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Paul Wrayno Date

On the Number of Edges in 2-factor Isomorphic Graphs

By

Paul M. Wrayno
Doctor of Philosophy

Mathematics

Ronald Gould, Ph.D.
Advisor

Dwight Duffus, Ph.D.
Committee Member

Michelangelo Grgni, Ph.D.
Committee Member

Accepted:

Lisa A. Tedesco, Ph.D.
Dean of the James T. Laney School of Graduate Studies

Date

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Paul M. Wrayno
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An abstract of
A dissertation submitted to the Faculty of the
James T. Laney School of Graduate Studies of Emory University
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Abstract

On the Number of Edges in 2-factor Isomorphic Graphs
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A 2-factor is a collection of disjoint cycles in a graph that cover all vertices of that graph. A graph is called 2-factor isomorphic if all of its 2-factors are the same when viewed as a multiset of unlabeled cycles.

In this dissertation, we find the maximum size of 2-factor isomorphic graphs that contain a desired 2-factor. We are also able to give general bounds when no 2-factor is specified or any 2-factor with a fixed number of cycles is desired. We also find similar results for the special case where the underlying graph is bipartite. In each case we provide constructions that attain the maximum size.

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In loving memory of my grandmother, Betty Lawrence

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Chapter 1

Introduction

Hamiltonian problems owe their name to Sir William Rowan Hamilton and the "icosian game," played on the vertices of a dodecahedron, that he introduced in 1856. In graph theoretic terms, the objective of the game was to find a hamiltonian cycle on the 20 vertices of the dodecahedron [5]. Hamiltonian problems are those that attempt to characterize which graph properties or combinations of properties imply the existence of a hamiltonian cycle in the graph. Commonly used properties include the order or size of the graph, minimum degree or degree sum, forbidden or required subgraphs, and connectivity. Graph factor problems date to the same era with a result by Reiss on a factorization of K_{2n} into 1-factors [3]. While 1-factors or matchings are the most heavily studied factor problems, 2-factors are also worthy of examination due to many hamiltonian problems being special cases of 2-factor problems, namely the case where the 2-factor consists of a single hamiltonian cycle. Thus, one may view the structure of cycles in a graph as one problem, rather than as a variety of unrelated problems.

In the remainder of this chapter, we begin with the necessary definitions and set the stage with some related results. In Chapter 2, we find the maximum size of 2-factor isomorphic bipartite graphs and provide a construction for obtaining graphs of this size. In Chapter 3, we extend this result to general graphs and are able to find the maximum size of 2-factor isomorphic graphs. In both chapters we begin by providing constructions for the hamiltonian case and proceed to use these to build large general 2-factor isomorphic graphs. Finding the maximum size mirrors this construction by finding the maximum number of edges between cycles in the 2-factor, the maximum number of chords within the cycles in the 2-factor, where we are again able to build from hamiltonian results. After verifying sharpness in each chapter, we are able to find upper and lower bounds when less is known about the 2-factor. In Chapter 4, we examine several related questions that build on the results of this dissertation.

1.1 Definitions

A *graph*, $G = (V, E)$, consists of set of vertices, V, and a set of edges between pairs of vertices, E. For this dissertation, we will restrict ourselves to *simple graphs*, graphs with finite vertex sets and with no loops (an edge between a vertex and itself) or parallel edges (multiple edges between the same pair of vertices). The cardinality of the vertex set is the *order* of G and is denoted by $|V(G)|$, or simply $|V|$ when the relevant G is clear. The cardinality of the edge set, is the *size* of G and is denoted by $|E(G)|$, or simply $|E|$ when the relevant G is clear. A vertex and an edge are called *incident* if the vertex is one end of the edge. Two vertices, u, and v, are called *adjacent* if the edge uv is in $E(G)$. Similarly, two edges e and f are *adjacent* if they are both incident with a common vertex.

The *degree* of a vertex, v, is the number of vertices adjacent to v and is denoted by $d(v)$. The *neighborhood* of a vertex, v, is the set of vertices adjacent to v and is denoted by $N(v)$. Similarly, the *neighborhood* of a vertex set U is the set of vertices adjacent to some vertex in U and is denoted by $N(U)$. We denote the minimum degree of G by $\delta(G) = \min\{d(v)|v \in V\}$. The minimum sum of degrees of nonadjacent vertices is denoted by $\sigma_2(G) = \min\{d(u) + d(v) \,|\, u, v \in V, uv \notin E\}$. An edge is called a *pendant edge* if one of its vertices has degree one. A *subgraph* $H = (V', E')$, of a graph $G = (V, E)$, is a graph with the property that the vertex and edge sets of H are subsets of the vertex and edge sets of G, i.e. $V' \subseteq V$ and $E' \subseteq E$. This relationship is denoted by $H \subseteq G$. A vertex, v, is said to be *covered* by a subgraph if v is in the vertex set of the subgraph. A subgraph of G is a *spanning subgraph* if it covers all vertices of G. A *walk* is an alternating sequence of vertices and edges from a vertex, u, to a vertex, v, in which each edge joins its preceding and succeeding vertices. A *path* is a walk where each vertex is distinct. A *cycle* is a walk where each vertex is distinct except for the first and last, which are the same vertex. A *chord* is an edge between nonconsecutive vertices of a cycle.

A graph of order n is called *complete* if there is an edge between every pair of vertices, and is denoted by K_n. A subgraph that is complete is called a *clique*. A graph, G, in which the vertex set V can be partitioned into two disjoint sets X and Y with each edge of G incident to a vertex in X and a vertex in Y is called *bipartite* and may be denoted as $G = (X \cup Y, E)$. A *complete bipartite* graph is a bipartite graph $G = (X \cup Y, E)$ that has an edge from every vertex $x \in X$ to every vertex $y \in Y$. Complete bipartite graphs are denoted by $K_{k,n-k}$, where $k = |X|$ and $n - k = n - |X| = |Y|$.

A *regular* graph is a graph for which the degree of each vertex is the same. In particular, if the degree of each vertex is k, then it is a k-*regular* graph. A *hamiltonian path* is a path in G that contains all vertices. Similarly, a *hamiltonian cycle* is a cycle in G that contains all vertices.

We say that two edges are *paired* with respect to cycles C_i and C_j if they, together with an edge on each of C_i and C_j form a C_4. We say that a graph G *covers* a graph H if H is a subgraph of G. We denote by $\lceil x \rceil$ the *ceiling* of x, the least integer greater than or equal to x. Similarly we denote by $\lfloor x \rfloor$ the *floor* of x, the greatest integer less than or equal to x.

1.2 Hamiltonian Results that Lead to 2-Factor Results

The hamiltonian problem seeks to classify what graph properties imply the existence of a hamiltonian cycle. Well known results of this type include Dirac's and Ore's theorems:

Theorem 1.1 (*Dirac*)[7] *For any graph G of order n, if $\delta(G) \geq n/2$ then G has a hamiltonian cycle.*

Theorem 1.2 (*Ore*)[*11*] *For any graph G of order n, $n \geq 3$, if $\sigma_2(G) \geq n$ then G has a hamiltonian cycle.*

For any positive integer k, a *k-factor* of a graph, G, is a k-regular spanning subgraph of G. For this dissertation we will be most interested in 2-factors, which can also be thought of as a collection of vertex disjoint cycles in the graph that cover all vertices. A hamiltonian cycle can be thought of as a 2-factor consisting of a single cycle, and so many hamiltonian questions can be generalized to 2-factor questions. Dirac's theorem for instance can be extended to any 2-factor at the price of a higher minimum degree in the following result of Aigner and Brandt:

Theorem 1.3 [*2*] *For any graph G of order n, if $\delta(G) \geq (2n-1)/3$ then G contains any 2-factor.*

If we instead restrict our class of 2-factors we are able to extend Dirac's and Ore's theorems with their original conditions to

Theorem 1.4 [*6*] *If G is a graph of order n, satisfying either*

(1) $\delta(G) \geq n/2$ and $n \geq 4k$ or

(2) $\sigma_2(G) \geq n$ *and* $n \geq 4k$

then G *contains a 2-factor with exactly* k *cycles.*

We turn now to a variant on the traditional hamiltonian problem.

1.3 2-Factor Hamiltonian and 2-factor Isomorphic Graphs

A graph is said to be *2-factor hamiltonian* if it has a 2-factor and all of its 2-factors are hamiltonian cycles. A graph is said to be *2-factor isomorphic* if it has a 2-factor and all of its 2-factors are isomorphic. In other words, a graph is 2-factor isomorphic if all of its 2-factors have the same multiset of unlabeled cycle lengths, e.g. all 2-factors are of the form $\{C_3, C_3, C_4\}$.

Funk, Jackson, Labbate, and Sheehan [9] determined that a k-regular graph can only be 2-factor hamiltonian if $k = 2$ or 3. With the addition of Aldred, the same group [4] were able to show that k-regular bipartite 2-factor isomorphic graphs share the same requirement for k. Together with Abreu, they [1] also showed that every graph which contains a 2-factor and has minimum degree at least eight has two non-isomorphic 2-factors. Faudree, Gould, and

Jacobson [8] examined the maximum size of 2-factor hamiltonian graphs. They were able to show the next two theorems.

Theorem 1.5 *If G is a bipartite 2-factor hamiltonian graph of order n, $n \geq 8$, then*

$$|E(G)| \leq \left\lceil \frac{n(n+4)}{8} \right\rceil$$

and the bound is sharp.

Theorem 1.6 *If G is a 2-factor hamiltonian graph of order n, $n \geq 7$, then*

$$|E(G)| \leq \left\lceil \frac{n(n+1)}{4} \right\rceil$$

and the bound is sharp.

These last two results and the constructions that attain these sizes will be used heavily as we generalize these results to 2-factor isomorphic graphs.

Chapter 2

Bipartite Graphs

Before turning our attention to the general case, it is informative and interesting in its own right to examine what happens when we restrict ourselves to bipartite graphs.

2.1 Constructions

Here we give a construction for bipartite 2-factor isomorphic graphs of maximum size. These constructions are not exhaustive, but will be used later to demonstrate the sharpness of the calculated maximum size. We seek to build such graphs by using bipartite 2-factor hamiltonian graphs on each cycle of the 2-factor and carefully joining these smaller graphs to form a single large bipartite 2-factor isomorphic graph.

2.1.1 Small Hamiltonian Constructions

We start with the graphs of maximum size for small n. For $n = 4, 6$, the only 2-factors that consist of even cycles, and so the only 2-factors possible in a bipartite graph, are the hamiltonian cycles. Therefore the complete balanced bipartite graphs $K_{2,2}, K_{3,3}$ are 2-factor isomorphic graphs of maximum size. These graphs have 4 and 9 edges respectively. For convenience, we label the partite sets of each as U and V and refer to the labeled $K_{2,2}$ and $K_{3,3}$ as $B(4, V)$ and $B(6, V)$ to emphasize this labeling and maintain a unified naming structure with the larger constructions.

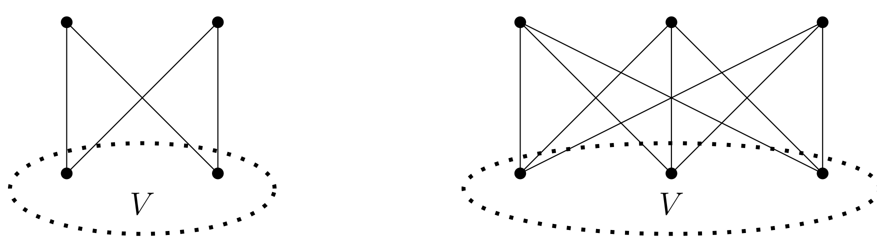

Figure 2.1: $B(4, V)$ and $B(6, V)$.

2.1.2 Larger Hamiltonian Constructions

To construct a 2-factor hamiltonian graph for $n \geq 8$, we use the constructions of Faudree, Gould, and Jacobson [8] and condition on whether

$n \equiv 0$ or 2 mod 4. Both cases give bipartite 2-factor isomorphic graphs with the maximum number, $\lceil \frac{n(n+4)}{8} \rceil$, of edges.

Case 1 *Suppose* $n \equiv 0$ *mod* 4.

Let $B(n, V)$ be the bipartite graph of order $n = 4m$ with partite sets $U = \{u_1, u_2, \ldots, u_{2m}\}$ and $V = \{v_1, v_2, \ldots, v_{2m}\}$. Now define the adjacencies in $B(n, V)$ as follows:

$N(u_1) = \{v_1, v_2\}$,

$N(u_2) = \{v_1, v_2, v_3\}, N(u_3) = \{v_1, v_2, v_4\}$,

$N(nu_4) = \{v_1, v_2, v_3, v_4, v_5\}, N(u_5) = \{v_1, v_2, v_3, v_4, v_6\}, \ldots,$

$N(u_{2j}) = \{v_1, v_2, \ldots v_{2j}, v_{2j+1}\}, N(u_{2j+1}) = \{v_1, v_2, \ldots v_{2j}, v_{2j+2}\}, \ldots,$

$N(u_{2m-2}) = \{v_1, v_2, \ldots, v_{2m-2}, v_{2m-1}\}$,

$N(u_{2m-1}) = \{v_1, v_2, \ldots, v_{2m-2}, v_{2m}\}$,

$N(u_{2m}) = \{v_1, v_2, \ldots v_{2m}\}$

Case 2 *Suppose* $n \equiv 2$ *mod* 4.

Let $B(n, V)$ be the bipartite graph of order $n = 4m+2$ with partite sets $U = \{u_1, u_2, \ldots, u_{2m+1}\}$ and $V = \{v_1, v_2, \ldots v_{2m+1}\}$. Now define the adjacencies in $B(n, V)$ as follows:

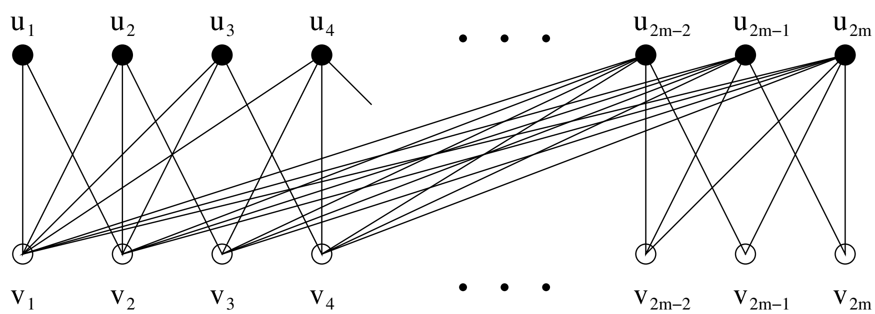

Figure 2.2: The Bipartite Construction for $n \equiv 0 \mod 4$ from [8].

