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Moonshine and Elliptic Curves

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Moonshine and Elliptic Curves

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M.S., Emory University, 2020

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An abstract of  
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# Abstract

Maryam Khaqan

*Moonshine and Elliptic Curves*

In this dissertation, we characterize all infinite-dimensional graded virtual modules for Thompson's sporadic simple group, whose graded traces are weight  $\frac{3}{2}$  weakly holomorphic modular forms satisfying certain special properties. We then use these modules to detect the non-triviality of Mordell-Weil, Selmer and Tate-Shafarevich groups of quadratic twists of certain elliptic curves. Thus proving the existence of a new kind of moonshine as well as establishing applications of moonshine to number theory.



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# Moonshine and Elliptic Curves

*Dissertation*

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By

MARYAM KHAQAN



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Advisor: John Duncan, Ph.D.

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<sup>1</sup>Future grad students, I leave it up to you to use this information for your advantage ;)

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teachers and mentors,  
counting the two who raised  
me, this one is for you.

راویار ہم نے قدم قدم تجھے یادگار بنا دیا

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# Chapter 1

## Introduction

### 1.1 Moonshine

In 1978, McKay and Thompson observed [Thompson 1979b] that the first few coefficients of the normalized elliptic modular invariant  $J(\tau) = q^{-1} + 196884q + 21493760q^2 + 864299970q^3 + O(q^4)$ , a central object in the theory of modular forms, can be written as sums involving the first few dimensions of irreducible representations of the monster group  $\mathbb{M}$ , e.g.,

$$\begin{aligned} 196884 &= 1 + 196883 \\ 21493760 &= 1 + 196883 + 21296876 \\ 864299970 &= 2 \cdot 1 + 2 \cdot 196883 + 21296876 + 842609326. \end{aligned} \tag{1.1.1}$$

This coincidence inspired Thompson's conjecture [Thompson 1979a] that there is an infinite-dimensional  $\mathbb{M}$ -module  $V = \bigoplus_{n \geq -1} V_n$  whose graded dimension is  $J(\tau)$  and whose McKay–Thompson series

$$T_g(\tau) := \sum_{n \geq -1} \text{tr}(g|V_n)q^n \tag{1.1.2}$$

are distinguished functions on the upper half-plane. Conway and Norton [Conway and Norton 1979] explicitly described the relevant McKay-Thompson series, and also christened this phenomenon “monstrous moonshine.” Their conjecture was proven by Borcherds [Borcherds 1992], building on work by Frenkel, Lepowsky and Meurman [Frenkel et al. 1988] (see also, [Frenkel et al. 1985; 1984]) in 1992. In the few decades since the first observations of McKay and Thompson, it has become clear that monstrous moonshine is just the first of a series of similar phenomena encompassing several finite groups and their counterparts in the world of modular forms.

Generalized moonshine [Carnahan 2012] (see also, [Norton 1987; Conway and Norton 1979; Queen 1981]), for example, relates various subquotients of the Monster to other weight zero modular forms. Umbral moonshine [Cheng et al. 2014a;b] (see also [Duncan et al. 2015; Gannon 2016] and [Cheng et al. 2018]), on the other hand, relates the 23 umbral groups (each of which is a quotient of the automorphism group of one of the 23 Niemeier lattices) to weight  $\frac{1}{2}$  *mock modular forms*. Thompson moonshine, conjectured by Harvey and Rayhaun [Harvey and Rayhaun 2015] in 2015 and proven by Griffin and Mertens in [Griffin and Mertens 2016], involves Thompson’s sporadic simple group  $Th$ , and certain weight  $\frac{1}{2}$  modular forms. (We remark here that the Thompson group, being a subgroup of the Monster, also appears in the generalized moonshine setting mentioned above. For the purpose of this dissertation, “Thompson moonshine” refers to the Harvey and Rayhaun version.)

Recently, in [Duncan et al. 2017; to appear], Duncan, Mertens, and Ono discovered the first instance of moonshine for the O’Nan group, one of the so-called pariah groups (i.e., a sporadic simple group which is not a subquotient of the monster group), where the functions involved are modular forms of weight  $\frac{3}{2}$ . Their work is not only a contribution to the theory of

moonshine, it also serves another important purpose: In the same paper, they use their O’Nan-module to study properties of quadratic twists of certain elliptic curves and thus use moonshine to provide insight into objects that are central to current research in number theory.

## 1.2 Weight $\frac{3}{2}$ moonshine for $Th$

While number theory’s contribution to moonshine is ubiquitous and irrefutable, O’Nan moonshine is one of the first instances where we see moonshine’s direct contribution to number theory. Such a role-reversal is our primary motivation for this work.

In this dissertation, we begin by proving the existence of a family of infinite-dimensional graded  $Th$ -modules whose McKay–Thompson series are weight  $\frac{3}{2}$  modular forms that satisfy certain properties (cf. Theorem 3.1.2). The techniques we use to prove this are similar to ones used in Griffin and Mertens’ work [Griffin and Mertens 2016] to prove the Thompson moonshine conjecture [Harvey and Rayhaun 2015]. (These techniques were first suggested by Thompson, and subsequently used by Atkin, Fong and Smith [Fong 1980; Smith 1985] to prove monstrous moonshine abstractly.) On the other hand, our McKay–Thompson series are weight  $\frac{3}{2}$  modular forms (in contrast to the weight  $\frac{1}{2}$  forms of [Harvey and Rayhaun 2015]) and the role played by theta functions in their paper is taken up by weight  $\frac{3}{2}$  cusp forms in ours. The involvement of weight  $\frac{3}{2}$  cusp forms allows us to employ an approach similar to Duncan, Mertens, and Ono (in [Duncan et al. 2017; to appear]): We exploit the existing relationship between these forms and elliptic curves to study geometric invariants of various elliptic curves. This is the content of Theorems 4.1.1 and 4.1.2.

Our result regarding the existence of a family of Thompson modules is, in fact, a classification result. We classify all infinite-dimensional graded modules  $W = \bigoplus W_n$  (cf. Theorem 3.1.2) for the Thompson group whose McKay–Thompson series take the form

$$\mathcal{F}_g(\tau) := 6q^{-5} + \sum_{n>0} \text{tr}(g|W_n)q^n \quad (1.2.1)$$

and satisfy the following properties (cf. Proposition 3.1.1):

1. For each  $g \in Th$ , the corresponding McKay–Thompson series  $\mathcal{F}_g(\tau)$  is a weight  $\frac{3}{2}$  weakly holomorphic modular form of a specific level and multiplier system, and satisfies the Kohnen plus space condition.
2. Each McKay–Thompson series  $\mathcal{F}_g(\tau)$  has integer coefficients and is uniquely determined — up to the addition of certain cusp forms — by its polar parts at the cusps, which are specified in a uniform way. (See Section 3.1 for details.)

We note here that properties (1) and (2) listed above ensure that the functions  $\mathcal{F}_g(\tau)$  are, up to the addition of cusp forms, simply Rademacher sums projected to the Kohnen plus space (see Section 2.1 for background on Rademacher sums).

The connection between Rademacher sums and moonshine was first proposed in [Duncan and Frenkel 2011], where the McKay–Thompson series that appear in monstrous moonshine were characterized completely in terms of Rademacher sums of weight 0. In particular, it was shown that the so-called *genus-zero property* of monstrous moonshine is equivalent to the fact that the McKay–Thompson series of the Monster module coincide (up to a

constant) with corresponding Rademacher sums of weight 0. It was later argued in [Cheng and Duncan 2014; Cheng et al. 2014a] that the correct analogue of the genus zero property in the case of Umbral (and Mathieu) moonshine is that the corresponding McKay–Thompson series must coincide with the relevant Rademacher sums in each case (see also [Cheng et al. 2018; Duncan 2019]). Here we take this perspective and hence consider it natural, from the point of view of moonshine, to ask for our McKay–Thompson series to satisfy the properties listed above.

To prove our classification result, we first construct spaces of weakly holomorphic modular forms of the appropriate level and multiplier for each  $g \in Th$ . We use Rademacher sums and eta-quotients to do this. Since we can explicitly compute the Fourier coefficients of these forms at various cusps, we can restrict our attention to the subspace of forms that satisfy properties (1) and (2). For a collection of these forms to be the McKay–Thompson series of a virtual module (as in Theorem 3.1.2), they must satisfy congruences modulo certain powers of primes that divide the order of the Thompson group (cf. Section 3.3). A complete description of these congruences can be obtained using Thompson’s reformulation (cf. [Smith 1985]) of Brauer’s characterization of generalized characters. We prove that our alleged McKay–Thompson series satisfy the congruences mentioned above in Section 3.3. We note here that it would be interesting to consider the analogous classification for the O’Nan group, building on the work already done in [Duncan et al. to appear]. Once we have proven the existence of the Thompson modules, we will use their properties to help detect the non-triviality of Mordell–Weil, Selmer, and Tate–Shafarevich groups of quadratic twists of certain elliptic curves. See Chapter 4 for details on theorems related to elliptic curves.





## Chapter 2

# Background and Notation

### 2.1 Notation

Throughout this dissertation, we use the notation  $e(x) = e^{2\pi ix}$  and  $q = e(\tau)$  with  $\tau$  in the upper half-plane, which we denote  $\mathbb{H}$ . We also use  $\left(\frac{m}{n}\right)$  to denote the Kronecker symbol [Cohen 1993; Algorithm 1.4.10]. We will use the ATLAS [Conway et al. 1985] notation for conjugacy classes of  $Th$ , and understand  $nAB$  to mean  $nA \cup nB$ .

### 2.2 Rational Characters.

We define the *rational conjugacy class* of an element  $g \in Th$ , denoted by  $[g]$ , to be the set of all elements conjugate to an  $n^{\text{th}}$  power of  $g$  where  $n$  is relatively prime to the order of  $g$ . (In particular, this contains the conjugacy class of  $g$  as a subset.)

We recall that a *rational character* of a group  $G$  is a character afforded by a  $\mathbb{Q}G$ -module. By [Curtis and Reiner 1988; Lemma 39.4], if  $g$  and  $h$  are in the same rational conjugacy class, and  $\phi : G \rightarrow \mathbb{Q}$  is a rational character, then

$\phi(g) = \phi(h)$ . We note that if  $\phi$  is a rational character of  $Th$ , then  $\phi(g)$  is an algebraic integer lying in  $\mathbb{Q}$ , so in fact  $\phi(g) \in \mathbb{Z}$ .

In this paper, we will consider the irreducible rational characters of  $Th$ . To describe these, we first define a few things. Let  $G$  be a finite group and  $V$  be a  $\mathbb{C}G$ -module. For each field automorphism  $\gamma : \mathbb{C} \rightarrow \mathbb{C}$ , there exists a unique (up to isomorphism) representation  $V^\gamma$  with character  $\chi_{V^\gamma(g)} = \gamma\chi_V(g)$ . We call  $V^\gamma$  a Galois-conjugate of  $V$ . Then, we have the following proposition.

**Proposition 2.2.1.** (see [Curtis and Reiner 1988; Theorem 74.5], for example.) Let  $V_1 (= V), V_2, \dots, V_n$  be the distinct Galois-conjugates of an irreducible  $\mathbb{C}G$ -module  $V$ . Then there exists a natural number  $m_V$  such that  $m_V(V_1 \oplus \dots \oplus V_n)$  is the complexification of an irreducible  $\mathbb{Q}G$ -module. Furthermore, each irreducible  $\mathbb{Q}G$ -module  $W$  arises in this way from a unique Galois-class of irreducible  $\mathbb{C}G$ -modules, i.e.

$$W \otimes_{\mathbb{Q}} \mathbb{C} \simeq m_V(V_1 \oplus \dots \oplus V_n). \quad (2.2.1)$$

The number  $m_V$  is called the (rational) Schur index of  $V$ . By [Feit 1983; Section 7], the Schur index is 1 for each irreducible representation of  $Th$ . Thus, we can read off the 39 irreducible rational characters of  $Th$  directly from the character table. We denote these by  $\chi_1, \chi_2, \dots, \chi_{39}$ .

## 2.3 Mock Modular Forms

To prove the existence of the Thompson modules in Theorem 3.1.2, we first have to construct weakly holomorphic modular forms of weight  $\frac{3}{2}$  with the appropriate level and multiplier. Recall that a *weakly holomorphic modular form* is a function on the upper half-plane that transforms like a modular form, is holomorphic on the upper half-plane and meromorphic at the

cusps. One way of constructing spaces of weakly holomorphic forms is to use Rademacher sums, which are *a priori* mock modular forms, and then restrict to the subspace of forms with vanishing shadow. Here we recall the definitions and basic facts that we will need from the theory of mock modular forms and Rademacher sums to describe these functions. We refer the reader to [Bringmann et al. 2017; Dabholkar et al. 2012; Ono 2009] for more on mock modular forms.

Let  $k \in \frac{1}{2}\mathbb{Z}$  and  $\Gamma$  be a subgroup of  $\mathrm{SL}_2(\mathbb{R})$  containing  $\pm I$  such that  $\Gamma$  is commensurable with  $\mathrm{SL}_2(\mathbb{Z})$ . For  $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ , write  $\gamma\tau$  for  $\frac{a\tau+b}{c\tau+d}$ , and define  $j(\gamma, \tau) = (c\tau + d)^{-2}$ . We call a function  $\psi : \Gamma \rightarrow \mathbb{C}$  a *multiplier system* for  $\Gamma$  of weight  $k$  if

$$\psi(\gamma_1\gamma_2)j(\gamma_1\gamma_2, \tau)^{\frac{k}{2}} = \psi(\gamma_1)j(\gamma_1, \gamma_2\tau)^{\frac{k}{2}}\psi(\gamma_2)j(\gamma_2, \tau)^{\frac{k}{2}} \quad (2.3.1)$$

for each  $\gamma_1, \gamma_2 \in \Gamma$ , where we choose the principal branch of the logarithm to define the exponential  $x \mapsto x^s$  in case  $s$  is not an integer.

In this paper we will consider multiplier systems of the form

$$\psi_{4N,v,h}(\gamma) := e\left(-\frac{vcd}{4Nh}\right), \quad (2.3.2)$$

where  $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(4N)$  and  $v, h$  are integers with  $h \mid \gcd(N, 24)$ .

