a HOSE phase. During this deformation, the edge states of the Chern insulator detach from the bulk, evolving into the non-Hermitian (NH) edge states with nontrivial spectral winding characteristic of the HOSE phase. However, this detachment occurs only along one pair of opposite edges; at the other pair, the states merge into the bulk at the transition. Under full OBC, the nontrivial winding of the detached edge states manifests as a skin effect, collapsing these states into the $O(L)$ corner-localized states that characterize this phase.

Next, we propose new NH topological phases, which we term “higher-order topological knot” (HOTEK) phases. These phases are characterized by bulk states with a line gap and edge states with point gaps and nontrivial windings *along all edges*. Unlike the edge states of HOSE phases, the edge states in HOTEK phases under OBC circulate along the entire perimeter of the sample. We show that the minimal HOTEK phase can be generated from lattices with “complex Chern bands” via topological phase transitions that close a real-line gap.

We consider NH Hamiltonians obeying C_n symmetry,

$$\hat{r}_n h(\mathbf{k}) \hat{r}_n^{-1} = h(R_n \mathbf{k}), \quad (1.8)$$

where $\hat{r}_n$ is the rotation operator satisfying $\hat{r}_n^n = 1$ (or $\hat{r}_n^n = -1$ due to the presence of magnetic fields threading the lattice), and R_n is the matrix that rotates the crystal momentum $\mathbf{k}$ by $2\pi/n$ rad.

We will see that HOTK phases exhibit bands in the complex energy plane with line gaps, yet they undergo phase transitions characterized by point gaps. These “critical point gaps” bifurcate the topological edge states with point gaps and nontrivial winding as the system transitions from a trivial to a HOTK phase. In the bulk, the HOTK topology is diagnosed by symmetry indicator invariants under C_n symmetry, for which we build the complete classification. The analysis of the symmetry indicator invariants revealed that HOTK phases result from multiband topology, which is possible in NH systems due to the simultaneous presence of multiple line gaps in the complex energy plane.

The structure of the paper is as follows. In Chapter 2, we review previous results and important concepts that would benefit later discussion. More specifically, we define complex Chern insulators in the presence of TRS; we also provide an overview of the higher-order skin effect; Finally, we draw connections between Chern insulators and HOSE phases to motivate our study. Chapter 3 and 4 present our main findings. In Chapter 3, we classify the NH Hamiltonians according to C_n symmetry; Chapter 4 introduces models with HOTK phases, one for each of the $C_{2,3,4,6}$ symmetries. Finally, in Chapter 5 we conclude with a discussion and outlook of our work.

Chapter 2

Background

In this chapter, we review the Chern insulator under time-reversal symmetry, higher-order skin effect, and the connection between higher-order skin effect and Chern insulator.

2.1 Complex Chern insulators under time-reversal symmetry

The Chern number of a Hermitian system vanishes under TRS (1.6). However, this is not always the case for NH systems. Consider a NH Bloch Hamiltonian $h(\mathbf{k})$. The right and left eigenstates obey

$$\begin{aligned} h(\mathbf{k}) |u_{\mathbf{k}}^n\rangle &= \epsilon_n(\mathbf{k}) |u_{\mathbf{k}}^n\rangle \\ h^\dagger(\mathbf{k}) |v_{\mathbf{k}}^n\rangle &= \epsilon_n^*(\mathbf{k}) |v_{\mathbf{k}}^n\rangle, \end{aligned} \tag{2.1}$$

respectively. They can be made to obey $\langle u_{\mathbf{k}}^m | v_{\mathbf{k}}^n \rangle = \delta_{mn}$ and $\sum |u_{\mathbf{k}}^m\rangle \langle v_{\mathbf{k}}^m| = 1$. A NH Hamiltonian in class AI obeys (1.6). Applying $h(-\mathbf{k})\mathcal{T}$ on a right eigenstate of $h(\mathbf{k})$

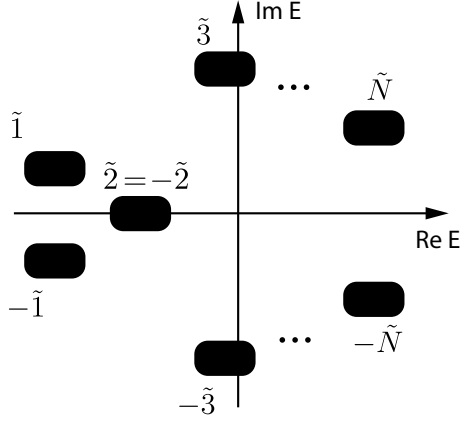


Figure 2.1: Schematic of the energy spectrum of a generic NH Hamiltonian in class AI. Energy bands that map into one another by the action of time-reversal are labeled in pairs $-\tilde{n}, \tilde{n}$.

we have

$$h(-\mathbf{k})\mathcal{T}|u_{\mathbf{k}}^n\rangle = \mathcal{T}h(\mathbf{k})|u_{\mathbf{k}}^n\rangle = \epsilon_n^*(\mathbf{k})\mathcal{T}|u_{\mathbf{k}}^n\rangle. \quad (2.2)$$

Hence, $\mathcal{T}|u_{\mathbf{k}}^n\rangle$ is an eigenstate of $h(-\mathbf{k})$ with the eigenvalue $\epsilon_n^*(\mathbf{k})$ and, as such, is proportional to $|u_{-\mathbf{k}}^n\rangle$, which has energy $\epsilon_n(-\mathbf{k})$. Thus, under TRS (1.6), the eigenvalues come in pairs $\{\epsilon_n(-\mathbf{k}), \epsilon_n^*(\mathbf{k})\}$. This allows us to label the energy bands using the following notation: let $\tilde{1}, \tilde{2}, \dots, \tilde{N}$ denote energy bands above $\text{Im}E = 0$ and $-\tilde{1}, -\tilde{2}, \dots, -\tilde{N}$ denote the corresponding bands below $\text{Im}E = 0$, such that bands $\tilde{n}$ and $-\tilde{n}$ are related by TRS. If a band $\tilde{l}$ lies on the real energy line, we say that $\tilde{l} = -\tilde{l}$. This notation is schematically represented in Fig. 2.1.

At a given $\mathbf{k}$, let the pair of indices $n \in \tilde{n}$ and $-n \in -\tilde{n}$ denote two states on opposite sides of the imaginary line gap. Choosing a gauge in which the sewing matrix $V_{\mathbf{k}}^{-n,n} = \langle v_{-\mathbf{k}}^{-n} | \mathcal{T} | u_{\mathbf{k}}^n \rangle$ is diagonal, we have $\mathcal{T}|u_{\mathbf{k}}^n\rangle = |u_{-\mathbf{k}}^{-n}\rangle$; that is, $\mathcal{T}$ takes an eigenstate at $\mathbf{k}$ and relates it to its time-reversal partner eigenstate at $-\mathbf{k}$ across the imaginary gap.

Now consider the biorthogonal Berry curvature [68] for band $\tilde{n}$,

$$[\Omega_{\tilde{n}}^{\text{RL}}(\mathbf{k})]^{n_1, n_2} = i \left(\langle \partial_{k_x} u_{\mathbf{k}}^{n_1} | \partial_{k_y} v_{\mathbf{k}}^{n_2} \rangle - \langle \partial_{k_y} u_{\mathbf{k}}^{n_1} | \partial_{k_x} v_{\mathbf{k}}^{n_2} \rangle \right), \quad (2.3)$$

where the superscript RL labels the order of the biorthogonal basis. States $n_1, n_2 \in \tilde{n}$.

Then

$$\begin{aligned} [\Omega_{\tilde{n}}^{\text{RL}}(-\mathbf{k})]^{n_1, n_2} &= i \left(\langle \partial_{k_x} u_{-\mathbf{k}}^{n_1} | \partial_{k_y} v_{-\mathbf{k}}^{n_2} \rangle - \langle \partial_{k_y} u_{-\mathbf{k}}^{n_1} | \partial_{k_x} v_{-\mathbf{k}}^{n_2} \rangle \right) \\ &= i \left(\langle \partial_{k_y} v_{-\mathbf{k}}^{n_2*} | \partial_{k_x} u_{-\mathbf{k}}^{n_1*} \rangle - \langle \partial_{k_x} v_{-\mathbf{k}}^{n_2*} | \partial_{k_y} u_{-\mathbf{k}}^{n_1*} \rangle \right) \\ &= i \left(\langle \partial_{k_y} \mathcal{T} v_{-\mathbf{k}}^{n_2} | \partial_{k_x} \mathcal{T} u_{-\mathbf{k}}^{n_1} \rangle - \langle \partial_{k_x} \mathcal{T} v_{-\mathbf{k}}^{n_2} | \partial_{k_y} \mathcal{T} u_{-\mathbf{k}}^{n_1} \rangle \right) \\ &= i \left(\langle \partial_{k_y} v_{\mathbf{k}}^{-n_2} | \partial_{k_x} u_{\mathbf{k}}^{-n_1} \rangle - \langle \partial_{k_x} v_{\mathbf{k}}^{-n_2} | \partial_{k_y} u_{\mathbf{k}}^{-n_1} \rangle \right) \\ &= -[\Omega_{-\tilde{n}}^{\text{LR}}(\mathbf{k})]^{-n_2, -n_1}, \end{aligned} \quad (2.4)$$

where in the first step we used $\langle \partial_{k_x} u_{-\mathbf{k}}^{n_1} | \partial_{k_y} v_{-\mathbf{k}}^{n_2} \rangle = \langle \partial_{k_y} v_{-\mathbf{k}}^{n_2*} | \partial_{k_x} u_{-\mathbf{k}}^{n_1*} \rangle$ and similarly for the second term. We see that $-\Omega_{-\tilde{n}}^{\text{LR}}(-\mathbf{k}) = \Omega_{\tilde{n}}^{\text{RL}}(\mathbf{k})$. These two versions of Berry connections, Ω^{LR} and Ω^{RL} , result in the same Chern number [68]. Accordingly, the Chern number for bands across an imaginary line gap related to one another by TRS obey

$$\begin{aligned} C_{\tilde{n}} &= \frac{1}{2\pi} \int \text{Tr} [\Omega_{\tilde{n}}^{\text{RL}}(\mathbf{k})] d^2\mathbf{k} \\ &= -\frac{1}{2\pi} \int \text{Tr} [\Omega_{-\tilde{n}}^{\text{LR}}(-\mathbf{k})] d^2\mathbf{k} = -C_{-\tilde{n}}. \end{aligned} \quad (2.5)$$

The Chern number only must vanish for energy bands lying on the real energy axis, such that $\tilde{n} = -\tilde{n}$. For a minimal model with two bands, $\tilde{1}$ and $-\tilde{1}$, related by TRS (1.6), we have $C_{\tilde{1}} = -C_{-\tilde{1}}$. When such a system is in contact with the vacuum, edge states must close the imaginary line gap that separates bands $\tilde{1}$ and $-\tilde{1}$ [Fig. 2.2(a)]. A less intuitive example occurs for a NH Hamiltonian with 4 bands

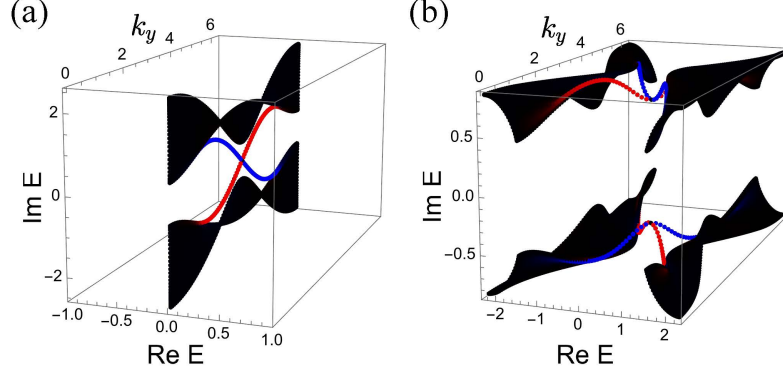


Figure 2.2: Energies of time-reversal symmetric NH Hamiltonians with “complex Chern bands” in the presence of an imaginary line gap. Both spectra are calculated under OBC along x and PBC along y . Black bands represent bulk bands, while blue and red lines denote edge states at opposite edges. In (a), the edge states cross an imaginary line gap for Hamiltonian (2.6) with $m = 0.5$, and in (b), they cross a real line gap for Hamiltonian (2.7) with $t = 1.2$. Both systems belong to class AI. Time-reversal symmetry relates the energy bands across the imaginary gap.

separated by both real and imaginary line gaps, all having nontrivial Chern numbers; in this case, the edge states can close the real line gap instead [Fig. 2.2(b)]. In both cases, the imaginary line gap closes for a Hermitian system, leading only to trivial phases.

As examples of these phases, first consider the Bloch Hamiltonian

$$h_{CC}^{\text{Im}}(\mathbf{k}) = -i\sigma_z \sin k_x + i\sigma_x \sin k_y + i\sigma_y(\cos k_y + \cos k_x + m), \quad (2.6)$$

where $\mathbf{k} = (k_x, k_y)$ is the crystal momentum, $\sigma_{x,y,z}$ are the Pauli matrices, and m sets the Chern number C , with $C = 1$ ($C = -1$) for $0 < m < 2$ ($-2 < m < 0$) or $C = 0$ else. This model obeys TRS (1.6) with $\mathcal{T} = \mathcal{K}$. A plot of its energy bands is shown in Fig. 2.2(a) for $m = 0.5$.

Next, consider the Bloch Hamiltonian

$$\begin{aligned}
h_{\text{CC}}^{\text{Re}}(\mathbf{k}) = & (\cos k_x + t/2)\sigma_x\tau_x + (\cos k_y + t/2)\sigma_z\tau_x \\
& - \sin k_x\sigma_y\tau_x - \sin k_y\sigma_0\tau_y \\
& + i(t/2)(\sigma_x\tau_y - \sigma_z\tau_y),
\end{aligned} \tag{2.7}$$

where σ_0 is the 2×2 identity matrix. This model obeys TRS (1.6) with $\mathcal{T} = \mathcal{K}$. When $0 < t < 1$, edge states cross the imaginary line gap, as in Hamiltonian (2.6). At $t = 1$, all edge bands touch. When $1 < t < 2$, topological edge states close the real line gap instead. For $t > 2$, this model enters the trivial phase, where no edge states exist. Figure 2.2(b) shows the energy bands of Hamiltonian (2.7) for $t = 1.2$. Note that there are two copies of Chern insulators with opposite Chern numbers above and below the imaginary line gap. This is the key feature of a real-line-gap Chern insulators under TRS (1.6). The Hamiltonian (2.7) obeys C_4 symmetry. Appendix A.7 describes a model on a C_6 -symmetric hexagonal lattice in class AI with a real-line gap Chern insulator phase.

2.2 Higher-order skin effect

In addition to bands separated by line gaps, NH systems also exhibit point gaps. In 1D, systems with a point gap exhibit the NH skin effect (NHSE), by which all eigenstates in a crystal with OBC exponentially localize at one of its edges [68, 92, 88, 76, 60, 97, 96, 91, 93, 29, 8].

This effect is associated with a bulk topological invariant, the winding number $W \in \mathbb{Z}$ of the complex energy spectrum with respect to a constant reference point E_p inside the point gap. For a translation invariant system with PBC, the winding

number is defined by

$$\begin{aligned} W &= \frac{1}{2\pi i} \int_{BZ} \text{Tr} (H'(k)^{-1} dH'(k)) \\ &= \frac{1}{2\pi i} \int_{BZ} dk \frac{d}{dk} \log \det H'(k), \end{aligned} \quad (2.8)$$

where $H'(k) = H(k) - E_p$ and E_p is any energy inside the point gap. Since the BZ is periodic, the complex spectra of NH Hamiltonians with point gaps form knots (or loops) in $(\text{Re}E, \text{Im}E, k)$ space. When more than one energy band wind, the winding is associated with the braid group B_N , where N is the number of separable energy bands [41, 63, 46, 87, 90, 27, 26]. The braid group classifies topologically inequivalent knots or loops. Such classification of 1D NH crystals was developed in Ref. [26]. In this and the next section, we focus on the windings and brading configurations at the 1D boundary of 2D NH lattices for systems with PBC along one direction and OBC along the other, so that, for example, edge states localized at edges $x = 1$ or $x = L$ can still be parametrized by the crystal momentum k_y along y . As shown in Fig. 1.2(b), there are NH lattices in which edge states spectrally disconnect from the 2D bulk bands. Remarkably, while the bulk bands present a line gap, the disconnected edge states present a point gap and carry nontrivial windings [Fig. 1.2(b)]. Now, consider the case in which edge windings occur for PBC along x and OBC along y , but not vice versa. In that case, under full OBC (i.e., OBC along both x and y) only one pair of edges manifests a 1D skin effect, collapsing its edge states to a pair of opposite corners of the 2D crystal. Since for a crystal of $L \times L$ unit cells, $O(L^2)$ states remain distributed across the bulk, and only $O(L)$ states localize at a corner, such phases have been referred to as possessing a “higher-order skin effect” [33, 45, 47, 58], in analogy with the existence of $O(1)$ corner states in 2D second-order topological phases [4, 5, 43, 77, 89, 28, 16].

A minimal model of a HOSE phase is given by the Bloch Hamiltonian

$$h_{\text{HOSE}}(\mathbf{k}) = -i(\gamma + \cos k_x)\sigma_0 + \sin k_x\sigma_z + (\gamma + \cos k_y)\sigma_y + \sin k_y\sigma_x, \quad (2.9)$$

proposed in Ref. [33]. The Hamiltonian (2.9) can be obtained from the quadrupole topological insulator (QTI) [4], which is chiral symmetric and can be written as $H_{\text{QTI}} = ((0, h_{\text{HOSE}}), (h_{\text{HOSE}}^\dagger, 0))$ ¹. Physically, the Bloch Hamiltonian (2.9) represents 1D horizontal Hatano-Nelson chains with alternating winding numbers stacked along the vertical direction. The chains are coupled via alternating vertical hopping terms with amplitudes 1 and γ . The phases of Hamiltonian (2.9) are controlled by the single parameter γ . When $0 < \gamma < 1$, there is a HOSE phase with $O(L)$ corner states at the top-left and bottom-right corners of a square lattice. A phase transition occurs at $\gamma = 1$. For $\gamma > 1$, the Hamiltonian (2.9) enters the trivial phase where the HOSE vanishes.

2.3 Deforming Chern insulators into HOSE phases

Both the edge states of Chern insulators and some of the edges in HOSE phases circumvent the no-go theorem by Nielsen and Ninomiya (NN), which forbids the existence of a net chirality in the states of crystals with noninteracting Hermitian Hamiltonians. The NN theorem stems from a topological consideration; in 1D, it is impossible to make a real spectrum periodic in the BZ without having an equal number of right- and left-moving states.

Chern insulators circumvent the limitations of the NN theorem by a topological bulk-boundary connection, by which edge states must merge into the bulk for part of

¹An extra term $-i\gamma\sigma_0$ will appear in this derivation, but this term only shifts the spectrum of the Hamiltonian (2.9) in the complex energy plane, with no consequences for its topological phase, and can therefore be discarded.

the spectrum, as in the region around $k_x = 0$ in Fig. 1.2(a). There is thus a topological obstruction to spectrally “peeling off” the chiral edge states of a Hermitian Chern insulator. At most, a bulk phase transition will eliminate the chiral states altogether.

Phases exhibiting the skin effect circumvent the limitations of the NN theorem by breaking Hermiticity. As a result, their energy spectra are complex, and can be made periodic in the complex energy plane as the crystal momentum traverses the 1D BZ while preserving a chirality. Since this nontrivial topology exists in a 1D manifold, 1D systems with a point gap topology do not need a bulk of a higher dimension to sustain it, nor a bulk-boundary correspondence. In HOSE phases, the edge states with point-gap topology are spectrally separated from the bulk bands. Yet, the existence of these edge states themselves is a manifestation of a (weak) nontrivial topological configuration of the 2D bulk bands.

The inequivalent ways in which the spectra of a periodic system connect across the BZ – some of which circumvent the NN theorem and lead to nonreciprocity – correspond to different topological classes, and thus, crystals that implement them correspond to distinct topological phases of matter, separated by bulk phase transitions. In particular, we are concerned with the relation between Chern insulators and HOSE phases, both of which have chiral edge states. Specifically, we ask whether the edge states of a Chern insulator can be peeled off its bulk if we promote its energy spectrum to the complex plane, in a similar way as the 1D edge states of the HOSE phase do. Such a connection would allow us to establish a relationship between Chern insulators and the HOSE, and consider these different systems on equal footing.

