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Elliptic Curves, eta-quotients, and Weierstrass mock modular forms

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B.A., Scripps College, 2012

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An abstract of

A dissertation submitted to the Faculty of the

James T. Laney School of Graduate Studies of Emory University

in partial fulfillment of the requirements for the degree of

Doctor of Philosophy

in Mathematics

2016

Abstract

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By Amanda Susan Clemm

The relationship between elliptic curves and modular forms informs many modern mathematical discussions, including the solution of Fermat's Last Theorem and the Birch and Swinnerton-Dyer Conjecture. In this thesis we explore properties of elliptic curves, a particular family of modular forms called eta-quotients and the relationships between them. We begin by discussing elliptic curves, specifically considering the question of which quadratic fields have elliptic curves with everywhere good reduction. By revisiting work of Setzer, we expand on congruence conditions that determine the real and imaginary quadratic fields with elliptic curves of everywhere good reduction and rational j-invariant. Using this, we determine the density of such real and imaginary fields. In the next chapter, we begin investigating the properties of eta-quotients and use this theory to prove a conjecture of Han related to the vanishing of coefficients of certain combinatorial functions. We prove the original conjecture that relates the vanishing of the hook lengths of partitions and the number of 3-core partitions to the coefficients of a third series by proving a general theorem about this phenomenon. Lastly, we will see how these eta-quotients relate to the Weierstrass mock modular forms associated with certain elliptic curves. Alfes, Griffin, Ono, and Rolen have shown that the harmonic Maass forms arising from Weierstrass ζ-functions associated to modular elliptic curves "encode" the vanishing and nonvanishing for central values and derivatives of twisted Hasse-Weil L-functions for elliptic curves. We construct a canonical harmonic Maass form for the five curves proven by Martin and Ono to have weight 2 newforms with complex multiplication that are eta-quotients. The holomorphic part of this harmonic Maass form is referred to as the Weierstrass mock modular form. We prove that the derivative of the Weierstrass mock modular form for these five curves is itself an eta-quotient or a twist of one.

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Acknowledgments

First, I would like to thank my adviser Ken Ono for his continuous support of my research. This dissertation would not be possible without his guidance, and immense knowledge.

I would like to thank the rest of my committee: David Zureick-Brown and John Duncan, for their insightful comments and encouragement. I would also like to thank Andrew Sutherland for his support throughout my graduate career (from encouraging me to apply to Emory to helping me study on a train through Korea).

I also thank my fellow graduate students, especially the Partition Partition and my co-author Sarah Trebat-Leder.

Last but not least, I would like to thank my family and friends: my parents and my brother for supporting me throughout this entire process, everyone in Front House and Back House, my first Atlanta family, and my puppies, Maisie and Scully.

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Chapter 1

Introduction

The relationship between elliptic curves and modular forms informs many modern mathematical discussions, including the solution of Fermat's Last Theorem and the Birch and Swinnerton-Dyer Conjecture. In this thesis we explore properties of elliptic curves, a particular family of modular forms called eta-quotients, and the relationships between them. We begin by discussing elliptic curves, specifically considering the question of which quadratic fields have elliptic curves with everywhere good reduction. By revisiting work of Setzer, we expand on congruence conditions that determine the real and imaginary quadratic fields over which there exists elliptic curves of everywhere good reduction and rational j-invariant. Using this, we determine the density of such real and imaginary fields. In the next chapter, we begin investigating the properties of eta-quotients and use this theory to prove a conjecture of Han related to the vanishing of coefficients of certain combinatorial functions. We prove the original conjecture that relates the vanishing of the hook lengths of partitions and the number of 3-core partitions to the coefficients of a third series by proving a general theorem about this phenomenon. Lastly, we will see how these eta-quotients relate to the Weierstrass mock modular forms associated with certain elliptic curves. Alfes, Griffin, Ono, and Rolen have shown that the harmonic Maass forms arising from

Weierstrass ζ-functions associated to modular elliptic curves "encode" the vanishing and nonvanishing for central values and derivatives of twisted Hasse-Weil L-functions for elliptic curves. We construct a canonical harmonic Maass form for the five isogeny classes of elliptic curves proven by Martin and Ono to have weight 2 newforms with complex multiplication that are eta-quotients. The holomorphic part of this harmonic Maass form is referred to as the Weierstrass mock modular form. We prove that the derivative of the Weierstrass mock modular form for these five curves is itself an eta-quotient or a twist of one.

1.1 Elliptic curves with everywhere good reduction

(Joint work with S. Trebat-Leder)

An elliptic curve E over a field K is a smooth projective curve of genus 1 (defined over K) with a distinguished (K-rational) point.

Up to isomorphism, every elliptic curve over K can be described using the *general Weierstrass equation*:

$$E : y^2 + a_1xy + a_3y = x^3 + a_2x^2 + a_4x + a_6. \tag{1.1}$$

This defines a model for a smooth (non-singular) projective genus 1 curve over K with the rational point $(0 : 1 : 0)$. We can take this model, reduce its coefficients modulo some prime and obtain another (possibly singular) curve $\tilde{E}$. If $\tilde{E}$ is non-singular, we say E has *good reduction at* p. We say that an elliptic curve E/K has *everywhere good reduction*, or $\mathrm{EGR}(K)$, if E has good reduction at every prime. We say that an elliptic curve E/K has $\mathrm{EGR}_{\mathbb{Q}}(K)$ if it additionally has $\mathbb{Q}-$rational j-invariant. Similarly, we say a number field has EGR if there exists a $\mathrm{EGR}(K)$ elliptic curve and a number field has $\mathrm{EGR}_{\mathbb{Q}}$ if there exists a $\mathrm{EGR}_{\mathbb{Q}}(K)$ elliptic curve.

It is a well-known result that over $\mathbb{Q}$ there are no elliptic curves E with everywhere good reduction. However, the same is not true over general number fields. For example, if $K = \mathbb{Q}(\sqrt{29})$ where $a = \frac{5+\sqrt{29}}{2}$, the elliptic curve

$$E : y^2 + xy + a^2 y = x^3$$

has everywhere good reduction over K. This leads to the natural question: over which number fields do there exist elliptic curves with everywhere good reduction?

This question has often been approached by studying E/K with everywhere good reduction which satisfy additional properties, such as those which have a K-rational torsion point or admit a global minimal model. For many real and imaginary quadratic fields K of small discriminant, explicit examples of elliptic curves E/K with everywhere good reduction can be found in the literature, such as [Kid99] and [Ish86]. There are also many known examples of such fields for which there do not exist any elliptic curves E/K with everywhere good reduction; see [Kid99], [KK97], [Kag00] for example. In Chapter 3 we will be considering the following question: if there are infinitely many real (respectively imaginary) quadratic fields K with EGR, does there exist a positive proportion of K with EGR? We obtain our result by revisiting a consequence of Setzer that shows there are infinitely many quadratic fields with $\mathrm{EGR}_{\mathbb{Q}}$. Letting $R(X)$ be the number of real quadratic number fields K with discriminant at most X for which there exists an elliptic curve E/K with $\mathrm{EGR}_{\mathbb{Q}}$, we prove the following.

Theorem 1.1.1. *With $R(X)$ as above, we have that*

$$R(X) \gg \frac{X}{\sqrt{\log(X)}}.$$

If $I(X)$ is the number of imaginary quadratic number fields K with absolute discriminant at most X for which there exists an elliptic curve E/K with $\mathrm{EGR}_{\mathbb{Q}}$, we

also obtain the result below.

Theorem 1.1.2. *With $I(X)$ as above, we have that*

$$I(X) \gg \frac{X}{\sqrt{\log(X)}}.$$

To prove Theorem 1.1.1 and Theorem 1.1.2, we first show that all real (resp. imaginary) quadratic fields of a certain form have $\mathrm{EGR}_{\mathbb{Q}}$, and then count these fields. Using this approach we were also able to determine nonexistence of $\mathrm{EGR}_{\mathbb{Q}}$ quadratic fields.

Theorem 1.1.3. *Let $p \equiv 3 \pmod 8$ be prime.*

1. *Let $K = \mathbb{Q}(\sqrt{p})$. Then there are no E/K with $EGR_{\mathbb{Q}}$.*

2. *let $K = \mathbb{Q}(\sqrt{-p})$. Then there are no E/K with $EGR_{\mathbb{Q}}$.*

Remark. While we have only looked at curves with rational j-invariant, Noam Elkies' computations [Elk] suggest that very few E/K with EGR have $j(E) \notin \mathbb{Q}$ and unit discriminant. Therefore Theorem 1.1.3, which to the best of our knowledge has not previously appeared in the literature, suggests that most fields of the form $K = \mathbb{Q}(\sqrt{\pm p})$ for primes $p \equiv 3 \pmod 8$ are not EGR. This is consistent with Elkies' data.

Remark. In [Kag00], Kagawa showed that if p is a prime number such that $p \equiv 3(4)$ and $p \neq 3, 11$, then there are no elliptic curves with EGR over $K = \mathbb{Q}(\sqrt{3p})$ whose discriminant is a cube in K. Since all $\mathrm{EGR}(K)$ curves have cubic discriminant as shown in Setzer [Set81], this gives a result similar to Theorem 1.1.3.

1.2 3-cores and modular forms

In their study of supersymmetric gauge theory, Nekrasov and Okounkov discovered a striking infinite product identity [NO06]. This surprising theorem relates the sum over products of partition hook lengths to the powers of Euler products and has been

generalized in many ways to give expressions for many infinite product q-series. The original identity is given by

$$F_z(x) := \sum_{\lambda} x^{|\lambda|} \prod_{h \in \mathcal{H}(\lambda)} \left(1 - \frac{z}{h^2}\right) = \prod_{n=1}^{\infty} (1 - x^n)^{z-1},$$

where the sum is over integer partitions, $|\lambda|$ is the integer partitioned by λ, and $\mathcal{H}(\lambda)$ denotes the multiset of classical hook lengths associated to a partition λ.

In Chapter 4, we will describe other specializations of the Nekrasov-Okounkov formula. The connection to eta-quotients arises from the work of Han and Ono in [HO11]. In earlier work [Han10], Han conjectured a relation between numbers $a(n)$ that are given in terms of hook lengths of partitions, with numbers $b(n)$ from the generating function for the t-core partitions of n. This generating function is given by the following formula,

$$C_t(x) = \sum_{n=0}^{\infty} b(n) x^n := \prod_{n=1}^{\infty} \frac{(1 - x^{tn})^t}{1 - x^n}.$$

Specifically, in [Han09], Han conjectured, based on numerical evidence, that the non-zero coefficients of $F_9(x)$ and $C_3(x)$ are supported on the same terms. These coefficients have the following generating functions:

$$C_3(x) = \sum_{n=0}^{\infty} b(n) x^n := \prod_{n=1}^{\infty} \frac{(1 - x^{3n})^3}{1 - x^n},$$
$$F_9(x) = \sum_{n=0}^{\infty} a(n) x^n := \prod_{n=1}^{\infty} (1 - x^n)^8.$$

Assuming the notation above, Han conjectured $a(n) = 0$ if and only if $b(n) = 0$.

This conjecture is proved in a joint paper with Ono, [HO11]. Han and Ono normalized the two functions and showed both were eta-quotients. Using properties of these particular modular forms, they proved the original conjecture.

Recently Han discovered another series that appears to be supported on the same terms as $C_3(x)$ and $F_9(x)$. This series is given by,

$$C(x) = \sum_{n=0}^{\infty} c(n)x^n := \prod_{n=1}^{\infty}(1-x^n)^2(1-x^{3n})^2.$$

Based on numerical evidence, Han conjectured $a(n) = 0$ if and only if $b(n) = 0$ if and only if $c(n) = 0$.

Here we prove the following general theorem that produces infinitely many modular forms, including $F_9(x)$ and $C(x)$, that are supported precisely on the same terms as $C_3(x)$.

It is convenient to normalize $C_3(x)$ as shown below:

$$\mathcal{B}(z) = \frac{\eta(9z)^3}{\eta(3z)} = \sum_{n=1}^{\infty} b^*(n)q^n := \sum_{n=0}^{\infty} b(n)q^{3n+1}.$$

Theorem 1.2.1. *Suppose that* $f(z) = \sum_{n=1}^{\infty} A(n)q^n$ *is an even weight newform with trivial Nebentypus that has complex multiplication by* $\mathbb{Q}(\sqrt{-3})$ *and a level of the form* 3^s*, where* $s \geq 2$*. Then the coefficients* $A(n) = 0$ *if and only if* $b(n) = 0$*. More precisely,* $A(n) = b^*(n) = 0$ *for those non-negative integers* n *for which* $\mathrm{ord}_p(n)$ *is odd for some prime* $p \equiv 2 \pmod 3$*.*

Remark. Here we let $q := e^{2\pi iz}$ and $\sum_{n=1}^{\infty} A(n)q^n$ is the usual Fourier expansion at infinity.

