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The Cayley-Bacharach Condition

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B.A., Lawrence University, WI, 2019

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Abstract

The Cayley-Bacharach Condition
By Rohan Nair

A set of points Γ in n-dimensional complex projective space is said to satisfy the *Cayley-Bacharach condition with respect to degree* r *hypersurfaces*, or is $CB(r)$, if any degree r hypersurface containing all but one point of Γ contains the final point. In recent literature, the condition has played an important role in computing a birational invariant called the degree of irrationality. However, the condition itself has not been studied extensively, and surprisingly little is known about the behavior of $CB(r)$ sets. We will discuss a new approach to studying $CB(r)$ sets, using combinatorial methods from matroid theory, and present some results to demonstrate how matroid theory can help us understand this condition in greater depth.

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Contents

Chapter 1

Introduction

1.1 This Dissertation: Context and Motivation

Let Γ be a finite set of points in projective space. If a degree r hypersurface that contains all but one point of Γ necessarily contains the final point, then we say that Γ satisfies the *Cayley-Bacharach condition with respect to degree r hypersurfaces*, or is $CB(r)$ for short. Point sets satisfying the Cayley-Bacharach condition occur most prominently in birational geometry, where their geometric properties can be exploited to study a birational invariant called the *degree of irrationality* of a complex projective variety. Thus, one major motivation for studying $CB(r)$ sets is to aid in the classification problem which occupies much of modern algebro-geometric research.

The Cayley-Bacharach condition is also interesting from a historical point of view. The condition is named after one of the great theorems of 19th century algebraic geometry, the famous Cayley-Bacharach theorem, which states that under certain circumstances, the finite intersection set of two curves in the projective plane satisfies $CB(r)$, for some r dependent on the degree of the curves in question. Furthermore, the Cayley-Bacharach theorem is a generalization of a much older theorem from the 4th century CE, Pappus's Theorem, which in modern terms says that point sets which arise as intersections of certain line configurations in the projective plane must be $CB(3)$.

Given the condition's origins in mathematical antiquity, one might be surprised to discover that very little is known about how $CB(r)$ sets behave. To emphasize this point, this dissertation was initially motivated by trying to answer the following simple question: *What conditions guarantee that a $CB(r)$ set also contains a $CB(r)$ subset?* The primary obstruction to answering this question and others like it is that many of the proof techniques developed to date are *ad hoc* in nature. For example, some techniques only work when we make certain restrictive assumptions about our point sets, such as requiring the points be collinear or contained on some configuration of linear spaces , and therefore resist generalization.

The primary objective of this dissertation is to present a new tool for studying $CB(r)$ sets, a tool which is fundamentally combinatorial in nature. The first inkling that such methods might apply to the Cayley-Bacharach condition comes from Levinson and Ullery [6], who observed that many of their proofs could be rewritten using the language of *matroid theory*. Matroids are combinatorial structures that simultaneously generalize notions of independence, dependence, and rank (or dimension) found in both linear algebra and graph theory. In exploring this possible connection, we discovered that the Cayley-Bacharach condition is elegantly described in matroid-theoretic terms and that many useful facts about Cayley-Bacharach sets which eluded prior proof follow naturally from the combinatorics.

1.2 Outline

Chapter 2 is expository and begins with some historical background on the Cayley-Bacharach theorem, followed by a section about the condition itself, including relevant definitions and examples. We then elaborate on the connections between $CB(r)$ sets and birational geometry, and conclude with an overview of what is currently known about $CB(r)$ sets to-date. **Chapter 3** is also expository, and covers some basic matroid theory, with an emphasis on material that is most relevant to our results. We present these results in **Chapter 4**, where we construct a matroid on point sets in projective space, and use this combinatorial structure to study $CB(r)$ sets, specifically. We conclude with a discussion of possible future avenues of research.

Chapter 2

The Cayley-Bacharach Condition

2.1 Some Historical Background

Our story begins long before the development of modern algebraic geometry, with a theorem attributed to the late antiquity mathematician Pappus of Alexandria:

Theorem 2.1.1 (Pappus). *Fix two lines in the plane, pick 6 points on these lines, and label them A, B, C, A', B', C'. Also label the points: $P = (\overline{AB'} \cap \overline{A'B}), Q = (\overline{AC'} \cap \overline{A'C})$, and $R = (\overline{BC'} \cap \overline{B'C})$. Then P, Q, and R are collinear.*

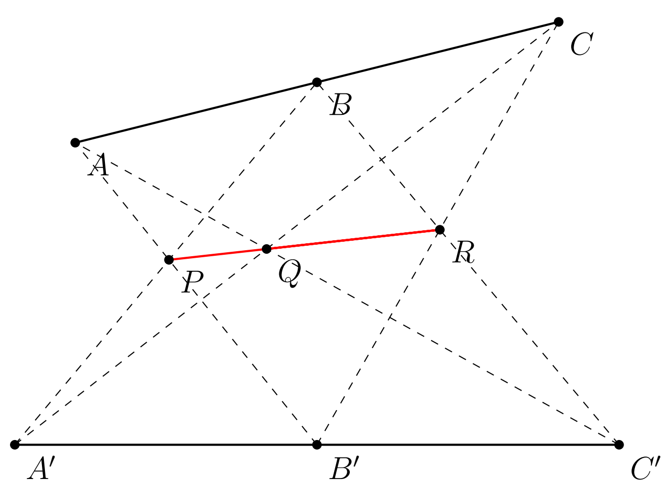

Figure 2.1: Pappus's Theorem

Those familiar with the history of mathematics will recognize Pappus's Theorem as the first known theorem in projective geometry [1]. Indeed, since we have not specified that the lines be skew nor that the points be chosen in order to guarantee the existence of the intersection points P, Q, and R, the statement as written above can only be true in the projective plane over a field. If we restrict our attention to the case of algebraically closed fields, then we find ourselves in the 19th century, closer to the birth of algebraic geometry as we understand it today, with the Cayley-Bacharach theorem:

Theorem 2.1.2 (Cayley-Bacharach)**.** *Let k be an algebraically closed field, and let X and Y be two curves in $\mathbb{P}^2_k$ of degrees d and e respectively, intersecting at de points. Then any curve Z of degree $d+e-3$ containing all but one of these intersection points also contains the final point.*

The Cayley-Bacharach theorem, as stated above, is not an obvious generalization of Pappus's Theorem, but the latter indeed follows from the former by setting $d = e = 3$, and letting $X = (\overline{AB'} \cup \overline{A'C} \cup \overline{BC'})$, $Y = (\overline{A'B} \cup \overline{AC'} \cup \overline{BC'})$, and $Z = (\overline{AC} \cup \overline{AC'} \cup \overline{PR})$ be our three (degenerate) cubic curves. In both cases, the key observation is that certain special point configurations in the projective plane impose very strong restrictions on the curves that can contain them. This phenomenon, in both the projective plane and generalized to projective space of higher dimension, is at the heart of the Cayley-Bacharach condition. For the remainder of this dissertation, we will focus our attention away from the theorem and towards the condition which bears its name. Curious readers who wish to know more about the Cayley-Bacharach theorem, its history, and its generalizations should read Eisenbud, Green, and Harris's excellent account in [5].

[1] This theorem is often referred to as Pappus's Hexagon Theorem, to avoid confusion with another geometric theorem which bears his name.

2.2 Definitions

Throughout the rest of this chapter, we assume we are working over $\mathbb{C}$, and that when we refer to finite sets of points, we are treating them as zero-dimensional reduced closed subschemes of some given scheme. Furthermore, although we will focus our attention on $CB(r)$ sets, we first state a more general form of the Cayley-Bacharach condition:

Definition. Let $\mathcal{L}$ be a line bundle on a smooth projective variety X, and let $\Gamma = \{p_1, \ldots, p_m\}$ be a finite set of points on X. We say that Γ *satisfies the Cayley-Bacharach condition with respect to the line bundle* $\mathcal{L}$ if any section $f \in H^0(X, \mathcal{L})$ vanishing on all but one point of Γ necessarily vanishes at the final point.

This definition restricts to $CB(r)$ in the following manner:

Definition. Let $X = \mathbb{P}^n$, let r be a fixed integer, and let $\mathcal{L} = \mathcal{O}_{\mathbb{P}^n}(r)$. A finite set of points $\Gamma = \{p_1, \ldots, p_m\} \subset X$ is said to be $CB(r)$ if Γ satisfies the Cayley-Bacharach condition with respect to $\mathcal{O}_{\mathbb{P}^n}(r)$.

Since we can interpret global sections of $\mathcal{O}_{\mathbb{P}^n}(r)$ as either polynomials or hypersurfaces when $r \geq 0$, we have two more ways of formulating the definition of $CB(r)$ in these cases:

1. (Algebraically) A finite set of points $\Gamma \subset \mathbb{P}^n$ *satisfies the Cayley-Bacharach condition with respect to degree r polynomials*, or is $CB(r)$, if any degree r homogeneous polynomial F vanishing on all but one point of Γ vanishes at the final point.

2. (Geometrically) A finite set of points $\Gamma \subset \mathbb{P}^n$ *satisfies the Cayley-Bacharach condition with respect to degree r hypersurfaces*, or is $CB(r)$, if any degree r hypersurface containing all but one point of Γ contains the final point.

Although all three of these definitions are equivalent, each provides a unique and useful perspective on $CB(r)$ sets, and we will often switch between these perspectives throughout the course of this dissertation.

