Distribution Agreement

In presenting this thesis or dissertation as a partial fulfillment of the requirements for an advanced degree from Emory University, I hereby grant to Emory University and its agents the non-exclusive license to archive, make accessible, and display my thesis or dissertation in whole or in part in all forms of media, now or hereafter known, including display on the world wide web. I understand that I may select some access restrictions as part of the online submission of this thesis or dissertation. I retain all ownership rights to the copyright of the thesis or dissertation. I also retain the right to use in future works (such as articles or books) all or part of this thesis or dissertation.

Annika Poerschke   Date

On Algorithmic Hypergraph Regularity

By

Annika Poerschke

Doctor of Philosophy

Mathematics

Vojtěch Rödl, Ph.D.
Adviser

Brendan Nagle, Ph.D.
Co-Adviser

Dwight Duffus, Ph.D.
Committee Member

Ron Gould, Ph.D.
Committee Member

Mathias Schacht, Ph.D.
Committee Member

Accepted:

Lisa A. Tedesco, Ph.D.
Dean of the Graduate School

Date

On Algorithmic Hypergraph Regularity

By

Annika Poerschke
Master of Science, Emory University, 2006

Advisor: Vojtěch Rödl, Ph.D.
Advisor: Brendan Nagle, Ph.D.

An abstract of
A dissertation submitted to the Faculty of the Graduate School of Emory University
in partial fulfillment of the requirements for the degree of
Doctor of Philosophy
in Mathematics
2008

Abstract

On Algorithmic Hypergraph Regularity

By Annika Poerschke

Thomason and Chung, Graham and Wilson were the first to systematically study *quasi-random* graphs and hypergraphs and showed that several properties of random graphs imply each other in a deterministic sense. In particular, they showed that ε-regularity from Szemerédi's regularity lemma is equivalent to their concepts. Over recent years several hypergraph regularity lemmas were established.

In this dissertation, we focus on two regularity lemmas for 3-uniform hypergraphs one due to Gowers, and one due to Haxell, Nagle, and Rödl. Their lemmas are based on different notions of quasirandom hypergraphs and we show that their concepts are in fact equivalent. Since the regularity lemma of Haxell, Nagle, and Rödl is algorithmic, we also obtain an algorithmic version of Gowers' regularity lemma. Further, we use Gowers' analytic approach to the hypergraph regularity lemma to give a more direct proof of the algorithmic version of his regularity lemma.

On Algorithmic Hypergraph Regularity

By

Annika Poerschke
Master of Science, Emory University, 2006

Advisor: Vojtěch Rödl, Ph.D.
Advisor: Brendan Nagle, Ph.D.

A dissertation submitted to the Faculty of the Graduate School of Emory University
in partial fulfillment of the requirements of the degree of
Doctor of Philosophy
in Mathematics
2008

Acknowledgments

I would like to thank both of my advisors Vojtěch Rödl and Brendan Nagle for the mathematics they taught me and for their valuable advice. In addition, I would like to thank Vojtěch Rödl for providing the topic of this dissertation and for sharing his ideas with me. I owe special thanks to Mathias Schacht who served as a member on my committee. I want to thank him for his multiple careful reading of this thesis. It is with no doubt that his comments and suggestions have improved the quality of this thesis. Thanks to his commitment this dissertation was finished on time.

I am also very grateful to the committee members, Dwight Duffus and Ron Gould for their valuable comments.

Moreover, I would like to express my appreciation to the people who invented and continue the exchange program with the Technical University Berlin in Germany. In particular, I would like to thank Vladimir Oliker, Ron Gould, and James G. Nagy from Emory University and Udo Simon and Alexander I. Bobenko from the Technical University Berlin. Furthermore, I would like to thank James G. Nagy for his extraordinary work as a Director of Graduate Studies.

I am very grateful to the graduate students and the staff of the Mathematics Department at Emory University for providing such a stimulating work environment. In particular, I would like to give special thanks to Silke Gehrke for her friendship and support in finishing this dissertation.

Last but not least, I would like to thank my family. In particular, I want to thank my common-law spouse Nader for his patience and unconditional support during this stressful time. Finally, I owe many thanks to my newborn daughter Leopoldina who was always able to put a smile on my face even in the most difficult moments.

to Leopoldina Poerschke

Contents

List of Figures

List of Tables

Chapter 1

Introduction

In 1975, Szemerédi proved his well-known *Regularity Lemma* [34] that is of fundamental importance in combinatorics and graph theory. His lemma helped in proving many results especially in extremal graph theory. See [22] for an excellent survey on the applications of the regularity lemma. Roughly speaking, the lemma says that we can decompose the vertex set of any graph into a fixed number of classes such that the graphs induced on almost every pair of the partition classes behave like random graphs. The essential concept involved in Szemerédi's regularity lemma is the notion of an ε-regular pair. For a graph $G = (V, E)$ (we refer the reader to [10] for basic graph theory definitions and concepts) with $A, B \subseteq V$ nonempty and disjoint, we let $e(A, B) = |E(A, B)|$ be the number of edges between A and B in G. Furthermore, we set $d(A, B) = e(A, B)/|A||B|$ as the *density* of the bipartite graph $(A \cup B, E(A, B))$. We say the pair (A, B) is *ε-regular* if for some positive ε and any $A' \subseteq A$, $B' \subseteq B$ satisfying $|A'| \geq \varepsilon|A|$ and $|B'| \geq \varepsilon|B|$ we have $|d(A', B') - d(A, B)| < \varepsilon$. Now, we are able to state the regularity lemma.

Theorem 1.1 (Regularity Lemma [34]) *For all $\varepsilon > 0$ and $t_0 \in \mathbb{N}$, there are two integers $T_0 = T_0(\varepsilon, t_0)$ and $n_0 = n_0(\varepsilon, t_0)$, such that for every graph $G = (V, E)$ with $|V| = n \geq n_0$, V admits a partition into $t + 1$ classes $V = V_0 \cup V_1 \cup \cdots \cup V_t$ with $t_0 \leq t \leq T_0$ satisfying the following:*

(i) $|V_0| \leq \varepsilon|V|$,

(ii) $|V_1| = \cdots = |V_t|$, *and*

(iii) all but at most εt^2 of the pairs (V_i, V_j) with $1 \leq i < j \leq t$ are ε-regular.

The original proof of Theorem 1.1 is non-constructive. But in 1994 Alon, Duke, Lefmann, Rödl, and Yuster [1, 2] were able to give an algorithmic proof of the regularity lemma which has already been applied to design algorithms for various combinatorial problems. Applications include for instance approximation algorithms for the max-cut problem [13] and a fast algorithm for computing the frequency of a subgraph [11]. Many of these problems have generalizations for hypergraphs. Therefore, the question whether we can extend the regularity concept to hypergraphs arises naturally.

In the hypergraph case regularity can be measured differently. Various authors were able to establish several hypergraph regularity lemmas including Frankl and Rödl [12], Haxell, Nagle, Rödl, Schacht and Skokan [17, 24, 30, 31, 32], Gowers [14, 15] and Tao [36]. Applications of these hypergraph extensions have been considered in [3, 8, 9, 12, 14, 15, 20, 23, 25, 24, 26, 29, 30, 27, 28, 33, 35]. Gowers used the notion of a (functional) quasirandom hypergraph in his proof of the hypergraph regularity lemma." In fact, Thomason [37] and Chung, Graham and Wilson [6] were the first to investigate the properties of so-called quasirandom graphs. In particular, the latter authors considered several properties of random-like graphs of density 1/2 and showed that they are all equivalent in a deterministic sense.

In this thesis we focus on 3-uniform hypergraphs and in particular, we consider the hypergraph regularity lemmas of Gowers [14] and Haxell, Rödl, and Nagle [17]. In fact we show that their regularity concepts are equivalent, i.e. we show that minimality (used by Haxell et al.) and (functional)-quasirandomness (introduced by Gowers) are equivalent (see Theorem 3.10). Our proof relies, in both directions, on the so-called hypergraph counting lemmas (see Theorem 3.11 and Theorem 3.12), which correspond to the regularity lemmas of Gowers and Haxell et al. [14, 17]. As a consequence, we infer an algorithmic version of Gowers' regularity lemma, using that the lemma of Haxell et al. is algorithmic. But we also use Gowers' elegant, analytic approach to the hypergraph regularity lemma to get a somewhat more direct proof of the algorithmic version of Gowers' regularity lemma for 3-uniform hypergraphs. For that we revisit his original proof that contains probabilistic arguments and provide a derandomized version thereof (see Theorem 4.1).

Organization of this thesis. In Chapter 2, we consider graphs and recall the equivalence of Gowers' (functional) quasirandomness concept for graphs [14] and ε-regularity [34]. In Chapter 3 we introduce 3-uniform hypergraphs and discuss the relation of Gowers' δ-quasirandomness and δ-minimality used in the paper of Haxell et al. [17] (see Theorem 3.10). In Chapter 4 we provide a constructive version of Gowers' quasirandom lemma (see Theorem 4.1).

Chapter 2

Graphs

2.1 Quasirandomness

2.1.1 Basic Definitions and Concepts

We start this section with a review of various notions of quasirandomness for graphs, and will point out their relations. Our objects of interest in this chapter are simple graphs, without loops and multiple edges, and we refer the reader to [10] for basic graph theoretic definitions and concepts. Thomason [37] was the first to study quasirandom graphs systematically. This research was continued by Chung, Graham, and Wilson [6] who investigated several properties of random-like graphs of density $d = 1/2$ and proved that they are all equivalent. For our purpose we will restrict ourselves to bipartite graphs here.

Definition 2.1 (($K^{(2)}_{2,2}, \varepsilon$)-minimality) *Let G be a bipartite graph with vertex partition $V_1 \cup V_2$ and density d_{12}. G is called ε-*quasirandom *if its number of labelled 4-cycles is at most $(d_{12}^4 + \varepsilon)|V_1|^2|V_2|^2$.*

We call this definition $(K^{(2)}_{2,2}, \varepsilon)$-minimality since it is always the case that for a bipartite graph G, defined as in Definition 2.1, the following holds:

$$|K^{(2)}_{2,2}(G)| \geq (d_{12}^4 - o(1))|V_1|^2|V_2|^2$$

where $o(1) \to 0$ as $\min\{|V_1|, |V_2|\} \to \infty$ and $K^{(2)}_{2,2}(G)$ denotes the family of all 4-cycles in G.

The next concept of quasirandomness was introduced by Gowers in [14].

Definition 2.2 *Let V_1 and V_2 be sets with $|V_1| = m_1$ and $|V_2| = m_2$, respectively. A* function $f : V_1 \times V_2 \to [-1, 1]$ *is called* γ-quasirandom *if*

$$\sum_{v_1, v_1' \in V_1} \sum_{v_2, v_2' \in V_2} f(v_1, v_2) f(v_1', v_2) f(v_1, v_2') f(v_1', v_2') \leq \gamma m_1^2 m_2^2.$$

The above definition gives rise to the concept of functional quasirandomness for graphs.

