Distribution Agreement

Characterization of Quasiconformal Mappings and Extremal Length Decomposition

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Abstract

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Quasiconformal mappings have abundant subtle analytic and geometric properties, which can be used widely in various contexts. The reason probably lies in that there exists several equivalent definitions for quasiconformal mappings. While conformal mappings preserve measures of angles, quasiconformal mappings are their natural generalizations. Geometrically, a quasiconformal mapping maps infinitesimal balls to infinitesimal ellipsoids with uniformly controlled eccentricity in space. This suggests that it is reasonable to use measures of angles to characterize quasiconformal mappings. In the first part of this dissertation, a measure of angle called topological angle is used to characterize quasiconformal mappings in higher dimensional Euclidean space, generalizing a similar result in the plane.

The second part of the dissertation deals with some important conformal invariants in the study of geometric function theory, such as quasiextremal distance (or QED) constant and extremal length. QED domains are a class of domains closely connected to quasiconformal mapping theory. The QED constant is a naturally defined conformal invariant on a domain whose values reflect the geometry of a domain. In this part, a sharp upper bound for the QED constant in terms of boundary dilatation is obtained for a finitely connected domain on the complex plane. Furthermore, the extremal length (or its reciprocal called modulus) of a curve family plays an essential role in studying quasiconformal mappings. In the second part of this dissertation, a decomposition result is established for the extremal length of a curve family in a finitely connected domain. This can be regarded as a natural generalization of subadditivity of extremal length. It is also a key ingredient in obtaining the sharp upper bound for the QED constant mentioned above.

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Contents

Chapter 1

Introduction

1.1 Quasiconformal mappings

Quasiconformal mappings are natural generalization of conformal mappings. They are used to solve problems on which conformal mappings turn out to be too restrictive. Quasiconformal mappings in the plane were first introduced by H.Grötzsch in 1920's. Then important results were developed by O.Teichmüller and L.V.Ahlfors in 1930's [3]. The systematic study of quasiconformal mappings in $\mathbb{R}^n$ was begun by F.W.Gehring [8] and J.Väisälä [14] in 1960's. Since then, its generalization has been actively studied, see [12] [11] [4], and [5].

While a conformal map preserves both angles and shape of infinitesimal small figures, a quasiconformal mapping maps infinitesimal balls to infinitesimal ellipsoids with uniformly controlled eccentricity in space. Quasiconformal mappings are characterized by the property that there exists a constant greater than 1 such that the infinitesimally small spheres are mapped onto infinitesimally small ellipsoids with the ratio of the largest "semiaxis" to the smallest one bounded from above by the constant. This naturally gives rise to the metric definition. We thus define linear dilatation first.

Let $f : \Omega \to \Omega'$ be a homeomorphism between domains in the Euclidean space $\mathbb{R}^n$, the linear dilatation of f at x is defined as:

$$H(x, f) = \limsup_{r \to 0} \frac{L(x, f, r)}{l(x, f, r)},$$

where

$$L(x, f, r) = \max_{|y-x|=r} |f(y) - f(x)|,$$
$$l(x, f, r) = \min_{|y-x|=r} |f(y) - f(x)|.$$

Definition 1.1. (Metric definition) *A homeomorphism $f : \Omega \to \Omega'$ is a quasiconformal mapping, where Ω, Ω' are domains in $\mathbb{R}^n$, if and only if $H(x, f)$ is bounded.*

The metric definition of quasiconformal mapping is classical and simple, but it is hard to deduce basic facts of quasiconformal mappings simply by the boundedness of $H(x, f)$. There are two other well-known equivalent definitions of quasiconformality, namely, geometric definition and analytic definition. The geometric definition is based on the concept of conformal modulus of a curve family. It provides the most direct approach to a large part of the quasiconformal mapping theory. Three dilatation parameters of homeomorphism f of a domain Ω are needed in order to give the geometric definition. The inner and outer dilatations of f are defined by

$$K_I(f) = \sup \frac{M(f(\Gamma))}{M(\Gamma)}, K_O(f) = \sup \frac{M(\Gamma)}{M(f(\Gamma))},$$

where the supremum is taken over all curve families Γ in Ω. The quantities $M(\Gamma), M(f(\Gamma))$ are the conformal moduli of the curve families Γ and $f(\Gamma)$, respectively. The reader is referred to section 1.2 for the definition of modulus. The maximal dilatation of f is

$$K(f) = \max\{K_I(f), K_O(f)\}.$$

Definition 1.2. (Geometric definition) *A homeomorphism $f : \Omega \to \Omega'$ is a quasiconformal mapping if its maximal dilatation $K(f) < \infty$. If $K(f) \leq K$ then f is said to be K-QC. Equivalently, f is K-QC if and only if*

$$\frac{M(\Gamma)}{K} \leq M(f(\Gamma)) \leq KM(\Gamma)$$

for every curve family Γ in Ω.

The following properties follow from the definition immediately:

1. If f is a K-quasiconformal mapping, then its inverse f^{-1} is also a K-quasiconformal mapping.

2. The composite of a K_1 quasiconformal mapping and a K_2 quasiconformal mapping is a K_1K_2 quasiconformal mapping.

An analytic definition for quasiconformal mappings was first considered by Lavrentiev in connection with elliptic systems of partial differential equations.

Definition 1.3. (Analytic definition) *A homeomorphism $f : \Omega \to \Omega'$ between domains in $\mathbb{R}^n$, $n \geq 2$, is said to be K-quasiconformal if the following conditions are satisfied:*

1. *The first distributional partial derivatives of f are locally in the Lebesgue space L^n.*

2. *The formal differential matrix $Df = (\partial_i f_j)$ satisfies*

$$\sup_{h\in\mathbb{R}^n, |h|\leq 1} |Df(x)(h)|^n \leq K|detDf(x)|$$

for almost every $x \in \Omega$.

In the alternative analytic definition given by J.Väisälä [16], absolute continuity and differentiability for almost all points are assumed and the boundedness of the dilatation quotient is required almost everywhere. Although the analytic definition given above is different from that by J.Väisälä, it follows indirectly that f is a.e. differentiable [13]. The reader is referred to section 2.9 in [13] for details.

1.2 The modulus of a curve family

The modulus of curve families is the main tool when studying the properties of quasiconformal mappings. Moreover, the extremal length method can also be applied to problems of conformal mappings and Teichmüller spaces [10].

1.2.1 The definition of modulus

Definition 1.4. *Let Γ be a curve family in $\mathbb{R}^n$ and denote by $F(\Gamma)$ the set of all nonnegative Borel functions $\rho : \mathbb{R}^n \to \mathbb{R}$ such that*

$$\int_\gamma \rho ds \geq 1$$

for every locally rectifiable curve $\gamma \in \Gamma$. For each $p \geq 1$, set

$$M_p(\Gamma) = \inf_{\rho \in F(\Gamma)} \int_{R^n} \rho^p dm$$

$M_p(\Gamma)$ is called the p-modulus of Γ.

For the case $p = n$, we call it the modulus of Γ and denote it $M(\Gamma)$. In the literature, one often uses the extremal length of Γ. It is denoted as $\lambda(\Gamma)$ and is simply equal to $M(\Gamma)^{\frac{1}{1-n}}$. Extremal length is a sort of average minimal length of curves in a curve family and the set of fewer and longer curves has larger extremal length. It is invariant under conformal mappings and quasi-invariant under quasiconformal mappings.

1.2.2 Properties of modulus

For a given curve family Γ, it is usually very difficult to compute $M(\Gamma)$ directly. We usually find the estimate of modulus instead. After listing several basic properties of the modulus, we list several examples about the estimation of modulus in some special domains, which can be used readily.

Theorem 1.5. [[16], Theorem 6.2] *M is an outer measure in the space of all curves in $\bar{\mathbb{R}}^n$:*

1. *$M(\emptyset) = 0$;*
2. *(Monotonicity) $\Gamma_1 \subset \Gamma_2$ implies $M_(\Gamma_1) < M(\Gamma_2)$.*
3. *(Subadditivity) $M(\cup_{i=1}^{\infty}\Gamma_i) \leq \sum_{i=1}^{\infty} M(\Gamma_i)$.*

Theorem 1.6. [Symmetry Principle] *For any γ let $\bar{\gamma}$ be its reflection in the real axis, and let γ^+ be obtained by reflecting the part below the real axis and retaining the part above it $(\gamma \cup \bar{\gamma}) = \gamma^+ \cup (\gamma^+)^-$. If $\Gamma = \bar{\Gamma}$, then*

$$\lambda(\Gamma) = \frac{1}{2}\lambda(\Gamma^+).$$

Definition 1.7. *Let Γ_1 and Γ_2 be curve families in $\bar{\mathbb{R}}^n$. We say that Γ_2 is minorized by Γ_1 and denote $\Gamma_2 > \Gamma_1$ if every $\gamma \in \Gamma_2$ has a subcurve which belongs to Γ_1.*

Theorem 1.8. [[16], Theorem 6.4] *If $\Gamma_1 < \Gamma_2$, then $M(\Gamma_1) \geq M(\Gamma_2)$.*

Example 1.9. [The module of a rectangle] Γ *is the set of all arcs in a closed rectangle* R *with length* a *and width* b *which joins a pair of opposite sides in length* b*, then*

$$\lambda(\Gamma) = \frac{a}{b}.$$

Example 1.10. [The module of an annulus] *Let* $G = r_1 \leq |z| \leq r_2$ *be a doubly connected region in the finite plane with* c_1 *the bounded,* c_2 *the unbounded component of the complement. Let* Γ *be the family of closed curves in* G *which separate* c_1 *and* c_2*. Then the module*

$$\lambda(\Gamma) = \frac{1}{2\pi} \log(\frac{r_2}{r_1}).$$

Example 1.11. [[10] Teichimüller ring] *Let* $R = R(E, F)$ *denote a ring domain with the property that its complement has exactly two components* E *and* F*. The Teichimüller ring is the ring domain defined by* $R_T(t) = R([-1, 0], [t, \infty])$ *for some* $t > 0$*. Let the Teichimüller function* $\Psi(t) : (0, \infty] \to (1, \infty]$ *be defined by*

$$\frac{2\pi}{\log \Psi(t)} = M(\triangle([-1, 0], [t, \infty]; R_T(t))).$$

It can be shown that $\Psi(t)$ *is continuous, non-decreasing and that* $\lim_{t\to 0} \Psi(t) = 1$ *and* $\Psi(\infty) = \infty$*. Moreover, it is known that* $\lim_{t\to\infty} \frac{\Psi(t)}{t} = 16$.

1.3 QED Domain and QED Constant

QED domains were first introduced by Gehring and Martio in the study of quasiconformal mappings [9].

Definition 1.12. *A domain* Ω *in* $\bar{\mathbb{R}}^n$ *is said to be an* $M - QED$ *domain, with* $1 \leq M < \infty$*, if for each pair of disjoint continua* A *and* B *in* Ω*,*

$$mod(A, B; \mathbb{R}^n) \leq M mod(A, B; \Omega).$$

On the complex plane, a quasicircle is a Jordan curve that is the image of a circle under a quasiconformal mapping of the plane onto itself. We say that $\Omega \subset \bar{\mathbb{C}}$ is a K-quasidisk if Ω is the image of an open disk or half plane under a K-quasiconformal self mapping of $\bar{\mathbb{C}}$ and that its boundary is called a K-quasicircle. A domain $\Omega \subset \bar{\mathbb{C}}$ is said to be a K-quasicircle domain if each component of $\partial\Omega$ is either a point or a k-quasicircle [9].

The following Theorem implies that QED domains are closely related to uniform domains, quasisphere domains and linearly locally connected domains [9].

Theorem 1.13. *If Ω is a finitely connected domain in $\mathbb{C}$, then the following conditions are equivalent.*

1. *Ω is a QED domain.*
2. *Ω is linearly locally connected.*
3. *Ω is a quasicircle domain.*
4. *Ω is uniform.*

By the symmetry principle for moduli of curve families, it follows that if domain Ω is a ball or a half space, then Ω is 2-QED. To better understand the geometry of a QED domain, we define the QED constant as follows [17].

Definition 1.14. [QED constant] *For domain Ω, its quasi-extremal distance constant $M(\Omega)$ is defined as:*

$$M(\Omega) = \sup_{A,B\in\bar{\Omega}} \frac{mod(A,B;\mathbb{R}^n)}{mod(A,B;\Omega)},$$

where the supremum is taken over all pairs of disjoint continua A and B in $\bar{\Omega}$ such that $mod(A,B;\mathbb{C})$ and $mod(A,B;\Omega)$ are not simultaneously 0 or ∞.

The properties of QED constant is well discussed in [17]. For example, complete geometric characterizations are given for 1-QED domains and 2-QED domains in $\mathbb{R}^n$. Some sharp lower and upper bounds of $M(\Omega)$ for different kinds of domains are also derived. In general, the value of $M(\Omega)$ reflects the geometry of a domain Ω in some sense.

1.4 Outline and summary of results

The thesis is organized in three parts. Chapter 2 concerns the characterization of quasiconformality using the measure of topological angle which generalizes a corresponding result in the plane [2] into higher dimensional Euclidean space. Similar results were obtained by Agard in [1], but this part of the thesis was done independently of [1]. In particular, quasisymmetry is used to estimate the lower bound for topological angles under quasiconformal mappings. In chapter 3, a QED constant called QED reflection constant is defined. Several basic properties are discussed and it is shown that the QED reflection constant can only be obtained by a pair of nondegenerate continua for a smooth Jordan domain other than a disk or half plane. The last two chapters are devoted to establishing a sharp upper bound for the QED constant $M(\Omega)$ of a finitely connected planar domain in terms of local boundary dilatation of its boundary components, which is a generalization of a result in [6] about Jordan domains. In particular, one of the lemmas, decomposition of extremal length on finitely connected domain, is formulated in Chapter 4 independently since it gives rise to its own applications and interests.

Chapter 2

Characterization of quasiconformal mappings using topological angle

2.1 Introduction

Recall that there are three equivalent definitions of quasiconformality. They all involve selecting a certain property of conformal mappings and then studying the class of homeomorphisms which enjoy a slightly weakened form of this property [2]. The fact that a conformal mapping is an angle preserving diffeomorphism could be generalized to the class of quasiconformal mappings. It provides the point of view of "angle preserving" to characterize quasiconformal mappings on the complex plane. This chapter is devoted to generalizing this characterization to higher dimensional Euclidean space.

To circumvent the exceptional set of zero measure where quasiconformal mapping is not differentiable, a form of measure called topological angle is defined in section 2.1. In section 2.2 and 2.3, we study how the measure of topological angle changes under various mappings. In section 2.4, using the local quasisymmetry of a quasiconformal mapping, we estimate lower bound of the topological angle under quasiconformal mappings. Section 2.5 establishes the main theorem about the characterization of quasiconformal mapping using topological angle in $\mathbb{R}^3$. And finally, we deduce similar results in Euclidean space $\mathbb{R}^{n+1}$ by providing some parallel theorems and lemmas.

2.2 Definition of topological angle

It is known that a conformal mapping preserves the measure of angles and that quasiconformal mappings are natural generalizations of conformal mappings. It is conceivable that we could define some measure of angle to characterize quasiconformal mappings and we may also obtain some interesting results in studying the behavior of topological angles under quasiconformal mappings. Before that, since quasiconformal mapping is only differentiable almost everywhere, we need to define some kind of measure of an angle which can be measured under quasiconformal mappings.

With the inspiration of trigonometric function of angles on the complex plane, the topological angle is defined as follows (see [2]).

Definition 2.1. *Let γ_1 and γ_2 be two arcs in $\mathbb{R}^n$. We say γ_1 and γ_2 form a topological angle at a point x_0 if both γ_1 and γ_2 have x_0 as an end-point and if x_0 is the only point γ_1 and γ_2 have in common in its neighborhood. Define the inner measure $A(\gamma_1, \gamma_2)$ of this topological angle as follows:*

$$A(\gamma_1, \gamma_2) = \lim_{x_1, x_2 \to x_0} \inf 2 \arcsin\left(\frac{|x_1 - x_2|}{|x_1 - x_0| + |x_2 - x_0|}\right),\ x_i \in \gamma_i. \tag{2.1}$$

It is easy to check that $0 \leq A(\gamma_1, \gamma_2) \leq \pi$, and that $A(f(\gamma_1), f(\gamma_2)) = A(\gamma_1, \gamma_2)$ when f is a similarity mapping or a reflection in a plane.