2.3 Cohomological Interpretation

Since the Cayley-Bacharach condition is defined in terms of line bundles, we can characterize $CB(r)$ sets via sheaf cohomology. Given a finite set of points $i : \Gamma \hookrightarrow \mathbb{P}^n$, we have the following standard closed subscheme short exact sequence:

$$0 \to \mathcal{I}_\Gamma \to \mathcal{O}_{\mathbb{P}^n} \to i_* \mathcal{O}_\Gamma \to 0$$

where $\mathcal{I}_\Gamma$ is the ideal sheaf corresponding to the closed subscheme Γ. Twisting by an integer r and taking global sections gives us the following four-term exact sequence on cohomology groups [2]:

$$0 \to H^0(\mathbb{P}^n, \mathcal{I}_\Gamma(r)) \to H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(r)) \to H^0(\mathbb{P}^n, i_*\mathcal{O}_\Gamma(r)) \to H^1(\mathbb{P}^n, \mathcal{I}_\Gamma(r)) \to 0$$

We thus get the following characterization of $CB(r)$ sets in terms of cohomology groups:

Proposition 2.3.1. *Let $\Gamma \subset \mathbb{P}^n$ be a finite set of points, and let $r \in \mathbb{Z}$ be fixed. Then Γ is $CB(r)$ if and only if $H^0(\mathbb{P}^n, \mathcal{I}_\Gamma(r)) = H^0(\mathbb{P}^n, \mathcal{I}_{\Gamma\setminus\{p\}}(r))$ for all $p \in \Gamma$.*

Proof. Suppose Γ is $CB(r)$, and let $p \in \Gamma$ be given. Since any section of $H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(r))$ vanishing on Γ must vanish on $\Gamma\setminus\{p\}$, we have $H^0(\mathbb{P}^n, \mathcal{I}_\Gamma(r)) \subset H^0(\mathbb{P}^n, \mathcal{I}_{\Gamma\setminus\{p\}}(r))$. To get the other inclusion, note that if $f \in H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(r))$ vanishes on $\Gamma\setminus\{p\}$, then f must vanish on Γ by $CB(r)$, so $H^0(\mathbb{P}^n, \mathcal{I}_{\Gamma\setminus\{p\}}(r)) \subset H^0(\mathbb{P}^n, \mathcal{I}_\Gamma(r))$. Hence, $H^0(\mathbb{P}^n, \mathcal{I}_\Gamma(r)) = H^0(\mathbb{P}^n, \mathcal{I}_{\Gamma\setminus\{p\}}(r))$.

[2] The final term of this sequence comes from the fact that $H^1(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(r)) = 0$.

Conversely, suppose $H^0(\mathbb{P}^n, \mathcal{I}_\Gamma(r)) = H^0(\mathbb{P}^n, \mathcal{I}_{\Gamma\setminus\{p\}}(r))$ for all $p \in \Gamma$. Then any section $f \in H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(r))$ vanishing on $\Gamma\setminus\{p\}$ also vanishes on Γ, so Γ is $CB(r)$. $\square$

The next corollary follows trivially, but is a useful enough fact to state explicitly:

Corollary 2.3.2. *Let $\Gamma \subset \mathbb{P}^n$ be a finite set of points, and let $r \in \mathbb{Z}$ be fixed. Then Γ is $CB(r)$ if and only if* $\dim H^0(\mathbb{P}^n, \mathcal{I}_\Gamma(r)) = \dim H^0(\mathbb{P}^n, \mathcal{I}_{\Gamma\setminus\{p\}}(r))$ *for all $p \in \Gamma$.*

One fact the cohomological interpretation makes clear is that a set Γ can be $CB(r)$ without being contained in a degree r hypersurface:

Definition. We say a $CB(r)$ set Γ is *vacuously* $CB(r)$ if $\dim H^0(\mathbb{P}^n, \mathcal{I}_\Gamma(r)) = 0$. Otherwise, we say that Γ is *nonvacuously* $CB(r)$.

2.4 Examples

We now present examples of $CB(r)$ sets in $\mathbb{P}^2$.

Example. Let $\Gamma = \{A, B, C\} \subset \mathbb{P}^2$ be three collinear points. Then Γ is $CB(1)$, since collinearity guarantees that any line passing through two points in Γ must pass through the third.

In $\mathbb{P}^2$, the collinearity of at least three points is equivalent to $CB(1)$. In higher dimensions, nonvacuously $CB(1)$ implies cohyperplanarity, but the converse does not hold.

Our next example is the simplest non-collinear case:

Example. Let $\Gamma = \{A, B, C, D, E, F\} \subset \mathbb{P}^2$ be six points on a conic X. Let Γ' be a five-point subset of Γ, and let Y be a conic (i.e., a degree 2 hypersurface in $\mathbb{P}^2$) containing Γ'. Suppose X and Y are distinct curves. Then we have $\Gamma' \subseteq X \cap Y$, but by Bezout's Theorem, $|X \cap Y| = 4 < 5 = |\Gamma'|$, a contradiction. Thus $Y = X$ and Y contains every point in Γ, so Γ is $CB(2)$.

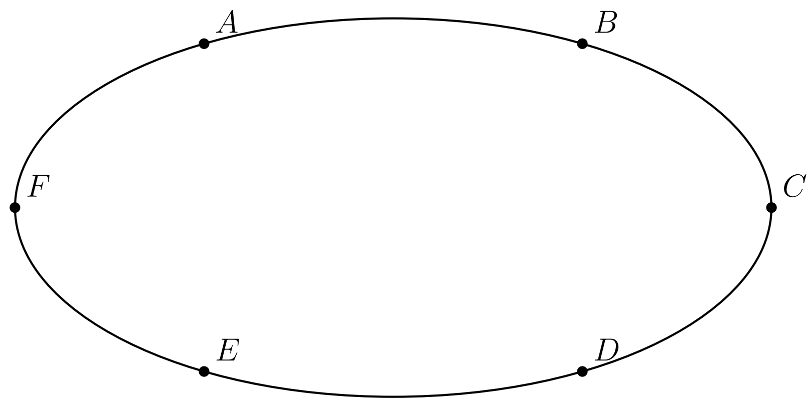

Figure 2.2: Six points on a conic X

The configuration in Figure 2.2 can be extended to a configuration that is also $CB(2)$:

Example. Let $\Gamma_1 = \{A, B, C, D, E, F\} \subset \mathbb{P}^2$ be six points on a conic X, and let $\Gamma_2 = \{G, H\} \subset \mathbb{P}^2$ be two points away from X. Let $\Gamma = \Gamma_1 \cup \Gamma_2$, and suppose, without loss of generality, that Y is a conic containing $\Gamma_1 \cup \{G\}$. Then $\Gamma_1 \subset X \cap Y$, but by Bezout's Theorem, $|X \cap Y| = 4 < 6 = |\Gamma_1|$. Therefore, no such conic Y can exist, so Γ is vacuously $CB(2)$.

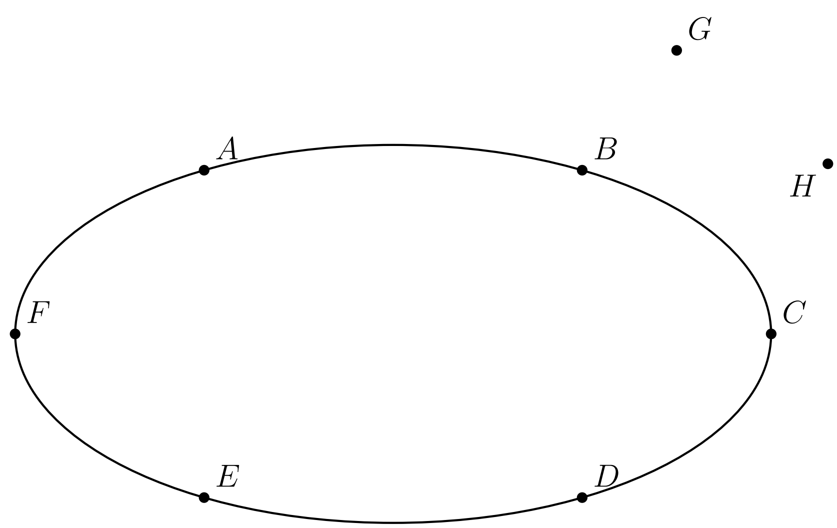

Figure 2.3: Six points on a conic X, two points away from X

The next example illustrates what happens when we remove the point H from Figure 2.3:

Example. Let $\Gamma_1 = \{A, B, C, D, E, F\} \subset \mathbb{P}^2$ be six points on a conic X, and let $\Gamma_2 = \{G\} \subset \mathbb{P}^2$ be one point away from X. Then $\Gamma = \Gamma_1 \cup \Gamma_2$ is *not* $CB(2)$, since we've exhibited a degree 2 hypersurface containing all but one point of Γ.

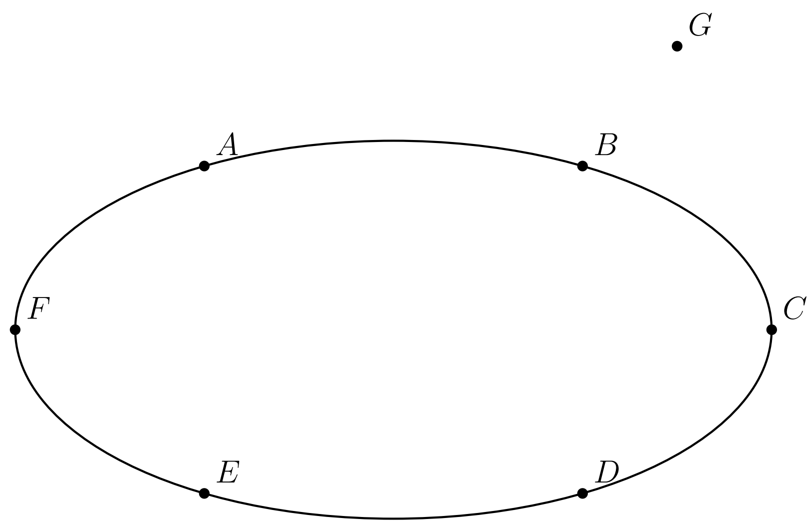

Figure 2.4: Six points on a conic X, one point away from X

The three examples above demonstrate the major issue with determining when $CB(r)$ sets contain $CB(r)$ subsets. Specifically, non-$CB(r)$ sets can be nested inside of $CB(r)$ sets in such a way that simply deleting points arbitrarily from a configuration does not guarantee that the resulting point set will also be $CB(r)$.