Consider the normalized functions of $F_9(x)$ and $C(x)$ given by:

$$\mathcal{A}(z) = \sum_{n=1}^{\infty} a^*(n)q^n := \sum_{n=0}^{\infty} a(n)q^{3n+1},$$
$$\mathcal{C}(z) = \sum_{n=1}^{\infty} c^*(n)q^n := \sum_{n=0}^{\infty} c(n)q^{3n+1}$$

Remark. Theorem 1.2.1 implies the original work of Han and Ono. As explained

in [HO11], $\mathcal{A}(z)$ is a weight 4 newform with complex multiplication by $\mathbb{Q}(\sqrt{-3})$ with level 9. Theorem 1.2.1 also implies Han's new conjecture because $\mathcal{C}(z)$ is the weight 2 complex multiplication form for the elliptic curve with complex multiplication by $\mathbb{Q}(\sqrt{-3})$ given by $y^2 + y = x^3 - 7$ with level $3^3 = 27$ [MO97].

Remark. It turns out that more is true about the relationship between the two series $\mathcal{A}(z)$ and $\mathcal{C}(z)$. If $p \equiv 1 \pmod 3$ is prime, then we have that $c^*(p)$ divides $a^*(p)$.

To prove Theorem 1.2.1, we make use of the known description of $C_3(x)$, the generating function for the 3-core partition function, and then generalize the work in [HO11] regarding $F_9(x)$ to extend to this situation.

1.3 Weierstrass mock modular forms and eta-quotients

In the 1980s, Waldspurger [Wal81], and Kohnen and Zagier [Koh82, Koh85, KZ81] used the Shimura correspondence to relate the square roots of central values of quadratic twists of modular L-functions to certain coefficients of 1/2-integral weight cusp forms. When the weight of these 1/2-integral weight cusp forms is 3/2, Gross, Zagier and Kohnen [GKZ87, GZ86] utilized these results for their work on the Birch and Swinnerton-Dyer conjecture.

Ono and Bruinier [BO10b], generalized a theorem of Walspurger and Kohnen to relate weight 1/2 harmonic Maass forms to the vanishing and non-vanishing of $L(E_D, 1)$ and $L'(E_D, 1)$ for quadratic twists E_D of all modular elliptic curves. However, these harmonic Maass forms are very difficult to compete. There are infinitely many weight 3/2 modular forms that map onto the weight 2 newform via the Shimura correspondence and these harmonic Maass forms are preimages under $\xi_{1/2}$ of the weight 3/2 modular forms. Additionally, these harmonic Maass forms were originally constructed using the theory of Poincare series.

Instead of attempting to construct these harmonic Maass forms via preimages of

certain weight 3/2 forms, in a recent paper, Alfes, Griffin, Ono, and Rolen [AGOR14] obtain canonical weight 0 harmonic Maass forms that arise from Eisenstein's corrected Weierstrass zeta-function for elliptic curves over $\mathbb{Q}$. Guerzhoy [Gue15] had previously studied the construction of harmonic Maass forms using the Weierstrass zeta-function in his work on the Kaneko-Zagier hypergeometric differential equation. This canonical harmonic Maass form encodes the central L-values and L-derivatives that occur in the Birch and Swinnerton-Dyer Conjecture for elliptic curves in a family of quadratic twists [AGOR14], [BOR09]. For more information on these mock modular forms, see [BO06,BO10a].

After the proof of Fermat's Last Theorem and the subsequent expository articles describing the modularity thoerem, Martin and Ono wrote an article compiling the complete list of all weight 2 newforms that are eta-quotients. In [MO97], Martin and Ono prove that there are exactly twelve weight 2 newforms $F_E(\tau)$ that are products and quotients of functions of the form $\eta(\delta\tau)$ where $\eta(\tau)$ is the Dedekind eta function

$$\eta(\tau) := q^{1/24}\prod_{n=1}^{\infty}(1-q^n),$$

where $q := e^{2\pi i\tau}$. By the modularity of elliptic curves, there is an isogeny class of $E/\mathbb{Q}$ for each of these eta-quotients. Martin and Ono present a table of elliptic curves E corresponding to these cusp forms and describe the Grössencharacters for the five curves with complex multiplication.

Let E be one of the five elliptic curves with complex multiplication whose associated newform, $F_E(\tau)$, is an eta-quotient. Let N_E denote the conductor of this curve and label its coefficients a_i such that they belong to the Weierstrass model

$$E : y^2 + a_1xy + a_3y = x^3 + a_2x^2 + a_4x + a_6.$$

The following table contains a strong Weil curve for each of the weight 2 newforms

with complex multiplication that are eta-quotients.

N_E	$F_E(\tau)$	a_1	a_2	a_3	a_4	a_6
27	$\eta^2(3\tau)\eta^2(9\tau)$	0	0	1	0	-7
32	$\eta^2(4\tau)\eta^2(8\tau)$	0	0	0	4	0
36	$\eta^4(6\tau)$	0	0	0	0	1
64	$\dfrac{\eta^8(8\tau)}{\eta^2(4\tau)\eta^2(16\tau)}$	0	0	0	-4	0
144	$\dfrac{\eta^{12}(12\tau)}{\eta^4(6\tau)\eta^4(24\tau)}$	0	0	0	0	-1

Table 1.1: Table of five elliptic curves

The holomorphic part of the cannonical harmonic Maass form is a mock modular form, referred to as the *Weierstrass mock modular form*. Let $\widehat{\mathfrak{Z}}_E^+(\tau)$ denote the Weierstrass mock modular form of E, and let $Z_{N_E}(\tau) := q \cdot \frac{d}{dq}\widehat{\mathfrak{Z}}_E^+(\tau)$ denote the derivative of the Weierstrass mock modular form. Let $\chi_D := \left(\frac{D}{\cdot}\right)$ denote the usual Kronecker symbol so that $\left(\sum a(n)q^n\right) |_{\chi_D} = \sum \chi_D(n)a(n)q^n$.

Theorem 1.3.1. *The derivative of the Weierstrass mock modular form for each of the five elliptic curves E given in Table 1.1 is an eta-quotient or a twist of one, as described below.*

$$
\begin{aligned}
Z_{27}(\tau) &= -\eta(3\tau)\eta^6(9\tau)\eta^{-3}(27\tau),\\
Z_{32}(\tau) &= -\eta^2(4\tau)\eta^6(16\tau)\eta^{-4}(32\tau),\\
Z_{36}(\tau) &= -\eta^3(6\tau)\eta(12\tau)\eta^3(18\tau)\eta^{-3}(36\tau),\\
Z_{64}(\tau) &= -\eta^2(4\tau)\eta^6(16\tau)\eta^{-4}(32\tau) \mid_{\chi_8},\\
Z_{144}(\tau) &= -\eta^3(6\tau)\eta(12\tau)\eta^3(18\tau)\eta^{-3}(36\tau) \mid_{\chi_{12}}.
\end{aligned}
$$

We also obtain p-adic formulas for the corresponding weight 2 newform using Atkin's U-operator,

$$\sum a(n)q^n \mid U(m) := \sum a(mn)q^n.$$

By taking a p-adic limit, we can retrieve the coefficients of the original cusp form, $F_E(\tau)$, of the elliptic curve. Let $Z_{N_E}(\tau) = \sum_{n=-1}^{\infty} d(n)q^n$ be the derivative of the Weierstrass mock modular form as before.

Theorem 1.3.2. *For each of the five elliptic curves listed in Table 1.1, if p is inert in the field of complex multiplication, then as a p-adic limit we have*

$$F_E(\tau) = \lim_{\omega \to \infty} \frac{Z_{N_E}(\tau) \mid U(p^{2\omega+1})}{d(p^{2\omega+1})}.$$

We prove this theorem using techniques outlined in [GKO10]. Similar results can be found in both [EGO10] and [AGOR14].

Chapter 2

Background

2.1 Good Reduction

An *elliptic curve* E over a field K is a smooth projective curve of genus 1 (defined over K) with a distinguished (K-rational) point.

Up to isomorphism, every elliptic curve over K can be described using the *general Weierstrass equation*:

$$E : y^2 + a_1xy + a_3y = x^3 + a_2x^2 + a_4x + a_6. \tag{2.1}$$

This defines a smooth projective genus 1 curve over K with the rational point $(0 : 1 : 0)$. This rational point is the point at infinity. If the characteristic of K is not 2 or 3, using a change of variables we can reduce the general Weierstrass equation to the following:

$$E : y^2 = x^x + Ax + B. \tag{2.2}$$

Assuming the characteristic of K is not 2 or 3, we can define the *discriminant* of an elliptic curve as $\Delta = -16(4A^3 + 27B^2)$.

A curve given by a Weierstrass equation is nonsingular if and only if $\Delta \neq 0$. Dif-

ferent models of the same elliptic curve result in different values for the discriminant.

If K is a local field, complete with a discrete valuation v and uniformizer π, we define reduction modulo π by reducing coefficients modulo π to obtain a (possibly singular) curve $\tilde{E}$. We say E has *good reduction* if $\tilde{E}$ is nonsingular. If there exists a model such that $v(\Delta) = 0$ then E has good reduction at v. If the class number of K is 1, we can define the minimal Weierstrass equation, the equation such that $v(\Delta)$ is minimized subject to the condition that the coefficients are still in $\mathcal{O}_K$. We say an equation for E is a *global minimal model* if and only if the equation is minimal with respect to all discrete valuations of K.

If E is given by the minimal Weierstrass equation, then E has *good reduction at* v if and only if $v(\Delta) = 0$. If E is an elliptic curve over a number field K, then E has good reduction at v if E has good reduction when considered over the completion K_v. If E has good reduction at every valuation v, then we say E has *everywhere good reduction* or EGR. Therefore, if the discriminant of an elliptic curve is a unit when considered over a number field K, it is possible for that curve to have good reduction everywhere. One way to prove there are no elliptic curves over $\mathbb{Q}$ with everywhere good reduction is to look at the congruence conditions related to curves with discriminant $\Delta = \pm 1$. However, for $K = \mathbb{Q}(\sqrt{29})$, $a = \frac{5+\sqrt{29}}{2}$, and $E : y^2 + xy + a^2y = x^3$, we see Δ is a unit (i.e. it has norm 1) and so E has everywhere good reduction over K.

2.2 Harmonic Maass forms

Let E be an elliptic curve over $\mathbb{Q}$ such that $E \simeq \mathbb{C}/\Lambda_E$, where Λ_E is a two-dimensional lattice in $\mathbb{C}$. By the modularity of elliptic curves over $\mathbb{Q}$, we have the modular parameterization

$$\phi_E : X_0(N_E) \to \mathbb{C}/\Lambda_E \simeq E,$$

where N_E is the conductor of E. Suppose E is a strong Weil curve and let

$$F_E(z) = \sum_{n=1}^{\infty} a_E(n) q^n \in S_2(\Gamma_0(N_E))$$

be the associated newform where $q = e^{2\pi i z}$.

Let $\wp(\Lambda_E; \mathfrak{z})$ be the usual Weierstrass $\wp$-function given by

$$\wp(\Lambda_E; \mathfrak{z}) := \frac{1}{\mathfrak{z}^2} + \sum_{\omega \in \Lambda_E \setminus \{0\}} \left(\frac{1}{(\mathfrak{z} - \omega)^2} - \frac{1}{\omega^2} \right).$$

All elliptic functions with respect to Λ_E are naturally generated from the Weierstrass $\wp$-functions. While there can never be a single-order elliptic function, Eisenstein constructed a simple function with a single pole that can be modified, at the expense of holomorphicity, to become lattice-invariant (see [Wei99]). Eisenstein began with the Weierstrass zeta-function, the holomorphic function defined for $\mathfrak{z} \notin \Lambda_E$ by

$$\zeta(\Lambda_E; \mathfrak{z}) := \frac{1}{\mathfrak{z}} + \sum_{\omega \in \Lambda_E \setminus \{0\}} \left(\frac{1}{\mathfrak{z} - \omega} + \frac{1}{\omega} + \frac{\mathfrak{z}}{\omega^2} \right) = \frac{1}{\mathfrak{z}} - \sum_{\omega \in \Lambda_E \setminus \{0\}} G_{2n+2}(\Lambda_E) \mathfrak{z}^{2n+1}.$$

The derivative of this function is $-\wp(\Lambda_E; \mathfrak{z})$. Eisenstein's corrected zeta-function is given by

$$\mathfrak{Z}_E(\mathfrak{z}) := \zeta(\Lambda_E; \mathfrak{z}) - S(\Lambda_E)\mathfrak{z} - \frac{\deg(\phi_{\mathrm{E}})}{4\pi ||F_E||^2} \bar{\mathfrak{z}},$$

where $S(\Lambda_E) := \lim_{s \to 0^+} \sum_{0 \neq \omega \in \Lambda_E} \frac{1}{\omega^2 |\omega|^{2s}}$, $\deg(\phi_{\mathrm{E}})$ is the degree of the modular parameterization and $||F_E||$ is the Petersson norm of F_E. In [Rol16], Rolen provides a new, direct proof of the lattice-invariance of $\mathfrak{Z}_E(\mathfrak{z})$ using the standard theory of differential operators for Jacobi forms.