2.3 Topological angle under linear mappings

We quantify topological angle under two linear mappings in this section.

Lemma 2.2. *Suppose that f is a homeomorphism of a neighborhood U of the origin, that*

$$f(x) = x + o(|x|)$$

near the origin, and that γ_1 and γ_2 are two arcs in U which form a topological angle at the origin. Then $f(\gamma_1)$ and $f(\gamma_2)$ form a topological angle and

$$A(f(\gamma_1), f(\gamma_2)) = A(\gamma_1, \gamma_2)$$

Proof. Given that $0 < \epsilon < 1$, we may choose $\delta > 0$ such that $|f(x) - x| \leq \epsilon |x|$ for $|x| < \delta$. This implies $|f(x)| \geq (1-\epsilon)|x|$. Choose x_i in γ_i so that $0 < |x_i| < \delta,\ i = 1, 2$.
Then

$$\begin{aligned}
\frac{|f(x_1) - f(x_2)|}{|f(x_1)| + |f(x_2)|} &\leq \frac{|f(x_1) - x_1| + |x_1 - x_2| + |f(x_2) - x_2|}{(1-\epsilon)|x_1| + (1-\epsilon)|x_2|} \\
&\leq \frac{\epsilon(|x_1| + |x_2|) + |x_1 - x_2|}{(1-\epsilon)(|x_1| + |x_2|)} \\
&= \frac{|x_1 - x_2|}{|x_1| + |x_2|} + \frac{\epsilon}{1-\epsilon}\frac{|x_1 - x_2|}{|x_1| + |x_2|} + \frac{\epsilon}{1-\epsilon} \\
&\leq \frac{|x_1 - x_2|}{|x_1| + |x_2|} + \frac{2\epsilon}{1-\epsilon}.
\end{aligned}$$

It follows that

$$\sin\frac{1}{2}A(f(\gamma_1), f(\gamma_2)) \leq \frac{|f(x_1) - f(x_2)|}{|f(x_1)| + |f(x_2)|} \leq \frac{|x_1 - x_2|}{|x_1| + |x_2|} + \frac{2\epsilon}{1-\epsilon}.$$

Letting $x_1, x_2 \to 0$ as in Definition (2.1) yields

$$\sin\frac{1}{2}A(f(\gamma_1), f(\gamma_2)) \leq \sin\frac{1}{2}A(\gamma_1, \gamma_2) + \frac{2\epsilon}{1-\epsilon}.$$

Since ϵ is arbitrary, we obtain

$$A(f(\gamma_1), f(\gamma_2)) \leq A(\gamma_1, \gamma_2).$$

The reverse inequality follows by symmetry.

$\square$

Lemma 2.3. *Suppose that $f(x) = Ax$, and that A is a non-degenerate $n \times n$ matrix which can be diagonalized in the form :*

$$\begin{pmatrix} a_1 & 0 & \cdots & 0 & 0 \\ 0 & a_2 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & a_{n-1} & 0 \\ 0 & 0 & \cdots & 0 & a_n \end{pmatrix}$$

where $a_1 \geq a_2 \geq \cdots \geq a_n > 0$. If

$$K \geq \frac{a_1}{a_n},$$

then

$$KA(\gamma_1, \gamma_2) \geq A(f(\gamma_1), f(\gamma_2)) \geq \frac{1}{K} A(\gamma_1, \gamma_2) \tag{2.2}$$

for each pair of arcs γ_1 and γ_2 which form a topological angle at the origin.

Conversely, if

$$A(f(\gamma_1), f(\gamma_2)) \geq \frac{1}{K} A(\gamma_1, \gamma_2)$$

holds for each pair of segments γ_1 and γ_2 which form an angle at the origin, then

$$K \geq \frac{a_1}{a_n}.$$

Proof. Fix any pair of arcs γ_1 and γ_2 which form a topological angle at the origin. Choose $x = (x_1, \cdots, x_n)$ in γ_1 and $y = (y_1, \cdots, y_n)$ in γ_2 so that $x, y \neq 0$, and set

$$\varphi = \arcsin(\frac{|x - y|}{|x| + |y|}),\ \varphi' = \arcsin(\frac{|f(x) - f(y)|}{|f(x)| + |f(y)|}).$$

Then

$$
\begin{aligned}
(\tan\varphi')^2 &= \frac{|f(x)-f(y)|^2}{(|f(x)|+|f(y)|)^2-|f(x)-f(y)|^2} \\
&= \frac{\sum_{i=1}^n a_i^2(x_i-y_i)^2}{(\sqrt{\sum_{i=1}^n a_i^2x_i^2}+\sqrt{\sum_{i=1}^n a_i^2y_i^2})^2-(\sqrt{\sum_{i=1}^n a_i^2(x_i-y_i)^2})^2} \\
&= \frac{\sum_{i=1}^n a_i^2(x_i-y_i)^2}{2\sum_{i=1}^n a_i^2x_iy_i+2\sqrt{\sum_{i=1}^n a_i^2x_i^2}\sqrt{\sum_{i=1}^n a_i^2y_i^2}} \\
&\geq \frac{a_n^2}{a_1^2}\frac{\sum_{i=1}^n(x_i-y_i)^2}{2\sum_{i=1}^n x_iy_i+2\sqrt{\sum_{i=1}^n x_i^2}\sqrt{\sum_{i=1}^n y_i^2}} \\
&= \frac{a_n^2}{a_1^2}(\tan\varphi)^2.
\end{aligned}
$$

Hence

$$\varphi' \geq \arctan(\frac{a_n}{a_1}\tan\varphi) \geq \frac{a_n}{a_1}\varphi.$$

This yields by the definition of $A(\gamma_1,\gamma_2)$ in (2.1),

$$2\arcsin(\frac{|f(x)-f(y)|}{|f(x)|+|f(y)|}) \geq \frac{2a_n}{a_1}\arcsin(\frac{|x-y|}{|x|+|y|}) \geq \frac{a_n}{a_1}A(\gamma_1,\gamma_2).$$

Since this is true for all $x\in\gamma_1$ and $y\in\gamma_2$ with $x,y\neq 0$, taking the infimum limit as $x\to 0$ and $y\to 0$ as in the definition (2.1) yields

$$A(f(\gamma_1),f(\gamma_2)) \geq \frac{a_n}{a_1}A(\gamma_1,\gamma_2).$$

Thus, if $K\geq\frac{a_1}{a_n}$,

$$A(f(\gamma_1),f(\gamma_2)) \geq \frac{a_n}{a_1}A(\gamma_1,\gamma_2) \geq \frac{1}{K}A(\gamma_1,\gamma_2)$$

which gives the second inequality in (2.2).

For the rest part in (2.2), we note that

$$\begin{aligned}
(\tan\varphi')^2 &= \frac{\sum_{i=1}^n a_i^2(x_i-y_i)^2}{2\sum_{i=1}^n a_i^2 x_i y_i + 2\sqrt{\sum_{i=1}^n a_i^2 x_i^2}\sqrt{\sum_{i=1}^n a_i^2 y_i^2}} \\
&\le \frac{a_1^2}{a_n^2}\frac{\sum_{i=1}^n (x_i-y_i)^2}{2\sum_{i=1}^n x_i y_i + 2\sqrt{\sum_{i=1}^n x_i^2}\sqrt{\sum_{i=1}^n y_i^2}} \\
&= \frac{a_1^2}{a_n^2}(\tan\varphi)^2
\end{aligned}$$

and that

$$\varphi' \le \arctan(\frac{a_1}{a_n}\tan\varphi) \le \frac{a_1}{a_n}\varphi.$$

Similarly, the following inequality holds,

$$A(f(\gamma_1), f(\gamma_2)) \le \frac{a_1}{a_n}A(\gamma_1,\gamma_2).$$

Thus if $K \ge \frac{a_1}{a_n}$,

$$A(f(\gamma_1), f(\gamma_2)) \le KA(\gamma_1,\gamma_2).$$

Conversely, let $A(f(\gamma_1), f(\gamma_2)) \ge \frac{1}{K}A(\gamma_1,\gamma_2)$ holds for each pair of segments γ_1 and γ_2 which form an angle at the origin. Fix $\theta > 0$ and let γ_1,γ_2 be the line segments connecting the origin and $\bar{x} = (\cos\theta, 0, \cdots, 0, \sin\theta)$, $\bar{y} = (\cos\theta, 0, \cdots, 0, -\sin\theta)$, respectively.
Then

$$\begin{aligned}
A(f(\gamma_1), f(\gamma_2)) &= 2\arcsin\frac{|f(\bar{x})-f(\bar{y})|}{|f(\bar{x})|+|f(\bar{y})|} \\
&= 2\arcsin\frac{2a_n\sin\theta}{2\sqrt{(a_1\cos\theta)^2+(a_n\sin\theta)^2}} \\
&= 2\arctan(\frac{a_n}{a_1}\tan\theta) \\
&\longrightarrow \frac{a_n}{a_1}A(\gamma_1,\gamma_2) \qquad \text{as } \theta\to 0,
\end{aligned}$$

and hence $A(f(\gamma_1), f(\gamma_2)) \ge \frac{1}{K}A(\gamma_1,\gamma_2)$ implies $\frac{a_n}{a_1}A(\gamma_1,\gamma_2) \ge \frac{1}{K}A(\gamma_1,\gamma_2)$. Therefore, $K \ge \frac{a_1}{a_n}$ holds. $\square$

2.4 Topological angle under differentiable homeomorphism

Using Lemma 2.2 and Lemma 2.3, we study the behavior of topological angles under differentiable homeomorphisms.

Theorem 2.4. *Suppose that f is a homeomorphism on a domain G, that f has a differential at x_0 and that*

$$\max_{\theta} |D_\theta f(x_0)| > 0 \tag{2.3}$$

where $D_\theta f$ denotes the directional derivative of f. If

$$\max_{\theta} |D_\theta f(x_0)|^n \leq K|J(x_0)| \tag{2.4}$$

where J denotes the Jacobian of f, then

$$A(f(\gamma_1), f(\gamma_2)) \geq \frac{1}{K} A(\gamma_1, \gamma_2) \tag{2.5}$$

for each pair of arcs γ_1 and γ_2 which form a topological angle in G at x_0.

Conversely, if (2.5) *holds for each pair of segments γ_1 and γ_2 which form an angle in G at x_0, then the following inequality holds*

$$\max_{\theta} |D_\theta f(x_0)|^n \leq K^{n-1}|J(x_0)|. \tag{2.6}$$

Proof. By performing preliminary similarity mappings, we may assume that $x_0 = f(x_0) = 0$ and that, since $f(x)$ has a differential at $x_0 = 0$,

$$f(x) = f(x_0) + J(x_0)(x - x_0) + \circ(|x - x_0|)$$

which is

$$f(x) = J(x_0)x + \circ(|x|),$$

where $J(x_0)$ is an $n \times n$ matrix and $x = (x_1, x_2, \cdots, x_n)^T$. Suppose inequality (2.4) holds. Then (2.3) implies that $|J(x_0)| \neq 0$. Hence the matrix $J(x_0)$ can be diagonalized in the form as in Lemma 2.3. Thus by performing preliminary similarity mappings and reflections again, we may assume, near $x_0 = 0$,

$$f(x) = g(x) + \circ|g(x)| = Ax + \circ(|Ax|),$$

where $g(x) = Ax$ and A is as in Lemma 2.3. Note that,

$$|J(x_0)| = \det(A) = \prod_{i=1}^{n} a_i \tag{2.7}$$

$$|D_\theta f(x_0)| = |A\bar{x}| = |\sum_{i=1}^{n} a_i \bar{x}_i|.$$

Where $\bar{x} = (\bar{x}_1, \bar{x}_2, \cdots, \bar{x}_n)^T$ and $|\bar{x}| = 1$. Set $\bar{x} = (1, 0, \cdots, 0)$, we can have

$$\max_\theta |D_\theta f(x_0)| = a_1. \tag{2.8}$$

By (2.7) and (2.8), inequality (2.4) implies that $a_1^n \leq K \prod_{i=1}^n a_i$, and since $a_1 \geq a_2 \geq \cdots \geq a_n$, $\frac{a_1}{a_n} \leq K$.

For any pair of arcs γ_1 and γ_2 in G which form a topological angle at the origin. Notice that topological angle is unchanged under similarity mappings. By lemma 2.2, $A(f(\gamma_1), f(\gamma_2)) = A(g(\gamma_1), g(\gamma_2))$. By lemma 2.3, inequality $K \geq \frac{a_1}{a_n}$ implies that $A(g(\gamma_1), g(\gamma_2)) \geq \frac{1}{K} A(\gamma_1, \gamma_2)$. Thus,

$$A(f(\gamma_1), f(\gamma_2)) \geq \frac{1}{K} A(\gamma_1, \gamma_2),$$

for each pair of arcs γ_1 and γ_2 which form a topological angle in G at x_0.

For the other direction, first suppose that $|J(x_0)| \neq 0, x_0 = f(x_0) = 0$ and that $f(x) = g(x) + \circ|g(x)| = Ax + \circ(|Ax|)$ holds, where function $g(x)$ and matrix A are defined as above. Fix any pair of segments γ_1 and γ_2 which form an angle in G at the origin. By Lemma 2.2, $A(f(\gamma_1), f(\gamma_2)) = A(g(\gamma_1), g(\gamma_2))$. Thus (2.5) yields

$$A(g(\gamma_1), g(\gamma_2)) \geq \frac{1}{K} A(\gamma_1, \gamma_2).$$

By Lemma 2.3, it implies that $\frac{a_1}{a_n} \leq K$. Noticing that $a_1 \geq a_2 \geq \cdots \geq a_n$, then

$$a_1 \leq Ka_n, a_1 \leq Ka_2, a_1 \leq Ka_3, \cdots, a_1 \leq Ka_{n-1}.$$

Multiplying these inequalities, we get $a_1^{n-1} \leq K^{n-1} \prod_{i=2}^{n} a_i$ and

$$a_1^n \leq K^{n-1} \prod_{i=1}^{n} a_i.$$

Thus

$$\max_{\theta} |D_\theta f(x_0)|^n \leq K^{n-1} |J(x_0)|.$$

Finally, (2.3) and (2.5) imply that $|J(x_0)| \neq 0$. If not, suppose $|J(x_0)| = 0$. By performing preliminary similarity mappings, we may assume $x_0 = f(x_0) = 0$ and that, near $x_0 = 0$,

$$f(x) = Bx + \circ(|Bx|).$$

Where B is in the form

$$B = \begin{pmatrix} b_1 & 0 & \cdots & 0 & 0 \\ 0 & b_2 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & b_{n-1} & 0 \\ 0 & 0 & \cdots & 0 & 0 \end{pmatrix}.$$

For $r > 0$, let γ_1, γ_2 denote the segments joining the origin to $(r, 0, \cdots, 0, r)$ and $(r, 0, \cdots, 0, -r)$. Then segments γ_1 and γ_2 lie in G for small r, we can see that

$$A(\gamma_1, \gamma_2) = \frac{\pi}{2} > A(f(\gamma_1), f(\gamma_2)) = 0.$$

We get a contradiction. It completes the proof of the reverse direction. $\square$

2.5 Lower bound of topological angles under quasiconformal mappings

2.5.1 Quasisymmetric mappings

The definition of quasisymmetric mapping is a stronger and global version of quasiconformal mapping. A quasisymmetric mapping between reasonable spaces has many strong properties such as it is also Hölder continuous, inverse mappings are also quasisymmetric etc. Much of the classical quasiconformal theory can be done by exploiting the definition of quasisymmetry. In particular, it is well known that a quasiconformal mapping $f : \mathbb{R}^n \to \mathbb{R}^n, n \geq 2$, is also quasisymmetric. It is also known that a quasiconformal mapping of a domain in $\mathbb{R}^n$ is locally quasisymmetric [**?**].

Definition 2.5. *A homeomorphism $f : A \to B$ of domains $A, B \subset \mathbb{R}^n$ is called quasisymmetric if there is a homeomorphism $\eta : [0,\infty) \to [0,\infty)$ so that*

$$|x-a| \leq t|x-b| \text{ implies } |f(x)-f(a)| \leq \eta(t)|f(x)-f(b)|$$

for each $t > 0$ and for each triple points $x, a, b \in A$.

2.5.2 Lower bound of angles under quasiconformal mappings

By the equivalence between quasisymmetry and quasiconformality in $\mathbb{R}^n$ for $n \geq 2$, we can derive the following estimate of the measure of topological angles under quasiconformal mappings.