$N(u_1) = \{v_1, v_2\}$,

$N(u_2) = \{v_1, v_2, v_3\}, N(u_3) = \{v_1, v_2, v_4\}$,

$N(u_4) = \{v_1, v_2, v_3, v_4, v_5\}, N(u_5) = \{v_1, v_2, v_3, v_4, v_6\}, \ldots,$

$N(u_{2j}) = \{v_1, v_2, \ldots v_{2j}, v_{2j+1}\}, N(u_{2j+1}) = \{v_1, v_2, \ldots v_{2j}, v_{2j+2}\}, \ldots,$

$N(u_{2m-2}) = \{v_1, v_2, \ldots, v_{2m-2}, v_{2m-1}\}$,

$N(u_{2m-1}) = \{v_1, v_2, \ldots, v_{2m-2}, v_{2m}\}$,

$N(u_{2m}) = N(u_{2m+1}) = \{v_1, v_2, \ldots v_{2m+1}\}$

2.1.3 General Construction

With a specified F as the 2-factor, we will now construct a bipartite 2-factor isomorphic graph of maximum size recursively.

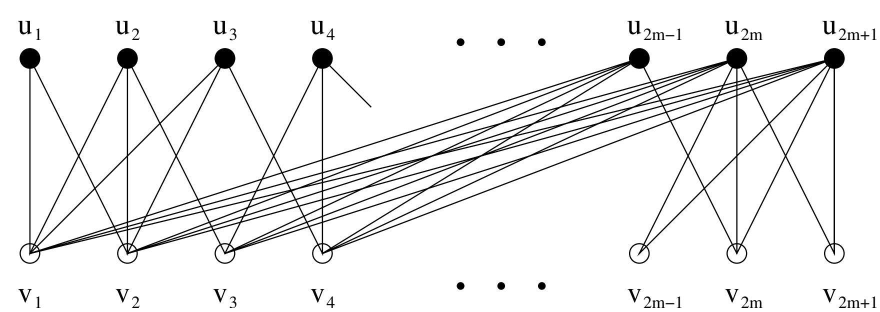

Figure 2.3: The Bipartite Construction for $n \equiv 2 \bmod 4$ [8].

Case 1 *Suppose that F consists of a single cycle.*

The graph G must then be hamiltonian, and we can use the hamiltonian construction for the appropriate n.

Case 2 *Suppose that F consists of multiple cycles.*

Remove a cycle C_k from F and construct a bipartite 2-factor isomorphic graph of maximum size, B' that has $F - C_k$ as its 2-factor. Construct a $B(k, V)$ that covers the removed C_k.

Let U, V be the parts in the bipartition of $B(k, V)$. Let X' and Y' be the parts in the bipartition of B'. Let $X = X' \cup U$, and $Y = Y' \cup V$. Let

$$E = E(B') \cup E(B(k, V)) \cup \{xv \mid x \in X', v \in V\}.$$

Then $B = (X \cup Y, E)$ is our desired graph.

The graph B is bipartite because all edges are between X and Y. Also, B is 2-factor isomorphic because in any 2-factor, there must be $2|U|$ edges from U to V, and so conversely $2|U| = 2|V|$ edges from V to U. This exhausts the degrees of V in the 2-factor and prevents any X' to V edges from appearing in the 2-factor. All edges in the 2-factor must then be from B' and $B(k, V)$. Both B' and $B(k, V)$ were 2-factor isomorphic, so the unions of their 2-factors will be isomorphic as well.

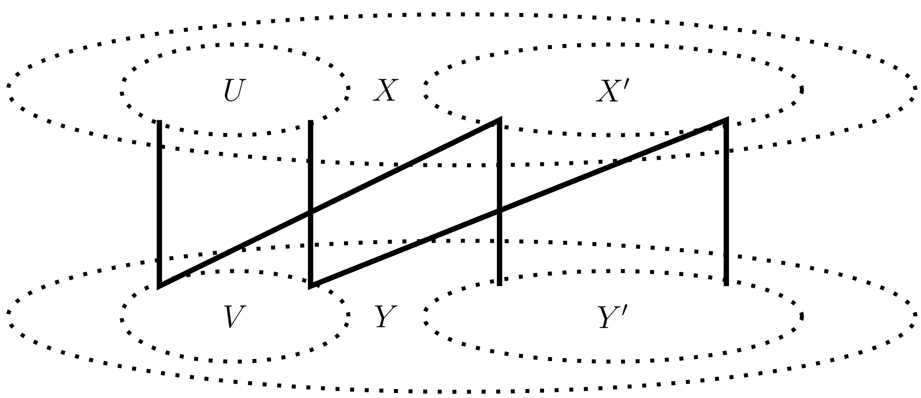

Figure 2.4: The General Construction

2.2 Bounding the Size in Bipartite 2-factor Isomorphic Graphs

Fix a 2-factor F and let c_j be the number of cycles of length j in F. Any edge in a bipartite 2-factor isomorphic graph must be on a cycle in F, a chord of a cycle in F or between two cycles of F.

2.2.1 Bound for Chords of Cycles in F and Edges on Cycles in F

The hamiltonian case gives an upper bound on the number of chords of a cycle in F because any edge in the hamiltonian case that is not on a cycle is a chord of that cycle. Combining the bound over all cycles gives an upper bound of:

$$c_6 + \sum_{C_j \in F} \left\lceil \frac{j(j+4)}{8} \right\rceil \cdot c_j$$

chords and cycle edges. The leading c_6 term accounts for $B(6, V)$ having one edge more than the general formula gives for $j = 6$. Fundamentally, we are just multiplying the bound for each cycle by the number of times it occurs in the 2-factor and adding.

2.2.2 Bound for Edges Between Cycles

We say that two edges are *paired* with respect to cycles C_i and C_j if they, together with an edge on each of C_i and C_j form a C_4. If there exist paired edges between two cycles then we can merge the cycles by replacing the other cycle edges in the C_4 with the paired edges in the 2-factor, creating a non-isomorphic 2-factor with one cycle on these vertices instead of two (see Figure 2.5). Therefore no paired edges can exist between cycles.

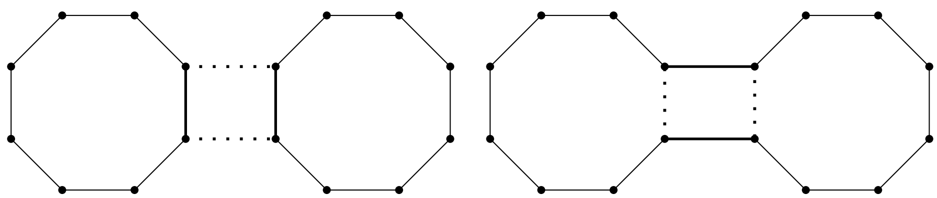

Figure 2.5: Paired Edges Allow Non-Isomorphic 2-factors.

Since the underlying graph is bipartite, a single edge between cycles C_i and C_j forbids half of the possible edges because adding any of those edges would form an odd cycle. Let u and v be consecutive vertices in C_i and $x_1, \ldots x_j$ be the vertices of C_j in the order they appear in the cycle. Without loss of generality, we may assume that ux_1 is present. The edges vx_1 or ux_2 would cause an odd cycle, and so are forbidden. vx_2 and ux_1 are paired, so vx_2 is forbidden. The edge ux_3 is allowable, but vx_3 would cause an odd cycle and so is forbidden. In general, if ux_k is present, ux_{k+2} has the next smallest index for a u-edge and vx_{k+3} has the next smallest index for a v-edge. The same holds if we reverse the roles of u and v. From this we can conclude that $|(N(u) \cup N(v)) \cap C_j| \leq \frac{j}{2}$. This implies that the maximum average degree from C_i to C_j is $\frac{j}{4}$, and implies that there are at most $\frac{i \cdot j}{4}$ edges between C_i and C_j.

Taking this bound over all pairs of cycles C_i, C_j gives an upper bound of

$$\sum_{\{C_i, C_j\} \subset F} \frac{i \cdot j}{4}$$

edges between vertices in different cycles.

2.2.3 Combining the Bounds

Adding the two bounds together gives us an upper bound on the total number of edges:

$$c_6 + \sum_{C_j \in F} \left\lceil \frac{j(j+4)}{8} \right\rceil + \sum_{\{C_i, C_j\} \subset F} \frac{i \cdot j}{4}.$$

Rewriting the sum over pairs of cycles as a double sum over cycles and removing the overcount from initially counting $\{C_j, C_j\}$ as a pair:

$$c_6 + \sum_{C_j \in F} \left\lceil \frac{j(j+4)}{8} \right\rceil + \sum_{C_j \in F} \left[\left(\sum_{C_i \in F} \frac{i \cdot j}{8} \right) - \frac{j^2}{8} \right].$$

Condensing terms with the same cycle length gives:

$$c_6 + \sum_{j=4}^{n} \left\lceil \frac{j(j+4)}{8} \right\rceil \cdot c_j + \sum_{j=4}^{n} \left[\left(\sum_{i=4}^{n} \frac{i \cdot j}{8} \cdot c_i \right) - \frac{j^2}{8} \right] \cdot c_j.$$

The ceiling function can be removed by counting the number of cycles, $c_2^* = \sum_{k=1}^n c_{4k+2}$, for which we round up the term $\left\lceil \frac{j(j+4)}{8} \right\rceil$. Summing over i and simplifying then yields:

$$c_6 + \frac{c_2^*}{2} + \sum_{j=4}^{n} \left(\frac{j(j+4)}{8} + \frac{(n-j) \cdot j}{8} \right) \cdot c_j = c_6 + \frac{c_2^*}{2} + \sum_{j=4}^{n} \frac{(n+4) \cdot j}{8} \cdot c_j.$$

Summing over j then yields:

$$\frac{n(n+4)}{8} + \frac{c_2^*}{2} + c_6.$$

2.3 Sharpness

To show that the bound is sharp, we return to our construction. If F is a single cycle, this agrees with the hamiltonian case, with the last two terms accounting for the ceiling function and $B(6, V)$ having one more edge than the general formula gives. If F consists of multiple cycles, the number of edges on cycles of F and chords of these cycles is precisely that used in the bound. The number of edges between cycles also achieves the bound because we are joining half of the vertices of the new cycle to half of the vertices in each of the previously added cycles, attaining the upper bound of $\frac{i \cdot j}{4}$ edges between cycles C_i and C_j.

Theorem 2.1 *The maximum size of a bipartite 2-factor isomorphic graph of order n with 2-factor F consisting of c_i cycles of length i for $3 \leq i \leq n$ is*

$$\frac{n(n+4)}{8} + \frac{c_2^*}{2} + c_6,$$

where c_2^* is the number of C_i's with $i \equiv 2 \mod 4$, including C_6's.

2.4 Extensions to unknown F

Using this result on fixed 2-factors, we would like to find upper and lower bounds on the maximum size of G when either less information about the fixed F is known or when all that is known is that a 2-factor is present.

2.4.1 The Maximum Size of 2-factor Isomorphic Graphs with a 2-factor Consisting of k Cycles

Suppose that instead of knowing the full form of the 2-factor, we only know that it consists of k cycles. We can still find the maximum size of such a graph by identifying a 2-factor with k cycles that maximizes the formula given by Theorem 2.1. Only the $c_2^*/2$ and c_6 terms are dependent on the structure of the 2-factor, so we may focus on maximizing these.

We begin with any 2-factor consisting of k cycles. If there are two large cycles, C_i and C_j, with $i, j > 6$, then by replacing them with a C_6 and a C_{i+j-6} we increase c_6 and do not reduce c_2^*. If there is a C_4 and a C_i, $i > 6$ in the 2-factor, then by replacing a C_4 and C_i with a C_6 and C_{i-2} we again increase c_6 and do not reduce c_2^*. Both of these replacement operations increase the maximum size of a 2-factor isomorphic graph. Repeating these processes as much as possible yields three final forms depending on the relationship between n and $6k$.

If $n < 6k$, large cycles are exhausted before C_4's, and the final state consists entirely of C_4's and C_6's. If instead, $n > 6k$, then the C_4's are exhausted before the large cycle(s), the number of large cycles is further reduced to a single large cycle, and the final state consists entirely of C_6's and one large C_i. Finally, if $n = 6k$, large cycles and C_4's are exhausted simultaneously, and the final state consists entirely of C_6's. More precisely, we are left with $\dfrac{6k-n}{2}$ C_4's and $\dfrac{n-4k}{2}$ C_6's if $n < 6k$, $k-1$ C_6's and a C_{n-6k+6} if $n > 6k$, or k C_6's if $n = 6k$. Evaluating Theorem 2.1 for these 2-factors gives:

Corollary 2.2 *If G is a bipartite 2-factor isomorphic graph with a 2-factor consisting of k cycles, then the maximum size of G and the 2-factor that* *allows attainment of this maximum are given by:*

Max Size	*Domain*	*2-factor*
$\frac{n^2+10n-24k}{8}$	$n < 6k$	$\{C_4, \ldots, C_4, C_6, \ldots, C_6\}$
$\left\lceil \frac{n^2+4n+12k-12}{8} \right\rceil$	$n > 6k$	$\{C_6, \ldots, C_6, C_{n-6k+6}\}$
$\frac{n^2+6n}{8}$	$n = 6k$	$\{C_6, \ldots, C_6\}$

2.4.2 The Maximum Size of 2-Factor Isomorphic Graphs with Unspecified 2-factors

Examining the results for the 2-factor consisting of k cycles we see that, for fixed n, the maximum size is a decreasing function of k if $n < 6k$ and an increasing function of k if $n > 6k$. Thus for $n < 6k$ the size is maximized when k is minimized, that is, when $k = \lfloor \frac{n}{6} \rfloor + 1$ and for $n > 6k$ the size is maximized when k is maximized, that is, when $k = \lceil \frac{n}{6} \rceil - 1$. The overall maximum size is either attained at one of these values of k or potentially at $k = n/6$ if 6 divides k. Rewriting n in terms of these potential k's and applying Theorem 2.1 gives a size of

$$\frac{n^2+6n}{8} + c$$

where c is given by:

$n \equiv$	$n < 6k$		$n > 6k$		$n = 6k$	
	$n =$	c	$n =$	c	$n =$	c
$0 \mod 6$	$6k - 6$	-3	$6k + 6$	-3	$6k$	0
$2 \mod 6$	$6k - 4$	-2	$6k + 2$	-2	$*$	$*$
$4 \mod 6$	$6k - 2$	-1	$6k + 4$	-3	$*$	$*$

The largest c in each row determines the overall maximum size and the relationship between k and n for that c determines which 2-factor(s) allow attainment of this bound.

Corollary 2.3 *If G is a bipartite 2-factor isomorphic graph with an unspecified 2-factor, then the maximum size of G and the 2-factor(s) that attains this maximum are given by:*

n	*Max*	*2-factor(s)*
$n \equiv 0 \mod 6$	$\frac{n^2 + 6n}{8}$	$\{C_6, \ldots, C_6\}$
$n \equiv 2 \mod 6$	$\frac{n^2 + 6n - 16}{8}$	$\{C_4, C_4, C_6, \ldots, C_6\}, \{C_6, \ldots, C_6, C_8\}$
$n \equiv 4 \mod 6$	$\frac{n^2 + 6n - 8}{8}$	$\{C_4, C_6 \ldots, C_6\}$

2.4.3 The Lower Bounds of the Maximum Size of 2-factor Isomorphic Graphs with a 2-factor Consisting of k Cycles or Unspecified 2-factor

In addition to the overall maximum size, it is useful to examine how much variation there is in maximum size depending on the form of the 2-factor. To this end, we now attempt to identify which 2-factor minimizes the formula given by Theorem 2.1. Only the $c_2^*/2$ and c_6 terms are dependent on the structure of the 2-factor, so we may focus on minimizing these.