A *harmonic weak Maass form of weight* $k \in \frac{1}{2}\mathbb{Z}$ *on* $\Gamma_0(N)$ (see [AGOR14]) is a smooth function on $\mathbb{H}$ which satisfies:

1. $f|_k \gamma = f$ for all $\gamma \in \Gamma_0(N)$;

2. $\Delta_k f = 0$, where Δ_k is weight k hyperbolic Laplacian on $\mathbb{H}$;

3. There is a polynomial $P_f \in \mathbb{C}[q^{-1}]$ such that

$$f(z) - P_f(z) = O(e^{-\epsilon y}),$$

as $v \to \infty$ for some $\epsilon > 0$. Similar conditions must hold at all cusps.

The canonical harmonic Maass form arises from the corrected Weierstrass zeta-function. Define $\mathfrak{Z}_E^+(\mathfrak{z}) := \zeta(\Lambda_E; \mathfrak{z}) - S(\Lambda_E)\mathfrak{z}$. Let $\mathcal{E}_E(z)$ be its Eichler integral defined

$$\mathcal{E}_E(z) := -2\pi i \int_z^{i\infty} F_E(\tau) d\tau = \sum_{n=1}^{\infty} \frac{a_E(n)}{n} q^n.$$

The nonholomorphic function $\widehat{\mathfrak{Z}}_E(z)$ is given by

$$\widehat{\mathfrak{Z}}_E(z) = \widehat{\mathfrak{Z}}_E^+(z) + \widehat{\mathfrak{Z}}_E^-(z) = \mathfrak{Z}_E(\mathcal{E}(z)).$$

Alfes, Griffin, Ono, and Rolen proved the following.

Theorem 2.2.1 (Theorem 1.1 of [AGOR14]). *Assume the notation and hypotheses above. Then the following are true:*

1. *The poles of $\widehat{\mathfrak{Z}}_E^+(z)$ are precisely those points z for which $\mathcal{E}_E(z) \in \Lambda_E$.*

2. *If $\widehat{\mathfrak{Z}}_E^+(z)$ has poles in $\mathbb{H}$, then there is a canonical modular form $M_E(z)$ with algebraic coefficients on $\Gamma_0(N_E)$ for which $\widehat{\mathfrak{Z}}_E^+(z) - M_E(z)$ is holomorphic on $\mathbb{H}$.*

3. *We have that $\widehat{\mathfrak{Z}}_E(z) - M_E(z)$ is a weight 0 harmonic Maass form on $\Gamma_0(N_E)$.*

In particular, the holomorphic part of $\widehat{\mathfrak{Z}}_E(z)$ is $\widehat{\mathfrak{Z}}_E^+(z) = \mathfrak{Z}_E^+(\mathcal{E}_E(z))$, where $\widehat{\mathfrak{Z}}_E^+(z)$ is a weight 0 mock modular form known as the *Weierstrass mock modular form* for E.

2.3 Eta-Quotients

One important example of a modular form is the Dedekind eta function, denoted $\eta(z)$. This function is defined by the infinite product

$$\eta(z) := q^{1/24} \prod_{n=1}^{\infty} (1 - q^n).$$

Using Jacobi's Triple Product Identity, we can deduce Dedekind's eta function is a modular form of weight $1/2$, more precisely,

Theorem 2.3.1 (Theorem 1.61 of [Ono04]). *For $\tau \in \mathbb{H}$, we have*

$$\eta(\tau + 1) = e^{\pi i/12} \eta(\tau),$$

and

$$\eta(-1/\tau + 1) = (-i\tau)^{1/2} \eta(\tau).$$

An *eta-quotient* is any function $f(\tau)$ of the form

$$f(\tau) = \prod_{\delta | N} \eta(\delta\tau)^{r_\delta},$$

where $N \geq 1$ and each r_δ is an integer. If each $r_\delta \geq 0$ then $f(\tau)$ is an eta-product.

In [Ono04], Ono described the following result of Gordon, Hughes, and Newman on eta-quotients.

Theorem 2.3.2 (Theorem 1.64 of [Ono04]). *If $f(\tau) = \prod_{\delta | N} \eta(\delta\tau)^{r_\delta}$ is an eta-quotient with $k = \frac{1}{2} \sum_{\delta | N} r_\delta \in \mathbb{Z}$, with the additional properties that*

$$\sum_{\delta | N} \delta r_\delta \equiv 0 \pmod{24}$$

and

$$\sum_{\delta|N} \frac{N}{\delta} r_\delta \equiv 0 \pmod{24},$$

then $f(\tau)$ *satisfies*

$$f\left(\frac{az+b}{cz+d}\right) = \chi(d)(cz+d)^k f(z).$$

Here the character χ *is defined for* $\chi(d) := \left(\frac{(-1)^k s}{d}\right)$, *where* $s = \prod_{\delta|N} \delta^{r_\delta}$.

The following formula can be used to determine the order of vanishing of an eta-quotient at each cusp c/d.

Theorem 2.3.3 (Theorem 1.65 of [Ono04])**.** *Let* c, d *and* N *be positive integers with* $d|N$ *and* $gcd(c,d) = 1$. *If* $f(z)$ *is an eta-quotient satisfying the conditions of Theorem 1.64 for* N, *then the order of vanishing of* $f(z)$ *at the cusp* $\frac{c}{d}$ *is*

$$\frac{N}{24} \sum_{\delta|N} \frac{gcd(d,\delta)^2 r_\delta}{gcd(d, \frac{N}{d}) d\delta}.$$

2.4 Newforms

2.4.1 Newforms with complex multiplication

We now briefly recall the theory of newforms with complex multiplication (see Chapter 12 of [Iwa97] or Section 1.2 of [Ono04]). Let $D < 0$ be the fundamental discriminant of an imaginary quadratic field $K = \mathbb{Q}(\sqrt{D})$. Let $\mathcal{O}_K$ be the ring of integers of K and $\chi_K := \left(\frac{D}{\bullet}\right)$ be the usual Kronecker character associated to K. Let Λ be a nontrivial ideal in $\mathcal{O}_K$ and $I(\Lambda)$ denote the group of fractional ideals prime to Λ. Then ϕ defines a homomorphism

$$\phi : I(\Lambda) \to \mathbb{C}^\times$$

such that for each $\alpha \in K^\times$ with $\alpha \equiv 1 \pmod{\Lambda}$, we have

$$\phi(\alpha \mathcal{O}_K) = \alpha^{k-1}.$$

Let ω_ϕ be the Dirichlet character defined as

$$\omega_\phi(n) := \phi((n))/n^{k-1}$$

for every integer n coprime to Λ. The cusp form $\Psi(z)$ is defined as

$$\Psi(z) := \sum_{\mathfrak{a}} \phi(\mathfrak{a}) q^{N(\mathfrak{a})},$$

where the sum is over the integral ideals $\mathfrak{a}$ that are prime to Λ and $N(\mathfrak{a})$ is a norm of the ideal $\mathfrak{a}$. This cusp form is a "newform" in the sense of Atkin and Lehner. The Atkin-Lehner theory of newforms for modular forms with trivial Nebentypus categorizes the relationship between spaces of modular forms of weight k on different congruence subgroups [AL70].

2.4.2 Weight 2 newforms

In [MO97], Martin and Ono compile the complete list of all weight 2 newforms that are eta-quotients along with their strong Weil curves. Five of these curves have complex multiplication, and using q-series infinite product identities, they described the Grössencharacters for these curves. The curves with conductors 27, 36, and 144 have complex multiplication by $\mathbb{Q}(\sqrt{-3})$ and the curves with conductors 32 and 64 have complex multiplication by $\mathbb{Q}(i)$. In addition, Martin and Ono in [MO97] proved the curves with $N = 36$ and $N = 144$ are quadratic twists of each other. Let E be one of the 12 elliptic curves whose associated newform, $F_E(\tau)$, is an eta-quotient. Let N_E denote the conductor of this curve and label its coefficients a_i such that they belong

to the Weierstrass model

$$E : y^2 + a_1xy + a_3y = x^3 + a_2x^2 + a_4x + a_6.$$

The following table contains a strong Weil curve for each of the weight 2 newforms that are eta-quotients.

N_E	$F_E(\tau)$	a_1	a_2	a_3	a_4	a_6
11	$\eta^2(\tau)\eta^2(11\tau)$	0	-1	1	-10	-20
14	$\eta(\tau)\eta(2\tau)\eta(7\tau)\eta(14\tau)$	1	0	1	4	-6
15	$\eta(\tau)\eta(3\tau)\eta(5\tau)\eta(15\tau)$	1	1	1	-10	-10
20	$\eta^2(2\tau)\eta^2(10\tau)$	0	1	0	4	4
24	$\eta(2\tau)\eta(4\tau)\eta(6\tau)\eta(12\tau)$	0	-1	0	-4	4
27	$\eta^2(3\tau)\eta^2(9\tau)$	0	0	1	0	-7
32	$\eta^2(4\tau)\eta^2(8\tau)$	0	0	0	4	0
36	$\eta^4(6\tau)$	0	0	0	0	1
48	$\dfrac{\eta^4(4\tau)\eta^4(12\tau)}{\eta(2\tau)\eta(6\tau)\eta(8\tau)\eta(24\tau)}$	0	1	0	-4	-4
64	$\dfrac{\eta^8(8\tau)}{\eta^2(4\tau)\eta^2(16\tau)}$	0	0	0	-4	0
80	$\dfrac{\eta^6(4\tau)\eta^6(20\tau)}{\eta^2(2\tau)\eta^2(8\tau)\eta^2(10\tau)\eta^2(40\tau)}$	0	1	0	-4	-4
144	$\dfrac{\eta^{12}(12\tau)}{\eta^4(6\tau)\eta^4(24\tau)}$	0	0	0	0	-1

Table 2.1: Table of five elliptic curves

Chapter 3

Elliptic curves with everywhere good reduction

3.1 Introduction

In this first chapter, we investigate the following question: over which number fields do there exist elliptic curves with everywhere good reduction?

For many real and imaginary quadratic fields K of small discriminant, explicit examples of elliptic curves E/K with everywhere good reduction can be found in the literature, such as [Kid99] and [Ish86]. There are also many known examples of such fields for which there do not exist any elliptic curves E/K with everywhere good reduction; see [Kid99], [KK97], [Kag00] for example.

This question has often been approached by studying E/K with everywhere good reduction which satisfy additional properties (as defined in Chapter 2), such as those which have a K-rational torsion point, admit a global minimal model, or have rational j-invariant. We say E/K is *admissible* if it has everywhere good reduction and a K-rational point of order 2. A curve E/K is *g-admissible* if it is admissible and has a global minimal model.

For example, Kida [Kid99] showed that if K satisfies certain hypotheses, every E/K with EGR has a K-rational point of order two. This condition led to a series of non-existence results for particular real quadratic fields with small discriminant. In [Set81], Setzer classified elliptic curves with $\mathrm{EGR}_{\mathbb{Q}}$ over real quadratic number fields. Kida extended Setzer's approach by giving a more general method suitable for computing elliptic curves with EGR over certain real quadratic fields with rational or singular j-invariants in [Kid00]. Comalada [Com90] showed that there exists E/K with EGR, a global minimal model, and a K-rational point of order two if and only if one of his sets of diophantine equations has a solution. Ishii supplemented this theorem by studying $K-$rational 2 division points in [Ish86] to demonstrate specific real quadratic fields without EGR elliptic curves. Later Kida and Kagawa in [KK97] generalized Ishii's result to obtain non-existence results for $\mathbb{Q}(\sqrt{17})$, $\mathbb{Q}(\sqrt{73})$ and $\mathbb{Q}(\sqrt{97})$. Yu Zhao determined criteria for real quadratic fields to have elliptic curves with EGR and a non-trivial 3-division point. In [Zha13], he provides a table for all such fields with discriminant less than 10,000.

For imaginary quadratic fields, Stroeker [Str83] showed that no E/K with EGR admits a global minimal model. In [Set78], Setzer showed that there exist elliptic curves with EGR and a K-rational point of order two if and only if $K = \mathbb{Q}(\sqrt{-m})$ with m satisfying certain congruence conditions. Comalada and Nart provided criteria to determine when elliptic curves have EGR in [CN92]. Kida combined this result with a method of computing the Mordell-Weil group in [Kid01] to prove there are no elliptic curves with EGR over the fields $\mathbb{Q}(\sqrt{-35}), \mathbb{Q}(\sqrt{-37}), \mathbb{Q}(\sqrt{-51})$ and $\mathbb{Q}(\sqrt{-91})$. There are no elliptic curves with $\mathrm{EGR}_{\mathbb{Q}}(K)$ for $-37 < m < -1$. However, there are elliptic curves with small discriminant and $\mathrm{EGR}_{\mathbb{Q}}(K)$ for real quadratic fields K.

Cremona and Lingham [CL07] described an algorithm for finding all elliptic curves over any number field K with good reduction outside a given set of primes. However, this procedure relies on finding integral points on certain elliptic curves over K, which

can limit its practical implementation.

Table 3.1 shows what is known for $K = \mathbb{Q}(\sqrt{m})$ with square-free positive integers $m \leq 47$. We stop at 47 because to the best of our knowledge, the $m = 51$ case is still unknown.