Theorem 2.6. *Suppose that $f : \bar{\mathbb{R}}^n \to \bar{\mathbb{R}}^n$ is a K-quasiconformal mapping with $f(\infty) = \infty$. Then for each triple of distinct finite points x_1, x_0, x_2,*

$$\sin\beta \geq \min\left\{\frac{1}{2}c^{r^2-1}\eta(\frac{1}{\sin\alpha})^{-r^2}, \frac{1}{2}\eta(\frac{1}{\sin\alpha})^{-1}\right\},$$

where

$$\alpha = \arcsin(\frac{|x_1 - x_2|}{|x_1 - x_0| + |x_2 - x_0|})$$

$$\beta = \arcsin(\frac{|f(x_1) - f(x_2)|}{|f(x_1) - f(x_0)| + |f(x_2) - f(x_0)|}),$$

η is a homeomorphism $\eta : [0, \infty) \longmapsto [0, \infty)$, $c \geq 1$ and $r \in (0, 1]$ which depend only on f.

Proof. Since $f : \mathbb{R}^n \to \mathbb{R}^n$ is a K-quasiconformal mapping, f is quasisymmetrc. Fix a triple of distinct finite points x_1, x_0, $x_2 \in \mathbb{R}^n$. Let $\alpha = \arcsin(\frac{|x_1-x_2|}{|x_1-x_0|+|x_2-x_0|})$, $\beta = \arcsin(\frac{|f(x_1)-f(x_2)|}{|f(x_1)-f(x_0)|+|f(x_2)-f(x_0)|})$. By the definition of quasisymmetric mapping, there is a homeomorphism $\eta : [0, \infty) \to [0, \infty)$ such that if

$$t_1 = \frac{|x_1 - x_0|}{|x_1 - x_2|}, t_2 = \frac{|x_2 - x_0|}{|x_2 - x_1|},$$

then we have

$$\frac{|f(x_1) - f(x_0)|}{|f(x_1) - f(x_2)|} \leq \eta(t_1), \frac{|f(x_2) - f(x_0)|}{|f(x_1) - f(x_2)|} \leq \eta(t_2).$$

Since for quasisymmetric mappings on $\mathbb{R}^n$, η is of the form [**?**]

$$\eta(t) = c \max\left\{t^r, t^{\frac{1}{r}}\right\},$$

where $c \geq 1$ and $r \in (0, 1]$ which depends only on f. Assume that $t_2 \leq t_1$. There are three cases to be considered.

Case 1. If $0 < t_1 < 1, 0 < t_2 < 1$:

$$\eta(t_1) + \eta(t_2) = ct_1^r + ct_2^r = ct_1^r(1 + (\frac{t_2}{t_1})^r) \leq 2ct_1^r \leq 2c(t_1 + t_2)^r.$$

subcase 1, if $0 < t_1 + t_2 < 1$,

$$\eta(t_1) + \eta(t_2) \leq 2c(t_1 + t_2)^r = 2\eta(t_1 + t_2).$$

subcase 2, if $t_1 + t_2 \geq 1$,

$$\eta(t_1) + \eta(t_2) \leq 2c(t_1 + t_2)^r = 2c(\frac{1}{c}\eta(t_1 + t_2))^{r^2} = 2c^{1-r^2}(\eta(t_1 + t_2))^{r^2}.$$

Case 2. If $0 < t_2 < 1$, $t_1 \geq 1$:

$$\eta(t_1) + \eta(t_2) = ct_1^{\frac{1}{r}} + ct_2^r \leq ct_1^{\frac{1}{r}} + c \leq 2ct_1^{\frac{1}{r}} \leq 2c(t_1 + t_2)^{\frac{1}{r}} = 2\eta(t_1 + t_2).$$

Case 3. If $t_1 \geq 1$, $t_2 \geq 1$:

$$\eta(t_1) + \eta(t_2) = ct_1^{\frac{1}{r}} + ct_2^{\frac{1}{r}} \leq c(t_1 + t_2)^{\frac{1}{r}} = \eta(t_1 + t_2).$$

Summarizing the three cases, an upper bound of $\eta(t_1) + \eta(t_2)$ is given by maximum $\left\{2c^{1-r^2}\eta(\frac{1}{\sin\alpha})^{r^2}, 2\eta(\frac{1}{\sin\alpha})\right\}$. Thus,

$$\begin{aligned}
\frac{1}{\sin\beta} &= \frac{|f(x_1) - f(x_0)| + |f(x_2) - f(x_0)|}{|f(x_1) - f(x_2)|} \\
&\leq \eta(t_1) + \eta(t_2) \\
&\leq \max\left\{2c^{1-r^2}\eta(t_1 + t_2)^{r^2}, 2\eta(t_1 + t_2)\right\} \\
&= \max\left\{2c^{1-r^2}\eta(\frac{1}{\sin\alpha})^{r^2}, 2\eta(\frac{1}{\sin\alpha})\right\}.
\end{aligned}$$

Noticing

$$\frac{1}{sin\alpha} = \frac{|x_1 - x_0| + |x_2 - x_0|}{|x_1 - x_2|} = t_1 + t_2,$$

one concludes that

$$\sin\beta \geq \min\left\{\frac{1}{2}c^{r^2-1}\eta(\frac{1}{\sin\alpha})^{-r^2}, \frac{1}{2}\eta(\frac{1}{\sin\alpha})^{-1}\right\}.$$

□

2.6 Characterization of quasiconformal mappings by topological angles in $\mathbb{R}^3$

2.6.1 Preliminary

Before proceeding to the main result, we present the definition and a lemma on the regular Caratheodory outer measure.

Definition 2.7. *Given a set $E \subseteq \mathbb{R}^n$ and $d > 0$, we define*

$$\wedge(E, d) = \inf \sum_{\alpha} diaE_\alpha,$$

where $\{E_\alpha\}$ is any covering of E with $diaE_\alpha \leq d$. Clearly $\wedge(E, d)$ is non-increasing in d, and we may define the regular Caratheodory outer measure:

$$\wedge(E) = \lim_{d \to 0} \wedge(E, d).$$

Lemma 2.8. *[2] If F is a bounded perfect linear set, then for each $\epsilon > 0$ there exists a $\delta > 0$ with the following property: given $0 < t < \delta$, there exist N non-overlapping intervals I_n, with end-points in F and lengths not greater than t, such that*

$$F \subseteq \cup_1^N I_n \quad \textit{and} \quad Nt \leq \wedge(F) + \epsilon.$$

2.6.2 Characterization Theorem

Now we are ready to show how we can characterize quasiconformal mappings by making use of topological angles. We first state the theorem and then separate the proof into several parts.

Theorem 2.9. *Let f be a homeomorphism of domain $G \subseteq \mathbb{R}^3$. If f is K-quasiconformal mapping, $1 \leq K < \infty$, then we have:*

1. *For all ζ_0 in G and for all arcs γ_1 and γ_2 which form an angle in G at ζ_0,*

$$A(f(\gamma_1), f(\gamma_2)) > 0.$$

2. *For almost all ζ_0 in G and for all arcs γ_1 and γ_2 which form an angle in G at ζ_0,*

$$A(f(\gamma_1), f(\gamma_2)) \geq \frac{1}{K} A(\gamma_1, \gamma_2).$$

Conversely, if conditions 1 and 2 are satisfied, then f is K_1-quasiconformal with $K_1 = K^2$.

2.6.3 Derivative of measure

Definition 2.10. *Let A be a Borel set in $U = (0,1) \times (0,1)$ and L be a compact set in $(0,1)$. Let f be a homeomorphism of domain $G \subseteq \mathbb{R}^3$ containing the unit cube. Define a set function in U:*

$$\varphi_L(A) := m(f(A \times L)).$$

It is easy to check that conditions in Lebesgue's Theorem are all satisfied: For all Borel sets A in U, $\varphi_L(A) \geq 0$. Since $A \times L \in$ unit cube C, $m(f(C)) < \infty$ implies $\varphi_L(A) < \infty$. Let $A_1, A_2, \cdots$ be a sequence of disjoint Borel sets in U, then

$$m(f(\cup(A_i \times L))) = \sum_i m(f(A_i \times L))$$

i.e.

$$\varphi_L(\cup A_i)) = \sum_i \varphi_L(A_i).$$

Furthermore, by Lebesgue's Theorem, the set function φ has a finite derivative in U almost everywhere. We may assume φ has a derivative at the point (x,y) and denote by A_n the closed rectangles $\{(x',y')|x - \frac{1}{n} \leq x' \leq x + \frac{1}{n}, y - \frac{1}{n} \leq y' \leq y + \frac{1}{n}\}$ $n = 1, 2, \cdots$. Then,

$$\begin{aligned} \varphi'_L(x,y) &= \lim_{n\to\infty} \frac{\varphi_L(A_n)}{m(A_n)} \\ &= \lim_{n\to\infty} \frac{m(f(A_n \times L))}{m(A_n)}. \end{aligned} \tag{2.9}$$

Lemma 2.11. *Let $C(x_0, y_0, L)$ denote the cube $\{(x,y,z)|0 \leq x \leq x_0, 0 \leq y \leq y_0, 0 \leq z \leq L\}$. Let $G(x,y,L) = mf(C(x,y,L))$ for $(x,y) \in (0,1) \times (0,1)$. Then*

$$\lim_{t\to 0} \frac{(G(x+t,y+t,L) - G(x+t,y-t,L)) - (G(x-t,y+t,L) - G(x-t,y-t,L))}{t^2}$$

is finite a.e. and equals $4\varphi'_L(x,y)$ a.e. on the domain $(0,1) \times (0,1)$.

Proof. Let ω denote the cube $\{(x', y', z')|x - t \leq x' \leq x + t, y - t \leq y' \leq y + t, 0 \leq z' \leq L\}$, then

$$m(f(\omega)) = (G(x+t, y+t, L) - G(x+t, y-t, L)) - (G(x-t, y+t, L) - G(x-t, y-t, L)).$$

It is equivalent to show that $\lim_{t\to 0} \frac{m(f(\omega))}{t^2}$ is finite a.e. on $(0,1) \times (0,1)$. Note that

$$\lim_{t\to 0} \frac{m(f(\omega))}{t^2} = 4 \lim_{n\to\infty} \frac{m(f(A_n \times L))}{m(A_n)} = 4\varphi'_L(x, y),$$

where φ' is the derivative of the set function and A_n are rectangles as above. Since φ' is finite a.e. , the proof is completed. $\square$

2.6.4 f is ACL on G

Let C be a closed cube in G with each face parallel to a coordinate plane. After performing preliminary similarity mappings, we may assume $C = \{(x, y, z)|0 \leq x \leq 1, 0 \leq y \leq 1, 0 \leq z \leq 1\}$. We shall show that f is absolutely continuous on the interval $0 \leq z \leq 1$ for almost all pairs $\{(x, y)|0 \leq x \leq 1, 0 \leq y \leq 1\}$. By symmetry, we can thus conclude that f is absolutely continuous on almost all horizontal and vertical segments in C.

Let $I(x_0, y_0)$ denote the interval $0 \leq z \leq 1, x = x_0, y = y_0$. Let $C(x_0, y_0)$ denote the cube $\{(x, y, z)|0 \leq x \leq x_0, 0 \leq y \leq y_0, 0 \leq z \leq 1\}$. Fix s with $0 < s < \frac{1}{2}\rho(\partial C, \partial G)$, where ρ denotes the distance between ∂G and ∂C. For each $\zeta_o = (x_0, y_0, z_0)$ in C, assume

$$\zeta_1 = (x_0+s, y_0, z_0), \zeta_2 = (x_0, y_0, z_0-s), \zeta_3 = (x_0, y_0, z_0+s), \zeta_4 = (x_0+s, y_0, z_0+s).$$

and let γ_i be segments jointing ζ_0 and ζ_i, $\quad i = 1, 2, 3, 4.$

Condition 1 in Theorem 2.9 implies

$$A(f(\gamma_1), f(\gamma_2)) > 0, A(f(\gamma_3), f(\gamma_4)) > 0.$$

This implies

$$\limsup_{\zeta_1,\zeta_2\to\zeta_0} \frac{|f(\zeta_1)-f(\zeta_0)|+|f(\zeta_2)-f(\zeta_0)|}{|f(\zeta_1)-f(\zeta_2)|} < \infty,$$

$$\limsup_{\zeta_3,\zeta_4\to\zeta_0} \frac{|f(\zeta_3)-f(\zeta_0)|+|f(\zeta_4)-f(\zeta_0)|}{|f(\zeta_3)-f(\zeta_4)|} < \infty,$$

where $\zeta_i \in \gamma_i, i = 1, \cdots, 4$.

For each pair of integers p, q, with $p > 0, 0 < \frac{1}{q} < s$. Let $H(p,q)$ denote the set of ζ_0 in C, which satisfies

$$|f(\zeta_1)-f(\zeta_0)|+|f(\zeta_2)-f(\zeta_0)| \leq p|f(\zeta_1)-f(\zeta_2)| \tag{2.10}$$

$$|f(\zeta_3)-f(\zeta_0)|+|f(\zeta_4)-f(\zeta_0)| \leq p|f(\zeta_3)-f(\zeta_4)| \tag{2.11}$$

whenever $|\zeta_i - \zeta_0| \leq \frac{1}{q}$, and $\zeta_i \in \gamma_i(\zeta_0)$. Then $H(p,q)$ is compact and

$$C = \cup_{p,q} H(p,q),$$

where the sum is taken over all relevant p, q.

Lemma 2.12. *Suppose that* $0 < x < 1$, $0 < y < 1$ *and* F *is compact in* $I(x,y) \cap H(p,q)$, *then*

$$\wedge(f(F))^3 \leq \frac{6p}{\pi}\varphi'_F(x,y) \wedge (F)^2$$

where φ_F *is the set function defined in Definition* 2.10.

Proof. Let I be a closed subinterval of $I(x,y)$ with end points $\zeta_1(x,y,z_1)$ and $\zeta_2(x,y,z_2)$ in F with $|z_1 - z_2| \leq \frac{\sqrt{2}}{2}\min(\frac{1}{q}, 1-x, x, y)$. Let T be the cone generated by the triangle with vertices ζ_1, ζ_2 and $\zeta_3 = (x + z_1 - z_2, y, z_1)$ rotated along the line through the points ζ_1, ζ_2. We call T the associated cone with the interval I. We will show that

$$|f(\zeta_1) - f(\zeta_2)|^3 \leq \frac{3p}{2\pi} m(f(T)). \tag{2.12}$$

By performing a change of variables, we may assume $f(\zeta_1) = (0,0,l)$, $f(\zeta_2) = (0,0,0)$. Pick some positive number u_0 between 0 and l. Use a horizontal plane to slice $f(T)$ through point $(0,0,u_0)$. The image of I is a curve lying in $f(T)$ with end points $f(\zeta_1)$ and $f(\zeta_2)$, and it intersects the plane at a point, say ω_1. Let ω_2 be the point on the curve formed by the intersection of the plane and the surface of $f(T)$ such that $d(\omega_1, \omega_2)$ is shortest. Notice that ω_2 is either on $f(\alpha)$ or on $f(\beta)$, where α, β are faces rotated by interval $\zeta_1\zeta_3$ and interval $\zeta_2\zeta_3$.