We begin with any 2-factor consisting of k cycles. If there are two cycles, C_i and C_j, with $i, j \equiv 2 \mod 4$, then by replacing them with a C_{i-2} and a C_{i+2} we do not increase c_6 and we reduce c_2^*. If there is a C_6 and a C_i, $i > 6$ in the 2-factor, then by replacing a C_6 and C_i with a C_4 and C_{i+2} we decrease c_6 and do not increase c_2^*. Repeating these processes as much as possible yields three final forms: a special form if $n = 4k+2$ and two general forms depending on whether $n \equiv 0 \mod 4$ or $n \equiv 2 \mod 4$.

For $n = 4k + 2$, only one 2-factor is possible, $\{C_4 \ldots, C_4, C_6\}$, for which $c_6 = c_2^* = 1$. For $n \neq 4k+2$ and $n \equiv 0 \mod 4$, the 2-factor consists of cycles with lengths that are all multiples of 4, for which $c_6 = c_2^* = 0$. For $n \neq 4k+2$

and $n \equiv 2 \mod 4$, the 2-factor consists of cycles with lengths that are all multiples of 4 and one large cycle with a length at least 10 and $\equiv 2 \mod 4$, for which $c_6 = 0$ and $c_2^* = 1$. Theorem 2.1 then gives the following:

Corollary 2.4 *If G is a bipartite 2-factor isomorphic graph with a 2-factor consisting of k cycles, then the lower bound of the maximum size of G and the 2-factor that allows attainment of this bound are given by:*

Max Size	n
$\frac{n^2+4n+12}{8}$	$n = 4k + 2$
$\frac{n^2+4n}{8}$	$n \equiv 0 \mod 4$
$\frac{n^2+4n+4}{8}$	$n \neq 4k + 2$ *and* $n \equiv 2 \mod 4$

Note that the only form dependent on k is the first form, so the lower bound for the unspecified 2-factor case can differ only for this form and any change in k shifts the 2-factor to the third form. The only n for which k cannot changed is $n = 6$ which retains the special form.

Corollary 2.5 *If G is a bipartite 2-factor isomorphic graph with an unspecified 2-factor, then the lower bound of the maximum size of G and the 2-factor* *that allows attainment of this bound are given by:*

Max Size	n
9	$n = 6$
$\frac{n^2+4n}{8}$	$n \equiv 0 \mod 4$
$\frac{n^2+4n+4}{8}$	$n \equiv 2 \mod 4.$

Chapter 3

General Graphs

We now turn our attention to the general case, and determine the maximum size of general 2-factor isomorphic graphs.

3.1 Constructions

Here we give a construction for 2-factor isomorphic graphs of maximum size. We seek to build such graphs by using 2-factor hamiltonian graphs on each cycle of the 2-factor and carefully joining these smaller graphs to form a single large 2-factor isomorphic graph.

We begin with the construction of small, $n \leq 6$, hamiltonian graphs, then reference constructions given by Faudree, Gould, and Jacobson [8] for the general hamiltonian case. We will then use these constructions to build graphs containing a desired 2-factor. These newly constructed graphs will be shown to attain the maximum size and to be 2-factor isomorphic in Sections 3.3 and 3.4.

3.1.1 Small Order Hamiltonian Constructions

For $n \leq 5$, there are insufficient vertices to find multiple disjoint cycles. Therefore the complete graph, K_n, is the 2-factor hamiltonian graph of maximum size. For $n = 6$, we take the complete bipartite graph $K_{3,3}$ with partite sets U and V and add in all possible edges into the partite set V.

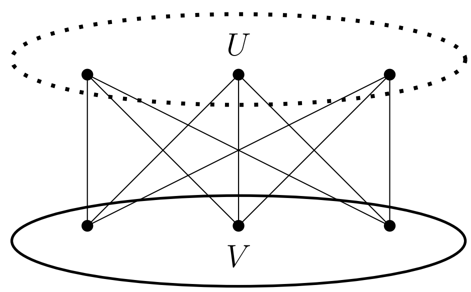

Figure 3.1: $G(6, V)$.

3.1.2 Larger Order Hamiltonian Constructions

To construct a 2-factor hamiltonian graph for $n \geq 7$, we once again turn to constructions of Faudree, Gould, and Jacobson [8] that build on those used in Chapter 2.

For even n, we take the bipartite graph $B(n,V)$, and add all edges between pairs of vertices in V to form the graph $G(n,V)$. For odd n, we form $G(n,V)$ by adding a new vertex, x, to $G(n-1,V)$ and joining x to the vertex set $\{v_1, v_2, \ldots v_{2m}\}$ and to $u_{(n-1)/2}$.

If we extend this construction to $n < 7$, we still get useful graphs. The graph $G(6,V)$ is the same 2-factor hamiltonian graph of maximum size that we found in the last subsection. We will see later that while the graphs $G(5,V)$ and $G(4,V)$ are not of maximum size, they generally serve as better building blocks when constructing general 2-factor isomorphic graphs of maximum size. There is no $B(2,V)$ from which to build a $G(3,V)$, so for convenience and uniformity of notation, we now define $G(3,V)$ as a K_3 with a single vertex as V.

3.1.3 Construction Of Non-Hamiltonian 2-factor Isomorphic Graphs

Given a 2-factor F, we will construct a 2-factor isomorphic graph, G, of maximum size recursively. We consider three cases.

Case 1 *Suppose that F consists of a single cycle.*

The graph G must then be hamiltonian, and we can use the hamiltonian construction for the appropriate n.

Case 2 *Suppose that F consists of multiple cycles, but that all of these cycles are odd.*

If $F = \{C_3, C_5\}$, then by an exhaustive analysis, G consists of $K_3 \cup K_5 \cup E'$, where

$$E' = \{uv \mid u \text{ is a particular vertex of } K_3 \text{ and } v \in K_5\}.$$

This construction yields a graph of order 8 and size 18. If F is not this special case, then remove the largest cycle C_{2k+1} from F and construct the 2-factor isomorphic graph, G' of maximum size and 2-factor $F - C_{2k+1}$. Construct a $G(2k+1, V)$ that covers the removed C_{2k+1}. Join the V set in $G(2k+1, V)$ to all vertices in G' and call this set of new edges E'. The graph

$$G = (V(G') \cup V(G(2k+1, V), E(G') \cup E(G(2k+1, V)) \cup E')$$

is our desired graph.

The graph G remains 2-factor isomorphic because in any 2-factor, there must be $2|U| - 1$ edges from U to V, and so conversely $2|U| - 1 = 2|V| - 1$ edges from V to U. The odd vertex, x, in neither U nor V must use up an adjacency to U and V in the 2-factor as well. The edges from U and x

exhaust the vertices of V and prevents any V to G' edges from appearing in the 2-factor. All edges in the 2-factor must then be from G' and $G(k, V)$. Both G' and $G(k, V)$ were 2-factor isomorphic, so any union of their 2-factors will be isomorphic as well.

Case 3 *Suppose that F consists of multiple cycles, at least one of which is even.*

Remove the largest cycle C_{2k} from F and construct the 2-factor isomorphic graph of maximum size, G' that has $F - C_{2k}$ as its 2-factor. Construct a $G(2k, V)$ that covers the removed C_{2k}. Join the V set in $G(2k, V)$ to all vertices in G' and call this set of new edges E'. The graph

$$G = (V(G') \cup V(G(2k, V), E(G') \cup E(G(2k, V)) \cup E')$$

is our desired graph.

The graph G remains 2-factor isomorphic because in any 2-factor, there must be $2|U|$ edges from U to V, and so conversely $2|U| = 2|V|$ edges from V to U. The edges from U exhaust the degrees of V and prevents any V to G' edges from appearing in the 2-factor. All edges in the 2-factor must then be from G' and $G(k, V)$. Both G' and $G(k, V)$ were 2-factor isomorphic, so the union of their 2-factors will be isomorphic as well.

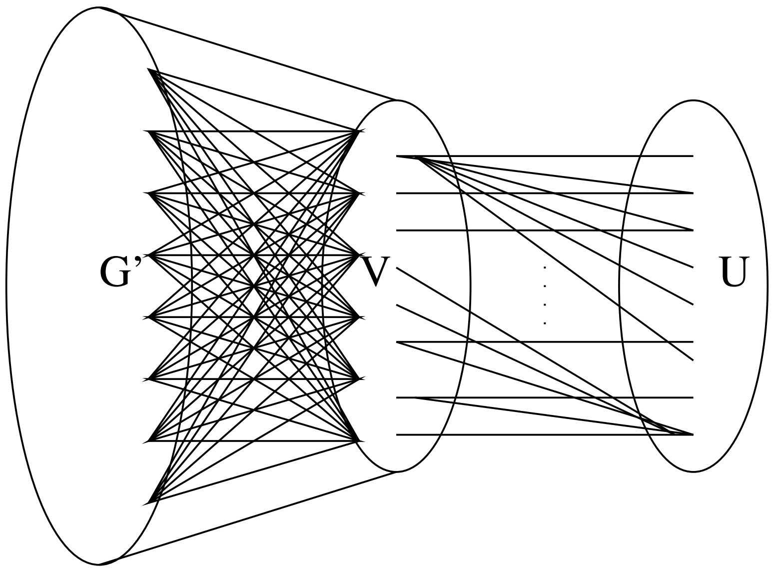

Figure 3.2: $V \subset G(2k, V)$ or $G(2k+1, V)$ is joined to G'.

3.2 Bounding the Size of 2-factor Hamiltonian Graphs

3.2.1 Bound for $n \leq 5$

If G is a graph on $n \leq 5$ vertices, then there are insufficient vertices to form two or more disjoint cycles, the only 2-factors are hamiltonian cycles, and the complete graph, K_n is the 2-factor isomorphic graph of maximum size. The maximum size of G is then $\dfrac{n(n-1)}{2}$.

3.2.2 Bound for $n = 6$

When $n = 6$ there are sufficient vertices to form disjoint cycles, namely two disjoint C_3's. Instead of seeing how many edges can be present, we examine how few edges need to be missing from a complete graph for a graph to be 2-factor hamiltonian. If only two edges are missing they are either disjoint or incident.

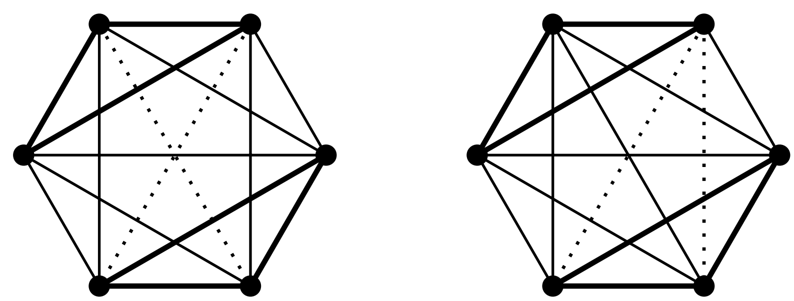

Figure 3.3: Graphs with Two Missing Edges

In either case, a $\{C_3, C_3\}$ 2-factor remains so at least three edges must be missing and the maximum size is at most 12. The graph $G(6, V)$ attains this bound.

3.2.3 Bound for $n \geq 7$

For $n \geq 7$, Theorem 1.6 gives $|E(G)| \leq \left\lceil \frac{n(n+1)}{4} \right\rceil$ and this bound is attained by $G(n, V)$.

3.3 Bounding the size of Non-Hamiltonian 2-factor Isomorphic Graphs

Fix a 2-factor F and for $3 \leq j \leq n$, let c_j be the number of cycles of length j in F. Any edge in a 2-factor isomorphic graph must be a chord of a cycle in F, an edge on a cycle in F or an edge between two cycles in F. We now determine bounds on the number of edges of each type.

3.3.1 Bound for Chords of Cycles in F and Edges on Cycles in F

The hamiltonian result provides an upper bound on the number of chords of a cycle in F and edges on that cycle. To see this, observe that any edge in the hamiltonian case that is not on a particular hamiltonian cycle is a chord of that cycle, so the hamiltonian result bounds the sum of these two types of edges. Combining this bound over all cycles gives an upper bound of:

$$c_6 + \sum_{C_j \in F} \left\lceil \frac{j(j+1)}{4} \right\rceil = c_6 + \sum_{j=3}^{n} \left\lceil \frac{j(j+1)}{4} \right\rceil \cdot c_j.$$

The leading c_6 term accounts for $G(6, V)$ having one more edge than the general formula gives for $j = 6$.

3.3.2 Bound for Edges Between Cycles

As with the bipartite case, we say that two edges are *paired* with respect to cycles C_i and C_j if they, together with an edge on each of C_i and C_j form a C_4. If there exist paired edges between two cycles (see Figure 3.4) then we can merge the cycles by replacing the other cycle edges in the C_4 with the paired edges in the 2-factor, creating a non-isomorphic 2-factor with one cycle on these vertices instead of two. Therefore no paired edges can exist between cycles.

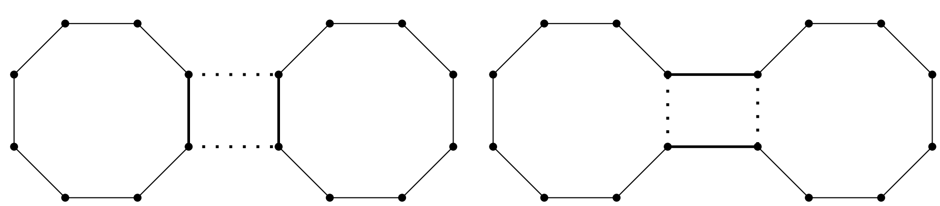

Figure 3.4: Paired Edges Allow Non-Isomorphic 2-factors.

Let u and v be consecutive vertices in C_i and $y_1, \ldots y_j$ be the vertices of

C_j in the order they appear in the cycle. Without loss of generality, we may assume that uy_1 is present. The edges vy_2 and uy_1, and the edges vy_j and uy_1 are paired, so vy_2 and vy_j are forbidden. In general, if uy_k is present, vy_{k+1} and vy_{k-1} are forbidden. Each present edge forbids two edges from u and v and is in turn forbidden by either of those same two edges. Therefore, the maximum number of edges from u and v to C_j is j. This bound applies to all pairs of consecutive vertices in C_i, so the average degree from C_i to C_j is at most $j/2$. There are therefore at most $\left\lfloor \frac{i \cdot j}{2} \right\rfloor$ total edges between C_i and C_j.

To prove sharpness when i is even, label the vertices of C_i as $x_1, \dots x_i$ and partition them into

$$U = \{x_{2k-1} \mid k \in [1, i/2]\} \text{ and } V = \{x_{2k} \mid k \in [1, i/2]\}.$$

Take as edges $\{xy \mid x \in V,\ y \in C_j\}$.