For this purpose, consider the two-band Bloch Hamiltonian

$$\begin{aligned}
 h_{\text{def}}^{\text{Re}}(\mathbf{k}) &= \sin k_x \sigma_x + \sin k_y \sigma_y \\
 &\quad + (m + t \cos k_x + \cos k_y) \sigma_z \\
 &\quad + i \cos k_x (1 - g) \sigma_0.
 \end{aligned} \tag{2.10}$$

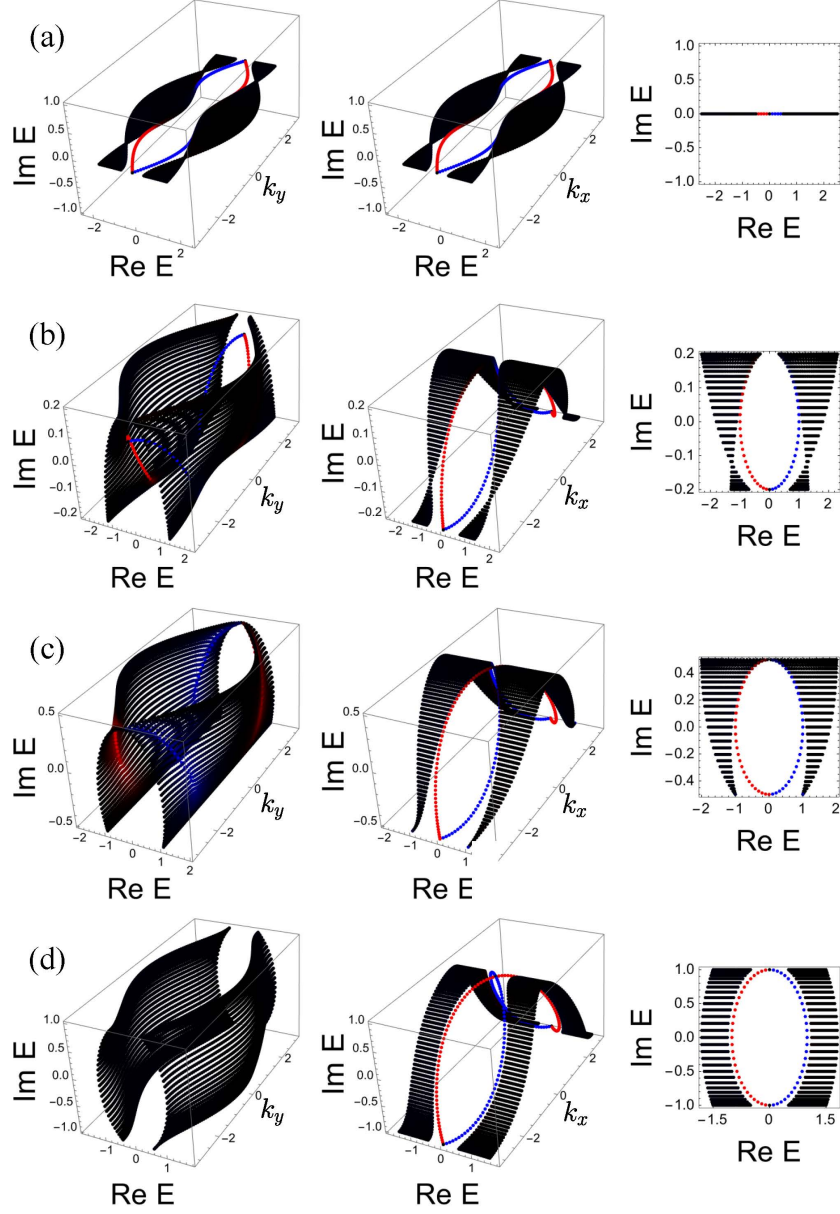


Figure 2.3: Deforming a real-line-gap Chern insulator into a HOSE phase in the tight-binding model with Bloch Hamiltonian (2.10). The deformation path chosen in parameter space is $(g, t) = (1, 1) - \theta(1, 1)$, for $\theta : 0 \rightarrow 1$. Panels (a)-(d) correspond to $\theta = 0, 0.2, 0.5, 1$, respectively. (a) A Hermitian Chern insulator phase. (b) A real-line-gap complex Chern insulator phase. (c) Bulk phase transition between the Chern insulator phase and the HOSE phase. (d) HOSE phase. In all panels, left column: OBC along x , PBC along y ; middle column: OBC along y , PBC along x ; right column: band projections of the plots on the middle column onto the complex energy plane. Black represents bulk states, while blue and red represent states localized at opposite edges.

When $g = 1$ and $t = 1$, this model is the Qi-Wu-Zhang (QWZ) Hamiltonian [62, 2], a minimal model for a Chern insulator, with Chern number $C = 1$ for $0 < m < 2$ or $C = 0$ otherwise. We fix the mass to $m = 0.5$. Its spectrum is shown in Fig. 2.3(a). To deform model (2.10) from the Chern insulator phase to the HOSE phase, we continuously vary the parameters g and t according to $(g, t) = \theta(1, 1)$ for $\theta : 1 \rightarrow 0$.

Note that $g \neq 1$ makes (2.10) non-Hermitian, promoting its spectrum to the complex plane. At $\theta = 0.2$, the model is in a NH Chern insulator phase [Fig. 2.3(b)], smoothly connected to the QWZ model in Fig. 2.3(a). As the deformation continues, a bulk phase transition at $\theta = 0.5$ changes the Chern number from $C = 1$ to $C = 0$ [Fig. 2.3(c)]. However, not all the chiral edge states merge and disappear into the bulk; instead, along one direction [middle panel in Fig. 2.3(c)], the edge states nontrivially reconnect, changing their topology to now wind in the complex plane. This is evident on the other side of the phase transition, as shown in Fig. 2.3(d) for $\theta = 1$. In Appendix A.3, we present complementary plots to those in Fig. 2.3.

During this deformation, the nontrivial topology in the bulk of the Chern insulator phase that gives rise to the chiral edge states is transferred to a nontrivial topological winding of the edge states themselves, leaving the bulk topologically trivial (from the point of view of strong topology). Since this skin effect occurs only at one pair of edges, a system in this phase with full OBC will necessarily manifest $O(L)$ corner states, i.e., it will manifest a HOSE. In fact, at the end of deformation process, $(g, t) = (0, 0)$, the Hamiltonian (2.10) is smoothly deformable to Hamiltonian (2.9) up to a transformation $\sigma_x \rightarrow \sigma_z, \sigma_y \rightarrow \sigma_x, \sigma_z \rightarrow \sigma_y$.

A similar deformation can connect the complex Chern insulator with an imaginary line gap (2.6) into a HOSE phase, (2.9). In this case, the entire deformation process is within class AI, obeying TRS (1.6). This process is shown in Appendix A.3.

Chapter 3

Approach

In this chapter, we introduce the classification of C_n -symmetric NH Hamiltonians in class AI in 2D. As we will see in chapter 4, this classification method can also help us classify higher-order topological knot phases. This chapter only describes general results. More detailed derivation can be found in Appendix A.4.

3.1 Classification of Higher-order topological knots

Since not all the edges in HOSE phases exhibit nonreciprocal transport, these phases possess $O(L)$ states exponentially localized at corners where edges with trivial and nontrivial winding intersect. In contrast, Chern insulators exhibit chiral edge states that extend continuously around the sample, enabling nonreciprocal propagation even in the presence of defects or corners [10, 86]. Thus, even though the edges in Chern insulators and some of the edges in HOSE phases circumvent the NN theorem, there are important differences among them, resulting in distinct densities of states and associated transport phenomena. These differences raise a question: can NH lattices sustain nontrivial windings along all edges of a sample, as Chern insulators do, rather than only along some edges, as in HOSE phases? In the following, we explore lattices with these properties. Specifically, we consider NH bulk-boundary correspondence

mechanisms that generate edge states with uniform nontrivial winding across all edges. This approach eliminates the $O(L)$ corner-localized states characteristic of HOSE phases, provided the protecting symmetries remain intact. To this effect, we first construct the topological classification of NH phases with C_n symmetries.

3.1.1 Classification of C_n -symmetric NH Hamiltonians in class AI

The presence of crystalline symmetries enriches the classification of topological phases [40, 74, 50, 79, 3, 70, 71, 72, 6, 95, 61, 9, 82]. In 2D NH Hamiltonians, HOSE phases have been understood by drawing a correspondence between the NH Hamiltonian in question h_{NH} and an associated chiral-symmetric Hermitian Hamiltonian $h_{\text{H}} = [0, h_{\text{NH}}; h_{\text{NH}}^\dagger, 0]$ [51, 52], both of which carry identical topological information [33]. This correspondence then makes use of existing crystalline topological classifications of h_{H} to diagnose topological phenomena in h_{NH} [69, 15, 83, 59, 44, 99, 84].

Here, we consider the crystalline classification of h_{NH} itself, i.e., without appealing to its corresponding h_{H} . The utility of such a classification was recently presented in Ref. [99] for a C_3 -symmetric NH Hamiltonian, and here we extend this classification to all C_n symmetries. Specifically, consider NH Hamiltonians obeying TRS (1.6) and C_n symmetry,

$$\hat{r}_n h(\mathbf{k}) \hat{r}_n^{-1} = h(R_n \mathbf{k}), \quad (3.1)$$

where $\hat{r}_n$ is the rotation operator acting on the unit cell degrees of freedom, obeying $[\hat{r}_n]^n = 1$ or $[\hat{r}_n]^n = -1$ (the latter case due to, e.g., a magnetic flux threading the lattice), and R_n is the n -fold rotation matrix acting on the crystal momentum $\mathbf{k}$. The high symmetry points (HSPs) in the BZ zone, $\mathbf{\Pi}_m$, for $m \leq n$, are crystal momenta that remain invariant under the little-group C_m rotation (modulo a reciprocal lattice

vector $\mathbf{G}$), i.e., $\mathbf{\Pi}_m = R_m \mathbf{\Pi}_m \pmod{\mathbf{G}}$. For example, setting the length of the unit cell to unity, in C_2 symmetric lattices, $\mathbf{X} = (\pi, 0)$, $\mathbf{Y} = (0, \pi)$, and $\mathbf{M} = (\pi, \pi)$ are all invariant under C_2 rotations, while in C_4 symmetric lattices, $\mathbf{X} = (\pi, 0)$ and $\mathbf{X}' = (0, \pi)$ are C_2 invariant and $\mathbf{M} = (\pi, \pi)$ is C_4 invariant. Also, notice that trivially, $\mathbf{\Gamma} = (0, 0)$ is invariant under the full group of a C_n symmetric lattice. Appendix A.4 describes the HSPs for all C_n -symmetric lattices.

At the HSPs, Eq. (3.1) implies that $[\hat{r}_m, h(\mathbf{\Pi}_m)] = 0$ and thus $\hat{r}_m$ and $h(\mathbf{\Pi}_m)$ have simultaneous eigenstates; the Bloch eigenstates of energy band l , $|u_{\mathbf{\Pi}_m}^l\rangle$, which obey $h(\mathbf{\Pi}_m) |u_{\mathbf{\Pi}_m}^l\rangle = \epsilon_l(\mathbf{\Pi}_m) |u_{\mathbf{\Pi}_m}^l\rangle$, simultaneously obey

$$\hat{r}_m |u_{\mathbf{\Pi}_m}^l\rangle = r_{\mathbf{\Pi}_m}^l |u_{\mathbf{\Pi}_m}^l\rangle, \quad (3.2)$$

where $r_{\mathbf{\Pi}_m}^l$ is the rotation eigenvalue associated with energy band l at HSP $\mathbf{\Pi}_m$, which can take the values

$$\Pi_p^{(m)} = \begin{cases} e^{2\pi i(p-1)/m}, & \text{for } [\hat{r}_n]^n = 1 \\ e^{2\pi i(p-1/2)/m}, & \text{for } [\hat{r}_n]^n = -1 \end{cases} \quad (3.3)$$

for $p = 1, 2, \dots, m$. Extending previous studies on the classification of crystalline topological phases [79, 3, 6, 66, 82], we define the symmetry indicator invariants for energy band l as

$$[\Pi_p^{(m)}] = \#_l \Pi_p^{(m)} - \#_l \Gamma_p^{(m)}, \quad (3.4)$$

where $\#_l \Pi_p^{(m)}$ is the number of eigenstates in the band l with rotation eigenvalue $\Pi_p^{(m)}$ at HSP $\mathbf{\Pi}_m$. The set of symmetry indicator invariants (3.4) across all HSPs of the BZ, along with the Chern number, provide a topological classification for the energy band l . However, some of these invariants are related to one another by symmetry, and thus there is redundancy in the topological information (Appendix A.4). We collect the set of non-redundant indicators in a vector index $\chi^{(n)}$ that uniquely identifies

the topological class of complex energy bands in C_n -symmetric lattices (Appendix A.4 contains the complete derivation of the $\chi^{(n)}$ indices for all C_n -symmetric Bloch Hamiltonians). These $\chi^{(n)}$ indices are

$$\begin{aligned}\chi^{(2)} &= (C|[X_1^{(2)}], [Y_1^{(2)}], [M_1^{(2)}]) \\ \chi^{(4)} &= (C|[X_1^{(2)}], [M_1^{(4)}], [M_2^{(4)}], [M_3^{(4)}]) \\ \chi^{(3)} &= (C|[K_1^{(3)}], [K_2^{(3)}], [K_1'^{(3)}], [K_2'^{(3)}]) \\ \chi^{(6)} &= (C|[M_1^{(2)}], [K_1^{(3)}], [K_2^{(3)}]).\end{aligned}\tag{3.5}$$

NonHermitian C_n -symmetric Hamiltonians in class AI with different $\chi^{(n)}$ indices belong to different topological phases, as they cannot be deformed into one another without closing the bulk energy gaps or breaking the symmetry.

The $\chi^{(n)}$ indices obey an algebraic structure. If two bands a and b in classes $\chi_a^{(n)}$ and $\chi_b^{(n)}$ are combined, the resulting Hamiltonian is in class $\chi_{a \cup b}^{(n)} = \chi_a^{(n)} + \chi_b^{(n)}$. As a consequence, if two bands have complementary topological indices, i.e., if $\chi_a^{(n)} = -\chi_b^{(n)}$, such that $\chi_{a \cup b}^{(n)} = 0$, boundary states will exist in the gap between them under OBC.

As an example, consider the Bloch Hamiltonian $h_{\text{HOSE}}(\mathbf{k})$ of Eq. (2.9) for the minimal model of a HOSE phase. While $h_{\text{HOSE}}(\mathbf{k})$ does not obey TRS (1.6), $ih_{\text{HOSE}}(\mathbf{k})$ does, with $\mathcal{T} = \mathcal{K}$. Additionally, $ih_{\text{HOSE}}(\mathbf{k})$ obeys C_2 symmetry, with rotation operator $\hat{r}_2 = \sigma_y$. The $\chi^{(2)}$ indices for its two bulk energy bands are shown in Appendix A.5. The two indices trivialize in pairs, leading to boundary states in the gap between these two bands under OBC. These are the skin effect modes of the HOSE phase.

This pairwise trivialization of bands is the simplest case of a more general topological connectivity between bands. In particular, there are cases in which no pairwise trivialization occurs; instead, bands trivialize only in groups of three or four. As we will see, these cases can undergo unusual phase transitions in which the line gaps close, generating a point gap, from which edge-localized states with nontrivial windings

emerge. These are the novel phases we refer to as “higher-order topological knot” (HOTEK) phases, for reasons to become apparent in the next chapter.

Chapter 4

Analysis

In this chapter, we give explicit lattice models and Hamiltonians for complex-Chern insulator and higher-order topological knots phases. We classify them using $\chi^{(n)}$ index introduced in the previous chapter.

4.1 Model Hamiltonians with HOTK phases

Higher-order topological knot phases are NH topological phases with *bulk line gaps* and nontrivial weak topology that generate edge states with *edge point gaps* with nontrivial windings along all edges. Note that similar characteristics are exhibited by HOSE phases. However, in HOSE phases, nontrivial edge states exist only at certain edges. In Section 2.3, we showed that a two-band model with a line gap and nonzero Chern numbers can only transition into a HOSE phase because, while the chiral edge states deform into a pair of edge states with a nontrivial winding number in the complex energy plane, the other pair of chiral edges merge into the bulk, resulting in no topological states in the HOSE phase. This suggests that the minimal model that generates topological edge states along all edges requires more than two line-gap energy bands.

In what follows, we will first present a four-band model for a C_4 -symmetric lattice

that exists in either a complex Chern insulator phase or in a HOTK phase. We will then describe HOTK models obeying C_3 and C_6 crystalline symmetries. Figure 4.1 illustrates the lattices we consider. The Hamiltonians for these lattices are written in a “maximally nonreciprocal limit”, where each of the arrows in Fig. 4.1 represent hoppings only along their directions. We do this to provide the simplest, minimal realizations of these phases; however, more generally, the arrows can represent an imbalance in the amplitude of hoppings along their direction vs their opposite direction. The lines with no arrow represent reciprocal hoppings.

Protected by C_4 symmetry

Consider the following Bloch Hamiltonian, which hosts both a complex Chern insulator phase and a HOTK phase,

$$\begin{aligned} h^{(4)}(\mathbf{k}) = & t\sigma_x(\tau_x + \tau_z) \\ & + \frac{1}{2}(\sigma_x + i\sigma_y)[\tau_x \cos(gk_x) - \tau_y \sin(gk_x)] \\ & + \frac{1}{2}(\sigma_x - i\sigma_y)[\tau_z \cos(gk_y) - i\tau_0 \sin(gk_y)], \end{aligned} \quad (4.1)$$

where both σ_i and τ_i , for $i = x, y, z$, are Pauli matrices and σ_0 and τ_0 are the 2×2 identity matrix. This model is parametrized by $t \in \mathbb{R}^+$, the amplitude of reciprocal hoppings within the unit cell, and $g \in \mathbb{Z}^+$, the nonreciprocal hopping distance between unit cells. We first consider the case $g = 1$, which corresponds to nearest-neighbor inter-cell hopping. The lattice of this model is shown in Fig. 4.1(a) and carries a π flux per plaquette, which is accounted for by a -1 sign in the hopping terms corresponding to dotted lines in Fig. 4.1(a). This model resembles the quadrupole topological insulator model of Ref. [5], but with the crucial distinction that its inter-cell hopping terms are nonreciprocal. It obeys TRS (1.6) with $\mathcal{T} = \mathcal{K}$ and C_4 symmetry

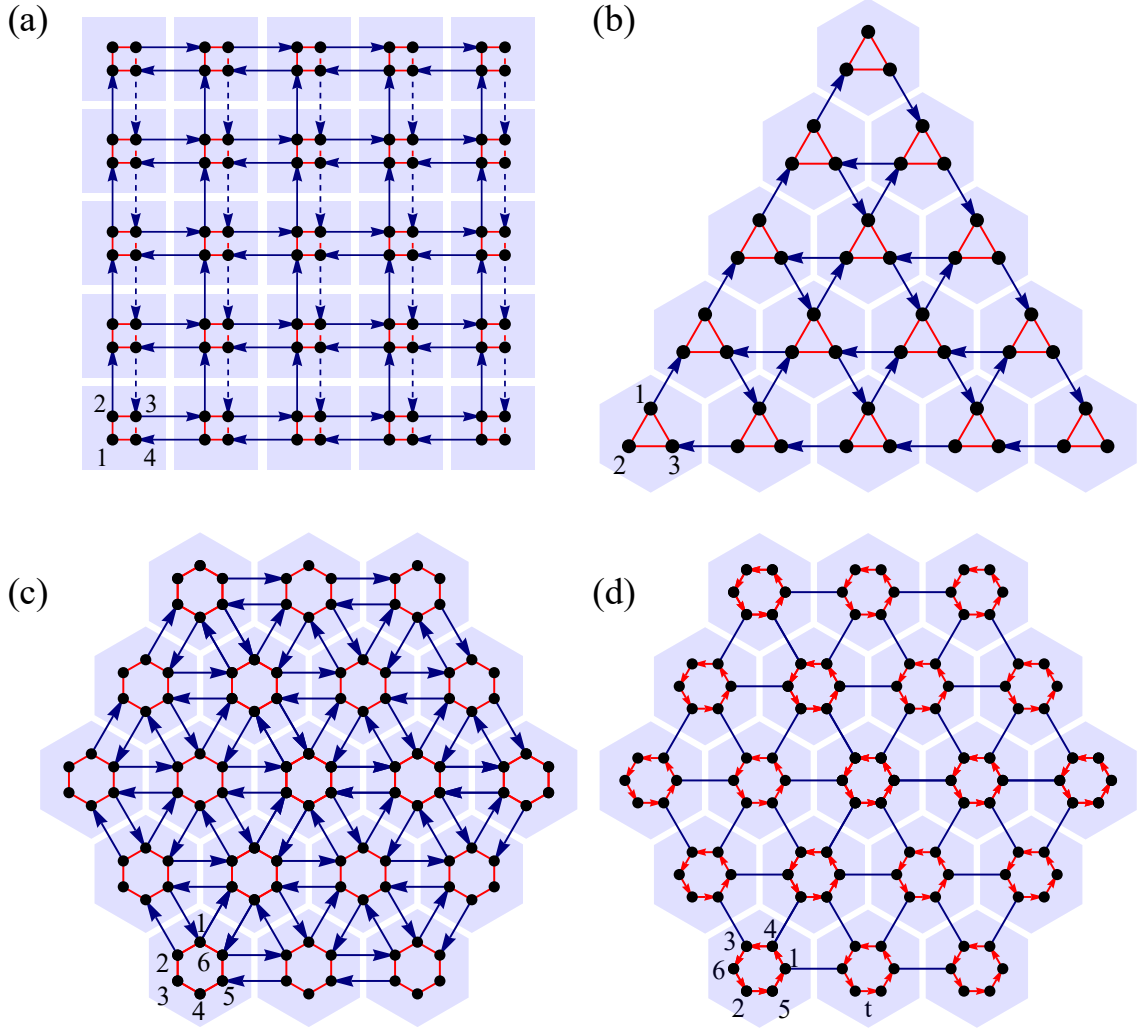


Figure 4.1: Tight-binding lattices that support various NH topological phases. Arrows indicate the direction of nonreciprocal hopping. (a) The C_4 -symmetric lattice with the Bloch Hamiltonian given in Eq. (4.1) hosts both a HOTK phase and an imaginary-line-gap complex Chern insulator phase. The hopping terms corresponding to dotted lines carry a -1 sign, a gauge choice to account for a π flux per plaquette. (b,c) The C_3 -symmetric (C_6 -symmetric) lattice with the Bloch Hamiltonian in Eq. (4.2) [Eq. (4.3)], which hosts a HOTK phase. The C_3 -symmetric model also supports an imaginary-line-gap complex Chern insulator phase. (d) The C_6 -symmetric lattice with the Bloch Hamiltonian in Eq. (A.51), which supports a real-line-gap complex Chern insulator phase.