Suppose $\omega_2 \in f(\alpha)$, denote $\eta_i = f^{-1}(\omega_i)$, then $|\eta_i - \zeta_1| \le \frac{1}{q}$. Then by equation (2.10),

$$2(l-u_0) \le |f(\zeta_1) - \omega_1| + |f(\zeta_1) - \omega_2| \le p|\omega_1 - \omega_2|.$$

Similarly, if $\omega_2 \in f(\beta)$, it follows from (2.11) that

$$2u_0 \le |\omega_1| + |\omega_2| \le p|\omega_1 - \omega_2|.$$

By Fubini's Theorem,

$$m(f(T)) \ge \frac{2}{p}\int_o^l \pi(\min\{u_0, l-u_0\})^2 du$$

and hence

$$|f(\zeta_1) - f(\zeta_2)|^3 \le \frac{3p}{2\pi} m(f(T)).$$

Since F is closed. We can write F as the union of F_1 and F_2, where F_1 is countable and F_2 is either empty or perfect. It is easy to see that

$$\wedge(F) = \wedge(F_2),\ \wedge(f(F)) = \wedge(f(F_2)).$$

We may from now assume F is perfect, otherwise the inequality which needs to prove is trivial. Fix $\varepsilon > 0$, choose δ as in Lemma 2.8, s.t, for $0 < t < \delta$,

$$t \le \frac{\sqrt{2}}{2}\min(\frac{1}{q}, 1-x, x, y).$$

Let $I_1, I_2, \cdots, I_N$ be the covering of F as in Lemma 2.8 and let T_n be the associated cones. Then for each pair of points ζ_1 and ζ_2 in $F \cap I_n$, let $T \subset T_n$ be the associated cone of the closed interval with end points ζ_1 and ζ_2. Since $|\zeta_1 - \zeta_2| \leq t$, we have $|\zeta_1 - \zeta_2| \leq \frac{\sqrt{2}}{2} \min(\frac{1}{q}, 1-x, x, y)$. Then inequality (2.12) applies, i.e.

$$|f(\zeta_1) - f(\zeta_2)|^3 \leq \frac{3p}{2\pi} m(f(T)) \leq \frac{3p}{2\pi} m(f(T_n)).$$

It follows that

$$(diaf(E_n))^3 = d_n^3 \leq \frac{3p}{2\pi} m(f(T_n)), \qquad \text{where } E_n = F \cap I_n.$$

Let $d = \max\{d_1, d_2, \cdots, d_N\}$. Note that $\{f(E_n)\}$ forms a covering of $f(F)$ and $dia(f(E_n)) \leq d$. Hence by Schwarz Inequality,

$$\begin{aligned} \wedge(f(F), d)^3 &\leq (\sum_1^N dia(f(E_n)))^3 \\ &\leq (\sum_1^N 1^{\frac{3}{2}})^2 (\sum_1^N (dia(f(E_n))^3) \\ &\leq \frac{3p}{2\pi} (Nt)^2 \frac{\sum_1^N m(f(T_n))}{t^2}. \end{aligned}$$

Without loss of generality, we may assume that $L = \cup I_n$ is an interval. Since $\sum_1^N m(f(T_n))$ is contained in the cube $\omega = \{(x', y', z') | x - t \leq x' \leq x + t, y - t \leq y' \leq y + t, 0 \leq z' \leq L\}$ and by Lemma 2.11,

$$\lim_{t \to 0} \frac{\sum_1^N m(f(T_n))}{t^2} \leq 4\varphi_L'(x, y).$$

Letting $t \to 0$ and then $d \to 0$, by Lemma 2.8, it follows that

$$\wedge(f(F))^3 \leq \frac{6p}{\pi} \varphi_F'(x, y) (\wedge(F))^2.$$

$\square$

Lemma 2.13. *Suppose that* $0 < x < 1, x < y < 1$ *and that* E *is a subset of* $I(x,y)$ *with* $\wedge(E) = 0$*. Then* $\wedge(f(E)) = 0$.

Proof. First suppose E is compact, then $F_{p,q} = E \cap H(p,q)$ is compact for each pair of p, q. By Lemma 2.12,

$$\wedge(f(F_{p,q}))^3 \leq \frac{6p}{\pi}\varphi'_{F_{p,q}}(x,y)(\wedge(F_{p,q}))^2.$$

Then since cube $C = \cup_{p,q} H(p,q)$, we have

$$\wedge(f(E)) \leq \sum_{p,q} f(F_{p,q}) \leq \sum_{p,q} (\frac{6p}{\pi}\varphi'_{F_{p,q}}(x,y)(\wedge(F_{p,q}))^2)^{\frac{1}{3}} = 0.$$

Next suppose E is a $G_\delta-$ Borel set. Then since $F = I(x,y) \cap H(p,q)$ is compact,

$$\wedge(f(E \cap H(p,q)))^3 \leq \frac{6p}{\pi}\varphi' \wedge (I(x,y) \cap H(p,q))^2 \leq \frac{6p}{\pi}\varphi' < \infty.$$

This shows $f(E)$ is of $\sum -$finite linear measure. Since $f(E)$ is a $G_\delta-$Borel set,

$$\wedge(f(E)) = \sup\{\wedge(F') : F' is\, compact\, subest\, in\, f(E)\}.$$

Let F' be any compact subset of $f(E)$ and set $F = f^{-1}(F')$, then F is compact and $F \subseteq E$, hence $\wedge(F) = 0$ and $\wedge(F') = 0$. Thus, we conclude that $\wedge(f(E)) = 0$.

In general, we can find a $G_\delta-$Borel set H s.t, $E \subseteq H \subseteq I(x,y)$ and $\wedge(E) = \wedge(H) = 0$. Then $\wedge(f(E)) \leq \wedge(f(H)) = 0$.

□

We finally complete the proof of ACL property of f by using Lemma 2.12 and Lemma 2.13. For each integer $p > 0$, set $H(p) = \cup_q H(p,q)$, where the sum is taken over relevant q. Condition 2 in Theorem 2.9 implies $m(C \setminus H(p)) = 0$, whenever $p > \csc(\pi/8k)$. Fix such a number p, by Fubini's Theorem,

$$\wedge(I(x,y) \setminus H(p)) = 0, \text{ for almost all pairs } (x,y) \in [0,1] \times [0,1].$$

Let E be any compact set in $I(x,y)$, fix such a pair (x,y) s.t, $\varphi'_E(x,y)$ exists and is finite. Write $E=(E\cap H(p))\cup(E\setminus H(p))$, where $\wedge(E\setminus H(p))=0$. Hence by Lemma 2.12 and Lemma 2.13,

$$\begin{aligned}\wedge(f(E))^3 &= \wedge(f(E\cap H(p)))^3\\ &= \lim_{q\to\infty}\wedge(f(E\cap H(p,q)))^3\\ &\le \frac{6p}{\pi}\varphi'_E(x,y)\lim_{q\to\infty}\wedge(E\cap H(p,q))^2\\ &\le \frac{6p}{\pi}\varphi'_E(x,y)\wedge(E)^2.\end{aligned}$$

Then $f(x,y,z)$ is absolutely continuous in $0\le z\le 1$. Therefore, f has the desired ACL property.

2.6.5 f_z is ACL^3 on G

By performing similarity mappings, it is sufficient to show that f_z is ACL^3 on compact set $K\subset C\setminus\{\infty, f^{-1}(\infty)\}$, where C denote the unit cube. Since it is already shown that f is ACL in z, f has finite partial derivative f_z a.e. in K.

For every line segment L on the fixed vertical line segment $I(x,y)\subset C$, where f is absolutely continuous, define

$$\wedge(L)=\int_L |f_z(x,y,\zeta)|d\zeta$$

and

$$g_n(x,y,z)=\frac{n}{2}\int_{I_n(z)}|f_z(x,y,\zeta)|d\zeta,$$

where $I_n(z)=\{\zeta : z-\frac{1}{n}\le\zeta\le z+\frac{1}{n}\}$ is an interval on $I(x,y)$ where f_z exists for $n=1,2,\ldots$. Let

$$g(x,y,z)=\liminf_{n\to\infty} g_n(x,y,z).$$

Then it is easy to see that $g(x,y,z) = |f_z(x,y,z)|$. On compact set K,

$$g_n^3(x,y,z) = (\frac{n}{2} \wedge (I_n(z)))^3.$$

Taking the integral of $g_n^3(x,y,z)$ over the compact set K and using Lemma 2.12 for $I_n(z)$, we obtain that

$$\begin{aligned}
\iint_K g_n^3(x,y,z)dm(x,y) &\leq \frac{1}{8}\iint_K \frac{6p}{\pi}\varphi'_{I_n(z)}(x,y) \wedge (I_n(z))^2 n^3 dm(x,y) \\
&\leq \frac{3p}{\pi}\iint_K \varphi'_{I_n(z)}(x,y) n dm(x,y) \\
&= \frac{3p}{\pi}\varphi_{I_n(z)}(K)n.
\end{aligned}$$

Letting $n \to \infty$, by Fatou's lemma, it yields that

$$\iint_K g^3(x,y,z)dm(x,y) \leq \liminf_{n\to\infty} \frac{3p}{\pi}\varphi_{I_n(z)}(K)n.$$

By Lebesgue Theorem, $\lim_{n\to\infty} \frac{n}{2}\varphi_{I_n(z)}(K) = \varphi''(z)$ where $\varphi''(z)$ is measurable and $\int_I \varphi''(z)d\zeta \leq \varphi_I(K)$. Therefore,

$$\iint_K g^3(x,y,z)dm(x,y) \leq \frac{6p}{\pi}\varphi''(z).$$

By Fubini's theorem, we have

$$\iint_{K\times L} |f_z|^3 dm < \infty,$$

i.e, f_z is locally ACL^3 in z and, by the symmetry, the partial derivatives of f are locally ACL^3 in G.

2.6.6 Completion of proof of Characterization Theorem 2.9

Proof. Suppose f is a K-quasiconformal mapping on domain G, we show that f satisfies condition 1 and 2 in Theorem 2.9. For any point $\zeta_0 \in G$, let

γ_1, γ_2 be a pair of arcs which form a topological angle at ζ_0, then condition 1 follows immediately from a local version of Theorem 2.6 because, by Theorem 11.14 in [**?**], f is locally quasisymmetric in G.

Let E be the set of all points ζ_0, such that f is differentiable at ζ_0 and satisfies

$$0 < \max_{\theta} |D_\theta f(\zeta_0)|^3 \leq k|J(\zeta_0)|.$$

Since quasiconformal mapping is differentiable a.e., $m(G \setminus E) = 0$. For any point $\zeta_0 \in E$, let γ_1 and γ_2 be a pair of arcs which form an angle at ζ_0, we have $A(f(\gamma_1), f(\gamma_2)) \geq \frac{1}{k} A(\gamma_1, \gamma_2)$ by Theorem 2.4. Thus condition 2 is proved given that f is a K-quasiconformal mapping.

Next suppose f is a homeomorphism on domain G satisfying conditions 1 and 2, we show f is K-quasiconformal on domain G. we show this through the analytic definition of quasiconformality. From deduction in section 2.6.5, f is locally ACL^3 in G.

Due to a result by Väisälä [15], if f is ACL^3, then f is differentiable a.e. We then show that $\max_\theta |D_\theta f(\zeta_0)|^3 \leq K_1 |J(\zeta_0)|$ for almost all $\zeta_0 \in G$. Fix $\zeta_0 \in G$ such that f is differentiable at ζ_0 and the inequality $A(f(\gamma_1), f(\gamma_2)) \geq \frac{1}{K} A(\gamma_1, \gamma_2)$ holds at ζ_0. If $\max_\theta |D_\theta f(\zeta_0)| > 0$, by Theorem 2.4,

$$\max_{\theta} |D_\theta f(\zeta_0)|^3 \leq K^2 |J(\zeta_0)|.$$

If $\max_\theta |D_\theta f(\zeta_0)| = 0$, the inequality holds trivially. The proof is complete.

□

2.7 Generalization to $\mathbb{R}^{n+1}$

To conclude this chapter, we give parallel results in $n+1$ dimensional Euclidean space with $n \geq 3$.

Theorem 2.14. *Let f be a homeomorphism on domain $G \subseteq \mathbb{R}^{n+1}$. If f is a K-quasiconformal mapping, $1 \leq K < \infty$, then we have:*

1. *For all* ζ_0 *in* G *and for all arcs* γ_1 *and* γ_2 *which form an angle in* G *at* ζ_0

$$A(f(\gamma_1), f(\gamma_2)) > 0.$$

2. *For almost all* ζ_0 *in* G *and for all arcs* γ_1 *and* γ_2 *which form an angle in* G *at* ζ_0

$$A(f(\gamma_1), f(\gamma_2)) \geq \frac{1}{K} A(\gamma_1, \gamma_2).$$

Conversely, if condition 1 and 2 are satisfied, then f *is* K_1*-quasiconformal with* $K_1 = K^n$.

Proof. Since most part of the proof of Theorem 2.9 is valid in higher dimensional space, we just outline the major difference in proving the ACL property. First of all, we should notice the change of symbols. We will only write out the coordinates relevant to the proof here. Denote by C the closed cube $\{(x_1, x_2, \cdots, x_{n+1}) | 0 \leq x_i \leq 1, i = 1, \cdots, n+1\}$ in domain $G \subseteq \mathbb{R}^{n+1}$ and we need to show f is absolutely continuous on $0 \leq x_{n+1} \leq 1$ for almost all n-dimensional points $\{(x_1, x_2, \cdots, x_n) | 0 \leq x_i \leq 1, i = 1, \cdots, n\}$.

Let I be a closed subinterval of $I(x_1, x_2, \cdots, x_n)$ with end points ζ_1 and ζ_2 in a compact set F on $I(x_1, x_2, \cdots, x_n) \cap H(p, q)$. And let T be the associated cone with interval I. Similar to (2.12), we obtain that

$$|f(\zeta_1) - f(\zeta_2)|^{n+1} \leq cm(f(T)),$$

where $c = \frac{n(n+1)\Gamma(\frac{1}{2}n)p^n}{2^{n+1}\pi^{\frac{n}{2}}}$. Furthermore,

$$\begin{aligned}
\wedge(f(F),d)^{n+1} &\leq (\sum_1^N dia(f(E_n)))^{n+1} \\
&\leq (\sum_1^N 1^{\frac{n+1}{n}})^n(\sum_1^N (dia(f(E_n))^{n+1}) \\
&\leq cN^n \sum_1^N m(f(T_n)) \\
&\leq c(Nt)^n \frac{\sum_1^N m(f(T_n))}{t^n} \\
&\leq c(\wedge(F)+\varepsilon)^n \frac{\sum_1^N m(f(T_n))}{t^n},
\end{aligned}$$

where, as in the proof of Lemma 2.12, t is selected sufficiently small which makes the associated cone T being contained in cube C and, d is the maximum diameter of the intersection of F and its finite coverings under f. Letting $t \to 0$, it follows that

$$\wedge(f(F))^{n+1} \leq c(\wedge(F))^n \sum_1^N \lim_{t\to 0} \frac{m(f(T_n))}{t^n}.$$

Next, we will give a more general result which is similar to Lemma 2.11. Let A be a Borel set in n-dimensional unit cube in $\mathbb{R}^n$ and $L(x_1, x_2, \cdots, x_n)$ be the vertical segment$\{(x_1, x_2, \cdots, x_n) \times L\}$. We introduce the set function μ defined as

$$\mu[A, L] = mf(A \times L).$$

Clearly, μ is a measure. By Lebesgue's theorem, for fixed segment L, the function $\mu'(x_1, x_2, \cdots, x_n, L)$ given by

$$\mu'(x_1, x_2, \cdots, x_n, L) = \lim_{m\to\infty} \frac{\mu[A_m(x_1, \cdots, x_n), L]}{m(A_m)},$$

where A_m is n dimensional cube $\{(x'_1, x'_2, ...x'_n)|x_1 - \frac{1}{m} \leq x'_1 \leq x_1 + \frac{1}{m}, x_2 - \frac{1}{m} \leq x'_2 \leq x_2 + \frac{1}{m}, ..., x_n - \frac{1}{m} \leq x'_n \leq x_n + \frac{1}{m}\}$ $m = 1, 2, \ldots$. $\mu'(x_1, x_2, \cdots, x_n, L)$

is well defined and finite almost everywhere.

Moreover,

$$\sum_{1}^{M} \lim_{t\to 0} \frac{m(f(T_m))}{t^n} \leq \lim_{t\to 0} \frac{\mu[A_m(x_1, ...x_n), L]}{t^n} = \frac{1}{2^n}\mu'(x_1, x_2, ...x_n, L).$$

Therefore

$$\wedge(f(F))^{n+1} \leq c\mu'(x_1, x_2, ...x_n, L)(\wedge(F))^n.$$

Thus, for any compact set E in $I(x_1, x_2, \cdots, x_n)$,

$$\wedge(f(E))^{n+1} \leq c\mu'(x_1, x_2, \cdots, x_n, L)(\wedge(E))^n.$$

Therefore accordingly, f has the desired ACL property in G. $\square$

Chapter 3

QED reflection constant

Quasiextremal distance (QED) constant $M(\Omega)$ was introduced in [17] and [18]. The QED constant is a conformal invariant because it is invariant under Möbius transformations or conformal mappings on the extended plane $\bar{\mathbb{C}}$. It reflects the geometry of a domain.

In this chapter, we will define a new QED constant, the quasiextremal distance reflection constant. We then list some fundamental properties, and show that for a smooth Jordan domain other than a disk or a half plane, the reflection QED constant can only be obtained by a pair of disjoint non-degenerate continua.