Recall that  $\Gamma_0(N)$  is the congruence subgroup

$$\Gamma_0(N) := \left\{ \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) \mid c \equiv 0 \pmod{N} \right\} \quad (2.3.3)$$

of the full modular group  $\mathrm{SL}_2(\mathbb{Z})$ . We can now define the  $(k, \psi)$ -action of  $\gamma \in \Gamma_0(N)$  on a smooth function  $f : \mathbb{H} \rightarrow \mathbb{C}$  by

$$(f|_{k, \psi} \gamma)(\tau) := \begin{cases} \psi(\gamma) j(\gamma, \tau)^{\frac{k}{2}} f(\gamma\tau) & \text{if } k \in \mathbb{Z} \\ \left(\frac{c}{d}\right) \varepsilon_d^{2k} \psi(\gamma) j(\gamma, \tau)^{\frac{k}{2}} f(\gamma\tau) & \text{if } k \in \frac{1}{2} + \mathbb{Z}, \end{cases} \quad (2.3.4)$$

where

$$\varepsilon_d := \begin{cases} 1 & d \equiv 1 \pmod{4}, \\ i & d \equiv 3 \pmod{4}. \end{cases} \quad (2.3.5)$$

and we assume  $4|N$  if  $k \notin \mathbb{Z}$ .

**Definition 2.3.1.** A *harmonic (weak) Maaß form* of weight  $k \in \frac{1}{2}\mathbb{Z}$ , level  $N$  and multiplier system  $\psi$ , is a smooth function  $f : \mathbb{H} \rightarrow \mathbb{C}$  on the upper half-plane that satisfies the following properties:

1. It is invariant under the  $(k, \psi)$ -action by all  $\gamma \in \Gamma_0(N)$  and  $\tau = u + iv \in \mathbb{H}$ .
2. It is annihilated by the weight  $k$  hyperbolic Laplacian,

$$\Delta_k f := \left[ -v^2 \left( \frac{\partial^2}{\partial u^2} + \frac{\partial^2}{\partial v^2} \right) + ikv \left( \frac{\partial}{\partial u} + i \frac{\partial}{\partial v} \right) \right] f \equiv 0. \quad (2.3.6)$$

3. There is a polynomial  $P_f(q^{-1})$  such that  $f(\tau) - P_f(e^{-2\pi i\tau}) = O(e^{-cv})$  for some  $c > 0$  as  $v \rightarrow \infty$ . Analogous growth conditions are required at all cusps of  $\Gamma_0(N)$ .

We denote the space of harmonic Maaß forms of weight  $k$ , level  $N$ , and multiplier  $\psi$  by  $H_k(\Gamma_0(N), \psi)$ , and we omit the multiplier if it is trivial.

Bruinier and Funke first introduced harmonic Maaß forms in [Bruinier and Funke 2004]. We are going to need the following two results from their

paper.

**Lemma 2.3.2.** [Bruinier and Funke 2004; equations (3.2a) and (3.2b)] Let  $f \in H_k(\Gamma_0(N), \psi)$  be a harmonic Maaß form of weight  $k \neq 1$  such that  $\psi\left(\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\right) = 1$ . Then there is a canonical splitting

$$f(\tau) = f^+(\tau) + f^-(\tau), \quad (2.3.7)$$

where for some  $m_0 \in \mathbb{Z}$  we have the holomorphic part,

$$f^+(\tau) := \sum_{n=m_0}^{\infty} c_f^+(n) q^n, \quad (2.3.8)$$

and the non-holomorphic part,

$$f^-(\tau) := \sum_{n=1}^{\infty} \overline{c_f^-(n)} n^{k-1} \Gamma(1-k; 4\pi n v) q^{-n}. \quad (2.3.9)$$

Here  $\Gamma(\alpha; x)$  denotes the upper incomplete Gamma function.

We call the holomorphic part of a harmonic Maaß form a *mock modular form*. Let  $M_k^!(\Gamma_0(N), \psi)$  denote the space of weakly holomorphic modular forms of weight  $k$ , level  $N$ , and multiplier system  $\psi$ . Then we have the following proposition.

**Proposition 2.3.3.** (See [Bruinier and Funke 2004; Proposition 3.2]) The operator

$$\xi_k : H_k(\Gamma_0(N), \psi) \rightarrow M_{2-k}((\Gamma_0(N), \overline{\psi})), f \mapsto \xi_k f := 2iv^k \frac{\overline{\partial f}}{\partial \overline{\tau}} \quad (2.3.10)$$

is well-defined and surjective with kernel  $M_k^!(\Gamma_0(N), \psi)$ . Moreover, we have that

$$(\xi_k f)(\tau) = -(4\pi)^{1-k} \sum_{n=1}^{\infty} c_f^-(n) q^n \quad (2.3.11)$$

and we call  $\xi_k f$  the *shadow* of (the holomorphic part of)  $f$ .

Thus, in particular, a mock modular form is a weakly holomorphic modular form if it has a vanishing shadow. We will construct the desired space of weakly holomorphic forms by first constructing mock modular forms of the appropriate level and weight and then showing that they have vanishing shadows.

## 2.4 Rademacher sums

To construct the relevant mock modular forms for the proof of Theorem 3.1.2, we need to recall some facts about Rademacher sums and Rademacher series. See [Cheng and Duncan 2012; 2014; Duncan and Frenkel 2011] for more details.

Let  $\Gamma_\infty := \{\pm \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix} : n \in \mathbb{Z}\}$  denote the stabilizer of  $\infty$  in  $\Gamma_0(N)$ . Then one can define the Rademacher sum of weight  $k \geq 1$ , level  $N$ , multiplier system  $\psi$  and index  $\mu$ , by

$$R_{k,N,\psi}^{[\mu]}(\tau) := \lim_{K \rightarrow \infty} \sum_{\gamma \in \Gamma_\infty \backslash \Gamma_{K,K^2}(N)} q^\mu |k, \psi \gamma \quad (2.4.1)$$

where

$$\Gamma_{K,K^2}(N) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N) : |c| < K \text{ and } |d| < K^2 \right\} \quad (2.4.2)$$

and  $\mu \in \mathbb{Z} + \frac{-i}{2\pi} \log(\psi \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix})$ .

When convergent, Rademacher sums define mock modular forms of level  $N$ , weight  $k$ , and multiplier system  $\psi$ . We will use the following important facts from the theory of Rademacher sums, which we condense in one lemma.

**Lemma 2.4.1.** (See [Duncan et al. to appear; Theorem 2.5], for example.) Let  $\mu \leq 0$ . Assuming locally uniform convergence, the Rademacher sum  $R_{k,4N,\psi}^{[\mu]}(\tau)$  for  $k \geq 1$  defines a mock modular form of weight  $k \in \mathbb{Z} + \frac{1}{2}$ , for  $\Gamma_0(4N)$  and multiplier  $\psi$  whose shadow is given by a constant multiple of the Rademacher sum  $R_{2-k,4N,\bar{\psi}}^{[-\mu]}(\tau)$ . The completion of  $R_{k,4N,\psi}^{[\mu]}(\tau)$  to a harmonic Maaß form has a pole of order  $\mu$  at the cusp  $\infty$  and vanishes at all other cusps.

In this paper, we will be looking at Rademacher sums of weight  $\frac{3}{2}$  for  $\Gamma_0(4N)$  with multiplier  $\psi_{4N,v,h}$  and index  $\mu < 0$ . In this particular case, it has been proven in [Cheng and Duncan 2012] that the sums converge locally uniformly and define holomorphic functions on  $\mathbb{H}$ .

## 2.5 Kohnen Plus Space Condition

One of the properties that we want our candidate McKay–Thompson series to satisfy is to lie in Kohnen’s plus space. Let  $S_k(\Gamma_0(N))$  denote the space of weight  $k$  cusp forms for  $\Gamma_0(N)$ . Kohnen’s plus space was first introduced by Kohnen (see [Kohnen 1980; 1982; 1985]) as the subspace of  $S_{k+\frac{1}{2}}(\Gamma_0(4N))$  which consists of all forms whose Fourier coefficients are supported on exponents  $n$  with  $n \equiv 0, (-1)^k \pmod{4}$ . We extend this idea to all modular forms and harmonic Maaß forms as follows: We say that a function  $f$  in  $H_{k+\frac{1}{2}}(\Gamma_0(4N))$  (resp. in  $M_{k+\frac{1}{2}}^!(\Gamma_0(4N))$ ) *satisfies the Kohnen plus space condition* if the Fourier coefficients of  $f$  are supported on exponents  $n$  with

$n \equiv 0, (-1)^k \pmod{4}$ . We denote the space of such forms  $H_{k+\frac{1}{2}}^+(\Gamma_0(4N))$  (resp.  $M_{k+\frac{1}{2}}^{+,!}(\Gamma_0(4N))$ ).

For odd  $N$ , there is a natural projection operator  $|_{\text{pr}} : S_{k+\frac{1}{2}}(\Gamma_0(4N)) \rightarrow S_{k+\frac{1}{2}}^+(\Gamma_0(4N))$  given in terms of slash operators which extends to spaces of weakly holomorphic modular forms and harmonic Maaß forms. Let  $f \in M_{k+\frac{1}{2}}^{+,!}(\Gamma_0(4N))$ , where  $N$  is an odd integer. Then the projection operator acts on  $f$  in the following way [Kohnen 1985] (see also, [Griffin and Mertens 2016])

$$(f|_{\text{pr}})(\tau) = (-1)^{\lfloor \frac{k+1}{2} \rfloor} \frac{1}{3\sqrt{2}} \sum_{v=-1}^2 (f| \left( \begin{smallmatrix} 4(1+Nv) & 1 \\ 8Nv & 4 \end{smallmatrix} \right))(\tau) + \frac{1}{3}f(\tau). \quad (2.5.1)$$

The action of this projection operator on principal parts of harmonic Maaß forms is described in the following lemma (see [Griffin and Mertens 2016; Lemma 2.9, 2.10] and [Duncan et al. to appear; Lemma 2.6]).

**Lemma 2.5.1.** Let  $N$  be odd and  $f \in H_{k+\frac{1}{2}}(\Gamma_0(4N))$  for some  $k \in \mathbb{N}$ , such that

$$f^+(\tau) = q^{-m} + \sum_{n=0}^{\infty} a_n q^n \quad (2.5.2)$$

for some  $m > 0$  with  $-m \equiv 0, (-1)^k \pmod{4}$ , and suppose that  $f$  has a non-vanishing principal part only at the cusp  $\infty$  and is bounded at the other cusps of  $\Gamma_0(4N)$ . Then the projection  $f|_{\text{pr}}$  of  $f$  to the plus space has a pole of order  $m$  at  $\infty$ , a pole of order  $\frac{m}{4}$  either at the cusp  $\frac{1}{N}$  if  $m \equiv 0 \pmod{4}$  or at the cusp  $\frac{1}{2N}$  if  $-m \equiv (-1)^k \pmod{4}$  and is bounded at all other cusps.

For even  $N$ , we have the following lemma for the Rademacher sums that we consider in this paper.



**Lemma 2.5.2.** For even  $N$ , the Rademacher sum  $R_{\frac{3}{2}, 4N, \psi}^{[-5]}(\tau)$  satisfies the Kohnen plus space condition.

*Proof.* This is an immediate consequence of Lemma 2.10 of [Griffin and Mertens 2016].  $\square$

For even  $N$ , we define the projection operator  $|_{\text{pr}}$  to be the following sieving operator: Let  $f(\tau) = \sum_{n=n_0}^{\infty} c(n)q^n$  be modular of weight  $k + \frac{1}{2}$  where  $k \in \mathbb{N}$ , and level  $4N$ , where  $N$  is even, then we define

$$f|_{\text{pr}} = \sum_{\substack{n=n_0 \\ n \equiv 0, (-1)^k \pmod{4}}}^{\infty} c(n)q^n. \quad (2.5.3)$$

By Lemma 2.5.2, we have  $R_{\frac{3}{2}, 4N, \psi}^{[-5]}(\tau)|_{\text{pr}} = R_{\frac{3}{2}, 4N, \psi}^{[-5]}(\tau)$  if  $N$  is even.

## 2.6 Eta-Quotients

For the rational conjugacy classes  $[g] \in \{21A, 30AB\}$ , it is convenient to use eta-quotients instead of Rademacher sums.

Recall that an eta-quotient is defined to be a function of the form

$$f(\tau) = \prod_{\delta|N} \eta(\delta\tau)^{r_\delta}, \quad (2.6.1)$$

where  $r_\delta \in \mathbb{Z}$  and  $\eta(\tau) := q^{\frac{1}{24}} \prod_{n=1}^{\infty} (1 - q^n)$  is the Dedekind eta function. As a consequence of the product definition for  $\eta(\tau)$ , any eta-quotient is non-vanishing on  $\mathbb{H}$ . We will need the following lemma from [Rouse and Webb 2015] to construct eta-quotients which vanish only at a specific cusp.

**Lemma 2.6.1.** ([Rouse and Webb 2015; Lemma 14]) Let  $N \in \mathbb{N}$ , then for each divisor  $d$  of  $N$ , there exists  $k_d \in \mathbb{N}$  and a corresponding eta-quotient  $E_{d,N}(\tau) \in M_{k_d}(\Gamma_0(N))$  such that  $E_{d,N}$  vanishes only at the cusp  $\frac{c}{d}$ .

The proof of Lemma 2.6.1 is constructive, and MAGMA [Bosma et al. 1997] code implementing it can be found at <http://users.wfu.edu/rouseja/eta/>. We write  $E_N(\tau)$  for the holomorphic eta-quotient  $E_{N,N}(\tau)$  that is produced by this code. In Section 3.1.2 we will use the explicit construction of eta-quotients  $E_N(\tau)$  to construct weakly holomorphic modular forms  $f_g^{wh}(\tau)$  for  $[g] \in \{21A, 30AB\}$ .

For completeness, we recall here the modular transformation law of the Dedekind eta-function,

$$\eta\left(\frac{az+b}{cz+d}\right) = e\left(\frac{a+d}{24c} + \frac{s(-d,c)}{2} + \frac{3}{8}\right) (cz+d)^{\frac{1}{2}} \eta(z). \quad (2.6.2)$$

Here,  $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$  and  $s(d,c)$  is the Dedekind sum,

$$s(d,c) = \sum_{r=1}^{c-1} \frac{r}{c} \left( \frac{dr}{c} - \left\lfloor \frac{dr}{c} \right\rfloor - \frac{1}{2} \right). \quad (2.6.3)$$

Using eq. (2.6.2), we can explicitly compute Fourier coefficients of  $(f|_{k,\psi\gamma})(\tau)$  whenever  $f$  is an eta-quotient,  $\gamma \in \Gamma_0(N)$  and  $\psi$  is a multiplier system of weight  $k$  and level  $N$ .