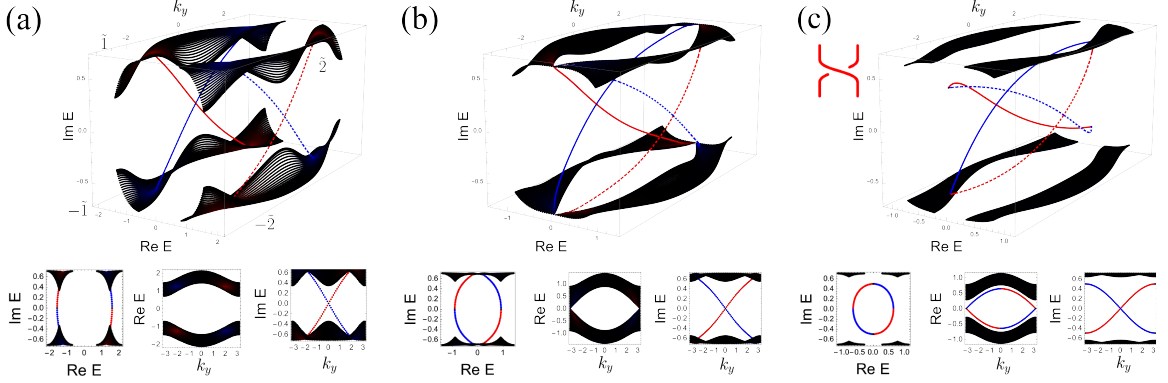


Figure 4.2: Energy spectra of Hamiltonian (4.1) in the $(\text{Re}E, \text{Im}E, k)$ space under OBC along x and PBC along y with (a) $t = 1$, (b) $t = 0.5$, and (c) $t = 0.3$. Due to C_4 symmetry, the energy spectra under OBC along y and PBC along x are identical to these plots. Black indicates bulk states; red and blue indicate states localized at opposite edges. The lower panels show projections of the 3D plots into 2D planes to aid visualization. The diagram at the top left corner of (c) indicates the braid group of each of the edge states across the BZ.

(3.1) with rotation operator

$$\hat{r}_4 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \end{pmatrix},$$

which obeys $\hat{r}_4^4 = -1$.

The Hamiltonian (4.1) has four bands, each occupying a quadrant on the complex plane. These four bands are labeled according to the rule introduced in Sec. 2.1 as shown in Fig. 4.2(a).

When $t > 0.5$, the Hamiltonian (4.1) is in a complex Chern insulator phase. The $\chi^{(4)}$ indices of each of the four bands in this phase are given in Table 4.1. Note that bands $\tilde{1}$ and $\tilde{2}$ have complementary topologies to those of bands $-\tilde{1}$ and $-\tilde{2}$, respectively. Consequently, the topological edge states cross the imaginary line gap [Fig. 4.2(a)].

A transition to a trivial phase would close the imaginary line gap to connect band

Phase	Band	C	$[X_1^{(2)}]$	$[M_1^{(4)}]$	$[M_2^{(4)}]$	$[M_3^{(4)}]$
Complex Chern	$\tilde{1}$	-1	1	0	-1	1
	$-\tilde{1}$	1	-1	0	1	-1
	$\tilde{2}$	1	-1	-1	0	0
	$-\tilde{2}$	-1	1	1	0	0
HOTK	$\tilde{1}$	0	1	0	-1	0
	$-\tilde{1}$	0	-1	1	0	-1
	$\tilde{2}$	0	-1	-1	0	1
	$-\tilde{2}$	0	1	0	1	0

Table 4.1: $\chi^{(4)}$ indices for the energy bands of Hamiltonian (4.1). Bands are labeled as indicated in Fig. 4.2(a). The Complex Chern insulator and HOTK phases correspond to $t > 0.5$ and $t < 0.5$, respectively.

Phase	Band	C	$[X_1^{(2)}]$	$[Y_1^{(2)}]$	$[M_1^{(2)}]$
Complex Chern	$\tilde{1}$	-1	1	1	1
	$-\tilde{1}$	1	-1	-1	-1
	$\tilde{2}$	1	-1	-1	-1
	$-\tilde{2}$	-1	1	1	1
HOTK	$\tilde{1}$	0	1	1	0
	$-\tilde{1}$	0	-1	-1	0
	$\tilde{2}$	0	-1	-1	0
	$-\tilde{2}$	0	1	1	0

Table 4.2: $\chi^{(2)}$ indices for the energy bands of Hamiltonian (4.1).

Phase	Band	C	$[K_1^{(3)}]$	$[K_2^{(3)}]$	$[K_1'^{(3)}]$	$[K_2'^{(3)}]$
HOTK	$\tilde{1}$	0	0	-1	1	-1
	$-\tilde{1}$	0	1	0	0	1
	$\tilde{2}$	0	-1	1	-1	0
Complex Chern	$\tilde{1}$	2	0	-1	0	-1
	$-\tilde{1}$	-2	0	1	0	1
	$\tilde{2}$	0	0	0	0	0

Table 4.3: $\chi^{(3)}$ indices for Hamiltonian (4.2). The HOTK and Complex Chern insulator phases correspond to $0 < t < 0.57$ and $0.57 < t < 1$, respectively.

Phase	Band	C	$[M_1^{(2)}]$	$[K_1^{(3)}]$	$[K_2^{(3)}]$
HOTK	$\tilde{1}$	0	-1	0	-1
	$\tilde{2}$	0	1	1	-1
	$\tilde{3}$	0	0	-2	1
	$-\tilde{1}$	0	1	1	1
	$-\tilde{2}$	0	-1	0	0

Table 4.4: $\chi^{(6)}$ indices for Hamiltonian (4.3) in the HOTK phase, which corresponds to $0 < t < 0.5$.

$\tilde{1}$ with band $-\tilde{1}$, and band $\tilde{2}$ with band $-\tilde{2}$, so that on the other side of the transition, $\chi^{(4)} = \mathbf{0}$ for all four bands, causing the edge states to disappear. However, a different phase transition occurs in this model when $t = 0.5$. This transition closes the real line gap instead, and connects band $\tilde{1}$ with band $\tilde{2}$ and band $-\tilde{1}$ with band $-\tilde{2}$. Along with these bulk gap closings, the edge states also close the real-line gap [Fig. 4.2(b)].

When $t < 0.5$, that is, on the other side of the nontrivial transition, the Hamiltonian (4.1) is in a HOTK phase [Fig. 4.2(c)]. The four bulk energy bands present real and imaginary line gaps, as before the transition. However, now separated from the bulk bands, edge states braid across the BZ in the complex energy plane around $E_p = 0$. Since there are two edge bands for each edge, the corresponding braid group is B_2 . The braid structure for one of the edge states is schematically illustrated at the top left corner of Fig. 4.2(c). Due to the periodicity of the BZ, the braid is also a knot, and for this phase, the knot has periodicity $4\pi/a$, where a is the unit cell length, i.e., it takes going around the BZ twice to get back to the original point in the edge spectrum. The plots of energy bands for OBC along y and PBC along x (not shown) are identical to those in Fig. 4.2, by C_4 symmetry. Thus, topological edges exist along the entire 1D boundary of the HOTK phase in Hamiltonian (4.1).

A plot of the density of edge states under full OBC for the HOTK phase is shown in Fig 4.3(a). Notably, the states have support along all edges. This is significantly different from the phenomenon of HOSE, in which eigenstates decay exponentially

from corners [33]. In the bulk bands, the nontrivial topology manifests in the nonzero $\chi^{(4)}$ indices shown in Table 4.1. Note that none of these bands can be trivialized in pairs; only the $\chi^{(4)}$ index for the ensemble of all four bands is trivial. This indicates that all four bands participate in the nontrivial topology of the HOTK phase.

The boundary states essentially constitute a boundary-localized 1D chain with PBC and an associated non-zero winding number protected by a point gap at $E_p = 0$, and thus, it lies in the same topological class as the 1D Hatano-Nelson model [22, 23, 21]. As such, the HOTK phase protects the nonreciprocal transport of wave packets along its 1D boundary, with unidirectional acceleration and amplification [48, 75, 98, 34].

Breaking C_4 symmetry and robustness of edge states – The complex Chern insulator phase of Hamiltonian (4.1) does not require crystalline symmetries; even in the absence of $\chi^{(4)}$ indices, the nonzero Chern numbers $C = \pm 1$ persists as long as the imaginary line gap is maintained. However, that is not the case for the HOTK phase, where all the bands have $C = 0$. It may be tempting to think that C_4 symmetry is crucial for protecting the topological phase. However, the nontrivial $\chi^{(4)}$ indices of the HOTK phase are sufficient, but not necessary, for the protection of the phase. Boundary states will persist as long as the bulk line gaps and the gap between bulk and boundary states remain open. In the absence of C_4 symmetry, however, the localization of boundary states may significantly change.

To examine the fate of the edge states in the HOTK phase, we break C_4 symmetry in three ways: (i) by making the hopping amplitude along y different than that along x , (ii) by adding onsite energy perturbations $\epsilon \text{diag}(1, -1, 1, -1)$ at each unit cell, and (iii) by adding a local onsite energy defect at an arbitrary point along the edge. While (i) and (ii) reduce C_4 symmetry to C_2 symmetry, (iii) breaks all global crystalline symmetries. The edge states for some of these cases are shown in Fig. 4.3. In case (i), the edge states do not extend along the edges anymore and instead exponentially localize

at corners, as shown in Fig. 4.3(b). In case (ii), although C_4 symmetry is broken, the resulting distribution of edge states is similar to that of Fig. 4.3(a). In case (iii), as shown in Fig. 4.3(c), the defect does affect the distribution of edge states, localizing some of them, but leaving most of them distributed along the entire boundary. This shows that HOTK phases are robust against disorders: while the distribution of the boundary modes may change, they generally exist across all boundaries. Under only C_2 symmetry, the $\chi^{(2)}$ indices of the bulk bands are shown in Table 4.2. Note that the bands can now be trivialized in pairs across either the real line gap or the imaginary line gap; thus, transitions that close either of the two line gaps could unwind the edge states or make them disappear.

Increasing the braiding of edge states – The braiding structure of the edge states across the BZ in the HOTK phase of Hamiltonian (4.1) is modified with increasing hopping distance g . For $g = 2$, the lattice has only next-nearest neighbor hoppings between unit cells. The spectra for the complex Chern insulator phase and the HOTK phase are shown in Fig. 4.4(a) and 4.4(b), respectively. Each of the bands in the Complex Chern insulator phase has $C = \pm 4$, and each edge state in Fig. 4.4(a) is two-fold degenerate. At each edge in the HOTK phase, the edge states braid forming a Hopf link, as shown in Fig. 4.4(c) for only one edge. Similarly, setting $g = 3$ fixes the Chern number of each band in the Complex Chern insulator phase to $C = \pm 9$, with each edge state in Fig. 4.4(d) being threefold degenerate. At each edge of the HOTK phase, the edge states braid forming a Trefoil knot, as shown in Fig. 4.4(f) for one of the edges.

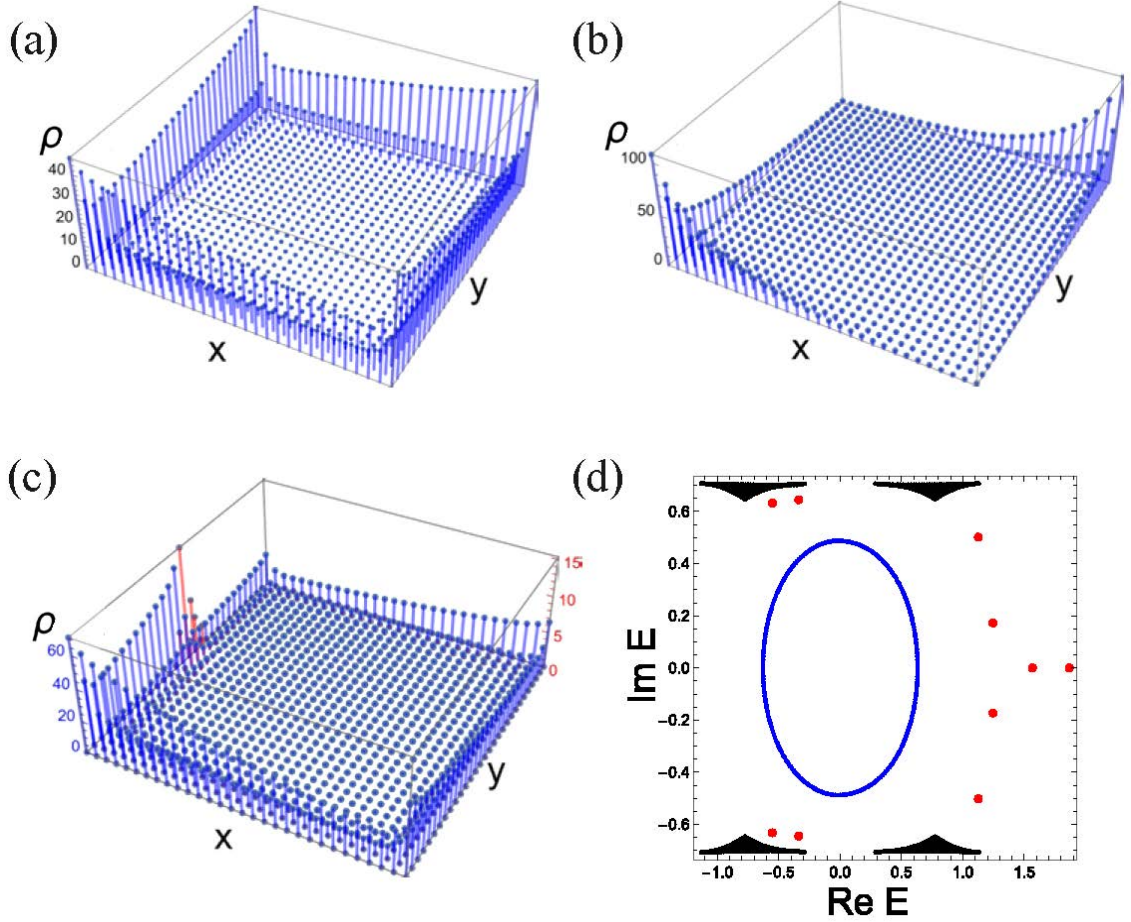


Figure 4.3: Local density of edge states ρ in the HOTK phase of Hamiltonian (4.1) under full OBC (30×30 unit cells) for three scenarios: (a) when C_4 symmetry is preserved, (b) when only C_2 symmetry is preserved by reducing hopping amplitudes along the y direction, and (c) when a defect is added to one edge. In (b), the C_4 symmetry of Hamiltonian (4.1) has been reduced to only C_2 symmetry by setting the inter-cell hopping along x to be 1 and the one along y to be 0.7, while setting the intra-cell hopping to $t = 0.3$. Panel (d) shows the spectrum corresponding to case (c) where the defect is present. In both the plots of the energy spectrum and local density of states, edge states are indicated in blue, while defect states are highlighted in red.

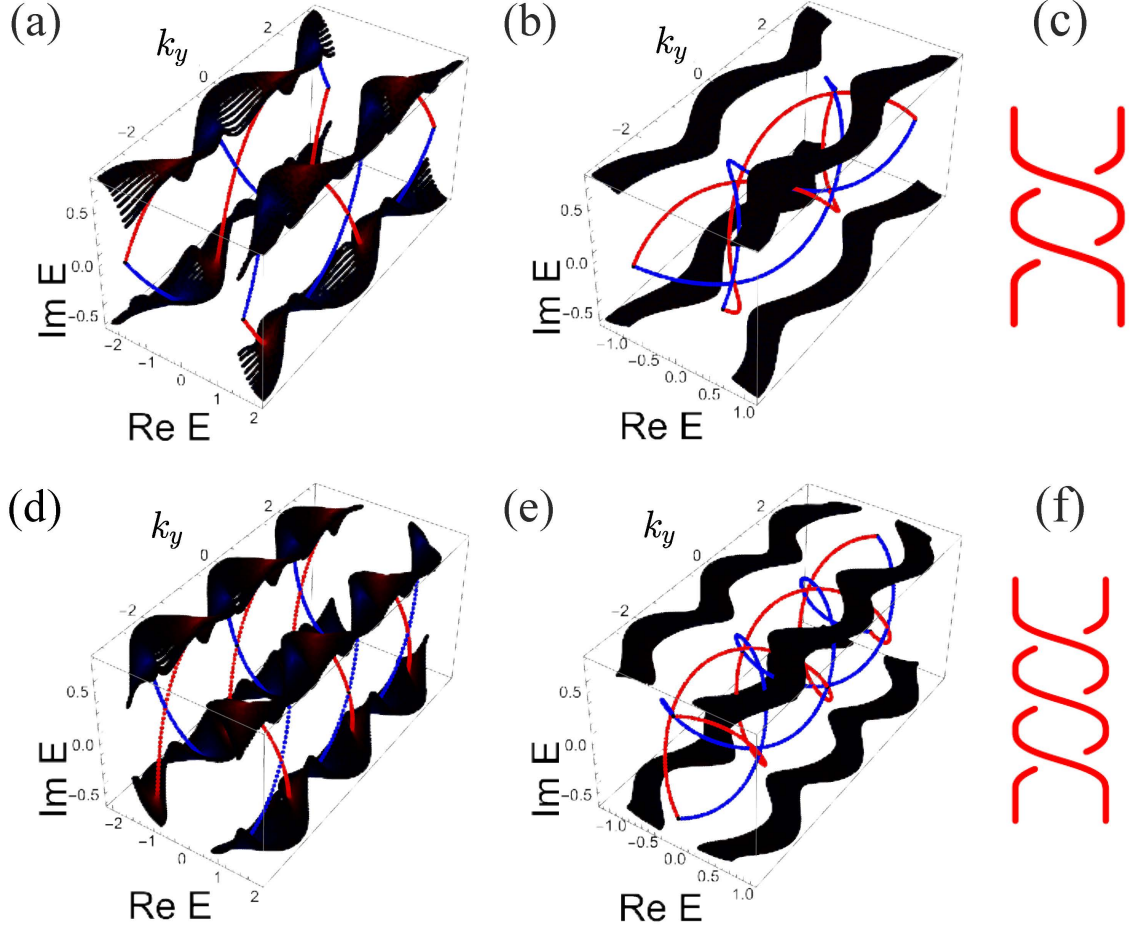


Figure 4.4: Energy spectra of Hamiltonian (4.1) under OBC along x and PBC along y for $g = 2$ (first row) and $g = 3$ (second row). First column: Complex Chern insulator phases with $t = 1$. Each bulk band has a Chern number of magnitude g^2 , specifically, $C = \pm 4$ in (a) and $C = \pm 9$ in (d). The edge states in (a) and (d) are twofold and threefold degenerate, respectively. Second column: HOTK phases, with $t = 0.3$. Third column: braid structure of the edge states at each edge of the HOTK phases.