We end this section by reformulating the Cayley-Bacharach theorem in terms of $CB(r)$ sets:

Theorem 2.4.1 (Cayley-Bacharach)**.** *Let X and Y be two curves in $\mathbb{P}^2$ of degrees d and e respectively, and let $\Gamma = X \cap Y$. Then Γ is $CB(d + e - 3)$.*

2.5 Measures of Irrationality

In this section, we briefly discuss one major source of recent motivation for studying the Cayley-Bacharach condition. We first recall the following definition from the study of algebraic curves:

Definition. Let C be a projective curve. The *gonality* of C, denoted $\text{gon}(C)$, is the minimum degree of a dominant rational map $C \dashrightarrow \mathbb{P}^1$.

Gonality is a birational invariant that quantifies the extent to which a curve fails to be rational. In other words, $\text{gon}(C) = 1$ if and only if C is birationally equivalent to $\mathbb{P}^1$ if and only if C is a rational curve.

In general, the gonality of a curve is not easily computable. In the case of plane curves, however, we have the following nice theorem, attributed to Max Nöether.

Theorem 2.5.1 (Max Nöether)**.** *Let C be a smooth projective plane curve of degree $d \geq 3$. Then the gonality of C, $gon(C) = d - 1$. Furthermore, any dominant rational map $C \dashrightarrow \mathbb{P}^1$ of degree $d - 1$ is obtained by projecting from a point p on C.* [3]

We can extend the notion of curve gonality to higher dimensions in a few different ways (see [4] for some examples), but the most straightforward generalization is the following:

Definition. Let X be a complex projective variety of dimension n. The *degree of irrationality* of X, denoted $\text{irr}(X)$, is the minimum degree of a dominant rational map $X \dashrightarrow \mathbb{P}^n$.

As with gonality, the degree of irrationality is a birational invariant that quantifies the extent to which an n-dimensional complex projective variety fails to be rational, i.e. $\text{irr}(X) = 1$ if and only if X is rational. We should expect computing degrees of

[3] In the case when $d < 3$, gonality can be computed with the help of the genus-degree formula for smooth plane curves.

irrationality of various varieties to be at least as difficult as computing curve gonality, if only because of the heuristic that computing invariants gets harder in higher dimensions. We might still ask, however, the following question: does there exist an extension of Max Nöether's theorem for sufficiently "nice" n-dimensional varieties? The following theorem from Bastianelli et al. gives us a (mostly) affirmative answer:

Theorem 2.5.2 ([4], Theorem C). *Let $X \subset \mathbb{P}^{n+1}$ be a very general smooth hypersurface of dimension n with degree $d \geq 2n+1$. Then the degree of irrationality of X, $irr(X) = d-1$. Additionally, if $d \geq 2n+2$, then any rational map $X \dashrightarrow \mathbb{P}^n$ of degree $d-1$ is obtained by projecting from a point p on X.*

The proof of this theorem relies on the following proposition from Bastianelli:

Proposition 2.5.3 ([2], Proposition 4.2). *Let X be a complex projective variety of dimension n. Then the general fiber of a generically finite dominant rational map $X \dashrightarrow \mathbb{P}^n$ satisfies the Cayley-Bacharach condition with respect to the canonical bundle ω_X.*

Thus, studying the geometry of points satisfying the Cayley-Bacharach condition can give us information about maps $X \dashrightarrow \mathbb{P}^n$, which we can leverage to bound or compute $\mathrm{irr}(X)$. Furthermore, for many types of varieties, such as hypersurfaces or complete intersections, the Cayley-Bacharach condition with respect to the canonical bundle restricts to $CB(r)$, thereby justifying our decision to focus on this special case.

2.6 An Overview of Known Results

The aim of this section is to present an overview of all major results about $CB(r)$ sets found to-date in the literature, with one qualification. The results found here are answers to the question: Given a $CB(r)$ set, what can we say about its geometry? We notably omit results about generating $CB(r)$ sets in the first place, which fall under the umbrella of generalized Cayley-Bacharach theorems. The purpose for this omission is that the motivation for developing our matroid-theoretic tools comes from the former type of problem, rather than the latter. Before we can state many of these results, however, we require the following definition:

Definition. A *plane configuration* $\mathcal{P}$ is a union of finitely-many distinct linear subspaces P_i in $\mathbb{P}^n$, with $\dim P_i > 0$ for each i. That is,

$$\mathcal{P} = \bigcup_{i=1}^{k} P_i$$

The dimension of $\mathcal{P}$, $\dim \mathcal{P} = \sum_i \dim P_i$ and the length of $\mathcal{P}$, $\ell(\mathcal{P}) = k$.

Many of the results in this section apply only to special types of plane configurations, called skew and split configurations:

Definition. A plane configuration $\mathcal{P} = \bigcup_{i=1}^{k} P_i$ is **skew** if $P_i \cap P_j = \emptyset$ for all pairs i, j. A skew plane configuration is called **split** if $\dim \mathrm{Span}\mathcal{P} = \dim(\mathcal{P}) + \ell(\mathcal{P}) - 1$.

In cases when P_i and P_j have a non-empty intersection, we can replace them in our configuration with $\mathrm{Span}(P_i, P_j)$, thereby often allowing us to focus our attention on skew configurations.

The most basic interaction between plane configurations and $CB(r)$ sets is known as the **excision property**:

Proposition 2.6.1 ([6], Proposition 2.5)**.** *Let $\Gamma \subset \mathbb{P}^n$ be a $CB(r)$ set, and let $\mathcal{P}$ be a plane configuration of length ℓ. Then $\Gamma \backslash \mathcal{P}$ is $CB(r - \ell)$.*

By taking $\mathcal{P}$ to be a single hyperplane disjoint from Γ, the excision property gives us the **descending property**:

Corollary 2.6.2. *If Γ is $CB(r)$, then Γ is $CB(d)$, for all integers $d \leq r$.*

The descending property illustrates the utility of our multiple perspectives on $CB(r)$ sets. From the perspective of polynomials or cohomology, it is not at all clear that $CB(r)$ sets actually satisfy *multiple* Cayley-Bacharach conditions simultaneously, but this fact follows relatively easily from the geometric perspective. Another fact which follows from the geometry is that the size of the $CB(r)$ set depends on r, via the **basic lower bound**:

Proposition 2.6.3 ([6], Proposition 2.6)**.** *If Γ is $CB(r)$, then $|\Gamma| \geq r+2$.*

Sometimes we know with certainty that subsets of $CB(r)$ sets are also $CB(r)$ sets, provided they are partitioned by split plane configurations:

Proposition 2.6.4 ([6], Proposition 4.2)**.** *Let Γ be a finite set of points contained in a split plane configuration $\mathcal{P} = \bigcup P_i$. Then Γ is $CB(r)$ if and only if $\Gamma \cap P_i$ is $CB(r)$, for each i.*

In the case our plane configuration is neither split nor skew, but has a single-point intersection, we have a weaker result of similar flavor:

Proposition 2.6.5 ([6], Lemma 4.5)**.** *Let $\mathcal{P} = A \cup B$ be a length 2 non-skew plane configuration, meeting at a point p. Suppose $\Gamma \subseteq A \cup B$ is $CB(r)$. Let $\Gamma_A = (\Gamma \cap A) \backslash \{p\}$ and let $\Gamma_B = (\Gamma \cap B) \backslash \{p\}$. Then:*

1. *At least one of Γ_A and $\Gamma_A \cup \{p\}$ is $CB(r)$.*
2. *At least one of Γ_B and $\Gamma_B \cup \{p\}$ is $CB(r)$.*
3. *If $p \notin \Gamma$, then at least one of Γ_A and $\Gamma_B \cup \{p\}$ is $CB(r)$.*
4. *If $p \notin \Gamma$, then at least one of Γ_B and $\Gamma_A \cup \{p\}$ is $CB(r)$.*
5. *If $p \in \Gamma$, then at least one of $\Gamma_A \cup \{p\}$ and $\Gamma_B \cup \{p\}$ is $CB(r)$.*

The next result demonstrates that $CB(r)$ sets restrict the geometry of skew plane configurations that contain them:

Proposition 2.6.6 ([6], Proposition 4.7)**.** *Let $\mathcal{P} = \bigcup P_i$ be a skew plane configuration containing a $CB(r)$ set Γ. Then either each plane P_i contains at least $\max(\ell(\mathcal{P}), r+2)$ points of Γ, or some plane P_i contains fewer than $\ell(\mathcal{P})$ points and $\ell(\mathcal{P}) \geq r+2$.*

In Propositions 2.6.4 and 2.6.5, we start with a $CB(r)$ set on a plane configuration, and determined when they contain $CB(r)$ subsets. Our next few results answer something of the opposite question: given a $CB(r)$ set, what conditions guarantee it lies on a linear subspace of $\mathbb{P}^n$ and/or a plane configuration of some given dimension and length? When our $CB(r)$ set contains sufficiently few elements, then surprisingly their geometry is restricted not just to a plane configuration, but to a line:

Proposition 2.6.7 ([3], Lemma 2.4)**.** *If Γ is $CB(r)$ and $|\Gamma| \leq 2r+1$, then the points of Γ are collinear.*

Under weaker restrictions on size, Levinson and Ullery proved in some cases that $CB(r)$ sets lie on "small" plane configurations:

Theorem 2.6.8 ([6], Theorem 1.3)**.** *Suppose $|\Gamma| \leq (d+1)r+1$. Then:*

(i) When $r \leq 2$ and for any d, Γ lies on a plane configuration with $\dim \mathcal{P} = 3$ and $\ell(\mathcal{P}) = 1$.