## Chapter 3

# Moonshine for Thompson's sporadic simple group

In this chapter, we will state and prove the first main result of this thesis. We will characterize all infinite-dimensional graded modules for the Thompson group whose graded traces are certain weight  $3/2$  weakly holomorphic modular forms satisfying special properties. This characterization serves as an example of moonshine for the Thompson group.

### 3.1 McKay–Thompson Series

To state our main theorem of this section, we associate to each rational conjugacy class  $[g]$  of the Thompson group  $Th$ , the following data:

1. Integers  $v_g$  and  $h_g$  as specified in Table A.1.1. We use these to define the character  $\psi_g := \psi_{4|g|,v_g,h_g}$  (cf. Equation (2.3.2)), where  $|g|$  denotes the order of  $g$  in  $Th$ .
2. The space of cusp forms  $S_g := S_{\frac{3}{2}}^+(\Gamma_0(4|g|), \psi_g)$  of weight  $\frac{3}{2}$  in the plus-space which transform under  $\Gamma_0(4|g|)$  with character  $\psi_g$ . We define

$d_g$  to be the dimension of this space and let  $f_g$  be the  $d_g$ -tuple  $f_g := (f_g^{(1)}, \dots, f_g^{(d_g)})$  where  $f_g^{(i)}$  is the  $i^{\text{th}}$  element of the canonical basis of  $S_g$ . A list of  $f_g^{(i)}$ 's can be found in Table A.3.1.

3. For each  $f_g^{(i)}$  defined as above, we let  $n_g^{(i)}$  and  $m_g^{(i)}$  be the integers listed in Table A.3.2. We define  $n_g$  to be the  $d_g$ -tuple  $n_g := (n_g^{(1)}, \dots, n_g^{(d_g)})$ , and define  $M_g$  to be the  $d_g \times d_g$  diagonal matrix with entries given by  $(m_g^{(1)}, \dots, m_g^{(d_g)})$ .
4. Finally, to each rational conjugacy class  $[g]$  of  $Th$ , we associate integers  $a_g(n)$  for each  $n > 0$ , where  $n \equiv 0, 3 \pmod{4}$ . For  $g \notin \{21A, 30AB\}$ , the integers  $a_g(n)$  are given by Equation (3.1.8). For  $g = 21A$ , we define  $a_{21A}(n)$  to be the Fourier coefficients of the unique modular form  $f_{21A}^{wh}$  in  $M_{\frac{3}{2}}^{+,1}(\Gamma_0(84), \psi_{21})$  whose Fourier expansion begins  $6q^{-5} - 2q^4 + 4q^7 - 8q^8 + O(q^{11})$ . Similarly, we define  $a_{30AB}(n)$  by setting

$$f_{30AB}^{wh} = 6q^{-5} + \sum_{\substack{n>0 \\ n \equiv 0,3 \pmod{4}}}^{\infty} a_{30AB}(n)q^n \quad (3.1.1)$$

where  $f_{30AB}^{wh} \in M_{\frac{3}{2}}^{+,1}(\Gamma_0(120), \psi_{30})$  is the unique form with the Fourier expansion  $6q^{-5} + 3q^3 + 3q^8 - 3q^{11} + O(q^{12})$ . We will prove the existence of  $f_{21A}^{wh}$  and  $f_{30AB}^{wh}$  in Lemma 3.1.4.

For each rational conjugacy class  $[g]$  we define  $\Lambda_g$  to be the set of all  $d_g$ -tuples  $\lambda_g := (\lambda_g^{(1)}, \dots, \lambda_g^{(d_g)}) \in \mathbb{Z}^{d_g}$  and obtain the following proposition.

**Proposition 3.1.1.** Fix a rational conjugacy class  $[g]$  in  $Th$ . Then for each  $\lambda_g \in \Lambda_g$ , the function

$$\mathcal{F}_g^\lambda(\tau) := 6q^{-5} + \sum_{0 < n} a_g(n)q^n + (\lambda_g M_g + n_g) \cdot f_g(\tau) \quad (3.1.2)$$

is a weakly holomorphic modular form that satisfies the following properties.

- (a) It lies in  $M_{\frac{3}{2}}^{+,!}(\Gamma_0(4|g|), \psi_g)$ , i.e.,  $\mathcal{F}_g^\lambda(\tau)$  has weight  $\frac{3}{2}$ , level  $4|g|$  with character  $\psi_g$ , and satisfies the Kohnen plus space condition.
- (b) It has a pole of order 5 at the cusp  $\infty$ , a pole of order  $\frac{5}{4}$  at the cusp  $\frac{1}{2|g|}$  if  $|g|$  is odd, and vanishes at all other cusps.
- (c) The Fourier coefficients of  $\mathcal{F}_g^\lambda(\tau)$  are integers.

Note that our multiplier system  $\psi_g$  is conjugate to the one used by [Griffin and Mertens 2016] and [Harvey and Rayhaun 2015]. This is necessary for Theorem 3.1.2 to be true, and is not unexpected since our functions are weight  $\frac{3}{2}$  as opposed to the weight  $\frac{1}{2}$  forms in [Griffin and Mertens 2016] and [Harvey and Rayhaun 2015].

We will prove Proposition 3.1.1 by constructing specific weakly holomorphic forms  $f_g^{wh}(\tau) = 6q^{-5} + \sum_{0 < n} a_g(n) \in M_{\frac{3}{2}}^{+,!}(\Gamma_0(4|g|), \psi_g)$  using the theory of Rademacher sums and eta-quotients. We will see from the explicit construction that each  $f_g^{wh}(\tau)$  already satisfies properties (a) – (c) listed in Proposition 3.1.1 even without the addition of any cusp forms. We need to add the cusp forms  $f_g(\tau)$  not for Proposition 3.1.1, but instead for the following theorem.

**Theorem 3.1.2.** Assume the above notation and let  $\Lambda$  be the set of functions  $\{\lambda : g \mapsto \lambda_g \in \Lambda_g\}$ . Then, for each  $\lambda \in \Lambda$ , there exists an infinite-dimensional graded virtual  $Th$ -module

$$W^\lambda := \bigoplus_{\substack{n > 0 \\ n \equiv 0, 3 \pmod{4}}} W_n^\lambda \quad (3.1.3)$$

such that for each rational conjugacy class  $[g]$  of  $Th$ , the corresponding McKay–Thompson series,

$$6q^{-5} + \sum_{\substack{n>0 \\ n \equiv 0,3 \pmod{4}}} \text{tr}(g|W_n^\lambda) q^n \quad (3.1.4)$$

is the specific weakly holomorphic modular form  $\mathcal{F}_g^\lambda(\tau) \in M_{\frac{3}{2}}^{+,!}(\Gamma_0(4|g|), \psi_g)$  described in Proposition 3.1.1. Furthermore, for every infinite-dimensional graded virtual  $Th$ -module  $W = \bigoplus_{n>0} W_n$  for which the McKay–Thompson series,

$$\mathcal{F}_g(\tau) = 6q^{-5} + \sum_{\substack{n>0 \\ n \equiv 0,3 \pmod{4}}} \text{tr}(g|W_n) q^n \quad (3.1.5)$$

satisfies the properties listed in Proposition 3.1.1, there exists a  $\lambda \in \Lambda$  for which  $W = W^\lambda$  described as above.

We will now construct the relevant spaces of modular forms required for the proof of Proposition 3.1.1.

### 3.1.1 Using Rademacher Sums

For each rational conjugacy class  $[g] \notin \{21A, 30AB\}$ , consider the function

$$f_g^{wh}(\tau) = 6R_{\frac{3}{2}, 4|g|, \psi_g}^{[-5],+}(\tau) := 6 \left( R_{\frac{3}{2}, 4|g|, \psi_g}^{[-5]} | \text{pr} \right) (\tau) \quad (3.1.6)$$

where  $| \text{pr}$  is the projection onto the Kohnen plus-space (cf. Section 2.5). Then by Lemma 2.4.1, Lemma 2.5.1 and Lemma 2.5.2, each  $f_g^{wh}(\tau)$  is a mock modular form in the plus-space of weight  $\frac{3}{2}$ , level  $|g|$ , and multiplier  $\psi_g$ , has a pole of order 5 at the cusp at infinity, a pole of order  $\frac{5}{4}$  at the cusp  $\frac{1}{2|g|}$  if  $|g|$  is odd (forced by the projection to the plus-space, cf. Lemma 2.5.1), and vanishes at all other cusps. The only thing left to prove here is that

each  $f_g^{wh}(\tau)$  is, in fact, weakly holomorphic (i.e., has vanishing shadow, cf. Proposition 2.3.3).

**Lemma 3.1.3.** For each rational conjugacy class  $[g] \notin \{21A, 30AB\}$  of the Thompson group, the function  $f_g^{wh}(\tau)$  (defined in eq. (3.1.6)) is in fact a weakly holomorphic modular form, and has Fourier expansion given by

$$f_g^{wh}(\tau) = 6q^{-5} + \sum_{\substack{n>0 \\ n \equiv 0,3 \pmod{4}}}^{\infty} a_g(n)q^n, \quad (3.1.7)$$

where for  $N = |g|$ , we have

$$a_g(n) := \frac{-3\pi}{N} \left(\frac{-n}{5}\right)^{\frac{1}{4}} \sum_{c=1}^{\infty} \frac{1 + \delta_{\text{odd}}(Nc)}{c} K_{\frac{3}{2},\psi}(-5, n, 4Nc) I_{\frac{1}{2}}\left(\frac{\pi\sqrt{5n}}{Nc}\right). \quad (3.1.8)$$

Here,  $I_{\frac{1}{2}}$  is the modified Bessel function of the first kind of order  $\frac{1}{2}$ ,

$$\delta_{\text{odd}}(k) := \begin{cases} 1 & k \text{ odd,} \\ 0 & k \text{ even,} \end{cases} \quad (3.1.9)$$

and  $K_{\frac{3}{2},\psi}$  is the twisted Kloosterman sum

$$K_{\frac{3}{2},\psi}(m, n, c) := \sum_{d \pmod{c}} \psi\left(\left(\frac{*}{c} \frac{*}{d}\right)\right) \left(\frac{c}{d}\right) \varepsilon_d^3 e\left(\frac{m\bar{d} + nd}{c}\right). \quad (3.1.10)$$

The sum here runs over primitive residue classes modulo  $c$ , and  $\bar{d}$  denotes the multiplicative inverse of  $d$  modulo  $c$ .

*Proof.* We have already established that each  $f_g^{wh}(\tau)$  is a mock modular form of weight  $\frac{3}{2}$  for the group  $\Gamma_0(4|g|, \psi_g)$  and by Proposition 2.3.3,

$$\xi_{\frac{3}{2}}(f_g^{wh}) \in S_{\frac{1}{2}}(\Gamma_0(4|g|), \overline{\psi_g}) \subset S_{\frac{1}{2}}(\Gamma_0(4|g|h_g)), \quad (3.1.11)$$

where  $S_{\frac{1}{2}}(\Gamma_0(4|g|h_g))$  is the (usually) larger space whose dimension can be computed using the built-in functions for spaces of modular forms in for example MAGMA [Bosma et al. 1997]. In our case, it turns out to be zero-dimensional for all  $g \in Th$ . Thus, each  $f_g^{wh}(\tau)$  is in fact weakly holomorphic. Computing the coefficients of Rademacher sums in terms of Kloosterman sums and Bessel functions is a standard computation, see for example [Cheng and Duncan 2014] (or Proposition 2.7 in [Duncan et al. to appear].)  $\square$

### 3.1.2 Using Eta-Quotients.

For  $[g] \in \{21A, 30AB\}$ , we will use the eta-quotients  $E_{|g|}(\tau)$  in Lemma 2.6.1 to compute spaces of weakly holomorphic forms of the desired weight and level. The main result of this section is the following lemma.

**Lemma 3.1.4.** For  $m \in \{21, 30\}$ , there exists a weakly holomorphic modular form with integer Fourier coefficients in  $M_{\frac{1}{2}}^{+,+}(\Gamma_0(4m, \psi_{mA}))$  which has a pole of order 5 at  $\infty$ , a pole of order  $\frac{5}{4}$  at the cusp  $\frac{1}{42}$  if  $m = 21$ , and vanishes at all other cusps.

Note that if Lemma 3.1.4 is true, such a form satisfies all properties of Proposition 3.1.1, and we can thus define  $f_{21A}^{wh}(\tau)$  (resp.  $f_{30AB}^{wh}(\tau)$ ) to be the unique such form with Fourier expansion  $6q^{-5} - 2q^4 + 4q^7 - 8q^8 + O(q^{11})$  (resp.  $6q^{-5} + 3q^3 + 3q^8 - 3q^{11} + O(q^{12})$ )



*Remark.* We could have written an analogous statement for each rational conjugacy class  $[g]$  in the Thompson group and forgone the discussion about Rademacher sums completely. This would not affect the proof of Theorem 3.1.2 at all. However, we need an expression for  $f_g^{wh}(\tau)$  for certain classes  $[g] \notin \{21A, 30B\}$  in terms of Rademacher sums for the application to elliptic curves. In particular, such expressions for  $g \in \{14A, 19A\}$  play key roles in the proofs of Theorems 4.1.1 and 4.1.2.

*Proof of Lemma 3.1.4.* By Lemma 2.6.1, for each  $N > 0$  we can construct an eta-quotient  $E_N(\tau)$  that vanishes only at the cusp  $\infty$ . We use  $N = 21$  and  $N = 30$  to get eta-quotients:

$$E_{21}(\tau) = \frac{\eta(\tau)\eta(21\tau)^{21}}{\eta(3\tau)^3\eta(7\tau)^7} \quad \text{and} \quad E_{30}(\tau) = \frac{\eta(2\tau)^2\eta(3\tau)^3\eta(5\tau)^5\eta(30\tau)^{30}}{\eta(\tau)\eta(6\tau)^6\eta(10\tau)^{10}\eta(15\tau)^{15}} \quad (3.1.12)$$

of weight 6 and 4, respectively. Now, consider the cusp form space  $S_{\frac{15}{2}}(\Gamma_0(84), \psi_{21A})$ , and suppose for now that we can compute a basis for this space explicitly. If so, we can divide each element of the basis by  $E_{21}(4\tau)$  to get a generating set  $B_{21}$  of forms in  $M_{\frac{3}{2}}^1(\Gamma_0(84), \psi_{21A})$  whose only (possible) pole is at the cusp  $\infty$ . Then, we apply the projection operator to each element of  $B_{21}$  to get a generating set of forms in  $M_{\frac{3}{2}}^{1,+}(\Gamma_0(84), \psi_{21A})$  that are holomorphic away from the cusps at  $\infty$  and  $\frac{1}{42}$ . This generating set turns out to be non-empty.