Definition 2.3 ((d_{12}, γ)- Quasirandomness) *Let G be a bipartite graph with vertex partition $V_1 \cup V_2$ where $|V_1| = m_1$, $|V_2| = m_2$, and density d_{12}. G is called* (d_{12}, γ)-quasirandom *if the function $g : V_1 \times V_2 \to [-1, 1]$ with $g(v_1, v_2) = G(v_1, v_2) - d_{12}$ is γ-quasirandom, where $G(v_1, v_2)$ is the characteristic function on the edges of G, i.e. $G(V_1, V_2) = 1$ if $\{v_1, v_2\} \in E(G)$ and* 0 *otherwise.*

Whenever we talk about quasirandom graphs in the later sections we will assume that these graphs are actually (d_{12}, γ)-quasirandom and we will omit the term (d_{12}, γ) for convenience if the underlying parameters are clear from the context.

2.1.2 Relation of the Concepts

In this section, we recall the equivalence of the two quaisrandom notions from above. More precisely, we show that $(K_{2,2}^{(2)}, \varepsilon)$-minimality (Definition 2.1) and the (d, γ)-quasirandomness (Definition 2.3) are equivalent (see Lemma 2.5).For the meaning of 'equivalence' here we refer the reader to the discussion after Lemma 2.4. Before we state the lemma let us refer to a result by Chung et al. As mentioned before these authors were able to show that several random-like properties are equivalent. We will state here the one that is of particular interest for us. Although the authors showed it only for graphs of density $d = 1/2$, one can easily generalize it for any arbitrary but fixed density d. Since we only consider bipartite graphs here the result is slightly different from that of the above authors, but the same techniques are applied to prove it.

Lemma 2.4 *Let G be a bipartite graph with vertex partition $V_1 \cup V_2$. Suppose further that G has density d_{12}. Then the following properties are equivalent:*

(1) $(K^{(2)}_{2,2}, \varepsilon)$*-minimality;*

(2) if $V'_1 \subseteq V_1$ *and* $V'_2 \subseteq V_2$ *then* $|e(V'_1, V'_2) - d_{12}|V'_1||V'_2|| \leq \varepsilon' |V_1||V_2|$.

For fixed density d_{12} the equivalence here means that for any $\varepsilon' > 0$ there exists an $\varepsilon > 0$ such that if (i) holds for ε then (ii) holds for ε' and vice versa for every $\varepsilon > 0$ exists $\varepsilon' > 0$ such that if (2) holds for ε', then (1) holds for ε. We refer to (2) as ε'*-regularity.* Although it is slightly different from Szemerédi's definition of an ε-regular pair (see Introduction), i.e. we do not have the restrictions on the size of the subsets, however one can easily show that both concepts are equivalent in the sense defined above.

The next lemma states the equivalence of $(K^{(2)}_{2,2}, \varepsilon)$-minimality and (d_{12}, γ)-quasirandomness.

Lemma 2.5 *Let G be a bipartite graph with vertex partition $V_1 \cup V_2$ where $|V_1| = m_1$ and $|V_2| = m_2$. Suppose further that G has density d_{12}. Then the following properties are equivalent:*

(i) G is $(K^{(2)}_{2,2}, \varepsilon)$*-minimal;*

(ii) G is ε'*-regular;*

(iii) G is (d_{12}, γ)*-quasirandom.*

Proof. Properties (i) and (ii) are equivalent due to Lemma 2.4. In order to prove that $(ii) \Leftrightarrow (iii)$ we refer to a result by Gowers. In fact, in [14] he proved that several properties are equivalent for two sets. Here we will only state the relevant parts needed to conclude the proof of Lemma 2.5.

Lemma 2.6 *Let V_1 and V_2 be sets where $|V_1| = m_1$ and $|V_2| = m_2$, and let* $g : V_1 \times V_2 \to [-1, 1]$. *Then the following are equivalent:*

(a) $\sum_{v_1, v'_1 \in V_1} \sum_{v_2, v'_2 \in V_2} g(v_1, v_2) g(v'_1, v_2) g(v_1, v'_2) g(v'_1, v'_2) \leq \gamma m_1^2 m_2^2$;

(b) For any pair of sets $V_1' \subseteq V_1$ and $V_2' \subseteq V_2$ we have the inequality

$$\left| \sum_{v_1' \in V_1'} \sum_{v_2' \in V_2} g(v_1', v_2') \right| \leq \varepsilon' m_1 m_2.$$

Suppose we are given a bipartite graph G with vertex partition $V_1 \cup V_2$, where $|V_1| = m_1$ and $|V_2| = m_2$, and suppose that the density of G is d_{12}. Let further $g : V_1 \times V_2 \to [-1, 1]$ with $g(v_1, v_2) = G(v_1, v_2) - d_{12}$. With this definition of g we also have

$$e(V_1', V_2') = \sum_{v_1' \in V_1'} \sum_{v_2' \in V_2'} G(v_1', v_2') = \sum_{v_1' \in V_1'} \sum_{v_2' \in V_2'} g(v_1', v_2') + d_{12}|V_1'||V_2'|.$$

Consequently,

$$\begin{aligned} \left| e(V_1', V_2') - d_{12}|V_1'||V_2'| \right| &\leq \sum_{v_1' \in V_1'} \sum_{v_2' \in V_2'} g(v_1', v_2') \\ &\leq \varepsilon' m_1 m_2 \end{aligned}$$

and, therefore, ε'-regularity is equivalent to property (a) of Lemma 2.6, which for the choice of g coincides with (d_{12}, γ)-quasirandomness.

■

2.2 Regularity Lemma

In this section we state an algorithmic version of Gowers' regularity lemma for graphs. Due to the equivalence of regularity and quasirandomness stated in Lemma 2.5 the algorithmic version is a direct consequence of the result of Alon et al., which establishes an algorithmic version of Szemerédi's regularity lemma. Before we state the algorithm let us introduce some notation. For a given t-partite graph $G = (V_1, \ldots, V_t, E)$, we denote by $G[V_i, V_j]$ the induced bipartite subgraph on V_i and V_j. Let further $K[V_i, V_j]$ denote the complete bipartite graph on V_i and V_j. We are now ready to state the algorithmic regularity lemma for graphs.

Algorithm 2.7 (Algorithmic Regularity Lemma for Graphs)

Input:

(i) $\varepsilon > 0$ *and integers* ℓ_0, t_0*;*

(ii) $U = U_1 \cup \cdots \cup U_{t_0}$, $m = |U_1| \leq \cdots \leq |U_{t_0}| \leq m + 1$*;*

(iii) $K[U_i, U_j] = G_1^{ij} \cup \cdots \cup G_{\ell_{ij}}^{ij}$, $1 \leq i < j \leq t_0$, *where* $\ell_{ij} \leq \ell_0$.

Output:

(1) *Constants* $T_0 = T_0(\varepsilon, \ell_0, t_0)$ *and* $N_0 = N_0(\varepsilon, \ell_0, t_0)$*;*

(2) *refined vertex partition* $U_i = U_{i1} \cup \cdots \cup U_{it}$, $1 \leq i \leq t_0$ *so that if* $m > N_0$, *then*

(a) *for each* $1 \leq i < j \leq t_0$ *and* $1 \leq i', j' \leq t$, $||U_{ii'}| - |U_{jj'}|| \leq 1$*;*

(b) *all but* $\varepsilon(t_0 t)^2 \ell_0$ *bipartite subgraphs* $G_a^{ij}[U_{ii'}, U_{jj'}]$, $1 \leq i < j \leq t_0$, $1 \leq a \leq \ell_{ij}$, $1 \leq i', j' \leq t$, *are* ε*-quasirandom.*

Complexity: $O(m^{2.376})$.

The running time of this algorithm can be improved to $O(m^2)$ as shown in [19].

2.3 Counting Lemma

In this section we state the graph counting lemma for the special case of triangles for ε-regularity introduced by Szemerédi. Roughly speaking it says that a sufficiently ε-regular tripartite graph contains approximately as many triangles as one would expect in a random graph with the corresponding density. Gowers formally showed in [14] a graph counting lemma for quasirandom graphs but due to the equivalence of ε-regularity and ε-quasirandomness discussed earlier we may simply apply the well-known graph counting lemma for ε-regular graphs although we consider ε-quasirandom graphs.

Theorem 2.8 (Triangle Counting Lemma) *Let $\eta > 0$, then there exists $\varepsilon > 0$ such that if G is a 3-partite graph with vertex 3-partition $V_1 \cup V_2 \cup V_3$ and $|V_1| = |V_2| = |V_3| = m$ where the bipartite graphs $G^{ij} = G[V_i, V_j]$ are (d_{ij}, ε)-regular $1 \leq i < j \leq 3$, then:*

$$|K_3(G)| = (1 \pm \eta) d_{12} d_{23} d_{13} m^3,$$

where $K_3(G)$ denotes the set of vertex triplets of G, which span a triangle in G.

Chapter 3

Equivalence Relations of Quasirandomness for 3-uniform Hypergraphs

3.1 Quasirandomness

3.1.1 Basic Definitions and Concepts

We will start this section with some basic definitions. From now on we will only consider 3-uniform hypergraphs. Therefore, let us introduce some notation. We denote by $[n]$ the set $\{1, \ldots, n\}$. Suppose V is a set, then $[V]^3$ denotes all 3-element subsets of V. A *3-uniform hypergraph* (also called *3-graph*) $\mathcal{H}$ is a pair (V, E) where $E(\mathcal{H}) \subseteq [V]^3$. The central objects in this thesis are tripartite 3-uniform hypergraphs with vertex partition $V = V_1 \cup V_2 \cup V_3$, $|V_1| = m_1$, $|V_2| = m_2$, and $|V_3| = m_3$. As for graphs we also want to define quasirandomness for 3-uniform hypergraphs. Similar to Definition 2.1 where we considered labelled 4-cycles in a graph we define for tripartite 3-uniform hypergraphs a so-called octahedron $\mathcal{O}$, where $V(\mathcal{O})$ corresponds to (distinct) vertices $v_1, v_1' \in V_1$, $v_2, v_2' \in V_2$, and $v_3, v_3' \in V_3$ that span an 'ordered' copy of $K^{(3)}_{2,2,2}$, i.e., the complete tripartite 3-uniform hypergraph with vertex classes $\{v_1, v_1'\}, \{v_2, v_2'\}$, and $\{v_3, v_3'\}$.

Our main objects of interests are so-called *quasirandom* 3-uniform hypergraphs. You can find many different definitions of quasirandom 3-uniform hypergraphs in the literature. Here we will mention three definitions of a quasirandom 3-uniform hypergraph. These concepts can be viewed as generalizations of the concepts for graphs discussed in Chapter 2. Moreover, as in Chapter 2 we will show that these definitions are all equivalent.