If i and j are both odd, we can sharpen this bound slightly. We first assume, via symmetry, that $i \geq j$ and note that we can reach $j \cdot \left\lfloor \frac{i}{2} \right\rfloor$ edges by following the same construction that we used for the even case, but noting that vertex x_i is in neither U nor V and any additional edge from x_i is forbidden because $x_i y_k$ and $x_{i-1} y_{k-1}$ are paired and $x_{i-1} y_{k-1}$ is already present. If we are to improve upon this construction, there must be two adjacent vertices in C_i

whose sum of degrees from C_i to C_j is

$$d_{ij}(x_k) + d_{ij}(x_{k+1}) > \frac{2}{i} \cdot \left(j \cdot \left\lfloor \frac{i}{2} \right\rfloor \right)$$

implying that $d_{ij}(x_k) + d_{ij}(x_{k+1}) \geq j$. Note though, that each neighbor of x_k forbids two others from being neighbors of x_{k+1} and each of these can be forbidden by up to two neighbors of x_k. In order to attain $d_{ij}(x_k)+d_{ij}(x_{k+1}) = j$, every forbidden neighbor must be doubly forbidden. This state can only happen if x_k has no neighbors, or it has all of $V(C_j)$ as neighbors, since doubly forbidding forces $y_{l+2} \in N_{C_j}(x_k)$ whenever $y_l \in N_{C_j}(x_k)$ and j being odd transforms an iterated version of this condition into $y_{l+(j+1)} = y_{l+1} \in N_{C_j}(x_k)$ whenever $y_l \in N_{C_j}(x_k)$. In either case, we get one of x_k and x_{k+1} adjacent to $V(C_j)$ and the other adjacent to nothing in $V(C_j)$. Without loss of generality, we can assume the former.

By symmetry, when $N_{C_j}(x_k) = V(C_j)$, $N_{C_j}(x_{k-1}) = \emptyset$ as well. So

$$d_{ij}(x_{k-1}) + d_{ij}(x_k) + d_{ij}(x_{k+1}) = j.$$

As previously noted, the average degree from C_i to C_j is at most $j/2$, so the sum of the degrees of the $i-3$ other vertices in C_i is at most $\frac{i-3}{2}j$ for a total of $\frac{i-1}{2} \cdot j$ edges between C_i and C_j, and we cannot improve on the

$$\left\lfloor \frac{i}{2} \right\rfloor \cdot j = \frac{i-1}{2} \cdot j$$

edges given by the construction. This final bound holds for all C_i and C_j under the assumption that i is either even or $i \geq j$ since it becomes $\frac{i \cdot j}{2}$ when i is even.

This bound on the number of edges between cycles can be generalized to a bound on the number of edges between one cycle and the rest of the graph to give:

Lemma 3.1 *The number of edges from a cycle C_i to the rest of the graph is at most*

$$\left\lfloor \frac{i(n-i)}{2} \right\rfloor$$

and this bound is sharp when i is even.

Proof. The sum of the orders of all other cycles is $n-i$, so summing $\left\lfloor \frac{i \cdot j}{2} \right\rfloor$ over all cycles C_j gives an upper bound of $\left\lfloor \frac{i(n-i)}{2} \right\rfloor$. If i is even, there is no rounding down in any of the terms, and so no rounding down in the upper bound. □

3.3.3 Combining the Bounds and Refining

The hamiltonian case provides a bound for the total number of chords within cycles of F and edges on cycles of F and we now have a bound for the number of edges between cycles while avoiding non-isomorphic 2-factors. Combining the two bounds gives us an overall upper bound on the number of edges in the 2-factor isomorphic graph, but we can improve this bound slightly and achieve sharpness via our construction.

To improve this bound, we will remove one cycle at a time, examining the maximum number of edges that could have been removed at each step, until the remaining graph is hamiltonian. After a cycle has been removed, the remaining graph must be 2-factor isomorphic because a non-isomorphic 2-factor in the smaller graph can be extended to a non-isomorphic 2-factor in the original graph by restoring the removed cycle. The size of the remaining graph must then be at most the maximum size of a 2-factor isomorphic graph with the remaining 2-factor. As in the hamiltonian case, cycles of lengths 4 and 5 behave a bit differently from those of other lengths. Therefore, we use the following lemmas to sharpen the upper bound on the number of edges incident to cycles of these lengths.

Lemma 3.2 *Using a $G(4, V)$ rather than a K_4 to cover a C_4 allows more edges in a 2-factor isomorphic graph that contains an odd cycle, C_{2k+1} in its 2-factor.*

Proof. If two consecutive vertices in C_{2k+1} have edges going to distinct vertices of K_4, we can form C_{2k+5}. Thus we can have either $2k+1$ edges from C_{2k+1} to a single vertex of the K_4, or $4\left\lfloor\frac{2k+1}{2}\right\rfloor = 4k$ edges between the C_4 and C_{2k+1}, by alternating neighborhoods of $V(K_4)$ and $\emptyset$ among the vertices of C_{2k+1}. This last case yields a maximum of $4k+2$ edges between the C_{2k+1} and C_4 and chords of the C_4. If a $G(4,V)$ is used to cover the C_4 instead, we get $2(2k+1)+1 = 4k+3$ edges between the C_4 and C_{2k+1} and chords of the C_4. The $G(4,V)$ can also attain the maximum of $2(n-2k-5)$ edges to the rest of the cycles by joining V to the $(n-2k-5)$ other vertices, so the K_4 cannot make up its deficiency elsewhere. □

Lemma 3.3 *Using a $G(5,V)$ instead of a K_5 or K_5-e (a K_5 with an edge removed) allows more edges in a 2-factor isomorphic graph that contains another K_5 or K_5-e on a C_5 in its 2-factor.*

Proof. Let H_1, H_2, respectively be isomorphic to K_5 or K_5-e. Let u_i and v_i, $i \in \{1,2,3,4,5\}$ be the vertices of H_1 and H_2 respectively. If u_iv_j and u_iv_k are present, we can find a C_4 and a C_6 in H_1-u_i and $H_2\cup(\{u_i\},\{u_iv_j,u_iv_k\})$ respectively, since $C_4 \subset K_4-e \subseteq H_1-u_i$ and there exists a hamiltonian path P from v_j to v_k in H_2 to which we can add edges v_ku_i and u_iv_j to form the C_6. Thus there can be at most one edge from u_i to $V(H_2)$. If u_iv_j, u_kv_l, with $i \neq k$, and $j \neq l$, are present, then we can find hamiltonian paths P_1, and P_2 from u_i to u_k, and v_l to v_j, in H_1, and in H_2, respectively. Joining these paths with u_iv_j and u_kv_l creates a C_{10}. Therefore, at most one edge can be present between H_1 and H_2 for a maximum of 21 edges on the subgraph induced by the vertices of H_1 and H_2. Compare this result to a $G(5, V)$ and a K_5 or $K_5 - e$, where we can have a total of 28 or 27 edges respectively; 8 from the $G(5, V)$, 10 from the K_5 or 9 from the $K_5 - e$ and 10 between the $G(5, V)$ and the K_5 or $K_5 - e$. Since $G(5, V) \subset K_5 - e \subset K_5$, any permissible collection of edges from the K_5 or $K_5 - e$ to the rest of the graph is also a permissible collection of edges from the $G(5, V)$ to the rest of the graph. Therefore the maximum number of edges from a $G(5, V)$ to the rest of the graph is at least as large as from a K_5 or $K_5 - e$ to the rest of the graph. □

Lemma 3.4 *Using a $G(4, V)$ rather than a K_4 on a C_4 allows more edges in a 2-factor isomorphic graph that already has a K_4 on a C_4.*

Proof. If two edges are present between the K_4's we can construct either a C_3 and a C_5, or a C_8. If these edges are incident to each other at a vertex, v, then we can construct a C_3 and a C_5. Let the vertex set of the K_4 not containing v be $\{u_1, u_2, u_3, u_4\}$, where u_1, u_2 are adjacent to v, we have

$u_1vu_2u_3u_4 \cong C_5$ and $K_4 - v \cong C_3$. If v_1u_1, v_2u_2 are the two edges between the K_4's $\{u_1, u_2, u_3, u_4\}$ and $\{v_1, v_2, v_3, v_4\}$ we have $v_1u_1u_3u_4u_2v_2v_3v_4v_1 \cong C_8$. This result gives us a maximum of 13 edges for the two C_4's in the 2-factor versus 18 edges available via two $G(4, V)$'s and 19 available via one K_4 and one $G(4, V)$. Further, $G(4, V)$ can attain the maximum of $2(n-8)$ edges to the rest of the cycles by joining V to the $(n-8)$ other other vertices, so the K_4 cannot make up its deficiency elsewhere. □

Lemma 3.5 *Using a $G(5, V)$ instead of a K_5 or $K_5 - e$ allows at least as many edges in a 2-factor isomorphic graph that contains at least two C_3's in its 2-factor.*

Proof. If there exist edges from two distinct vertices u_1, and u_2 in a K_3 to two distinct vertices v_1, and v_2, respectively, in the K_5, or $K_5 - e$, then we can form a C_8 by taking a hamiltonian path P from v_1 to v_2 in $K_5 - e \subset K_5$ to which we can append edges v_2u_2, u_2u_3, u_3u_1, and u_1v_1, where u_3 is the third vertex of the K_3, to complete the cycle. Therefore, there can be at most five edges between a K_3 and a K_5 or $K_5 - e$, and this maximum can be achieved by joining a vertex of the K_3 to all vertices of the K_5 or $K_5 - e$. Each K_3 has three edges, and there can be at most three edges between the two K_3's.

This analysis gives a maximum of 29 edges in the subgraph induced on the vertices of the C_5 and two C_3's. If $G(5, V)$ is used on the C_5 instead, we can join V to each K_3 for 6 edges between each K_3 and $G(5, V)$. This choice can also attain the maximum of 29 edges in the subgraph induced on the vertices of the C_5 and two C_3's. Also, because $G(5, V) \subset K_5 - e \subset K_5$, the maximum number of edges from a $G(5, V)$ to the rest of the graph is at least as large as from a K_5 to the rest of the graph. □

3.3.4 Finding and Attaining the Bound

Let G be a 2-factor isomorphic graph of maximum size on n vertices. If at any stage in our analysis, G is hamiltonian, we use the earlier formulas for the maximum size of 2-factor hamiltonian graphs. Assume instead that the 2-factor contains multiple cycles. We remove cycles one at a time from the 2-factor, keeping careful track of the maximum number of edges that can be removed at each step.

Let C_{2k} be the largest even cycle in the 2-factor. If no such cycle exists, skip ahead to dealing with only odd cycles. From Lemma 3.1 we know that there can be at most $(n - 2k)k$ edges between the C_{2k} and the rest of the graph. If $2k \neq 4$, then there are at most $|E(G(2k, V))|$ induced on the vertices of

C_{2k}. If $2k = 4$, then any other cycles in the graph are either odd or also of length 4. If there is an odd cycle, Lemma 3.2 implies that removing the C_{2k} removes at most $5 + 2(n - 4) = 2n - 3$ edges. If there is another C_4, then Lemma 3.4 implies that more edges are removed if at least one of the C_4's is covered by a $G(4, V)$. Remove a C_4 that is covered by a $G(4, V)$. Removing this cycle removes at most $2n - 3$ edges. Repeat the process with the next largest even cycle in the 2-factor until the remaining 2-factor consists of only one cycle or consists of all odd cycles.

Let C_{2k+1} be the largest odd cycle in the 2-factor. From Section 3.3.2, we know that there can be at most $(n - 2k - 1)k$ edges between the C_{2k+1} and the rest of the graph. If $2k + 1 \neq 5$, there are at most $|E(G(2k, V))|$ edges induced on the vertices of C_{2k+1}. If $2k + 1 = 5$ and there is another C_5 in the 2-factor, then Lemma 3.3 implies that at least one must be a $G(5, V)$. Remove a C_5 that is covered by a $G(5, V)$, then there are 8 internal edges and up to $2(n - 5)$ edges from the C_5 to the rest of the graph for a total of $2n - 2$ edges. If $2k + 1 = 5$ and there are at least two C_3's in the 2-factor, then the maximum number of edges that can be removed are the 8 edges of a $G(5, V)$ and the $2(n - 5)$ possible edges between the C_5 and the rest of the graph for a total of $2n - 2$ edges. Repeat the process for the next largest cycle in the 2-factor until the remaining 2-factor consists of only one cycle or none of the conditions are met, in which case, the remaining 2-factor must consist of a C_3 and a C_5. If there is a K_5 or $K_5 - e$ on the C_5, then there are at most 5 edges between the two cycles from the analysis in Lemma 3.5, giving a maximum of 18 edges in all. If there is a subgraph of $G(5, V)$ on the C_5, there are at most 6 edges between the two cycles, giving a maximum of 17 edges in all, so this situation cannot occur.

Tracing through the construction, we see that if we remove a newly added cycle, we remove precisely the maximum number of edges calculated here. Since the hamiltonian constructions are already known to attain their bounds, the general case constructions will also attain the bound.

3.3.5 Computing the Bound

We now compute the maximum size of a 2-factor isomorphic graph by counting the maximum number of edges added at each step in our construction of the 2-factor isomorphic graph.

Recall that for a specified 2-factor F, c_j is the number of cycles of length j in F for $3 \leq j \leq n$. We initially assume that no K_4 or K_5 is used in the construction and adjust for their possible presence at the end. When cycle

C_{2i} is added to the 2-factor, a $G(2i, V)$ is constructed on the C_{2i} and V in $G(2i, V)$ is joined to the G' previously constructed. This adds $i \cdot j$ edges between C_{2i} and any cycle C_j in the 2-factor of G' and adds $\left\lceil \frac{(2i)(2i+1)}{4} \right\rceil$ edges from the $G(2i, V)$, with an extra edge from $G(2i, V)$ when $i = 3$. Let $C_k \prec C_l$ denote that C_k precedes C_l in the construction. This ordering of cycles induces an ordering of cycle lengths, namely $k \prec l$ if $k < l$ and k and l are of the same parity, or if k is odd and l is even. The former ordering allows us to express the number of edges added when C_{2i} is added to the 2-factor as

$$\left\lceil \frac{(2i)(2i+1)}{4} \right\rceil + \sum_{C_j \prec C_{2i+1}} i \cdot j$$

when $i \neq 3$ and one edge more when $i = 3$. Summing over all C_{2i} then gives a total of

$$c_6 + \sum_{C_{2i} \in F} \left[\left\lceil \frac{(2i)(2i+1)}{4} \right\rceil + \sum_{C_j \prec C_{2i}} i \cdot (j) \right]$$

edges incident to an even cycle in the construction. Those edges not incident to an even cycle must only be incident to an odd cycle.

To count these we use a similar analysis. When cycle C_{2i+1} is added to the 2-factor, a $G(2i+1, V)$ is constructed on the C_{2i+1} and V in $G(2i+1, V)$ is joined to the G' previously constructed. This adds $i \cdot j$ edges between C_{2i+1} and any cycle C_j in the 2-factor of G' and adds $\left\lceil \frac{(2i+1)(2i+2)}{4} \right\rceil$ edges from the $G(2i+1, V)$. The total number of edges added when C_{2i+1} is added to the 2-factor is then

$$\left\lceil \frac{(2i+1)(2i+2)}{4} \right\rceil + \sum_{C_j \prec C_{2i+1}} i \cdot j.$$

Summing over all C_{2i+1} gives a total of

$$\sum_{C_{2i+1} \in F} \left[\left\lceil \frac{(2i+1)(2i+2)}{4} \right\rceil + \sum_{C_j \prec C_{2i+1}} i \cdot j \right]$$

edges induced on the vertices of the odd cycles of F.