We can now construct  $f_{21A}^{wh}(\tau)$  as a suitable linear combination of elements of this space determined completely by its Fourier expansion  $6q^{-5} - 2q^4 + 4q^7 - 8q^8 + O(q^{11})$ . The same argument works for  $f_{30AB}^{wh}(\tau)$  if we start with  $S_{\frac{11}{2}}(\Gamma_0(120), \psi_{30AB})$  instead. Here again, the set of forms in  $M_{\frac{3}{2}}^{1,+}(\Gamma_0(120), \psi_{30AB})$  which are holomorphic away from the cusp at  $\infty$  turns out to be non-empty.

## 3.2 Proof of Proposition 3.1.1

We now describe how to compute the bases for  $S_{\frac{15}{2}}(\Gamma_0(84), \psi_{21A})$  and  $S_{\frac{11}{2}}(\Gamma_0(120), \psi_{30AB})$  in some detail. We will essentially follow the method described in Proposition 3.1 of [Griffin and Mertens 2016]. Let  $(m, k) \in \{(21, \frac{15}{2}), (30, \frac{11}{2})\}$ . Let  $f \in S_k(\Gamma_0(4m), \psi_{mA})$  and let

$$\vartheta(\tau) := \sum_{n \in \mathbb{Z}} q^{n^2} \in M_{\frac{1}{2}}^{+,1}(\Gamma_0(4)). \quad (3.2.1)$$

Then  $f\vartheta$  lies in  $M_{k+\frac{1}{2}}(\Gamma_0(4m), \psi_{mA}) \subset M_{k+\frac{1}{2}}(\Gamma_0(4mh_g))$ . Using programs (available at <http://users.wfu.edu/rouseja/eta/>) written by Rouse and Webb one can verify that the space  $M_{k+\frac{1}{2}}(\Gamma_0(12m))$  is generated by eta-quotients ( $h_g = 3$ , for both values of  $m$ ). Since we can explicitly compute Fourier expansions of  $(g|_k\gamma)(\tau)$  for any eta-quotient  $g(\tau)$  and  $\gamma \in \mathrm{SL}_2(\mathbb{Z})$  (cf. Section 2.6), we can thus compute a basis for  $M_{k+\frac{1}{2}}(\Gamma_0(4m), \psi_{mA})$  and hence for  $S_k(\Gamma_0(4m), \psi_{mA})$ . Alternatively, we can also compute a basis for  $M_{k+\frac{1}{2}}(\Gamma_0(4m), \psi_{mA})$  using in-built functions in PARI/GP [PAR 2019]. This concludes the proof of Lemma 3.1.4.

□

We now have an explicit description of  $f_g^{wh}(\tau)$  for each rational conjugacy class  $[g]$ . The next steps in the proof are showing that each  $f_g^{wh}(\tau)$  (and thus each  $\mathcal{F}_g^\lambda(\tau)$ ) satisfies all properties listed in Proposition 3.1.1 (we do this in Section 3.2.1), and that the only cusp forms we can add for Theorem 3.1.2 to be true are appropriate integer multiples of the elements of  $S_g$  for each  $g \in Th$ , respectively (cf. Proposition 3.1.1). The latter will follow from our work in Section 3.3.

### 3.2.1 Cusp forms

We begin by noting that the weakly holomorphic forms  $f_g^{wh}(\tau)$  described in Sections 3.1.1 and 3.1.2 satisfy the properties (a) – (c) listed in Proposition 3.1.1. Also, said properties uniquely determine a weakly holomorphic form up to cusp forms [Duncan et al. to appear; Lemma 2.4]. Thus to specify the functions  $\mathcal{F}_g^\lambda(\tau)$  completely we have to compute the cusp form spaces  $S_{\frac{3}{2}}^+(\Gamma_0(4|g|), \psi_g)$  for each  $[g]$  in  $Th$ .

**Lemma 3.2.1.** For each rational conjugacy class  $[g]$  of  $Th$ , the corresponding cusp form space  $S_g$  is spanned by the cusp forms given in Table A.3.1.

*Proof.* We use the same method as in the proof of Lemma 3.1.4 to compute the cusp form spaces. Let  $f \in S_{\frac{3}{2}}^+(4|g|, \psi_g)$  be any cusp form. Then  $f\vartheta$  lies in  $M_2(4|g|, \psi_g) \subset M_2(4|g|h_g)$ , where the larger space is spanned by eta quotients for each  $[g]$ . This can be verified using MAGMA code written by Rouse and Webb [Rouse and Webb 2015]. We can then use the modular properties of the eta-quotients and the projection onto the plus-space to determine a basis for  $M_2(4|g|, \psi_g)$  and hence  $S_g = S_{\frac{3}{2}}^+(4|g|, \psi_g)$ . The space  $S_g$  turns out to be trivial for every

$$[g] \notin \{12D, 14A, 18B, 19A, 20A, 21A, 24AB, 24CD, 28A, 30AB, 31AB, 39AB\}. \quad (3.2.2)$$

The Fourier coefficients given in Table A.3.1 are enough to determine each  $f_g^{(i)}(\tau)$  completely for all other rational conjugacy classes  $[g]$ .  $\square$

### 3.2.2 Integer Coefficients.

The last thing we need to check in order to prove Proposition 3.1.1 is that the functions

$$\mathcal{F}_g^\lambda(\tau) = f_g^{wh}(\tau) + (\lambda_g M_g + n_g) \cdot f_g(\tau) \quad (3.2.3)$$

constructed in the preceding section have integer coefficients. We will use Sturm's theorem [Sturm 1987] for this. Note that each of these functions lies in  $M_{\frac{3}{2}}^{+,!}(\Gamma_0(4|g|), \psi_g) \subset M_{\frac{3}{2}}^{+,!}(\Gamma_0(4|g|h_g))$ , thus if  $\nu(\tau) = q^5 + O(q^8)$  is a cusp form with integer coefficients in  $S_{2k-\frac{3}{2}}^+(\Gamma_0(4|g|h_g))$ , then,  $\mathcal{F}_g^\lambda(\tau)\nu(\tau)$  lies in  $M_{2k}(\Gamma_0(4|g|h_g))$ , so we can apply Sturm's theorem to it. Thus,  $\mathcal{F}_g^\lambda(\tau)$  has integer coefficients if the first  $\frac{k}{6}[\mathrm{SL}_2(\mathbb{Z}) : \Gamma_0(4|g|h_g)]$  coefficients of  $\mathcal{F}_g^\lambda(\tau)\nu(\tau)$  are integers. The largest bound we have to check is less than 1200. The author used PARI/GP [PAR 2019] to do this computation.

This concludes the proof of Proposition 3.1.1.

### 3.3 Proof of Theorem 3.1.2

To prove Theorem 3.1.2, we have to show that the  $\mathcal{F}_g^\lambda(\tau)$ 's we described in Section 3.1 are indeed the McKay–Thompson series of a virtual module of the Thompson group.

This is equivalent to proving that there exist integers  $m_1^\lambda(n), \dots, m_{39}^\lambda(n)$  such that if  $\mathcal{F}_g^\lambda(\tau) = 6q^{-5} + \sum_{n \geq 3} \alpha_g^\lambda(n)q^n$ , then for each  $n \geq 3$  the Fourier coefficient  $\alpha_g^\lambda(n)$  can be written in the form,

$$\alpha_g^\lambda(n) = \sum_{j=1}^{39} m_j^\lambda(n) \chi_j(g), \quad (3.3.1)$$

where  $\chi_1, \dots, \chi_{39}$  are the irreducible rational characters of  $Th$  (See Section 2.2 for a definition of rational character). We say that the function  $\omega_n^\lambda : Th \rightarrow \mathbb{C}$ , defined by  $g \mapsto \alpha_g^\lambda(n)$ , is a *virtual rational character* of  $Th$  if the above condition is satisfied. Thus, the goal of this section is to prove that  $\omega_n^\lambda$  is a virtual rational character of  $Th$  for every  $n \geq 3$  and choice of  $\lambda \in \Lambda$ .

As explained in [Griffin and Mertens 2016], this is computationally infeasible to prove directly using only Sturm bounds [Sturm 1987]. However, it can be reduced to a finite computation using a variant of Thompson’s reformulation (cf. [Smith 1985]) of Brauer’s characterization of generalized characters. (For another example of a similar computation, see [Gannon 2016].) To state the result, we first have to define a few things.

### 3.3.1 Generalized Characters

For the rest of this section, let  $G$  be a finite group and  $p$  a fixed prime dividing the order of  $G$ . Let  $\mathcal{C}_G$  denote the set of all rational conjugacy classes of  $G$ . We call  $[g] \in \mathcal{C}_G$  *p-regular* if the order of  $g$  is coprime to  $p$ . Let  $K_p$  denote the set of all  $p$ -regular classes in  $G$  whose centralizer in  $G$  has order divisible by  $p$ . For a fixed  $[g] \in K_p$ , we will let  $\alpha$  denote the highest power of  $p$  dividing the order of the centralizer of  $g$  in  $G$ . (This  $\alpha$  should not be confused with the  $\alpha_g^\lambda(n)$  of eq. (3.3.1).)

Let  $h \in G$  be any element in  $G$  and let  $|h| = n = p^k m$  where  $k \geq 0$  and  $(p, m) = 1$ . Then we can write  $h$  as a product  $h = ab$ , where  $a$  and  $b$  commute and  $a$  has order  $m$ . (Both  $a$  and  $b$  can be expressed as powers of  $h$ .) We call  $a$  the *p-regular part* of  $h$ . We note here that if  $h' \in G$  is in the same rational conjugacy class as  $h$ , then their corresponding  $p$ -regular parts  $a'$  and  $a$  are also in the same rational conjugacy class, i.e if  $h' \in [h]$  then  $a' \in [a]$ . This allows us to make the following definition.

**Definition 3.3.1.** For a fixed  $p$  and  $[g] \in K_p$  as above, the *p-regular section*  $R_{p,g}$  of  $[g]$  is the set of rational conjugacy classes  $[h] \in \mathcal{C}_G$  such that the  $p$ -regular part of  $h$  lies in  $[g]$ .

For  $G = Th$ , and for each prime  $p$  dividing  $|G|$ , Table A.2.1 lists the rational conjugacy classes  $[g]$  in  $K_p$ , along with their  $p$ -regular section  $R_{p,g}$  and the

highest power  $\alpha$  such that  $p^\alpha$  divides the order of the centralizer  $C_G(g)$ .

For a fixed group  $G$ , and prime  $p$  dividing  $|G|$ , fix a rational conjugacy class  $[g] \in K_p$ . Let  $\mathbb{Z}_{(p)} = \{\frac{a}{b} : a, b \in \mathbb{Z}, p \nmid b\}$  denote the localization of  $\mathbb{Z}$  at the prime ideal  $(p)$ , and let  $I := p^\alpha \mathbb{Z}_{(p)}$ . We define  $m := |R_{p,g}|$ , and let  $M_{p,g}$  denote the set of all  $m$ -tuples  $(l_1, l_2, \dots, l_m) \in \mathbb{Z}_{(p)}^{\oplus m}$  such that

$$\sum_{i=1}^m l_i \chi([h]_i) \equiv 0 \pmod{I^m} \quad (3.3.2)$$

for all irreducible rational characters  $\chi$  of  $G$  and all rational conjugacy classes  $[h]_i$  in  $R_{p,g}$ .

We are now ready to state the following important lemma.

**Lemma 3.3.2.** Assuming the above notation, an integer-valued class function  $c : G \rightarrow \mathbb{Z}$  of  $G$  is a virtual rational character of  $G$  if and only if for all primes  $p$  and rational conjugacy classes  $[g]$ ,

$$\sum_{i=1}^m l_i c([h]_i) \equiv 0 \pmod{I^m} \quad (3.3.3)$$

for all  $(l_1, l_2, \dots, l_m) \in M_{p,g}$ .

*Proof.* This is a direct application of [Smith 1985; Theorem 1.1].  $\square$

Lemma 3.3.2 reduces the problem of checking whether the multiplicities are integral to a  $p$ -local computation. We illustrate this with an example.

**Example 1.** Let  $p = 19$ . Then,  $K_{19} = \{1A\}$  and  $R_{19,1A} = \{1A, 19A\}$ . We have  $\alpha = 1$ , and  $M_{19,1A}$  is the set of ordered pairs  $(x, y) \in \mathbb{Z}_{(19)}^{\oplus 2}$  such that

$$x\chi(1A) + y\chi(19A) \equiv 0 \pmod{19} \quad (3.3.4)$$

for each irreducible rational character  $\chi$  of the Thompson group. Plugging in values for  $\chi(1A)$  and  $\chi(19A)$ , we find that  $M_{19,1A} = \{(x, y) \in \mathbb{Z}_{(19)}^{\oplus 2} \mid x + y \equiv 0 \pmod{19}\}$ . So in order to prove that  $\omega_n^\lambda : Th \rightarrow \mathbb{C}$  is a virtual rational character for each  $\lambda \in \Lambda$  and  $n \in \mathbb{Z}$ , we need to check that for each  $(x, y) \in M_{19,1A}$ ,

$$x\alpha_{1A}^\lambda(n) + y\alpha_{19A}^\lambda(n) \equiv 0 \pmod{19} \quad (3.3.5)$$

where  $\mathcal{F}_g^\lambda(\tau) = 6q^{-5} + \sum_{n \geq 3} \alpha_g^\lambda(n)$  (cf. eq. (3.3.1).) Thus, we have to show that the following congruence is satisfied for every  $n \in \mathbb{N}$  and  $\lambda \in \Lambda$

$$\alpha_{1A}^\lambda(n) - \alpha_{19A}^\lambda(n) \equiv 0 \pmod{19}. \quad (3.3.6)$$

This is a doubly infinite set of congruences (for each fixed  $\lambda \in \Lambda$ , we have a congruence for every integer  $n$ ), but we can get rid of the dependence on  $\lambda$  as follows: Note that  $\alpha_{1A}^\lambda(n)$  is independent of  $\lambda$  since the cusp form space  $S_{1A}$  is empty so we can write  $a_{1A}(n)$  for  $\alpha_{1A}^\lambda(n)$  (cf. Proposition 3.1.1 for notation). Also by Proposition 3.1.1,  $\alpha_{19A}^\lambda(n) = a_{19A}(n) + (m_{19A}\lambda_{19A} + n_{19A})b_{19A}(n)$  where  $b_{19A}(n)$  is the  $n^{\text{th}}$  coefficient of  $f_{19A}(\tau) \in S_{19A}$ . From Table A.3.2,  $m_{19A} = 18$  and  $n_{19A} = 19$ , so checking eq. (3.3.6) reduces to checking that

$$a_{1A}(n) = a_{19A}(n) + 18b_{19A}(n) \pmod{19} \quad (3.3.7)$$

for all  $n \in \mathbb{N}$ .