Protected by C_3 symmetry

We now consider the NH Kagome lattice shown in Fig. 4.1(b). It has Bloch Hamiltonian

$$h^{(3)}(\mathbf{k}) = \begin{pmatrix} 0 & t & t + e^{-i\mathbf{k}\cdot\mathbf{a}_3} \\ t + e^{-i\mathbf{k}\cdot\mathbf{a}_2} & 0 & t \\ t & t + e^{i\mathbf{k}\cdot\mathbf{a}_1} & 0 \end{pmatrix}, \quad (4.2)$$

where $\mathbf{a}_1 = (1, 0)$, $\mathbf{a}_2 = (1/2, \sqrt{3}/2)$, $\mathbf{a}_3 = (1/2, -\sqrt{3}/2)$. This model obeys TRS (1.6) with $\mathcal{T} = \mathcal{K}$ and C_3 symmetry with rotation operator

$$\hat{r}_3 = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$

For $0 < t < 0.57$, this model is in a HOTK phase. Its spectrum at $t = 0.3$ is shown in Fig. 4.5(a). The $\chi^{(3)}$ index in this phase is shown in Table 4.3, where we have labeled the bands as indicated in Fig. 4.5(a). Note that no pairs of bands lead to a trivial index; a trivial $\chi^{(3)}$ index is obtained only for the set of all three bands, as expected for a HOTK phase. Under full OBC, a crystal that preserves C_3 symmetry, such as the triangular crystal shown in Fig. 4.1(b), hosts point-gapped edge states along all edges, as shown in Fig. 4.5(c). These edges are then associated with nonreciprocal propagation.

When a crystal under full OBC breaks C_3 symmetry, the edge states can collapse into corners. This phenomenon was recently presented in Ref. [99] as the realization of a HOSE phase in the Kagome lattice. The exponential localization of states to a corner is similar to what we found in the HOTK phase of Hamiltonian (4.1) when C_4 symmetry is reduced to C_2 symmetry [Fig. 4.3(b)]. However, the corner states in the configuration of Fig. 4.5(d) were demonstrated to have a correspondence with

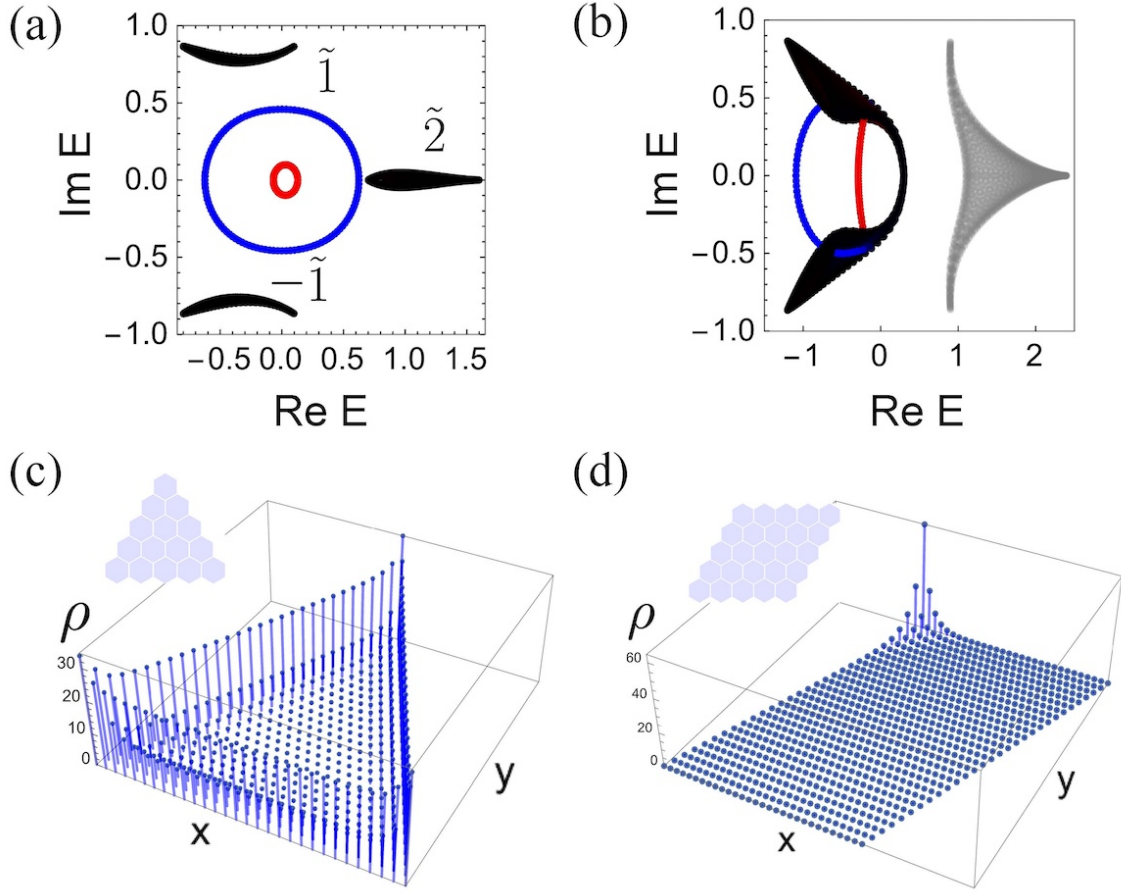


Figure 4.5: Energy spectra under OBC along y and PBC along x (a,b) and edge density of states under full OBC (c,d) of Hamiltonian (4.2). (a,c,d) are in the HOTK phase, with $t = 0.3$. (b) is in the complex Chern insulator phase, with $t = 0.7$. The insets in (c) and (d) schematically indicate the shape of the lattice.

the zero energy corner states of a chiral symmetric hermitian higher-order topological insulator, predicted and demonstrated experimentally in Ref. [57].

For $0.57 < t < 1$, the Hamiltonian (4.2) is in a complex Chern insulator phase with the $\chi^{(3)}$ indices shown in Table 4.3. As expected, the $\chi^{(3)}$ indices of bands $-\tilde{1}$ and $\tilde{1}$ are complementary in this phase, which necessarily leads to a trivial $\chi^{(3)}$ index for band $\tilde{2}$. The energy spectrum for $t = 0.7$ is shown in Fig. 4.5(b). The chiral edge states cross the imaginary line gap between bands $-\tilde{1}$ and $\tilde{1}$. Although no direct line gap exists between $-\tilde{1}$ and $\tilde{1}$, these two bands are indeed separated everywhere in the BZ (Appendix A.6).

Protected by C_6 symmetry

Finally, we consider the NH Hamiltonian for the hexagonal lattice in Fig. 4.1(c), described by the Hamiltonian

$$h^{(6)}(\mathbf{k}) = \begin{pmatrix} 0 & t & 0 & 0 & e^{-i\mathbf{k}\cdot\mathbf{a}_3} & t \\ t & 0 & t & 0 & 0 & e^{-i\mathbf{k}\cdot\mathbf{a}_1} \\ e^{-i\mathbf{k}\cdot\mathbf{a}_2} & t & 0 & t & 0 & 0 \\ 0 & e^{i\mathbf{k}\cdot\mathbf{a}_3} & t & 0 & t & 0 \\ 0 & 0 & e^{i\mathbf{k}\cdot\mathbf{a}_1} & t & 0 & t \\ t & 0 & 0 & e^{i\mathbf{k}\cdot\mathbf{a}_2} & t & 0 \end{pmatrix}. \quad (4.3)$$

This model obeys TRS (1.6) and C_6 symmetry with a rotation operator $\hat{r}_6$ that permutes the sites within the unit cells of the lattice in Fig. 4.1(c) upon rotation by $2\pi/6$ about the center of the unit cell.

In the range $0 < t < 0.5$, the Hamiltonian (4.3) is in a HOTK phase. A phase transition happens at $t = 0.5$. At $0.5 < t < 1$, Eq. (4.3) enters a gapless phase in which edge states merge into the bulk. A real line gap will re-open at $t > 1$. However, all $t > 1$ phases are trivial. The $\chi^{(6)}$ indices for the HOTK phase are shown in Table 4.4 [we labeled each band according to Fig. 4.6(a)].

Figure 4.6(a) shows the energy spectrum of Hamiltonian (4.3) in the HOTK phase, at a value of $t = 0.2$. Once again, none of the $\chi^{(6)}$ indices for each band can be trivialized in pairs or in any combination other than the one involving all bands. Under full OBC that preserves C_6 symmetry, such as the hexagon shown in Fig. 4.1(c), edge states with point gaps and nontrivial winding exist at along all edges [Fig. 4.6(b)]. In fact, these states perdure even in lattices that preserve only C_3 symmetry [Fig. 4.6(c)] or C_2 symmetry [Fig. 4.6(d)]. We show a more detailed spectrum of Hamiltonian (4.3) in Appendix A.6.

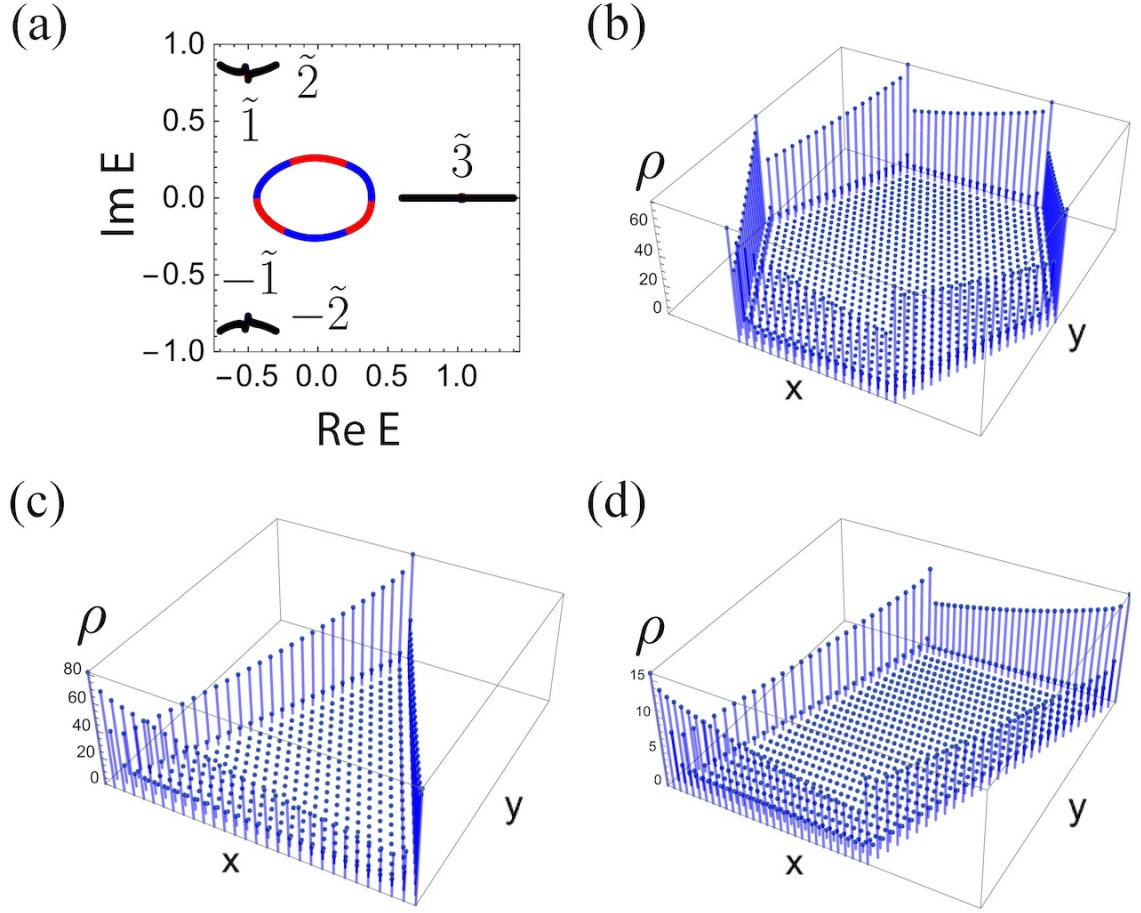


Figure 4.6: Energy spectra under PBC along x and OBC along y with a zig-zag edge (a) and edge density of states under full OBC (b-d) of Hamiltonian (4.3) in the HOTK phase with $t = 0.2$.

Chapter 5

Conclusion

We have introduced and characterized new topological phases, which we term “higher-order topological knot” (HOTK) phases. These are NH phases defined by bulk bands with line gaps that support edge states with point gaps and nontrivial winding numbers. While these spectral characteristics are present in HOSE phases, there are two crucial distinctions: (i) in HOSE phases, the point-gap boundary states are always localized at corners; and (ii) these corner-localized states are skin states arising from nontrivial point-gap topology present only at certain edges of the lattice. In contrast, HOTK phases present boundary point-gap states extended across all edges, with no skin effect.

In this work, we focused on HOTK phases protected by C_n symmetries. As markers of their topology, we identified symmetry indicator invariants for class AI under C_n symmetry. Specifically, HOTK phases are characterized by bands with nonzero $\chi^{(n)}$ indices that cannot be trivialized in pairs, indicating that these phases arise from multi-band topology involving three or more bands, all separated by bulk line gaps. This topological connectivity is unique to NH systems, as multiple line band gaps are required to support it.

In a specific example, we have shown how, once the phase has been established

under a pristine C_n symmetry, disorder that breaks the symmetry will not disrupt the existence of the topological states (unless disorder is strong enough to induce a bulk phase transition). We have also shown how breaking the protecting symmetries of a HOTK phase can result in $O(L)$ corner states, similar to those in HOSE phases. In this regard, it is worth distinguishing two cases: First, in Hamiltonian (4.1), reducing C_4 symmetry to C_2 symmetry results in $O(L)$ corner states that do not have corresponding zero-energy corner states in its counterpart chiral-symmetric Hermitian Hamiltonian. These states are, to the best of our knowledge, novel, and the mechanism behind their localization remains an open question for future work. Second, in Hamiltonian (4.2), modifying the lattice geometry to break C_3 symmetry – by shifting from a triangular to a parallelogram shape – transforms the extended edge states into $O(L)$ corner states, localized exclusively at one of the two $2\pi/3$ rad corners. In this second case, the $O(L)$ states do correspond to zero-energy corner states in its counterpart chiral-symmetric Hermitian Hamiltonian, as recently identified in Ref. [99]. We point out, however, that Hamiltonian (4.2) possesses edge point gaps along all edges under partial PBC, which contrasts with the paradigmatic HOSE phase of Ref. [33], where the corner states owe their localization to the absence of point-gap topology at one of the two pairs of opposite edges.

In 1D NH Hamiltonians, point-gap topology is naturally characterized by the braid group B_N , which captures the winding of energy bands around each other in the complex energy plane as the BZ is traversed [26]. The periodicity of the BZ allows these braids to be identified with knots. By analogy with higher-order topological insulators in Hermitian systems, where edge-state topology is determined by 2D bulk bands [5], we named our phases “higher-order topological knot” phases, reflecting the fact that the point-gap knot topology of the 1D edge states is governed by the 2D bulk bands of the Hamiltonian. In this work, the HOTK phases we have presented produced the edge braids and knots listed in Table 5.1. A key task for future research is to

Name	Unlink	Hopf link	Trefoil knot	(3,2)Torus knot
Braids				
BWs	σ_1	σ_1^2	σ_1^3	$\sigma_1\sigma_2\sigma_1\sigma_2$
Knots				
Models	(4.1) $[g = 1]$	(4.1) $[g = 2]$, (4.2)	(4.1) $[g = 3]$	(4.3)

Table 5.1: Braids, braid words (BWs), and corresponding knots of the edge states in the HOTK phases described in Section 3.1.

understand how different classes of edge knots influence the nonreciprocal dynamics in these systems.

Overall, we have shown that the interplay of crystalline symmetries and internal symmetries results in novel topological phases in non-Hermitian systems. Another recent discovery of breaking the NN theorem in a non-Hermitian setting is in Ref [42]. This paper shows that by introducing non-symmorphic symmetry to non-Hermitian system, the Hamiltonian can hold an odd number of exceptional points (EPs), which directly **bypassed** NN theorem. These examples show great potential in the field of non-Hermiticity and topological phases of matters in general. There is a large amount of novel phenomenon that has yet to be discovered that can break the NN theorem in one way or another.

Besides **bypassing** the NN theorem, the application of non-Hermiticity is diverse. For example, through the understanding of the non-Hermitian Hamiltonian, it is possible to revisit the effective edge Hamiltonian of open quantum system [20]; gain new insights into the Hermitian topological phenomenon, such as Wannierizability of electron wave function [53, 73]; application to interacting systems [14, 39]. All of these directions have immense potential that could broaden our horizon of physics phenomenon.

Appendix A

Appendix

A.1 Winding number under TRS and TRS[†]

In this section, we demonstrate that in 1D, the winding number vanishes under TRS[†] (1.7), while it does not under TRS (1.6). To simplify our notation, we use $\mathcal{T}_a$ ($\mathcal{T}_b$) for the TRS[†] (TRS) operator in this section.

First consider an 1D Hamiltonian that obeys TRS[†],

$$\begin{aligned} h(-k)^\dagger &= \mathcal{T}_a h(k) \mathcal{T}_a^{-1} \\ &= U_a h(k)^* U_a^\dagger, \end{aligned} \tag{A.1}$$

where we used $\mathcal{T}_a = U_a \mathcal{K}$. Then, taking complex conjugate in both sides in the above equation, we have

$$h(-k)^T = U_a^* h(k) U_a^T. \tag{A.2}$$

Then note that for the determinant, we have $\det h(k) = \det U_a^* h(k) U_a^T = \det h(-k)^T =$

$\det h(-k)$. We can see that the winding number of $h(k)$ obeys

$$\begin{aligned}
W &= \frac{1}{2\pi i} \int_{BZ} dk \frac{d}{dk} \log \det h(k) \\
&= \frac{1}{2\pi i} \int_{BZ} dk \frac{d}{dk} \log \det h(-k) \\
&= -\frac{1}{2\pi i} \int_{BZ} dk \frac{d}{dk} \log \det h(k) = -W,
\end{aligned} \tag{A.3}$$

and thus, $W = 0$.

Next, consider the constraint due to TRS,

$$h(-k) = \mathcal{T}_b h(k) \mathcal{T}_b^{-1}. \tag{A.4}$$

Using $\mathcal{T}_b = U_b \mathcal{K}$, we arrive at

$$h(-k)^* = U_b^* h(k) U_b^T. \tag{A.5}$$

Then, the determinant of $h(k)$ obeys $\det h(k) = \det h(-k)^*$. We can see that the winding number obeys

$$\begin{aligned}
W &= \frac{1}{2\pi i} \int_{BZ} dk \frac{d}{dk} \log \det h(-k)^* \\
&= -\frac{1}{2\pi i} \int_{BZ} dk \frac{d}{dk} \log \det h(k)^* = W^* = W,
\end{aligned} \tag{A.6}$$

where in the last step, we used the fact that the winding number is a real number. Therefore, W does not vanish under TRS (1.6).

A.2 $\mathbb{Z}_2$ quantization of the Berry phase under TRS

In this section, we prove that the $\mathbb{Z}_2$ invariant in Table 1.1 is the Berry phase quantized to 0 or π . The constraints of TRS for a generic Hamiltonian are discussed in Sec. 2.1

and Appendix A.4. Now consider a 1D NH Bloch Hamiltonian $h(k)$. As discussed in Sec. 2.1, TRS forces the eigenvalues of a Hamiltonian $h(k)$ come in $\{\epsilon_n(-k), \epsilon_n^*(k)\}$ pairs. In the presence of an imaginary line gap, this allows us to label bands of $h(k)$ in pairs $\{\tilde{n}, -\tilde{n}\}$ for bands above and below the imaginary line gap, respectively.

Let us now consider the biorthogonal Wilson line defined by

$$\begin{aligned} \mathcal{W}_{+,k_f \leftarrow k_i}^{\tilde{n}} &= G_{k_f - \Delta}^{\tilde{n}} G_{k_f - 2\Delta}^{\tilde{n}} \cdots \\ &\cdots G_{k_i + \Delta}^{\tilde{n}} G_{k_i}^{\tilde{n}}, \end{aligned} \quad (\text{A.7})$$

where the biorthogonal Wilson line element is defined as $[G_k^{\tilde{n}}]^{mn} = \langle u_{k+\Delta}^m | v_k^n \rangle$. The superscript $\tilde{n}$ labels the band or group of bands over which the Wilson line is calculated, so that $m, n \in \tilde{n}$. Δ is the spacing between adjacent Wilson line elements in k space. The sign of Δ determines the direction in which the Wilson line is calculated. In this section, we choose $\Delta > 0$, which corresponds to the subscript $+$ on the left-hand side of (A.7).