To make this resemble the even version more closely, we add and subtract $j/2$ edges within the double sum:

$$\sum_{C_{2i+1} \in F} \left[\left\lceil \frac{(2i+1)(2i+2)}{4} \right\rceil + \sum_{C_j \prec C_{2i+1}} \frac{(2i+1)j}{2} - \frac{j}{2} \right]$$

and separate out the subtracted term into its own sum:

$$\sum_{C_{2i+1} \in F} \left[\left\lceil \frac{(2i+1)(2i+2)}{4} \right\rceil + \sum_{C_j \prec C_{2i+1}} \frac{(2i+1)j}{2} \right] - \sum_{C_{2i+1} \in F} \sum_{C_j \prec C_{2i+1}} \frac{j}{2}.$$

The first sum over odd cycles and the sum over even cycles agree as functions of cycle length, so the two expressions can be combined for a total edge count of:

$$c_6 + \sum_{C_i \in F} \left[\left\lceil \frac{i(i+1)}{4} \right\rceil + \sum_{C_j \prec C_i} \frac{i \cdot j}{2} \right] - \sum_{C_{2i+1} \in F} \sum_{C_j \prec C_{2i+1}} \frac{j}{2}.$$

The inner sum $\sum_{C_j \prec C_i} \frac{i \cdot j}{2}$ is symmetric in i and j, so can be replaced with $\sum_{C_j \in F} \left(\frac{i \cdot j}{4}\right) - \frac{i^2}{4}$, with the last term correcting for when C_i and C_j are the same cycle, where there are no edges between C_i and C_j. This gives:

$$c_6 + \sum_{C_i \in F} \left[\left\lceil \frac{i(i+1)}{4} \right\rceil + \sum_{C_j \in F} \left(\frac{i \cdot j}{4}\right) - \frac{i^2}{4} \right] - \sum_{C_{2i+1} \in F} \sum_{C_j \prec C_{2i+1}} \frac{j}{2}.$$

Summing over C_j gives:

$$c_6 + \sum_{C_i \in F} \left[\left\lceil \frac{i(i+1)}{4} \right\rceil + \frac{i \cdot n}{4} - \frac{i^2}{4} \right] - \sum_{C_{2i+1} \in F} \sum_{C_j \prec C_{2i+1}} \frac{j}{2}.$$

We can remove the ceiling function by noting that we round up by 0.5 edges whenever $i \equiv 1, 2 \mod 4$, so we can account for these extra edges by counting the number cycles with these properties. Let $c_1^* = \sum_{k=1}^{n} c_{4k+1}$ and $c_2^* = \sum_{k=1}^{n} c_{4k+2}$ Then we can rewrite the total number of edges as:

$$\frac{c_1^* + c_2^*}{2} + c_6 + \sum_{C_i \in F} \left[\frac{i(i+1)}{4} + \frac{i \cdot n}{4} - \frac{i^2}{4} \right] - \sum_{C_{2i+1} \in F} \sum_{C_j \prec C_{2i+1}} \frac{j}{2}.$$

Combining the fractions in the first sum gives:

$$\frac{c_1^* + c_2^*}{2} + c_6 + \sum_{C_i \in F} \left[\frac{i \cdot (n+1)}{4} \right] - \sum_{C_{2i+1} \in F} \sum_{C_j \prec C_{2i+1}} \frac{j}{2}.$$

Summing over i gives a bound of:

$$\frac{c_1^* + c_2^*}{2} + c_6 + \frac{n(n+1)}{4} - \sum_{C_{2i+1} \in F} \sum_{C_j \prec C_{2i+1}} \frac{j}{2}.$$

The remaining double sum can be rewritten to be independent of the order in which cycles were added giving:

$$\frac{c_1^* + c_2^*}{2} + c_6 + \frac{n(n+1)}{4} - \sum_{\{C_{2i+1}, C_{2j+1}\} \subset F} \min\left(\frac{2i+1}{2}, \frac{2j+1}{2}\right).$$

We now account for the possibility of having a single K_4 or K_5 instead of one of the $G(l,V)$'s. Let C be the constant number of edges by which switching one $G(5,V)$ to a K_5 or one $G(4,V)$ to a K_4 increases the number of edges. We can determine this C by applying the analysis used in proving Lemmas 3.2-3.5. If $c_3 = 1$ and $c_5 > 0$ then $C = 1$. If $c_3 = 0$ and $c_5 > 0$ then $C = 2$. If $c_{2i+1} = 0$ for all i and $c_4 > 0$ then $C = 1$. If none of these, then $C = 0$.

Theorem 3.6 *The size of a 2-factor isomorphic graph, G, on n vertices is*

$$|E| \leq C + \frac{c_1^* + c_2^*}{2} + c_6 + \frac{n(n+1)}{4} - \sum_{\{C_{2i+1}, C_{2j+1}\} \subset F} \min\left(\frac{2i+1}{2}, \frac{2j+1}{2}\right),$$

where $C = 0, 1,$ or 2 is a correction for the possible presence of a clique on a single cycle of the 2-factor, $c_1^* = \sum_{k=1}^{n} c_{4k+1}$, $c_2^* = \sum_{k=1}^{n} c_{4k+2}$, and c_6 is the number of cycles of length 6.

3.4 Demonstrating that the Constructions are 2-factor Isomorphic

3.4.1 Showing that K_n is 2-factor Isomorphic for $n \leq 5$

As mentioned in Section 3.2, if $n \leq 5$, there are insufficient vertices to form two or more disjoint cycles, so only the hamiltonian 2-factor can exist, thereby making K_n 2-factor isomorphic.

3.4.2 Showing that $G(6, V)$ is 2-factor Isomorphic

The only non-isomorphic 2-factors on six vertices are a C_6 and two C_3's. That C_6 is a 2-factor can be seen by taking the cycle $u_1v_1u_2v_2u_3v_3$ where $\{u_1, u_2, u_3\} = U$ and $\{v_1, v_2, v_3\} = V$. Suppose that there is a 2-factor consisting of two C_3's. Then by the pigeonhole principle, one of the C_3's must contain at least two vertices in U. An examination of $G(6, V)$ shows that there are no edges between vertices of U, which contradicts our supposition, and therefore $G(6, V)$ is 2-factor isomorphic.

3.4.3 Showing that $G(n, V)$ is 2-factor Isomorphic for $n \geq 7$

The construction due to Faudree, Gould, and Jacobson was shown to be 2-factor hamiltonian [8], and so is also 2-factor isomorphic .

3.4.4 Demonstrating that the Construction for the Non-Hamiltonian Case is 2-factor Isomorphic

We will prove that the non-hamiltonian construction is 2-factor isomorphic by induction on the number of cycles, k, in the 2-factor. If $k = 1$, then we are merely constructing the graph $G(n, V)$ if $n \geq 6$ or K_n if $n \leq 5$ which we have just established to be 2-factor isomorphic. Assume that, for all 2-factors with fewer than k cycles, the construction gives a 2-factor isomorphic graph. Let H_k be the graph covering the last cycle added in the construction. Then $H_k = G - G'$, and H_k is one of the following: a K_3, a K_4, a K_5, or $G(l, V)$ for some l.

Case 1 *Suppose H_k is a K_3.*

Returning to the construction, we note that all cycles in the 2-factor must in fact be C_3's. By construction, H_k has a vertex, v, adjacent to all other vertices, and its other two vertices u and x are adjacent only to each other and v. Thus in any 2-factor, the cycle u, x, v must be present and any two non-isomorphic 2-factors must differ in G'. By induction however, G' is 2-factor isomorphic, so G must also be 2-factor isomorphic.

Case 2 *Suppose H_k is a K_4.*

Returning to the construction, we note that a K_4 is used to cover a C_4 only if there are no odd cycles, no other C_4's are already present, and C_4 is the largest even cycle and so the largest cycle. These observations imply that there are no other cycles in the 2-factor, and we have already shown that K_4 is 2-factor isomorphic.

Case 3 *Suppose H_k is a K_5.*

Returning to the construction, we note that a K_5 is used to cover a C_5 only if there is at most one C_3 and no other C_5's in the 2-factor. We also note that, because odd cycles were added in increasing order, there are no larger odd cycles and that, because even cycles are added after odd cycles in the construction, there can be no even cycles. These observations imply that $k = 1$ or 2. If $k = 2$, then G' is a K_3 and all edges between H_k and G' are, by construction, incident to a single vertex $x \in G'$. Therefore, the only cycle that $y \in G' - x$ lies in is the C_3 of G', and so this C_3 must be one of the cycles in the 2-factor. The graph $H_k \cong K_5$ is 2-factor hamiltonian, implying that all 2-factors of G are isomorphic to $\{C_3, C_5\}$ and G is 2-factor isomorphic.

Case 4 *Suppose H_k is a $G(l, V)$.*

In this case, H_k is itself 2-factor hamiltonian, and, by construction, has vertex sets U, V, and, if l is odd, $\{x\}$. If l is even, then for all $u \in U, N_G(u) \in V$ and thus in any 2-factor, each vertex $u \in U$ is adjacent to 2 vertices of V, thereby exhausting the degrees of the vertices $v \in V$ because $|U| = |V|$ and $d(u) = d(v) = 2$ in the 2-factor. If l is odd, the result is similar except that $N_{2-factor}(x) = \{u_{\lfloor \frac{l}{2} \rfloor}, v\}$ for some $v \in V$ in order for all $u \in U$ to be degree 2. In both of these cases, $\forall v \in V, N_{2-factor}(v) \subseteq U \cup \{x\}$. Therefore, any 2-factor in G must consist of the union of a 2-factor in G', which, by assumption, is 2-factor isomorphic, and a 2-factor in H_k, which is also 2-factor isomorphic. All unions of these 2-factors must then also be isomorphic. Therefore, G is 2-factor isomorphic.

For any choice of last cycle H_k, G is 2-factor isomorphic, so by induction the construction always gives a 2-factor isomorphic graph.

3.5 Maximum Size of 2-factor Isomorphic Graphs with Unspecified 2-factors

Now that we have a formula for the maximum number of edges in a 2-factor isomorphic graph based on the 2-factor it contains, we may ask what is the overall maximum size over all possible 2-factors, or over all possible 2-factors with a fixed number of cycles. We can also find lower bounds for the maximum size of a 2-factor isomorphic graph over all possible 2-factors or over all possible 2-factors with a fixed number of cycles. To find such bounds, we examine the changes to the maximum size of a 2-factor isomorphic graph caused by slight alterations in the 2-factor. Throughout this section, we will make heavy use of the Lemma 3.1 bound on edges from a cycle to the rest of the graph.

Lemma 3.7 *Replacing two C_3's with a C_6 in the 2-factor allows more edges to be present in a 2-factor isomorphic graph with that 2-factor. Equivalently, replacing a C_6 with 2 C_3's in the 2-factor allows fewer edges to be present in a 2-factor isomorphic graph with that 2-factor.*

Proof. There are a total of 9 edges induced on the vertices of the two C_3's and at most $3(n-6)$ edges between the vertices of the two C_3's and the rest of the graph for a total of at most $3n-9$ edges incident to the vertices of the C_3's. From Section 3.2.2, we know that there are 12 edges induced on the vertices of the C_6 and $3(n-6)$ edges between the vertices of the C_6 and the rest of the graph for a total of $3n-6$ edges incident to the vertices of the C_6. This number is more than for two C_3's. Tracing through the construction, we can see that the number of edges not incident to the changed vertices remains the same, or if the number of C_3's is reduced enough, the number may increase due to a C_5 being covered by a K_5 rather than a $G(5, V)$ or a C_4 being covered by a K_4 rather than a $G(4, V)$, further increasing the number of edges in the 2-factor isomorphic graph. Replacing a C_6 with two C_3's can similarly force a K_5 or K_4 to become a $G(5, V)$ or $G(4, V)$ covering C_5 or C_4, further decreasing the number of edges in the 2-factor isomorphic graph. $\square$

Lemma 3.8 *Replacing two odd cycles, not both* C_3*'s, in the 2-factor with two even cycles on the same vertices allows at least as many edges to be present in a 2-factor isomorphic graph. Equivalently, replacing two even cycles in a 2-factor with two odd cycles on the same vertices allows at most as many edges to be present in a 2-factor isomorphic graph.*

Proof. Suppose that there are two odd cycles in the 2-factor, say a C_{2k+1} and a C_{2l+1}.

Case 1 *Neither C_{2k+1} nor C_{2l+1} is covered by a K_5.*

Without loss of generality, we may assume that $k \leq l$ and $2 \leq l$. From Theorem 1.6 we know that there are at most $\frac{(2k+1)(2k+2)+2}{4}$ edges in the graph induced by the vertices of the C_{2k+1}, and at most $\frac{(2l+1)(2l+2)+2}{4}$ edges in the graph induced by the vertices of the C_{2l+1}. From Section 3.3.2 we know that there are at most $\frac{(2k+1)(2l)}{2}$ edges between the two cycles for a total of

$$k^2 + 1.5k + l^2 + 2.5l + 2kl + 2$$

edges induced by the vertices of the C_{2k+1} and the C_{2l+1}. If instead we had a C_{2k+2} and a C_{2l} on these vertices, then there are at least $\frac{(2k+2)(2k+3)}{4}$ edges induced by the vertices of the C_{2k+2} and $\frac{2l(2l+1)}{4}$ edges induced by the vertices of the C_{2l}. There are $\frac{(2k+2)(2l)}{2}$ edges between the two cycles for a total of

$$k^2 + 2.5k + l^2 + 2.5l + 2kl + 1.5$$

edges induced by the vertices of the C_{2k+2} and the C_{2l}. Because $k \geq 1$, this total is larger than the total for the C_{2k+1} and C_{2l+1}. Further, there are at least as many edges incident to these two even cycles as there were incident to the two odd cycles because the $\left\lfloor \frac{(2k+2)(j)}{2} \right\rfloor + \left\lfloor \frac{(2l)(j)}{2} \right\rfloor$ edges from the even cycles is at least as many as the upper bound of $\left\lfloor \frac{(2k+1)(j)}{2} \right\rfloor + \left\lfloor \frac{(2l+1)(j)}{2} \right\rfloor$ edges from the odd cycles to any other cycle, C_j, in the 2-factor. Therefore replacing the two odd cycles with these two even cycles allows at least as many edges to be present in a 2-factor isomorphic graph.