Let us start with the most natural definition of an $(K^{(3)}_{2,2,2}, \eta)$-minimal 3-uniform hypergraph. All definitions we consider here are stated for tripartite 3-uniform hypergraphs. For convenience, let us define the (absolute) density of a 3-uniform hypergraph. Suppose we are given a 3-uniform hypergraph $\mathcal{H}$ with vertex partition $V_1 \cup V_2 \cup V_3$ and $|V_1| = m_1$, $|V_2| = m_2$, and $|V_3| = m_3$. Then the *(absolute) density of* $\mathcal{H}$ is defined as $d_{123}(\mathcal{H}) = |\mathcal{H}|/m_1m_2m_3$, where $|\mathcal{H}|$ denotes the number of edges in $\mathcal{H}$.

Definition 3.1 ($(K^{(3)}_{2,2,2}, \eta)$-minimality) *Let $\mathcal{H}$ be a tripartite 3-uniform hypergraph with vertex partition $V_1 \cup V_2 \cup V_3$, $|V_1| = m_1$, $|V_2| = m_2$, and $|V_3| = m_3$. Suppose that $\mathcal{H}$ has (absolute) density $d_{123}(\mathcal{H})$. Then $\mathcal{H}$ is* $(K^{(3)}_{2,2,2}, \eta)$-minimal $(\eta > 0)$ *if it contains at most $(d_{123}(\mathcal{H})^8 + \eta)(m_1m_2m_3)^2$ octahedra.*

Note that $\mathcal{H}$ always contains at least $(d_{123}(\mathcal{H})^8 - o(1))(m_1m_2m_3)^2$ many octahedra (cf. [17]) which motivates the name of this definition.

The next concept is called discrepancy in the literature. Before we state it, let us introduce another convenient notation. For a given graph G we write $K_3(G)$ for the family of triangles in G:

$$K_3(G) = \left\{\{v_1, v_2, v_3\} \in \binom{V}{3} : \{v_1, v_2\}, \{v_1, v_3\}, \{v_2, v_3\} \in G\right\}.$$

Definition 3.2 (Discrepancy) *Let $\mathcal{H}$ be a tripartite 3-uniform hypergraph with vertex partition $V_1 \cup V_2 \cup V_3$, of sizes $|V_1| = m_1$, $|V_2| = m_2$, and $|V_3| = m_3$. The* discrepancy $disc(\mathcal{H})$ of $\mathcal{H}$ *is defined to be*

$$disc(\mathcal{H}) = \frac{1}{m_1m_2m_3} \max_{G \subseteq [V]^2} ||\mathcal{H} \cap K_3(G)| - d_{123}(\mathcal{H})|K_3(G)||$$

where $d_{123}(\mathcal{H})$ is the (absolute) density of $\mathcal{H}$ and the maximum is taken over all graphs G with vertex set V.

As in the case of graphs we would like to establish a concept involving functions. Gowers formulated such definitions in [14] . But before we state the definitions let us introduce a convenient notation. Suppose we are given sets V_1, V_2, and V_3. If $f : V_1 \times V_2 \times V_3 \to \mathbb{R}$ is any function of three variables v_1, v_2, v_3 then

$$\begin{aligned} f_{v_1 v_1' v_2 v_2' v_3 v_3'} =& f(v_1, v_2, v_3) f(v_1', v_2, v_3) f(v_1, v_2', v_3) f(v_1, v_2, v_3') \\ & f(v_1', v_2', v_3) f(v_1, v_2', v_3') f(v_1', v_2, v_3') f(v_1', v_2', v_3'). \end{aligned}$$

We are now ready to define the notion of a quasirandom function.

Definition 3.3 *Let V_1, V_2, and V_3 be sets of sizes m_1, m_2, and m_3 and let $f : V_1 \times V_2 \times V_3 \to [-1, 1]$. We say f* is η-quasirandom *if*

$$\sum_{v_1, v_1' \in V_1} \sum_{v_2, v_2' \in V_2} \sum_{v_3, v_3' \in V_3} f_{v_1 v_1' v_2 v_2' v_3 v_3'} \leq \eta (m_1 m_2 m_3)^2.$$

The above definition allows to give a definition of a quasirandom hypergraph.

Definition 3.4 (η''-Functional Quasirandomness) *Let $\mathcal{H}$ be a tripartite 3-uniform hypergraph with vertex partition $V_1 \cup V_2 \cup V_3$, where $|V_1| = m_1$, $|V_2| = m_2$, and $|V_3| = m_3$. Suppose $\mathcal{H}$ has (absolute) density $d_{123}(\mathcal{H})$ and let*

$$h(v_1, v_2, v_3) = \mathcal{H}(v_1, v_2, v_3) - d_{123}(\mathcal{H}),$$

where $\mathcal{H}(v_1, v_2, v_3)$ is the characteristic function on the edges of $\mathcal{H}$. We say that $\mathcal{H}$ is η''-(functionally) quasirandom *if h is η''-quasirandom.*

3.1.2 The Equivalence of Several Versions of Quasirandomness for 3-uniform Hypergraphs

As in the graph case we would like to establish a relation among the concepts we introduced before. In fact, the next lemma establishes the equivalence of the three concepts of Section 3.1.1.

Lemma 3.5 *For a tripartite 3-uniform hypergraph $\mathcal{H}$ the following statements are equivalent:*

(i) $\mathcal{H}$ *is* $(octahedral, \eta)$*-minimal;*

(ii) $disc(\mathcal{H}) \leq \eta'$*;*

(iii) $\mathcal{H}$ *is* η''*-(functional) quasirandom.*

Proof. The equivalence of (i) and (ii) in Lemma 3.5 was shown by Kohayakawa, Rödl, and Skokan [21], who extended results of Chung and Graham [4, 5, 7]. The latter authors investigated hypergraphs of density 1/2 and proved some equivalences for that case. Kohayakawa et al. generalized those results for arbitrary densities. In particular, they considered six different properties (including $K^{(3)}_{2,2,2}$-minimality and discrepancy) of random-like hypergraphs and showed that they are all equivalent. Although they did not show it explicitly in the case of partite hypergraphs their techniques still apply. Consequently, it remains to show the equivalence of property (ii) and (iii). Again, we refer to an already established result. The proof of the lemma we use can be found in [14]. We only state the relevant parts for our purpose here.

Lemma 3.6 *Let* V_1*,* V_2*, and* V_3 *be sets, where* $|V_1| = m_1$*,* $|V_2| = m_2$*, and* $|V_3| = m_3$ *and let* $h : V_1 \times V_2 \times V_3 \to [-1, 1]$*. Then the following statements are equivalent.*

(a) $\sum_{v_1,v_1',v_2,v_2',v_3,v_3'} h_{v_1v_1'v_2v_2'v_3v_3'} \leq \eta'' m_1^2 m_2^2 m_3^2$,

(b) *For any tripartite graph* G *with vertex partition* $V_1 \cup V_2 \cup V_3$,

$$\sum_{(v_1,v_2,v_3)\in K_3(G)} h(v_1, v_2, v_3) \leq \eta' m_1 m_2 m_3.$$

For $h(v_1, v_2, v_3) = \mathcal{H}(v_1, v_2, v_3) - d_{123}(\mathcal{H})$, where $d_{123}(\mathcal{H})$ is the (absolute) density of a 3-uniform hypergraph $\mathcal{H}$ property (i) corresponds to the concept of η''-(functional) quasirandomness whereas property (iii) corresponds to the η'-discrepancy concept. Hence Lemma 3.5 follows.

∎

3.2 Relative quasirandomness

3.2.1 Definitions and Concepts

In order to state the basic definition behind the hypergraph regularity lemma, we will need to extend the concept of quasirandomness and define quasirandomness of a hypergraph relative to a graph. In what follows we shall consider a tripartite 3-uniform hypergraph $\mathcal{H} \subseteq K_3(G)$, where G is the underlying tripartite graph. In order to define what it means for a hypergraph $\mathcal{H}$ to be quasirandom relative to a graph G, we need to consider the (relative) density of $\mathcal{H}$ w.r.t. G instead of the (absolute) density. More precisely, if $G = G^{12} \cup G^{23} \cup G^{13}$ is a tripartite graph and $\mathcal{H}$ is a tripartite 3-uniform hypergraph with $\mathcal{H} \subseteq K_3(G)$, then the *relative density of* $\mathcal{H}$ *w.r.t.* G is

$$\alpha = d(\mathcal{H}|G) = |\mathcal{H}|/|K_3(G)|.$$

We can now define what it means for $\mathcal{H}$ to "sit quasirandomly" in G, where $G = G^{12} \cup G^{23} \cup G^{13}$ consists of three ε-quasirandom bipartite graphs.

Definition 3.7 *Let G be a tripartite graph as above with vertex partitions V_1, V_2, and V_3, where $|V_1| = m_1$, $|V_2| = m_2$, and $|V_3| = m_3$. Furthermore, let $f : V_1 \times V_2 \times V_3 \to [-1, 1]$ be a function that is supported on $K_3(G)$. Let the densities be $d(G^{ij}) = d_{ij}$, $1 \le i < j \le 3$. Then f is δ-quasirandom w.r.t. G if*

$$\sum_{v_1, v_1' \in V_1} \sum_{v_2, v_2' \in V_2} \sum_{v_3, v_3' \in V_3} f_{v_1 v_1' v_2 v_2' v_3 v_3'} < \delta (d_{12} d_{23} d_{13})^4 (m_1 m_2 m_3)^2.$$

Finally, we are ready to define what it means for a hypergraph to be quasirandom relative to a graph. As in the graph case we will consider here a special function f involving the (relative) density of the hypergraph.

Definition 3.8 ((α, δ)-quasirandomness) *Let G be defined as in Definition 3.7 and let $\mathcal{H}$ be a tripartite 3-uniform hypergraph with $\mathcal{H} \subseteq K_3(G)$ and (relative) density α. Furthermore, let $h(v_1, v_2, v_3) = \mathcal{H}(v_1, v_2, v_3) - \alpha$ for*

$(v_1, v_2, v_3) \in K_3(G)$ *and* 0 *otherwise where* $\mathcal{H}(v_1, v_2, v_3) = 1$ *if* $\{v_1, v_2, v_3\} \in \mathcal{H}$ *and* 0 *otherwise. Then* $\mathcal{H}$ *is* (α, δ)*-quasirandom w.r.t.* G *if* h *is* δ*-quasirandom w.r.t.* G*, i.e.*

$$\sum_{v_1, v_1' \in V_1} \sum_{v_2, v_2' \in V_2} \sum_{v_3, v_3' \in V_3} h_{v_1 v_1' v_2 v_2' v_3 v_3'} < \delta (d_{12} d_{23} d_{13})^4 (m_1 m_2 m_3)^2.$$

(Here and throughout the thesis, we use the notation $\sum_{x, x' \in X}$ to denote a sum of *ordered* pairs.)