Let us now consider the case in which the Wilson line traverses the entire BZ, i.e. $k_f = k_i + 2\pi$; this constitutes the biorthogonal Wilson loop

$$\begin{aligned} \mathcal{W}_{+,k}^{\tilde{n}} &= G_{k-\Delta}^{\tilde{n}} G_{k-2\Delta}^{\tilde{n}} \cdots \\ &\cdots G_{k+\Delta}^{\tilde{n}} G_k^{\tilde{n}}, \end{aligned} \quad (\text{A.8})$$

which is gauge-invariant. Furthermore, the eigenvalues of the biorthogonal Wilson loop are independent of the starting point k [94]. Since we are interested only on the Wilson loop spectrum, we will drop the subscript k in the Wilson loop. Wilson loops calculated by advancing k in opposite directions obey

$$\mathcal{W}_+^{\tilde{n}} = [\mathcal{W}_-^{\tilde{n}}]^{-1}. \quad (\text{A.9})$$

In the Hermitian case, the Wilson loop is unitary, and thus its eigenvalues take the form $\exp(i\gamma)$, where $\gamma \in \mathbb{R}$ is the *Berry phase*. In the NH case, the Wilson loop is no longer unitary, leading to complex values of γ . However, in this section, we will only consider real γ to simplify our argument. Under that consideration, Eq. (A.9) becomes

$$\mathcal{W}_+^{\tilde{n}} = [\mathcal{W}_-^{\tilde{n}}]^\dagger. \quad (\text{A.10})$$

When γ is real, the order in which we choose the biorthogonal basis in the definition of the Wilson loop will not affect the result Berry phases, i.e., using $\langle u_{k+\Delta}^m | v_k^n \rangle$ or $\langle v_{k+\Delta}^m | u_k^n \rangle$ for the Wilson line elements will result in the same value of γ for the eigenvalues of the Wilson loop.

Now we insert $\mathcal{T}^2 = 1$ into the Wilson line elements $G_k^{\tilde{n}}$ and apply Eq. (2.2)

$$\begin{aligned} G_k^{\tilde{n}} &= \langle u_{k+\Delta}^{m*} | \mathcal{T}^2 | v_k^{n*} \rangle \\ &= \sum_{-n, -m \in -\tilde{n}} [V_{k+\Delta}^{-m, m}]^T \langle u_{-k-\Delta}^{-m*} | v_{-k}^{-n*} \rangle [V_k^{-n, n}]^* \\ &= \sum_{-n, -m \in -\tilde{n}} [V_{k+\Delta}^{-m, m}]^T \langle u_{-k-\Delta}^{-m*} | v_{-k}^{-n*} \rangle [V_k^{-n, n}]^*, \end{aligned} \quad (\text{A.11})$$

where we defined the sewing matrix $V_{\mathbf{k}}^{-n, n} = \langle v_{-\mathbf{k}}^{-n} | \mathcal{T} | u_{\mathbf{k}}^n \rangle$. We see that $\langle u_{-k-\Delta}^{-m*} | v_{-k}^{-n*} \rangle$ is a Wilson line element for the bands in $-\tilde{n}$ and in the opposite direction. This allows us to exploit the property (A.10). Applying Eq. (A.8) and using the fact that $[V_k^{-n, n}]^\dagger V_k^{-n, n} = 1$, we get

$$\mathcal{W}_+^{\tilde{n}} = V_k^\dagger \mathcal{W}_+^{-\tilde{n}} V_k. \quad (\text{A.12})$$

This implies that the Berry phases $\gamma_{\tilde{n}}$ for bands in $\tilde{n}$ obey $\{e^{i\gamma_{\tilde{n}}}\} = \{e^{i\gamma_{-\tilde{n}}}\}$. Consider now a system with only two bands. The two Berry phases obey

$$\gamma_{\tilde{n}} = \gamma_{-\tilde{n}}. \quad (\text{A.13})$$

For such systems, there is the additional constraint that the Wilson loop for the combined bands $\tilde{n}$ and $-\tilde{n}$ is trivial, such that

$$\gamma_{\tilde{n}} + \gamma_{-\tilde{n}} = 0 \pmod{2\pi}. \quad (\text{A.14})$$

Eq. (A.13) and Eq. (A.14) lead to two possible values for the Berry phase, $\gamma_{\tilde{n}} = 0$ or π . This is the $\mathbb{Z}_2$ invariant in 38-fold classification table for 1D in class AI or Table 1.1 in this paper. In the case of systems with multiple bands, it is straightforward to generalize our findings and show that the quantized index is the polarization, $p = \frac{1}{2\pi} \log \det(\mathcal{W})$.

A.3 Details on the deformation of complex Chern insulators into HOSE phases

In this section, we consider a model that connects the HOSE phase [Eq. (2.9)] with the imaginary-line-gap Chern insulator [Eq. (2.6)]. Consider a lattice with the two-band Bloch Hamiltonian

$$\begin{aligned} h_{\text{def}}^{\text{Im}}(\mathbf{k}) = & -i\sigma_z \sin k_x + i\sigma_x \sin k_y \\ & + i\sigma_y(\cos k_y + t \cos k_x + m) \\ & + \sigma_0 \cos k_x(1 - g). \end{aligned} \quad (\text{A.15})$$

This model obeys TRS (1.6) with $\mathcal{T} = \mathcal{K}$. Just as in the deformation of the real-line-gap Chern insulator to a HOSE phase described in Section 2.3, we set $m = 0.5$ throughout the entire deformation. The main difference between the deformation in the main text and this one is that this one is in class AI. We first set the parameters to $(g, t) = (1, 1)$, which puts Hamiltonian (A.15) in the imaginary-line-gap Chern

insulator phase [Fig. 2.2(b)]. We then deform Hamiltonian (A.15) according to $(g, t) : (1, 1) - \theta(1, 1)$ for $\theta \in [0, 1]$.

We first continuously evolve θ from 0 to 0.2. The spectrum at $\theta = 0.2$ is shown in Fig. A.1(b). Next, we evolve θ from 0.2 to 0.5. A phase transition occurs at $\theta = 0.5$ as shown in Fig. A.1(c). Finally, we vary θ from 0.5 to 1. As shown in Fig. A.1(d), this model is in a HOSE phase.

Figures A.2 and A.3 show the spectrum for some points during the deformations in (2.10) and (A.15) that complement those shown in Figs. 2.3 and A.1. As we can confirm from (b) in both Fig. A.2 and A.3, the deformation is not smooth since line gaps close, causing a phase transition.

A.4 Construction of the topological classification of C_n -symmetric NH Hamiltonians in class AI

In this section, we build the topological indices of Eq. (3.5) that classify the energy bands of C_n -symmetric NH lattices in class AI of the 38-fold way. Due to non-Hermiticity, the classification is, in general, different from the one obtained for Hermitian systems. However, for “real” energy bands, additional constraints result in indices (A.48), which coincide with the indices of Hermitian energy bands [6]. We start by discussing the implications of TRS and C_n rotation symmetry on the energy bands and build the symmetry indicator invariants. Then, we discuss the constraints that these two symmetries impose on these indicators. These two steps then allow the construction of the $\chi^{(n)}$ indices in Eq. (3.5) and (A.48).

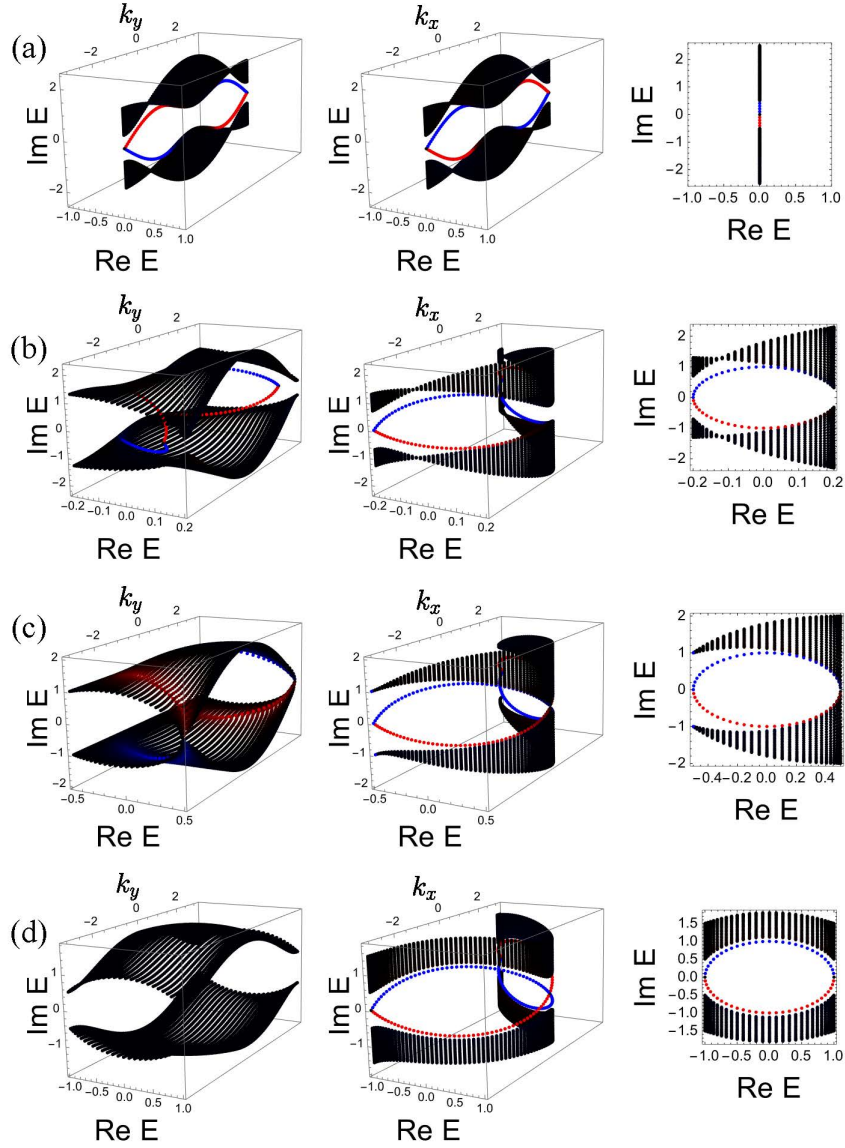


Figure A.1: Deforming an imaginary-line-gap complex Chern insulator into a HOSE phase in the Bloch Hamiltonian (A.15). (a)-(d) correspond to $\theta = 0, 0.2, 0.5, 1$, respectively. (a,b) An imaginary-line-gap complex Chern insulator. (c) Bulk phase transition between the imaginary-line-gap complex Chern insulator phase and the HOSE phase. (d) HOSE phase. Left panels: OBC along x and PBC along y ; middle panels: OBC along y and PBC along x ; right panels: band projections of the plots on the second column on the complex energy plane. Black represents bulk states, while blue and red denote states localized at opposite edges.

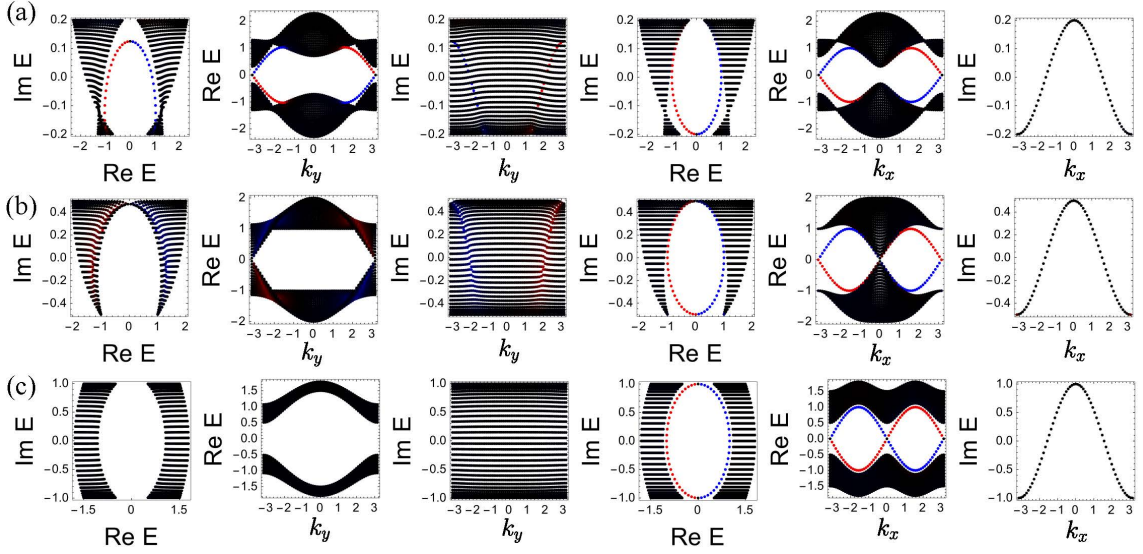


Figure A.2: Spectrum of model with Bloch Hamiltonian (2.10). For all plots, left (right) three panels are spectra for OBC only along x (y). The deformation path is $(g, t, m) = (1, 1, 0.5) - \theta(1, 1, 0)$. (a) Chern phase at $\theta = 0.2$. (b) Phase transition at $\theta = 0.5$. (c) HOSE phase at $\theta = 1$.

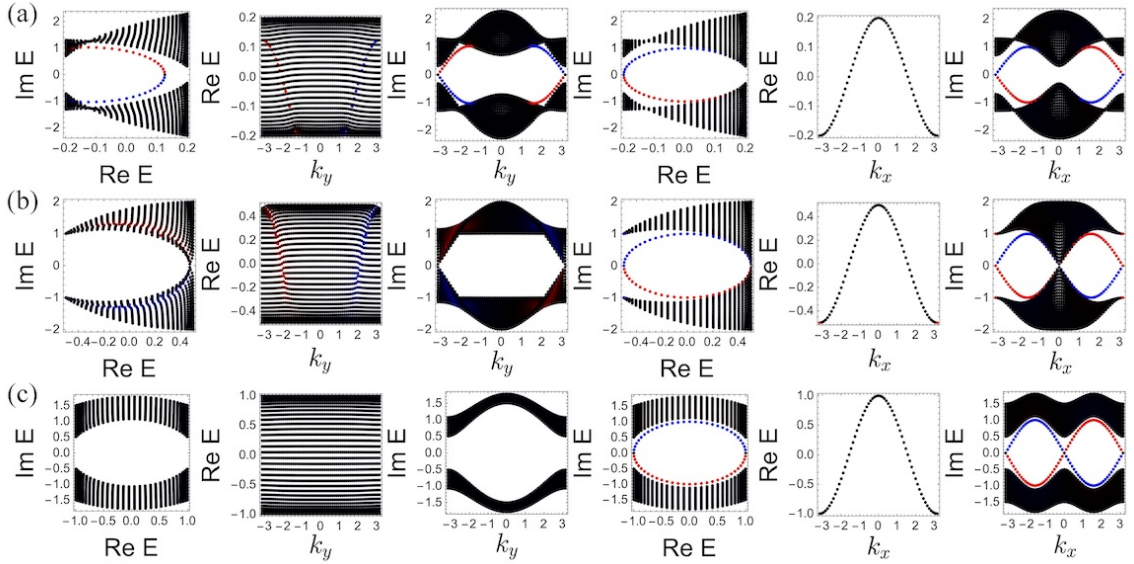


Figure A.3: Spectrum of model with Bloch Hamiltonian (A.15). For all plots, left(right) three panels are spectra for OBC only along x (y). The deformation path is $(g, t, m) = (1, 1, 0.5) - \theta(1, 1, 0)$. (a) Imaginary-line-gap complex Chern phase at $\theta = 0.2$. (b) Phase transition at $\theta = 0.5$. (c) HOSE phase at $\theta = 1$.

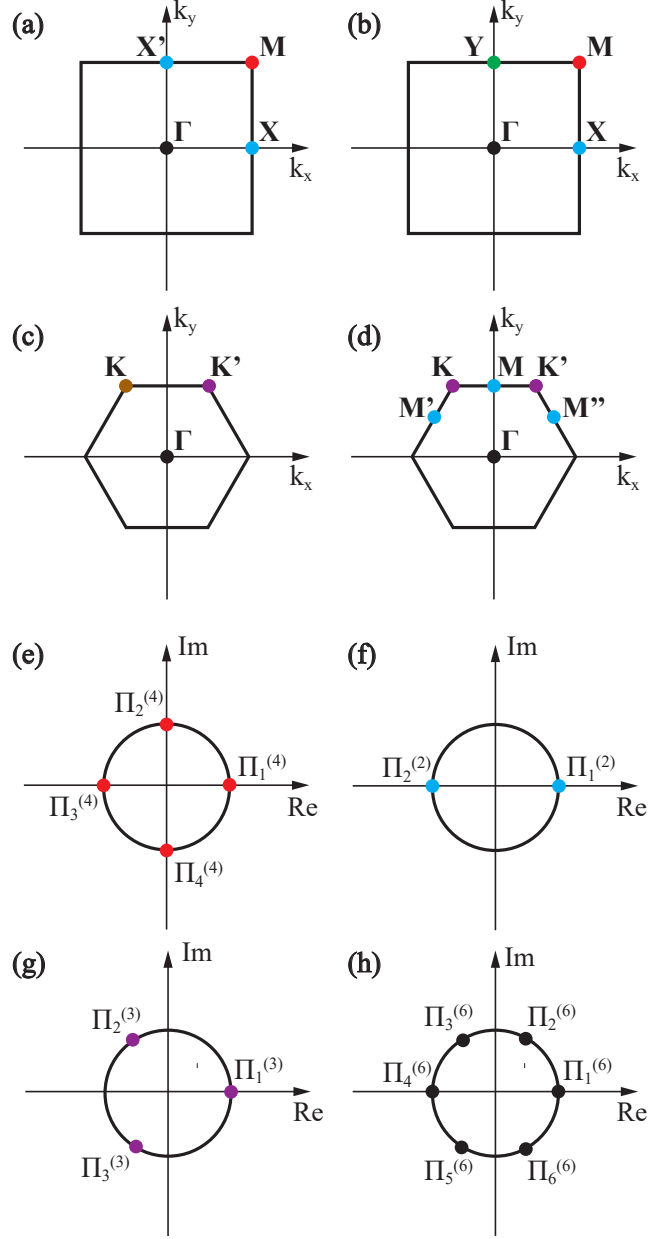


Figure A.4: (a)-(d) HSPs in the BZ of $C_{4,2,3,6}$ -symmetric lattices, respectively. Colored dots are the HSPs defined in Eq. (A.27). (e)-(h) Eigenvalues of the rotation operators $\hat{r}_4$, $\hat{r}_2$, $\hat{r}_3$, and $\hat{r}_6$, respectively. In (e)-(h), we consider only operators obeying $[\hat{r}_n]^n = 1$.

A.4.1 Time-Reversal symmetry

In Sec. 2.1, we saw that TRS enforces the relation

$$h(-\mathbf{k})\mathcal{T}|u_{\mathbf{k}}^l\rangle = \mathcal{T}h(\mathbf{k})|u_{\mathbf{k}}^l\rangle = \epsilon_l^*(\mathbf{k})\mathcal{T}|u_{\mathbf{k}}^l\rangle. \quad (\text{A.16})$$

Hence, $\mathcal{T}|u_{\mathbf{k}}^l\rangle$ is an eigenstate of $h(-\mathbf{k})$ with energy $\epsilon_l^*(\mathbf{k})$. To prove the constraint of TRS on the energy eigenvalues more rigorously, we project the state at $l \in \tilde{l}$ into the space spanned by states at band $-\tilde{l}$.

$$\mathcal{T}|u_{\mathbf{k}}^l\rangle = \sum_{-l \in -\tilde{l}} |u_{-\mathbf{k}}^{-l}\rangle \langle v_{-\mathbf{k}}^{-l} | \mathcal{T}|u_{\mathbf{k}}^l\rangle = \sum_{-l \in -\tilde{l}} V_{\mathbf{k}}^{-l,l} |u_{-\mathbf{k}}^{-l}\rangle, \quad (\text{A.17})$$

where we have defined sewing matrix $V_{\mathbf{k}}^{-l,l} = \langle v_{-\mathbf{k}}^{-l} | \mathcal{T}|u_{\mathbf{k}}^l\rangle$.

From Eq. (A.16), it follows that

$$h(-\mathbf{k})\mathcal{T}|u_{\mathbf{k}}^l\rangle = \epsilon_l^*(\mathbf{k})\mathcal{T}|u_{\mathbf{k}}^l\rangle = \epsilon_l^*(\mathbf{k}) \sum_{-\tilde{l}} V_{\mathbf{k}}^{-l,l} |u_{-\mathbf{k}}^{-l}\rangle. \quad (\text{A.18})$$

On the other hand, we have

$$\begin{aligned} h(-\mathbf{k})\mathcal{T}|u_{\mathbf{k}}^l\rangle &= h(-\mathbf{k}) \sum_{-\tilde{l}} V_{\mathbf{k}}^{-l,l} |u_{-\mathbf{k}}^{-l}\rangle \\ &= \sum_{-\tilde{l}} \epsilon_{-l}(-\mathbf{k}) V_{\mathbf{k}}^{-l,l} |u_{-\mathbf{k}}^{-l}\rangle. \end{aligned} \quad (\text{A.19})$$

Therefore, by subtracting these two equations, we have

$$\sum_{-\tilde{l}} V_{\mathbf{k}}^{-l,l} |u_{-\mathbf{k}}^{-l}\rangle (\epsilon_l^*(\mathbf{k}) - \epsilon_{-l}(-\mathbf{k})) = 0 \quad (\text{A.20})$$

for every $l \in \tilde{l}$. Applying $\langle v_{-\mathbf{k}}^{-l} |$ to the above expression, we get

$$V_{\mathbf{k}}^{-l,l}(\epsilon_l^*(\mathbf{k}) - \epsilon_{-l}(-\mathbf{k})) = 0 \quad (\text{A.21})$$

for every $l \in \tilde{l}$ and $-l \in -\tilde{l}$, which implies that the sewing matrix has elements $V_{\mathbf{k}}^{-l,l} \neq 0$ only for bands obeying $\epsilon_l^*(\mathbf{k}) = \epsilon_{-l}(-\mathbf{k})$.