Table 3.1: Real Quadratic Fields $\mathbb{Q}(\sqrt{m})$ with and without EGR

EGR	no EGR
6	2
7	3
14	5
22	10
26	11
29	13
33	15
37	17
38	19
41	21
	23
	30
	31
	34
	35
	39
	42
	43
	46
	47

The results listed above gives many methods to prove that a particular quadratic number field has an EGR elliptic curve. A method of Setzer regarding the classification of elliptic curves over both real and imaginary quadratic number fields with rational j-invariant, shows that there infinitely many quadratic fields which have an EGR elliptic curve.

Let $R(X)$ be the number of real quadratic number fields K with discriminant at most X and an $\mathrm{EGR}_{\mathbb{Q}}(K)$ elliptic curve. By revisiting the results of Setzer, we prove

the following.

Theorem 3.1.1. *With $R(X)$ as above, we have that*

$$R(X) \gg \frac{X}{\sqrt{\log(X)}}.$$

If $I(X)$ is the number of imaginary quadratic number fields K with $|\Delta_K| < X$ and an $\mathrm{EGR}_{\mathbb{Q}}(K)$ elliptic curve, we also obtain the result below.

Theorem 3.1.2. *With $I(X)$ as above, we have that*

$$I(X) \gg \frac{X}{\sqrt{\log(X)}}.$$

To prove Theorem 3.1.1, we first show that all real quadratic fields of the form described below in Theorem 3.1.3 have $\mathrm{EGR}_{\mathbb{Q}}$, and then count these fields.

Theorem 3.1.3. *Let $m = 2q$, where $q = q_1 \cdots q_n \equiv 3 \pmod 8$ with $q_j \equiv 1, 3 \pmod 8$ distinct primes. Then the real quadratic field $K = \mathbb{Q}(\sqrt{m})$ has $EGR_{\mathbb{Q}}$.*

Remark. If m is as described in Theorem 3.1.3, there exists E/K with $\mathrm{EGR}_{\mathbb{Q}}$ and $j(E) = 20^3$ as shown by Setzer in 3.2.1.

Similarly, to prove Theorem 3.1.2, we show all imaginary quadratic fields found below in Theorem 3.1.4 have $\mathrm{EGR}_{\mathbb{Q}}$.

Theorem 3.1.4. *Let $m = 37q$, where $q = -q_1 \cdots q_n \equiv 1 \pmod 8$ with q_j distinct primes such that $\left(\frac{q_j}{37}\right) = 1$. Then the imaginary quadratic field $K = \mathbb{Q}(\sqrt{m})$ has $EGR_{\mathbb{Q}}$.*

Remark. If m is as described in Theorem 3.1.4, there exists E/K with $\mathrm{EGR}_{\mathbb{Q}}$ and $j(E) = 16^3$ as shown by Setzer in 3.2.1.

We can achieve results like Theorem 3.1.3 and 3.1.4 for integers other than 2 and 37; these two cases are all is required to prove Theorem 3.1.1 and 3.1.2.

To obtain a density result for $m = qD$, where D is fixed and q varies, we define certain 'good' D. We say D is good if it is the square free part of $A^3 - 1728$, where A satisfies certain congruence conditions modulo powers of 2 and 3. Both $D = 2$ and $D = 37$ are examples of 'good' values of D. These congruence conditions will be described explicitly in Section 3.2. If D is good, then $K = \mathbb{Q}(\sqrt{Dq})$ has $\mathrm{EGR}_{\mathbb{Q}}$ whenever D and q satisfy certain explicit conditions, see Section 3.2. For any square-free D, define

$$\epsilon_D = \begin{cases} 1 & D \equiv 1 \pmod 4 \\ -1 & \text{otherwise} \end{cases}$$

When $\mathrm{sign}(D) = -\epsilon_D$, we get real quadratic fields $\mathbb{Q}(\sqrt{qD})$, and when $\mathrm{sign}(D) = \epsilon_D$, we get imaginary quadratic fields.

Using this, we show that $R_D(X)$, the number of $q \leq X$ such that $\mathbb{Q}(\sqrt{Dq})$ is a real $\mathrm{EGR}_{\mathbb{Q}}$ quadratic number field, satisfies the following lower bound:

Theorem 3.1.5. *Let D be good with r distinct prime factors and $R_D(X)$, the number of $EGR_{\mathbb{Q}}$ real quadratic number fields $\mathbb{Q}(\sqrt{Dq})$ with $q \leq X$. Assume that sign$(D) = -\epsilon_D$. Then*

$$R_D(X) \gg \frac{X}{\log^{1-1/2^r} X}.$$

We obtain a similar result to show that $I_D(X)$, the number of $\mathrm{EGR}_{\mathbb{Q}}$ imaginary quadratic number fields $\mathbb{Q}(\sqrt{Dq})$ satisfies the following lower bound.

Theorem 3.1.6. *Let D be good with r distinct prime factors and $I_D(X)$, the number of $EGR_{\mathbb{Q}}$ imaginary quadratic number fields $\mathbb{Q}(\sqrt{Dq})$ with $q \leq X$. Assume that sign$(D) = \epsilon_D$. Then*

$$I_D(X) \gg \frac{X}{\log^{1-1/2^r} X}.$$

Remark. While we have only looked at curves with rational j-invariant, Noam Elkies' computations [Elk] suggest that very few E/K with EGR have $j(E) \notin \mathbb{Q}$ and unit discriminant. Therefore, the theorem below, which to the best of our knowledge

has not previously appeared in the literature, suggests that most fields of the form $K = \mathbb{Q}(\sqrt{\pm p})$ for primes $p \equiv 3 \pmod 8$ are not EGR. This is consistent with Elkies' data.

Using this approach we were also able to determine nonexistence of $\mathrm{EGR}_{\mathbb{Q}}$ quadratic fields.

Theorem 3.1.7. *Let $p \equiv 3 \pmod 8$ be prime.*

1. *Let $K = \mathbb{Q}(\sqrt{p})$. Then there are no E/K with $EGR_{\mathbb{Q}}$.*

2. *let $K = \mathbb{Q}(\sqrt{-p})$. Then there are no E/K with $EGR_{\mathbb{Q}}$.*

Remark. In [Kag00], Kagawa showed that if p is a prime number such that $p \equiv 3(4)$ and $p \neq 3, 11$, then there are no elliptic curves with EGR over $K = \mathbb{Q}(\sqrt{3p})$ whose discriminant is a cube in K. Since all $\mathrm{EGR}(K)$ curves have cubic discriminant as shown in Setzer [Set81], this gives a result similar to Theorem 3.1.7.

In Section 3.2, we describe conditions arising from Setzer to define when we have $\mathrm{EGR}_{\mathbb{Q}}$ quadratic fields. In Section 3.3, we use these conditions to find a lower bound based on an example of Serre. In Section 3.4, we will give examples of $\mathrm{EGR}_{\mathbb{Q}}$ real quadratic fields and $\mathrm{EGR}_{\mathbb{Q}}$ imaginary quadratic fields.

3.2 Constructing $\mathrm{EGR}_{\mathbb{Q}}$ Quadratic Fields

In [Set81], given a rational j-invariant, Setzer determines whether there exists an elliptic curve and number field over which this curve has everywhere good reduction. Following his notation, we make the following definitions. Let $\mathcal{R}$ be the following set:

$$\mathcal{R} = \{A \in \mathbb{Z} : 2|A \Rightarrow 16|A \text{ or } 16|A-4, \text{ and } 3|A \Rightarrow 27|A-12\}.$$

Note that by the Chinese Remainder Theorem, $\mathcal{R}$ is then the union of the following congruence classes:

- $1, 5 \pmod 6$
- $4, 16, 20, 32 \pmod{48}$
- $39 \pmod{54}$
- $228, 336 \pmod{432}$

We say that D is good if it is in the following set:

$$\{D : Dt^2 = A^3 - 1728, D \text{ square-free}, A \in \mathcal{R}, t \in \mathbb{Z}\}.$$

For example, the good D with $|D| < 100$ are exactly

$$-91, -67, -43, -26, -19, -11, -7, 2, 7, 37, 65, 79.$$

Remark. We note that ± 1 are not good, as the elliptic curves $Y^2 = X^3 - 1728, -Y^2 = X^3 - 1728$ have no integral points with $Y \neq 0$.

By Setzer [Set81], the only candidates for elliptic curves E with $\mathrm{EGR}_{\mathbb{Q}}(K)$ over a quadratic field K have $j(E) = A^3$ with $A \in \mathcal{R}$.

Theorem 3.2.1 (See [Set81].). *Let $K = \mathbb{Q}(\sqrt{m})$ be a quadratic field with m square-free. Then there exists an elliptic curve E/K with $EGR_{\mathbb{Q}}$ if and only if the following conditions are satisfied for some good $D \mid \Delta_K$.*

1. *$\epsilon_D D$ is a rational norm from K.*
2. *If $D \equiv \pm 3 \pmod 8$, then $m \equiv 1 \pmod 4$.*
3. *If D is even then $m \equiv 4 + D \pmod{16}$.*

To prove the theorem, Setzer shows that given a pair (m, D) satisfying the conditions of the theorem, there exists $u \in K^\times$ such that

$$E_{u,A} : y^2 = x^3 - 3A(A^3 - 1728)u^2x - 2(A^3 - 1728)^2u^3$$

has j-invariant A^3 and $\mathrm{EGR}_{\mathbb{Q}}$ over K.

Remark. We correct a mistake in Condition (2) of this theorem as written in [Set81].

We note that if $u \equiv v \pmod{4\mathcal{O}_K}$ and $m \equiv 2, 3, \pmod 4$, then we must have that $N(u) \equiv N(v) \pmod 8$. However, if $m \equiv 1 \pmod 4$, we only know that $N(u) \equiv N(v) \pmod 4$. Moreover, we can pick $w \in 4\mathcal{O}_K$ such that $N(u + w) \equiv N(u) + 4 \pmod 8$.

Condition (2) as written in Setzer's paper states that if $D \equiv \pm 3 \pmod 8$, then $m \equiv 5 \pmod 8$. $D \equiv \pm 3 \pmod 8$ implies that a certain element $u \in \mathcal{O}_K$ has $N(u) \equiv 5 \pmod 8$. But for the curve to have good reduction at primes dividing 2, it is necessary that u is congruent to a square modulo $4\mathcal{O}_K$. For $m \equiv 2, 3 \pmod 4$ this is not possible, as no squares can have norm equivalent to 5 modulo 8. However, if $m \equiv 1 \pmod 4$, the condition that $N(u) \equiv 5 \pmod 8$ is not an obstacle, as u is congruent modulo $4\mathcal{O}_K$ to elements of norm 1 modulo 8. Setzer mistakenly assumes that this can only happen when $m \equiv 5 \pmod 8$.

In proving that fields do and do not have elliptic curves with $\mathrm{EGR}_{\mathbb{Q}}$, the following equivalent version of Setzer's theorem will be useful.

Theorem 3.2.2. *Fix D good, and $m = qD$ square-free. $K = \mathbb{Q}(\sqrt{m})$ has $EGR_{\mathbb{Q}}$ if and only if the following conditions are satisfied:*

1. $(-\epsilon_D q/p_i) = 1$ *for all odd primes p_i dividing D;*
2. $(\epsilon_D D/q_j) = 1$ *for all odd primes q_j dividing q;*
3. $m > 0$ *if* $\epsilon_D D < 0$;

4. *If* $D \equiv \pm 3 \pmod 8$ *then* $q \equiv D \pmod 4$*;*

5. *If* D *is even then* $q \equiv D + 1 \pmod 8$.

Proof of Theorem 3.2.2. We need to show that the conditions in Theorem 3.2.1 are equivalent to those in Theorem 3.2.2.

Assume that $K = \mathbb{Q}(\sqrt{m})$ where m is square-free.

Clearly if $m = qD$, D divides Δ_K. We need to show that if $D \mid \Delta_K$ then $D \mid m$. This is trivial for $m \equiv 1 \pmod 4$, as then $\Delta_K = m$. If $m \equiv 3 \pmod 4$, then D cannot be even because of condition (3) of Theorem 3.2.1, so $D \mid m$. If $m \equiv 2 \pmod 4$, then D must be square-free, so $D \mid m$.

Now, $\epsilon_D D$ is a rational norm from K if and only if there exists a rational solution to $\epsilon_D D = a^2 - b^2 Dq$. Since $D \mid a$, the above is equivalent to the existence of a rational solution to $\epsilon_D = D(a')^2 - b^2 q$, which is equivalent to the existence of a nontrivial integer solution to $\epsilon_D x^2 - Dy^2 + qz^2 = 0$. By Legendre's Theorem [IR90], this equation has a nontrivial integral solution if and only if the following hold:

1. $\epsilon_D, -D$, and q do not all have the same sign, which is equivalent to condition (3).

2. $\epsilon_D D$ is a square modulo $|q|$, which is equivalent to condition (2).

3. $-\epsilon_D q$ is a square modulo $|D|$, which is equivalent to condition (1).