Definition 3.1. *For a Jordan domain Ω in the complex plane $\mathbb{C}$ we define quasiextremal distance reflection constant (QED reflection constant) $M^*(\Omega)$ as follows:*

$$M^*(\Omega) = \sup_{A,B\subset\partial\Omega} \frac{mod(A,B;\Omega^*)}{mod(A,B;\Omega)}, \tag{3.1}$$

where A, B are disjoint non-degenerate continua on $\partial\Omega$, and $mod(A,B;\Omega^*)$ denotes the modulus of family of curves connecting A and B in $\Omega^* = \mathbb{C}/\bar{\Omega}$.

3.1 Fundamental properties about the QED reflection constant

3.1.1 Properties about the QED reflection constant

We study several basic facts of the QED reflection constant.

Lemma 3.2. $M^*(\Omega) \geq 1$.

Proof. On $\partial\Omega$, denote two non-degenerate disjoint continua by A, B, their complements by A', B'. By Riemann mapping theorem and Elliptic integral, Ω is mapped onto a rectangle and continua A, B are mapped onto a pair of parallel sides and their compliments A', B' onto the other pair. Since modulus is invariant under conformal mappings, using the module of rectangle (see Example 1.9), one can deduce that

$$mod(A, B; \Omega) \cdot mod(A', B'; \Omega) = 1,$$

$$mod(A, B; \Omega^*) \cdot mod(A', B'; \Omega^*) = 1.$$

Thus,

$$\frac{mod(A, B; \Omega)}{mod(A, B; \Omega^*)} = \frac{mod(A', B'; \Omega^*)}{mod(A', B'; \Omega)}. \tag{3.2}$$

This gives rise to

$$M^*(\Omega) \geq 1.$$

□

Lemma 3.3. $M^*(\Omega) = M^*(\Omega^*)$.

Proof. Using the same notation as in the proof of Lemma 3.2, identity (3.2) holds. Taking the supremum over all pairs $A, B \subset \partial\Omega$ in (3.2), we obtain

$$\sup_{A,B\subset\partial\Omega} \frac{mod(A, B; \Omega)}{mod(A, B; \Omega^*)} = \sup_{A',B'\subset\partial\Omega} \frac{mod(A', B'; \Omega^*)}{mod(A', B'; \Omega)},$$

which gives the identity

$$M^*(\Omega) = M^*(\Omega^*).$$

$\square$

Lemma 3.4. *$M^*(\Omega) = 1$ if and only if Ω is a disk or a half plane in $\mathbb{C}$.*

Proof. We first assume that Ω is a disk (or a half plane). Let A, B be two non-degenerate disjoint continua on $\partial\Omega$. By symmetry principle,

$$\mathrm{mod}(A, B; \Omega) = \mathrm{mod}(A, B; \Omega^*) = \frac{1}{2}\mathrm{mod}(A, B; \mathbb{C}).$$

Thus,

$$M^*(\Omega) = \sup_{A,B\subset\partial\Omega} \frac{\mathrm{mod}(A, B, \Omega^*)}{\mathrm{mod}(A, B; \Omega)} = 1.$$

Next we show that if $M^*(\Omega) = 1$, then Ω is a disk or a half plane. Suppose A and B are two disjoint non-degenerate continua on $\partial\Omega$. Denote two complement disjoint continua as A' and B'. As same as (3.2),

$$\frac{mod(A, B; \Omega)}{mod(A, B; \Omega^*)} = \frac{mod(A', B'; \Omega^*)}{mod(A', B'; \Omega)}.$$

This, together with $M^*(\Omega) = 1$, yields that

$$\sup_{A,B\subset\partial\Omega} \frac{mod(A, B; \Omega^*)}{mod(A, B; \Omega)} = \sup_{A',B'\subset\partial\Omega} \frac{mod(A', B'; \Omega^*)}{mod(A', B'; \Omega)} = \inf_{A,B\subset\partial\Omega} \frac{mod(A, B; \Omega^*)}{mod(A, B; \Omega)} = 1.$$

Thus, for any pair of disjoint continua A and B on $\partial\Omega$,

$$\mathrm{mod}(A, B; \Omega) = \mathrm{mod}(A, B; \Omega^*).$$

Let p_1, p_2 and p_3 be three points on $\partial\Omega$. There exists a preliminary Möbius transformation $f : \bar{\Omega} \to \bar{\mathbb{H}}^+$, such that $f(p_1) = -1, f(p_2) = 0$ and $f(p_3) = \infty$ and that Ω is mapped onto the upper half plane. Similarly, there exists preliminary Möbius transformation $f^* : \bar{\Omega}^* \to \bar{\mathbb{H}}^-$ with $f^*(p_1) = -1, f^*(p_2) = 0$ and $f^*(p_3) = \infty$ such that Ω^* is mapped onto the lower half plane.

Denote the boundary curve between p_1 and p_2 as A such that $p_3 \notin A$. Select a point $z \in \partial\Omega$ such that $z \notin A$. It is obvious that $f(z)$ and $f^*(z)$ lie on the real axis. Denote the boundary curve between p_3 and z as B such that $A \cap B = \emptyset$. And let $A' = [-1, 0], B' = [f(z), \infty)$ and $C' = [f^*(z), \infty)$. Then it is easy to see that $f(A) = f^*(A) = A'$, $f(B) = B'$ and $f^*(B) = C'$. Since modulus is invariant under conformal mappings, we obtain

$$\mathrm{mod}(A, B; \Omega) = \mathrm{mod}(A', B'; \mathbb{H}^+),$$

$$\mathrm{mod}(A, B; \Omega^*) = \mathrm{mod}(A', C'; \mathbb{H}^-).$$

Since $\mathrm{mod}(A, B; \Omega) = \mathrm{mod}(A, B; \Omega^*)$,

$$\mathrm{mod}(A', B'; \mathbb{H}^+) = \mathrm{mod}(A', C'; \mathbb{H}^-).$$

By the monotonicity of modulus of Teichmüller ring domain in Example 1.10,

$$f(z) = f^*(z).$$

This implies that $f = f^*$ on $\partial\mathbb{H}^+$. Thus, by the Schwarz reflection principle, there exists a Möbius transformation $g(z), z \in \bar{\mathbb{C}}$ such that $g(z) = f(z)$ on Ω and $g(z) = f^*(z)$ on Ω^*. Moreover, Ω is Möbius equivalent to $\mathbb{H}^+$ under a global Möbius transformation on $\bar{\mathbb{C}}$. Therefore, Ω is a disk or a half plane. $\square$

3.2 The QED reflection constant for smooth domains

3.2.1 Preliminary

To study the QED reflection constant for a smooth Jordan domain, we first give some definitions and preliminary results.

Definition 3.5. (Reflection constant) *The quasiconformal reflection constant (or reflection constant) of Ω, denoted by $R(\Omega)$, is defined as*

$$R(\Omega) = \inf_f K(f),$$

where the infimum is taken over all homeomorphic reflections f in the boundary $\partial\Omega$ and $K(f)$ denotes the maximal dilatation of f. A homeomorphic reflection in a Jordan curve is a homeomorphism of $\bar{\mathbb{C}}$ that interchanges the two components of the complement of the curve taken with respect to the extended plane and fixes the curve pointwise.

Definition 3.6. (L-bi-Lipschitz) *A mapping $f : E \to E'$ is L-bi-Lipschitz if*

$$\frac{1}{L}|x-y| \leq |f(x) - f(y)| \leq L|x-y|$$

for $x, y \in E$; f is locally L-bi-Lipschitz if each $x \in E$ has a neighborhood U such that f is L-bi-Lipschitz in $E \cap U$.

Theorem 3.7. *[19] Let Ω be a smooth Jordan domain other than a disk or a half plane and let f be a conformal map of Ω onto the unit disk D. Then*

1. *f has a quasiconformal extension to $\mathbb{C}$ such that the complex dilatation*
$$\frac{\partial f}{\partial \bar{z}} / \frac{\partial f}{\partial z} \to 0$$
uniformly as $z \to \partial\Omega$;

2. *f has a bilipschitz extension to $\bar{\Omega}$;*

3. $$\sup_{A,B\subset\partial\Omega} \frac{mod(A,B;\Omega^*)}{mod(A,B;\Omega)} < R(\Omega).$$

Definition 3.8. (Condenser) *A condenser is a domain in $\bar{\mathbb{R}}^n$ whose complement consists of two disjoint compact sets F_0 and F_1. Condenser is usually denoted by $R(F_0, F_1)$ or R.*

Theorem 3.9. *[19] Let $R(A,B)$ be a condenser and f a homeomorphism defined on $A \cup B$ such that*

$$L_1|x-y| \leq |f(x)-f(y)| \leq L_2|x-y|$$

when $x, y \in A$ or $x, y \in B$ and such that

$$M_1|x-y| \leq |f(x)-f(y)| \leq M_2|x-y|$$

when $x \in A$ and $y \in B$, where L_1, L_2, M_1, M_2 are constants. Then

$$c_1 + \frac{2\pi}{mod(A,B;\mathbb{C})} \leq \frac{2\pi}{mod(f(A),f(B);\mathbb{C})} \leq c_2 + \frac{2\pi}{mod(A,B;\mathbb{C})},$$

where c_1 and c_2 are constants depending on L_1, L_2, M_1, M_2.

3.2.2 The QED reflection constant

In [19], on a smooth Jordan domain other than a disk or a half plane, the QED constant is well studied. We can also obtain a similar result for the QED reflection constant.

Theorem 3.10. *For any smooth Jordan domain Ω other than a disk or a half plane, the supremum in* (3.1) *is attained, that is there exists a pair of disjoint non-degenerate continua A and B on $\partial\Omega$, such that:*

$$M^*(\Omega) = \frac{mod(A,B;\Omega^*)}{mod(A,B;\Omega)}.$$

Proof. We assume that Ω is a bounded smooth Jordan domain other than a disk or a half plane. Let f be a conformal map of Ω onto the unit disk D and F be a conformal map from Ω^* to D^*. By Theorem 3.7, f and F have bilipschitz extension to $\bar{\Omega}$ and $\bar{\Omega}^*$. We still denote the extensions by f and F. For each $n \geq 1$, fix disjoint non-degenerate continua A_n and B_n on $\partial\Omega$, such that

$$M^*(\Omega) = \lim_{n\to\infty} \frac{mod(A_n,B_n;\Omega^*)}{mod(A_n,B_n;\Omega)},$$

where A_n and B_n converge to A and B in the Hausdorff metric. For convenience, let $A'_n = f(A_n), B'_n = f(B_n), A''_n = F(A_n), B''_n = F(B_n)$. If A and B are disjoint non-degenerate continua, by the continuity of moduli,

$$M^*(\Omega) = \frac{mod(A, B; \Omega^*)}{mod(A, B; \Omega)}.$$

Depending on the positions and sizes of continua A and B, we consider remaining three cases respectively:

Case 1: At least one of the two sets A, B is a single point and $A \cap B = \emptyset$.

From above notations,

$$\operatorname{mod}(A_n, B_n; \Omega) = \operatorname{mod}(A'_n, B'_n; D),$$

$$\operatorname{mod}(A_n, B_n; \Omega^*) = \operatorname{mod}(A''_n, B''_n; D^*).$$

Then we obtain

$$\frac{\operatorname{mod}(A_n, B_n; \Omega^*)}{\operatorname{mod}(A_n, B_n; \Omega)} = \frac{\operatorname{mod}(A''_n, B''_n; D^*)}{\operatorname{mod}(A'_n, B'_n; D)} = \frac{\operatorname{mod}(A''_n, B''_n; \mathbb{C})}{\operatorname{mod}(A'_n, B'_n; \mathbb{C})}.$$

Let $\varphi = f \circ F^{-1}$, then φ is a bilipschitz map from $\partial D \to \partial D$, where $\varphi(A''_n) = A'_n$ and $\varphi(B''_n) = B'_n$. We will show that

$$\lim_{n\to\infty} \frac{\operatorname{mod}(A''_n, B''_n; \mathbb{C})}{\operatorname{mod}(A'_n, B'_n; \mathbb{C})} = 1.$$

By Theorem 3.9,

$$c_1 + \frac{2\pi}{\operatorname{mod}(A''_n, B''_n; \mathbb{C})} \leq \frac{2\pi}{\operatorname{mod}(A'_n, B'_n; \mathbb{C})} \leq c_2 + \frac{2\pi}{\operatorname{mod}(A''_n, B''_n; \mathbb{C})}.$$

Multiplying this by $\operatorname{mod}(A''_n, B''_n; \mathbb{C})$ and letting $n \to \infty$, the fact that $\lim_{n\to\infty} \operatorname{mod}(A''_n, B''_n; \mathbb{C}) = 0$ yields

$$\lim_{n\to\infty} \frac{\operatorname{mod}(A''_n, B''_n; \mathbb{C})}{\operatorname{mod}(A'_n, B'_n; \mathbb{C})} = 1.$$

And by the continuity of modulus, we obtain

$$M^*(\Omega) = 1.$$

Case 2: At least one of the two sets A, B is a single point and $A \cap B \neq \emptyset$; We consider two subcases. First assume that

$$\lim_{n\to\infty} \mathrm{mod}(A''_n, B''_n; \mathbb{C}) = 0,$$

then similar to Case 1, $\lim_{n\to\infty} \frac{\mathrm{mod}(A''_n,B''_n;\mathbb{C})}{\mathrm{mod}(A'_n,B'_n;\mathbb{C})} = 1$, thus $M^*(\Omega) = 1$.

Next assume that

$$\lim_{n\to\infty} \mathrm{mod}(A''_n, B''_n; \mathbb{C}) = a > 0,$$

and we keep the same indices for subsequences of $\{A_n\}$ and $\{B_n\}$.

Without loss of generality, assume that A is a single point. Choose $a_n, b_n \in A$ such that $|b_n - a_n|$ is the diameter of A_n. Fix some $\varepsilon > 0$. Since f is K-quasiconformal in Ω^* and its complex dilatation converges to 0 uniformly as $z \to \partial\Omega$, there exists a domain $\Omega_\varepsilon \supset \bar{\Omega}$, such that f is $(1+\varepsilon)$-quasiconformal in Ω_ε and the image $f(\Omega_\varepsilon)$ is a disk. By Theorem 3.7, F can be extended to a K-quasiconformal mapping on $\mathbb{C}$. Restrict F on Ω_ε. Let $D_\varepsilon = F(\Omega_\varepsilon), a''_n = F(a_n), b''_n = F(b_n)$ and $|b''_n - a''_n| =$ dia $(F(A_n))$. Again by Theorem 3.7, since F has a bilipschitz extension to $\bar{\Omega}^*$, $|b''_n - a''_n| \to 0$ as $n \to \infty$. Let $\varphi = F \circ f^{-1}$, which is quasiconformal from disk $f(\Omega_\varepsilon)$ to D_ε. By quasi-invariance of modulus,

$$\begin{aligned} \mathrm{mod}(A''_n, B''_n; D_\varepsilon) &\leq (1+\varepsilon)^2 \mathrm{mod}(A'_n, B'_n; \varphi(D_\varepsilon)) \\ &\leq 2(1+\varepsilon)^2 \mathrm{mod}(A'_n, B'_n; D) \\ &= 2(1+\varepsilon)^2 \mathrm{mod}(A_n, B_n; \Omega). \end{aligned} \tag{3.3}$$

Thus,

$$\frac{1}{\mathrm{mod}(A_n, B_n; \Omega)} \leq 2(1+\varepsilon)^2 \frac{1}{\mathrm{mod}(A''_n, B''_n; D_\varepsilon)}.$$

Multiplying by $\mathrm{mod}(A_n, B_n; \Omega^*)$ and noticing $\mathrm{mod}(A_n, B_n; \Omega^*) = \mathrm{mod}(A''_n, B''_n; D^*)$, we obtain

$$\begin{aligned} \frac{\mathrm{mod}(A_n, B_n; \Omega^*)}{\mathrm{mod}(A_n, B_n; \Omega))} &\leq 2(1+\varepsilon)^2 \frac{\mathrm{mod}(A''_n, B''_n; D^*)}{\mathrm{mod}(A''_n, B''_n; D_\varepsilon)} \\ &\leq (1+\varepsilon)^2 \frac{\mathrm{mod}(A''_n, B''_n; \mathbb{C})}{\mathrm{mod}(A''_n, B''_n; D_\varepsilon)}. \end{aligned} \tag{3.4}$$

Next, we show

$$\lim_{n\to\infty} \frac{\mathrm{mod}(A_n'', B_n''; \mathbb{C})}{\mathrm{mod}(A_n'', B_n''; D_\varepsilon)} = 1.$$

Choose $\delta > 0$, such that $A_n'' \subset D(b_n'', \delta) \subset D_\varepsilon$ for large n. Then, it follows that

$$\mathrm{mod}(A_n'', B_n''; \mathbb{C}) \leq \mathrm{mod}(A_n'', B_n''; D_\varepsilon) + \frac{2\pi}{\log \frac{\delta}{|b_n'' - a_n''|}}.$$

Since $\lim_{n\to\infty} |b_n'' - a_n''| = 0$ and $\lim_{n\to\infty} \mathrm{mod}(A_n'', B_n''; \mathbb{C}) = a > 0$, it follows that

$$\begin{aligned} 1 &\geq \frac{\mathrm{mod}(A_n'', B_n''; D_\varepsilon)}{\mathrm{mod}(A_n'', B_n''; \mathbb{C})} \\ &\geq \frac{\mathrm{mod}(A_n'', B_n''; \mathbb{C}) - \frac{2\pi}{\log \frac{\delta}{|b_n'' - a_n''|}}}{\mathrm{mod}(A_n'', B_n''; \mathbb{C})}. \end{aligned} \tag{3.5}$$

Letting $n \to \infty$ in inequality (3.5), we obtain $\lim_{n\to\infty} \frac{\mathrm{mod}(A_n'', B_n''; \mathbb{C})}{\mathrm{mod}(A_n'', B_n''; D_\varepsilon)} = 1$, and letting $n \to \infty$ in inequality (3.4), it follows that

$$M^*(\Omega) \leq (1+\varepsilon)^2.$$

Since ε is arbitrary, letting $\varepsilon \to 0$, we obtain $M^*(\Omega) \leq 1$.