A.4.2 Rotation symmetry

Rotation symmetry is expressed as

$$\hat{r}_n h(\mathbf{k}) \hat{r}_n^\dagger = h(R_n \mathbf{k}), \quad (\text{A.22})$$

where $\hat{r}_n$ is the n -fold rotation operator, which obeys $[\hat{r}_n]^n = \pm 1$ and R_n is the n -fold rotation matrix acting on the crystal momentum $\mathbf{k}$. Let Eq. (A.22) act on the energy eigenstate of band $l \in \tilde{l}$. We have

$$h(R_n \mathbf{k}) \hat{r}_n |u_{\mathbf{k}}^l\rangle = \hat{r}_n h(\mathbf{k}) |u_{\mathbf{k}}^l\rangle = \epsilon_l(\mathbf{k}) \hat{r}_n |u_{\mathbf{k}}^l\rangle. \quad (\text{A.23})$$

Thus $\hat{r}_n |u_{\mathbf{k}}^l\rangle$ is an eigenstate of $h(R_n \mathbf{k})$ with eigenvalue $\epsilon_l(\mathbf{k})$. We can make the expansion

$$\hat{r}_n |u_{\mathbf{k}}^l\rangle = \sum_{q \in \tilde{l}} |u_{R_n \mathbf{k}}^q\rangle B_{\mathbf{k}}^{ql} \quad (\text{A.24})$$

where the sewing matrix of the rotation operator is defined as

$$B_{\mathbf{k}}^{ql} = \langle v_{R_n \mathbf{k}}^q | \hat{r}_n |u_{\mathbf{k}}^l\rangle. \quad (\text{A.25})$$

High-symmetry points (HSPs) $\mathbf{\Pi}_m$ remain invariant under the little group C_m rotation modulo a reciprocal lattice vector $\mathbf{G}$ (for $m \leq n$), i.e., they obey

$$R_m \mathbf{\Pi}_m = \mathbf{\Pi}_m \pmod{\mathbf{G}}. \quad (\text{A.26})$$

The HSPs in the BZ of C_n -symmetric lattices are

$$\begin{aligned} C_2 : \quad & \mathbf{X} = \{\pi, 0\}, \mathbf{Y} = \{0, \pi\}, \mathbf{M} = \{\pi, \pi\} \\ C_4 : \quad & \mathbf{X} = \{\pi, 0\}, \mathbf{X}' = \{0, \pi\}, \mathbf{M} = \{\pi, \pi\} \\ C_3 : \quad & \mathbf{K} = \left\{-\frac{2\pi}{3}, \frac{2\pi}{\sqrt{3}}\right\}, \mathbf{K}' = \left\{\frac{2\pi}{3}, \frac{2\pi}{\sqrt{3}}\right\} \\ C_6 : \quad & \mathbf{K} = \left\{-\frac{2\pi}{3}, \frac{2\pi}{\sqrt{3}}\right\}, \mathbf{K}' = \left\{\frac{2\pi}{3}, \frac{2\pi}{\sqrt{3}}\right\} \\ & \mathbf{M} = \left\{0, \frac{2\pi}{\sqrt{3}}\right\}, \mathbf{M}' = \left\{-\pi, \frac{\pi}{\sqrt{3}}\right\}, \mathbf{M}'' = \left\{\pi, \frac{\pi}{\sqrt{3}}\right\}, \end{aligned} \quad (\text{A.27})$$

and $\mathbf{\Gamma} = \{0, 0\}$ for all cases. These HSPs are indicated in Fig. A.4(a)-(d). In C_2 -symmetric lattices, $\mathbf{X}$, $\mathbf{Y}$, $\mathbf{M}$ are invariant under C_2 rotations; in C_4 -symmetric lattices, $\mathbf{M}$ is invariant under C_4 while $\mathbf{X}$ and $\mathbf{X}'$ are invariant under C_2 ; in C_3 -symmetric lattices, $\mathbf{K}$ and $\mathbf{K}'$ are invariant under C_3 ; and in C_6 -symmetric lattices, $\mathbf{K}$ and $\mathbf{K}'$ are invariant under C_3 while $\mathbf{M}$, $\mathbf{M}'$, and $\mathbf{M}''$ are invariant under C_2 . Finally, $\mathbf{\Gamma}$ is invariant under the full group C_n rotation for all C_n symmetric lattices.

From Eq. (A.26) and (A.22) it follows that $[h(\mathbf{\Pi}_m), \hat{r}_m] = 0$. Thus, at HSPs, we also have

$$\hat{r}_m |u_{\mathbf{\Pi}_m}^{\tilde{l}}\rangle = r_{\mathbf{\Pi}_m}^{\tilde{l}} |u_{\mathbf{\Pi}_m}^{\tilde{l}}\rangle, \quad (\text{A.28})$$

where $r_{\mathbf{\Pi}_m}^{\tilde{l}}$ is the rotation eigenvalue associated with energy band $\tilde{l}$ at HSP $\mathbf{\Pi}_m$, which

can take the values

$$\Pi_p^{(m)} = \begin{cases} e^{2\pi i(p-1)/m}, & \text{for } [\hat{r}_n]^n = 1 \\ e^{2\pi i(p-1/2)/m}, & \text{for } [\hat{r}_n]^n = -1 \end{cases} \quad (\text{A.29})$$

for $p = 1, 2, \dots, m$. We now define the symmetry indicator invariants

$$[\Pi_p^{(m)}]_{\tilde{l}} = \#_{\tilde{l}} \Pi_p^{(m)} - \#_{\tilde{l}} \Gamma_p^{(m)}, \quad (\text{A.30})$$

where $\#_{\tilde{l}} \Pi_p^{(m)}$ is the number of energy bands in the band group $\tilde{l}$ with eigenvalue $\Pi_p^{(m)}$. Note that if there is an equal number of bands of a given rotation representation of $\hat{r}_m$ at both $\mathbf{\Pi}_m$ and $\mathbf{\Gamma} = (0, 0)$, the symmetry indicator invariants are zero. Hence, these symmetry indicator invariants signal an imbalance in the number of representations across a generic HSP and those at $\mathbf{\Gamma}$ at energy bands $\tilde{l}$. Not all these invariants are independent. In a C_4 symmetric crystal, rotation symmetry forces the representation at $\mathbf{X}$ and $\mathbf{X}'$ to be equal [Fig. A.4(a)]. Similarly, C_6 symmetry forces equal representations at $\mathbf{M}$, $\mathbf{M}'$, and $\mathbf{M}''$, as well as at $\mathbf{K}$ and $\mathbf{K}'$ [Fig. A.4(d)]. We will demonstrate this in the following sections.

A.4.3 Constraints due to rotation

Consider a crystal with C_n symmetry with operator $\hat{r}_n$. C_n symmetry relates some of the HSPs (A.27) that are invariant under little group C_m , where $m < n$. We are interested in the eigenvalues of the C_m rotation operator at $\mathbf{\Pi}_m$ and $R_n \mathbf{\Pi}_m$. We now demonstrate that the symmetry indicator invariants for C_m of a band group $\tilde{l}$ at m -fold HSPs are identical due to C_n .

Since $R_n \mathbf{\Pi}_m$ is invariant under R_m , we have

$$\hat{r}_m |u_{R_n \mathbf{\Pi}_m}^l\rangle = r_{R_n \mathbf{\Pi}_m}^l |u_{R_n \mathbf{\Pi}_m}^l\rangle, \quad (\text{A.31})$$

for band $l \in \tilde{l}$. Since $R_n \mathbf{\Pi}_m$ and $\mathbf{\Pi}_m$ are related by C_n symmetry, we can make the expansion

$$\hat{r}_n |u_{\mathbf{\Pi}_m}^l\rangle = \sum_{q \in \tilde{l}} |u_{R_n \mathbf{\Pi}_m}^q\rangle B_{\mathbf{\Pi}_m}^{ql}, \quad (\text{A.32})$$

where $B_{\mathbf{\Pi}_m}^{ql} = \langle v_{R_n \mathbf{\Pi}_m}^q | \hat{r}_n |u_{\mathbf{\Pi}_m}^l\rangle$ is the sewing matrix of rotation at HSP $\mathbf{\Pi}_m$. By applying $\hat{r}_m$ to the above expression and using the fact that $[\hat{r}_m, \hat{r}_n] = 0$, we have

$$(r_{R_n \mathbf{\Pi}_m}^q - r_{\mathbf{\Pi}_m}^l) B_{\mathbf{\Pi}_m}^{ql} = 0 \quad (\text{A.33})$$

for all $q, l \in \tilde{l}$. Thus the rotation eigenvalues of the little group at $R_n \mathbf{\Pi}_m$ and $\mathbf{\Pi}_m$ are equal at any given band groups $\tilde{l}$,

$$\boxed{\{r_{R_n \mathbf{\Pi}_m}^l\}_{\tilde{l}}} \stackrel{C_n}{=} \{r_{\mathbf{\Pi}_m}^l\}_{\tilde{l}} \quad (\text{A.34})$$

More explicitly,

$$\begin{aligned} \{r_{\mathbf{X}}^l\} &\stackrel{C_4}{=} \{r_{\mathbf{X}'}^l\} \\ \{r_{\mathbf{K}}^l\} &\stackrel{C_6}{=} \{r_{\mathbf{K}'}^l\} \\ \{r_{\mathbf{M}}^l\} &\stackrel{C_6}{=} \{r_{\mathbf{M}'}^l\} \stackrel{C_6}{=} \{r_{\mathbf{M}''}^l\}, \end{aligned} \quad (\text{A.35})$$

This implies that the invariants (A.30) obey

$$\begin{aligned} [X_p^{(2)}] &\stackrel{C_4}{=} [X_p'^{(2)}] \\ [K_p^{(3)}] &\stackrel{C_6}{=} [K_p'^{(3)}] \\ [M_p^{(2)}] &\stackrel{C_6}{=} [M_p'^{(2)}] \stackrel{C_6}{=} [M_p''^{(2)}] \end{aligned} \quad (\text{A.36})$$

in the same band group $\tilde{l}$. This conclusion applies for both $[\hat{r}_n]^n = \pm 1$ cases.

A.4.4 Constraints due to TRS

TRS will add another constraint to the rotation invariants (A.30) $[\Pi_p^{(m)}]$. The TRS operator and rotation operator in general commute

$$[\mathcal{T}, \hat{r}_m] = 0. \quad (\text{A.37})$$

Thus, we have

$$\begin{aligned} \mathcal{T}(\hat{r}_m |u_{\mathbf{k}}^l\rangle) &= \mathcal{T} \sum_{q \in \tilde{l}} |u_{R_m \mathbf{k}}^q\rangle B_{\mathbf{k}}^{q,l} \\ &= \sum_{-q \in -\tilde{l}, q \in \tilde{l}} |u_{R_m \mathbf{k}}^{-q}\rangle V_{R_m \mathbf{k}}^{-q,q} B_{\mathbf{k}}^{q,l*}. \end{aligned} \quad (\text{A.38})$$

Here, $B_{\mathbf{k}}^{q,l} = \langle v_{R_m \mathbf{k}}^q | \hat{r}_m |u_{\mathbf{k}}^l\rangle$ is the sewing matrix of $\hat{r}_m$ from the little group C_m . On the other hand, we have

$$\begin{aligned} \hat{r}_m(\mathcal{T} |u_{\mathbf{k}}^l\rangle) &= \hat{r}_m \sum_{-l \in -\tilde{l}} |u_{-\mathbf{k}}^{-l}\rangle V_{\mathbf{k}}^{-l,l} \\ &= \sum_{-q, -l \in -\tilde{l}} |u_{-R_m \mathbf{k}}^{-q}\rangle B_{-\mathbf{k}}^{-q,-l} V_{\mathbf{k}}^{-l,l}. \end{aligned} \quad (\text{A.39})$$

Therefore, by subtracting these two equations and acting $\langle v_{-R_m \mathbf{k}}^{-q} |$ on the left, we have

$$\sum_{q \in \tilde{l}, -l \in -\tilde{l}} V_{R_m \mathbf{k}}^{-q,q} B_{\mathbf{k}}^{q,l*} - B_{-\mathbf{k}}^{-q,-l} V_{\mathbf{k}}^{-l,l} = 0. \quad (\text{A.40})$$

At HSPs Π_m , we choose the gauge in which $B_{\Pi_m}^{l,q} = r_{\Pi_m}^l \delta_{l,q}$ is diagonal, we have

$$V_{\Pi_m}^{-q,l} (r_{\Pi_m}^{l*} - r_{-\Pi_m}^{-l}) = 0. \quad (\text{A.41})$$

Hence, TRS imposes the constraint

$$\boxed{\{r_{\mathbf{\Pi}_m}^l\}_{\tilde{l}} \stackrel{TRS}{=} \{r_{-\mathbf{\Pi}_m}^{-l*}\}_{-\tilde{l}}} \quad (\text{A.42})$$

For the symmetry indicator invariants, this constraint implies the following relations:

For $[\hat{r}_n]^n = 1$,

$$\begin{aligned} [M_2^{(4)}]_{\pm\tilde{l}} &\stackrel{C_4}{=} [M_4^{(4)}]_{\mp\tilde{l}} \\ [K_1^{(3)}]_{\pm\tilde{l}} &\stackrel{C_{3,6}}{=} [K_1^{'(3)}]_{\mp\tilde{l}} \\ [K_2^{(3)}]_{\pm\tilde{l}} &\stackrel{C_{3,6}}{=} [K_3^{'(3)}]_{\mp\tilde{l}} \\ [K_3^{(3)}]_{\pm\tilde{l}} &\stackrel{C_{3,6}}{=} [K_2^{'(3)}]_{\mp\tilde{l}}. \end{aligned} \quad (\text{A.43})$$

For $[\hat{r}_n]^n = -1$,

$$\begin{aligned} [M_1^{(4)}]_{\pm\tilde{l}} &\stackrel{C_4}{=} [M_4^{(4)}]_{\mp\tilde{l}} \\ [M_2^{(4)}]_{\pm\tilde{l}} &\stackrel{C_4}{=} [M_3^{(4)}]_{\mp\tilde{l}} \\ [X_1^{(2)}]_{\pm\tilde{l}} &\stackrel{C_{2,4}}{=} [X_2^{(2)}]_{\mp\tilde{l}} \\ [M_1^{(2)}]_{\pm\tilde{l}} &\stackrel{C_6}{=} [M_2^{(2)}]_{\mp\tilde{l}} \\ [K_1^{(3)}]_{\pm\tilde{l}} &\stackrel{C_{3,6}}{=} [K_3^{'(3)}]_{\mp\tilde{l}} \\ [K_3^{(3)}]_{\pm\tilde{l}} &\stackrel{C_{3,6}}{=} [K_1^{'(3)}]_{\mp\tilde{l}} \\ [K_2^{(3)}]_{\pm\tilde{l}} &\stackrel{C_{3,6}}{=} [K_2^{'(3)}]_{\mp\tilde{l}}. \end{aligned} \quad (\text{A.44})$$

For the rest of the symmetry indicator invariants, i.e., those corresponding to real-valued rotation eigenvalues, the constraint is

$$[\Pi_p^{(m)}]_{\pm\tilde{l}} \stackrel{C_n}{=} [\Pi_p^{(m)}]_{\mp\tilde{l}} \quad (\text{A.45})$$

for both $[\hat{r}_n]^n = \pm 1$.

A.4.5 $\chi^{(n)}$ indices for C_n -symmetric NH crystals

Since the number of bands within each band group $\tilde{l}$ is constant across the BZ, we have the constraint

$$\boxed{\sum_p [\Pi_p^{(m)}]_{\tilde{l}} = 0.} \quad (\text{A.46})$$

The three boxed equations above give the full set of constraints on the symmetry indicator invariants $\{[\Pi_p^{(m)}]\}$. The only difference between the non-Hermitian and the Hermitian cases lies in Eq. (A.42) for “non-real bands”, i.e., when $\tilde{l} \neq -\tilde{l}$, in which case TRS relates energy eigenstates across the imaginary line gap. If $\tilde{l} = -\tilde{l}$, i.e., if the energy band groups are real, the classification is exactly the same as in the Hermitian case. Based on the above discussion, we can now generalize the $\chi^{(n)}$ indices introduced in Ref. [6] to NH systems:

Case $\tilde{l} \neq -\tilde{l}$

The full classification is given by

$$\begin{aligned} \chi^{(2)} &= (C|[X_1^{(2)}], [Y_1^{(2)}], [M_1^{(2)}]) \\ \chi^{(4)} &= (C|[X_1^{(2)}], [M_1^{(4)}], [M_2^{(4)}], [M_3^{(4)}]) \\ \chi^{(3)} &= (C|[K_1^{(3)}], [K_2^{(3)}], [K_1'^{(3)}], [K_2'^{(3)}]) \\ \chi^{(6)} &= (C|[M_1^{(2)}], [K_1^{(3)}], [K_2^{(3)}]), \end{aligned} \quad (\text{A.47})$$

where, for a set of non-redundant invariants, it suffices to determine the $\chi^{(n)}$ only for energy band groups $\tilde{l}$ such that $\text{Im}[\epsilon_{\tilde{l}}] > 0$.

Case $\tilde{l} = -\tilde{l}$

For real-energy bands, i.e., those for which $\tilde{l} = -\tilde{l}$, some of the symmetry indicator invariants are redundant, i.e., they can be obtained from other symmetry indica-

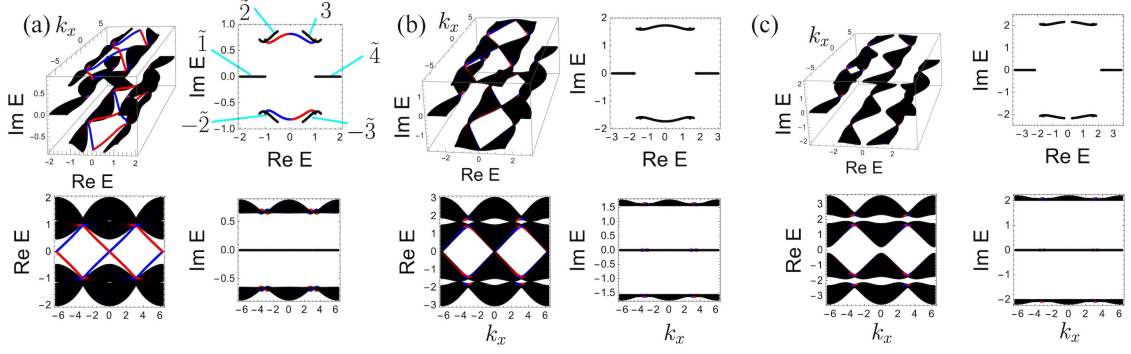


Figure A.5: NH Breathing honeycomb lattice described by Eq. (A.51). (a), (b), (c) energy spectra under OBC along y , PBC along x for $t = 1$ (topological phase), $t = 2$ (phase transition), $t = 2.5$ (trivial phase) with zig-zag edge, respectively.

tor invariants at the same HSP. Specifically, while for real bands with $[\hat{r}_n]^n = 1$, $[M_3^{(4)}] = -2[M_2^{(4)}] - [M_1^{(4)}]$ in $\chi^{(4)}$, $([K_1'^{(3)}], [K_2'^{(3)}]) = ([K_1^{(3)}], -[K_1^{(3)}] - [K_2^{(3)}])$ in $\chi^{(3)}$, and $[K_2^{(3)}] = -[K_1^{(3)}]/2$ in $\chi^{(6)}$; for real bands with $[\hat{r}_n]^n = -1$, all two-fold rotation symmetry indicator invariants are 0, $[M_3^{(4)}] = [M_2^{(4)}]$, $[M_2^{(4)}] = [M_1^{(4)}]$ in $\chi^{(4)}$, $([K_1'^{(3)}], [K_2'^{(3)}]) = (-[K_1^{(3)}] - [K_2^{(3)}], [K_2^{(3)}])$ in $\chi^{(3)}$, and $[K_2^{(3)}] = -[K_1^{(3)}]/2$ in $\chi^{(6)}$. Dropping redundant indicators, and noting that $C = 0$ for these bands, the classification for real-energy bands with $[\hat{r}_n]^n = 1$ is reduced to

$$\begin{aligned}
 \chi_{\text{Real}}^{(2)} &= (0|[X_1^{(2)}], [Y_1^{(2)}], [M_1^{(2)}]) \\
 \chi_{\text{Real}}^{(4)} &= (0|[X_1^{(2)}], [M_1^{(4)}], [M_2^{(4)}]) \\
 \chi_{\text{Real}}^{(3)} &= (0|[K_1^{(3)}], [K_2^{(3)}]) \\
 \chi_{\text{Real}}^{(6)} &= (0|[M_1^{(2)}], [K_1^{(3)}]),
 \end{aligned} \tag{A.48}$$

which coincides with the classification of Hermitian Hamiltonians [6]. For real-energy bands with $[\hat{r}_n]^n = -1$, the classification is reduced to

$$\begin{aligned}
\chi_{\text{Real}}^{(2)} &= (0|0) \\
\chi_{\text{Real}}^{(4)} &= (0|[M_1^{(4)}]) \\
\chi_{\text{Real}}^{(3)} &= (0|[K_1^{(3)}], [K_2^{(3)}]) \\
\chi_{\text{Real}}^{(6)} &= (0|[K_1^{(3)}]).
\end{aligned} \tag{A.49}$$

The subscript Real indicates these invariants are calculated for energy bands on the real energy line, i.e., those obeying $\tilde{l} = -\tilde{l}$.