4. $-Dq$ is a square modulo $|\epsilon_D|$, which is always the case.

Lastly, conditions (4) and (5) are directly equivalent to conditions in Theorem 3.2.1.

□

To prove Theorem 3.1.1, the lower bound for $R_D(X)$ and Theorem 3.1.2, the lower bound for $I_D(X)$, we require Theorem 3.1.3 (which considers the case $D = 2$) and Theorem 3.1.4 (which considers the case $D = 37$). Below, we prove both those theorems using the result above.

Proof of Theorem 3.1.3. Let $A = 20 \in \mathcal{R}$. This shows that $D = 2$ is good. For $m = 2q$ with $q = q_1 \cdots q_n \equiv 3 \pmod 8$ and $q_j \equiv 1, 3 \pmod 8$ distinct primes, all of the conditions in Theorem 3.2.2 are satisfied, and so $K = \mathbb{Q}(\sqrt{m})$ has $\mathrm{EGR}_{\mathbb{Q}}$. □

Proof of Theorem 3.1.4. Let $A = 16 \in \mathcal{R}$. This that shows that $D = 37$ is good. For $m = 37q$ with $q = -q_1 \cdots q_n \equiv 1 \pmod 8$ and q_j distinct primes such that $\left(\frac{q_j}{37}\right) = 1$, all of the conditions in Theorem 3.2.2 are satisfied, and so $K = \mathbb{Q}(\sqrt{m})$ has $\mathrm{EGR}_{\mathbb{Q}}$. □

We also can use Theorem 3.2.2 to prove nonexistence results about $\mathrm{EGR}_{\mathbb{Q}}$ quadratic fields.

Proof of Theorem 3.1.7. Let $p \equiv 3 \pmod 8$ be prime.

To show that there are no $E/\mathbb{Q}(\sqrt{p})$ with $\mathrm{EGR}_{\mathbb{Q}}$, we must show that neither of the pairs $(D, q) = (p, 1), (-p, -1)$ satisfy the conditions of Theorem 3.2.2. We note that since $p = D \equiv \pm 3 \pmod 8$, condition (d) implies that $q \equiv 5D \equiv \pm 1 \pmod 8$, which is a contradiction.

Similarly, to show that there are no $\mathrm{EGR}_{\mathbb{Q}}(\mathbb{Q}(\sqrt{-p})$, we have to show that neither of the pairs $(D, q) = (p, -1), (-p, 1)$ satisfy the conditions of the theorem. We note that in both cases, condition (a) implies that $\left(\frac{-1}{p}\right) = 1$, which is a contradiction. □

3.3 Finding Lower Bounds

To prove the lower bounds, we use an example of Serre [Ser72] as a reference. Let $K/\mathbb{Q}$ be a Galois extension and $C \subset \mathrm{Gal}(K/\mathbb{Q})$ be a conjugacy class. Let $\pi(K/\mathbb{Q}, C)$ denote the set of primes p that are unramified in $K/\mathbb{Q}$ which Frobenius conjugacy class C.

Definition. We call a set of primes a Chebotarev set if there are finitely many finite Galois extensions $K_i/\mathbb{Q}$ and conjugacy classes $C_i \subset \mathrm{Gal}(K_i/\mathbb{Q})$ such that up to finite sets, $P = \cup_i \pi(K_i/\mathbb{Q}, C_i)$.

Definition. We define a set $E \subset \mathbb{N}_{>0}$ to be multiplicative if for all pairs n_1, n_2 relatively prime, we have that $n_1 n_2 \in E$ if and only if $n_1 \in E$ or $n_2 \in E$.

Given a multiplicative set E, let $P(E)$ be the set of primes p in E. Let $\bar{E} := \mathbb{N}_{>0} - E$, and $\bar{E}(X) := \{m \in \bar{E}, m \leq X\}$.

Theorem 3.3.1 (See [Ser72].). *Suppose that E is multiplicative and $P(E)$ is a Chebotarev set with density $0 < \alpha < 1$. Then*

$$\bar{E}(X) \sim cX/\log^{\alpha} X$$

for some $c > 0$.

We will use the theorem above to prove Theorem 3.1.5 and Theorem 3.1.6. As shown in Section 3.2, the special cases with $D = 2, 37$ will then imply Theorem 3.1.1 and 3.1.2.

Proof of Theorem 3.1.5 and Theorem 3.1.6. Let D be good. Let D' be the odd part of D, and $\delta = \epsilon_D \epsilon_{D'} D/D'$. Note that if D is odd, then $\delta = 1$.

Also define

$$\bar{E}_D := \{q_1^{a_1} \cdots q_n^{a_n} : q_j \text{ is prime}, a_j \geq 0, \left(\frac{q_j}{p}\right) = 1, \left(\frac{\delta}{q_j}\right) = 1\},$$

for all odd primes $p \mid D$.

The compliment $E_D = \mathbb{N} - \bar{E}_D$ is then multiplicative and $P(E_D)$ has Chebotarev density $\alpha = 1 - 1/2^r$, where r is the number of prime factors of D. Therefore, by Theorem 3.3.1, we have

$$\bar{E}_D(X) \sim cX/\log^{\alpha} X.$$

Now, we have to relate $\bar{E}_D(X)$ to $R_D(X)$ and $I_D(X)$. We do this by showing that if $\pm q \in \bar{E}(X)$ is squarefree and satisfies congruence conditions coming from (d) and (e) of Theorem 2.2, then $m = qD$ has $\mathrm{EGR}_{\mathbb{Q}}$.

Let C_D be the set of $q \in \mathbb{Z}$ that satisfy the congruence conditions (d) and (e) of Theorem 3.2.2, so that

$$C_D = \begin{cases} \{q \in \mathbb{Z} : q \equiv D \,(\mathrm{mod}\, 4)\} & \text{if } D \equiv \pm 3 \,(\mathrm{mod}\, 8) \\ \{q \in \mathbb{Z} : q \equiv D+1 \,(\mathrm{mod}\, 8)\} & \text{if } D \equiv 0 \,(\mathrm{mod}\, 2) \\ \{q \in \mathbb{Z}\} & \text{otherwise} \end{cases}$$

We define

$$R_D^E(X) := \{Dq : \mathrm{sgn}(D)q \in \bar{E}_D(X/D), q \text{ squarefree}, q \in C_D\}$$
$$I_D^E(X) := \{Dq : -\mathrm{sgn}(D)q \in \bar{E}_D(X/D), q \text{ squarefree}, q \in C_D\}$$

Lemma 3.3.2. *For good D, $R_D^E(X) \subset R_D(X)$ if $\epsilon_D = -sgn(D)$ and $I_D^E(X) \subset I_D(X)$ if $\epsilon_D = sgn(D)$.*

Proof. We need to check (a) and (b) of Theorem 3.2.2. They follows from the properties of the Jacobi Symbol. Let D be good. If either $\epsilon_D = -\mathrm{sgn}(D)$ with $0 < qD$ or $\epsilon_D = \mathrm{sgn}(D)$ with $0 > qD$, we have that $0 < -\epsilon_D q = \prod q_j$, so

$$\left(\frac{-\epsilon_D q}{p}\right) = \prod \left(\frac{q_j}{p}\right) = 1.$$

Note that then we always have that $\epsilon_{D'} D' \equiv 1 \,(\mathrm{mod}\, 4)$ and $\epsilon_D D = \delta D' \epsilon_{D'}$. So

$$\left(\frac{\epsilon_D D}{q_j}\right) = \left(\frac{\delta}{q_j}\right)\left(\frac{\epsilon_{D'} D'}{q_j}\right) = \left(\frac{q_j}{|\epsilon_{D'} D'|}\right) = \prod_{p|D \text{ odd}} \left(\frac{q_j}{p}\right) = 1$$

□

Since a positive proportion of $\pm q \in \overline{E}_D(X/D)$ satisfy the extra conditions of being

squarefree and in C_D, we have that

$$R_D^E(X), I_D^E(X) >> \frac{X}{\log^\alpha X},$$

and hence the same is true of the bigger sets $R_D(X), I_D(X)$. □

Proof of Theorem 3.1.1. The theorem follows immediately from Theorem 3.1.5 and Theorem 3.1.3. Theorem 3.1.3 shows $D = 2$ is good with $r = 1$ distinct factors and the real quadratic field $K = \mathbb{Q}(\sqrt{qD})$ has $\mathrm{EGR}_{\mathbb{Q}}$. If $R(X)$ is the number of these fields, Theorem 3.1.5 shows

$$R(X) \gg \frac{X}{\sqrt{\log(X)}}.$$

□

Proof of Theorem 3.1.2. The theorem follows immediately from Theorem 3.1.6 and Theorem 3.1.4. Theorem 3.1.4 shows $D = 37$ is good with $r = 1$ distinct factors and the imaginary quadratic field $K = \mathbb{Q}(\sqrt{qD})$ has $\mathrm{EGR}_{\mathbb{Q}}$. If $I(X)$ is the number of these fields, Theorem 3.1.6 shows

$$I(X) \gg \frac{X}{\sqrt{\log(X)}}.$$

□

3.4 Examples

In this section, we explain how to find elliptic curves with $\mathrm{EGR}_{\mathbb{Q}}$ when the conditions of Theorem 3.2.2 are satisfied, and give examples of elliptic curves with $\mathrm{EGR}_{\mathbb{Q}}$. The results in this section are based on Setzer's construction in 3.2.1.

We start with a quadratic field $K = \mathbb{Q}(\sqrt{m})$ and a factorization $m = Dq$ with D

good which satisfies the conditions of Theorem 3.2.2. We want to find u such that

$$E_{u,A} : y^2 = x^3 - 3A(A^3 - 1728)u^2x - 2(A^3 - 1728)^2u^3$$

has $\mathrm{EGR}_{\mathbb{Q}}(K)$. Let $\alpha \in K$ have norm $\epsilon_D D$, and pick n odd such that $\beta := n\alpha = a+b\sqrt{m} \in \mathcal{O}_K$. Let $A \in \mathcal{R}$ be such that D is the square-free part of A^3-1728. Define d_1, d_2 such that $3^2(A^3 - 1728) = Dd_1^2d_2^4$ with d_1 square-free. If $m \equiv 1, 2 \pmod 4$, then one of $u = \pm\beta d_1$ works. If $m \equiv 3 \pmod 4$, then either $u = \pm\beta d_1$ both work or $u = \pm\beta d_1\rho$ both work, where $\rho = \frac{1}{2}(m+1) + \sqrt{m}$.

The table below has some examples.

A	D	d_1	q	α	u
20	2	42	3	$2+\sqrt{6}$	$-d_1\alpha = -84 - 42\sqrt{6}$
-15	-7	1	-11	$35+4\sqrt{77}$	$-d_1\alpha = -35 - 4\sqrt{77}$
-32	-11	42	-15	$77+6\sqrt{165}$	$d_1\alpha = 3234 + 252\sqrt{165}$
-32	-11	42	-3	$11+2\sqrt{33}$	$-d_1\alpha = -462 - 84\sqrt{33}$
39	79	1	5	$79+4\sqrt{395}$	$\pm d_1\alpha\rho = \pm(17222 + 871\sqrt{395})$
16	37	6	-7	$37+6\sqrt{-259}$	$\pm d_1\alpha = \pm(222 + 36\sqrt{-259})$

Chapter 4

A conjecture of Han on 3-cores and modular forms

4.1 Introduction

We begin this chapter by providing a brief background into some of the combinatorial identities involved in Han's conjectures.

A *partition* λ *of* n is an ordered tuple $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)$ with $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k$, such that $\lambda_1 + \lambda_2 + \cdots + \lambda_k = n$. To each partition λ of n, we can associate a frame with λ_i boxes in the i^{th} row such that the rows of boxes are lined up on the left. This is called a *Young diagram of shape* λ (or a Ferrers diagram). The partitions of n are in a one-to-one correspondence with Young diagrams of size n. A Young diagram of shape λ is a *standard Young tableau* if each cell contains a distinct positive integer $1 \leq i \leq n$ such that each column and row form an increasing sequence. The *hook length*, $h_v(\lambda)$, for a cell v is computed by adding the number of the cells to the right of v and the number of cells below v, counting v only once. The *hook length multi-set* of λ, $\mathcal{H}(\lambda)$ is the multi-set of all hook lengths of λ.

The original identity of Nekrasov and Okounkov relates the sum over products of

partition hook lengths to the powers of Euler products and is given by

$$F_z(x) := \sum_{\lambda} x^{|\lambda|} \prod_{h \in \mathcal{H}(\lambda)} \left(1 - \frac{z}{h^2}\right) = \prod_{n=1}^{\infty} (1-x^n)^{z-1},$$

where the sum is over integer partitions, $|\lambda|$ the integer partitioned by λ, and $\mathcal{H}(\lambda)$ the multiset of classical hook lengths associated to a partition λ.