Case 2 *One of the odd cycles, without loss of generality C_{2k+1}, is covered by a K_5.*

There are $\left\lceil \frac{(2l+1)(2l+2)}{4} \right\rceil$ edges in the graph induced by the vertices of the C_{2l+1}. From the proof of Lemma 3.5 for $l = 1$, and from Lemma 3.1 for $l > 1$, there can be at most $5l$ edges between the two cycles. This analysis gives a total of

$$\left\lceil \frac{(2l+1)(2l+2)}{4} \right\rceil + 5l + 10$$

edges induced by the vertices of the C_5 and the C_{2l+1}. If instead, we had a C_4 and a C_{2l+2} on these vertices, then there are $\left\lceil \frac{(2l+2)(2l+3)}{4} \right\rceil$ edges in the graph induced by the vertices of the C_4 and the C_{2l+2}. There are at least 5 edges in the graph induced by the vertices of the C_4. There are $2(2l+2)$

edges between the two cycles for a total of

$$\left\lceil \frac{(2l+2)(2l+3)}{4} \right\rceil + 4l + 9 = \left\lceil \frac{(2l+1)(2l+2)}{4} \right\rceil + 5l + 10$$

edges induced by the vertices of the C_4 and the C_{2l+2}. Further, there are at least as many edges incident to these two even cycles as there were incident to the two odd cycles because there are

$$\left\lfloor \frac{(4)(j)}{2} \right\rfloor + \left\lfloor \frac{(2l+2)(j)}{2} \right\rfloor \geq \left\lfloor \frac{(5)(j)}{2} \right\rfloor + \left\lfloor \frac{(2l+1)(j)}{2} \right\rfloor$$

edges between the even cycles and any other cycle in the 2-factor. Therefore, replacing the two odd cycles with these two even cycles allows at least as many edges to be present in a 2-factor isomorphic graph. □

Corollary 3.9 *For each n, there exists a 2-factor isomorphic graph attaining the maximum size that has at most one odd cycle in the 2-factor.*

Proof. Lemma 3.7 implies that any such graph has at most one C_3 in the 2-factor. Under that condition, Lemma 3.8 permits us to freely replace pairs of odd cycles with pairs of even cycles without decreasing the number of edges and so arrive at a graph that has at most one odd cycle in the 2-factor. □

3.5.1 Maximum Size of 2-factor Isomorphic Graphs with an Unspecified 2-factor

In seeking to find the maximum number over all possible choices of 2-factors, it is useful to examine the equivalent problem of maximizing the average degree of the 2-factor isomorphic graph. Corollary 3.9 allows us to examine only those 2-factor isomorphic graphs with at most one odd cycle. For these graphs, in every pair of cycles C_i and C_j, at least one of i and j is even, giving us $\frac{i(n-i)}{2}$ edges from C_i by Lemma 3.1. This reduction allows us to compute the average degree for vertices in various $G(l,V)$'s, K_4's or K_5's.

- The average degree for vertices in a K_5 is $4 + \frac{n-5}{2} = \frac{n+3}{2}$, 4 edges are within the K_5, the rest between the underlying C_5 and all other cycles.

- The average degree for vertices in a K_4 is $3 + \frac{n-4}{2} = \frac{n+2}{2}$, 3 edges are within the K_4, the rest between the underlying C_4 and all other cycles.

- The average degree for vertices in a $G(6,V)$ is $4 + \frac{n-6}{2} = \frac{n+2}{2}$, 4 edges are within the $G(6,V)$, the rest between the underlying C_6 and all other cycles.

- The average degree for vertices in a $G(l, V)$ is

$$\frac{l^2+l+2}{2l} + \frac{n-l}{2} = \frac{n+1}{2} + \frac{1}{l} \text{ if } l \equiv 1,2 \mod 4$$

$$\text{and } \frac{l^2+l}{2l} + \frac{n-l}{2} = \frac{n+1}{2} \text{ if } l \equiv 0,3 \mod 4$$

  the first term from within the $G(l, V)$, the rest between the underlying C_l and all other cycles.

The average degree for vertices in a $G(l, V)$ for $l \geq 7$, is at most the same as the average degree for vertices in $G(l-4, V) \cup G(4, V)$. This property implies that, for all n, there exist 2-factor isomorphic graphs that attain the maximum size with 2-factors that consist entirely of C_3's, C_4's, C_5's, and C_6's. The following table shows the maximum number of edges obtainable on a graph of order n in which at least one copy of the designated cycle appears in the 2-factor. A * indicates that no 2-factor containing the designated cycle exists, and the entries in the 2-factor column are the 2-factors that allow the maximum size to be attained.

n	C_3	C_4	C_5	C_6	Maximum Size	2-factor
3	3	$*$	$*$	$*$	3	$\{C_3\}$
4	$*$	6	$*$	$*$	6	$\{C_4\}$
5	$*$	$*$	10	$*$	10	$\{C_5\}$
6	9	$*$	$*$	12	12	$\{C_6\}$
7	14	14	$*$	$*$	14	$\{C_3, C_4\}$
8	18	19	18	$*$	19	$\{C_4, C_4\}$
9	24	25	25	24	25	$\{C_4, C_5\}$
10	26	30	28	30	30	$\{C_4, C_6\}$
11	33	33	37	37	37	$\{C_5, C_6\}$
12	39	39	39	42	42	$\{C_6, C_6\}$
13	47	48	48	47	48	$\{C_4, C_4, C_5\}$
14	54	55	54	55	55	$\{C_4, C_4, C_6\}$
15	60	64	64	64	64	$\{C_4, C_5, C_6\}$
16	69	72	70	72	72	$\{C_4, C_6, C_6\}$
17	78	79	82	82	82	$\{C_5, C_6, C_6\}$
18	88	88	88	90	90	$\{C_6, C_6, C_6\}$
19	95	99	99	99	99	$\{C_4, C_4, C_5, C_6\}$
20	108	109	107	109	109	$\{C_4, C_4, C_6, C_6\}$

Observe that the 2-factor for graphs of order 14 through 20 all contain a C_6. If $n > 20$, the 2-factor for graphs of order n that attain the maximum size must also contain a C_6 because removing the vertices of any C_l that is not covered by a K_4 or K_5 from the graph must yield a graph of maximum size for order $n - k$, and therefore, by induction, must contain a C_6. This observation leads to the following general formulas for the maximum size of a 2-factor isomorphic graph of order $n \geq 14$ and the general form of the 2-factor that allows attainment of this size.

$n \geq 14$	Maximum Size	2-factor
$n \equiv 0 \mod 6$	$(n^2 + 2n)/4$	$\{C_6, \ldots, C_6\}$
$n \equiv 1 \mod 6$	$(n^2 + 2n - 3)/4$	$\{C_4, C_4, C_5, C_6, \ldots, C_6\}$
$n \equiv 2 \mod 6$	$(n^2 + 2n - 4)/4$	$\{C_4, C_4, C_6, \ldots, C_6\}$
$n \equiv 3 \mod 6$	$(n^2 + 2n + 1)/4$	$\{C_4, C_5, C_6, \ldots, C_6\}$
$n \equiv 4 \mod 6$	$(n^2 + 2n)/4$	$\{C_4, C_6, \ldots, C_6\}$
$n \equiv 5 \mod 6$	$(n^2 + 2n + 5)/4$	$\{C_5, C_6, \ldots, C_6\}$

3.5.2 Lower Bound of the Maximum Size of 2-factor Isomorphic Graphs with Unspecified 2-factor

Lemma 3.10 *Replacing an odd cycle of length* $2i+2j+2k+3$, $1 \leq i \leq j \leq k$, *with 3 cycles of lengths* $2i+1$, $2j+1$, *and* $2k+1$ *in the 2-factor reduces the the maximum size of a 2-factor isomorphic graph containing the 2-factor.*

Proof. There are at least

$$\frac{(2i+2j+2k+3)(2i+2j+2k+4)}{4} =$$
$$i^2+j^2+k^2+2ij+2ik+2jk+3.5i+3.5j+3.5k+3$$

edges in the graph induced on the vertices of the $C_{2i+2j+2k+3}$. There are at most

$$\frac{(2i+1)(2i+2)+2}{4} + \frac{(2j+1)(2j+2)+2}{4} + \frac{(2k+1)(2k+2)+2}{4}$$
$$+(2i+1)j+(2i+1)k+(2j+1)k+2$$

edges in the graph induced on the vertices of the C_{2i+1}, C_{2j+1}, and C_{2k+1} - the 2's in the fractions accounting for potentially rounding up and the last 2 accounts for the possible presence of a K_5 rather than a $G(5,V)$. Simplifying this expression gives

$$i^2+j^2+k^2+2ij+2ik+2jk+1.5i+2.5j+3.5k+5.$$

Since $i, j, k \geq 1$, this value is less than the number of edges in the graph induced on the vertices of the $C_{2i+2j+2k+3}$. Further, the number of edges between the three cycles and the remaining cycles in the 2-factor will be at most the same as the number of edges between the $C_{2i+2j+2k+3}$ and the remaining cycles. This follows from the fact that if for a remaining cycle C_l, $l \geq (2i+2j+2k+3)$ or l even, there will be $\left\lfloor \frac{l}{2} \right\rfloor (2i+2j+2k+3)$ edges between either the three cycles and the C_l or the $C_{2i+2j+2k+3}$ and the C_l. If $l < (2i+2j+2k+3)$ and l is odd, then there will be

$$\max\left(\tfrac{l-1}{2}(2i+1), li\right) + \max\left(\tfrac{l-1}{2}(2j+1), lj\right)$$
$$+ \max\left(\tfrac{l-1}{2}(2k+1), lk\right)$$

edges between the three cycles and the C_l. This value is less than the $l(2i+2j+2k+2)$ edges between the $C_{2i+2j+2k+3}$ and the C_l. As these relations hold for all remaining C_l, the total must have the same relation. □

Lemma 3.11 *Replacing an even cycle of length $2k+4 > 4$ with odd cycles of lengths 3 and $2k+1$ in the 2-factor reduces the maximum size of a 2-factor isomorphic graph containing the 2-factor.*

Proof. There are at least $\dfrac{(2k+4)(2k+5)}{4}$ edges in the graph induced on the vertices of the C_{2k+4} and $(k+2)(n-2k-4)$ edges from C_{2k+4} to the rest of the graph. There are at most

$$3 + \frac{(2k+1)(2k+2)+2}{4} + 3k$$

edges in the graph induced on the vertices of the C_3 and C_{2k+1} and at most $(k+2)(n-2k-4)$ edges the C_3 and C_{2k+1} to the rest of the graph, no more than for C_{2k+4}. There is one fewer edge induced on the vertices of the C_3 and C_{2k+1} than induced on the vertices of the C_{2k+4} so this replacement reduces the maximum size by at least one edge. □

Putting these lemmas together allows us to search for the 2-factor of smallest maximum size with a limited selection of cycles in the 2-factor. Lemma 3.11 allows us to eliminate any even cycle of length at least 6 from appearing in the 2-factor, reducing us to odd cycles and C_4's. Lemma 3.8 permits us to replace pairs of C_4's with pairs of odd cycles, reducing us to odd cycles and possibly one C_4. Lemma 3.10 allows us to eliminate any odd cycles of length at least 9 from appearing in the 2-factor. Therefore there exists a 2-factor isomorphic graph that attains the lower bound for maximum size of a 2-factor isomorphic graph that consists entirely of C_3's, C_4's, C_5's, and C_7's.

The following table shows the lower bound of the maximum size obtainable on such a graph of order n, in which, the designated cycle is the last added by the construction. A * indicates that the designated cycle cannot be added last for any 2-factor of that order. The entries in the 2-factor column are the 2-factors obtaining the lower bound of the maximum size.

n	C_3	C_4	C_5	C_7	Lower Bound	2-factor(s)
3	3	$*$	$*$	$*$	3	$\{C_3\}$
4	$*$	6	$*$	$*$	6	$\{C_4\}$
5	$*$	$*$	10	$*$	10	$\{C_5\}$
6	9	$*$	$*$	$*$	9	$\{C_3, C_3\}$
7	$*$	14	$*$	14	14	$\{C_3, C_4\}, \{C_7\}$
8	18	19	18	$*$	18	$\{C_3, C_5\}$
9	18	25	$*$	$*$	18	$\{C_3, C_3, C_3\}$
10	$*$	26	28	26	26	$\{C_3, C_3, C_4\}, \{C_3, C_7\}$
11	29	33	$*$	$*$	29	$\{C_3, C_3, C_5\}$
12	30	39	$*$	39	30	$\{C_3, C_3, C_3, C_3\}$
13	$*$	41	42	41	41	$\{C_3, C_3, C_3, C_4\}, \{C_3, C_3, C_7\}$
14	$*$	49	44	49	44	$\{C_3, C_3, C_3, C_5\}$
15	45	56	56	56	45	$\{C_3, C_3, C_3, C_3, C_3\}$
16	$*$	59	59	59	59	$\{C_3, C_3, C_3, C_3, C_4\}$, $\{C_3, C_3, C_3, C_7\}$, $\{C_3, C_3, C_5, C_5\}$

All 2-factors attaining the lower bound of the maximum size for 2-factor isomorphic graphs of order 9 through 16 contain at least two C_3's. If $n > 16$, the 2-factor for graphs of order n attaining the lower bound of the maximum size must also contain at least two C_3's, since removing the vertices of the last cycle, C_k, added in the construction from the graph must yield a 2-factor isomorphic graph of order $n-k$, which attains the lower bound of the maximum size for this order, and so contains at least two C_3's by induction. This observation leads to the following general formulas for the minimal number of edges in a 2-factor isomorphic graph of order $n \geq 14$ and the general form(s) of the 2-factor(s) attaining this number of edges.

$n \geq 14$	Lower Bound	2-factor(s)
$n \equiv 0 \mod 3$	$\frac{n^2+3n}{6}$	$\{C_3, \ldots, C_3\}$
$n \equiv 1 \mod 3$	$\frac{n^2+7n-14}{6}$	$\{C_3, \ldots, C_3, C_4\}, \{C_3, \ldots, C_3, C_7\}, \{C_3, \ldots, C_3, C_5, C_5\}$
$n \equiv 2 \mod 3$	$\frac{n^2+5n-2}{6}$	$\{C_3, \ldots, C_3, C_5\}$

3.5.3 Maximum Size of 2-factor Isomorphic Graphs with a 2-factor consisting of k cycles

To find the maximum size over all possible choices of 2-factors consisting of k cycles, we first make use of Lemma 3.8 to assume that the 2-factor consists of either C_3's and even cycle(s), or of one odd cycle and even cycles.

Lemma 3.12 *If the 2-factor consists of C_3's and even cycles and there is an even cycle, C_l, of length 6 or greater, replacing one or more C_3's with C_4's and reducing the large cycle allows at least as many edges in a 2-factor isomorphic graph without changing the number of cycles in the 2-factor.*

Proof.

Case 1 *The 2-factor contains 2 or more C_3's.*

The number of edges induced on the vertices of a C_{l-2} and two C_4's is at least

$$\frac{l^2 - 3l + 2}{4} + 4l + 10,$$

which is the most that the number of edges induced on the vertices of a C_l and two C_3's can be. This maximum is only achieved if $l = 6$, with the last constant falling to 9 for $l \neq 6$. Because all the cycles in the former case are even, the average degree to any remaining cycle, C_i, is $i/2$, while in the latter, the average degree to any remaining cycle, C_i, is $i/2$ for the C_l, and $\lfloor i/2 \rfloor$ for the vertices in the C_3's. Replacing the two C_3's and a C_l with two C_4's and a C_{l-2} thus allows at least as many edges in the 2-factor isomorphic graph.

Case 2 *The 2-factor contains only one C_3 and $l \neq 6$*

The number of edges induced on the vertices of a C_{l-1} and a C_4 is $\left\lceil \frac{l^2 - l}{4} \right\rceil +$ $2l + 3$. The number of edges induced on the vertices of a C_l and a C_3 is $\left\lceil \frac{l^2 + l}{4} \right\rceil + \left\lfloor \frac{3l}{2} \right\rfloor + 3$. These quantities are the same and, because all other cycles were assumed to be even, the average degree to any remaining cycle, C_i is $i/2$, implying that the number of edges remains the same after the replacement.