We can do the same thing for every pair  $(p, [g])$  where  $[g] \in K_p$ , and get a list of congruences that we need to check in order to show that the function  $\omega_n^\lambda$  is a virtual rational character in every case. An inspection of Table A.3.2 and Table A.3.1 confirms that we can always get rid of the dependence on  $\lambda$ . This still isn't a finite computation because at the moment, we need to check each congruence for all positive integers  $n$ . However, that can be easily resolved

in the following way: Let  $v(\tau)$  be the unique cusp form in  $S_{\frac{37}{2}}^+(\Gamma_0(4))$  whose Fourier expansion is of the form  $q^5 - 56q^8 + O(q^9)$ . Then for each  $[g]$  and  $\lambda$ ,  $\mathcal{F}_g^\lambda(\tau)v(\tau)$  is a holomorphic modular form of weight 20 and level  $|g|h_g$  so Sturm's theorem [Sturm 1987] applies. Thus, it suffices to check that the congruences hold for the first  $M$  Fourier coefficients of the holomorphic modular form where  $M$  is the Sturm bound which in the worst case is just shy of 4000. As before, we used [PAR 2019] to check these.

We conclude this section with another example of this procedure, for clarity.

**Example 2.** Let  $p = 3$ , then  $K_3 = \{1A, 2A, 4A, 4B, 5A, 7A, 8A, 8B, 10A, 13A\}$ . Pick  $[g] = 1A$ . Then,  $\alpha = 10$ ,  $R_{p,g} = \{1A, 3A, 3B, 3C, 9A, 9B, 9C, 27A, 27BC\}$  and  $M_{3,1A}$  is the set of 9-tuples  $(y_1, y_2, \dots, y_9)$  in  $\mathbb{Z}_{(3)}^{\oplus 9}$  such that

$$y_1\chi(1A) + y_2\chi(3A) + \dots + y_9\chi(27BC) \equiv 0 \pmod{3^{10}}$$

for each irreducible rational character  $\chi$  of the Thompson group. As before, in order to prove that  $\omega_n^\lambda : Th \rightarrow \mathbb{C}$  is a virtual rational character for each  $\lambda \in \Lambda$  and  $n \in \mathbb{Z}$ , we need to check that for each  $(y_1, \dots, y_9) \in M_{3,1A}$  we have,

$$y_1\alpha_{1A}^\lambda + y_2\alpha_{3A}^\lambda + \dots + y_9\alpha_{27BC}^\lambda \equiv 0 \pmod{3^{10}}. \quad (3.3.8)$$

This is easier to manage as a matrix computation. We let  $X$  denote the  $39 \times 9$  matrix

$$X = [\chi_i(h)]_{0 < i \leq 39, h \in R_{p,g}},$$

and let  $\mathbf{a} := (a_{1A}, a_{3A}, a_{3B}, \dots, a_{27BC})$ . For each  $(y_1, \dots, y_9) \in M_{3,1A}$ , we denote by  $\mathbf{y}$  the corresponding column vector whose entries are  $y_1, y_2, \dots, y_9$ .

Note that for all rational conjugacy classes  $[g]$  in  $R_{3,1A}$  the corresponding cusp form space  $S_g$  is empty, so we can in fact reduce to checking that  $\mathbf{a} \cdot \mathbf{y} \equiv 0 \pmod{3^{10}}$  for all  $\mathbf{y}$  such that  $X\mathbf{y} \equiv 0 \pmod{3^{10}}$ . In order to check this, we



first compute a basis for the  $\mathbb{Z}_{(3)}$ -span of the row vectors of  $X$ . We can use the GAP [GAP] command *BaseIntMat* to do this computation. It turns out that the  $\mathbb{Z}_{(3)}$ -span of the row vectors of  $X$  is the same as that of the row vectors of the following  $9 \times 9$  matrix:

$$M := \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 9 & 1944 & 72 & 0 & 45 & 24 & 0 & 15 \\ 0 & 0 & 2187 & 0 & 0 & 27 & 0 & 3 & 3 \\ 0 & 0 & 0 & 81 & 0 & 27 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 27 & 27 & 0 & 3 & 3 \\ 0 & 0 & 0 & 0 & 0 & 81 & 9 & 6 & 15 \\ 0 & 0 & 0 & 0 & 0 & 0 & 27 & 0 & 18 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 9 & 9 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 27 \end{bmatrix}. \quad (3.3.9)$$

We can solve  $My \equiv 0 \pmod{3^{10}}$  for  $y$  and then compute  $a \cdot y$  modulo  $3^{10}$  to see that the congruences we need to check are:

$$\begin{aligned} a_{1A} - a_{3A} &\equiv 0 \pmod{3^2} \\ a_{1A} - a_{9A} &\equiv 0 \pmod{3^3} \\ 7a_{1A} - 8a_{3A} + a_{3C} &\equiv 0 \pmod{3^4} \\ 215a_{1A} - 216a_{3A} + a_{3B} &\equiv 0 \pmod{3^7} \\ a_{1A} + 27a_{3A} - a_{3B} - 27a_{3C} - 81a_{9A} + 81a_{9B} &\equiv 0 \pmod{3^8} \\ 1214a_{1A} - 1971a_{3A} + a_{3B} + 27a_{3C} + 81a_{9A} - 81a_{9B} + 729a_{9C} &\equiv 0 \pmod{3^9} \\ 2591a_{1A} - 594a_{3A} + a_{3B} - 54a_{3C} + 81a_{9A} + 162a_{9B} - 2187a_{27A} &\equiv 0 \pmod{3^9} \\ 1214a_{1A} + 216a_{3A} + a_{3B} + 27a_{3C} + 81a_{9A} - 81a_{9B} - 1458a_{9C} + \dots \\ &\quad - 2187a_{27A} + 2187a_{27BC} \equiv 0 \pmod{3^{10}}. \end{aligned} \quad (3.3.10)$$

(Alternatively, we can use GAP to check that any given vector  $\mathbf{a}$  is in the aforementioned span, by using the following code: `IsContainedInSpan(MutableBasis(Integers,M),a mod 310).`)

As in Example 1, we can use PARI [PAR 2019] to check these congruences up to the Sturm bound, which in this case comes out to be less than 1100.

We can continue in this manner and check that all multiplicities are integral and hence for each  $\mathcal{F}_g^\lambda(\tau)$  as described in Proposition 3.1.1, there exists a virtual  $Th$ -module  $W^\lambda$  such that for each  $[g]$  in  $Th$ ,

$$\mathcal{F}_g^\lambda(\tau) = 6q^{-5} + \sum_{\substack{n>0 \\ n \equiv 0,3 \pmod{4}}} \text{tr}(g|W_n^\lambda) q^n \quad (3.3.11)$$

This proves Theorem 3.1.2.

## Chapter 4

# Elliptic Curves

### 4.1 Statements of Theorems

Now that we have proven the existence of the Thompson modules, we use their properties to help detect the non-triviality of Mordell–Weil, Selmer, and Tate–Shafarevich groups of quadratic twists of certain elliptic curves (cf. Theorems 4.1.1 and 4.1.2).

To state our main results, we let

$$\mathcal{F}(\tau) = 6(q^{-5} + 85995q^3 - 565760q^4 + 52756480q^7 - 190356480q^8 + O(q^{11})) \quad (4.1.1)$$

be the unique weakly holomorphic modular form of weight  $\frac{3}{2}$  and level 4 in the plus space whose Fourier expansion is of the form  $\mathcal{F}(\tau) = 6q^{-5} + O(q)$ . (Note that  $\mathcal{F}(\tau)$  is relevant for us because for each of the graded  $Th$ -modules  $W = \bigoplus_{n>0} W_n$  described in Theorem 3.1.2, the graded dimension of  $W$  is  $\mathcal{F}(\tau)$ .) We denote by  $\binom{m}{n}$  the usual Kronecker symbol [Cohen 1993], then we have the following theorem.

**Theorem 4.1.1.** Let  $d < 0$  be a fundamental discriminant which satisfies  $\left(\frac{d}{19}\right) = -1$ . Let  $E$  be an elliptic curve of conductor 19, and let  $E^d$  denote the  $d^{\text{th}}$  quadratic twist of  $E$ . Let  $c(d)$  denote the coefficient of  $q^{-d}$  in the Fourier expansion of  $\mathcal{F}(\tau)$ . If  $c(d) \not\equiv 0 \pmod{19}$ , then the Mordell–Weil group  $E^d(\mathbb{Q})$  is finite.

Here,  $E(\mathbb{Q})$  denotes the set of  $\mathbb{Q}$ -rational points on  $E$ . We can state a stronger result for elliptic curves of conductor 14, one which depends on a local version of the strong form of Birch and Swinnerton-Dyer Conjecture.

Let  $E$  be an elliptic curve over  $\mathbb{Q}$ , and for each  $d < 0$  a fundamental discriminant, let  $E^d$  denote the  $d^{\text{th}}$  quadratic twist of  $E$ . As we prove in Theorem 3.1.2, there exists an infinite-dimensional graded  $Th$ -module  $W = \bigoplus_{n>0} W_n$  whose McKay–Thompson series  $\mathcal{F}_g(\tau)$  satisfy properties (1) and (2) as above. Then we have the following theorem.

**Theorem 4.1.2.** Let  $d < 0$  be a fundamental discriminant for which  $\left(\frac{d}{7}\right) = -1$  and  $\left(\frac{d}{2}\right) = 1$ . Let  $E$  be an elliptic curve of conductor 14, and let  $g$  denote an element of order 14 in  $Th$ . If  $\text{tr}(g|W_{|d|}) \not\equiv 0 \pmod{49}$ , then the Mordell–Weil group  $E^d(\mathbb{Q})$  is finite and  $\text{III}(E^d)[7]$  is trivial. If, on the other hand,  $\text{tr}(g|W_{|d|}) \equiv 0 \pmod{49}$  and  $\text{tr}(g|W_4) \not\equiv 43 \pmod{56}$ , then  $\text{Sel}_7(E^d)$  is non-trivial, and if  $L_{E^d}(1)$  is non-zero then so is  $\text{III}(E^d)[7]$ .

This is akin to Theorem 1.4 of [Duncan et al. to appear], and our proof will follow along similar lines. One notable difference is that there is no dependence on (generalized) class numbers in the corresponding congruences in our case. We can also write down an analogous statement for elliptic curves of conductor 19, but the techniques we use to prove Theorem 4.1.2 do not apply in this case (cf. Theorem 4.5.1), so it is conditional upon the (strong form of the) Birch and Swinnerton-Dyer Conjecture.

We now describe a sketch of the proof of Theorems 4.1.1 and 4.1.2. Let  $p \in \{7, 19\}$  be the relevant prime in either statement and fix  $W = \bigoplus_{n>0} W_n$  to be a virtual Thompson module whose McKay–Thompson series  $\mathcal{F}_g(\tau)$  satisfies the properties listed in Theorem 3.1.2. We then write each  $\mathcal{F}_g(\tau)$  for  $[g] \in \{14A, 19A\}$  as a sum of traces of singular moduli (cf. Section 4.2) and weight  $\frac{3}{2}$  cusp forms. This expression combined with the condition on  $d$  in the statement of Theorem 4.1.1 gives us that the congruence in the statement holds if and only if the relevant cusp form coefficient is divisible by  $p = 19$ . Thus, if the congruence in the statement of Theorem 4.1.1 does not hold, then the cusp form coefficient is not divisible by 19, and we can employ a corollary (cf. Lemma 4.3.1) of Kohlen’s work [Kohlen 1985] to show that this means  $L_{E^d}(1) \neq 0$ . Finally, Kolyvagin’s work shows that  $E^d(\mathbb{Q})$  is finite. This completes the proof of Theorem 4.1.1. For Theorem 4.1.2, we first consider the case that  $\text{tr}(g_{14}|W_{|d|}) \not\equiv 0 \pmod{49}$ . The expression for  $\mathcal{F}_g(\tau)$  in terms of traces of singular moduli and cusp forms implies that the relevant cusp form coefficient is not divisible by  $p = 7$ . We can utilize Kohlen’s work again to conclude that  $\text{ord}_p\left(\frac{L_{E^d}(1)}{\Omega(E^d)}\right) > 0$ . At this point, we use work of Skinner and Urban (cf. Theorem 4.5.1) which connects  $\text{ord}_p\left(\frac{L_{E^d}(1)}{\Omega(E^d)}\right)$  to the non-triviality of the  $p$ -Selmer and Tate-Shafarevich groups of  $E^d$  to prove the theorem. A similar argument applies if we assume that  $\text{tr}(g_{14}|W_{|d|}) \equiv 0 \pmod{49}$  and  $\text{tr}(g_{14}|W_{|d|}) \not\equiv 43 \pmod{49}$ .

The family of *Th*-modules that we get from Theorem 3.1.2 encodes arithmetic information about quadratic twists of elliptic curves with conductors 14 and 19. This is the content of Theorems 4.1.1 and 4.1.2. We will prove these theorems in this section, but first, we have to develop some background. We recall next some basic notation and facts about traces of singular moduli, which were studied by Zagier in [Zagier 2002] and have since been examined extensively.