As in the sections before, there is also an equivalent concept, called (α, δ)-minimality, to Gowers (α, δ)-quasirandomness. Next, we will define the (α, δ)-minimality concept considered by Haxell, Nagle, and Rödl [17]. To that end, let $G = G^{12} \cup G^{23} \cup G^{13}$ be a tripartite graph, consisting of three (d_{ij}, ε)-regular (or equivalently $((d_{ij}, \varepsilon')$-quasirandom), $1 \leq i < j \leq 3$, bipartite graphs, and let $\mathcal{H} \subseteq K_3(G)$ be a tripartite, 3-uniform hypergraph with (relative) density α. Let $K^{(3)}_{2,2,2}(\mathcal{H})$ denote a copy of the complete tripartite, 3-uniform hypergraph with two vertices in each partition class in $\mathcal{H}$.

Now we are ready to define the (α, δ)-minimality concept.

Definition 3.9 ((α, δ)-minimality) *Let* G *and* $\mathcal{H}$ *be defined as above. For* $\delta > 0$*, the hypergraph* $\mathcal{H}$ *is* (α, δ)-minimal *w.r.t.* G *if*

$$|K^{(3)}_{2,2,2}(\mathcal{H})| \leq \alpha^8 d_{12}^4 d_{23}^4 d_{13}^4 \binom{m_1}{2} \binom{m_2}{2} \binom{m_3}{2} (1 + \delta).$$

Note that since each G^{ij} is (d_{ij}, ε)-regular, $1 \leq i < j \leq 3$, standard convexity and double-counting arguments (see Proposition 4.1 (p.1744) of [17]) show that

$$|K^{(3)}_{2,2,2}(\mathcal{H})| \geq \alpha^8 d_{12}^4 d_{23}^4 d_{13}^4 \binom{m_1}{2} \binom{m_2}{2} \binom{m_3}{2} (1 - f(\varepsilon)), \tag{3.1}$$

where $f(\varepsilon) \to 0$ as $\varepsilon \to 0$.

3.2.2 The Equivalence Statement

In this section we state the relation of the concepts of (α, δ)-quasirandomness and (α, δ)-minimality that were introduced in the previous section. It is one of the main results of this thesis.

Theorem 3.10 *For all $\alpha, \delta_1 > 0$, there exists $\delta_2 > 0$ so that for all $d_0 > 0$, there exist an $\varepsilon > 0$ and an integer m_0 such that the following holds:*

(i) Let $G = G^{12} \cup G^{23} \cup G^{13}$ be a tripartite graph with vertex partition $V(G) = V_1 \cup V_2 \cup V_3$ with $|V_1| = |V_2| = |V_3| = m \geq m_0$.

(ii) Let each $G^{ij} = G[V_i, V_j]$, $1 \leq i < j \leq 3$ be (d_{ij}, ε)-quasirandom for some $d_{ij} \geq d_0$.

(iii) Let $\mathcal{H} \subseteq K_3(G)$ be a 3-uniform hypergraph with (relative) density $d(\mathcal{H}|G) = \alpha$.

Then,

(1) if $\mathcal{H}$ is (α, δ_2)-quasirandom w.r.t. G, then $\mathcal{H}$ is also (α, δ_1)-minimal w.r.t. G;

(2) if $\mathcal{H}$ is (α, δ_2)-minimal w.r.t. G, then $\mathcal{H}$ is also (α, δ_1)-quasirandom w.r.t. G.

Observe that the main difference between Lemma 3.5 and Theorem 3.10 is that Theorem 3.10 includes the case when the underlying graph G is fairly sparse, i.e., if $d_{ij} \ll \delta$.

3.2.3 Counting Lemmas and Auxiliary Facts

The proof of Theorem 3.10 relies mainly on Counting Lemmas established by Gowers (c.f. [14]) and Haxell et al. (c.f. [17]). Both Counting Lemmas will estimate the number of hypercliques $K_k^{(3)}$ $(k \geq 3)$ in an appropriately quasirandom (respectively minimal) environment.

We now state Gowers' Counting Lemma for the concept of (α, δ)-quasirandom hypergraphs.

Theorem 3.11 (Quasirandom Counting Lemma) *For all integers $k \geq 3$, $\mu > 0$, and $\alpha_0 > 0$, there exists $\delta > 0$ so that for all $d_0 > 0$, there exist $\varepsilon > 0$ and an integer n_0 so that the following holds:*

(i) Let $P = \bigcup_{1 \leq i < j \leq k} P^{ij}$ be a k-partite graph with vertex partition $V(P) = U_1 \cup \cdots \cup U_k$ with $|U_1| = \cdots = |U_k| = n \geq n_0$.

(ii) Let each $P^{ij} = P[U_i, U_j]$, $1 \leq i < j \leq k$ be (d_{ij}, ε)-quasirandom for some $d_{ij} \geq d_0$.

(iii) Let $\mathcal{J} = \bigcup_{1 \leq h < i < j \leq k} \mathcal{J}^{hij} \subseteq K_3(G)$ be a k-partite 3-uniform hypergraph that satisfies for each $1 \leq h < i < j \leq k$, $\mathcal{J}^{hij}$ is (α_{hij}, δ)-quasirandom w.r.t. $P^{hi} \cup P^{ij} \cup P^{hj}$, where $\alpha_{hij} \geq \alpha_0$.

Then,

$$|K_k^{(3)}(\mathcal{J})| = (1 \pm \mu) \prod_{1 \leq h < i < j \leq k} \alpha_{hij} \times \prod_{1 \leq i < j \leq k} d_{ij} \times n^k.$$

We will now state a generalized version of Haxell's et al. Counting Lemma for the concept of (α, δ)-minimality.

Theorem 3.12 (Minimality Counting Lemma) *For all integers $k \geq 3$, $\mu > 0$, and $\alpha_0 > 0$, there exists $\delta > 0$ so that for all $d_0 > 0$, there exist $\varepsilon > 0$ and an integer n_0 so that the following holds:*

(i) Let $P = \bigcup_{1 \leq i < j \leq k} P^{ij}$ be a k-partite graph with vertex partition $V(P) = U_1 \cup \cdots \cup U_k$ with $|U_1| = \cdots = |U_k| = n \geq n_0$.

(ii) Let each $P^{ij} = P[U_i, U_j]$, $1 \leq i < j \leq k$ be (d_{ij}, ε)-regular for some $d_{ij} \geq d_0$.

(iii) Let $\mathcal{J} = \bigcup_{1 \leq h < i < j \leq k} \mathcal{J}^{hij} \subseteq K_3(G)$ be a k-partite 3-uniform hypergraph that satisfies for each $1 \leq h < i < j \leq k$, $\mathcal{J}^{hij}$ is (α_{hij}, δ)-minimal w.r.t. $P^{hi} \cup P^{ij} \cup P^{hj}$, where $\alpha_{hij} \geq \alpha_0$.

Then,

$$|K_k^{(3)}(\mathcal{J})| = (1 \pm \mu) \prod_{1 \le h < i < j \le k} \alpha_{hij} \times \prod_{1 \le i < j \le k} d_{ij} \times n^k.$$

As mentioned before, this is a generalized version of the counting lemma found in [17]. There, Haxell, Nagle, and Rödl proved the special case where each $\alpha_{hij} = \alpha = \alpha_0$ for all $1 \le h < i < j \le k$ and each $d_{ij} = d = 1/\ell$, for an integer ℓ, for all $1 \le i < j \le k$. The fact that each d_{ij} can be taken as the reciprocal of a common integer ℓ is a convenience afforded by the corresponding regularity lemma in [17], as well as the original regularity lemma of Frankl and Rödl [12]. With only symbolic alterations, the proof of Haxell, Nagle, and Rödl [17] would establish Theorem 3.12. However, one can actually deduce Theorem 3.12 from the special case that all $\alpha_{hij} = \alpha$ and all $d_{ij} = 1/\ell$ (see Theorem 3.13). In this thesis, we will deduce Theorem 3.12 from Theorem 3.13 and refer the reader to [17] for a proof of Theorem 3.13 stated below.

Theorem 3.13 (Minimality Counting Lemma-Special Case) *For all integers $k \ge 3$, $\gamma > 0$, and $\alpha \in (0, 1]$, there exists $\delta > 0$ so that for all integers l, there exist $\varepsilon > 0$ and an integer n_0 so that the following holds:*

(i) Let $P = \bigcup_{1 \le i < j \le k} P^{ij}$ be a k-partite graph with vertex partition $V(P) = U_1 \cup \cdots \cup U_k$ with $|U_1| = \cdots = |U_k| = n \ge n_0$.

(ii) Let each $P^{ij} = P[U_i, U_j]$, $1 \le i < j \le k$ be $(1/\ell, \varepsilon)$-regular.

(iii) Let $\mathcal{J} = \bigcup_{1 \le h < i < j \le k} \mathcal{J}^{hij} \subseteq K_3(G)$ be a k-partite 3-uniform hypergraph that satisfies for each $1 \le h < i < j \le k$, $\mathcal{J}^{hij}$ is (α, δ)-minimal w.r.t. $P^{hi} \cup P^{ij} \cup P^{hj}$.

Then,

$$|K_k^{(3)}(\mathcal{J})| = (1 \pm \gamma) \frac{\alpha^{\binom{k}{3}}}{\ell^{\binom{k}{2}}} n^k.$$

The deduction of Theorem 3.12 from Theorem 3.13 consists of two steps:

S1: We first prove Theorem 3.12 with the (relative) hypergraph densities α_{hij}, $1 \le h < i < j \le k$, allowed to vary while the graph densities $d_{ij} = d = 1/\ell$, $1 \le i < j \le k$, remain constant.

S2: We use S1 to imply Theorem 3.12 then with the graph densities d_{ij}, $1 \le i < j \le k$ also allowed to vary.

Note that we chose the order of the above steps arbitrarily. We could have also performed them in reverse order. The proofs of both steps are very similar, both implementing standard 'random slicing' arguments (Chernoff and Chebyshev applications) together with subsequent applications of Theorem 3.13.