Case 3: A, B are both non-degenerate and $A \cap B \neq \emptyset$. (3.3) implies that the following inequality is true:

$$\begin{aligned} \mathrm{mod}(A_n, B_n; \Omega^*) &= \mathrm{mod}(A_n'', B_n''; D^*) \\ &= \frac{1}{2}\mathrm{mod}(A_n'', B_n''; \mathbb{C}) \\ &\leq \frac{1}{2}(\mathrm{mod}(A_n'', B_n''; D_\varepsilon) + \mathrm{mod}(\partial D_\varepsilon, \partial D; \mathbb{C})) \\ &\leq (1+\varepsilon)^2 \mathrm{mod}(A_n, B_n; \Omega) + \frac{1}{2}\mathrm{mod}(\partial D_\varepsilon, \partial D; \mathbb{C}). \end{aligned} \tag{3.6}$$

Note that $\mathrm{mod}(\partial D_\varepsilon, \partial D; \mathbb{C})$ is finite and $\mathrm{mod}(A_n, B_n; \Omega) \to \infty$. Letting $n \to \infty$ after dividing by $\mathrm{mod}(A_n, B_n; \Omega)$ in (3.6) yields that

$$M^*(\Omega) \leq (1+\varepsilon)^2,$$

for any ε. Thus, we get $M^*(\Omega) \leq 1$.

Finally, by lemma 3.2 and lemma 3.4, all three degenerate cases imply that Ω is a disk or a half plane. Thus only the non-degenerate case can occur: the limit sets A and B are disjoint non-degenerate continua and

$$M^*(\Omega) = \frac{mod(A, B; \Omega^*)}{mod(A, B; \Omega)}.$$

$\square$

Remark: From the above proof, one can conclude that, if Ω is a smooth Jordan domain other than a disk or a half plane, then the QED reflection constant $M^*(\Omega)$ can only be achieved by a pair of disjoint non-degenerate continua.

Chapter 4

Decomposition of extremal length on finitely connected domains

The method of extremal length has had a profound influence on the theory of comformal mappings and the more general theory of quasiconformal mappings. In this chapter, we will explore the decomposition of extremal length within multiply connected domains. More specifically, the Decomposition theorem shows the extremal length of a certain family equals to the sum of the extremal lengths of its decomposed parts. It can be regarded as a strengthened version of the subadditivity for extremal length.

The decomposition theorem is a major ingredient in the proof of the main result in chapter 5. It is also a generalization of the result from paper [6]. We state the Decomposition Theorem first.

Theorem 4.1. (Decomposition Theorem) *Suppose Ω is an n-connected domain in the plane and A, B are two disjoint continua in Ω. Then there exist two simple closed curves γ_A and γ_B in $\bar{\Omega}$, such that,*

1. *The number of intersection points of γ_A with $\partial\Omega$ and γ_B with $\partial\Omega$ are at least 1, i.e, $\sharp(\gamma_A \cap \partial\Omega) \geq 1$ and $\sharp(\gamma_B \cap \partial\Omega) \geq 1$.*

2. $\lambda(A, B; \Omega) = \lambda(A, \gamma_A) + \lambda(B, \gamma_B) + \lambda(\gamma_A, \gamma_B; \Omega)$.

3. *The components of* $\Omega \backslash (\gamma_A \cup \gamma_B)$ *which contain continua* A *or* B *are simply connected.*

4.1 Reduction

For the proof of Theorem 4.1, we make the following reduction. First we may assume that $G = \Omega \backslash (A \cup B)$ is connected and note that $\lambda(A, B; \Omega) = \lambda(A, B; G)$. Next, by Koebe's circle domain theorem (see [7]), G can be conformally mapped onto a domain bounded by circles. Since extremal length is invariant under conformal mappings, without loss of generality, we may assume that the domain G is bounded by analytic Jordan curves β_i $(i = 0, 1, ..., n-1)$, ∂A and ∂B, and assume that Ω is bounded by β_i.

4.2 Preliminaries

Theorem 4.2. (The Generalized Argument Principle) *Let* Ω *be a bounded domain with* $\mathcal{C}^\infty$ *smooth boundary. Suppose* h *is meromorphic in a neighborhood of* $\bar{\Omega}$ *and that* h *is not identically zero. Let* $\{z_i\}_{i=1}^N$ *be the set of zeros of* h *that lie in* Ω, $\{p_i\}_{i=1}^Q$ *be the set of poles of* h *that lie in* Ω, $\{b_i\}_{i=1}^M$ *be the set of zeros of* h *that lie on* $\partial\Omega$ *and* $\{B_i\}_{i=1}^R$ *be the set of poles of* h *that lie on* $\partial\Omega$. *Then zeros and poles of* h *are isolated and*

$$\sum_{i=1}^{N} m_h(z_i) + \frac{1}{2}\sum_{i=1}^{M} m_h(b_i) - \sum_{i=1}^{Q} m_h(p_i) - \frac{1}{2}\sum_{i=1}^{R} m_h(B_i) = \frac{1}{2\pi} \triangle \arg h.$$

Theorem 4.3. (Green's Theorem) *Let* C *be a positively oriented, piecewise-smooth, simple closed curve in the plane and let* D *be the region bounded by* C. *if* P *and* Q *have continuous partial derivatives on an open region that* *contains D, then*

$$\int_C pdx + Qdy = \int\int_D (\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y})dA.$$

Green's Theorem can be extended to multiply-connected domains. It is called Extended Green's Theorem.

4.3 Mixed Dirichlet-Newmann problem

Due to Ahlfors [3] , extremal distance $\lambda(A, B; G)$ can be computed by Dirichlet integral $D(u)$. Let G be the previously discussed domain in 4.1.

Theorem 4.4. *The extremal distance $\lambda(A, B; G)$ is the reciprocal of the Dirichlet integral $D(u)$, i.e,*

$$\lambda(A, B; G) = \frac{1}{D(u)},$$

where

$$D(u) = \iint_G (u_x^2 + u_y^2)dxdy.$$

The function $u(z)$ is called the solution of a mixed Dirichlet-Neumann problem with the following properties:

1. u is bounded and harmonic in G.
2. u has a continuous extension to ∂G, which is equal to 0 on ∂A and 1 on ∂B.
3. The normal derivative $\partial u/\partial \mathbf{n}$ exists and the normal derivative vanishes on ∂G.

Moreover, function $u(z)$ is unique in G by the maximum principle and $0 < u(z) < 1$ in G.

4.4 Critical points

We now consider critical points of $u(z)$ on $\overline{G}$. Note that critical points of u are zeros of the analytic function $u_x - iu_y$. The reflection principle implies that u has a harmonic extension across $\partial\Omega \cup \partial A \cup \partial B$. Thus, $u_x - iu_y$ has an analytic extension. Since A, B are continua in Ω, zeros of analytic function $u_x - iu_y$ have no accumulation points in $\overline{G}$. So the number of its zeros in $\overline{G}$ is finite. Let C_u be the critical point set of u. Suppose there are p zeros of $u_x - iu_y$ (counting multiplicity) in the interior of G and q zeros of $u_x - iu_y$, also counting multiplicity, on the boundary of G, i.e., on $\left(\cup_{i=0}^{n-1}\beta_i\right) \cup \partial A \cup \partial B$.

By using theorem 4.2, the generalized argument principle, we have

$$\int_{\partial G} d\arg(u_x - iu_y) = 2\pi(p + \frac{1}{2}q).$$

Let $v(z)$ be a locally single-valued harmonic conjugate of $u(z)$. If we write $w = u + iv$, then

$$u_x - iu_y = \frac{dw}{dz},$$

and

$$\int_{\partial G} d\arg(u_x - iu_y) = \int_{\partial G} d\arg(dw) - \int_{\partial G} d\arg(dz).$$

Since ∂G is composed by analytic curves, on ∂G, we have

$$\begin{aligned} dw &= (u_x dx + u_y dy) + i(u_x dy - u_y dx) \\ &= \frac{\partial u}{\partial \mathbf{T}} ds + \frac{\partial u}{\partial \mathbf{n}} ds, \end{aligned}$$

where $\mathbf{T}$ is the tangent vector and $\mathbf{n}$ is the outer normal vector. Thus

$$\int_{\partial\Omega\cup\partial A\cup\partial B} d\arg(dw) = \int_{\partial\Omega\cup\partial A\cup\partial B} d\arg(\frac{\partial u}{\partial \mathbf{T}} ds) + \int_{\partial\Omega\cup\partial A\cup\partial B} d\arg(\frac{\partial u}{\partial \mathbf{n}} ds) = 0.$$

Since $u = 0$ for $z \in \partial A$ and $u = 1$ for $z \in \partial B$, we have

$$\frac{\partial u}{\partial \mathbf{T}} = 0 \text{ on } \partial A \cup \partial B.$$

Taking into account of

$$\frac{\partial u}{\partial \mathbf{n}} = 0 \text{ on } \partial\Omega,$$

we conclude that on ∂G,

$$d\arg(dw) = 0.$$

Moreover, since Ω is an n-connected domain,

$$\int_{\partial G} d\arg(dz) = \int_{\beta_0\cup\beta_1\cup\cdots\cup\beta_{n-1}\cup\partial A\cup\partial B} d\arg(dz) = 2n\pi.$$

Therefore,

$$2p + q = 2n.$$

4.5 Level curves and critical points

Note that u can not be constant on any subarc of $\partial\Omega$. Otherwise, on such a subarc, $\frac{\partial u}{\partial \mathbf{T}} = 0$ and $\frac{\partial u}{\partial \mathbf{n}} = 0$, which imply $u_x - iu_y$ has infinitely many zeros on $\partial\Omega$, a contradiction.

Let

$$\gamma_k = \{z \in \bar{\Omega} : u(z) = k, k \in [0, 1]\}$$

and

$$\Gamma = \{\gamma_k : k \in [0, 1]\}.$$

Easy to see $\gamma_0 = \partial A$, $\gamma_1 = \partial B$. We call γ_k the level curve of harmonic function u in $\bar{\Omega}$. Define

$$\Gamma^* = \{\gamma \in \Gamma : \gamma \text{ contains at least one critical point of } u\}.$$

We call the element of Γ^* the critical level curve. Since C_u is a finite set, Γ^* contains finite elements.

Now we distinguish the level curve into two sets. Suppose $\gamma \in \Gamma$ and a_i is one of the intersection points of γ with some β_i. Then there are two and only two cases:

1. There exists a neighborhood U_i of a_i and a homeomorphism ϕ_i such that
$$\phi_i(U_i) = \mathbb{D},\ \phi_i(a_i) = 0,$$
and
$$\phi_i(\gamma|_{U_i}) = [0, 1).$$
In this case, we call a_i the regular intersection point.

2. There exists a neighborhood U_i of a_i and a homeomorphism ϕ_i such that
$$\phi_i(U_i) = \mathbb{D},\ \phi_i(a_i) = 0,$$
and
$$\phi_i(\gamma|_{U_i}) = (-1, 1).$$
In this case, we call a_i the non-regular intersection point.

It is obvious that all the intersection points of a level curve and $\partial\Omega$ are either regular intersection points or non-regular intersection points.

If a_i is a non-regular intersection point on level curve γ and boundary component β_i, it is obvious that $\frac{\partial u}{\partial \mathbf{T}} = 0$ on a_i since γ is a level curve. Note that u is the unique solution of mixed Dirichlet-Newmann problem, so $\frac{\partial u}{\partial \mathbf{n}} = 0$ on β_i, i.e. $\frac{\partial u}{\partial \mathbf{n}} = 0$ on a_i. We conclude, from above analysis, a_i is a critical point of u. That means, all the non-regular intersection points are critical points.

4.6 No interior critical points

Suppose $\gamma_i (i \in \Lambda)$ be the level curves of u intersecting with some boundary curve β_i. Let
$$\bar{i} = \sup_{i \in \Lambda} i, \quad \underline{i} = \inf_{i \in \Lambda} i.$$

Since β_i is a closed Jordan curve, $\gamma_{\bar{i}}$ and $\gamma_{\underline{i}}$ also intersect with β_i. Suppose $a_{\bar{i}}$ is a point of $\gamma_{\bar{i}} \cap \beta_i$ and $a_{\bar{i}}$ is a regular intersecting point. Then there exists a neighborhood $U_{\bar{i}}$ of $a_{\bar{i}}$ and a homeomorphism $\phi_{\bar{i}}$ such that

$$\phi_{\bar{i}}(U_{\bar{i}}) = \mathbb{D}, \quad \phi_{\bar{i}}(a_{\bar{i}}) = 0,$$

and

$$\phi_{\bar{i}}(\gamma_{\bar{i}} \cap U_{\bar{i}}) = [0, 1).$$

Therefore, there exists some point $a^*_{\bar{i}}$ in $\beta_i \cap U_{\bar{i}}$ such that the level curve γ^*_i passing through $a^*_{\bar{i}}$ satisfies

$$u(\gamma^*_i) > u(\gamma_{\bar{i}}).$$

It is contradictive to the definition of $\bar{i}$. So $a_{\bar{i}}$ is a non-regular intersection point. The same reason can deduce that $\gamma_{\underline{i}}$ also contains a non-regular intersection point. Since non-regular intersecting points are critical points, $\gamma_{\bar{i}}$ and $\gamma_{\underline{i}}$ contain at least one critical point, respectively. That means, β_i contains at least two critical points.

Note that we have $2p+q = 2n$ and each boundary curve β_i $(i = 0, 1, ..., n-1)$ contains at least two critical points, so all the critical points lie on boundary curves of Ω. Furthermore, each β_i contains two and only two critical points. From above analysis, we can also deduce that all the critical points are non-regular intersection points and no regular intersection point is the critical point. That is to say, a point is a critical point of u if and only if it is a non-regular intersection point.

If critical points a_{i_1} and a_{i_2} in β_i are located on the same level curve γ_i, then there exist level curves joining at least one of the two components of $\beta_i \backslash \{a_{i_1}, a_{i_2}\}$ across γ_i. This is a contradiction. Therefore, if some $\gamma_i \in \Gamma^*$ contains k critical points, then there exist disjoint boundary curves $\beta_{i_1}, \beta_{i_2}, ..., \beta_{i_k} \subset \partial\Omega$ such that

$$\sharp(\beta_{i_j} \cap \gamma_i) = 1, \ (j = 1, 2, ..., k)$$

and the point $\beta_{i_j} \cap \gamma_i$ is a critical point of u.

4.7 Domain Decomposition

From the above discussion, the intersection points of level curve γ with some boundary curve β_i is either a critical point or two regular intersection points. So

$$2 \leq \sharp\Gamma^* = l \leq 2n.$$

Therefore, there exist $\gamma_{_A}, \gamma_{_B} \in \Gamma^*$ such that

$$u(\gamma_{_A}) = \inf\{u(\gamma) : \gamma \in \Gamma^*\},$$

$$u(\gamma_{_B}) = \sup\{u(\gamma) : \gamma \in \Gamma^*\}.$$

It is easy to see that $\gamma_{_A}$ and $\gamma_{_B}$ are unique.