A.5 $\chi^{(2)}$ index of the minimal model for a HOSE phase

As mentioned in the Main Text, the crystal with Hamiltonian $h_{\text{HOSE}}(\mathbf{k})$ in (2.9) obeys C_2 symmetry with operator $\hat{r}_2 = \sigma_y$. However, it does not obey TRS (1.6), and thus it does not belong to class AI. The Hamiltonian $ih_{\text{HOSE}}(\mathbf{k})$, on the other hand, obeys TRS (1.6), with $\mathcal{T} = \mathcal{K}$. This Hamiltonian is

$$\begin{aligned}
ih_{\text{HOSE}}(\mathbf{k}) &= \cos k_x \sigma_0 + i \sin k_x \sigma_z \\
&\quad + i(\gamma + \cos k_y) \sigma_y + i \sin k_y \sigma_x,
\end{aligned} \tag{A.50}$$

where the $-i\gamma\sigma_0$, present in Eq. (2.9), has been removed, with no consequence for the Hamiltonian's topological phase. $ih_{\text{HOSE}}(\mathbf{k})$ also possesses C_2 symmetry with $\hat{r}_2 = \sigma_y$. Multiplying h_{HOSE} by i rotates its energy spectrum in the complex energy plane by 90 degrees counter-clockwise. Therefore, $ih_{\text{HOSE}}(\mathbf{k})$ has the same topological properties as $h_{\text{HOSE}}(\mathbf{k})$. The Hamiltonian $ih_{\text{HOSE}}(\mathbf{k})$ is equivalent to Hamiltonian (A.15) at the

end of deformation process (i.e. at $g = t = 0$) by identifying γ in Eq. (A.50) with m in Eq. (A.15).

Phase	Band	C	$[X_1^{(2)}]$	$[Y_1^{(2)}]$	$[M_1^{(2)}]$
HOSE	$\tilde{1}$	0	0	-1	-1
	$-\tilde{1}$	0	0	1	1
Trivial	$\tilde{1}$	0	0	0	0
	$-\tilde{1}$	0	0	0	0

Table A.1: $\chi^{(2)}$ index for Hamiltonian (A.50) in the HOSE phase ($0 < \gamma < 1$) and trivial phase ($\gamma > 1$).

The spectrum of $ih_{\text{HOSE}}(\mathbf{k})$ under OBC along x and PBC along y is shown in Fig. A.1(d) for $\gamma = 0.5$. We label the top (bottom) energy band in the complex plane with $\tilde{1}$ ($-\tilde{1}$).

For $0 < \gamma < 1$, the model is in a HOSE phase. For $\gamma > 1$, it is in a trivial phase. The $\chi^{(2)}$ indices for this model are shown in Table A.1 for both phases.

A.6 Spectra of HOTK phases with C_3 and C_6 symmetries

This section contains Fig. A.6 and Fig. A.7 with plots of the energy bands of Hamiltonians (4.2) and (4.3).

A.7 A C_6 -symmetric real-line-gap Chern insulator

In this section, we provide a C_6 symmetric lattice Hamiltonian that realizes a real-line-gap complex Chern insulator [a minimal, 4-band model for this phase is shown in Eq. (2.7)]. Consider the NH breathing honeycomb lattice of Fig. 4.1(d). It has Bloch

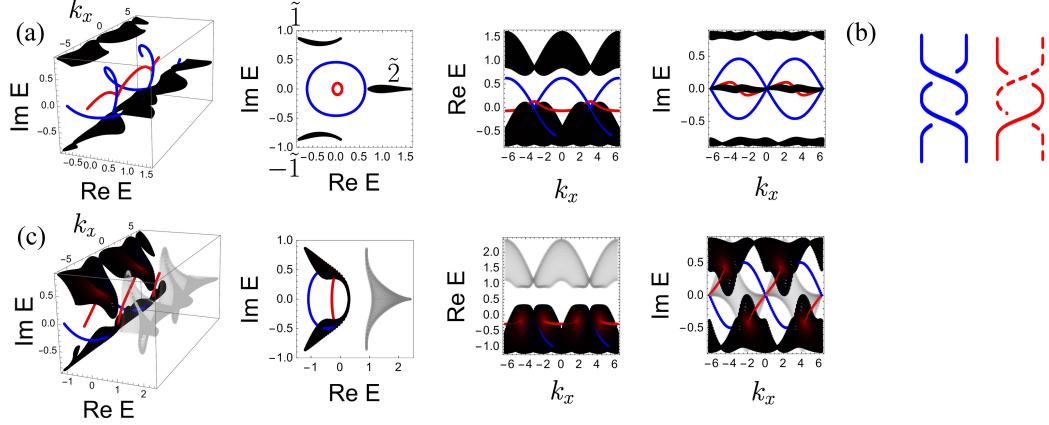


Figure A.6: (a)[(c)] Spectra of Eq. (4.2) plotted under PBC along x , OBC along y plotted at $t = 0.3$ [$t = 0.7$] with zig-zag edge. In (c), band $\tilde{2}$ is plotted in gray to facilitate the presentation of edge states. (b) Braid structures for the two-band edge states.

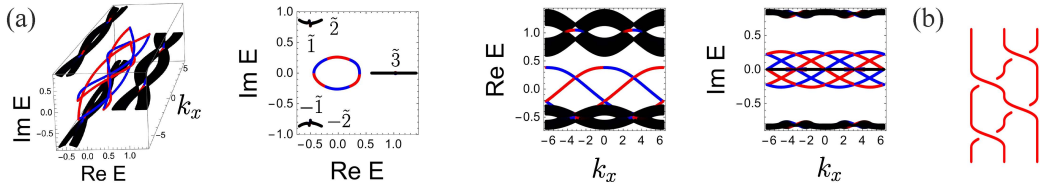


Figure A.7: Energy spectrum of Eq. (4.3). (a) PBC along x , OBC along y , for $t = 0.2$ with zig-zag edge. (b) Braid structure of edge states.

Hamiltonian

$$h_{\text{BH}}(\mathbf{k}) = \begin{pmatrix} 0 & 0 & 0 & 0 & t & e^{i\mathbf{k} \cdot \mathbf{a}_1} \\ 0 & 0 & 0 & e^{-i\mathbf{k} \cdot \mathbf{a}_2} & 0 & t \\ 0 & 0 & 0 & t & e^{-i\mathbf{k} \cdot \mathbf{a}_3} & 0 \\ t & e^{i\mathbf{k} \cdot \mathbf{a}_2} & 0 & 0 & 0 & 0 \\ 0 & t & e^{i\mathbf{k} \cdot \mathbf{a}_3} & 0 & 0 & 0 \\ e^{-i\mathbf{k} \cdot \mathbf{a}_1} & 0 & t & 0 & 0 & 0 \end{pmatrix}. \quad (\text{A.51})$$

Hamiltonian (A.51) obeys TRS (1.6) with $\mathcal{T} = \mathcal{K}$ and C_6 symmetry with a rotation operator $\hat{r}_6$ represented by the matrix that permutes the sites within the unit cells in the lattice in Fig. 4.1(d) upon rotation by $2\pi/6$ about the center of the unit cell.

Phase	Band	C	$[M_1^{(2)}]$	$[K_1^{(3)}]$	$[K_2^{(3)}]$
Complex Chern	$\tilde{2}$	2	1	0	1
	$\tilde{3}$	-2	-1	0	-1
	$\tilde{1}$	0	0	0	0
	$\tilde{4}$	0	0	0	0
	$-\tilde{2}$	-2	1	0	-1
	$-\tilde{3}$	2	-1	0	1

Table A.2: $\chi^{(6)}$ indices for Hamiltonian (A.51) in the real-line-gap complex Chern insulator phase, for $t < 2$. The bands are labeled as indicated in Fig. A.5(a). For $t > 2$, all the bands are trivial, with $\chi^{(6)} = \mathbf{0}$.

For $0 < t < 2$, the Hamiltonian (A.51) is in the topological phase, with energy bands across the real line gap having non-trivial $\chi^{(6)}$ indices that come in opposite pairs [Table A.2]. As a result, topological edge states cross the line gaps that separate the energy bands with opposite $\chi^{(6)}$ indices.

For $t > 2$, the system enters the trivial phase, with $\chi^{(6)} = \mathbf{0}$ for all bands and with no edge states between them.

Bibliography

- [1] Alexander Altland and Martin R. Zirnbauer. Nonstandard symmetry classes in mesoscopic normal-superconducting hybrid structures. *Phys. Rev. B*, 55(2): 1142–1161, 1997. doi: 10.1103/PhysRevB.55.1142. URL <https://link.aps.org/doi/10.1103/PhysRevB.55.1142>.
- [2] János K. Asbóth, László Oroszlány, and András Pályi. *A Short Course on Topological Insulators*. Springer International Publishing, 2016. ISBN 9783319256078. doi: 10.1007/978-3-319-25607-8. URL <http://dx.doi.org/10.1007/978-3-319-25607-8>.
- [3] Wladimir A. Benalcazar, Jeffrey C. Y. Teo, and Taylor L. Hughes. Classification of two-dimensional topological crystalline superconductors and majorana bound states at disclinations. *Phys. Rev. B*, 89(22):224503. doi: 10.1103/PhysRevB.89.224503. URL <https://link.aps.org/doi/10.1103/PhysRevB.89.224503>.
- [4] Wladimir A. Benalcazar, B. Andrei Bernevig, and Taylor L. Hughes. Electric multipole moments, topological multipole moment pumping, and chiral hinge states in crystalline insulators. *Phys. Rev. B*, 96:245115, Dec 2017. doi: 10.1103/PhysRevB.96.245115. URL <https://link.aps.org/doi/10.1103/PhysRevB.96.245115>.
- [5] Wladimir A. Benalcazar, B. Andrei Bernevig, and Taylor L. Hughes. Quantized

- electric multipole insulators. *Science*, 357(6346):61–66, Jul 2017. doi: 10.1126/science.aah6442.
- [6] Wladimir A. Benalcazar, Tianhe Li, and Taylor L. Hughes. Quantization of fractional corner charge in c_n -symmetric higher-order topological crystalline insulators. *Phys. Rev. B*, 99(24):245151, 2019. doi: 10.1103/PhysRevB.99.245151. URL <https://link.aps.org/doi/10.1103/PhysRevB.99.245151>.
- [7] Denis Bernard and André LeClair. *A Classification of Non-Hermitian Random Matrices*, page 207–214. Springer Netherlands, 2002. ISBN 9789401005142. doi: 10.1007/978-94-010-0514-2_19. URL http://dx.doi.org/10.1007/978-94-010-0514-2_19.
- [8] Dan S. Borgnia, Alex Jura Kruchkov, and Robert-Jan Slager. Non-hermitian boundary modes and topology. *Phys. Rev. Lett.*, 124:056802, Feb 2020. doi: 10.1103/PhysRevLett.124.056802. URL <https://link.aps.org/doi/10.1103/PhysRevLett.124.056802>.
- [9] Barry Bradlyn, L. Elcoro, Jennifer Cano, M. G. Vergniory, Zhijun Wang, C. Felser, M. I. Aroyo, and B. Andrei Bernevig. Topological quantum chemistry. *Nature*, 547(7663):298–305. ISSN 1476-4687. doi: 10.1038/nature23268. URL <https://www.nature.com/articles/nature23268>.
- [10] Cui-Zu Chang, Chao-Xing Liu, and Allan H. MacDonald. Colloquium: Quantum anomalous hall effect. *Rev. Mod. Phys.*, 95:011002, Jan 2023. doi: 10.1103/RevModPhys.95.011002. URL <https://link.aps.org/doi/10.1103/RevModPhys.95.011002>.
- [11] M N Chernodub. The nielsen–ninomiya theorem, η -invariant non-hermiticity and single 8-shaped dirac cone. *Journal of Physics A: Mathematical and Theoretical*,

- 50(38):385001, August 2017. ISSN 1751-8121. doi: 10.1088/1751-8121/aa809a. URL <http://dx.doi.org/10.1088/1751-8121/aa809a>.
- [12] Ching-Kai Chiu, Jeffrey C.Y. Teo, Andreas P. Schnyder, and Shinsei Ryu. Classification of topological quantum matter with symmetries. *Rev. Mod. Phys.*, 88(3):035005. doi: 10.1103/RevModPhys.88.035005. URL <https://link.aps.org/doi/10.1103/RevModPhys.88.035005>.
- [13] Supriyo Datta. *Electronic Transport in Mesoscopic Systems*. Cambridge University Press, Cambridge, 1995. ISBN 9780521599436. doi: 10.1017/CBO9780511805776. URL <https://www.cambridge.org/core/books/electronic-transport-in-mesoscopic-systems/1E55DEF5978AA7B843FF70337C220D8B>.
- [14] M. Michael Denner, Anastasiia Skurativska, Frank Schindler, Mark H. Fischer, Ronny Thomale, Tomáš Bzdušek, and Titus Neupert. Exceptional topological insulators. *Nature Communications*, 12:5681, 2021. doi: 10.1038/s41467-021-25947-z. URL <https://www.nature.com/articles/s41467-021-25947-z>.
- [15] Marco Michael Denner and Frank Schindler. Magnetic flux response of non-Hermitian topological phases. *SciPost Phys.*, 14:107, 2023. doi: 10.21468/SciPostPhys.14.5.107. URL <https://scipost.org/10.21468/SciPostPhys.14.5.107>.
- [16] Joana Fraxanet, Alexandre Dauphin, Maciej Lewenstein, Luca Barbiero, and Daniel González-Cuadra. Higher-order topological peierls insulator in a two-dimensional atom-cavity system. *Phys. Rev. Lett.*, 131:263001, Dec 2023. doi: 10.1103/PhysRevLett.131.263001. URL <https://link.aps.org/doi/10.1103/PhysRevLett.131.263001>.

- [17] L. Fu and C. L. Kane. Topological insulators with inversion symmetry. *Phys. Rev. B*, 76:045302, 2007. doi: 10.1103/PhysRevB.76.045302.
- [18] L. Fu, C. L. Kane, and E. J. Mele. Topological insulators in three dimensions. *Phys. Rev. Lett.*, 98:106803, 2007. doi: 10.1103/PhysRevLett.98.106803.
- [19] F. D. M. Haldane. Model for a quantum hall effect without landau levels: Condensed-matter realization of the "parity anomaly". *Phys. Rev. Lett.*, 61: 2015–2018, Oct 1988. doi: 10.1103/PhysRevLett.61.2015. URL <https://link.aps.org/doi/10.1103/PhysRevLett.61.2015>.
- [20] Shu Hamanaka, Tsuneya Yoshida, and Kohei Kawabata. Non-hermitian topology in hermitian topological matter. *Phys. Rev. Lett.*, 133:266604, Dec 2024. doi: 10.1103/PhysRevLett.133.266604. URL <https://link.aps.org/doi/10.1103/PhysRevLett.133.266604>.
- [21] Naomichi Hatano. Localization in non-hermitian quantum mechanics and flux-line pinning in superconductors. *Physica A: Statistical Mechanics and its Applications*, 254(1):317–331, 1998. ISSN 0378-4371. doi: [https://doi.org/10.1016/S0378-4371\(98\)00010-7](https://doi.org/10.1016/S0378-4371(98)00010-7). URL <https://www.sciencedirect.com/science/article/pii/S0378437198000107>.
- [22] Naomichi Hatano and David R. Nelson. Vortex pinning and non-hermitian quantum mechanics. *Phys. Rev. B*, 56:8651–8673, Oct 1997. doi: 10.1103/PhysRevB.56.8651. URL <https://link.aps.org/doi/10.1103/PhysRevB.56.8651>.
- [23] Naomichi Hatano and David R. Nelson. Non-hermitian delocalization and eigenfunctions. *Phys. Rev. B*, 58:8384–8390, Oct 1998. doi: 10.1103/PhysRevB.58.8384. URL <https://link.aps.org/doi/10.1103/PhysRevB.58.8384>.

- [24] Allen Hatcher. *Algebraic Topology*. Cambridge University Press, Cambridge, 2002. ISBN 9780521795401. URL <https://pi.math.cornell.edu/~hatcher/AT/AT.pdf>.
- [25] D. Hsieh, D. Qian, L. Wray, Y. Xia, Y. S. Hor, R. J. Cava, and M. Z. Hasan. A topological dirac insulator in a quantum spin hall phase. *Nature*, 452:970–974, 2008. doi: 10.1038/nature06843.
- [26] Haiping Hu and Erhai Zhao. Knots and non-hermitian bloch bands. *Phys. Rev. Lett.*, 126(1):010401. doi: 10.1103/PhysRevLett.126.010401. URL <https://link.aps.org/doi/10.1103/PhysRevLett.126.010401>.
- [27] Haiping Hu, Shikang Sun, and Shu Chen. Knot topology of exceptional point and non-hermitian no-go theorem. *Phys. Rev. Res.*, 4(2):L022064. doi: 10.1103/PhysRevResearch.4.L022064. URL <https://link.aps.org/doi/10.1103/PhysRevResearch.4.L022064>.
- [28] Jinbing Hu, Songlin Zhuang, and Yi Yang. Higher-order topological insulators via momentum-space nonsymmorphic symmetries. *Phys. Rev. Lett.*, 132:213801, May 2024. doi: 10.1103/PhysRevLett.132.213801. URL <https://link.aps.org/doi/10.1103/PhysRevLett.132.213801>.
- [29] Wen-Cheng Jiang, Hong Wu, Qing-Xu Li, Jian Li, and Jia-Ji Zhu. Tunable non-hermitian skin effect via gain and loss. *Phys. Rev. B*, 110:155144, Oct 2024. doi: 10.1103/PhysRevB.110.155144. URL <https://link.aps.org/doi/10.1103/PhysRevB.110.155144>.
- [30] C. L. Kane and E. J. Mele. Quantum spin hall effect in graphene. *Physical Review Letters*, 95(22):226801, 2005. doi: 10.1103/PhysRevLett.95.226801. URL <https://doi.org/10.1103/PhysRevLett.95.226801>.

- [31] C. L. Kane and E. J. Mele. Z_2 topological order and the quantum spin hall effect. *Phys. Rev. Lett.*, 95:146802, Sep 2005. doi: 10.1103/PhysRevLett.95.146802. URL <https://link.aps.org/doi/10.1103/PhysRevLett.95.146802>.
- [32] Kohei Kawabata, Ken Shiozaki, Masahito Ueda, and Masatoshi Sato. Symmetry and topology in non-hermitian physics. *Phys. Rev. X*, 9(4):041015. doi: 10.1103/PhysRevX.9.041015. URL <https://link.aps.org/doi/10.1103/PhysRevX.9.041015>.
- [33] Kohei Kawabata, Masatoshi Sato, and Ken Shiozaki. Higher-order non-hermitian skin effect. *Phys. Rev. B*, 102(20):205118, 2020. ISSN 2469-9950, 2469-9969. doi: 10.1103/PhysRevB.102.205118. URL <https://link.aps.org/doi/10.1103/PhysRevB.102.205118>.
- [34] Kohei Kawabata, Ken Shiozaki, and Shinsei Ryu. Many-body topology of non-hermitian systems. *Phys. Rev. B*, 105:165137, Apr 2022. doi: 10.1103/PhysRevB.105.165137. URL <https://link.aps.org/doi/10.1103/PhysRevB.105.165137>.
- [35] A Yu Kitaev. Unpaired majorana fermions in quantum wires. *Physics-Uspekhi*, 44(10S):131, oct 2001. doi: 10.1070/1063-7869/44/10S/S29. URL <https://dx.doi.org/10.1070/1063-7869/44/10S/S29>.
- [36] Alexei Kitaev. Periodic table for topological insulators and superconductors. *AIP Conference Proceedings*, 1134(1):22–30. ISSN 0094-243X. doi: 10.1063/1.3149495. URL <https://doi.org/10.1063/1.3149495>.
- [37] K. v. Klitzing, G. Dorda, and M. Pepper. New method for high-accuracy determination of the fine-structure constant based on quantized hall resistance. *Phys. Rev. Lett.*, 45:494–497, Aug 1980. doi: 10.1103/PhysRevLett.45.494. URL <https://link.aps.org/doi/10.1103/PhysRevLett.45.494>.