Proof of Theorem 3.12 (S1). We work with the following hierarchy of constants which is consistent with the quantifications of both Theorems 3.12 and 3.13:

$$\min\left\{\frac{1}{k}, \mu, \alpha_0\right\} \gg \gamma, \alpha \gg \delta \ge \min\{\delta, d_0 = d = 1/\ell\} \gg \varepsilon \gg \frac{1}{n_0} \ge \frac{1}{n} > 0, \tag{3.2}$$

where γ, $\alpha > 0$ are auxiliary constants defined in the context. With these constants, let the 3-uniform hypergraph $\mathcal{J}$ and graph P be given as in Theorem 3.12, where all $d_{ij} = d = d_0 = 1/\ell$, $1 \le i < j \le k$, for a fixed integer ℓ. Our goal is to find an estimate of $|K_k^{(3)}(\mathcal{J})|$. To that end, fix $1 \le h < i < j \le k$ and let $p = p_{hij} = \alpha/\alpha_{hij}$ and $s = s_{hij} = \lfloor 1/p_{hij} \rfloor$. Consider the following random partition

$$\mathcal{J}^{hij} = \mathcal{J}_0^{hij} \cup \mathcal{J}_1^{hij} \cup \cdots \cup \mathcal{J}_s^{hij}$$

obtained by independently including each triple (hyperedge) $g \in \mathcal{J}^{hij}$ in $\mathcal{J}_a^{hij}$ with probability

$$\mathbb{P}(g \in \mathcal{J}_a^{hij}) = \begin{cases} p & \text{if } 1 \le a \le s, \\ 1 - ps & \text{if } a = 0. \end{cases}$$

Next, we will apply Chernoff's inequality, see for example [18], to show that

$$\mathbb{P}(d(\mathcal{J}_a^{hij}|P^{hi}\cup P^{ij}\cup P^{hj})=\alpha\pm o(1))=1-o(1), \tag{3.3}$$

for every $1\le a\le s$. In fact, for fixed $1\le a\le s$ let $X_a=X_a^{hij}=|\mathcal{J}_a^{hij}|$. Observe that

$$\mathbb{E}(X_a)=p|\mathcal{J}^{hij}|=\alpha|K_3(P^{hi}\cup P^{ij}\cup P^{hj})|.$$

Using Chernoff's inequality and the triangle counting lemma applied to $P^{hi}\cup P^{ij}\cup P^{hj}$ now proves equation (3.3) for sufficiently large n.
We now use Chebyshev's inequality, see for example [18], to show that

$$\mathbb{P}(\mathcal{J}_a^{hij}\text{ is }(\alpha\pm o(1),2\delta)\text{-minimal w.r.t. }P^{hi}\cup P^{ij}\cup P^{hj})=1-o(1), \tag{3.4}$$

for every $1\le a\le s$. Therefore, for fixed $1\le a\le s$ it remains to determine $|K_{2,2,2}^{(3)}(\mathcal{J}_a^{hij})|$. Let $Y_a=Y_a^{hij}=|K_{2,2,2}^{(3)}(\mathcal{J}_a^{hij})|$. Then,

$$\begin{aligned}\mathbb{E}(Y_a)&=p^8|K_{2,2,2}^{(3)}(\mathcal{J}^{hij})|\overset{(3.1)}{=}\Omega(n^6)\quad\text{ and}\\ \mathrm{Var}(Y_a)&=(1-p^8)p^8|K_{2,2,2}^{(3)}(\mathcal{J}^{hij})|=O(n^9).\end{aligned}$$

By using Chebyshev's inequality we get that

$$\mathbb{P}(Y_a\ge(1+\delta^2)\mathbb{E}(Y_a))=\frac{O(n^9)}{\Omega(n^{12})}=O(n^{-3}).$$

Therefore, with probability $1-o(1)$, every $1\le a\le s$ satisfies

$$\begin{aligned}|K_{2,2,2}^{(3)}(\mathcal{J}_a^{hij})|&\le(1+\delta^2)p^8|K_{2,2,2}^{(3)}(\mathcal{J}^{hij})|\\ &\le(1+\delta+\delta^2+\delta^3)\alpha^8d^{12}\binom{n}{2}^3\\ &\le(1+2\delta)(\alpha\pm o(1))^8d^{12}\binom{n}{2}^3,\end{aligned}$$

proving (3.4).

Using (3.4) we now find a lower bound on the number of $K_k^{(3)}$'s in $\mathcal{J}$. Therefore, consider the set $\vec{A}=\prod_{1\le h<i<j\le k}[s_{hij}]$ and let π_{hij} be the projection onto the hij-th coordinate. For $\vec{a}\in\vec{A}$, set

$$\mathcal{J}_{\vec{a}}=\bigcup\{\mathcal{J}_{\pi_{hij}(\vec{a})}^{hij}:1\le h<i<j\le k\}.$$

By (3.4) we know, with high probability all $\mathcal{J}_{\vec{a}}$, $\vec{a} \in \vec{A}$, satisfy with the graph P the hypothesis of Theorem 3.13. As such,

$$\begin{aligned}
|K_k^{(3)}(\mathcal{J})| &\geq \sum_{\vec{a}\in\vec{A}} |K_k^{(3)}(\mathcal{J}_{\vec{a}})| \\
&\geq (1-\gamma)\alpha^{\binom{k}{3}} d^{\binom{k}{2}} n^k \prod_{1\leq h<i<j\leq k} s_{hij} \\
&\geq (1-\gamma)\alpha^{\binom{k}{3}} d^{\binom{k}{2}} n^k \prod_{1\leq h<i<j\leq k} \left(\frac{\alpha_{hij}}{\alpha} - 1\right) \\
&\geq (1-\gamma) d^{\binom{k}{2}} n^k \prod_{1\leq h<i<j\leq k} \alpha_{hij} \times \prod_{1\leq h<i<j\leq k} \left(1 - \frac{\alpha}{\alpha_{hij}}\right) \\
&\geq d^{\binom{k}{2}} n^k \prod_{1\leq h<i<j\leq k} \alpha_{hij} \times \left(1 - \frac{\alpha}{\alpha_0}\right)^{\binom{k}{3}} (1-\gamma) \\
&\overset{(3.2)}{\geq} (1-\mu) d^{\binom{k}{2}} n^k \prod_{1\leq h<i<j\leq k} \alpha_{hij},
\end{aligned}$$

which proves the lower bound of Theorem 3.12 for (S1).

To give an upper bound on the number of $K_k^{(3)}$'s in $\mathcal{J}$ we need to account for those $K_k^{(3)}$'s that contain a triple $g \in \mathcal{J}_0^{hij}$ for some $1 \leq h < i < j \leq k$. To that end, for each $1 \leq h < i < j \leq k$, let $[s_{hij}]_0 = \{0\} \cup [s_{hij}]$ and let $\vec{A}_0 = \prod_{1\leq h<i<j\leq k} [s_{hij}]_0$. For $\vec{a} \in \vec{A}_0 \backslash \vec{A}$, $\mathcal{J}_{\vec{a}}$ is defined analogously. Then by Theorem 3.13 and (3.4),

$$\begin{aligned}
|K_k^{(3)}(\mathcal{J})| &= \sum_{\vec{a}\in\vec{A}_0} |K_k^{(3)}(\mathcal{J}_{\vec{a}})| \\
&= \sum_{\vec{a}\in\vec{A}} |K_k^{(3)}(\mathcal{J}_{\vec{a}})| + \sum_{\vec{a}\in\vec{A}_0\backslash\vec{A}} |K_k^{(3)}(\mathcal{J}_{\vec{a}})| \\
&\leq (1+\gamma) d^{\binom{k}{2}} n^k \prod_{1\leq h<i<j\leq k} \alpha_{hij} + \sum_{\vec{a}\in\vec{A}_0\backslash\vec{A}} |K_k^{(3)}(\mathcal{J}_{\vec{a}})|. \qquad (3.5)
\end{aligned}$$

It remains to find an upper bound for $\sum_{\vec{a}\in\vec{A}_0\backslash\vec{A}} |K_k^{(3)}(\mathcal{J}_{\vec{a}})|$. For that we use Theorem 2.8. For sufficiently small $\varepsilon > 0$ we know that for fixed $1 \leq h <$

$i < j \leq k$, all but $6\varepsilon n^3$ triangles $t \in K_3(P^{hi} \cup P^{ij} \cup P^{hj})$ belong to at most $2d^{\binom{k}{2}-3}n^{k-3}$ K_k's in P (see for example Lemma 15 in [16]). As such,

$$\sum_{\vec{a} \in \vec{A}_0 \setminus \vec{A}} |K_k^{(3)}(\mathcal{J}_{\vec{a}})| \leq \binom{k}{3} \left[\max_{1 \leq h < i < j \leq k} |\mathcal{J}_0^{hij}| \times 2d^{\binom{k}{2}-3} n^{k-3} + 6\varepsilon n^3 n^{k-3} \right]. \tag{3.6}$$

By the definition of $\mathcal{J}_{\vec{a}}$ and since $1 - ps \leq p$ we have the following upper bound,

$$\begin{aligned} \max_{1 \leq h < i < j \leq k} |\mathcal{J}_0^{hij}| &\leq \max_{1 \leq h < i < j \leq k} p_{hij} |\mathcal{J}^{hij}| \\ &= \max_{1 \leq h < i < j \leq k} p_{hij} \alpha_{hij} |K_3(P^{hi} \cup P^{ij} \cup P^{hj})| \\ &\leq 2\alpha d^3 n^3, \end{aligned}$$

where, in the last inequality, we applied Theorem 2.8 for sufficiently small $\varepsilon > 0$ (satisfied by inequality (3.2)). Combining our last inequality and (3.6) we get,

$$\begin{aligned} \sum_{\vec{a} \in \vec{A}_0 \setminus \vec{A}} |K_k^{(3)}(\mathcal{J}_{\vec{a}})| &\leq \binom{k}{3} \left(4\alpha d^{\binom{k}{2}} n^k + 6\varepsilon n^k \right) \\ &\leq 2\alpha k^3 d^{\binom{k}{2}} n^k. \end{aligned}$$

We infer from (3.5) that

$$\begin{aligned} |K_k^{(3)}(\mathcal{J})| &\leq \left[1 + \gamma + 2\alpha k^3 \prod_{1 \leq h < i < j \leq k} \alpha_{hij}^{-1} \right] d^{\binom{k}{2}} n^k \prod_{1 \leq h < i < j \leq k} \alpha_{hij} \\ &\leq \left[1 + \gamma + 2\alpha k^3 \alpha_0^{-\binom{k}{3}} \right] d^{\binom{k}{2}} n^k \prod_{1 \leq h < i < j \leq k} \alpha_{hij} \\ &\overset{(3.2)}{\leq} (1 + \mu) d^{\binom{k}{2}} n^k \prod_{1 \leq h < i < j \leq k} \alpha_{hij}, \end{aligned}$$

which proves the upper bound and therefore concludes the proof of (S1) of Theorem 3.12.