## 4.2 Traces of Singular Moduli

Let  $\mathcal{Q}_D^{(N)}$  be the set of positive definite quadratic forms  $Q = [a, b, c] := ax^2 + bxy + cy^2$  of discriminant  $-D = b^2 - 4ac < 0$  such that  $N|a$ . Then,  $\Gamma_0(N)$  acts on  $\mathcal{Q}_D^{(N)}$  with finitely many orbits. For  $Q = [a, b, c] \in \mathcal{Q}_D^{(N)}$ , we denote by  $\tau_Q := \frac{-b+i\sqrt{D}}{2a}$  the unique root of  $Q(x, 1)$  in the upper half-plane  $\mathbb{H}$ . Let  $f : \mathbb{H} \rightarrow \mathbb{C}$  be a function invariant under the action of  $\Gamma_0(N)$ , and  $n \equiv 0, 3 \pmod{4}$  be a positive integer. Then we can define,

$$\mathrm{Tr}_D^{(N)}(f; n) := \sum_{Q \in \mathcal{Q}_{nD}^{(N)}/\Gamma_0(N)} \chi_D(Q) \frac{f(\tau_Q)}{\omega^{(N)}(Q)}, \quad (4.2.1)$$

where  $\omega^{(N)}(Q)$  is the order of the stabilizer of  $Q$  in  $\Gamma_0(N)/\{\pm 1\}$  and  $\chi_D(Q)$  is the genus character for positive definite binary quadratic forms whose discriminants are multiples of  $D$ , defined as follows (see for example [Miller and Pixton 2010]):

$$\chi_D([a, b, c]) = \begin{cases} 0 & \text{if } (a, b, c, D) > 1 \\ \left(\frac{D}{r}\right) & \text{if } (a, b, c, D) = 1 \text{ and } Q \text{ represents } r \text{ with } (r, D) = 1. \end{cases} \quad (4.2.2)$$

For  $N \in \{14, 19\}$ , let  $J^{(N,+)}$  be the normalized Hauptmodul for the group  $\Gamma_0^+(N)$ . (We know this exists because the corresponding modular curve  $X_0^{(+)}(N)$  has genus 0. See [Ford et al. 1994], or [Duncan et al. to appear; Table 5.2].)

**Proposition 4.2.1.** Let  $N \in \{14, 19\}$  and let  $J^{(N,+)}$  as above. Then,

$$R_{\frac{3}{2}, 4N}^{[-5], +}(\tau) = q^{-5} + \frac{-2}{3\sqrt{5}} \sum_{\substack{n > 0 \\ n \equiv 0, 3 \pmod{4}}} \mathrm{Tr}_5^{(N)}(J^{(N,+)}; n) q^n \quad (4.2.3)$$

*Proof.* This is a direct application of Corollary 1.3 of [Miller and Pixton 2010].  $\square$

In particular, this means that  $f_g^{wh}(\tau)$  for  $o(g) = N \in \{14, 19\}$  is given by

$$f_g^{wh}(\tau) = 6q^{-5} + \sum_{0 < n} a_g(n)q^n = 6q^{-5} - \frac{4}{\sqrt{5}} \sum_{\substack{n > 0 \\ n \equiv 0,3 \pmod{4}}} \text{Tr}_5^{(N)}(J^{(N,+)}; n)q^n. \quad (4.2.4)$$

Writing  $f_g^{wh}(\tau)$  in the above form turns out to be essential for the proofs of Theorems 4.1.1 and 4.1.2. We give here another key lemma which we will use in both proofs.

**Lemma 4.2.2.** Let  $N \in \{14, 19\}$  and let  $d < 0$  be a fundamental discriminant that satisfies the respective conditions of Theorems 4.1.1 and 4.1.2; then

$$\text{Tr}_5^{(N)}(f, |d|) = 0 \quad (4.2.5)$$

for any  $\Gamma_0(N)$  invariant function  $f$ , and hence, in particular, for  $f = J^{(N,+)}(\tau)$ .

*Proof.* For  $N \in \{14, 19\}$ , the conditions of the theorems ensure that  $5d$  is not a square mod  $4N$ , which means that there are no quadratic forms  $[a, b, c]$  of discriminant  $b^2 - 4ac = 5d$  such that  $N|a$ . Thus  $\mathcal{Q}_{|5d|}^{(N)}$  is empty for all such  $d$  and thus

$$\text{Tr}_5^{(N)}(f; |d|) = \sum_{Q \in \mathcal{Q}_{|5d|}^{(N)}/\Gamma_0(N)} \chi_5(Q) \frac{f(\tau_Q)}{\omega^{(N)}(Q)} = 0 \quad (4.2.6)$$

for any function  $f$  that is  $\Gamma_0(N)$  invariant.  $\square$

We now recall facts about elliptic curves that we will use in order to prove Theorems 4.1.1 and 4.1.2.

### 4.3 Background on Elliptic Curves

To prove our main results, we let  $E$  be an elliptic curve over  $\mathbb{Q}$ . For  $d < 0$  a fundamental discriminant, we let  $E^d$  denote the  $d^{\text{th}}$  quadratic twist of  $E$ . We let  $N$  denote the conductor,  $\Omega(E)$  denote the real period and  $\text{Reg}(E)$  denote the regulator of  $E$ . We refer the reader to standard texts on elliptic curves, e.g. [Silverman 2009] for the definitions of these invariants. We let  $L_E(s)$  denote the  $L$ -function associated to  $E$ . Then, by the modularity theorem [Breuil et al. 2001] (see also [Wiles 1995; Taylor and Wiles 1995]), there exists a unique weight 2 newform  $\mathcal{G}_E = \sum_{n=1}^{\infty} a_E(n)q^n$  of level equal to the conductor of  $E$  such that

$$L_E(s) = \sum_{n=1}^{\infty} a_E(n)n^{-s}, \quad (4.3.1)$$

where the right-hand side extends to a holomorphic function on  $\mathbb{C}$  [Atkin and Lehner 1970]. We let  $g_E(\tau) = \sum_{n=3}^{\infty} b_E(n)q^n \in S_{\frac{3}{2}}^+(\Gamma_0(4N))$  be the weight  $\frac{3}{2}$  cusp form associated to  $\mathcal{G}_E$  under the Shintani lift (see [Hofmann 2017] for an overview of the Shintani lift). For  $N \in \{14, 19\}$  the dimension of  $S_{\frac{3}{2}}^+(\Gamma_0(4N))$  is 1, so for an elliptic curve of conductor  $N$ , the weight  $\frac{3}{2}$  cusp forms  $g_E(\tau)$  defined as above are the same as the cusp form  $f_g(\tau)$  associated to  $g \in \{14A, 19A\}$  in Section 3.1. This is the key fact that we employ in order to prove Theorems 4.1.1 and 4.1.2.

Let  $E/\mathbb{Q}$  be an elliptic curve with square-free conductor  $N$ , and for each  $\ell|N$ , let  $\omega_\ell$  denote the eigenvalue of the newform  $\mathcal{G}_E \in S_2(\Gamma_0(N))$  associated to  $E$  and the Atkin–Lehner involution  $W_\ell$ .

Then we have the following lemma of Duncan, Mertens, and Ono, [Duncan et al. to appear] (based on results due to Agashe [Agashe 2010] and Kohnen [Kohnen 1985], and the generalization of Kohnen’s work by Ueda



and Yamana [Ueda 1988; Ueda and Yamana 2010]) which connects the  $p$ -divisibility of the cusp form coefficient to  $L_{E^d}(1)$ .

**Lemma 4.3.1.** (see [Duncan et al. to appear; Lemma 6.5]) Assume the notation above, and let  $p \geq 3$  be a prime. Let  $d < 0$  be a fundamental discriminant satisfying  $\left(\frac{d}{\ell}\right) = \omega_\ell$  for each  $\ell$ . Denote by  $d_0$  the smallest such discriminant. Then we have that

$$\text{ord}_p \left( \frac{L_{E^d}(1)}{\Omega(E^d)} \right) = \text{ord}_p \left( \frac{L_{E^{d_0}}(1)}{\Omega(E^{d_0})} \right) + \text{ord}_p (b_E(|d|)^2), \quad (4.3.2)$$

where  $E^d$  denotes the  $d^{\text{th}}$  quadratic twist of  $E$ .

Both our proofs of Theorems 4.1.1 and 4.1.2 depend on the above lemma. We are now ready to prove Theorem 4.1.1.

## 4.4 Proof of Theorem 4.1.1

Fix  $W = W^\lambda$  to be an infinite-dimensional graded  $Th$ -module that satisfies all the properties listed Theorem 3.1.2. Then, for  $g$  an element of order 19 in  $Th$ , we can combine Proposition 3.1.1 and Proposition 4.2.1 to get the following expression for the coefficients of  $\mathcal{F}_{19A}(\tau)$  :

$$\text{tr}(g|W_n) \equiv \frac{-4}{\sqrt{5}} \text{Tr}_5^{(19)}(J^{(19,+)}; n) + (n_{19A} + \lambda_{19A} m_{19A}) b_{19A}(n), \quad (4.4.1)$$

where  $b_{19A}(n)$  denotes the  $n^{\text{th}}$  coefficient of the weight  $\frac{3}{2}$  cusp form  $f_{19A} \in S_{19A}$ . Since  $W$  is a virtual module for the Thompson group, we know the following congruence holds for each  $p \nmid \#Th$  (and in particular for  $p = 19$ ) and for all  $n > 0$  (cf. Section 3.3)

$$\dim(W_n) \equiv \text{tr}(g_p|W_n) \pmod{p}. \quad (4.4.2)$$

where  $g_p$  denotes an element of order  $p$ . Plugging in the values of  $n_g$  and  $m_g$  from Table A.3.2, we get,

$$\dim(W_n) \equiv \frac{-4}{\sqrt{5}} \operatorname{Tr}_5^{(19)}(J^{(19,+)}; n) + 18b_{19A}(n) \pmod{19}. \quad (4.4.3)$$

Thus for  $n = |d|$  where  $d$  is a fundamental discriminant that satisfies the properties of Theorem 4.1.1, we use Lemma 4.2.2 to get:

$$\dim(W_{|d|}) \equiv \operatorname{tr}(g_{19}|W_{|d|}) \equiv 18b_{19A}(|d|) \pmod{19}. \quad (4.4.4)$$

This shows that the congruence in the statement of our theorem holds if and only if  $19|b_{19A}(|d|)$ , or by Lemma 4.3.1, if and only if

$$\operatorname{ord}_{19} \left( \frac{L_{E^d}(1)}{\Omega(E^d)} \right) > \operatorname{ord}_{19} \left( \frac{L_{E^{d_0}}(1)}{\Omega(E^{d_0})} \right). \quad (4.4.5)$$

A quick MAGMA computation for  $d_0 = -4$  shows that the right-hand side is 0. Thus, if  $\dim(W_{|d|}) \not\equiv 0 \pmod{19}$ , then  $L_{E^d}(1) \not\equiv 0 \pmod{19}$ , and in particular,  $L_{E^d}(1) \neq 0$ . By Kolyvagin's work [Kolyvagin 1989], this means that  $E^d(\mathbb{Q})$  is finite. This completes the proof of Theorem 4.1.1.

## 4.5 Proof of Theorem 4.1.2

We need to develop some more background before proving Theorem 4.1.2. For  $\ell$  prime, we let  $c_\ell(E)$  denote the Tamagawa number of  $E$  at  $\ell$ , defined as the finite index

$$c_\ell = [E(\mathbb{Q}_\ell) : E^0(\mathbb{Q}_\ell)], \quad (4.5.1)$$

where  $E^0(\mathbb{Q}_\ell)$  is the subgroup of points which have good reduction at  $\ell$ . If  $E$  has good reduction at  $\ell$ , then  $E(\mathbb{Q}_\ell) = E^0(\mathbb{Q}_\ell)$  and  $c_\ell = 1$ . In particular

for a general elliptic curve defined over  $\mathbb{Q}$ , we have that  $c_\ell = 1$  for all but finitely many primes  $\ell$ . The following result of C. Skinner (see also [Skinner and Urban 2014]) gives a local version of the Birch and Swinnerton-Dyer Conjecture for certain elliptic curves.

**Theorem 4.5.1** ([Skinner 2016], Theorem C). Let  $E/\mathbb{Q}$  be an elliptic curve and  $p \geq 3$  a prime of good ordinary or multiplicative reduction. Assume that the  $\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ -representation  $E[p]$  is irreducible and that there exists a prime  $p' \neq p$  at which  $E$  has multiplicative reduction and  $E[p]$  ramifies. If  $L_E(1) \neq 0$ , then we have that

$$\text{ord}_p \left( \frac{L_E(1)}{\Omega_E} \right) = \text{ord}_p \left( \#\text{III}(E) \prod_{\ell} c_{\ell}(E) \right). \quad (4.5.2)$$

If  $L_E(1) = 0$ , then we have  $\text{Sel}_p(E) \neq \{0\}$ .

In order to use Theorem 4.5.1 in our proof of Theorem 4.1.2, we first show that each elliptic curve  $E$  of conductor 14 satisfies the hypotheses of Theorem 4.5.1 in the following lemma.

**Lemma 4.5.2.** Let  $d < 0$  be a fundamental discriminant for which  $\left(\frac{d}{7}\right) = -1$  and  $\left(\frac{d}{2}\right) = 1$ ; then for each elliptic curve  $E$  of conductor 14 the following are true:

- (a) The  $d^{\text{th}}$  quadratic twist of  $E$  has multiplicative reduction at  $p \in \{2, 7\}$ ;
- (b) The  $\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ -representation  $E^d[7]$  is irreducible; and
- (c)  $E^d[7]$  ramifies at 2.

*Proof.* Let  $E/\mathbb{Q}$  be an elliptic curve given by a minimal Weierstrass model

$$E : y^2 + a_1xy + a_3y = x^3 + a_2x^2 + a_4x + a_6 \quad (4.5.3)$$

and define the discriminant of  $E$  by the equation

$$\Delta(E) := -b_2^2 b_8 - 8b_4^3 - 27b_6^2 + 9b_2 b_4 b_6, \quad (4.5.4)$$

where  $b_2 := a_1^2 + 4a_4$ ,  $b_4 := 2a_4 + a_1 a_3$ ,  $b_6 := a_3^2 + 4a_6$  and  $b_8 := a_1^2 a_6 + 4a_2 a_6 - a_1 a_3 a_4 + a_2 a_3^2 - a_4^2$ . Then  $E$  has multiplicative reduction at  $p$  if and only if  $p$  divides the discriminant of  $E$  but not the quantity  $c_4(E) := (a_1^2 + 4a_4)^2 - 24(2a_4 + a_1 a_3)$ . For each elliptic curve of conductor 14, we have that  $a_1 = a_3 = 1$ ,  $a_2 = 0$  and  $a_3 \in \{-2731, -171, -36, -11, -1, 4\}$  (cf. [The LMFDB Collaboration 2013; Elliptic Curve 14.a]). Thus, for each elliptic curve  $E$  of conductor 14,  $E$  has multiplicative reduction at  $p \in \{2, 7\}$ . Since twisting by a fundamental discriminant  $d$  only changes  $\Delta(E)$  and  $c_4(E)$  up to a power of  $d$ , and  $d$  is coprime to 14, this proves part (a).

Part (b) follows from a lemma of Serre [Serre 1971] which shows that the Galois representation  $E^d[7]$  is surjective and hence irreducible. Finally, part (c) follows from part (b) and (the contrapositive of) Theorem 1.1 of [Ribet 1990].  $\square$

We are now ready to prove Theorem 4.1.2.