(ii) For any r and when $d \leq 3$, Γ lies on a plane configuration with $\dim \mathcal{P} = d$, and for $r \leq 4$, $\ell(\mathcal{P}) \leq 2$.

(iii) When $d = 4$ and $r = 3$, Γ lies on a plane configuration with $\dim \mathcal{P} = 4$ and $\ell(\mathcal{P}) \leq 2$.

Banerjee was able to prove a similar result, but with a different relation between r and d:

Theorem 2.6.9 ([1], Theorem 1.15)**.** *Suppose $|\Gamma| \leq (d+1)r+1$. If $r >> d$, then Γ lies on a plane configuration of length d.*

Proposition 2.6.7, Theorem 2.6.8, and Theorem 2.6.9 all represent partial progress towards the following conjecture:

Conjecture 2.6.10. *If $|\Gamma| \leq (r+1)d+1$, then Γ lies on a plane configuration of length d.*

If this conjecture is true, then we also have the following result:

Proposition 2.6.11 ([6], Propositions 4.3 and 4.6)**.** *Fix $r \in \mathbb{Z}$. Suppose Conjecture 2.6.10 holds up to $d-1$, and let $|\Gamma| \leq (d+1)r+1$. Let A and B be positive dimensional linear spaces, and suppose $\Gamma \subset A \cup B$. Then if $A \cup B = \emptyset$ or $A \cup B = \{p\}$ for some point p, then Γ lies on a plane configuration of dimension d.*

Given the results about plane configurations, an immediate question one might ask is: under what conditions do $CB(r)$ sets lie on curves or higher-dimensional schemes? To date, we only have answers in the case of curves. Stapleton and Ullery were able to slightly adjust the bound from Proposition 2.6.7 to prove the following result:

Theorem 2.6.12 ([9], Theorem 1.9)**.** *Let Γ be $CB(r)$. If $|\Gamma| \leq (5/2)r+1$, then Γ is contained in either a line or on a (not necessarily irreducible) curve of degree 2.*

Picoco was able to extend this result to curves of up to degree 4:

Theorem 2.6.13 ([8], Theorem A)**.** *Let Γ be $CB(r)$. For $3 \leq d \leq 5$, if $|\Gamma| \leq d(r-d+3)-1$, then Γ lies on a curve C of degree $d-1$.*

Our final result of this section is also from Banerjee, which extends these results to curves of any degree, but with a caveat that r must be sufficiently large:

Theorem 2.6.14 ([1], Theorem 1.13)**.** *Let $\Gamma \subset \mathbb{P}^n$ be $CB(r)$, and let $r >> d \geq 1$. Then there exists a function f, which depends on n, such that $|\Gamma| < rd - f(d)$ implies that Γ lies on a curve C with $\deg C \leq d$.*

As evidenced by this overview, there is still much work to be done on the geometry of $CB(r)$ sets. For posterity, we can broadly categorize most of the results in this section as answering one of two questions, each of which corresponds to a future avenue of research. These questions are:

1. Given a $CB(r)$ set Γ lying on some scheme X, when does there exist a subscheme X' of X such that $\Gamma \cap X'$ is also $CB(r)$?

2. Given a $CB(r)$ set Γ and some extra conditions imposed on Γ (such as upper bounds on its size), are the points of Γ contained in some scheme X, the nature of which is determined by said conditions?

The remainder of this dissertation is dedicated to our matroid-theoretic tools which might one day help produce new answers to these questions and others.

Chapter 3

Some Relevant Matroid Theory

The goal of this chapter is to provide the requisite amount of matroid theory to understand the content of Chapter 4. The reference for this section is James Oxley's comprehensive book on the subject [7]. Many of the results we require are either relegated to the exercises or are not stated explicitly in the text, and in these cases we have furnished the necessary proofs.

3.1 Matroids as Rank Functions

One unique feature of matroid theory is the fact that there exist several equivalent, but not obviously equivalent, definitions of a matroid. This equivalence is known as the "matroid cryptomorphism". The cryptomorphic definition we will adopt for this dissertation is expressed in terms of an abstract notion of rank:

Definition. A **matroid** M is a pair (E, ρ), where E is a finite set and $\rho : 2^E \to \mathbb{Z}$ is a function satisfying the following conditions:

1. (Boundedness) $0 \leq \rho(U) \leq |U|$ for all $U \subseteq E$
2. (Monotonicity) If $S \subseteq U$, then $\rho(S) \leq \rho(U)$
3. (Submodularity) $\rho(S \cup U) + \rho(S \cap U) \leq \rho(S) + \rho(U)$ for all $S, U \subseteq E$

The function ρ is called the **rank function** of the matroid, and the integer $\rho(E)$ is called the matroid's **rank**.

One important class of matroids are called **linear matroids**, which come from vector spaces in the following manner:

Proposition 3.1.1. *Let V be a vector space over a field k, and let $E = \{v_1, \ldots, v_n\}$ be a collection of vectors in V. For each subset $U \subset E$, we let $V_U = Span\{v : v \in U\}$. Define a function $\rho : 2^E \to \mathbb{Z}$ by $U \mapsto \dim(V_U)$. Then (E, ρ) is a matroid.*

Proof. Let U and S be subsets of E.

1. (Boundedness) Since $\rho(U)$ is a dimension of a subspace of V, $\rho(U) \geq 0$. If U is linearly independent, then $\rho(U) = \dim(V_U) = |U|$. If U is linearly dependent, then $\rho(U) = \dim(V_U) < |U|$. In either case, we have $0 \leq \rho(U) \leq |U|$.

2. (Monotonicity) Suppose $S \subseteq U$. Then $V_S \subseteq V_U$, and since dimension is monotonic on vector subspaces, $\rho(S) \leq \rho(U)$.

3. (Submodularity) Let $v \in V_{S\cap U}$. Then $v = \sum_{i=1}^{|S\cap U|} a_i v_i$, where each $a_i \in k$ and each $v_i \in S \cap U$. Thus, $v \in V_S$ and $v \in V_U$, so $V_{S\cap U} \subseteq V_S \cap V_U$, which implies that $\rho(S \cap U) \leq \dim(V_S \cap V_U)$. Since $V_{S\cup U} = V_S + V_U$, we have:

$$
\begin{aligned}
\rho(S \cap U) \leq \dim(V_S \cap V_U) &= \dim(V_S) + \dim(V_U) - \dim(V_S + V_U) \\
&= \dim(V_S) + \dim(V_U) - \dim(V_{S\cup U}) \\
&= \rho(S) + \rho(U) - \rho(S \cup U)
\end{aligned}
$$

$\square$

On an arbitrary matroid, it is not guaranteed that the rank function will behave the way it does on a vector space. Our first lemma establishes, however, that just as removing a vector from a subspace can only drop the dimension by at most one, removing an element from a subset of a matroid will not drop the rank by too much:

Lemma 3.1.2. *Let $M = (E, \rho)$ be a matroid, let $U \subseteq E$, and let $p \in U$. Then:*

$$\rho(U) - 1 \leq \rho(U \backslash \{p\}) \leq \rho(U)$$

Proof. We immediately have $\rho(U \backslash \{p\}) \leq \rho(U)$ by monotonicity. For the lower bound, applying submodularity to the sets $U \backslash \{p\}$ and $\{p\}$ gives us the following inequality:

$$\rho((U \backslash \{p\}) \cup \{p\}) + \rho((U \backslash \{p\}) \cap \{p\}) \leq \rho(U \backslash \{p\}) + \rho(\{p\})$$

Boundedness tells us that $0 \leq \rho(\{p\}) \leq 1$ and that $\rho((U \backslash \{p\}) \cap \{p\}) = \rho(\emptyset) = 0$. Combining these facts, we have:

$$\rho((U \backslash \{p\}) \cup \{p\}) = \rho(U) \leq \rho(U \backslash \{p\}) + \rho(\{p\}) \leq \rho(U \backslash \{p\}) + 1$$

Thus we have our result. $\square$

For the remainder of this chapter, we will assume that M is a matroid on a set E with rank function ρ, unless stated otherwise.

3.2 Independent and Dependent Sets

Extending the vector space analogy, matroids also come equipped with a notion of independent and dependent sets. In fact, another common cryptomorphic definition of matroids begins with independent sets as the primary objects, and defines the rank function in terms of independent sets. We will do the opposite here:

Definition. A subset $U \subseteq E$ is **independent** if and only if $|U| = \rho(U)$, and is **dependent** if and only if $|U| > \rho(U)$.

The boundedness of rank implies that the above dichotomy is strict: all sets in a matroid are either independent or dependent. Another way to characterize independent and dependent sets is via the notion of nullity:

Definition. The **nullity** of a subset U of E, denoted $\nu(U)$, is defined as:

$$\nu(U) = |U| - \rho(U)$$

An independent set is therefore a set with zero nullity, and a dependent set is a set with positive nullity. As before, the boundedness of rank means that sets cannot have negative nullity.