- [38] M. König, S. Wiedmann, C. Brüne, A. Roth, H. Buhmann, L. W. Molenkamp, X.-L. Qi, and S.-C. Zhang. Quantum spin hall insulator state in hgte quantum wells. *Science*, 318(5851):766–770, 2007. doi: 10.1126/science.1148047. URL <https://www.science.org/doi/10.1126/science.1148047>.
- [39] Vladyslav Kozii and Liang Fu. Non-hermitian topological theory of finite-lifetime quasiparticles: Prediction of bulk fermi arc due to exceptional point, 2017. URL <https://arxiv.org/abs/1708.05841>.
- [40] Jorrit Kruthoff, Jan de Boer, Jasper van Wezel, Charles L. Kane, and Robert-Jan Slager. Topological classification of crystalline insulators through band structure combinatorics. *Phys. Rev. X*, 7:041069, Dec 2017. doi: 10.1103/PhysRevX.7.041069. URL <https://link.aps.org/doi/10.1103/PhysRevX.7.041069>.
- [41] J. Lukas K. König, Kang Yang, Jan Carl Budich, and Emil J. Bergholtz. Braid-protected topological band structures with unpaired exceptional points. *Phys. Rev. Res.*, 5(4):L042010. doi: 10.1103/PhysRevResearch.5.L042010. URL <https://link.aps.org/doi/10.1103/PhysRevResearch.5.L042010>.
- [42] J. Lukas K. König, Kang Yang, André Grossi Fonseca, Sachin Vaidya, Marin Soljačić, and Emil J. Bergholtz. Exceptional topology on nonorientable manifolds, 2025. URL <https://arxiv.org/abs/2503.04889>.
- [43] Josias Langbehn, Yang Peng, Luka Trifunovic, Felix von Oppen, and Piet W. Brouwer. Reflection-symmetric second-order topological insulators and superconductors. *Phys. Rev. Lett.*, 119:246401, Dec 2017. doi: 10.1103/PhysRevLett.119.246401. URL <https://link.aps.org/doi/10.1103/PhysRevLett.119.246401>.
- [44] Congcong Le, Zhesen Yang, Fan Cui, A. P. Schnyder, and Ching-Kai Chiu. Generalized fermion doubling theorems: Classification of two-dimensional nodal

- systems in terms of wallpaper groups. *Phys. Rev. B*, 106:045126, Jul 2022. doi: 10.1103/PhysRevB.106.045126. URL <https://link.aps.org/doi/10.1103/PhysRevB.106.045126>.
- [45] Ching Hua Lee, Linhu Li, and Jiangbin Gong. Hybrid higher-order skin-topological modes in nonreciprocal systems. *Phys. Rev. Lett.*, 123(1):016805. doi: 10.1103/PhysRevLett.123.016805. URL <https://link.aps.org/doi/10.1103/PhysRevLett.123.016805>.
- [46] Zhi Li and Roger S. K. Mong. Homotopical characterization of non-hermitian band structures. *Phys. Rev. B*, 103(15):155129. ISSN 2469-9950, 2469-9969. doi: 10.1103/PhysRevB.103.155129. URL <https://link.aps.org/doi/10.1103/PhysRevB.103.155129>.
- [47] Tao Liu, Yu-Ran Zhang, Qing Ai, Zongping Gong, Kohei Kawabata, Masahito Ueda, and Franco Nori. Second-order topological phases in non-hermitian systems. *Phys. Rev. Lett.*, 122(7):076801. doi: 10.1103/PhysRevLett.122.076801. URL <https://link.aps.org/doi/10.1103/PhysRevLett.122.076801>.
- [48] Stefano Longhi. Non-hermitian skin effect and self-acceleration. *Phys. Rev. B*, 105:245143, Jun 2022. doi: 10.1103/PhysRevB.105.245143. URL <https://link.aps.org/doi/10.1103/PhysRevB.105.245143>.
- [49] J. E. Moore and L. Balents. Topological invariants of time-reversal-invariant band structures. *Phys. Rev. B*, 75:121306, 2007. doi: 10.1103/PhysRevB.75.121306.
- [50] Takahiro Morimoto and Akira Furusaki. Topological classification with additional symmetries from clifford algebras. *Phys. Rev. B*, 88:125129, Sep 2013. doi: 10.1103/PhysRevB.88.125129. URL <https://link.aps.org/doi/10.1103/PhysRevB.88.125129>.

- [51] Christopher Mudry, P. W. Brouwer, B. I. Halperin, V. Gurarie, and A. Zee. Density of states in the non-hermitian lloyd model. *Phys. Rev. B*, 58:13539–13543, Nov 1998. doi: 10.1103/PhysRevB.58.13539. URL <https://link.aps.org/doi/10.1103/PhysRevB.58.13539>.
- [52] Christopher Mudry, B. D. Simons, and Alexander Altland. Random dirac fermions and non-hermitian quantum mechanics. *Phys. Rev. Lett.*, 80:4257–4260, May 1998. doi: 10.1103/PhysRevLett.80.4257. URL <https://link.aps.org/doi/10.1103/PhysRevLett.80.4257>.
- [53] Daichi Nakamura, Ken Shiozaki, Kenji Shimomura, Masatoshi Sato, and Kohei Kawabata. Non-hermitian origin of wannier localizability and detachable topological boundary states, 2024. URL <https://arxiv.org/abs/2407.09458>.
- [54] H. B. Nielsen and M. Ninomiya. Absence of neutrinos on a lattice: (II). intuitive topological proof. *Nuclear Physics B*, 193(1):173–194, . ISSN 0550-3213. doi: 10.1016/0550-3213(81)90524-1. URL <https://www.sciencedirect.com/science/article/pii/0550321381905241>.
- [55] H. B. Nielsen and M. Ninomiya. Absence of neutrinos on a lattice: (i). proof by homotopy theory. *Nuclear Physics B*, 185(1):20–40, . ISSN 0550-3213. doi: 10.1016/0550-3213(81)90361-8. URL <https://www.sciencedirect.com/science/article/pii/0550321381903618>.
- [56] H. B. Nielsen and M. Ninomiya. A no-go theorem for regularizing chiral fermions. *Physics Letters B*, 105(2):219–223, . ISSN 0370-2693. doi: 10.1016/0370-2693(81)91026-1. URL <https://www.sciencedirect.com/science/article/pii/0370269381910261>.
- [57] Jiho Noh, Wladimir A Benalcazar, Sheng Huang, Matthew J Collins, Kevin P

- Chen, Taylor L Hughes, and Mikael C Rechtsman. Topological protection of photonic mid-gap defect modes. *Nature Photonics*, 12(7):408–415, 2018.
- [58] Ryo Okugawa, Ryo Takahashi, and Kazuki Yokomizo. Second-order topological non-hermitian skin effects. *Phys. Rev. B*, 102(24):241202. ISSN 2469-9950, 2469-9969. doi: 10.1103/PhysRevB.102.241202. URL <https://link.aps.org/doi/10.1103/PhysRevB.102.241202>.
- [59] Ryo Okugawa, Ryo Takahashi, and Kazuki Yokomizo. Non-hermitian band topology with generalized inversion symmetry. *Phys. Rev. B*, 103:205205, May 2021. doi: 10.1103/PhysRevB.103.205205. URL <https://link.aps.org/doi/10.1103/PhysRevB.103.205205>.
- [60] Nobuyuki Okuma, Kohei Kawabata, Ken Shiozaki, and Masatoshi Sato. Topological origin of non-hermitian skin effects. *Phys. Rev. Lett.*, 124(8):086801. doi: 10.1103/PhysRevLett.124.086801. URL <https://link.aps.org/doi/10.1103/PhysRevLett.124.086801>.
- [61] Hoi Chun Po, Ashvin Vishwanath, and Haruki Watanabe. Symmetry-based indicators of band topology in the 230 space groups. *Nature Communications*, 8(1):50. ISSN 2041-1723. doi: 10.1038/s41467-017-00133-2. URL <https://www.nature.com/articles/s41467-017-00133-2>.
- [62] Xiao-Liang Qi, Yong-Shi Wu, and Shou-Cheng Zhang. Topological quantization of the spin hall effect in two-dimensional paramagnetic semiconductors. *Phys. Rev. B*, 74:085308, Aug 2006. doi: 10.1103/PhysRevB.74.085308. URL <https://link.aps.org/doi/10.1103/PhysRevB.74.085308>.
- [63] W. B. Rui, Y. X. Zhao, and Z. D. Wang. Hermitian topologies originating from non-hermitian braidings. *Phys. Rev. B*, 108(16):165105. doi: 10.1103/

- PhysRevB.108.165105. URL <https://link.aps.org/doi/10.1103/PhysRevB.108.165105>.
- [64] Shinsei Ryu, Andreas P Schnyder, Akira Furusaki, and Andreas W Ludwig. Topological insulators and superconductors: Tenfold way and dimensional hierarchy. *New Journal of Physics*, 12(6):065010, Jun 2010. doi: 10.1088/1367-2630/12/6/065010.
- [65] Masatoshi Sato and Yoichi Ando. Topological superconductors: a review. *Reports on Progress in Physics*, 80(7):076501, 2017. doi: 10.1088/1361-6633/aa6ac7. URL <https://iopscience.iop.org/article/10.1088/1361-6633/aa6ac7/meta>.
- [66] Frank Schindler, Marta Brzezińska, Wladimir A. Benalcazar, Mikel Iraola, Adrien Bouhon, Stepan S. Tsirkin, Maia G. Vergniory, and Titus Neupert. Fractional corner charges in spin-orbit coupled crystals. *Phys. Rev. Res.*, 1:033074, Nov 2019. doi: 10.1103/PhysRevResearch.1.033074. URL <https://link.aps.org/doi/10.1103/PhysRevResearch.1.033074>.
- [67] Andreas P. Schnyder, Shinsei Ryu, Akira Furusaki, Andreas W. Ludwig, Vladimir Lebedev, and Mikhail Feigel'man. Classification of topological insulators and superconductors. *AIP Conference Proceedings*, 2009. doi: 10.1063/1.3149481.
- [68] Huitao Shen, Bo Zhen, and Liang Fu. Topological band theory for non-hermitian hamiltonians. *Phys. Rev. Lett.*, 120(14):146402. doi: 10.1103/PhysRevLett.120.146402. URL <https://link.aps.org/doi/10.1103/PhysRevLett.120.146402>.
- [69] Ken Shiozaki and Seishiro Ono. Symmetry indicator in non-hermitian systems. *Phys. Rev. B*, 104:035424, Jul 2021. doi: 10.1103/PhysRevB.104.035424. URL <https://link.aps.org/doi/10.1103/PhysRevB.104.035424>.

- [70] Ken Shiozaki and Masatoshi Sato. Topology of crystalline insulators and superconductors. *Phys. Rev. B*, 90:165114, Oct 2014. doi: 10.1103/PhysRevB.90.165114. URL <https://link.aps.org/doi/10.1103/PhysRevB.90.165114>.
- [71] Ken Shiozaki, Masatoshi Sato, and Kiyonori Gomi. Topology of nonsymmorphic crystalline insulators and superconductors. *Phys. Rev. B*, 93:195413, May 2016. doi: 10.1103/PhysRevB.93.195413. URL <https://link.aps.org/doi/10.1103/PhysRevB.93.195413>.
- [72] Ken Shiozaki, Masatoshi Sato, and Kiyonori Gomi. Topological crystalline materials: General formulation, module structure, and wallpaper groups. *Phys. Rev. B*, 95:235425, Jun 2017. doi: 10.1103/PhysRevB.95.235425. URL <https://link.aps.org/doi/10.1103/PhysRevB.95.235425>.
- [73] Ken Shiozaki, Daichi Nakamura, Kenji Shimomura, Masatoshi Sato, and Kohei Kawabata. k -theory classification of wannier localizability and detachable topological boundary states, 2024. URL <https://arxiv.org/abs/2407.18273>.
- [74] Robert-Jan Slager, Andrej Mesaros, Vladimir Juričić, and Jan Zaanen. The space group classification of topological band-insulators. *Nature Physics*, 9(2): 98–102, Dec 2012. doi: 10.1038/nphys2513.
- [75] Fei Song, Shunyu Yao, and Zhong Wang. Non-hermitian skin effect and chiral damping in open quantum systems. *Phys. Rev. Lett.*, 123:170401, Oct 2019. doi: 10.1103/PhysRevLett.123.170401. URL <https://link.aps.org/doi/10.1103/PhysRevLett.123.170401>.
- [76] Fei Song, Shunyu Yao, and Zhong Wang. Non-hermitian skin effect and chiral damping in open quantum systems. *Phys. Rev. Lett.*, 123:170401, Oct 2019. doi: 10.1103/PhysRevLett.123.170401. URL <https://link.aps.org/doi/10.1103/PhysRevLett.123.170401>.

- [77] Zhida Song, Zhong Fang, and Chen Fang. $(d - 2)$ -dimensional edge states of rotation symmetry protected topological states. *Phys. Rev. Lett.*, 119:246402, Dec 2017. doi: 10.1103/PhysRevLett.119.246402. URL <https://link.aps.org/doi/10.1103/PhysRevLett.119.246402>.
- [78] Michael Stone, Ching-Kai Chiu, and Abhishek Roy. Symmetries, dimensions and topological insulators: The mechanism behind the face of the bott clock. *Journal of Physics A: Mathematical and Theoretical*, 44(4):045001, Dec 2010. doi: 10.1088/1751-8113/44/4/045001.
- [79] Jeffrey C. Y. Teo and Taylor L. Hughes. Existence of majorana-fermion bound states on disclinations and the classification of topological crystalline superconductors in two dimensions. *Phys. Rev. Lett.*, 111:047006, Jul 2013. doi: 10.1103/PhysRevLett.111.047006. URL <https://link.aps.org/doi/10.1103/PhysRevLett.111.047006>.
- [80] Jeffrey C. Y. Teo and C. L. Kane. Topological defects and gapless modes in insulators and superconductors. *Phys. Rev. B*, 82:115120, Sep 2010. doi: 10.1103/PhysRevB.82.115120. URL <https://link.aps.org/doi/10.1103/PhysRevB.82.115120>.
- [81] D. J. Thouless, M. Kohmoto, M. P. Nightingale, and M. den Nijs. Quantized hall conductance in a two-dimensional periodic potential. *Phys. Rev. Lett.*, 49: 405–408, Aug 1982. doi: 10.1103/PhysRevLett.49.405. URL <https://link.aps.org/doi/10.1103/PhysRevLett.49.405>.
- [82] Sachin Vaidya, Ali Ghorashi, Thomas Christensen, Mikael C. Rechtsman, and Wladimir A. Benalcazar. Topological phases of photonic crystals under crystalline symmetries. *Phys. Rev. B*, 108:085116, Aug 2023. doi: 10.1103/PhysRevB.108.085116. URL <https://link.aps.org/doi/10.1103/PhysRevB.108.085116>.

- [83] Pascal M. Vecsei, M. Michael Denner, Titus Neupert, and Frank Schindler. Symmetry indicators for inversion-symmetric non-hermitian topological band structures. *Phys. Rev. B*, 103:L201114, May 2021. doi: 10.1103/PhysRevB.103.L201114. URL <https://link.aps.org/doi/10.1103/PhysRevB.103.L201114>.
- [84] Yifan Wang. Classifying order-two spatial symmetries in non-hermitian hamiltonians: Point-gapped az and $az^\dagger$ classes, 2024. URL <https://arxiv.org/abs/2411.03410>.
- [85] Yifan Wang and Wladimir A. Benalcazar. Higher-order topological knots in non-hermitian lattices, 2024. URL <https://arxiv.org/abs/2412.05809>.
- [86] Zheng Wang, Yidong Chong, J. D. Joannopoulos, and Marin Soljačić. Observation of unidirectional backscattering-immune topological electromagnetic states. *Nature*, 461(7265):772–775, Oct 2009. doi: 10.1038/nature08293.
- [87] Charles C. Wojcik, Xiao-Qi Sun, Tomáš Bzdušek, and Shanhui Fan. Homotopy characterization of non-hermitian hamiltonians. *Phys. Rev. B*, 101(20):205417. ISSN 2469-9950, 2469-9969. doi: 10.1103/PhysRevB.101.205417. URL <https://link.aps.org/doi/10.1103/PhysRevB.101.205417>.
- [88] Ye Xiong. Why does bulk boundary correspondence fail in some non-hermitian topological models. *J. Phys. Commun.*, 2(3):035043, March 2018. doi: 10.1088/2399-6528/AAB64A. URL <https://robots.iopscience.iop.org/article/10.1088/2399-6528/aab64a/pdf>.
- [89] Yang Xue, Hao Huan, Bao Zhao, Youhua Luo, Zhenyu Zhang, and Zhongqin Yang. Higher-order topological insulators in two-dimensional dirac materials. *Phys. Rev. Res.*, 3:L042044, Dec 2021. doi: 10.1103/PhysRevResearch.3.L042044. URL <https://link.aps.org/doi/10.1103/PhysRevResearch.3.L042044>.

- [90] Kang Yang, Zhi Li, J Lukas König, Lukas Rødland, Marcus Stålhammar, and Emil J Bergholtz. Homotopy, symmetry, and non-hermitian band topology. *Reports on Progress in Physics*, 87(7):078002, Jul 2024. doi: 10.1088/1361-6633/ad4e64.
- [91] Zhesen Yang, Kai Zhang, Chen Fang, and Jiangping Hu. Non-hermitian bulk-boundary correspondence and auxiliary generalized brillouin zone theory. *Phys. Rev. Lett.*, 125:226402, Nov 2020. doi: 10.1103/PhysRevLett.125.226402. URL <https://link.aps.org/doi/10.1103/PhysRevLett.125.226402>.
- [92] Shunyu Yao and Zhong Wang. Edge states and topological invariants of non-hermitian systems. *Phys. Rev. Lett.*, 121(8):086803. ISSN 0031-9007, 1079-7114. doi: 10.1103/PhysRevLett.121.086803. URL <https://link.aps.org/doi/10.1103/PhysRevLett.121.086803>.
- [93] Kazuki Yokomizo and Shuichi Murakami. Non-bloch band theory of non-hermitian systems. *Phys. Rev. Lett.*, 123:066404, Aug 2019. doi: 10.1103/PhysRevLett.123.066404. URL <https://link.aps.org/doi/10.1103/PhysRevLett.123.066404>.
- [94] Rui Yu, Xiao Liang Qi, Andrei Bernevig, Zhong Fang, and Xi Dai. Equivalent expression of z_2 topological invariant for band insulators using the non-abelian berry connection. *Phys. Rev. B*, 84:075119, Aug 2011. doi: 10.1103/PhysRevB.84.075119. URL <https://link.aps.org/doi/10.1103/PhysRevB.84.075119>.
- [95] J. Zak. Berry’s phase for energy bands in solids. *Phys. Rev. Lett.*, 62(23): 2747–2750. doi: 10.1103/PhysRevLett.62.2747. URL <https://link.aps.org/doi/10.1103/PhysRevLett.62.2747>.
- [96] Kai Zhang, Zhesen Yang, and Chen Fang. Correspondence between winding numbers and skin modes in non-hermitian systems. *Phys. Rev. Lett.*, 125:126402,

- Sep 2020. doi: 10.1103/PhysRevLett.125.126402. URL <https://link.aps.org/doi/10.1103/PhysRevLett.125.126402>.
- [97] Kai Zhang, Zhesen Yang, and Chen Fang. Universal non-hermitian skin effect in two and higher dimensions. *Nature Communications*, 13(1), May 2022. doi: 10.1038/s41467-022-30161-6.
- [98] Zhi-Xu Zhang, Ji Cao, Jing-Quan Li, Yu Zhang, Shutian Liu, Shou Zhang, and Hong-Fu Wang. Topological skin modes and intensity amplification in a nonlinear non-hermitian lattice. *Phys. Rev. B*, 108:125402, Sep 2023. doi: 10.1103/PhysRevB.108.125402. URL <https://link.aps.org/doi/10.1103/PhysRevB.108.125402>.
- [99] Jia-Xin Zhong, Pedro Fittipaldi de Castro, Tianhong Lu, Jeewoo Kim, Mourad Oudich, Jun Ji, Li Shi, Kai Chen, Jing Lu, Yun Jing, and Wladimir A. Benalcazar. Higher-order skin effect through a hermitian-non-hermitian correspondence and its observation in an acoustic kagome lattice, 2024. URL <https://arxiv.org/abs/2409.01516>.
- [100] Hengyun Zhou and Jong Yeon Lee. Periodic table for topological bands with non-hermitian symmetries. *Phys. Rev. B*, 99:235112, Jun 2019. doi: 10.1103/PhysRevB.99.235112. URL <https://link.aps.org/doi/10.1103/PhysRevB.99.235112>.