Proof of (S2). Since the proof is almost the same (uses the same techniques) we will only sketch it here. Let the 3-uniform hypergraph $\mathcal{J}$ and graph P be given as in Theorem 3.12 with constants satisfying

$$\min\left\{\frac{1}{k},\mu,\alpha_0\right\} \gg \gamma \gg \delta \geq \min\{\delta, d_0\} \gg \frac{1}{l} \gg \varepsilon \gg \frac{1}{n_0} \geq \frac{1}{n} > 0,$$

where l is an auxiliary integer. As before, we want to estimate the number of K_k's in $\mathcal{J}$. This time, for each $1 \leq i < j \leq k$, $p_{ij} = 1/(ld_{ij})$, and $s_{ij} = \lfloor 1/p_{ij} \rfloor$, we randomly partition the bipartite parts of the graph P as follows

$$P^{ij} = P_0^{ij} \cup P_1^{ij} \cup \dots \cup P_{s_{ij}}^{ij},$$

where P_a^{ij} is obtained by independently including each edge $e \in P^{ij}$ with probability

$$\mathbb{P}(e \in P_a^{ij}) = \begin{cases} p_{ij} & \text{if } 1 \leq a \leq s, \\ 1 - p_{ij}s_{ij} & \text{if } a = 0. \end{cases}$$

As before, we use Chernoff's inequality to show for each P_a^{ij}, $1 \leq a \leq s_{ij}$ that

$$\mathbb{P}(P_a^{ij} \text{ is } (1/l, 2\varepsilon)\text{-regular}) = 1 - o(1).$$

Again, as before we apply Chebyshev's inequality to prove that for each $1 \leq h < i < j \leq k$, $1 \leq a \leq s_{hi}$, $1 \leq b \leq s_{ij}$, $1 \leq c \leq s_{hj}$, and $\mathcal{J}_{abc}^{hij} = \mathcal{J}^{hij} \cap K_3(P_a^{hi} \cup P_b^{ij} \cup P_c^{hj})$

$$\mathbb{P}\Big(\mathcal{J}_{abc}^{hij} \text{ is } (\alpha_{hij} \pm o(1), 2\delta)\text{-minimal w.r.t. } P_a^{hi} \cup P_b^{ij} \cup P_c^{hj}\Big) = 1 - o(1).$$

The following expectations and variances are necessary in order to apply Chebyshev's inequality and can easily be verified

$$\begin{aligned} \mathbb{E}(|\mathcal{J}_{abc}^{hij}|) &= p_{hi}p_{ij}p_{hj}|\mathcal{J}^{hij}| = \Omega(n^3), \\ \mathrm{Var}(|\mathcal{J}_{abc}^{hij}|) &= O(n^4), \\ \mathbb{E}(|K_{2,2,2}^{(3)}(\mathcal{J}_{abc}^{hij})|) &= p_{hi}^4 p_{ij}^4 p_{hj}^4 |K_{2,2,2}^{(3)}(\mathcal{J}^{hij})| = \Omega(n^6), \\ \mathrm{Var}(|K_{2,2,2}^{(3)}(\mathcal{J}_{abc}^{hij})|) &= O(n^{10}). \end{aligned}$$

Similar as before, we estimate a lower and upper bound on the number of K_k's in $\mathcal{J}$, using Theorem 3.13 (assumptions are satisfied with high probability as shown above). Since we only give a sketch here we combine both cases. To that end, let's define $\vec{A}' = \prod_{1\le i<j\le k}[s_{ij}]$ and $\vec{A}'_0 = \prod_{1\le i<j\le k}[s_{ij}]_0$. As before, let π_{ij} be the projection onto the ij-th coordinate. For $\vec{a} \in \vec{A}'_0$ we set

$$P_{\vec{a}} = \bigcup_{1\le i<j\le k} P^{ij}_{\pi_{ij}(\vec{a})} \qquad \text{and} \qquad \mathcal{J}_{\vec{a}} = \mathcal{J} \cap K_3(P_{\vec{a}}).$$

Having defined $P_{\vec{a}}$ and $\mathcal{J}_{\vec{a}}$ for $\vec{a} \in \vec{A}'$ Theorem 3.13 applies with high probability. As mentioned before, we consider here the lower and upper bound of the size of $K_k^{(3)}(\mathcal{J})$ simultaneously and therefore introduce '$\sim$' to hide errors from the terms $\vec{a} \in \vec{A}'_0 \backslash \vec{A}'$. Then,

$$\begin{aligned}
|K_k^{(3)}(\mathcal{J})| &= \sum_{\vec{a}\in\vec{A}'_0} |K_k^{(3)}(\mathcal{J}_{\vec{a}})| \\
&\sim \sum_{\vec{a}\in\vec{A}'} |K_k^{(3)}(\mathcal{J}_{\vec{a}})| \\
&= (1\pm\gamma) \prod_{1\le h<i<j\le k} \alpha_{hij} \times \frac{1}{l^{\binom{k}{2}}} \times n^k \times \prod_{1\le i<j\le k} s_{ij} \\
&\sim (1\pm\mu) \prod_{1\le h<i<j\le k} \alpha_{hij} \times \prod_{1\le h<i<j\le k} d_{ij} \times n^k,
\end{aligned}$$

which finishes the proof of Theorem 3.12.

■

In order to prove Theorem 3.10 we need two more facts, one considering the (α,δ)-quasirandomness concept and the other one regarding the (α,δ)-minimality concept. For both facts let $G = G^{12} \cup G^{23} \cup G^{13}$ be a tripartite graph with three partition $V_1 \cup V_2 \cup V_3$ where each G^{ij} is (d_{ij},ε)-quasirandom (or (d_{ij},ε)-regular respectively) for $1 \le i < j \le 3$. Suppose furthermore that $\mathcal{H} \subset K_3(G)$ is a 3-uniform hypergraph. Let us first state the fact regarding the (α,δ)-quasirandomness concept.

Fact 3.14 *Suppose $\mathcal{H}$ is (α, δ)-quasirandom w.r.t. G and assume that $m = |V_1| = |V_2| = |V_3| = 2n$ is even. Then for all but $o(\binom{m}{n}^3)$ subpartitions $V_1 = V_{11} \cup V_{12}$, $V_2 = V_{21} \cup V_{22}$, and $V_3 = V_{31} \cup V_{32}$ with $|V_{11}| = \cdots = |V_{32}| = n$, we have that $\mathcal{H}[V_{1h}, V_{2i}, V_{3j}]$ is $(\alpha \pm o(1), 64\delta)$-quasirandom w.r.t. $G[V_{1h}, V_{2i}, V_{3j}]$ for all $1 \le h, i, j \le 2$.*

We will now state an analogous fact regarding the (α, δ)-minimality concept.

Fact 3.15 *Suppose $\mathcal{H}$ is (α, δ)-minimal w.r.t. G and assume that $m = |V_1| = |V_2| = |V_3| = 2n$ is even. Then for all but $o(\binom{m}{n}^3)$ subpartitions $V_1 = V_{11} \cup V_{12}$, $V_2 = V_{21} \cup V_{22}$, and $V_3 = V_{31} \cup V_{32}$ with $|V_{11}| = \cdots = |V_{32}| = n$, we have that $\mathcal{H}[V_{1h}, V_{2i}, V_{3j}]$ is $(\alpha \pm o(1), 2\delta)$-minimal w.r.t. $G[V_{1h}, V_{2i}, V_{3j}]$ for all $1 \le h, i, j \le 2$.*

We prove the following statement, which implies both Facts 3.15 and 3.14.

Fact 3.16 *Let $\mathcal{F}$ be a 3-partite 3-uniform hypergraph with 3-partition $W_1 \cup W_2 \cup W_3$, where $|W_a| = m_a$, $1 \le a \le 3$. Suppose $\mathcal{F}$ has $c_1 m_1 m_2 m_3$ triples and $c_2 m_1^2 m_2^2 m_3^2$ unlabelled copies of $K^{(3)}_{2,2,2}$, where $c_1, c_2 > 0$. Then all but $o(\binom{m_1}{\lceil m_1/2 \rceil}\binom{m_2}{\lceil m_2/2 \rceil}\binom{m_3}{\lceil m_3/2 \rceil})$ subpartitions $W_1 = W_{11} \cup W_{12}$, $W_2 = W_{21} \cup W_{22}$, $W_3 = W_{31} \cup W_{32}$, $|W_{a1}| = n_{a1} = \lceil m_a/2 \rceil$, $1 \le a \le 3$, satisfy that for each $1 \le h, i, j \le 2$:*

1. *$\mathcal{F}[W_{1h}, W_{2i}, W_{3j}]$ has $(c_1 \pm o(1)) n_{1h} n_{2i} n_{3j}$ triples, and more strongly, for each $w_1 \in W_1$ and $w_2 \in W_2$ for which*

$$\deg_{\mathcal{F}}(w_1, w_2) = \sum_{w_3 \in W_3} \mathcal{F}(w_1, w_2, w_3) \ge m_3 / \log m_3 \,,$$

we have

$$\deg_{\mathcal{F},j}(w_1, w_2) := \sum_{w_3 \in W_{3j}} \mathcal{F}(w_1, w_2, w_3) = \left(\tfrac{1}{2} \pm o(1)\right) \deg_{\mathcal{F}}(w_1, w_2) \,,$$

where $\mathcal{F}(w_1, w_2, w_3)$ is the characteristic function of $\mathcal{F}$;

2. *$\mathcal{F}[W_{1h}, W_{2i}, W_{3j}]$ contains $(c_2 \pm o(1)) n_{1h}^2 n_{2i}^2 n_{3j}^2$ copies of $K^{(3)}_{2,2,2}$,*

where $o(1)$ *above tends to* 0 *as* $\min\{m_1, m_2, m_3\} \to \infty$.

Fact 3.15 follows easily from Fact 3.16.

Proof of Fact 3.15. Fix a 'typical' subpartition $V_{11}, \ldots, V_{32}$ (typical in the sense of Fact 3.16). Then, for example with $V_{11} \cup V_{21} \cup V_{31}$, we have

$$\begin{aligned}\left|\mathcal{H}[V_{11}, V_{21}, V_{31}]\right| &= \left(\tfrac{1}{8} \pm o(1)\right)|\mathcal{H}| \quad \text{and similarly} \\ \left|K_3(G[V_{11}, V_{21}, V_{31}])\right| &= \left|K_3(G)[V_{11}, V_{21}, V_{31}]\right| \\ &= \left(\tfrac{1}{8} \pm o(1)\right)\left|K_3(G)\right|.\end{aligned}$$

Then $|\mathcal{H}| = \alpha|K_3(G)|$ implies that $\mathcal{H}[V_{11}, V_{21}, V_{31}]$ has density $\alpha \pm o(1)$ w.r.t. $G[V_{11}, V_{21}, V_{31}]$, as required. Moreover, by Fact 3.16, part 2,

$$\begin{aligned}\left|K^{(3)}_{2,2,2}(\mathcal{H}[V_{11}, V_{21}, V_{31}])\right| &= \left(\tfrac{1}{64} \pm o(1)\right)\left|K^{(3)}_{2,2,2}(\mathcal{H})\right| \\ &\le (1 + 2\delta)\alpha^8 d_{12}^4 d_{23}^4 d_{13}^4 \tbinom{n}{2}^3,\end{aligned}$$

as required.

■

We now prove that Fact 3.14 follows from Fact 3.16.