Let

$$\Gamma^* = \{\gamma_1 = \gamma_{_A}, \gamma_2, \gamma_3, ..., \gamma_{l-1}, \gamma_l = \gamma_{_B}\},$$

and

$$0 = u(\partial A) = u(\gamma_0) < u(\gamma_1) < u(\gamma_2) < ... < u(\gamma_{l-1})$$
$$< u(\gamma_l) < u(\gamma_{l+1}) = u(\partial B) = 1.$$

For any level curve δ_i, we decompose it by the regular intersection points and critical points. Let

$$\delta_i = \delta_i^1 \cup \delta_i^2 \cup ... \cup \delta_i^{n_i},$$

$$u(\gamma_k) < u(\delta_i) < u(\gamma_{k+1}).$$

It is easy to see that any level curve δ_j satisfying $u(\gamma_k) < u(\delta_j) < u(\gamma_{k+1})$ has the same number components of δ_i, i.e.

$$\delta_j = \delta_j^1 \cup \delta_j^2 \cup ... \cup \delta_j^{n_i}.$$

Furthermore, δ_j^m is homotopic to δ_i^m with respect to $\partial\Omega$ for any $m \in \{1, 2, ..., n_i\}$.

Using the curves in Γ^*, we can decompose Ω into finite subsets Δ_i $(0 \leq i \leq l)$

$$\Delta_i = \{z \in \Omega : u(\gamma_i) < u(z) < u(\gamma_{i+1})\}.$$

It is obvious that

$$\Omega \backslash \Gamma^* = \bigcup_{i=0}^{l} \Delta_i,$$

and each Δ_i is an open set. Note Δ_i need not to be a domain.

Let

$$\Omega_c = \{\Delta_i : \Delta_i \text{ is connected}\},$$

$$\Omega_d = \{\Delta_i : \Delta_i \text{ is disconnected}\}.$$

Since all the level curves in Δ_0 are homotopic to $\gamma_0 = \partial A$, $\Delta_0 \in \Omega_c$. The same reason deduces $\Delta_l \in \Omega_c$. So

$$2 \leq \sharp\Omega_c \leq l + 1.$$

4.8 Integration on critical level curves

Recall that the domain Δ_0 is a doubly connected domain bounded by $\gamma_0 = \partial A$ and γ_1. By using a conformal map of Δ_0 onto an annulus, we can construct a simple arc τ_A in Δ_0 joining ∂A and one critical point a_1 in $\gamma_{_A}$, such that τ_A is orthogonal to each level curve γ_t for $0 < u(\gamma_t) < u(\gamma_1)$ and that

$$\lambda(A, \gamma_1) = \lambda(\partial A, \gamma_1; \widetilde{\Delta}_0),$$

where $\widetilde{\Delta}_0 = \Delta_0 \backslash \tau_A$ is a simply connected domain. By basic properties of the harmonic function,

$$\int_{\partial G} \frac{\partial u}{\partial \mathbf{n}} ds = 0,$$

$$D(u) = \int_{\partial G} u \frac{\partial u}{\partial \mathbf{n}} ds.$$

Since $\frac{\partial u}{\partial \mathbf{n}} = 0$ on $\partial\Omega$ and $u = 0$ on ∂A, $u = 1$ on ∂B, so

$$-\int_{\partial A} \frac{\partial u}{\partial \mathbf{n}} ds = \int_{\partial B} \frac{\partial u}{\partial \mathbf{n}} ds = \int_{\partial G} u \frac{\partial u}{\partial \mathbf{n}} ds = D(u).$$

On the other hand, one can extend v continuously to prime ends of $\partial\Omega' = \partial(\Omega \backslash (\tau_A \cup \tau_B))$. Then simple calculation yields that

$$\int_{\partial A} dv = -\int_{\partial A} \frac{\partial u}{\partial \mathbf{n}} ds,$$

which implies

$$\int_{\partial A} dv = D(u).$$

The same procedure implies

$$\int_{\partial B} dv = D(u).$$

For any critical level curve $\gamma(\gamma \neq \gamma_A, \gamma_B)$, suppose $u(\gamma) = k$ $(0 < k < 1)$. Denote G^* be the component of $\Omega \backslash (\gamma \cup A)$ containing γ_A. Then ∂G^* is composed by ∂A, γ and a part of $\partial\Omega$. Write

$$u^*(z) = \frac{u(z)}{k}, \quad C_{G^*} = \partial G^* \backslash (\gamma \cup \partial A).$$

Since γ has finite components, $u^*(z)$ is the solution of mixed Dirichlet-Newmann problem on G^* which satisfies:

(i) u^* is bounded and harmonic in G^*;

(ii) u^* has a continuous extension to ∂G^*, which is equal to 0 on ∂A and 1 on γ;

(iii) the outer normal derivative $\frac{\partial u}{\partial \mathbf{n}}$ exists and vanishes on C_{G^*}.

Similarly, if v^* is a harmonic conjugate of u^*, we have

$$D(u^*) = \int_{\partial A} dv^* = \int_{\gamma} dv^*.$$

That is,

$$D(u) = \int_\gamma dv.$$

So, for any $\gamma \in \Gamma^*$, we have

$$\int_\gamma dv = D(u).$$

Since the normal derivative of u vanishes on $\partial\Omega$, the tangent derivative of its conjugate v vanishes on the critical points of u. Therefore, v takes constant values on the components of $\partial\Omega\backslash C_u$ and the difference of the values represents the value change of v along different level curves of u.

4.9 Completion of the Proof

Proof. For any $\Delta_i \in \Omega_d$, if

$$\Delta_i = \Delta_i^1 \cup \Delta_i^2 \cup ...\Delta_i^{k_i},$$

then by above analysis, Δ_i^m is disjoint from $\Delta_i^n (m \neq n)$. Denote by Γ_i the curve family composed by the curves in Ω connecting γ_i and γ_{i+1}, and denote by Γ_{ij} the sub-family of Γ_i composed by curves in Δ_i^j.

If we choose some $z \in \Omega\backslash C_u$ such that $v(z) = 0$, then the conformal map $w = u + iv$ maps Δ_i onto the rectangle

$$R_{\Delta_i} = (u(\gamma_i), u(\gamma_{i+1})) \times (0, D(u)).$$

Let $R_{\Delta_i^j}$ be the image of Δ_i^j under mapping $w = u + iv$. Since $\frac{\partial u}{\partial \mathbf{n}} = 0$ on $\partial\Omega$, $R_{\Delta_i^j}$ is also a rectangle in R_{Δ_i}. Furthermore, since

$$\int_{\gamma_i} dv = \int_{\gamma_{i+1}} dv = D(u),$$

we deduce that R_{Δ_i} is composed exactly by $R_{\Delta_i^j} (1 \leq j \leq k_i)$, that is

$$R_{\Delta_i} = \bigcup_{j=1}^{k_i} R_{\Delta_i^j}.$$

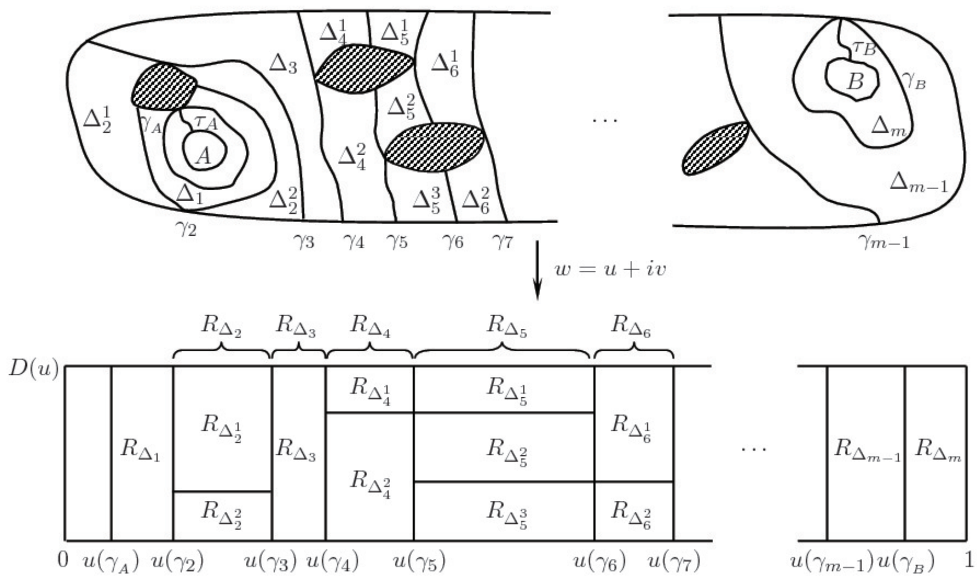

Figure 4.1: Decomposition of G

So

$$\lambda(\Gamma_i) = \frac{u(\gamma_{i+1}) - u(\gamma_i)}{D(u)} = \frac{1}{\frac{1}{\lambda(\Gamma_{i1})} + \frac{1}{\lambda(\Gamma_{i2})} + \cdots + \frac{1}{\lambda(\Gamma_{ik_i})}}.$$

Therefore

$$\sum_{i=0}^{l} \lambda(\Gamma_i) = \sum_{i=0}^{l} \frac{u(\gamma_{i+1}) - u(\gamma_i)}{D(u)} = \frac{1}{D(u)}.$$

Then

$$\lambda(A, B, \Omega) = \frac{1}{D(u)} = \sum_{i=0}^{l} \lambda(\Gamma_i) = \lambda(A, \gamma_A) + \lambda(B, \gamma_B) + \sum_{i=1}^{l-1} \lambda(\Gamma_i).$$

Using the same method as above, we obtain that

$$\lambda(\gamma_A, \gamma_B, \Omega) = \sum_{i=1}^{l-1} \frac{u(\gamma_{i+1}) - u(\gamma_i)}{D(u)} = \frac{1}{D(u)} = \sum_{i=1}^{l-1} \lambda(\Gamma_i).$$

Therefore

$$\lambda(A, B; \Omega) = \lambda(A, \gamma_A) + \lambda(B, \gamma_B) + \lambda(\gamma_A, \gamma_B; \Omega).$$

This completes the proof of Theorem 4.1. $\square$

Chapter 5

The QED constant and the Boundary dilatation on multiply-connected domains

For a Jordan QED domain Ω, a sharp upper bound of QED constant $M(\Omega)$ is obtained in terms of boundary dilatation $H(\Omega)$ in [6]. It is natural to consider the generalization of this result to finitely connected QED domains. We state our main theorem of this chapter:

Theorem 5.1. *Let Ω be a finitely connected QED domain in the extended complex plane. Then either $M(\Omega)$ is attained by a pair of disjoint non-degenerate continua or*

$$M(\Omega) \leq 1 + H(\Omega).$$

First we define the boundary dilatation $H(\Omega)$ for a multiply connected domain (both finite and infinite case) and prove the finiteness of $H(\Omega)$ for a QED domain. The reflection lemma and the comparison lemma given in section 5.2 are two major ingredients in proving the main theorem. In particular, the decomposition theorem in chapter 4 is used in dealing with the degenerate case in the comparison lemma. Finally, the proof of the main theorem is given in section 5.3.

5.1 Boundary dilatation of multiply-connected domains

Boundary $\partial\Omega$ of a quasidisk Ω admits a quasiconformal reflection which interchanges Ω and its exterior domain $\Omega^* = \bar{\mathbb{C}} \setminus \bar{\Omega}$ and keeps the boundary fixed pointwise. For a QED Jordan domain Ω, the quasiconformal reflection constant $R(\Omega)$ is defined as follows:

$$R(\Omega) = \inf\{K(f) : f \text{ is a quasiconformal reflection in } \partial\Omega\}.$$

In order to consider the degenerate case, the localized version of $R(\Omega)$, the boundary dilatation $H(\Omega)$ is defined as:

$$H(\Omega) = \inf\{k(f|\Omega \setminus E) : f \text{ is a quasiconformal reflection and}$$

$$E \text{ is a compact subset in } \Omega\}.$$

It is obvious that $1 \leq H(\Omega) \leq R(\Omega)$.

Since the boundary dilatation is defined only for a Jordan domain, it is necessary to provide a similar definition for a multiply-connected domain. We define boundary dilation along each boundary curve respectively and take the maximum of them.

Definition 5.2. (Boundary dilation of multiply-connected Domains) *Let Ω be a multiply-connected domain whose nondegenerate boundary components consist of disjoint Jordan curves β_j $j = 0, 1...$. Let Ω_j be the component of $C \setminus \beta_j$ containing Ω. Set*

$$H(\beta_j) = \inf\{K(f|\Omega_j \setminus E) : f \textit{ is homeomorphic reflection about}$$

$$\beta_j \textit{ and } E \textit{ is a compact subset in } \Omega_j\},$$

and define the boundary dilatation

$$H(\Omega) = \sup H(\beta_j),\ \ j = 0, 1...$$

To show the finiteness of boundary dilatation $H(\Omega)$ for a QED domain, the following theorem [17] about an upper bound for the constant K of quasicircle domain is needed.

Theorem 5.3. *If D is a QED domain in $\mathbb{C}$, then D is a K-quasicircle domain with*

$$K \leq L^2,$$

where

$$L = \Psi^{-1}(\Psi(1)^{(M(D)-1)}).$$

Using Theorem 5.3, we are ready to show the finiteness of boundary dilatation.

Lemma 5.4. *For any multiply-connected QED domain Ω, $H(\Omega) < \infty$.*

Proof. Given a QED domain Ω, let $\beta_j, j = 1, 2, ...,$ be the non-degenerate boundary components of Ω. Since Ω is a QED domain, then by Theorem 5.3, Ω is a K-quasicircle domain with $K \leq L^2$, where $L = \Psi^{-1}(\Psi(1)^{(M(D)-1)})$. Then each β_j is a K-quasicircle and moreover, each β_j admits quasiconformal reflections f on Ω_j as in Definition 5.2.

Taking the infimum with respect to all quasiconformal reflections f,

$$R(\Omega_j) = \inf_f K(f) \leq K.$$

Thus,

$$H(\beta_j) \leq R(\Omega_j) \leq K.$$

Therefore,

$$H(\Omega) = \sup_j H(\beta_j) \leq K.$$

This gives the finiteness of $H(\Omega)$. □

5.2 Two Lemmas

In this section, we establish two lemmas that are needed in the proof of Theorem 5.1. The first one, reflection lemma, can be regarded as a generalization of the symmetry principle for modulus.

Lemma 5.5. (Reflection Lemma) *Let Ω be a domain in $\mathbb{C}$ with boundary Γ consisting of disjoint quasicircles $\beta_i, i = 0, 1, ..., n-1$, $\Omega_i, i = 0, 1, ..., n-1$ be the component of $\overline{\mathbb{C}}\backslash\beta_i$ containing Ω, $\Omega_{i,\varepsilon}$ be a domain in $\overline{\mathbb{C}}$ with $\overline{\Omega}_i \subset \Omega_{i,\varepsilon}$. Denote $\Omega^*_{i,\varepsilon} = \Omega_{i,\varepsilon}\backslash\overline{\Omega}_i$ and let $f_i : \Omega^*_{i,\varepsilon} \to \Omega_i$ be K_i-quasiconformal reflection across β_i. Write*

$$\Omega_\varepsilon = \bigcap_{i=0}^{n-1} \Omega_{i,\varepsilon}.$$

Then for any disjoint non-degenerate continua $A, B \subset \overline{\Omega}$, we have

$$mod(A, B; \Omega_\varepsilon) \leq (1 + K)mod(A, B; \overline{\Omega}) = (1 + K)mod(A, B; \Omega),$$

where $K = \max_i K_i$.