Case 3 *The 2-factor contains only one C_3 and $l = 6$.*

The number of edges induced on the vertices of a C_3 and a C_6 is 24. The number of edges induced on the vertices of a C_4 and a C_5 covered by a K_5 is 25. The C_4 and C_5 allow more induced edges than the C_3 and C_6. Because all other cycles were assumed even, the average degree to any remaining cycle, C_i is $i/2$, implying the the number of edges to other cycles remains unchanged and so the replacement increases the overall size. □

From this lemma we can assume either that there are no C_3's in the 2-factor or that there is no cycle of length greater than 4 in the 2-factor. Based on this assumption, we can conclude that the 2-factor consists of C_3's and C_4's or that the 2-factor consists of even cycles and, at most, one odd cycle. In the former case, the 2-factor is uniquely determined by n and k.

In the latter case, we have the same conditions that were used to compute the average degree in Section 3.5.1. Note that all of the average degrees are at least $\frac{n+1}{2}$, and the number of edges present is thus at least $\frac{n+1}{4}$. From the degree averages, we add to this minimum 2.5 edges for a C_5 covered by a K_5, 1.5 edges per $G(6, V)$, and 0.5 edges per $G(4j+1, V)$ and $G(4j+2, V)$. These correspond to number edges beyond the minimum $\frac{i(i+1)}{4}$ in the graph covering C_i for a K_5, $G(6, V)$, $G(4j+1, V)$, and $G(4j+2, V)$. If there is an odd cycle, C_i, other than a C_5 and an even cycle C_l in the 2-factor, the number of edges can be increased by replacing them with a C_5 and a C_{l+i-5} because the former can have at most an additional 2 edges and the latter has at least 2.5 additional edges. Similarly, if there are two even cycles C_i, C_j, $i \neq 6$, $j > 6$, the number of edges can be increased by replacing them with a C_6 and a C_{i+j-6} because the former can have at most an additional 1 edge and the latter has at least 1.5 additional edges. To maximize the number of edges then, we need a K_5 if n is odd, but no other odd cycles and as many C_6's as possible in the 2-factor. After maximizing the number of C_6's, any remaining cycles must either be C_4's or a single large cycle. This classification gives us the following formulas for the number of edges based on the 2-factor.

Max Edges	Domain	2-factor
$\frac{-n^2}{2} + 6nk -$ $12k^2 - \frac{n}{2} + 3k$	$n < 4k$	$\{C_3, \ldots, C_3, C_4, \ldots, C_4\}$
$\frac{n^2}{4} + n - 3k + 1$	$4k \leq n < 6k$, n even	$\{C_4, \ldots, C_4, C_6, \ldots, C_6\}$
$\frac{n^2}{4} + n - 3k + \frac{7}{4}$	$4k \leq n < 6k$, n odd	$\{C_4, \ldots, C_4, C_5, C_6, \ldots, C_6\}$
$\frac{n^2}{4} + 2n$	$n = 6k$	$\{C_6, \ldots, C_6\}$
$\left\lceil \frac{n^2+n}{4} + \frac{3k-3}{2} \right\rceil$	$n > 6k$, n even	$\{C_6, \ldots, C_6, C_{n-6k+6}\}$
$\left\lceil \frac{n^2+n}{4} + \frac{3k-1}{2} \right\rceil$	$n > 6k$, n odd	$\{C_5, C_6, \ldots, C_6, C_{n-6k+7}\}$

3.5.4 Lower Bound of the Maximum Size of 2-factor Isomorphic Graphs with 2-factor consisting of k cycles

If $k = 1$ the hamiltonian 2-factor is the only option and we are done, so for the rest of this subsection we will assume that $k \geq 2$. To find the lower bound of the maximum size over all choices of 2-factors consisting of k cycles, we first make use of Lemma 3.8 to assume that the 2-factor consists either of only odd cycles or of one even cycle and odd cycles.

Lemma 3.13 *If there are two cycles, C_{2i+1} and C_{2j+3} in the two factor with $j \geq i+1$, then replacing them with a C_{2i+3} and a C_{j+1} will allow at most as many edges in the graph unless the C_{2i+1} is a C_3 and the C_{2i+3} that replaces it can be covered by a K_5.*

Proof. A K_5 is only used to cover a C_5 if it allows more edges than a $G(5, V)$. Therefore, if the lemma holds for C_{2i+1} being covered by a $G(5, V)$, it must hold for C_{2i+1} being covered by a K_5, and thus we can assume that the C_{2i+1} and C_{2j+3} are covered by a $G(2i + 1, V)$, and a $G(2j + 3, V)$ respectively.

A direct comparison of the number of edges incident to the vertices of the C_{2i+1} and C_{2j+3}, and the number of edges incident to the vertices of the C_{2i+3} and C_{2j+1} establishes this lemma. We start by counting the number of edges induced on the vertices of the two cycles, conditioning on whether $i, j \equiv 0$ or $1 \mod 2$.

$i, j \equiv$	C_{2i+1}, C_{2j+3}	C_{2i+3}, C_{2j+1}
1,1	$(i+j)^2 + 3.5i + 4.5j + 5$	$(i+j)^2 + 3.5i + 4.5j + 4$
1,0	$(i+j)^2 + 3.5i + 4.5j + 4.5$	$(i+j)^2 + 3.5i + 4.5j + 4.5$
0,1	$(i+j)^2 + 3.5i + 4.5j + 5.5$	$(i+j)^2 + 3.5i + 4.5j + 3.5$
0,0	$(i+j)^2 + 3.5i + 4.5j + 5$	$(i+j)^2 + 3.5i + 4.5j + 4$

In all four cases, the number of edges is not increased by the replacement.

Now we examine the effects on the number of edges between these vertices and other cycles. For any C_{2l+1}, $l \leq i$, there are $(2l+1)(i+j+1)$ edges between the vertices of the C_{2l+1} and the two cycles before and after the replacement. For any C_{2l+1}, $i < l \leq j$, there are $2il + 2jl + j + 3l + 1$ edges between the vertices of the C_{2l+1} and the vertices of the C_{2i+1} and C_{2j+3}, and there are $2il + 2jl + j + 3l$ edges between the vertices of the C_{2l+1} and the vertices of the C_{2i+3} and C_{2j+1}. For any C_{2l+1}, $l > j$, and any C_{2l}, there are $(2i + 2j + 4)l$ edges between the vertices of the C_{2l+1} and the two cycles before and after the replacement. In all four cases, the number of edges is not increased by the replacement. Further, the number of edges decreases by 1 for each cycle of odd length $2l + 1$, $i < l \leq j$. □

Lemma 3.14 *If there are two cycles, C_{2i+1} and C_{4j+2} in the two factor, then replacing them with a C_{2i+3} and a C_{4j} will allow at most as many edges in the graph unless the C_{2i+1} is a C_3 and the C_{2i+3} that replaces it can be covered by a K_5.*

Proof. As in the previous lemma, we can assume that if C_{2i+1} is a C_5, then it is covered by a $G(5, V)$. A direct comparison of the number of edges incident to the vertices of the C_{2i+1} and C_{4j+2}, and the number of edges incident to the vertices of the C_{2i+3} and C_{4j} establishes this lemma. We start by counting the number of edges induced on the vertices of the two cycles, conditioning on the parity of i.

The +[1] in the following analysis is present to adjust for $G(6, V)$ having 12 edges rather than the 11 computed from the general formula for $G(4j+2, V)$ by adding the extra edge only if $j = 1$.

i	C_{2i+1}, C_{4j+2}	C_{2i+3}, C_{4j}
odd	$(i+2j)^2 + 3.5i + 7j + 3.5 + [1]$	$(i+2j)^2 + 3.5i + 7j + 3.5$
even	$(i+2j)^2 + 3.5i + 7j + 4 + [1]$	$(i+2j)^2 + 3.5i + 7j + 3$

In both cases, the number of edges is not increased by the replacement.

Now we examine the effects on the number of edges between these vertices and other cycles. For any C_{2l+1}, $l \leq i$, there are $(2l+1)(i+2j+1)$ edges between the vertices of the C_{2l+1} and the two cycles before and after the replacement. For any C_{2l+1}, $i < l$, there are $2il + 4jl + 2j + 3l + 1$ edges between the vertices of the C_{2l+1} and the vertices of $C_{2i+1} \cup C_{4j+2}$, and there are $2il + 4jl + 2j + 3l$ edges between the vertices of the C_{2l+1} and the vertices of $C_{2i+3} \cup C_{4j}$. For and C_{2l}, there are $2il+4jl+3l$ edges between the vertices of C_{2l} and the vertices of the two cycles before and after replacement. In all three cases, the number of edges is not increased by the replacement. Further, the number of edges decreases by 1 for each cycle of odd length $2l+1$, $i < l$. □

Lemma 3.15 *If there are two cycles, C_{2i+1} and C_{4j+4} in the two factor, then* *replacing them with a* C_{2i+5} *and a* C_{4j} *will allow at most as many edges in the graph.*

Proof. As in the previous lemmas, we can assume that if C_{2i+1} is a C_5, then it is covered by a $G(5, V)$. A direct comparison of the number of edges incident to the vertices of the C_{2i+1} and C_{4j+4} and the number of edges incident to the vertices of the C_{2i+5} and C_{4j} establishes this lemma. There are $\lceil (i + 2j)^2 + 5.5i + 11j + 7.5 \rceil$ edges induced on the vertices of the two cycles before and after replacement.

Now we examine the effects on the number of edges between these vertices and other cycles. For any C_{2l+1}, $l < i$, there are $(2l + 1)(i + 2j + 2)$ edges between the vertices of the C_{2l+1} and the two cycles before and after the replacement. For any C_{2l+1}, $l = i + 1$, there are $2l^2 + 4jl + 2j + 3l + 2$ edges between the C_{2l+1} and $C_{2i+1} \cup C_{4j+4}$, and $2l^2 + 4jl + 2j + 3l + 1$ edges between the C_{2l+1} and $C_{2i+5} \cup C_{4j}$. For any C_{2l+1}, $l > i + 1$, there are $2il + 4jl + 2j + 5l + 2$ edges between the C_{2l+1} and $C_{2i+1} \cup C_{4j+4}$; and $2il + 4jl + 2j + 5l$ edges between the C_{2l+1} and $C_{2i+5} \cup C_{4j}$. For any C_{2l}, there are $il + 2jl + 5l$ edges between the vertices of the C_{2l} and the two cycles before and after the replacement. In all four cases, the number of edges is not increased by replacement. □

Using these three lemmas we can find a lower bound for the maximum size by temporarily assuming that all C_5's are covered by $G(5, V)$'s for a lower bound, and by noting that there are at most 2 additional edges if one of the C_5's is replaced by a K_5. Lemma 3.13 then implies that there can be at most two odd cycle lengths, i and $i+2$. The case where $k = 1$ is already addressed as the hamiltonian case, so we will assume $k > 1$. Lemmas 3.8, 3.14, and 3.15 then imply that there can be at most one even cycle in the 2-factor, and that we may, if it is present, assume that it is of length 4. Once again letting c_j be the number of cycles of length j, in the 2-factor. We can restate our constraints as a system of equations:

$$\begin{aligned} i \cdot c_i + (i+2) \cdot c_{i+2} + 4 \cdot c_4 \quad &= n \\ c_i + c_{i+2} + c_4 \quad &= k. \end{aligned}$$

Manipulating these equations gives

$$ik + 2c_{i+2} + (4 - i)c_4 = n.$$

For equality to hold, parity must hold, and because i and thus $4 - i$ are odd,

$$\begin{aligned} ik + 2c_{i+2} + (4 - i)c_4 \quad &\equiv n \mod 2 \\ k + c_4 \quad &\equiv n \mod 2 \\ c_4 \quad &\equiv n - k \mod 2. \end{aligned}$$

We are now almost ready to solve this system, but first we need a little extra notation. Let $\lfloor\lfloor x \rfloor\rfloor$ denote the greatest odd integer less than or equal to x. We already noted that c_4 is either 0 or 1 and so can solve for c_4 directly from n and k. Once we know c_4, there is only one choice for i, and from i we can determine c_i and c_{i+2}.

$$c_4 = \begin{cases} 0 & n-k \equiv 0 \mod 2 \\ 1 & n-k \equiv 1 \mod 2 \end{cases}$$

$$i = \begin{cases} \left\lfloor\left\lfloor \frac{n}{k} \right\rfloor\right\rfloor & n-k \equiv 0 \mod 2 \\ \left\lfloor\left\lfloor \frac{n-4}{k-1} \right\rfloor\right\rfloor & n-k \equiv 1 \mod 2 \end{cases}$$

$$c_i = \begin{cases} k - \frac{n-ki}{2} & n-k \equiv 0 \mod 2 \\ k - \frac{n-(k-1)i-4}{2} - 1 & n-k \equiv 1 \mod 2 \end{cases}$$

$$c_{i+2} = \begin{cases} \frac{n-ki}{2} & n-k \equiv 0 \mod 2 \\ \frac{n-(k-1)i-4}{2} & n-k \equiv 1 \mod 2 \end{cases}$$

Theorem 3.6 gives a maximum size of

$$C + \frac{c_1^*}{2} + \frac{n(n+1)}{4} - \sum_{\{C_{2j+1}, C_{2l+1}\} \subset F} \min\left(\frac{2j+1}{2}, \frac{2l+1}{2}\right).$$

Here the sum can be simplified a bit, giving:

$$C + \frac{c_1^*}{2} + \frac{n(n+1)}{4} - \left[\binom{c_i}{2} + c_i \cdot c_{i+2}\right]\frac{i}{2} - \binom{c_{i+2}}{2}\frac{i+2}{2}$$

or collecting the i terms and substituting $\frac{n-k\cdot i-c_4}{2}$ for c_{i+2}:

$$C+\frac{c_1^*}{2}+\frac{n(n+1)}{4}-\binom{k-c_4}{2}\frac{i}{2}-\binom{n-k\cdot i-c_4}{2}$$

Allowing K_5's is only relevant if there is a C_5 in the 2-factor that produces the minimal number of edges, which can only occur if $i = 3$ or 5. We will examine each case individually.

Suppose that $i = 3$. If $c_3 \geq 2$, no K_5's are present by construction of the 2-factor, so $C = 0$. If $c_3 = 1$ and $c_5 = 2$, we can replace the two C_5's with a C_3 and C_7 and remain at the same size, so $C = 0$. If $c_3 = 1$ and $c_5 = 1$ or $c_5 > 2$, then one of the C_5's is covered by a K_5, instead of a $G(5, V)$, increasing the size by 1. Avoiding the K_5 by replacing a C_5 and a C_4 with a C_3 and a C_6, or two C_5's with a C_3 and a C_7 in the latter case, does not reduce the size, so $C = 1$.