Proof of Fact 3.14. We've already argued the density assertion, so for the remainder (the quasirandomness), fix typical subpartition $V_{11}, \ldots, V_{32}$. In what follows, we shall only consider, w.l.o.g., $V_{11} \cup V_{21} \cup V_{31}$, so that $\mathcal{H}[V_{11}, V_{21}, V_{31}]$ has density $\alpha' = \alpha \pm o(1)$ w.r.t. $G[V_{11}, V_{21}, V_{31}]$. Consider the function $h' : V_{11} \times V_{21} \times V_{31} \to [-1, 1]$ given by

$$h'(v_1, v_2, v_3) = \begin{cases} \mathcal{H}(v_1, v_2, v_3) - \alpha' & \text{if } \{v_1, v_2, v_3\} \in K_3(G[V_{11}, V_{21}, V_{31}]), \\ 0 & \text{otherwise} \end{cases}$$

where, again, $\mathcal{H}(v_1, v_2, v_3)$ is the characteristic function of the hyperedges of $\mathcal{H}$. On account that $V_{11}, \ldots V_{31}$ is typical, we then have $h'(v_1, v_2, v_3) = (1 \pm o(1))h(v_1, v_2, v_3)$ for all $(v_1, v_2, v_3) \in V_{11} \times V_{21} \times V_{31}$, where h is the function for the original hypergraph $\mathcal{H}$. We write

$$\begin{aligned} h'_{v_1v_1'v_2v_2'v_3v_3'} &= h(v_1,v_2,v_3)h(v_1',v_2,v_3)h(v_1,v_2',v_3)h(v_1,v_2,v_3') \\ &\quad \times h(v_1',v_2',v_3)h(v_1',v_2,v_3')h(v_1,v_2',v_3')h(v_1',v_2',v_3')\,, \end{aligned}$$

so that, for each $v_1,v_1' \in V_{11}$, $v_2,v_2' \in V_{21}$ and $v_3,v_3' \in V_{31}$, we have that $h'_{v_1v_1'v_2v_2'v_3v_3'} = (1 \pm o(1))h_{v_1v_1'v_2v_2'v_3v_3'}$, or equivalently, $h_{v_1v_1'v_2v_2'v_3v_3'} = (1 \pm o(1))h'_{v_1v_1'v_2v_2'v_3v_3'}$. As such,

$$\begin{aligned} &(1 \pm o(1)) \sum_{v_1,v_1' \in V_{11}} \sum_{v_2,v_2' \in V_{21}} \sum_{v_3,v_3' \in V_{31}} h'_{v_1v_1'v_2v_2'v_3v_3'} \\ &= \sum_{v_1,v_1' \in V_{11}} \sum_{v_2,v_2' \in V_{21}} \sum_{v_3,v_3' \in V_{31}} h_{v_1v_1'v_2v_2'v_3v_3'} \\ &= \sum_{v_1,v_1' \in V_{11}} \sum_{v_2,v_2' \in V_{21}} \sum_{v_3,v_3' \in V_{31}} h(v_1,v_2,v_3)h(v_1',v_2,v_3)h(v_1,v_2',v_3)h(v_1',v_2',v_3) \\ &\qquad \times h(v_1,v_2,v_3')h(v_1',v_2,v_3')h(v_1,v_2',v_3')h(v_1',v_2',v_3') \\ &= \sum_{v_1,v_1' \in V_{11}} \sum_{v_2,v_2' \in V_{21}} \left(\sum_{v_3 \in V_{31}} h(v_1,v_2,v_3)h(v_1',v_2,v_3)h(v_1,v_2',v_3)h(v_1',v_2',v_3) \right)^2 \\ &\le \sum_{v_1,v_1' \in V_1} \sum_{v_2,v_2' \in V_2} \left(\sum_{v_3 \in V_{31}} h(v_1,v_2,v_3)h(v_1',v_2,v_3)h(v_1,v_2',v_3)h(v_1',v_2',v_3) \right)^2 \\ &= \sum_{v_3,v_3' \in V_{31}} \sum_{v_1,v_1' \in V_1} \sum_{v_2,v_2' \in V_2} h(v_1,v_2,v_3)h(v_1',v_2,v_3)h(v_1,v_2,v_3')h(v_1',v_2,v_3') \\ &\qquad \times h(v_1,v_2',v_3)h(v_1',v_2',v_3)h(v_1,v_2',v_3')h(v_1',v_2',v_3') \\ &= \sum_{v_3,v_3' \in V_{31}} \sum_{v_1,v_1' \in V_1} \left(\sum_{v_2 \in V_2} h(v_1,v_2,v_3)h(v_1',v_2,v_3)h(v_1,v_2,v_3')h(v_1',v_2,v_3') \right)^2 \\ &\le \sum_{v_3,v_3' \in V_3} \sum_{v_1,v_1' \in V_1} \left(\sum_{v_2 \in V_2} h(v_1,v_2,v_3)h(v_1',v_2,v_3)h(v_1,v_2,v_3')h(v_1',v_2,v_3') \right)^2 \\ &= \sum_{v_1,v_1' \in V_1} \sum_{v_2,v_2' \in V_2} \sum_{v_3,v_3' \in V_3} h_{v_1v_1'v_2v_2'v_3v_3'} \\ &\le \delta d_{12}^4 d_{13}^4 d_{23}^4 m^6 = 64\delta d_{12}^4 d_{13}^4 d_{23}^4 (m/2)^6 = 64\delta d_{12}^4 d_{13}^4 d_{23}^4 n^6. \end{aligned} \tag{3.7}$$

This proves Fact 3.14.

■

We now proceed to prove Fact 3.16.

Proof of Fact 3.16. The proofs of Statements (1) and (2) are standard applications of the Chernoff inequality. We prove Statement (2) only (Statement (1) is well-known, and is proved along similar, albeit simpler, lines).

Let $\mathcal{F}$ and $W_1 \cup W_2 \cup W_3$ be given as in Fact 3.16. We first prove that all but $o(\binom{m_3}{\lceil m_3/2 \rceil})$ subpartitions $W_3 = W_{31} \cup W_{32}$, $|W_{31}| = \lceil m_3/2 \rceil$, satisfy that for $i = 1, 2$,

$$\left|K^{(3)}_{2,2,2}(\mathcal{F}[W_1, W_2, W_{3i}])\right| = (c_2 \pm o(1)) m_1^2 m_2^2 m_{3i}^2 \,. \tag{3.8}$$

Iterating (3.8) for W_1 and W_2 renders Fact 3.16.

To set up the proof of ((3.8)), let $\mathcal{C}_4 = \mathcal{C}_4^{12}$ denote the family of all 4-cycles $\{w_1, w_1', w_2, w_2'\}$ in the complete bipartite graph $K[W_1, W_2]$. For a fixed $C_4 = \{w_1, w_1', w_2, w_2'\} \in \mathcal{C}_4$, write

$$N(C_4) = \big\{w_3 \in W_3 : \{w_1, w_1', w_2, w_2', w_3\} \text{ spans 4 triples in } \mathcal{F}\big\} \quad \text{and}$$
$$\deg(C_4) = |N(C_4)| \,.$$

We shall say that C_4 is *big* if $\deg(C_4) \geq m_3/\log m_3$, and *small* otherwise. We write $\mathcal{C}_4^+$ ($\mathcal{C}_4^-$) for the class of all big (small) $C_4 \in \mathcal{C}_4$. With this notation, we have

$$\begin{aligned} c_2 m_1^2 m_2^2 m_3^2 &= \left|K^{(3)}_{2,2,2}(\mathcal{F})\right| \\ &= \sum_{C_4 \in \mathcal{C}_4} \binom{\deg(C_4)}{2} \\ &= \sum_{C_4 \in \mathcal{C}_4^+} \binom{\deg(C_4)}{2} + \sum_{C_4 \in \mathcal{C}_4^-} \binom{\deg(C_4)}{2} \,. \end{aligned}$$

In this way,

$$\begin{aligned}\sum_{C_4 \in \mathcal{C}_4^+} \binom{\deg(C_4)}{2} &\leq c_2 m_1^2 m_2^2 m_3^2 \\ &\leq \tfrac{m_1^2 m_2^2 m_3^2}{2\log^2 m_3} + \sum_{C_4 \in \mathcal{C}_4^+} \binom{\deg(C_4)}{2}.\end{aligned}$$

Now, fix $C_4 \in \mathcal{C}_4^+$. For a subpartition $W_3 = W_{31} \cup W_{32}$, write $N_i(C_4) = N(C_4) \cap W_{3i}$ and $\deg_i(C_4) = |N_i(C_4)|$, where $i = 1, 2$. If $W_3 = W_{31} \cup W_{32}$ is a random subpartition with $|W_{31}| = \lceil |W_3|/2 \rceil$, then $\deg_1(C_4)$ has hypergeometric distribution with mean

$$\begin{aligned}\mathbb{E}[\deg_1(C_4)] &= \tfrac{\lceil m_3/2 \rceil}{m_3} \deg(C_4) \\ &= \left(\tfrac{1}{2} \pm o(1)\right) \deg(C_4) \\ &\geq \tfrac{1}{3} \tfrac{m_3}{\log m_3}.\end{aligned}$$

The Chernoff inequality therefore ensures

$$\begin{aligned}\mathbb{P}\Big[\deg_1(C_4) \neq \big(1 \pm \tfrac{1}{\log m_3}\big)\mathbb{E}[\deg_1(C_4)]\Big] &\leq 2\exp\Big\{-\tfrac{1}{3\log^2 m_3}\mathbb{E}[\deg_1(C_4)]\Big\} \\ &\leq 2\exp\Big\{-\tfrac{m_3}{9\log^3 m_3}\Big\}.\end{aligned}$$

As such, with high probability, all $C_4 \in \mathcal{C}_4^+$ satisfy

$$\begin{aligned}\deg_1(C_4) &= \left(\tfrac{1}{2} \pm o(1)\right) \deg(C_4) \\ &= \deg_2(C_4).\end{aligned}$$

We now approach the end of the proof. Fix $i = 1, 2$. For the random subpartition $W_3 = W_{31} \cup W_{32}$ above, we have

$$\begin{aligned}\big|K_{2,2,2}^{(3)}(\mathcal{F}[W_1, W_2, W_{3i}])\big| &= \sum_{C_4 \in \mathcal{C}_4} \binom{\deg_i(C_4)}{2} \\ &= \sum_{C_4 \in \mathcal{C}_4^+} \binom{\deg_i(C_4)}{2} + \sum_{C_4 \in \mathcal{C}_4^-} \binom{\deg_i(C_4)}{2}.\end{aligned}$$

As such, with high probability,

$$\begin{aligned}
\left|K^{(3)}_{2,2,2}(\mathcal{F}[W_1,W_2,W_{3i}])\right| &\geq \sum_{C_4\in\mathcal{C}_4^+} \tbinom{\deg_i(C_4)}{2} \\
&\geq \sum_{C_4\in\mathcal{C}_4^+} \binom{\big(\frac{1}{2}-o(1)\big)\deg_i(C_4)}{2} \\
&= \big(\tfrac{1}{4}-o(1)\big)\sum_{C_4\in\mathcal{C}_4^+} \tbinom{\deg(C_4)}{2} \\
&\geq \big(\tfrac{1}{4}-o(1)\big)m_1^2m_2^2m_3^2\big(c_2-\tfrac{1}{2\log^2 m_3}\big) \\
&= (c_2-o(1))m_1^2m_2^2m_{3i}^2\,,
\end{aligned}$$

which proves the lower bound of (3.8). Similarly, with high probability,

$$\begin{aligned}
\left|K^{(3)}_{2,2,2}(\mathcal{F}[W_1,W_2,W_{3i}])\right| &\leq \tfrac{m_1^2m_2^2m_3^2}{\log^2 m_3} + \sum_{C_4\in\mathcal{C}_4^+} \tbinom{\deg_i(C_4)}{2} \\
&\leq \tfrac{m_1^2m_2^2m_3^2}{2\log^2 m_3} + \sum_{C_4\in\mathcal{C}_4^+} \binom{\big(\frac{1}{2}+o(1)\big)\deg_i(C_4)}{2} \\
&= \tfrac{m_1^2m_2^2m_3^2}{2\log^2 m_3} + \big(\tfrac{1}{4}+o(1)\big)\sum_{C_4\in\mathcal{C}_4^+} \tbinom{\deg(C_4)}{2} \\
&\leq \tfrac{m_1^2m_2^2m_3^2}{2\log^2 m_3} + \big(\tfrac{1}{4}+o(1)\big)c_2m_1^2m_2^2m_3^2 \\
&= (c_2+o(1))m_1^2m_2^2m_{3i}^2\,,
\end{aligned}$$

which proves the upper bound of (3.8).