In the case of linear matroids, independent and dependent sets are precisely those subsets of E which are linearly independent and linearly dependent, respectively. We know from basic linear algebra that subsets of linearly independent sets are also linearly independent, and that linearly dependent sets can only be contained in other linearly dependent sets. The following proposition confirms that this behavior holds in the case of all matroids, thus strengthening the vector space analogy:

Proposition 3.2.1. *Let $M = (E, \rho)$ be a matroid, and let $U \subseteq E$.*

1. *If I is independent and $U \subseteq I$, then U is independent.*
2. *If D is dependent and $D \subseteq U$, then U is dependent.*

Proof. For the first claim, it suffices to prove that $I \backslash \{p\}$ is independent for all $p \in I$. For a contradiction, suppose that $I \backslash \{p\}$ is dependent. Then by the boundedness property of rank and the independence of I, $\rho(I \backslash \{p\}) < |I \backslash \{p\}| = |I| - 1 = \rho(I) - 1$. On the other hand, Lemma 3.1.2 gives us $\rho(I) - 1 \leq \rho(I \backslash \{p\}) < \rho(I) - 1$, a contradiction. Thus $I \backslash \{p\}$ is independent.

The second claim follows from the first, since if we suppose U is independent, then it would contradict the dependence of D. $\square$

Corollary 3.2.2. *The empty set is independent.*

Lemma 3.1.2 tells us that deleting an element from a set can only drop the rank by one at most. In fact, our next lemma establishes that independent sets can also be characterized by how their rank changes in response to removing an element:

Lemma 3.2.3. *I is independent if and only if, for any $p \in I$, $\rho(I \backslash \{p\}) = \rho(I) - 1$.*

Proof. Suppose I is independent. Then $|I| = \rho(I)$. For any $p \in I$, we have $\rho(I \backslash \{p\}) \leq |I \backslash \{p\}| = |I| - 1 < |I| = \rho(I)$. By Lemma 3.1.2, $\rho(I \backslash \{p\}) = \rho(I) - 1$.

Conversely, suppose there exists a $p \in I$ such that $\rho(I \backslash \{p\}) = \rho(I)$. Then by boundedness and monotonicity, $\rho(I) = \rho(I \backslash \{p\}) \leq |I \backslash \{p\}| < |I|$, so I is dependent.

□

Finally, independent sets are an especially useful notion from our perspective because they are "rank-regulating":

Lemma 3.2.4. *Let U be a subset of E, and let I be a independent subset of U of maximal size. Then $\rho(U) = \rho(I)$*

We omit the proof of the above proposition, because we would first have to prove the cryptomorphic equivalence of the rank and independent set definitions of a matroid, which is beyond the scope of this exposition. Interested readers can consult Chapter 1.3 in [7].

3.3 Bases, Circuits, and Connectivity

One major consequence of Proposition 3.2.1 is that a matroid must contain maximal independent and minimal dependent sets, in a manner made precise by the following definition:

Definition. A **basis** of a matroid is a maximal independent set, i.e. a subset B such that $B \cup \{p\}$ is dependent for all $P \in E \backslash B$. A **circuit** is a minimal dependent set, i.e. a subset C such that $C \backslash \{p\}$ is independent for all $p \in C$.

One can check that every independent set is contained in a basis, and that every dependent set contains a circuit. Conceiving of matroids in terms of bases and circuits is often useful, and in fact each notion corresponds to another cryptomorphic definition. Bases are specifically useful because, just as maximal independent subsets regulate the ranks of the sets that contain them, bases regulate the rank of the entire matroid:

Proposition 3.3.1. *Let B be a basis of a matroid M. Then $\rho(B) = \rho(E)$. Furthermore, for any other basis B' of M, $|B| = |B'|$*

Proof. Since B is a maximal independent subset of E, $\rho(B) = \rho(E)$ by Lemma 3.2.4. For the second part, note that $|B| = \rho(B) = \rho(E) = \rho(B') = |B'|$. □

Unlike bases, circuits do not all need to have the same rank and size. However, all circuits *do* have the same nullity:

Lemma 3.3.2. *Let C be a circuit in M. Then $\nu(C) = 1$.*

Proof. Since C is a circuit, $C \backslash \{p\}$ is independent for all $p \in C$, so $\nu(C \backslash \{p\}) = 0$. By Lemma 3.2.4, $\rho(C) = \rho(C \backslash \{p\})$. Combining these two facts, we have $\nu(C) = |C| - \rho(C)$ and $|C - \{p\}| = \rho(C \backslash \{p\}) = \rho(C)$. Thus, $\nu(C) = 1$. □

We can use the notion of a circuit to define some other distinguished subsets of a matroid.

Definition. Let p and q be elements in E.

1. We say p is a **loop** in M if $\{p\}$ is a circuit in M.

2. We say p is a **coloop** in M if $\{p\}$ is not contained in *any* circuits.

3. We say p and q are **parallel** in M if $\{p, q\}$ is a circuit.

Definition. A matroid containing no loops or parallel elements is called **simple**.

Coloops are of special interest to the results in the next chapter, and are related to bases as follows:

Lemma 3.3.3. *An element $p \in E$ is a coloop if and only if p is contained in every basis B of M.*

Proof. Suppose p is a coloop in M. For a contradiction, suppose there exists a basis B that does not contain p. Then $B \cup \{p\}$ is dependent, and thus contains a circuit C. Since B is independent, it cannot contain a circuit, so $p \in C$. This contradicts the fact that p is a coloop, so no such basis can exist.

Conversely, suppose p is contained in every basis B of M. If p is contained in a circuit C, then $C \backslash \{p\}$ is independent and thus contained in a basis B. But since $p \in B$, we have $C \backslash \{p\} \subset C \subseteq B$, which cannot occur since B is independent and C is dependent. Thus p is not contained in any circuits, and is hence a coloop.

□

We can also use circuits to import from graph theory the notion of connectivity:

Definition. A matroid M is **connected** if for any two points $p, q \in E$, there exists a circuit C of M such that $\{p, q\} \subset C$.

In some cases, it is also useful to think of connectivity in terms of the rank function. To see how this works, we first need a definition:

Definition. A **separator** of a matroid M is a subset U of E such that:

$$\rho(U) + \rho(E \backslash U) = \rho(E)$$

Proposition 3.3.4. *A matroid is not connected if and only if it contains a separator.*

Again, we omit the proof of the above proposition, since it requires invoking a notion called matroid duality, which is not required to understand the results of the next chapter. This result, however, follows immediately by combining Proposition 2.1.9. and Proposition 4.2.4. in [7].

3.4 Rank Hyperplanes

Rank-maximal sets, called **flats,** are also important in matroid theory. For our purposes, we only need to consider the following types of flats.

Definition. A **rank hyperplane**[1] is a subset H of E such that:

1. $\rho(H) = \rho(E) - 1$
2. For any element $p \in E \backslash H$, $\rho(H \cup \{p\}) = \rho(H) + 1 = \rho(E)$

Not all rank hyperplanes have the same cardinality. This distinction will be important later on:

Definition. A rank hyperplane H of M is **maximal** if $|H| \geq |H'|$ for all other rank hyperplanes H' of M.

However, all rank hyperplanes are "maximal" in the sense that no rank hyperplane can contain another:

Lemma 3.4.1. *If H and H' are rank hyperplanes such that $H \subseteq H'$, then $H = H'$.*

Proof. Let $p \in H' \backslash H$. Since H is a rank hyperplane, $\rho(H \cup \{p\}) = \rho(H) + 1 = \rho(E)$. By monotonicity, however, we have $\rho(E) = \rho(H \cup \{p\}) \leq \rho(H') = \rho(E) - 1$, a contradiction. Thus $H = H'$. $\square$

Coloops and rank hyperplanes are intimately related, as per the following lemma:

[1] In matroid theory, these are simply referred to as "hyperplanes". The term "rank hyperplane" is used here to distinguish these from geometric hyperplanes.

Lemma 3.4.2. *An element $p \in E$ is a coloop if and only if $E \backslash \{p\}$ is a rank hyperplane of M.*

Proof. Suppose p is a coloop of M. Then $\rho(E\backslash\{p\}) = \rho(E) - 1$ and $\rho(E\backslash\{p\} \cup \{p\}) = \rho(E)$, so $E\backslash\{p\}$ is a rank hyperplane of M.

Conversely, suppose $E\backslash\{p\}$ is a rank hyperplane of M. Suppose there exists a basis B disjoint from p. Then $B \subseteq E\backslash\{p\}$, which implies that $\rho(B) \leq \rho(E\backslash\{p\}) = \rho(E) - 1$, which contradicts the fact that B is a basis of M. Thus, no such basis exists. By Lemma 3.3.3, p is a coloop of M. $\square$

3.5 Deletion and Restriction

Finally, we discuss two operations we can perform on matroids to construct new matroids. The first operation constructs a new matroid by removing a subset from the original matroid and restricting the rank function accordingly, via a process called "deletion":

Definition. Let M be a matroid, and let $U \subseteq E$. The **deletion** of U from M is the matroid $M\backslash U = (E\backslash U, \rho_{E\backslash U})$, where for any subset $S \subseteq E\backslash U$,

$$\rho_{E\backslash U}(S) = \rho(S)$$

Using this operation, we can restrict a matroid to a subset:

Definition. The **restriction** of a matroid M to a subset U, denoted $M|_U$, is the deletion $M\backslash(E\backslash U)$. For notational simplicity, we will write $M|_U = (U, \rho_U)$, and if $S \subseteq U \subseteq \Gamma$, we will let $\nu_U(S) = |S| - \rho_U(S)$ denote the restriction of the nullity to U.