The Nekrasov-Okounkov formula specializes in the case $z = 2$ and $z = 4$ to two classical q-series identities. The first is a special case of Euler's Pentagonal Number Theorem, and the second gives Jacobi's famous identity for the product $\prod_{n=1}^{\infty}(1-x^n)^3$, [Han09].

$$F_2(x) = \prod_{n=1}^{\infty}(1-x^n) = \sum_{n=-\infty}^{\infty} (-1)^n x^{\frac{3n^2+n}{2}}, \qquad \text{(Euler)}$$
$$F_4(x) = \prod_{n=1}^{\infty}(1-x^n)^3 = \sum_{n=0}^{\infty}(-1)^n(2n+1)x^{\frac{n^2+n}{2}}, \qquad \text{(Jacobi)}.$$

In [Han10], Han extended the Nekrasov-Okounkov identity to consider the number of t-core partitions of n. A partition of n is called a *t-core partition of n* if none of its hook numbers are multiples of t. While working on this generalization, Han investigated the nonvanishing of infinite product coefficients. For example, he considers the infinite product,

$$\prod_{n \geq 1} \frac{(1-x^{sn})^{t^2/s}}{1-x^n},$$

and conjectures in [Han09] that the coefficient of x^n is not equal to 0 for $t \geq 5$, t, s positive integers such that $s|t$ and $t \neq 10$. Letting $s = 1$ and $t = 5$, Han reformulates the famous conjecture of Lehmer that the coefficients of

$$x\prod_{n \geq 1}(1-x^n)^{24} = \sum_{n \geq 1} \tau(n)x^n$$

never vanish.

In [Han09], Han formulated a conjecture comparing the nonvanishing of terms of $F_9(x)$ with terms of C_3. Recall C_3 is the series given by

$$\begin{aligned} C_3(x) = \sum_{n=0}^{\infty} b(n)x^n &:= \prod_{n=1}^{\infty} \frac{(1-x^{3n})^3}{1-x^n} \\ &= 1 + x + 2x^2 + 2x^4 + \cdots + 2x^{14} + 3x^{16} + 2x^{17} + 2x^{20} + \dots . \end{aligned} \tag{1.1}$$

and $F_9(x)$ is the series given by

$$\begin{aligned} F_9(x) = \sum_{n=0}^{\infty} a(n)x^n &:= \prod_{n=1}^{\infty} (1-x^n)^8 \\ &= 1 - 8x + 20x^2 - 70x^4 + \cdots - 520x^{14} + 57x^{16} + 560x^{17} + 182x^{20} + \dots \end{aligned} \tag{1.2}$$

Based on numerical evidence, Han conjectured that the non-zero coefficients of $F_9(x)$ and $C_3(x)$ are supported on the same terms; assuming the notation above, $a(n) = 0$ if and only if $b(n) = 0$.

This conjecture is proved in a joint paper with Ono [HO11]. In addition to proving the conjecture, Han and Ono proved $a(n) = b(n) = 0$ precisely for those non-negative integers n for which $\mathrm{ord}_p(3n+1)$ is odd for some prime $p \equiv 2 \pmod 3$.

Recently Han discovered another series that appears to be supported on the same terms as $C_3(x)$ and $F_9(x)$. This series is given by,

$$\begin{aligned} C(x) = \sum_{n=1}^{\infty} c(n)x^n &:= \prod_{n=1}^{\infty} (1-x^n)^2(1-x^{3n})^2 \\ &= 1 - 2x - x^2 + 5x^4 + \cdots + 8x^{14} - 6x^{16} - 10x^{17} - x^{20} + \dots \end{aligned} \tag{1.3}$$

Based on numerical evidence, Han conjectured $a(n) = 0$ if and only if $b(n) = 0$ if and only if $c(n) = 0$.

Here we prove the following general theorem that produces infinitely many mod-

ular forms, including those in Equations (1.2) and (1.3) that are supported precisely on the same terms as Equation (1.1).

It is convenient to normalize Equation (1.1) as shown below.

$$\begin{aligned}\mathcal{B}(z) = \frac{\eta(9z)^3}{\eta(3z)} &= \sum_{n=1}^{\infty} b^*(n)q^n := \sum_{n=0}^{\infty} b(n)q^{3n+1} \\ &= q + q^4 + 2q^7 + 2q^{13} + q^{16} + 2q^{19} + q^{25} + 2q^{28} + \dots.\end{aligned} \tag{2.1}$$

Theorem 4.1.1. *Suppose that* $f(z) = \sum_{n=1}^{\infty} A(n)q^n$ *is an even weight newform with trivial Nebentypus that has complex multiplication by* $\mathbb{Q}(\sqrt{-3})$ *and a level of the form* 3^s*, where* $s \geq 2$*. Then the coefficients* $A(n) = 0$ *if and only if* $b(n) = 0$*. More precisely,* $A(n) = b^*(n) = 0$ *for those non-negative integers* n *for which* $\mathrm{ord}_p(n)$ *is odd for some prime* $p \equiv 2 \pmod 3$*.*

Remark. Here we let $q := e^{2\pi iz}$ and $\sum_{n=1}^{\infty} A(n)q^n$ is the usual Fourier expansion at infinity.

Remark. Consider the normalized function of $a(n)$ and $c(n)$ given by,

$$\begin{aligned}\mathcal{A}(z) &= \sum_{n=1}^{\infty} a^*(n)q^n := \sum_{n=0}^{\infty} a(n)q^{3n+1} \\ &= q - 8q^4 + 20q^7 - 70q^{13} + 64q^{16} + 56q^{19} - 125q^{25} - 160q^{28} + \dots\end{aligned} \tag{2.2}$$

and

$$\begin{aligned}\mathcal{C}(z) &= \sum_{n=1}^{\infty} c^*(n)q^n := \sum_{n=0}^{\infty} c(n)q^{3n+1} \\ &= q - 2q^4 - q^7 + 5q^{13} + 4q^{16} - 7q^{19} - 5q^{25} + 2q^{28} + \dots\end{aligned} \tag{2.3}$$

Theorem 4.1.1 implies the original work of Han and Ono. As explained in [HO11], $\mathcal{A}(z)$ is a weight 4 newform with complex multiplication by $\mathbb{Q}(\sqrt{-3})$ with level 9. The-

orem 4.1.1 also implies Han's new conjecture concerning coefficients of (1.3) because $\mathcal{C}(z)$ is the weight 2 complex multiplication form for the elliptic curve with complex multiplication by $\mathbb{Q}(\sqrt{-3})$ given by $y^2 + y = x^3 - 7$ with level $3^3 = 27$ [MO97].

Remark. It turns out that more is true about the relationship between the two series in Equations (1.2) and (1.3). If $p \equiv 1 \pmod 3$ is prime, then we have that $c^*(p)$ divides $a^*(p)$. We will prove this statement in Section 3.1.

To prove Theorem 4.1.1, we make use of the known description of Equation (1.1), the generating function for the 3-core partition function, and then generalize the work in [HO11] to extend to this situation.

We begin by recalling the exact formula for the coefficients $b^*(n)$ of the modular form $\mathcal{B}(z)$, (2.1), defined below. Recall that the Dedekind's eta function, denoted $\eta(z)$, is defined by the infinite product

$$\eta(z) := q^{1/24} \prod_{n=1}^{\infty} (1 - q^n).$$

The coefficients $b^*(n)$ are given by

$$\begin{aligned} \mathcal{B}(z) = \frac{\eta(9z)^3}{\eta(3z)} &= \sum_{n=1}^{\infty} b^*(n) q^n := \sum_{n=0}^{\infty} b(n) q^{3n+1} \\ &= q + q^4 + 2q^7 + 2q^{13} + q^{16} + 2q^{19} + q^{25} + 2q^{28} + \dots. \end{aligned}$$

Lemma 4.1.2 (Lemma 2.5 of [HO11]). *Assuming the notation above, we have that*

$$\mathcal{B}(z) = \sum_{n=1}^{\infty} b^*(n) q^n = \sum_{n=0}^{\infty} b(n) q^{3n+1} = \sum_{n=0}^{\infty} \sum_{d \mid 3n+1} \left(\frac{d}{3} \right) q^{3n+1}.$$

The following lemma describes the nonvanishing conditions for the Equation (2.1) as described in [HO11].

Lemma 4.1.3. *Assume the notation above. Then $b^*(n) = 0$ if and only if n is a non-negative integer for which $\mathrm{ord}_p(n)$ is odd for some prime $p \equiv 2 \pmod 3$.*

To prove the original conjecture, Han and Ono recalled the exact formula for the coefficients $a^*(n)$ described in [HO11]. The modular form $\mathcal{A}(z)$, (2.2), is given by

$$\mathcal{A}(z) = \eta^8(3z) = \sum_{n=1}^{\infty} a^*(n)q^n := \sum_{n=0}^{\infty} a(n)q^{3n+1}$$

where $q := e^{2\pi i}$ and $z \in \mathcal{H}$, the upper half of the complex plane. This normalized series $\mathcal{A}(z)$, such that $a(n) \equiv a^*(3n+1)$, is an example of a newform with complex multiplication in $S_4(\Gamma_0(9))$, the space of weight 4 cusp forms on $\Gamma_0(9)$. Using the theory of newforms, Han and Ono proved the following theorem.

Theorem 4.1.4 (Theorem 2.1 of [HO11])**.** *Assume the notation above. Then the following are true:*

1. *If $p = 3$ or $p \equiv 2 \pmod 3$ is prime, then $a^*(p) = 0$.*
2. *If $p \equiv 1 \pmod 3$ is prime, then*

$$a^*(p) = 2x^3 - 18xy^2,$$

where x and y are integers for which $p = x^2 + 3y^2$ and $x \equiv 1 \pmod 3$.

The theorem above shows that $a^*(n)$ satisfies the same nonvanishing conditions demonstrated by $b^*(n)$ as noted in Lemma 4.1.3, proving the original conjecture.

4.2 Proof of Theorem 4.1.1

Consider the function $\Psi(z)$ defined by

$$\Psi(z) := \sum_{\mathfrak{a}} \phi(\mathfrak{a})q^{N(\mathfrak{a})} = \sum_{n=1}^{\infty} a(n)q^n,$$

where the sum is over the integral ideals $\mathfrak{a}$ that are prime to Λ and $N(\mathfrak{a})$ is the norm of the ideal $\mathfrak{a}$. This function $\Psi(z)$ is a cusp form in $S_k(\Gamma_0(D \cdot N(\Lambda)), \left(\frac{-D}{\bullet}\right)\omega_\phi)$. When p does not divide the level, notice that if p is inert in K, then $a(p) = 0$ [Ono04].

The cusp form $\Psi(z)$ is a "newform" in the sense of Atkin and Lehner [Ono04]. Therefore, $\Psi(z)$ is a normalized cusp form that is an eigenform of all the Hecke operators and all the Atkin-Lehner involutions $|_k W(Q_p)$ for primes $p|N$ and $|_k W(N)$. The following theorem describes the vanishing Hecke eigenvalues when there is a prime p such that p^2 divides the level.

Theorem 4.2.1 (Theorem 2.27 (3) of [Ono04]). *Suppose* $f(z) = \sum_{n=1}^{\infty} a(n)q^n \in S_k^{new}(\Gamma_0(N))$ *is a newform. If* p *is a prime for which* $p^2|N$, *then* $a(p) = 0$.

This information gives the following nonvanishing conditions on newforms with complex multiplication.

Lemma 4.2.2. *Suppose that* $f(z) = \sum_{n=1}^{\infty} A(n)q^n$ *is an even weight newform with trivial Nebentypus and complex multiplication by* $\mathbb{Q}(\sqrt{-3})$ *with level of the form* 3^s *where* $s \geq 2$. *Then* $A(p) = 0$ *if and only if* $p = 3$ *or* $p \equiv 2 \pmod 3$ *is prime.*

Proof of Lemma 4.2.2. The level of $f(z)$ is 3^s and therefore 3 is the only prime that divides the level. Since $k \geq 2$, we know 3^2 always divides the level, therefore by Theorem 4.2.1 in [Ono04], $A(3) = 0$. When $p \equiv 2 \pmod 3$ for $p \neq 3$ prime, p is inert and therefore $A(p) = 0$. □

Corollary 4.2.3. *The following are true about* $A(n)$.

1. *If* m *and* n *are coprime positive integers, then*

$$A(mn) = A(m)A(n).$$

2. *For every positive integer* t, *we have that* $A(3^t) = 0$.

3. *If* $p \equiv 2 \pmod 3$ *is prime and* t *is a positive integer, then* $A(p^t) = 0$ *if* t *is odd and* $A(p^t) \neq 0$ *if* t *is even.*

4. *If* $p \equiv 1 \pmod 3$*, then* $A(p^t) \neq 0$.

Proof of Corollary 4.2.3. Claim (1) is well known to hold for all normalized Hecke eigenforms.

Claim (2) follows as $A(3) = 0$.

To prove Claim (3), observe that every newform is a Hecke eigenform. Moreover, since $A(1) = 1$, the Hecke eigenvalue of $T(p)$ is $A(p)$.Therefore, for every integer n and prime $p \neq 3$, we have that

$$A(p)A(n) = A(pn) + p^{k-1}A(n/p).$$

The left hand side of the equation is the statement that $A(p)$ is the Hecke eigenvalue. The right hand side of the equation is the action of the Hecke operator $T(p)$. Let $n = p^t$ and $p \equiv 2 \pmod 3$ be prime. Since $A(p) = 0$ for $p \equiv 2 \pmod 3$, this equation becomes

$$0 = A(p^{t+1}) + p^{k-1}A(p^{t-1}).$$

Claim (3) follows from induction as $A(1) = 1$ and $A(p) = 0$.