Suppose that $i = 5$. If $c_5 = 2$, we can replace the two C_5's with a C_3 and C_7 and remain at the same size, so $C = 0$. If $c_5 = 1$ and $c_7 = 1$, replacing the C_5 and the C_7 with a C_3 and a C_9 reduces the maximum size. This size is 1 more than the lower bound ignoring K_5's, so $C = 1$. If $c_5 = 1, c_7 = 0$, and $c_4 = 1$, replacing the C_5 and C_4 with a C_3 and a C_6 achieves the smallest maximum size, and this size is 1 more than the lower bound ignoring K_5's, so $C = 1$. If $c_5 > 2$ or $c_7 \geq 2$, then replacing the C_5 with a C_3 creates a C_6, a C_7, or a C_9. Regardless of the replacement, there remains at least one C_5 or C_7. Tracing the effects of replacing the C_3 and the C_9 with a C_5 and a C_7 in the proof of Lemma 3.13 shows that the number of edges is reduced by 1 induced edge, and reduced by 1 edge for each cycle of length $2l+1$, $1 < l < 4$, and so reduced by at least 2 edges. The increase of 2 edges from replacing the $G(5, V)$ with a K_5 can at most offset this reduction. Therefore, the original 2-factor allows a maximum size that is the same or smaller than any 2-factor that avoids a K_5, and the smallest maximum size is 2 more than the lower bound ignoring K_5's, so $C = 2$.

This examination gives us a final formula for a sharp lower bound on the maximum size of a 2-factor isomorphic graph with a 2-factor consisting of k cycles, $k > 1$, of

$$C + \frac{c_1^*}{2} + \frac{n(n+1)}{4} - \binom{k-c_4}{2}\frac{i}{2} - \binom{n-k\cdot i-c_4}{2}$$

$$\text{where } C = \begin{cases} 0 & c_3 \geq 2 \text{ or } c_5 = 2 \\ 1 & c_3 = 1 \text{ and } c_5 > 2; c_5 = 1 \text{ and at least one of} \\ & c_3 = 1, c_7 = 1, \text{ or } k = 2 \\ 2 & i = 5 \text{ and either } c_5 > 2 \text{ or } c_7 > 2. \end{cases}$$

Chapter 4

Other Directions

In the earlier chapters we made use of the structure of the 2-factor that is present in 2-factor isomorphic graphs to help us build 2-factor isomorphic graphs of maximum size, and determine the size of such graphs. We were able to find the maximum size of a bipartite or general graph containing a particular 2-factor, the range of maximum sizes when we require that the 2-factor consist of k cycles, and the range of maximum sizes when we only require that a 2-factor is present. We now examine the case where a 2-factor is not required to be present and suggest possible extensions of this work and offer some related problems.

4.1 Vacuously 2-factor Isomorphic Graphs

In Chapter 1, we emphasized that the definitions of 2-factor hamiltonian and 2-factor isomorphic graphs required that the graph have a hamiltonian cycle and a 2-factor, respectively. If we remove this requirement, we get rather different results. We get the same maximum size for $n \leq 6$ because we cannot improve on a complete graph, nor can we avoid a $K_{3,3}$ subgraph when at least 12 edges are present on 6 vertices. For $n > 6$ however, we have other constructions that produce a larger maximum size.

4.1.1 Vacuously 2-factor Isomorphic Bipartite Graphs

For the bipartite graphs, the graph $K_{\lceil \frac{n+1}{2} \rceil, \lfloor \frac{n-1}{2} \rfloor}$ has a size of $\dfrac{n^2-4}{4}$ for even n, and $\dfrac{n^2-1}{4}$ for odd n. The only bipartite graph of larger size is $K_{\frac{n}{2},\frac{n}{2}}$ for even n, which has all possible 2-factors on n vertices and therefore cannot be 2-factor isomorphic for $n > 6$. Thus the maximum size for vacuously 2-factor isomorphic bipartite graphs is

$$\left\lfloor \frac{n^2-1}{4} \right\rfloor$$

for $n > 6$. It is however, somewhat unsatisfying to search for a 2-factor in an unbalanced bipartite graph. Therefore we now add the restriction that the partite sets be balanced.

For balanced bipartite graphs, the graph $K_{\frac{n}{2},\frac{n-2}{2}}$ with a pendant edge from the first part is balanced and has $\dfrac{n^2-2n+4}{4}$ edges (See Figure 4.1). To check that this is the maximum size for $n > 6$, we show that any graph of larger size has a 2-factor and, by our results in Chapter 2, cannot be 2-factor isomorphic. To show this, we first need an additional theorem:

Theorem 4.1 (*Hall*) *Let $G = (A \cup B, E)$ be a bipartite graph. Then G has a matching of A into B if and only if $|N(X)| \geq |X|$ for all $X \subseteq A$ [10].*

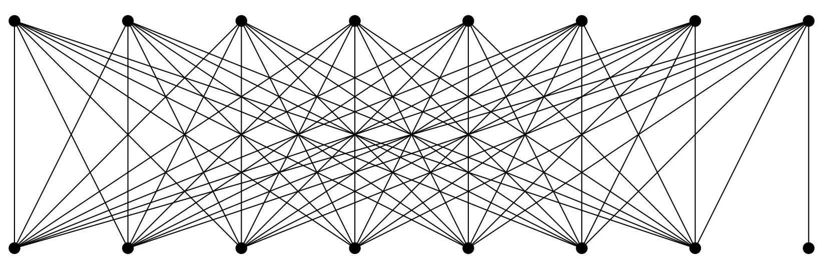

Figure 4.1: Bipartite Vacuously 2-factor Isomorphic Graph.

Let G be a balanced bipartite graph of order $n = 2m$ with partite sets A and B, and $|A| = |B| = m$ and size $m^2 - m + 2$. Note that a bipartite graph of size $m^2 - m + 2$ or larger is at most $m - 2$ edges away from the complete bipartite graph, $K_{m,m}$. This observation implies that for any $X \subset A$, there are at least $m|X| - m + 2$ edges from X to B. It also implies that for any $Y \subset B$, there are at least $m|X| - m + 2$ edges from Y to A. In particular, when $|X|$ or $|Y|$ is 1, there are at least two edges from X to B or Y to A respectively, implying that the minimum degree, $\delta(G)$, is at least 2. Any vertex in B has at most $|X|$ neighbors in X, combining this observation with the number of edges from X to B, we find that there are at least

$$\left\lceil \frac{m|X| - m + 2}{|X|} \right\rceil$$

distinct neighbors of X in B. We now verify that G meets the conditions of Hall's Theorem. For the conditions to be violated, there must be some X for which

$$|X| > |N(X)| \geq \left\lceil \frac{m|X| - m + 2}{|X|} \right\rceil \geq \frac{m|X| - m + 2}{|X|}.$$

That is, there must be some X for which $|X|^2 - |X|m + m - 2 > 0$. The roots of $|X|^2 - m|X| + m - 2 = 0$ are

$$\frac{m \pm \sqrt{m^2 - 4m + 8}}{2},$$

which are greater than $m - 1$ and less than 1 respectively. Testing the inequality on either side of the roots gives that for $1 \leq |X| \leq m - 1$, $|X| \leq |N(x)|$ For $|X| = m$, the minimum degree condition implies that there are no isolated vertices and so $|N(X)| = |X|$. Hall's Theorem then implies that there is a matching in G. We now delete the edges of this matching from G to form G', and search for another matching in G'. There are now at least $(m-1)|X| - m + 2$ edges from X to B, and so at least

$$\left\lceil \frac{(m-1)|X| - m + 2}{|X|} \right\rceil$$

distinct neighbors of X in B. Following our earlier logic, for the conditions of Hall's Theorem to be violated, there must be some X for which

$$|X| > \frac{(m-1)|X| - m + 2}{|X|}.$$

That is, there must be some X for which $|X|^2 + (1-m)|X| + m - 2 > 0$. The roots of $|X|^2 + (1-m)|X| + m - 2 = 0$ are

$$\frac{m - 1 \pm \sqrt{(m-1)^2 - 4m + 8}}{2} = m - 2,\ 1.$$

Testing the inequality on either side of the roots gives that for $1 \leq |X| \leq m - 2$, $|N(X)| \geq |X|$. For $|X| = m - 1$,

$$|N(X)| \geq \left\lceil \frac{(m-1)|X| - m + 2}{|X|} \right\rceil = \left\lceil \frac{(m-1)(m-2) + 2}{m-1} \right\rceil = |X|.$$

Finally for $|X| = m$, deleting the matching reduced the degree of each vertex by 1, so $\delta(G) \geq 2$ implies that there are no isolated vertices and so $|N(X)| =$

$|X|$. Hall's Theorem then implies that there is a matching in G'. Combining the two matchings produces a 2-factor in G, so G cannot be vacuously 2-factor isomorphic.

4.1.2 Vacuously 2-factor Isomorphic General Graphs

The graph of maximum size for general vacuously 2-factor isomorphic graphs is the complete graph K_{n-1} with a pendant edge. This graph has size $\dfrac{n^2-3n+4}{2}$. That this is the maximum size can be seen from the observation that any graph of larger size must be at most $n-3$ edges away from a complete graph, implying that $\sigma_2(G) \geq n$. Ore's Theorem then implies that such a graph has a hamiltonian cycle, and therefore a 2-factor, so the graph cannot be vacuously 2-factor isomorphic.

4.2 Other Possible Extensions

While investigating bipartite vacuously 2-factor isomorphic graphs, we examined graphs without restrictions on the form, and those graphs that possessed a necessary property to have a 2-factor, namely that the partite sets were

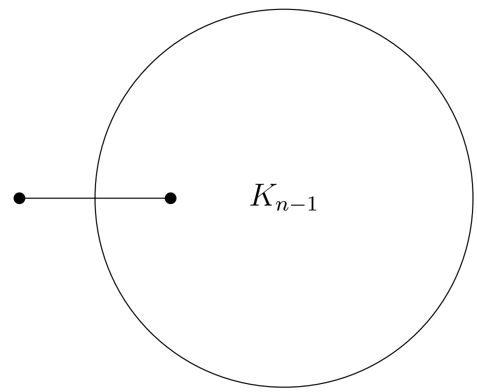

Figure 4.2: Vacuously 2-factor Isomorphic Graph.

balanced. There are several other properties that could also be required, and which might give different or more interesting results. Such properties include degree or connectivity constraints that would forbid the pendant vertex. Even a small minimum degree condition would force us to alter our general construction because it has minimum degree 3 or 2, depending on whether or not the final cycle added was a C_6. It seems less likely to have as strong an effect on the vacuously 2-factor isomorphic graphs because we can build an unbalanced bipartite graph that is only connected to the rest of the graph via the smaller partite set, and have no hope of finding a 2-factor.

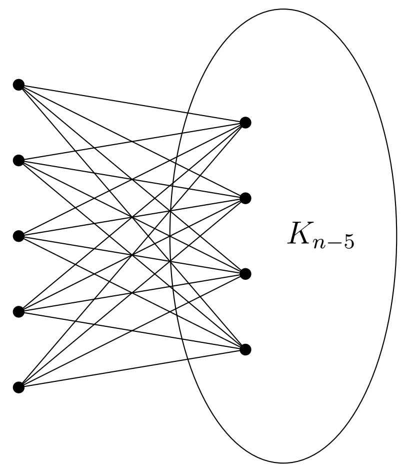

Figure 4.3: Vacuously 2-factor Isomorphic Graph with $\delta = 4$.

4.3 Related Questions

In addition to the direct extensions for maximum size, it is worth looking at 2-factor isomorphic graphs of *maximal* size. That is, rather than maximizing the size of the graph overall, we seek only to add edges until the addition of any edge violates the property of being 2-factor isomorphic. There are many questions that can be asked for maximal 2-factor isomorphic graphs:

- What is the smallest size of a maximal 2-factor isomorphic graph with a given 2-factor or family of 2-factors?
- For what sizes can we find maximal 2-factor isomorphic graphs with a given 2-factor or family of 2-factors?

- How does the answer to either of these change if we broaden the usual definition of 2-factor isomorphic to include vacuously 2-factor isomorphic graphs?
- What is the maximum size if all 2-factors are isomorphic to a member of a family of 2-factors rather than to a single 2-factor?

One possible candidate for the first question comes from the class of graphs known as *wheels*, which consist of a single vertex, the *hub*, joined to all the vertices of an $n-1$ cycle for the hamiltonian case, and wheels that are by a single edge to the rest of the graph (See Figure 4.4). A new edge in a wheel forms a cycle with the shorter path between its endpoints and this path is replaced by a path through the hub of the cycle to complete the non-isomorphic 2-factor. This construction works for $n \geq 6$ in the hamiltonian case. For 2-factors consisting of multiple cycles, this construction does not always give a maximal 2-factor isomorphic graph but for many 2-factors it will. For 2-factors that contain a C_3, this construction does not produce a maximal 2-factor isomorphic graph because additional edges can be added from the rest of the graph to the vertex in the C_3 that is the only connection to the rest of the graph without forming a 2-factor. The extent of 2-factors for which this construction gives a maximal 2-factor isomorphic graph is not yet known, but one example for which it does is 2-factors consisting of two cycles of the same length, l, $l \geq 6$ as in Figure 4.4. Any edge from a hub to the other cycle allows a 2-factor of the form $\{C_{l-1}, C_{l+1}\}$ by stealing the hub of one cycle and incorporating it into the other. Any edge within a wheel produces a non-isomorphic 2-factor within the wheel and extends to a non-isomorphic 2-factor of the graph. Any edge between non-hubs of the two wheels is paired with the edge between the wheels, allowing a hamiltonian 2-factor.

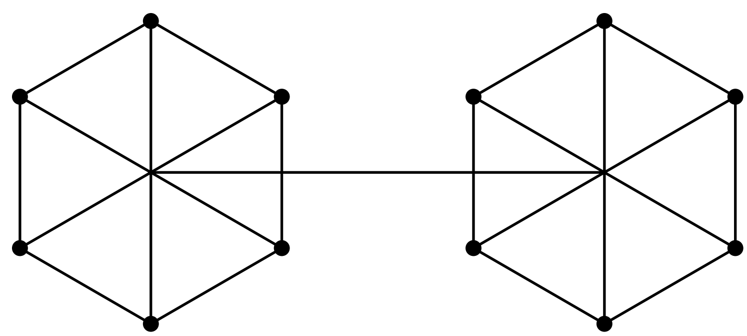

Figure 4.4: Two Wheels Connected Via Hubs

One idea for addressing the second question would be to choose a different way to insert edges between cycles, producing less than the maximum number of edges, and, hopefully, allowing a fine tuning to the desired size. Another idea would be to mix the wheel and $G(n, V)$ constructions to reduce the size in larger spurts, and then fine tune with the edge structure between cycles. While these approaches seem promising, much work remains before anything can be said with certainty for this problem, and even more so when the vacuously 2-factor isomorphic graphs are included.

Depending on the family, the last extension may be quite hard or more manageable. Families for which merging any two cycles in a 2-factor form a 2-factor not in the family are bounded by the member of the family allowing maximum size. The family of 2-factors consisting of all even cycles allows at least $\frac{3n^2-2n}{8}$ edges by completing one partite set of a balanced bipartite graph. It seems rather more difficult to maximize the size of arbitrary families, but those that consist of related 2-factors, e.g. those consisting of C_4's and C_8's or more generally C_{4i}'s for several i's hold promise as well.

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