We end this chapter with a lemma about how circuits behave with respect to matroid deletion:

Lemma 3.5.1. *Let U be a subset of E. If C is a circuit in $M\backslash U$, then C is a circuit in M*

Proof. Since C is a circuit of $M\backslash U$, we have $\rho_U(C) < |C|$ and $\rho_U(C\backslash\{q\}) = |C\backslash\{q\}|$ for all $q \in C$. Since both C and $C\backslash\{q\}$ are contained in $E\backslash U$, these two conditions hold when we replace ρ_U with ρ. Hence, C is a circuit of M. $\square$

Chapter 4

Hilbert Function Matroids

This chapter contains the novel results of this dissertation. As with most of Chapter 2, we will assume we are working over $\mathbb{C}$, and that when we refer to finite sets of points, we are implicitly treating them as zero-dimensional reduced closed subschemes of $\mathbb{P}^n$.

4.1 A Matroid on Projective Points

We first establish some notation. Let $\Gamma \subset \mathbb{P}^n$ be a finite set of points, and fix $r \in \mathbb{Z}$. For a subset $\Lambda \subseteq \Gamma$, let $h_\Lambda(r) = \operatorname{codim}(H^0(\mathbb{P}^n, \mathcal{I}_\Lambda(r)), H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(r)))$ denote the Hilbert function of Λ evaluated at r. Let $\rho : 2^\Gamma \to \mathbb{Z}$ be the function given by $\rho(\Lambda) = h_\Lambda(r)$. With this notation established, we state our first major theorem:

Theorem 4.1.1. *The pair $\Gamma_r = (\Gamma, \rho)$ is a matroid.*

We call this matroid the **degree r Hilbert function matroid** of Γ. Before we prove Theorem 4.1.1, we recall the following definition from linear algebra:

Definition. Let V be a vector space, and let W be a subspace of the dual space V^*. The **annihilated subspace** of W is

$$W^{-0} = \{v \in V : \phi(v) = 0 \text{ for all } \phi \in W\} \subseteq V$$

Lemma 4.1.2. *Let V be a vector space, and let W be a subspace of the dual V^*. Then* $\dim W + \dim W^{-0} = \dim V$

Proof. Fix a basis $\{\phi_1, \ldots, \phi_k\}$ for W, and consider the map $F : V \to \mathbb{C}^k$ given by $v \mapsto (\phi_1(v), \ldots, \phi_k(v))$. This map is surjective since the set $\{\phi_1, \ldots, \phi_k\}$ is linearly independent in V^*, and by construction, $\ker F = W^{-0}$. Then by the Rank-Nullity Theorem, we have $\dim V = k + \dim W^{-0} = \dim W + \dim W^{-0}$. $\square$

Proof of Theorem 4.1.1. For each $p \in \Gamma$, we fix coordinates $[p_0 : \cdots : p_n]$, and let $e_p \in H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(r))^*$ denote the map $f \mapsto f(p_0, \ldots, p_n)$. For each subset $\Lambda \subseteq \Gamma$, let $E_\Lambda = \{e_p : p \in \Lambda\}$. By Proposition 3.1.1, the pair $M = (E_\Gamma, \varrho)$ is a matroid, where $\varrho : 2^{E_\Gamma} \to \mathbb{Z}$ is the map $E_\Lambda \mapsto \dim \operatorname{Span}\{e_p : e_p \in E_\Lambda\}$.

We now want to relabel M to get our Hilbert function matroid. The first step is to establish that the map $\Gamma \to E_\Gamma$ given by $p \mapsto e_p$ is a bijection. It suffices to prove injectivity, since the definition of E_Γ guarantees surjectivity. Let p and q be distinct points in Γ, let $f \in H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1))$ be a hyperplane that contains p but not q, and let $g \in H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(r-1))$ be a hypersurface that avoids both p and q. Then $e_p(fg) = 0$ and $e_q(fg) \neq 0$, so $e_p \neq e_q$. Thus, our map is injective.

Next, we aim to show that ϱ is the same function as ρ. For each subset $\Lambda \subset \Gamma$, let $W_\Lambda = \operatorname{Span}\{e_p : e_p \in E_\Lambda\} \subseteq H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(r))^*$. We claim that the annihilated subspace of W_Λ, $W_\Lambda^{-0} = H^0(\mathbb{P}^n, \mathcal{I}_\Lambda(r))$[1], where $\mathcal{I}_\Lambda$ is the ideal sheaf of $\Lambda \hookrightarrow \mathbb{P}^n$. To

[1] Here we are identifying $H^0(\mathbb{P}^n, \mathcal{I}_\Lambda(r))$ with its image under the inclusion $H^0(\mathbb{P}^n, \mathcal{I}_\Lambda(r)) \hookrightarrow H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(r))$.

this end, let $h \in H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(r))$. We then have the following equivalences:

$$\begin{aligned} h \in W_\Lambda^{-0} &\iff \\ \phi(h) = 0 \text{ for all } \phi \in W_\Lambda &\iff \\ e_q(h) = 0 \text{ for all } q \in \Lambda &\iff \\ h(q) = 0 \text{ for all } q \in \Lambda &\iff \\ h \in H^0(\mathbb{P}^n, \mathcal{I}_\Lambda(r)) & \end{aligned}$$

Thus we have our equality. By Lemma 4.1.2:

$$\begin{aligned} \dim W_\Lambda &= h^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(r)) - \dim W_\Lambda^{-0} \\ &= h^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(r)) - h^0(\mathbb{P}^n, \mathcal{I}_\Lambda(r)) \\ &= h_\Lambda(r) \end{aligned}$$

Hence, $\varrho(\Lambda) = \rho(\Lambda)$ for all $\Lambda \in 2^\Gamma$, so $\Gamma_r = (\Gamma, \rho)$ is a matroid. □

Corollary 4.1.3. *Hilbert function matroids are simple.*

Proof. Γ_r is a linear matroid, so it contains no loops, since a singleton set in a vector space can never be linearly dependent. Now given two points p and q in Γ, we can always find a degree r hypersurface containing one but not the other, which is equivalent to the claim that $e_p \neq \lambda e_q$ for all non-zero $\lambda \in \mathbb{C}$. Thus $\{p, q\}$ is not a circuit. □

One useful consequence of our new matroid structure is that the notion of nullity has a very simple cohomological interpretation:

Lemma 4.1.4. *Let $\Gamma \subset \mathbb{P}^n$ be a finite set of points with Hilbert function matroid Γ_r, and let $\Lambda \subseteq \Gamma$. Then the nullity of Λ in Γ_r, $\nu(\Lambda) = h^1(\mathbb{P}^n, \mathcal{I}_\Lambda(r))$.*

Proof. Fixing a closed immersion $i : \Lambda \hookrightarrow \mathbb{P}^n$, recall we have the following exact sequence:

$$0 \to H^0(\mathbb{P}^n, \mathcal{I}_\Lambda(r)) \to H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(r)) \to H^0(\mathbb{P}^n, i_*\mathcal{O}_\Lambda(r)) \to H^1(\mathbb{P}^n, \mathcal{I}_\Lambda(r)) \to 0$$

Then by exactness, we have:

$$\begin{aligned} h^1(\mathbb{P}^n, \mathcal{I}_\Lambda(r)) &= h^0(\mathbb{P}^n, i_*\mathcal{O}_\Lambda(r)) - h^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(r)) + h^0(\mathbb{P}^n, \mathcal{I}_\Lambda(r)) \\ &= h^0(\mathbb{P}^n, i_*\mathcal{O}_\Lambda(r)) - h_\Lambda(r) \\ &= |\Lambda| - h_\Lambda(r) \\ &= |\Lambda| - \rho(\Lambda) \\ &= \nu(\Lambda) \end{aligned}$$

□

We now recall the following definition, which is often used to delineate between two types of configurations in projective space:

Definition. A set of points $\Gamma \subset \mathbb{P}^n$ is said to be in **general position** (with respect to degree r polynomials) if $h^1(\mathbb{P}^n, \mathcal{I}_\Gamma(r)) = 0$. If $h^1(\mathbb{P}^n, \mathcal{I}_\Gamma(r)) > 0$, then we say that Γ is r**-superabundant**.