*Proof of Theorem 1.3.* Fix  $W = W^\lambda$  to be an infinite-dimensional graded  $Th$ -module that satisfies all the properties listed Theorem 3.1.2. Let  $g$  denote an element of order 14 in  $Th$ . As before, we can combine Proposition 3.1.1 and Proposition 4.2.1 to get the following expression for the trace of  $g$  on  $W$  :

$$\mathrm{tr}(g|W_n) = \frac{-4}{\sqrt{5}} \mathrm{Tr}_5^{(14)}(J^{(14,+)}; n) + (n_{14A} + \lambda_{14A}^{(1)} m_{14A}) b_{14A}(n). \quad (4.5.5)$$

Here,  $b_{14A}(n)$  denotes the  $n^{\mathrm{th}}$  coefficient of the weight  $\frac{3}{2}$  cusp form  $f_{14A} \in S_{14A}$ . By Lemma 4.2.2, we get that for  $n = |d|$  where  $d$  is a fundamental

discriminant that satisfies the properties of Theorem 4.1.2, the first term on the right-hand side of the above equation is 0. Plugging in values of  $n_g^{(1)}$  and  $m_g^{(1)}$  from Table A.3.2, we get the following congruence

$$\mathrm{tr}(g|W_{|d|}) = \left(42 + 56\lambda_{14A}^{(1)}\right) b_{14A}(|d|) \pmod{49}. \quad (4.5.6)$$

Suppose first that  $\mathrm{tr}(g|W_{|d|}) \not\equiv 0 \pmod{49}$ . Then,  $b_{14A}(|d|) \not\equiv 0 \pmod{7}$ . By Lemma 4.3.1, this means that

$$\mathrm{ord}_7 \left( \frac{L_{E^d}(1)}{\Omega(E^d)} \right) = \mathrm{ord}_7 \left( \frac{L_{E^{d_0}}(1)}{\Omega(E^{d_0})} \right). \quad (4.5.7)$$

As before we can use MAGMA to check that the right-hand side of the above equation is 0 for each  $E$  of conductor 14. Thus, if  $\mathrm{tr}(g|W_{|d|}) \not\equiv 0 \pmod{49}$  then  $\mathrm{ord}_7 \left( \frac{L_{E^d}(1)}{\Omega(E^d)} \right) = 0$  and in particular,  $L_{E^d}(1) \neq 0$ . By Lemma 4.5.2 and Theorem 4.5.1, we have that

$$\mathrm{ord}_7 \left( \#\mathrm{III}(E^d) \prod_{\ell} c_{\ell}(E^d) \right) = 0. \quad (4.5.8)$$

Thus,  $\mathrm{III}(E^d)[7]$  is trivial. Furthermore, the Mordell–Weil group  $E^d(\mathbb{Q})$  is finite [Kolyvagin 1989].

We now consider the case that  $\mathrm{tr}(g|W_{|d|}) \equiv 0 \pmod{49}$  and assume that  $\mathrm{tr}(g|W_4) \not\equiv 43 \pmod{49}$ . We can once again use Proposition 3.1.1 and Proposition 4.2.1 to write

$$\mathrm{tr}(g|W_4) = \frac{-4}{\sqrt{5}} \mathrm{Tr}_5^{(14)}(J^{(14,+)}; 4) + \left(42 + 56\lambda_{14A}^{(1)}\right) b_{14A}(4) = -6 + (42 + 56\lambda_{14A}^{(1)}). \quad (4.5.9)$$

Our assumption on  $\mathrm{tr}(g|W_4)$  gives us the congruence  $(42 + 56\lambda_{14A}^{(1)}) \not\equiv 0 \pmod{49}$  and hence by Equation (4.5.6) we get that  $7 \mid b_{14A}(|d|)$ . By

Lemma 4.3.1 we get

$$\text{ord}_7 \left( \frac{L_{E^d}(1)}{\Omega(E^d)} \right) > 0. \quad (4.5.10)$$

First suppose that  $L_{E^d}(1) = 0$ , then  $\text{Sel}_p(E) \neq 0$  by Theorem 4.5.1. So we can reduce to the case where  $L_{E^d}(1) \neq 0$ . In that case, again by Theorem 4.5.1, we get,

$$\text{ord}_7 \left( \#\text{III}(E^d) \prod_{\ell} c_{\ell}(E^d) \right) > 0. \quad (4.5.11)$$

Thus the only thing left to check is that 7 does not divide any of the Tamagawa numbers  $c_{\ell}(E^d)$  for any choice of  $E$  and  $d$ . By Theorem VII.6.1 in Silverman I [Silverman 2009],  $c_{\ell}(E^d) \leq 4$  for most of these cases. The only other possibility is when  $E^d$  has split multiplicative reduction at  $\ell$ , in which case,  $c_{\ell}(E^d) = \text{ord}_{\ell}(\Delta(E^d)) = \text{ord}_{\ell}(|d|^6 \Delta(E))$ . The conditions on  $d$  in the theorem imply that  $|d|$  is square-free and coprime to  $\Delta(E)$  for all  $E$  of conductor 14. Thus, if  $7 \mid c_{\ell}(E^d)$ , for some  $\ell$  then  $\ell$  lies in  $\{2, 7\}$  and 7 divides  $\text{ord}_{\ell}(\Delta(E))$  which is independent of  $d$ . A quick check reveals that this is never the case for an elliptic curve of conductor 14.

□

# Appendix A

## Tables

### A.1 Multipliers

$[g]$	1A	2A	3A	3B	3C	4A	4B	5A	6A	6B
$v, h$	0,1	0,1	1,3	0,1	2,3	0,1	1,2	0,1	1,3	2,3

$[g]$	6C	7A	8A	8B	9A	9B	9C	10A	12AB	12 C
$v, h$	0,1	0,1	1,2	1,4	0,1	0,1	1,3	0,1	1,3	0,1

$[g]$	12D	13A	14A	15AB	18A	18B	19A	20A	21A	24AB
$v, h$	1,6	0,1	0,1	1,3	0,1	2,3	0,1	1,2	1,3	1,6

$[g]$	24CD	27A	27BC	28A	30AB	31AB	36A	36BC	39AB
$v, h$	1,12	1,3	1,3	0,1	2,3	0,1	0,1	0,1	1,3

TABLE A.1.1: Multipliers for each rational conjugacy class.





### A.3 Cusp forms

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$$f_{12D}(q) = q^4 - 2q^8 + 2q^{20} - 2q^{40} - 2q^{52} + 4q^{56} - 2q^{68} + 4q^{88} - q^{100} - 6q^{116} + 2q^{136} + 4q^{148} + O(q^{150})$$

---

$$f_{14A}(q) = q^4 - q^7 - q^8 + 2q^{15} - q^{16} + q^{28} + q^{32} - q^{36} - 2q^{39} + q^{56} - 2q^{60} + q^{63} + q^{64} - 2q^{71} + 3q^{72} + 2q^{79} - 2q^{84} - 2q^{88} + 2q^{95} - q^{100} - q^{112} + 2q^{119} + 2q^{120} - 4q^{127} - q^{128} - 4q^{135} + q^{144} + 4q^{148} + O(q^{151})$$

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$$f_{18B}(q) = q^4 + q^7 - q^{16} - q^{28} - 3q^{31} + q^{40} - 2q^{52} + q^{55} + q^{64} + 2q^{79} + q^{88} + 2q^{100} + 2q^{103} + q^{112} + 3q^{124} - q^{127} - 4q^{136} - 2q^{148} + O(q^{150})$$

---

$$f_{19A}(q) = q^4 - q^7 - q^{11} + q^{19} + q^{20} - 2q^{24} + q^{28} + q^{35} - q^{36} + 2q^{39} - q^{43} - q^{44} + q^{47} - q^{55} + q^{63} - 2q^{64} - q^{68} + q^{76} - q^{95} + 3q^{99} + 2q^{100} - 2q^{111} + 2q^{112} - 2q^{115} - q^{119} + 2q^{120} - 2q^{123} + q^{131} - 3q^{139} + q^{140} + O(q^{151})$$

---

$$f_{20A}^{(1)}(q) = q^4 - q^{20} - 2q^{24} - q^{36} + 2q^{40} + 2q^{56} + 2q^{84} - q^{100} - 2q^{120} - 4q^{136} + O(q^{151})$$

---

$$f_{20A}^{(2)}(q) = q^7 - q^{15} - q^{23} + q^{47} + q^{63} - 2q^{87} + 2q^{95} + q^{103} - 3q^{127} - 2q^{143} + O(q^{151})$$

---

$$f_{21A}^{(1)}(q) = q^4 - q^{11} - q^{16} + q^{23} - q^{28} - q^{32} + q^{35} + q^{44} + q^{56} + q^{64} + 2q^{67} - q^{71} - 2q^{79} - 2q^{91} - q^{92} + q^{100} - q^{107} + q^{112} - q^{116} - q^{119} + 2q^{127} - 2q^{148} + O(q^{150})$$

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$$f_{21A}^{(2)}(q) = q^7 + q^8 - q^{11} - 2q^{16} - q^{23} + q^{32} + 2q^{43} + q^{56} - q^{71} - 2q^{88} - 2q^{91} - 2q^{92} + 2q^{95} + 4q^{100} + q^{107} - 2q^{116} - 4q^{127} + q^{128} - 2q^{140} + O(q^{150})$$


---

$$f_{24AB}(q) = q^4 - q^8 - q^{20} + 2q^{40} - 2q^{52} + 2q^{56} + q^{68} - 4q^{88} - q^{100} + 3q^{116} - 2q^{136} + 4q^{148} + O(q^{150})$$


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$$f_{24CD}^{(1)}(q) = q^4 - q^{100} + O(q^{150})$$


---

$$f_{24CD}^{(2)}(q) = q^7 - 2q^{15} - q^{31} + 4q^{39} - 2q^{63} - 3q^{79} + 2q^{87} + q^{103} + q^{127} - 2q^{135} + O(q^{150})$$


---

$$f_{24CD}^{(3)}(q) = q^{16} + q^{20} - 2q^{32} - q^{52} + q^{68} + 2q^{80} - 3q^{116} + 2q^{148} + O(q^{150})$$


---

$$f_{28A}^{(1)}(q) = q^4 - q^8 - q^{16} + q^{28} + q^{32} - q^{36} + q^{56} - 2q^{60} + q^{64} + 3q^{72} - 2q^{84} - 2q^{88} - q^{100} - q^{112} + 2q^{120} - q^{128} + q^{144} + 4q^{148} + 2q^{156} - 2q^{168} - 4q^{184} + q^{196} - q^{200} + O(q^{201})$$


---

$$f_{28A}^{(2)}(q) = q^7 - 2q^{15} + 2q^{39} - q^{63} + 2q^{71} - 2q^{79} - 2q^{95} - 2q^{119} + 4q^{127} + 4q^{135} + O(q^{151})$$


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$$f_{30AB}^{(1)}(q) = q^4 + 2q^{15} - q^{16} + 2q^{24} - 2q^{36} - 4q^{39} - q^{40} + 2q^{55} - 2q^{60} + q^{64} - 4q^{79} + 4q^{84} - 2q^{96} - q^{100} + 4q^{111} - 4q^{120} + 2q^{135} + 2q^{136} + 2q^{144} + O(q^{150})$$


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$$f_{30AB}^{(2)}(q) = q^7 + q^8 + q^{20} - q^{28} - q^{32} - q^{40} - 2q^{47} + q^{52} - q^{55} - 2q^{68} - q^{80} + q^{88} + 2q^{95} - q^{103} + q^{112} - q^{127} + q^{128} + 2q^{143} + q^{148} + O(q^{150})$$

---


$$f_{30AB}^{(3)}(q) = q^{11} - q^{15} + q^{16} - q^{19} - q^{20} - q^{24} + q^{35} + q^{36} + 2q^{39} + q^{44} - q^{55} - q^{56} - 3q^{59} + q^{60} - q^{76} + 2q^{79} - q^{80} - 2q^{84} + q^{91} + q^{96} + 3q^{104} - 2q^{111} + 2q^{115} + 2q^{120} - q^{131} - q^{135} - 3q^{136} + q^{140} - q^{144} + O(q^{150})$$


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$$f_{31AB}^{(1)}(q) = q^4 - q^8 - q^{20} - q^{28} + q^{32} + 2q^{35} + q^{36} - 2q^{39} + q^{40} - 2q^{51} + q^{56} - 2q^{59} + 2q^{63} - q^{64} + 2q^{67} + 2q^{71} - q^{72} - q^{76} + 2q^{87} - 2q^{95} - 2q^{103} - 2q^{107} + q^{124} + 2q^{128} - 2q^{132} + q^{140} - 2q^{144} + O(q^{151}),$$


---

$$f_{31AB}^{(2)}(q) = q^7 - q^8 - q^{16} + q^{19} - q^{31} + q^{35} + 2q^{36} - q^{40} - 2q^{51} + q^{56} - q^{59} - q^{63} - q^{64} + q^{71} + q^{72} + 2q^{76} + q^{80} - q^{95} + q^{103} - q^{107} - 2q^{111} - q^{112} + q^{128} + q^{144} + O(q^{151})$$


---

$$f_{39AB}^{(1)}(q) = q^7 - q^{19} + q^{20} - q^{31} - q^{32} - q^{44} - q^{59} + q^{67} + q^{71} + q^{80} + q^{83} + q^{91} + q^{104} - q^{119} - 2q^{124} + 2q^{136} - q^{143} + 2q^{148} + O(q^{150})$$


---

$$f_{39AB}^{(2)}(q) = q^8 + q^{15} - q^{19} - q^{24} - q^{28} + q^{31} - q^{39} - q^{44} - q^{47} + q^{52} + q^{72} + q^{76} - q^{80} + q^{83} + 2q^{84} - q^{96} - q^{99} + q^{112} + 2q^{115} + q^{119} + q^{123} - q^{124} - q^{135} - q^{136} + O(q^{150})$$


---

$$f_{39AB}^{(3)}(q) = q^{11} - q^{15} - q^{19} + q^{24} - q^{28} + q^{31} - q^{32} + q^{39} + q^{44} + q^{52} - q^{59} - q^{71} - q^{72} + q^{76} - q^{80} - 2q^{84} + q^{96} + q^{99} + q^{104} + q^{112} + 2q^{115} - q^{119} - q^{123} - q^{124} + q^{128} + q^{135} - q^{136} + q^{143} + O(q^{150})$$


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TABLE A.3.1: List of non-zero cusp forms in  $S_g$  for each rational conjugacy class  $[g]$  of  $Th$ .