To prove Claim (4), let p be a prime such that $p \equiv 1 \pmod 3$. Suppose that $A(p) = 0$. This implies that α is totally imaginary, but then

$$p = (\beta\sqrt{-3}))(-\beta\sqrt{-3}) = 3\beta^2,$$

which is false. Claim (4) then follows by induction. □

Proof of Theorem 4.1.1. The theorem follows by combining Lemma 4.1.3, Corollary 4.2.3 and Lemma 4.2.2. □

4.3 Relating $\mathcal{A}(z)$ and $\mathcal{C}(z)$

We normalize the function $c(n)$ using the following series,

$$\mathcal{C}(z) = \sum_{n=1}^{\infty} c^*(n)q^n := \sum_{n=0}^{\infty} c(n)q^{3n+1}.$$

The series $\mathcal{C}(z)$ is a modular form given by

$$\mathcal{C}(z) = \eta^2(3z)\eta^2(9z) = \sum_{n=1}^{\infty} c^*(n)q^n.$$

In [MO97], Martin and Ono gave a complete description of all weight 2 newforms that are products and quotients of the Dedekind eta-function. The descriptions in [MO97] include formulas for the p^{th} coefficients. Since these coefficients are Hecke multiplicative, it suffices to give the formula for only p prime. Specifically, for $\mathcal{C}(z)$, we have the following theorem.

Theorem 4.3.1 (Theorem 2 in [MO97]). *Assuming the notation above, the following are true.*

1. *If* $p \equiv 2 \pmod 3$, *then* $c^*(p) = 0$.
2. *If* $p \equiv 1 \pmod 3$, *then* $c^*(p) = 2m+n$ *where* $p = m^2+mn+n^2$ *and* $m \equiv 1 \pmod 3$ *and* $n \equiv 0 \pmod 3$.

Recall 4.1.4 from [HO11] gave the following conditions on the coefficients of $\mathcal{A}(z)$: If $p \equiv 1 \pmod 3$ is prime, then

$$a^*(p) = 2x^3 - 18xy^2,$$

where x and y are integers for which $p = x^2 + 3y^2$ and $x \equiv 1 \pmod 3$.

Here we show that $c^*(p)|a^*(p)$ for primes $p \equiv 1 \pmod 3$ and n being even:

$$\begin{aligned} p &= m^2 + mn + n^2 \\ &= \left(m + \frac{n}{2}\right)^2 + 3\left(\frac{n}{2}\right)^2 \\ &= x^2 + 3y^2. \end{aligned}$$

Let $x = \left(m + \frac{n}{2}\right)$ and $y = \dfrac{n}{2}$. Then

$$\begin{aligned} a^*(p) &= 2x^3 - 18xy^2 \\ &= 2\left(m + \frac{n}{2}\right)^3 - 18\left(m + \frac{n}{2}\right)\left(\frac{n}{2}\right)^2 \\ &= (2m + n)(m + 2n)(m - n) \\ &= c^*(p)(m + 2n)(m - n). \end{aligned}$$

Since $m \equiv 1 \pmod 3$ and $n \equiv 0 \pmod 3$, we have $a^*(p) \equiv c^*(p) \pmod 3$ and as mentioned in a remark, $c^*(p)|a^*(p)$.

Chapter 5

Weierstrass mock modular forms

5.1 Introduction

In Chapter 2, we discussed results of Alfes, Griffin, Ono, and Rolen [AGOR14] that obtain a canonical weight 0 harmonic Maass forms arising from Eisenstein's corrected Weierstrass zeta-function for elliptic curves over $\mathbb{Q}$. Guerzhoy [Gue15] had also previously studied the construction of harmonic Maass forms using the Weierstrass zeta-function in his work on the Kaneko-Zagier hypergeometric differential equation. In this chapter, we will focus on the Weierstrass mock modular forms associated to 5 specific elliptic curves. These five elliptic curves are the curves Martin and Ono proved in [MO97] to be the only five elliptic curves with complex multiplication whose associated weight 2 newform is a product or quotient of Dedekind eta-functions. We prove that the derivative of the Weierstrass mock modular form of each such elliptic curve E is a weight 2 weakly holomorphic modular form which also turns out to be an eta-quotient or a twist of one. We also obtain p-adic formulas for the corresponding weight 2 newforms using Atkin's U-operator.

Let E be one of the five elliptic curves with complex multiplication whose associated newform, $F_E(\tau)$, is an eta-quotient. Let N_E denote the conductor of this curve

and label its coefficients a_i such that they belong to the Weierstrass model

$$E : y^2 + a_1xy + a_3y = x^3 + a_2x^2 + a_4x + a_6.$$

The following table contains a strong Weil curve for each of the weight 2 newforms with complex multiplication that are eta-quotients.

N_E	$F_E(\tau)$	a_1	a_2	a_3	a_4	a_6
27	$\eta^2(3\tau)\eta^2(9\tau)$	0	0	1	0	-7
32	$\eta^2(4\tau)\eta^2(8\tau)$	0	0	0	4	0
36	$\eta^4(6\tau)$	0	0	0	0	1
64	$\dfrac{\eta^8(8\tau)}{\eta^2(4\tau)\eta^2(16\tau)}$	0	0	0	-4	0
144	$\dfrac{\eta^{12}(12\tau)}{\eta^4(6\tau)\eta^4(24\tau)}$	0	0	0	0	-1

Table 5.1: Table of five elliptic curves

Let $\widehat{\mathfrak{Z}}_E^+(\tau)$ denote the Weierstrass mock modular form of E, and let $Z_{N_E}(\tau) := q \cdot \frac{d}{dq}\widehat{\mathfrak{Z}}_E^+(\tau)$ denote the derivative of the Weierstrass mock modular form (see Section 2.2 for details). Let $\chi_D := \left(\frac{D}{\cdot}\right)$ denote the usual Kronecker symbol so that $\left(\sum a(n)q^n\right) |_{\chi_D} = \sum \chi_D(n)a(n)q^n$.

Theorem 5.1.1. *The derivative of the Weierstrass mock modular form for each of the five elliptic curves E given in Table 5.1 is an eta-quotient or a twist of one, as described below.*

$$Z_{27}(\tau) = -\eta(3\tau)\eta^6(9\tau)\eta^{-3}(27\tau)$$

$$Z_{32}(\tau) = -\eta^2(4\tau)\eta^6(16\tau)\eta^{-4}(32\tau)$$

$$Z_{36}(\tau) = -\eta^3(6\tau)\eta(12\tau)\eta^3(18\tau)\eta^{-3}(36\tau)$$

$$Z_{64}(\tau) = -\eta^2(4\tau)\eta^6(16\tau)\eta^{-4}(32\tau) \mid_{\chi_8}$$

$$Z_{144}(\tau) = -\eta^3(6\tau)\eta(12\tau)\eta^3(18\tau)\eta^{-3}(36\tau) \mid_{\chi_{12}}$$

We also obtain p-adic formulas for the corresponding weight 2 newform using Atkin's U-operator,

$$\sum a(n)q^n \mid U(m) := \sum a(mn)q^n.$$

By taking a p-adic limit, we can retrieve the coefficients of the original cusp form, $F_E(\tau)$, of the elliptic curve. Let $Z_{N_E}(\tau) = \sum_{n=-1}^{\infty} d(n)q^n$ be the derivative of the Weierstrass mock modular form as before.

Theorem 5.1.2. *For each of the five elliptic curves listed in Table 5.1, if p is inert in the field of complex multiplication, then as a p-adic limit we have*

$$F_E(\tau) = \lim_{\omega \to \infty} \frac{Z_{N_E}(\tau) \mid U(p^{2\omega+1})}{d(p^{2\omega+1})}.$$

Example. Here we illustrate Theorem 5.1.2 for the prime $p = 5$ and the newform with conductor 27. Let

$$Z_{E,\omega}(p,\tau) = \frac{Z_{N_E}(\tau) \mid U(p^{2\omega+1})}{d(p^{2\omega+1})}.$$

If $p = 5$, then we have

$$\begin{aligned} Z_{E,0}(5,\tau) &= q + 8q^4 + 49q^7 + 75q^{10} + \ldots && \equiv F_E(z) \pmod 5 \\ Z_{E,1}(5,\tau) &= q + \frac{195040}{480}q^4 + \frac{6821395}{480}q^7 + \ldots && \equiv F_E(z) \left(\mathrm{mod}\, 5^2\right). \end{aligned}$$

We prove this theorem using techniques outlined in [GKO10]. Similar results can be found in both [EGO10] and [AGOR14]. In [EGO10], El-Guindy and Ono study a modular function that arises from Gauss's hypergeometric function that gives a modular parameterization of period integrals of E_{32}, the elliptic curve with conductor 32. In [AGOR14], Theorem 1.3 builds p-adic formulas for the corresponding weight 2 newforms using the action of the Hecke algebra on the Weierstrass mock modular

forms.

5.2 Weierstrass mock modular forms

Recall we are interested in computing the Weierstrass mock modular form for the elliptic curves with conductors 27, 32, 36, 64, and 144 given by Table 5.1. The value of $S(\Lambda_E)$ is 0 for each of these curves and so the Weierstrass mock modular form $\widehat{\mathfrak{Z}}_E^+(z)$ is $\zeta(\Lambda_E; \mathcal{E}_E(z))$. Bruinier, Rhoades, and Ono [BOR09], and Candelori [Can] proved that if a normalized newform has complex multiplication then the holomorphic part of a certain harmonic Maass form has algebraic coefficients; in particular, the coefficients of $\widehat{\mathfrak{Z}}_E^+(z)$ are algebraic.

Relabeling z as τ so that $q = e^{2\pi i\tau}$, we can now define the derivative of the Weierstrass mock modular form as $Z_{N_E}(\tau) = q \cdot \frac{d}{dq}\widehat{\mathfrak{Z}}_E^+(\tau)$. The list below gives the first few terms of the q-expansion for the derivative of the Weierstrass mock modular form for each of the five curves.

N_E	q-expansion for $Z_{N_E}(\tau)$
27	$-q^{-1} + q^2 + q^5 + 6q^8 - 6q^{11} - 7q^{14} - 9q^{17} + 8q^{20} + 15q^{23} + \ldots$
32	$-q^{-1} + 2q^3 + q^7 - 2q^{11} + 5q^{15} - 14q^{19} - 4q^{23} + 12q^{27} + \ldots$
36	$-q^{-1} + 3q^5 + q^{11} - 5q^{17} - 8q^{23} - q^{29} + 28q^{35} + \ldots$
64	$-q^{-1} - 2q^3 + q^7 + 2q^{11} + 5q^{15} + 14q^{19} - 4q^{23} - 12q^{27} + \ldots$
144	$-q^{-1} - 3q^5 + q^{11} + 5q^{17} - 8q^{23} + q^{29} + 28q^{35} + \ldots$

5.3 Eta-quotients

In Chapter 2, we described some of the properties of eta-quotients. We now want to relate these properties to the Weierstrass mock modular forms formulated above. If the derivative of the Weierstrass mock modular form, $Z_{N_E}(\tau)$, is an eta-quotient, certain properties must hold. In [Ono04], Ono described the following result of Gordon, Hughes, and Newman on eta-quotients.

Theorem 5.3.1 (Theorem 1.64 of [Ono04]). *If* $f(\tau) = \prod_{\delta|N} \eta(\delta\tau)^{r_\delta}$ *is an eta-quotient with* $k = \frac{1}{2}\sum_{\delta|N} r_\delta \in \mathbb{Z}$, *with the additional properties that*

$$\sum_{\delta|N} \delta r_\delta \equiv 0 \pmod{24}$$

and

$$\sum_{\delta|N} \frac{N}{\delta} r_\delta \equiv 0 \pmod{24},$$

then $f(\tau)$ *satisfies*

$$f\left(\frac{a\tau+b}{c\tau+d}\right) = \chi(d)(c\tau+d)^k f(\tau)$$

for all

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N).$$

Here the character χ *is defined for* $\chi(d) := \left(\frac{(-1)^k s}{d}\right)$, *where* $s = \prod_{\delta|N} \delta^{r_\delta}$.

In Section 5.5, we will prove that the derivative of the Weierstrass mock modular form $Z_{N_E}(\tau)$ is an eta-quotient or a twist of one. In order to help us identify plausible eta-quotients to describe $Z_{N_E}(\tau)$, note that any such eta-quotient $\prod_{\delta|N_E} \eta(\delta\tau)^{r_\delta}$ must satisfy the following:

$$\begin{aligned} &\sum_{\delta|N_E} r_\delta = 4, \\ &\sum_{\delta|N_E} \delta r_\delta = -24, \\ &\sum_{\delta|N_E} \frac{N_E}{\delta} r_\delta \equiv 0 \pmod{24}, \\ &\prod_{\delta|N_E} \delta^{r_\delta} = a^2 \text{ for some integer } a. \end{aligned} \tag{5.1}$$

This description follows from Theorem 5.3.1, together with the fact that $Z_{N_E}(\tau)$ has weight 2, level N_E and leading term q^{-1}.