It turns out that this delineation is the correct characterization of independent and dependent sets in a Hilbert function matroid:

Proposition 4.1.5. *Let $\Gamma_r = (\Gamma, \rho)$ be a Hilbert function matroid, and let $\Lambda \subseteq \Gamma$. Then:*

1. *Λ is independent if and only if it is in general position.*

2. *Λ is dependent if and only if it is r-superabundant.*

Proof. This follows immediately from Lemma 4.1.4. □

4.2 Matroids and *CB(r)* Sets

We now turn our attention again to the Cayley-Bacharach condition. Our first result demonstrates that special Hilbert function matroids precisely correspond to $CB(r)$ sets:

Proposition 4.2.1. *Let $\Gamma_r = (\Gamma, \rho)$ be a Hilbert function matroid. The point set Γ is $CB(r)$ if and only if $\rho(\Gamma) = \rho(\Gamma \backslash \{p\})$ for all $p \in \Gamma$.*

Proof. This follows from Corollary 2.3.2, replacing dimension with codimension. □

Furthermore, $CB(r)$ sets are dependent:

Proposition 4.2.2. *$CB(r)$ sets are r-superabundant.*

Proof. Let Γ be a $CB(r)$ set. Then $\nu(\Gamma) = |\Gamma| - \rho(\Gamma)$. Let $p \in \Gamma$. Then by Proposition 4.2.1:

$$\begin{aligned} \nu(\Gamma \backslash \{p\}) &= |\Gamma \backslash \{p\}| - \rho(\Gamma \backslash \{p\}) \\ &= |\Gamma| - 1 - \rho(\Gamma) \end{aligned}$$

Combining these two equations, we get $\nu(\Gamma) = \nu(\Gamma \backslash \{p\}) + 1 \geq 1$. Thus, Γ is r-superabundant. □

Hilbert function matroids also correspond to $CB(r)$ sets in another way, via coloops:

Proposition 4.2.3. *Γ is $CB(r)$ if and only if Γ_r has no coloops.*

Proof. Suppose Γ is $CB(r)$. Then for any point $p \in \Gamma$, $\rho(\Gamma \backslash \{p\}) = \rho(\Gamma)$, so $\Gamma \backslash \{p\}$ is *not* a rank hyperplane. By Lemma 3.4.2, p is not a coloop of Γ_r, so Γ_r contains no coloops.

Conversely, suppose Γ_r contains no coloops. Then $\Gamma \backslash \{p\}$ is *not* a rank hyperplane for all $p \in \Gamma$, so $\rho(\Gamma \backslash \{p\}) = \rho(\Gamma)$. Thus Γ is $CB(r)$. □

A simple consequence of Proposition 4.2.3 is that $CB(r)$ sets can be decomposed as follows:

Corollary 4.2.4. *Γ is $CB(r)$ if and only if $\Gamma = H \cup T$, where H is a maximal rank hyperplane of Γ_r and $|T| \geq 2$.*

Finally, we can use nullity to characterize $CB(r)$ sets in a third way:

Lemma 4.2.5. *Γ is $CB(r)$ if and only if $\nu(\Gamma) = \nu(\Gamma \backslash \{p\}) + 1$ for all $p \in \Gamma$.*

Proof. By Proposition 4.2.1, Γ is $CB(r)$ if and only if $\rho(\Gamma) = \rho(\Gamma \backslash \{p\})$ for all $p \in \Gamma$, which is true if and only if $|\Gamma| - \nu(\Gamma) = |\Gamma \backslash \{p\}| - \nu(\Gamma \backslash \{p\})$, which is true if and only if $\nu(\Gamma) = \nu(\Gamma \backslash \{p\}) + 1$. □

4.3 Circuits and *CB(r)* Sets

In this section, we examine the connection between circuits and $CB(r)$ sets, ultimately culminating in a partial answer to one of our earliest motivating questions: when do $CB(r)$ sets contain $CB(r)$ subsets?

Proposition 4.3.1. *Let Γ be a finite set of points with $\nu(\Gamma) = 1$. Then Γ is $CB(r)$ if and only if Γ is a circuit in Γ_r.*

Proof. Suppose Γ is $CB(r)$. By Proposition 4.2.2, Γ is dependent. Furthermore, by Lemma 4.2.5, $\nu(\Gamma \backslash \{p\}) = 0$ for all $p \in \Gamma$, so $\Gamma \backslash \{p\}$ is independent. Thus, Γ is a circuit.

Conversely, suppose Γ is a circuit in Γ_r. Then $\Gamma \backslash \{p\}$ is independent for all $p \in \Gamma$. Since Γ is dependent, $\Gamma \backslash \{p\}$ must be a basis for Γ_r, so by Proposition 3.3.1, $\rho(\Gamma \backslash \{p\}) = \rho(\Gamma)$. Thus, Γ is $CB(r)$. □

Our next result answers our question very broadly:

Proposition 4.3.2. *Let Γ be a finite set of points with $\nu(\Gamma) > 1$. Then Γ contains a proper subset that is $CB(r)$.*

Proof. Since $\nu(\Gamma) > 1$, we know Γ is dependent, and by Lemma 3.3.2, Γ is *not* a circuit in Γ_r. Thus it contains a circuit $C \neq \Gamma$. Restricting Γ_r to this circuit C gives us the matroid $\Gamma_r|_C = (C, \rho_C)$. By Lemma 3.5.1, the circuits of $\Gamma_r|_C$ are precisely those circuits of Γ_r contained in C, so C is a circuit in $\Gamma_r|_C$. Furthermore, because C is a circuit in Γ_r, Lemma 3.3.2 tells us that $\nu(C) = 1$. As $\rho(C) = \rho_C(C)$, C satisfies the nullity condition of Proposition 4.3.1. Hence, C is a $CB(r)$ set. $\square$

An interesting consequence is that circuits in Hilbert function matroids are actually bounded below in size, which is not usually the case for arbitrary matroids:

Corollary 4.3.3. *Let Γ be a finite set of points, and let C be a circuit in Γ_r. Then $|C| \geq r + 2$.*

Proof. Since C is a circuit in Γ_r, it also satisfies $CB(r)$. Then by Proposition 2.6.3, $|C| \geq r + 2$. $\square$

Though Proposition 4.3.2 technically answers our initial question, it turns out we can say more in the case of $CB(r)$ sets specifically. First, we will require the following lemma:

Lemma 4.3.4. *Let Γ be $CB(r)$. If H is a rank hyperplane in Γ_r, then for all $p \in \Gamma \backslash H$, H is a rank hyperplane in $\Gamma_r \backslash \{p\}$. Furthermore, if H is maximal in Γ_r, then H is maximal in $\Gamma_r \backslash \{p\}$.*

Proof. Since $H \subseteq \Gamma \backslash \{p\}$, $\rho_{\Gamma \backslash \{p\}}(H) = \rho(H) = \rho(\Gamma) - 1$. By $CB(r)$, $\rho_{\Gamma \backslash \{p\}}(\Gamma \backslash \{p\}) = \rho(\Gamma \backslash \{p\}) = \rho(\Gamma)$, so $\rho_{\Gamma \backslash \{p\}}(H) = \rho_{\Gamma \backslash \{p\}}(\Gamma \backslash \{p\}) - 1$.

Next, let $q \in (\Gamma \backslash \{p\}) \backslash H$. Then:

$$\begin{aligned}
\rho_{\Gamma \backslash \{p\}}(H \cup \{q\}) &= \rho(H \cup \{q\}) \\
&= \rho(\Gamma) \\
&= \rho(\Gamma \backslash \{p\}) \\
&= \rho_{\Gamma \backslash \{p\}}(\Gamma \backslash \{p\})
\end{aligned}$$

Thus, H is a rank hyperplane in $\Gamma_r \backslash \{p\}$.

Now, suppose H is maximal in Γ_r, and let H' be a hyperplane in $\Gamma_r \backslash \{p\}$. Observe that:

$$\begin{aligned}
\rho(H') &= \rho_{\Gamma \backslash \{p\}}(H') \\
&= \rho_{\Gamma \backslash \{p\}}(\Gamma \backslash \{p\}) - 1 \\
&= \rho(\Gamma \backslash \{p\}) - 1 \\
&= \rho(\Gamma) - 1
\end{aligned}$$

Thus, the set H' is either a rank hyperplane of Γ_r or contained in a rank hyperplane of Γ_r. In either case, $|H'| \leq |H|$, by the maximality of H, so H is maximal in $\Gamma_r \backslash \{p\}$. $\square$

Theorem 4.3.5. *Let Γ be a $CB(r)$ set satisfying the following two conditions:*

1. *$\Gamma = H \cup T$, where H is a maximal rank hyperplane and $|T| > 2$*
2. *$\nu(\Gamma) > 1$*

Then for any point $p \in T$, $\Gamma \backslash \{p\}$ is $CB(r)$.

Proof. The nullity condition guarantees that both Γ and $\Gamma \backslash \{p\}$ are dependent. Next we have the decomposition $\Gamma \backslash \{p\} = H \cup (T \backslash \{p\})$. By Lemma 4.3.4, H is still maximal in $\Gamma_r\backslash\{p\}$. Since $|T\backslash\{p\}| \geq 2$, Corollary 4.2.4 allows us to conclude that $\Gamma\backslash\{p\}$ is $CB(r)$. $\square$

Corollary 4.3.6. *Let Γ be a $CB(r)$ set satisfying the conditions of Theorem 4.3.5. Then there exist points $\{p_1, \ldots, p_k\}$ such that, for all subsets $I \subset \{1, \ldots, k\}$, $\Gamma\backslash\{p_i : i \in I\}$ is $CB(r)$.*

Proof. We can choose any $|T| - 2$ points from T and label them arbitrarily. We then delete one point at a time from Γ. At each step of this process, note that $\Gamma\backslash\{p_i : i \in I\} = H \cup (T\backslash\{p_i : i \in I)$. By Lemma 4.3.4, H remains a maximal rank hyperplane after each point deletion, and $|T\backslash\{p_i : i \in I\}| = |T| - |I| \geq |T| - (|T| - 2) = 2$, so $\Gamma\backslash\{p_i : i \in I\}$ is $CB(r)$ for all $I \subset \{1, \ldots, k\}$. Furthermore, $\Gamma\backslash\{p_i : i \in I\}$ only fails to satisfy the conditions of Theorem 4.3.5 once we've deleted all of our chosen points, thereby justifying our repeated application of the theorem. $\square$

4.4 Connected Hilbert Function Matroids

We now examine the relationship between $CB(r)$ sets and matroid connectivity:

Proposition 4.4.1. *If Γ_r is connected, then Γ is $CB(r)$.*

Proof. Suppose Γ is *not* $CB(r)$. Then there exists a point $p \in \Gamma$ such that $\rho(\Gamma - \{p\}) + \rho(\{p\}) = \rho(\Gamma)$. Thus, the set $\{p\}$ is a separator of Γ_r, so Γ_r is not connected. $\square$

Our next three results demonstrate that the converse is also true, under certain restrictions:

Proposition 4.4.2. *Let Γ be a $CB(r)$ set such that $\rho(\Gamma) \leq r + 1$. Then Γ_r is connected.*

Proof. Let p and q be two distinct points of Γ. Since Γ_r is simple, $\{p\}$ and $\{q\}$ are independent, and hence contained in some basis B. Since Γ is dependent, there exists a point $t \notin B$, with $B \cup \{t\}$ dependent. Thus $B \cup \{t\}$ contains a circuit C. However, we have:

$$r + 2 \leq |C| \leq |B \cup \{t\}| = \rho(\Gamma) + 1 \leq r + 2$$

Thus, $B \cup \{t\}$ is a circuit containing both p and q, so M_r is connected. □

Proposition 4.4.3. *Let Γ be a $CB(r)$ subset with $|\Gamma| = r+2$. Then Γ_r is connected.*

Proof. Suppose $|\Gamma| = r+2$. Let C be a circuit of M_r. Then $r+2 \leq |C| \leq |\Gamma| = r+2$, so Γ is a circuit. Thus, Γ_r is connected. □

Theorem 4.4.4. *Let Γ be a $CB(r)$ set satisfying the following two conditions:*

1. *$\Gamma = H \cup T$, where H is a maximal rank hyperplane in M_r, H is independent, and $|T| > 2$.*

2. *$\nu(\Gamma) > 1$*

Then Γ_r is connected.

Proof. We proceed by induction on $\nu(\Gamma)$. Suppose $\nu(\Gamma) = 2$. Let p and q be two points in Γ. Since $|T| > 2$, we know there exists a point $s \in T$ such that $s \neq p$ and $s \neq q$. By Lemma 4.2.5 and Theorem 4.3.5, $\Gamma \backslash \{s\}$ is $CB(r)$ with $\nu_{\Gamma\backslash\{s\}}(\Gamma\backslash\{s\}) = 1$. By Proposition 4.3.1, $\Gamma\backslash\{s\}$ is a circuit in $\Gamma_r\backslash\{s\}$. By Lemma 3.5.1, $\Gamma\backslash\{s\}$ is also circuit in Γ_r, and it contains both p and q. Since we can find a circuit containing any pair of points in Γ, Γ_r is connected.

Suppose our claim holds for $\nu(\Lambda) = k > 1$, where Λ is any $CB(r)$ set whose Hilbert function matroid satisfies the conditions above. Let Γ be a $CB(r)$ set also satisfying these conditions, and such that $\nu(\Gamma) = k + 1$. Fix two points p and q in Γ. Again, since $|T| > 2$, there exists a point $s \in T$ such that $s \neq p$ and $s \neq q$. Then $\Gamma\backslash\{s\}$ is $CB(r)$. Since we immediately know that $\nu_{\Gamma\backslash\{s\}}(\Gamma\backslash\{s\}) = k > 1$, we only need to show that $\Gamma\backslash\{s\}$ satisfies Condition 1.

We have $\Gamma\backslash\{s\} = H \cup (T\backslash\{s\})$. We know H is maximal in $\Gamma_r\backslash\{s\}$ since it is maximal in Γ_r, and that H is independent in $\Gamma_r\backslash\{s\}$ since $|H| = \rho(H) = \rho_{\Gamma\backslash\{s\}}(H)$. We now need to show that $|T\backslash\{s\}| > 2$. Since H is an independent rank hyperplane, $|H| = \rho_{\Gamma\backslash\{s\}}(H) = \rho_{\Gamma\backslash\{s\}}(\Gamma\backslash\{s\}) - 1$. Then:

$$\begin{aligned}
|T\backslash\{s\}| &= |(\Gamma\backslash\{s\})\backslash H| \\
&= |\Gamma\backslash\{s\}| - |H| \\
&= |\Gamma\backslash\{s\}| - (\rho_{\Gamma\backslash\{s\}}(\Gamma) - 1) \\
&= \nu_{\Gamma\backslash\{s\}}(\Gamma\backslash\{s\}) + 1 \\
&= k + 1 \\
&> 2
\end{aligned}$$

Thus, $\Gamma_r\backslash\{s\}$ is connected by our induction hypothesis, so there exists a circuit C in $\Gamma_r\backslash\{s\}$ that contains both p and q. Then C is also a circuit in Γ_r containing p and q, so Γ_r is connected. $\square$

4.5 Future Research

We now present some possible extensions of the work done in this dissertation. The first conjecture is similar to Theorem 4.3.5:

Conjecture 4.5.1. *Let Γ be a $CB(r)$ set satisfying the following two conditions:*

1. *$\Gamma = H \cup T$, where H is a maximal rank hyperplane and $|T| = 2$*
2. *$\nu(\Gamma) > 1$*

Then there exists a point $p \in H$ such that $\Gamma\backslash\{p\}$ is $CB(r)$.

Before stating the next conjecture, we need a definition:

Definition. Let $\Gamma \subset \mathbb{P}^n$ be a finite set of points. We say that Γ satisfies the *strong CB(r) condition* if Γ is $CB(r)$ and if, for any $p \in \Gamma$, $\Gamma \backslash \{p\}$ is not $CB(r)$.

If Conjecture 4.5.1 is true, then in tandem with Theorem 4.3.5, it should be possible to prove the next conjecture:

Conjecture 4.5.2. *Every strong $CB(r)$ set is of the form $\Gamma = H \cup T$, where H is an independent maximal rank hyperplane and $|T| = 2$.*

Next, we can consider matroids satisfying a more general notion of connectivity. Given a matroid $M = (E, \rho)$, we define a *connectivity function* $\lambda_M : 2^E \to \mathbb{Z}$, where for any subset U of E,

$$\lambda_M(U) = \rho(U) + \rho(E - U) - \rho(E)$$

We say that the pair $(U, E - U)$ is a j-separation of M if $\lambda_M(U) < j$ and $\min\{|U|, |E - U|\} \geq j$. A matroid M is *k-connected* if M contains no j-separations for $1 \leq j \leq k - 1$. An immediate consequence of this definition is that a matroid is connected if and only if it is 2-connected. However, this does not preclude a connected matroid from satisfying higher connectivity conditions.

As per Proposition 4.4.1, if Γ_r is 2-connected, then Γ is $CB(r)$. However, nothing is known about what happens when $k > 2$, and this suggests the possibility of a classification of all $CB(r)$ sets in terms of higher connectivity conditions:

Problem. *Classify $CB(r)$ sets by k-connectivity.*

Finally, we can examine generalized Cayley-Bacharach conditions. One generalization, suggested by Levinson and Ullery [6], is to consider a zero-dimensional subscheme $X \subseteq \mathbb{P}^n$ of finite length t. In this case, we say that X is $CB(r)$ if, whenever a homogeneous degree r polynomial F vanishes on a closed subscheme $X' \subseteq X$ of length $t - 1$, then F vanishes on all of X.

The second generalization is to higher dimensions, by considering the Cayley-Bacharach condition on algebraic k-cycles in $\mathbb{P}^n$ (or on some projective variety X). We say, in this case, that a k-cycle $Z = X_1 + \cdots + X_m$ is $CB(r)$ if any homogeneous degree r polynomial vanishing on $Z - X_i$ must vanish at X_i, for all $1 \leq i \leq m$.

Problem. *Determine if there exists an analogue for Hilbert function matroids in the case of these generalized $CB(r)$ sets.*

In the higher-dimensional generalization, a Hilbert function matroid on algebraic cycles would present many interesting opportunities to extend matroid-theoretic methods to problems in intersection theory and enumerative geometry.

Bibliography

[1] Ishan Banerjee. Error terms for the motives of discriminant complements and a cayley-bacharach theorem, 2024. URL `https://arxiv.org/abs/2403.07272`.

[2] Francesco Bastianelli. On symmetric products of curves. *Transactions of the American Mathematical Society*, 364(5):2493–2519, September 2012. ISSN 0021-8693.

[3] Francesco Bastianelli, Renza Cortini, and Pietro De Poi. The gonality theorem of noether for hypersurfaces. *Journal of Algebraic Geometry*, 23:313–339, 02 2011.

[4] Francesco Bastianelli, Pietro De Poi, Lawrence Ein, Robert Lazarsfeld, and Brooke Ullery. Measures of irrationality for hypersurfaces of large degree. *Compositio Mathematica*, 153(11):2368–2393, September 2017. ISSN 1570-5846.

[5] David Eisenbud, Mark Green, and Joe Harris. Cayley-bacharach theorems and conjectures. *Bulletin of the American Mathematical Society*, 33(3):295–324, July 1996.

[6] Jake Levinson and Brooke Ullery. A cayley-bacharach theorem and plane configurations. *Proceedings of the American Mathematical Society*, 150:4603–4618, 02 2022.

[7] James Oxley. *Matroid Theory*. Oxford Graduate Texts in Mathematics. Oxford University Press, 2nd edition, 2011. ISBN 9780199603398.

[8] Nicola Picoco. Geometry of points satisfying cayley–bacharach conditions and applications. *Journal of Algebra*, 631:332–354, September 2023. ISSN 0021-8693.

[9] David Stapleton and Brooke Ullery. The degree of irrationality of hypersurfaces in various fano varieties. *Manuscripta Math.*, 161:377–408, 03 2020.