$f_g^{(i)}$	$n_g^{(i)}$	$m_g^{(i)}$	$f_g^{(i)}$	$n_g^{(i)}$	$m_g^{(i)}$	$f_g^{(i)}$	$n_g^{(i)}$	$m_g^{(i)}$	$f_g^{(i)}$	$n_g^{(i)}$	$m_g^{(i)}$
$f_{12D}$	12	24	$f_{14A}$	42	56	$f_{18B}$	0	18	$f_{19A}$	18	19
$f_{20A}^{(1)}$	0	20	$f_{20A}^{(2)}$	0	20	$f_{21A}^{(1)}$	9	21	$f_{21A}^{(2)}$	17	21
$f_{24AB}^{(1)}$	0	48	$f_{24CD}^{(1)}$	0	12	$f_{24CD}^{(2)}$	0	12	$f_{24CD}^{(3)}$	0	12
$f_{28A}^{(1)}$	0	14	$f_{28A}^{(2)}$	0	28	$f_{30AB}^{(1)}$	3	30	$f_{30AB}^{(2)}$	15	30
$f_{30AB}^{(3)}$	21	30	$f_{31A}^{(1)}$	2	31	$f_{31A}^{(2)}$	19	31	$f_{39AB}^{(1)}$	21	39
$f_{39AB}^{(2)}$	6	39	$f_{39AB}^{(3)}$	6	39						

TABLE A.3.2: Integers  $n_g^{(i)}$  and  $m_g^{(i)}$  associated to each cusp form  $f_g^{(i)}(\tau)$ .

## Bibliography

- Amod Agashe. A visible factor of the special  $L$ -value. *Journal für die reine und angewandte Mathematik (Crelles Journal)*, 2010(644):159–187, 2010.
- Arthur O. L. Atkin and Joseph Lehner. Hecke operators on  $\Gamma_0(m)$ . *Mathematische Annalen*, 185(2):134–160, 1970.
- Richard E. Borcherds. Monstrous moonshine and monstrous Lie superalgebras. *Inventiones mathematicae*, 109(1):405–444, 1992.
- Wieb Bosma, John Cannon, and Catherine Playoust. The Magma algebra system. I. The user language. *J. Symbolic Comput.*, 24(3-4):235–265, 1997. ISSN 0747-7171. doi: 10.1006/jSCO.1996.0125. URL <http://dx.doi.org/10.1006/jSCO.1996.0125>. Computational algebra and number theory (London, 1993).
- Christophe Breuil, Brian Conrad, Fred Diamond, and Richard Taylor. On the modularity of elliptic curves over  $\mathbb{Q}$ : wild 3-adic exercises. *Journal of the American Mathematical Society*, 14(4):843–939, 2001.
- Kathrin Bringmann, Amanda Folsom, Ken Ono, and Larry Rolen. *Harmonic Maass forms and mock modular forms: theory and applications*, volume 64. American Mathematical Soc., 2017.
- Jan Hendrik Bruinier and Jens Funke. On two geometric theta lifts. *Duke Mathematical Journal*, 125(1):45–90, 2004.

- Scott Carnahan. Generalized moonshine iv: monstrous lie algebras. *arXiv preprint arXiv:1208.6254*, 2012.
- Miranda C. N. Cheng and John F. R. Duncan. On Rademacher sums, the largest Mathieu group and the holographic modularity of moonshine. *Commun. Number Theory Phys.*, 6(3):697–758, 2012. ISSN 1931-4523. doi: 10.4310/CNTP.2012.v6.n3.a4. URL <https://doi.org/10.4310/CNTP.2012.v6.n3.a4>.
- Miranda C. N. Cheng and John F. R. Duncan. Rademacher sums and Rademacher series. In *Conformal field theory, automorphic forms and related topics*, pages 143–182. Springer, 2014.
- Miranda C. N. Cheng, John F. R. Duncan, and Jeffrey A. Harvey. Umbral moonshine. *Commun. Number Theory Phys.*, 8(2):101–242, 2014a. ISSN 1931-4523. doi: 10.4310/CNTP.2014.v8.n2.a1. URL <https://doi.org/10.4310/CNTP.2014.v8.n2.a1>.
- Miranda C. N. Cheng, John F. R. Duncan, and Jeffrey A. Harvey. Umbral moonshine and the Niemeier lattices. *Res. Math. Sci.*, 1:Art. 3, 81, 2014b. ISSN 2522-0144. doi: 10.1186/2197-9847-1-3. URL <https://doi.org/10.1186/2197-9847-1-3>.
- Miranda C. N. Cheng, John F. R. Duncan, and Jeffrey A. Harvey. Weight one Jacobi forms and umbral moonshine. *J. Phys. A*, 51(10):104002, 37, 2018. ISSN 1751-8113. doi: 10.1088/1751-8121/aaa819. URL <https://doi.org/10.1088/1751-8121/aaa819>.
- Henri Cohen. *A course in computational algebraic number theory*, volume 138 of *Graduate Texts in Mathematics*. Springer-Verlag, Berlin, 1993. ISBN 3-540-55640-0. doi: 10.1007/978-3-662-02945-9. URL <https://doi.org/10.1007/978-3-662-02945-9>.

John H Conway and Simon P Norton. Monstrous moonshine. *Bulletin of the London Mathematical Society*, 11(3):308–339, 1979.

John H. Conway, R. T. Curtis, S. P. Norton, R. A. Parker, and R. A. Wilson. *Atlas of finite groups*. Oxford University Press, Eynsham, 1985. ISBN 0-19-853199-0. Maximal subgroups and ordinary characters for simple groups, With computational assistance from J. G. Thackray.

Charles W. Curtis and Irving Reiner. *Representation theory of finite groups and associative algebras*. Wiley Classics Library. John Wiley & Sons, Inc., New York, 1988. ISBN 0-471-60845-9. Reprint of the 1962 original, A Wiley-Interscience Publication.

Atish Dabholkar, Sameer Murthy, and Don Zagier. Quantum black holes, wall crossing, and mock modular forms. *arXiv preprint arXiv:1208.4074*, 2012.

John F. R. Duncan. From the Monster to Thompson to O’Nan. *arXiv preprint arXiv:1909.09684*, 2019.

John F. R. Duncan and Igor B. Frenkel. Rademacher sums, moonshine and gravity. *Commun. Number Theory Phys.*, 5(4):849–976, 2011. ISSN 1931-4523. doi: 10.4310/CNTP.2011.v5.n4.a4. URL <https://doi.org/10.4310/CNTP.2011.v5.n4.a4>.

John F. R. Duncan, Michael J. Griffin, and Ken Ono. Proof of the umbral moonshine conjecture. *Res. Math. Sci.*, 2:Art. 26, 47, 2015. ISSN 2522-0144. doi: 10.1186/s40687-015-0044-7. URL <https://doi.org/10.1186/s40687-015-0044-7>.

John F. R. Duncan, Michael H. Mertens, and Ken Ono. Pariah moonshine. *Nature Communications*, 9(1), 2017. doi: 10.1038/s41467-017-00660-y. URL <https://www.nature.com/articles/s41467-017-00660-y>.

- John F. R. Duncan, Michael H. Mertens, and Ken Ono. O’Nan moonshine and arithmetic. *Amer. J. Math.*, to appear. (arXiv:1702.03516).
- Walter Feit. The computations of some Schur indices. *Israel J. Math.*, 46 (4):274–300, 1983. ISSN 0021-2172. doi: 10.1007/BF02762888. URL <https://doi.org/10.1007/BF02762888>.
- Paul Fong. Characters arising in the Monster-modular connection. In *The Santa Cruz Conference on Finite Groups (Univ. California, Santa Cruz, Calif., 1979)*, volume 37 of *Proc. Sympos. Pure Math.*, pages 557–559. Amer. Math. Soc., Providence, R.I., 1980.
- David Ford, John McKay, and Simon Norton. More on replicable functions. *Communications in Algebra*, 22(13):5175–5193, 1994.
- I. B. Frenkel, J. Lepowsky, and A. Meurman. A natural representation of the Fischer-Griess Monster with the modular function  $J$  as character. *Proc. Nat. Acad. Sci. U.S.A.*, 81(10, , Phys. Sci.):3256–3260, 1984. ISSN 0027-8424. doi: 10.1073/pnas.81.10.3256. URL <https://doi.org/10.1073/pnas.81.10.3256>.
- Igor Frenkel, James Lepowsky, and Arne Meurman. *Vertex operator algebras and the Monster*, volume 134 of *Pure and Applied Mathematics*. Academic Press, Inc., Boston, MA, 1988. ISBN 0-12-267065-5.
- Igor B. Frenkel, James Lepowsky, and Arne Meurman. A moonshine module for the Monster. In *Vertex operators in mathematics and physics (Berkeley, Calif., 1983)*, volume 3 of *Math. Sci. Res. Inst. Publ.*, pages 231–273. Springer, New York, 1985. doi: 10.1007/978-1-4613-9550-8\_12. URL [https://doi.org/10.1007/978-1-4613-9550-8\\_12](https://doi.org/10.1007/978-1-4613-9550-8_12).
- Terry Gannon. Much ado about Mathieu. *Advances in Mathematics*, 301: 322–358, 2016.



- GAP. *GAP – Groups, Algorithms, and Programming, Version 4.10.1*. The GAP Group, 2019. URL <https://www.gap-system.org>.
- Michael J Griffin and Michael H Mertens. A proof of the Thompson moonshine conjecture. *Research in the Mathematical Sciences*, 3(1):36, 2016.
- Jeffrey Harvey and Brandon Rayhaun. Traces of singular moduli and moonshine for the Thompson group. *Communications in Number Theory and Physics*, 10, 04 2015. doi: 10.4310/CNTP.2016.v10.n1.a2.
- Eric Hofmann. Liftings and Borchers products. In *L-functions and automorphic forms*, volume 10 of *Contrib. Math. Comput. Sci.*, pages 333–366. Springer, Cham, 2017.
- Winfried Kohnen. Modular forms of half-integral weight on  $\Gamma_0(4)$ . *Math. Ann.*, 248(3):249–266, 1980. ISSN 0025-5831. doi: 10.1007/BF01420529. URL <https://doi.org/10.1007/BF01420529>.
- Winfried Kohnen. Newforms of half-integral weight. *J. Reine Angew. Math.*, 333:32–72, 1982. ISSN 0075-4102. doi: 10.1515/crll.1982.333.32. URL <https://doi.org/10.1515/crll.1982.333.32>.
- Winfried Kohnen. Fourier coefficients of modular forms of half-integral weight. *Mathematische Annalen*, 271(2):237–268, 1985.
- Victor A. Kolyvagin. Finiteness of and for a subclass of Weil curves. *Mathematics of the USSR-Izvestiya*, 32(3):523, 1989.
- Alison Miller and Aaron Pixton. Arithmetic traces of non-holomorphic modular invariants. *International Journal of Number Theory*, 6(01):69–87, 2010.
- SP Norton. Generalized moonshine. In *Proc. Symp. Pure Math*, volume 47, pages 208–209, 1987.

- Ken Ono. Unearthing the visions of a master: harmonic Maass forms and number theory. *Current developments in mathematics*, 2008:347–454, 2009.
- PARI/GP version 2.11.2. The PARI Group, Univ. Bordeaux, 2019. available from <http://pari.math.u-bordeaux.fr/>.
- Larissa Queen. Modular functions arising from some finite groups. *Math. Comp.*, 37(156):547–580, 1981. ISSN 0025-5718. doi: 10.2307/2007446. URL <https://doi.org/10.2307/2007446>.
- Kenneth A Ribet. On modular representations of arising from modular forms. *Inventiones mathematicae*, 100(1):431–476, 1990.
- Jeremy Rouse and John J Webb. On spaces of modular forms spanned by eta-quotients. *Advances in Mathematics*, 272:200–224, 2015.
- Jean-Pierre Serre. Propriétés galoisiennes des points d’ordre fini des courbes elliptiques. *Inventiones mathematicae*, 15(4):259–331, 1971.
- Joseph H Silverman. *The arithmetic of elliptic curves*, volume 106. Springer Science & Business Media, 2009.
- Christopher Skinner. Multiplicative reduction and the cyclotomic main conjecture for  $gl_2$ . *Pacific Journal of Mathematics*, 283(1):171–200, 2016.
- Christopher Skinner and Eric Urban. The Iwasawa main conjectures for  $GL_2$ . *Invent. Math.*, 195(1):1–277, 2014. ISSN 0020-9910. doi: 10.1007/s00222-013-0448-1. URL <https://doi.org/10.1007/s00222-013-0448-1>.
- Stephen D. Smith. On the head characters of the Monster simple group. In *Finite groups—coming of age (Montreal, Que., 1982)*, volume 45 of *Contemp. Math.*, pages 303–313. Amer. Math. Soc., Providence, RI, 1985. doi:

10.1090/conm/045/822245. URL <https://doi.org/10.1090/conm/045/822245>.

Jacob Sturm. On the congruence of modular forms. In *Number theory*, pages 275–280. Springer, 1987.

Richard Taylor and Andrew Wiles. Ring-theoretic properties of certain Hecke algebras. *Annals of Mathematics*, 141(3):553–572, 1995.

The LMFDB Collaboration. The  $L$ -functions and Modular Forms Database. <http://www.lmfdb.org>, 2013.

John G. Thompson. Finite Groups and Modular Functions. *Bulletin of the London Mathematical Society*, 11(3):347–351, 10 1979a. ISSN 0024-6093. doi: 10.1112/blms/11.3.347. URL <https://doi.org/10.1112/blms/11.3.347>.

John G Thompson. Some numerology between the Fischer-Griess Monster and the elliptic modular function. *Bulletin of the London Mathematical Society*, 11(3):352–353, 1979b.

Masaru Ueda. The decomposition of the spaces of cusp forms of half-integral weight and trace formula of Hecke operators. *J. Math.*, 11:505–555, 1988.

Masaru Ueda and Shunsuke Yamana. On newforms for Kohnen plus spaces. *Mathematische Zeitschrift*, 264(1):1, 2010.

Andrew Wiles. Modular elliptic curves and Fermat’s last theorem. *Annals of mathematics*, 141(3):443–551, 1995.

Don Zagier. Traces of singular moduli. In *Motives, Polylogarithms and Hodge Theory (Eds. F. Bogomolov, L. Katzarkov), Lecture Series 3*, pages 209–244. International Press, Somerville, 2002.