5.4 Examples

5.4.1 $N_E = 27$

Consider the curve $E : y^2 + y = x^3 - 7$, which has conductor $N_E = 27$. The eta-quotient $\eta(3\tau)\eta^6(9\tau)\eta^{-3}(27\tau)$ satisfies the four properties described in Equation(1) for $N_E = 27$ and its initial terms match with those of $Z_{27}(\tau)$, as shown below:

$$\eta(3\tau)\eta^6(9\tau)\eta^{-3}(27\tau) = q^{-1} - q^2 - q^5 - 6q^8 + 6q^{11} + 7q^{14} + 9q^{17} - O(q^{20})$$
$$Z_{27}(\tau) = -q^{-1} + q^2 + q^5 + 6q^8 - 6q^{11} - 7q^{14} - 9q^{17} + O(q^{20}).$$

Thus we define $\eta_{27} = -\eta(3\tau)\eta^6(9\tau)\eta^{-3}(27\tau)$ and guess that $Z_{27} = \eta_{27}$. This will be proven in Section 5.5 in order to establish Theorem 5.1.1.

5.4.2 $N_E = 32$

Consider the curve, $E : y^2 = x^3 + 4x$. The eta-quotient $\eta^2(4\tau)\eta^6(16\tau)\eta^{-4}(32\tau)$ satisfies the four properties described in Equation(1) for $N_E = 32$ and its initial terms match with those of $Z_{32}(\tau)$, as shown below:

$$\eta^2(4\tau)\eta^6(16\tau)\eta^{-4}(32\tau) = q^{-1} - 2q^3 - q^7 + 2q^{11} - 5q^{15} + 14q^{19} + O(q^{23})$$
$$Z_{32}(\tau) = -q^{-1} + 2q^3 + q^7 - 2q^{11} + 5q^{15} - 14q^{19} - O(q^{23}).$$

Letting $\eta_{32} = -\eta^2(4\tau)\eta^6(16\tau)\eta^{-4}(32\tau)$, we will later prove $Z_{32} = \eta_{32}$ in Section 5.5 to establish Theorem 5.1.1.

5.4.3 $N_E = 36$

Consider the curve with level 36, $E : y^2 = x^3+1$. The eta-quotient $\eta^3(6\tau)\eta(12\tau)\eta^3(18\tau)\eta^{-3}(36\tau)$ satisfies the four properties described in Equation(1) for $N_E = 36$ and its initial terms match with those of $Z_{36}(\tau)$, as shown below:

$$\eta^3(6\tau)\eta(12\tau)\eta^3(18\tau)\eta^{-3}(36\tau) = q^{-1} - 3q^5 - q^{11} + 5q^{17} + 8q^{23} + q^{29} - O(q^{35})$$
$$Z_{36}(\tau) = -q^{-1} + 3q^5 + q^{11} - 5q^{17} - 8q^{23} - q^{29} + O(q^{35}).$$

Letting $\eta_{36} = -\eta^3(6\tau)\eta(12\tau)\eta^3(18\tau)\eta^{-3}(36\tau)$, we will later prove $Z_{36} = \eta_{36}$.

5.4.4 $N_E = 64$

Consider the curve with level 64, $E : y^2 = x^3-4x$. The eta-quotient $\eta^2(4\tau)\eta^6(16\tau)\eta^{-4}(32\tau)$ satisfies the four properties described in Equation(1) for $N_E = 64$. Note $-\eta^2(4\tau)\eta^6(16\tau)\eta^{-4}(32\tau)$ η_{32}. The initial terms of this eta-quotient match with those of $Z_{64}(\tau)$, as shown below:

$$\eta^2(4\tau)\eta^6(16\tau)\eta^{-4}(32\tau) = q^{-1} - 2q^3 - q^7 + 2q^{11} - 5q^{15} + 14q^{19} + O(q^{23})$$
$$Z_{64}(\tau) = -q^{-1} - 2q^3 + q^7 + 2q^{11} + 5q^{15} + 14q^{19} - O(q^{23}).$$

Letting $\eta_{64} = \eta_{32} \mid_{\chi_8}$, we will later prove $Z_{64} = \eta_{64}$.

5.4.5 $N_E = 144$

Consider the curve with level 144, $E : y^2 = x^3-1$. The eta-quotient $\eta^3(6\tau)\eta(12\tau)\eta^3(18\tau)\eta^{-3}(36\tau)$ satisfies the four properties described in Equation(1) for $N_E = 144$. Note $-\eta^3(6\tau)\eta(12\tau)\eta^3(18\tau)\eta$ η_{36}. The initial terms of this eta-quotient match with those of $Z_{144}(\tau)$, as shown be-

low:

$$\eta^3(6\tau)\eta(12\tau)\eta^3(18\tau)\eta^{-3}(36\tau) = q^{-1} - 3q^5 - q^{11} + 5q^{17} + 8q^{23} + O(q^{29})$$
$$Z_{144}(\tau) = -q^{-1} - 3q^5 + q^{11} + 5q^{17} - 8q^{23} + O(q^{29}).$$

Letting $\eta_{144} = \eta_{36} \mid_{\chi_{12}}$, we will later prove $Z_{144} = \eta_{144}$.

5.5 Proof of Theorem 5.1.1 and Theorem 5.1.2

Proof of Theorem 5.1.1. When the conductor of E is 27, 32, and 36, the modular parameterization of these 3 curves has degree 1 (as computed in Sage [Dev15]) and each Weierstrass mock modular form has only a single pole at infinity. Let S_{N_E} denote Sturm's bound for the space of modular forms on $\Gamma_0(N_E)$ of weight 2, and let η_{N_E} denote the eta-quotient described in Section 5.3. For example, recall $\eta_{27} = -\eta(3\tau)\eta^6(9\tau)\eta^{-3}(27\tau)$. Consider the difference of the eta-quotients, η_{N_E}, and the derivatives of the Weierstrass mock modular form, $Z_{N_E}(\tau)$. Both q-expansions have a simple pole at infinity. The principal part of $Z_{N_E}(\tau)$ for $N_E = 27, 32, 36$ is constant at every cusp except infinity because the degree of modular parameterization for E_{27}, E_{32} and E_{36} is 1. Using the following formula, one can verify with a few Sage computations that the order of vanishing of η_{N_E} is nonnegative at each cusp c/d (except at infinity, where there is a simple pole) [Dev15].

Theorem 5.5.1 (Theorem 1.65 of [Ono04]). *Let c, d and N be positive integers with $d|N$ and $gcd(c, d) = 1$. If $f(z)$ is an eta-quotient satisfying the conditions of Theorem 1.64 for N, then the order of vanishing of $f(z)$ at the cusp $\frac{c}{d}$ is*

$$\frac{N}{24} \sum_{\delta|N} \frac{gcd(d, \delta)^2 r_\delta}{gcd(d, \frac{N}{d}) d\delta}.$$

Since the difference $Z_{N_E}(\tau) - \eta_{N_E}$ is holomorphic, as shown above, if $Z_{N_E}(\tau) - \eta_{N_E}$

is 0 for S_{N_E} coefficients, the identities claimed for $N_E = 27, 32, 36$ are correct. The following table gives Sturm's bound for the space of modular forms on $\Gamma_0(N_E)$ of weight 2.

N_E	S_{N_E}
27	13
32	17
36	25

After checking the coefficients of the expansions up to the corresponding bound, we see $Z_{27}(\tau) = \eta_{27} = -\eta(3\tau)\eta^6(9\tau)\eta^{-3}(27\tau)$, $Z_{32}(\tau) = \eta_{32} = -\eta^2(4\tau)\eta^6(16\tau)\eta^{-4}(32\tau)$, and $Z_{36}(\tau) = \eta_{36} = -\eta^3(6\tau)\eta(12\tau)\eta^3(18\tau)\eta^{-3}(36\tau)$, as claimed.

The modular parametrization for E_{64} has degree 2, and the modular parametrization for E_{144} has degree 4 (as computed in Sage [Dev15]; also see [Zag85]); therefore we cannot apply Sturm's bound to the difference of the associated Weierstrass mock modular forms and eta-quotients. Instead we prove Z_{64} is a twist of Z_{32} by χ_8, and Z_{144} is a twist of Z_{36} by χ_{12}. Consider first Z_{64}, Z_{32}, and χ_8, where χ_8 denotes the Kronecker symbol as before. We have already shown $(Z_{32} - \eta_{32}) \mid_{\chi_8} = 0$. Therefore, $Z_{32} \mid_{\chi_8} - \eta_{32} \mid_{\chi_8} = 0$. Since $Z_{32} \mid_{\chi_8} - \eta_{32} \mid_{\chi_8}$ is a twist of a holomorphic difference, we can use Sturm's bound to check up to S_{32} coefficients and confirm $Z_{32} \mid_{\chi_8} = \eta_{32} \mid_{\chi_8} = \eta_{64}$. To prove $Z_{32} \mid_{\chi_8} = Z_{64}$, note the q-expansions are equal up to 17 coefficients and their difference is holomorphic (as the principal part of each is constant at every cusp except infinity as shown before). Therefore, $Z_{32} \mid_{\chi_8} = Z_{64}$ so $Z_{64} = \eta_{32} \mid_{\chi_8} = \eta_{64} = -\eta^2(4\tau)\eta^6(16\tau)\eta^{-4}(32\tau) \mid_{\chi_8}$. The proof for $Z_{36} \mid_{\chi_{12}} = Z_{144}$ is similar, giving us the equality $Z_{144} = \eta_{36} \mid_{\chi_{12}} = \eta_{144} = -\eta^3(6\tau)\eta(12\tau)\eta^3(18\tau)\eta^{-3}(36\tau) \mid_{\chi_{12}}$. □

Proof of Theorem 5.1.2. Theorem 5.1.2 is a consequence of Theorem 5.5.2 of Guerzhoy, Kent, and Ono. Let $g(\tau) = \sum_{n=1}^{\infty} b(n)q^n \in S_2(\Gamma_0(N_E))$ denote the normalized newform and $\mathcal{E}_E(\tau)$ its Eichler integral. Recall, g has rational coefficients. Let $f = f^+ + f^-$ denote a weight-0 harmonic Maass form where f^+ is the holomorphic part. If $\xi = \xi_2 :=$

$2iy^2\frac{\bar{d}}{d\bar{\tau}}$, then we say that g is a shadow of f^+ if $\xi(f) = g$. We say $f \in H_0(\Gamma_0(N_E))$ is *good* for $g(\tau)$ if the following hold;

1. The principal part of f at the cusp ∞ belongs to $\mathbb{Q}[q^{-1}]$.

2. The principal part of f at other cusps is constant.

3. $\xi(f) = \frac{g}{<g,g>}$ where $< \cdot, \cdot >$ denotes the usual Petersson inner product.

Let D denote the operator $D := \frac{1}{2\pi i}\frac{d}{d\tau}$ so that $D(f^+) = \sum_{n=1}^{\infty} d(n)q^n$ is the derivative of the holomorphic part of the harmonic Maass form, i.e. the mock modular form. Guerzhoy, Kent, and Ono relate the coefficients of g and f using the following theorem.

Theorem 5.5.2 (Theorem 1.2 (2) of [GKO10]). *Suppose $g(\tau) \in S_2(\Gamma_0(N))$ has CM and g is good for f. If p is inert in the field of complex multiplication, then we have that*

$$g = \lim_{\omega \to \infty} \frac{D(f^+) \mid U(p^{2\omega+1})}{d(p^{2\omega+1})}.$$

Consider $F_E(\tau) \in S_2(\Gamma_0(N_E))$, the normalized newform equal to an eta-quotient for one of the elliptic curves E with complex multiplication listed in Table 5.1, $\widehat{\mathfrak{Z}}_E(z)$ the canonical harmonic Maass form and $Z_{N_E}(\tau)$ the derivative of the Weierstrass mock modular form. The harmonic Maass form $\widehat{\mathfrak{Z}}_E(z)$ is good for F_E as follows:

1. The principal part of $\widehat{\mathfrak{Z}}_E(z)$ at ∞ belongs to $\mathbb{Q}[q^{-1}]$.

2. There are no poles at other cusps for $N_E = 27, 32, 36$. Since Z_{64} is a twist of Z_{32} and Z_{144} is a twist of Z_{36}, the principal parts of $\widehat{\mathfrak{Z}}_E(z)$ for E_{64} and E_{144} must have constant principal parts at other cusps.

3. By definition of $\widehat{\mathfrak{Z}}_E(z)$, we have $\xi(f) = \frac{g}{<g,g>}$.

Therefore, $\widehat{\mathfrak{Z}}_E(z)$ is good for F_E and we can apply Theorem 5.5.2 to show the p-adic limit holds for the derivative of the Weierstrass mock modular form. $\square$

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