Proof. Let

$$\Gamma = \Gamma(A, B; \overline{\Omega}), \quad \Gamma_\varepsilon = \Gamma(A, B; \Omega_\varepsilon).$$

For any given admissible function $\rho \in \mathrm{adm}(\Gamma)$, define $\rho_\varepsilon : \mathbb{C} \to [0, \infty)$ as

$$\rho_\varepsilon(z) = \begin{cases} 0, & z \notin \Omega_\varepsilon; \\ \rho(g(z))|g'(z)|, & z \in \Omega_\varepsilon, \end{cases}$$

where

$$g(z) = \begin{cases} z, & z \in \overline{\Omega}; \\ f_j(z), & z \in \Omega^*_{j,\varepsilon}, \ (j = 0, 1, ..., n-1). \end{cases}$$

Let $\gamma_\varepsilon \in \Gamma_\varepsilon$ be a locally rectifiable curve such that g is absolutely continuous on each closed subcurve of γ_ε. Then $g(\gamma_\varepsilon) \in \Gamma$ is also locally rectifiable and

$$\begin{aligned}\int_{\gamma_\varepsilon} \rho_\varepsilon(z)ds &= \int_{\gamma_\varepsilon \cap \overline{\Omega}} \rho_\varepsilon ds + \sum_{j=0}^{n-1} \int_{\gamma_\varepsilon \cap \Omega^*_{j,\varepsilon}} \rho(f_j(z))|f_j'(z)|ds \\ &\geq \int_{\gamma_\varepsilon \cap \overline{\Omega}} \rho ds + \sum_{j=0}^{n-1} \int_{f_j(\gamma_\varepsilon \cap \Omega^*_{j,\varepsilon})} \rho(z)ds \\ &= \int_{g(\gamma_\varepsilon)} \rho ds \geq 1.\end{aligned}$$

Therefore $\rho_\varepsilon \in \mathrm{adm}(\Gamma'_\varepsilon)$ where

$$\Gamma'_\varepsilon = \{\gamma_\varepsilon \in \Gamma_\varepsilon : g \text{ is absolutely continuous on each closed subcurve of } \gamma_\varepsilon\}.$$

Since g is ACL for almost all curve $\gamma_\varepsilon \in \Gamma_\varepsilon$, g is absolutely continuous on each closed subcurve of almost all $\gamma_\varepsilon \in \Gamma_\varepsilon$. So

$$\mathrm{mod}(\Gamma_\varepsilon) = \mathrm{mod}(\Gamma'_\varepsilon).$$

Since f_j is a quasiconformal mapping from $\Omega_{j,\varepsilon}$ to Ω_j, by the analytic properties, we have

$$|f_j'(z)|^2/K_j \leq J(z, f_j),$$

where $J(z, f_j)$ denotes the Jacobian of f_j on z. So

$$\begin{aligned}\mathrm{mod}(\Gamma'_\varepsilon) &\leq \int_{\mathbb{C}} \rho_\varepsilon^2(z)dm \\ &= \int_{\overline{\Omega}} \rho^2 dm + \sum_{j=0}^{n-1} \int_{\Omega^*_{j,\varepsilon}} \rho^2(f_j(z))|f_j'(z)|^2 dm \\ &\leq \int_{\overline{\Omega}} \rho^2 dm + \sum_{j=0}^{n-1} K_j \int_{f_j(\Omega^*_{j,\varepsilon})} \rho^2(z)dm \\ &\leq (1+K) \int_{\overline{\Omega}} \rho^2(z)dm \\ &\leq (1+K) \int_{\mathbb{C}} \rho^2(z)dm.\end{aligned}$$

Taking the infimum over $\rho \in \mathrm{adm}(\Gamma)$ yields that

$$\mathrm{mod}(\Gamma'_\varepsilon) \le (1+K)\mathrm{mod}(\Gamma) = (1+K)\mathrm{mod}(A, B; \overline{\Omega}),$$

So

$$\mathrm{mod}(\Gamma_\varepsilon) \le (1+K)\mathrm{mod}(\Gamma) = (1+K)\mathrm{mod}(A, B; \overline{\Omega}).$$

The fact that the boundary of a QED domain has zero measure induces $\mathrm{mod}(A, B; \overline{\Omega}) = \mathrm{mod}(A, B; \Omega)$. This completes the proof of lemma 5.5. □

We further need to compare the moduli of curve families joining the same disjoint continua in different domains. The comparison lemma shows, in the degenerate case, the moduli of the curve families joining two disjoint continua in the whole plane and in a fixed domain, respectively, are asymptotically the same.

Lemma 5.6. (Comparison Lemma) *Let Ω be a finitely connected domain and Ω_ε be domain with $\bar{\Omega} \subset \Omega_\varepsilon$. Suppose that (A_n, B_n) is a sequence of pairs of disjoint non-degenerate continua in $\bar{\Omega}$ such that at least one of the two sequences $\{A_n\}$ and $\{B_n\}$ converges to a point. Then there exists a subsequence of (A_n, B_n), denoted again by (A_n, B_n), such that*

$$\lim_{n\to\infty} \frac{mod(A_n, B_n; \mathbb{C})}{mod(A_n, B_n; \Omega_\varepsilon)} = 1.$$

Proof. We fix such a domain Ω_ε such that $\overline{\Omega} \subset \Omega_\varepsilon$. Without loss of generality, we may assume that sequence $\{A_n\}$ converges to a point $\{a\}$. For sequence $\{(A_n, B_n)\}$, we will also use notation $\{(A_n, B_n)\}$ to denote its subsequence for our convenience. By the existence of limit of $\mathrm{mod}(A_n, B_n; \Omega_\varepsilon)$, we can assume

$$\lim_{n\to\infty} \mathrm{mod}(A_n, B_n; \Omega_\varepsilon) = L,$$

where $L = 0$ or $L > 0$ (including the case that $L = \infty$). We consider the two cases respectively.

Case 1. $L > 0$. Let

$$\delta = \min_j \ \text{dist } (\partial\Omega, \partial\Omega_\varepsilon) = \inf_{x\in\partial\Omega, y\in\partial\Omega_\varepsilon} |x-y|.$$

We choose a pair of points (a_n, b_n) such that $|a_n - b_n| =$ diam (A_n). Because $\{A_n\}$ converges to a point,

$$\lim_{n\to\infty} |a_n - b_n| = 0.$$

For each curve γ in $\Gamma(A_n, B_n; \mathbb{C})$, either $\gamma \subset \Omega_\varepsilon$ or it contains a subcurve which joins curve $|z - a_n| = |b_n - a_n|$ and curve $|z - a_n| = \delta$ when n is sufficiently large. It follows that

$$\text{mod}(A_n, B_n; \mathbb{C}) \leq \text{mod}(A_n, B_n; \Omega_\varepsilon) + \frac{2\pi}{\ln \frac{\delta}{|b_n - a_n|}}.$$

Taking into account of $|a_n - b_n| \to 0$ as $n \to \infty$, it follows that

$$\lim_{n\to\infty} \frac{\text{mod}(A_n, B_n; \mathbb{C})}{\text{mod}(A_n, B_n; \Omega_\varepsilon)} = 1.$$

Case 2. $L = 0$. Replacing Ω_ε by a subdomain if necessary, we can assume, without loss of generality, Ω_ε is bounded by quasicircles. That is, Ω_ε is a QED domain. For each fixed pair of disjoint non-degenerate continua A_n and B_n, apply theorem 4.1. Then there exist two simple closed curves $\gamma_{_{A_n}}$ and $\gamma_{_{B_n}}$ in $\bar{\Omega}_\varepsilon$ such that there exist two points a_n and b_n with $a_n \in \gamma_{_{A_n}} \cap \partial\Omega_\varepsilon$ and $b_n \in \gamma_{_{B_n}} \cap \partial\Omega_\varepsilon$, the components of $\Omega_\varepsilon \setminus (\gamma_{_{A_n}} \cup \gamma_{_{B_n}})$ which contain continua A_n or B_n are simply connected, and

$$\lambda(A_n, B_n; \Omega_\varepsilon) = \lambda(A_n, \gamma_{_{A_n}}) + \lambda(B_n, \gamma_{_{B_n}}) + \lambda(\gamma_{_{A_n}}, \gamma_{_{B_n}}; \Omega_\varepsilon).$$

On the other hand, by comparison principle of extremal length,

$$\lambda(A_n, B_n; \mathbb{C}) \geq \lambda(A_n, \gamma_{_{A_n}}) + \lambda(B_n, \gamma_{_{B_n}}).$$

So

$$\lambda(A_n, B_n; \mathbb{C}) \leq \lambda(A_n, B_n; \Omega_\varepsilon) \leq \lambda(A_n, B_n; \mathbb{C}) + \lambda(\gamma_{A_n}, \gamma_{B_n}; \Omega_\varepsilon).$$

Thus

$$1 \leq \frac{\operatorname{mod}(A_n, B_n; \mathbb{C})}{\operatorname{mod}(A_n, B_n; \Omega_\varepsilon)} \leq 1 + \operatorname{mod}(A_n, B_n; \mathbb{C})\lambda(\gamma_{A_n}, \gamma_{B_n}; \Omega_\varepsilon). \tag{5.1}$$

Since Ω_ε is a bounded QED domain, we obtain

$$\operatorname{mod}(A_n, B_n; \mathbb{C}) \leq M(\Omega_\varepsilon)\operatorname{mod}(A_n, B_n; \Omega_\varepsilon). \tag{5.2}$$

Since in case 2, it is assumed that $L = 0$, i.e, $\lim_{n\to\infty} \operatorname{mod}(A_n, B_n; \Omega_\varepsilon) = 0$. (5.2) implies that $\lim_{n\to\infty} \operatorname{mod}(A_n, B_n; \mathbb{C}) = 0$.

To complete the proof, it remains to show that $\lambda(\gamma_{A_n}, \gamma_{B_n}; \Omega_\varepsilon)$ is bounded as $n \to \infty$. To this end, fix $x_n \in A_n$ and $y_n \in B_n$. Since $a_n \in \gamma_{A_n} \cap \partial\Omega_\varepsilon$ and $b_n \in \gamma_{B_n} \cap \partial\Omega_\varepsilon$, it follows that

$$|a_n - x_n| \geq \text{ dist } (\bar{\Omega}, \partial\Omega_\varepsilon), \ \ |b_n - y_n| \geq \text{ dist } (\bar{\Omega}, \partial\Omega_\varepsilon),$$

and

$$|y_n - x_n| \leq \text{ diam } (\partial\Omega_\varepsilon), \ \ |b_n - a_n| \leq \text{ diam } (\partial\Omega_\varepsilon).$$

Thus the cross-ratio $[x_n, a_n, y_n, b_n]$ satisfies

$$[x_n, a_n, y_n, b_n] = \frac{|y_n - x_n||b_n - a_n|}{|a_n - x_n||b_n - y_n|} \leq \frac{(\text{ diam } (\partial\Omega_\varepsilon))^2}{(\text{ dist } (\bar{\Omega}, \partial\Omega_\varepsilon))^2}.$$

So an upper bound can be derived for $\lambda(\gamma_{A_n}, \gamma_{B_n}; \Omega_\varepsilon)$ by using extremal property of Teichmüller ring domain as follows.

$$\begin{aligned}
\lambda(\gamma_{A_n}, \gamma_{B_n}; \Omega_\varepsilon) &\leq M(\Omega_\varepsilon)\lambda(\gamma_{A_n}, \gamma_{B_n}; \mathbb{C}) \\
&\leq \frac{1}{2\pi} M(\Omega_\varepsilon) \ln \Psi([x_n, a_n, y_n, b_n]) \\
&\leq \frac{1}{2\pi} M(\Omega_\varepsilon) \ln \Psi\left(\frac{(\text{ diam } (\partial\Omega_\varepsilon))^2}{(\text{ dist } (\bar{\Omega}, \partial\Omega_\varepsilon))^2}\right),
\end{aligned}$$

where Ψ is the function defined by the Teichmüller ring domain as in Example 1.10. Since Ω_ε is a QED domain, $M(\Omega_\varepsilon) < \infty$. So $\lambda(\gamma_{A_n}, \gamma_{B_n}; \Omega_\varepsilon)$ is bounded as $n \to \infty$.

Letting $n \to \infty$ in (5.1), we have

$$\lim_{n\to\infty} \frac{\text{mod}(A_n, B_n; \mathbb{C})}{\text{mod}(A_n, B_n; \Omega_\varepsilon)} = 1.$$

□

5.3 Proof of Main Theorem

Proof. We are ready to prove the main theorem. Recall that quasiextremal distance constant for domain Ω is defined by

$$M(\Omega) = \sup_{A,B\subset\bar{\Omega}} \frac{\text{mod}(A, B; \mathbb{C})}{\text{mod}(A, B; \Omega)},$$

where A, B are disjoint continua in $\bar{\Omega}$. There exists a sequence of pairs of disjoint continua $\{(A_n, B_n)\}$, such that,

$$M(\Omega) = \lim_{n\to\infty} \frac{\text{mod}(A_n, B_n; \mathbb{C})}{\text{mod}(A_n, B_n; \Omega)}.$$

For our convenience, we will denote the subsequence the same as $\{(A_n, B_n)\}$. Under Hausdorf distance, by passing to a subsequence if necessary, we can assume

$$\lim_{n\to\infty} A_n = A, \quad \lim_{n\to\infty} B_n = B.$$

There are three cases need to be considered with respect to A and B.

Case 1. Both A and B are nondegenerate continua with $A \cap B = \emptyset$;

Case 2. At least one of A and B is a single point;

Case 3. Both A and B are nondegenerate continua with $A \cap B \neq \emptyset$.

For case 1, when A and B are disjoint nondegenerate continua,

$$M(\Omega) = \lim_{n\to\infty} \frac{\operatorname{mod}(A_n, B_n; \mathbb{C})}{\operatorname{mod}(A_n, B_n; \Omega)} = \frac{\operatorname{mod}(A, B; \mathbb{C})}{\operatorname{mod}(A, B; \Omega)}.$$

For the other two cases, we will show that $M(\Omega) \leq 1 + H(\Omega)$. First fix $\varepsilon > 0$. By the definition of $H(\Omega)$, there is a quasiconformal reflection f across $\partial\Omega$ such that $K(f) \leq H(\Omega)+\varepsilon$ in a neighborhood of $\partial\Omega$. Thus, we can choose a domain Ω_ε containing $\overline{\Omega}$ such that f is $(H(\Omega)+\varepsilon)$-quasiconformal reflection across $\partial\Omega$ satisfying the conditions in the reflection lemma. Thus,

$$\operatorname{mod}(A_n, B_n; \Omega_\varepsilon) \leq (1 + H(\Omega) + \varepsilon)\operatorname{mod}(A_n, B_n; \Omega). \tag{5.3}$$

In both case 2 and case 3, we claim that

$$\lim_{n\to\infty} \frac{\operatorname{mod}(A_n, B_n; \mathbb{C})}{\operatorname{mod}(A_n, B_n; \Omega_\varepsilon)} = 1. \tag{5.4}$$

In case 2, it follows from the comparison lemma directly since at least one of the two sequence A_n and B_n converges to a point.

In case 3, both continua A and B are nondegenerate with $A \cap B \neq \emptyset$. It is easy to see that $\operatorname{mod}(A_n, B_n; \Omega_\varepsilon) \to \infty$ and $\operatorname{mod}(A_n, \partial\Omega_\varepsilon; \Omega_\varepsilon)$ is bounded as $n \to \infty$. Moreover,

$$\operatorname{mod}(A_n, B_n; \mathbb{C}) \leq \operatorname{mod}(A_n, B_n; \Omega_\varepsilon) + \operatorname{mod}(A_n, \partial\Omega_\varepsilon; \Omega_\varepsilon).$$

Then

$$1 \leq \frac{\operatorname{mod}(A_n, B_n; \mathbb{C})}{\operatorname{mod}(A_n, B_n; \Omega_\varepsilon)} \leq 1 + \frac{\operatorname{mod}(A_n, \partial\Omega_\varepsilon; \Omega_\varepsilon)}{\operatorname{mod}(A_n, B_n; \Omega_\varepsilon)},$$

which yields (5.4).

To finish the proof, combing (5.3) and (5.4), we obtain that

$$\begin{aligned} M(\Omega) &= \lim_{n\to\infty} \frac{\operatorname{mod}(A_n, B_n; \mathbb{C})}{\operatorname{mod}(A_n, B_n; \Omega)} \\ &= \lim_{n\to\infty} \frac{\operatorname{mod}(A_n, B_n; \mathbb{C})}{\operatorname{mod}(A_n, B_n; \Omega_\varepsilon)} \cdot \frac{\operatorname{mod}(A_n, B_n; \Omega_\varepsilon)}{\operatorname{mod}(A_n, B_n; \Omega)} \\ &\leq 1 + H(\Omega) + \varepsilon. \end{aligned}$$

Finally, letting $\varepsilon \to 0$, we obtain

$$M(\Omega) \leq 1 + H(\Omega).$$

This completes the proof of the main theorem. $\square$

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