■

3.2.4 Proof of Theorem 3.10

Finally, we established all the necessary tools in order to give a proof of Theorem 3.10. We will first show assertion (1), i.e. that (α,δ_2)-quasirandomness implies (α,δ_1)-minimality.

Proof of Theorem 3.10, Part (1). We will work with the following hierarchy of constants:

$$0 < \frac{1}{m} \leq \frac{1}{m_0} \ll \varepsilon \ll \varepsilon' \ll \min\{\delta_2, d_0\} \leq \delta_2 \ll \alpha, \delta_1$$

where $\varepsilon' > 0$ is an auxiliary constant defined in the context.

Now, let the 3-partite graph G and the 3-partite, 3-uniform hypergraph $\mathcal{H}$ be given as in Theorem 3.10. Suppose that $\mathcal{H}$ is (α, δ_2)-quasirandom w.r.t. G. For simplicity we may also assume that the vertex partitions V_i, $1 \leq i \leq 3$ do not only have the same size but also have size $m = 2n$. Indeed, otherwise one can delete one vertex from each V_i, $1 \leq i \leq 3$ to obtain a graph G' with corresponding hypergraph $\mathcal{H}'$. Then we know that $\mathcal{H}'$ is $(\alpha \pm o(1), \delta_2 \pm o(1))$-quasirandom w.r.t. G' and $|K^{(3)}_{2,2,2}(\mathcal{H})| \leq |K'^{(3)}_{2,2,2}(\mathcal{H}')| + O(m^5)$.

We now consider the family $\mathbf{\Pi}$ of all subpartitions $V_1 = V_{11} \cup V_{12}$, $V_2 = V_{21} \cup V_{22}$, and $V_3 = V_{31} \cup V_{32}$, where $|V_{11}| \ldots |V_{32}| = n$. For a fixed element $\Pi \in \mathbf{\Pi}$, i.e. $\Pi = (V_{11}, \ldots, V_{32})$ we know that the graph $G[V_{ai}, V_{bj}]$, $1 \leq a < b \leq 3$ and $1 \leq i, j \leq 2$ is (d_{ab}, ε')-quasirandom. Indeed, in Chapter 2 we have shown that (d, ε)-quasirandomness is essentially equivalent to (d, ε)-regularity and for the latter it is known that $G[V_{ai}, V_{bj}]$, $1 \leq a < b \leq 3$ and $1 \leq i, j \leq 2$ is (d_{ab}, ε')-regular and therefore it is also (d_{ab}, ε')-quasirandom. Applying Fact 3.14 to $\mathbf{\Pi}$ we know that all but $o(\binom{m}{n}^3)$ elements $\Pi \in \mathbf{\Pi}$ satisfy that $\mathcal{H}[V_{1h}, V_{2i}, V_{3j}]$ is $(\alpha \pm o(1), 64\delta_2)$-quasirandom w.r.t. $G[V_{1h}, V_{2i}, V_{3j}]$ for each $1 \leq h, i, j \leq 2$. We call a partition $\Pi \in \mathbf{\Pi}$ 'good' if the above property is satisfied and we denote the set of all good partitions $\Pi \in \mathbf{\Pi}$ by $\mathbf{\Pi}^+$.

For a fixed $\Pi \in \mathbf{\Pi}^+$ we define a 6-partite graph $P = P_\Pi$ and a 6-partite hypergraph $\mathcal{J} = \mathcal{J}_\Pi$ with vertex partition $V_{11} \cup \cdots \cup V_{32}$. Further we will show that the assumptions for Theorem 3.11 are satisfied and apply it in order to estimate the number of $K^{(3)}_{2,2,2}$ in $\mathcal{H}$. First, let us define the graph P. For $1 \leq a \leq b \leq 3$ and $1 \leq i, j \leq 2$ and $V_{ai} \neq V_{bj}$ we set

$$P[V_{ai}, V_{bj}] = \begin{cases} G[V_{ai}, V_{bj}] \text{ if } a \neq b \\ K[V_{ai}, V_{bj}] \text{ otherwise.} \end{cases}$$

$K[V_{ai}, V_{bj}]$ denotes the complete bipartite graph with vertex bipartition $V_{ai} \cup V_{bj}$. As explained above, for $a \neq b$ $P[V_{ai}, V_{bj}]$ is (d_{ij}, ε')-quasirandom and for $a = b$ it is trivially $(1, 0)$-quasirandom. Now, let us define the corresponding hypergraph $\mathcal{J}$. For $1 \leq a \leq b \leq c \leq 3$ and $1 \leq i, j \leq 2$ and $V_{ah} \neq V_{bi} \neq V_{cj}$ we set

$$\mathcal{J}[V_{ah}, V_{bi}, V_{cj}] = \begin{cases} \mathcal{H}[V_{ah}, V_{bi}, V_{cj}] \text{ if } \{a, b, c\} = \{1, 2, 3\}, \\ K_3(P[V_{ah}, V_{bi}, V_{cj}]) \text{ otherwise.} \end{cases}$$

By the definition of $\mathbf{\Pi}^+$ we know that for $\{a, b, c\} = \{1, 2, 3\}$ $\mathcal{J}[V_{ah}, V_{bi}, V_{cj}]$ is $(\alpha, 64\delta_2)$-quasirandom and whereas otherwise it is $(1, 0)$-quasirandom.

Note that by the definition of $\mathcal{J}$ every copy of $K_6^{(3)}$ in $\mathcal{J}$ belongs to a copy of $K_{2,2,2}^{(3)}$ in $\mathcal{H}$ for a fixed $\Pi \in \mathbf{\Pi}^+$. Therefore, we first count the $K_6^{(3)}$ in $\mathcal{J}$. Applying Theorem 3.11 to P and $\mathcal{J}$ we get

$$|K_6^{(3)}(\mathcal{J})| = \Big(1 \pm \frac{\delta_1}{2}\Big)\alpha^8 d_{12}^4 d_{23}^4 d_{13}^4 n^6. \tag{3.9}$$

We now double-count the number of pairs (J, Π), where $J \in K_{2,2,2}^{(3)}(\mathcal{H})$, $\Pi \in \mathbf{\Pi}^+$ and Π 'splits' J, i.e. if J has vertices $\{v_{11}, \ldots, v_{32}\}$, then $(v_{11}, \ldots, v_{32}) \in V_{11} \times \cdots \times V_{32}$. Note that we can relabel the vertices, that is v_{i1} and v_{i2} could be swapped for $1 \leq i \leq 3$. Then this number is

$$\begin{aligned} |K_{2,2,2}^{(3)}(\mathcal{H})| \times 8 \times \binom{m-2}{n-1}^3 &= |(J, \Pi)| \\ &= \sum_{\Pi \in \mathbf{\Pi}} |K_6^{(3)}(\mathcal{J}_\Pi)| \\ &= \sum_{\Pi \in \mathbf{\Pi}^+} |K_6^{(3)}(\mathcal{J}_\Pi)| + \sum_{\Pi \in \mathbf{\Pi} \setminus \mathbf{\Pi}^+} |K_6^{(3)}(\mathcal{J}_\Pi)|. \end{aligned}$$

We now use Theorem 3.11 and Fact 3.14 to find an estimation for

$\sum_{\Pi\in\mathbf{\Pi}^+}|K_6^{(3)}(\mathcal{J}_\Pi)|$. We get

$$\begin{aligned}|K_{2,2,2}^{(3)}(\mathcal{H})| &\le \frac{1}{8}\binom{m-2}{n-1}^{-3}\Big[\sum_{\Pi\in\mathbf{\Pi}^+}|K_6^{(3)}(\mathcal{J}_\Pi)| + n^6\times o(\binom{m}{n}^3)\Big]\\ &\le \frac{1}{8}\binom{m-2}{n-1}^{-3}|\Pi|\Big(1+\frac{\delta_1}{2}\Big)\alpha^8 d_{12}^4 d_{23}^4 d_{13}^4 n^6 + o(n^6)\\ &\le \Big(1+\frac{\delta_1}{2}+o(1)\Big)\alpha^8 d_{12}^4 d_{23}^4 d_{13}^4\binom{m}{2}^3\\ &\le (1+\delta_1)\alpha^8 d_{12}^4 d_{23}^4 d_{13}^4\binom{m}{2}^3.\end{aligned}$$

Therefore we have shown that $\mathcal{H}$ is also (α,δ_1)-minimal as promised.

■

We now proceed with the proof of Theorem 3.10 by showing the second implication. This proof is very similar to the one before although slightly more complicated. We will need the following proposition which is a corollary of Theorem 3.12. We defer the proof of Proposition 3.17 to the next section.

Proposition 3.17 *For all* $0<\alpha<1$ *and* $\vartheta>0$*, there exists* $\delta>0$ *so that for all* $d_0\in(0,1]$*, there exist* $\varepsilon>0$ *and integer* m_0 *so that the following holds. Suppose that graph* G *and* 3*-uniform hypergraph* $\mathcal{H}$ *satisfy:*

(i) $V(G)=V(\mathcal{H})=V=V_1\cup V_2\cup V_3$*, where* $|V_1|=|V_2|=|V_3|=m\ge m_0$*;*

(ii) $G=G^{12}\cup G^{23}\cup G^{13}$ *is* 3*-partite with* 3*-partition above, where for each* $1\le i<j\le 3$*,* G^{ij} *is* (d_{ij},ε)*-regular, with* $d_{ij}\ge d_0$*;*

(iii) $\mathcal{H}\subseteq K_3(G)$ *is* (α,δ)*-minimal w.r.t.* G*;*

Then for each suboctahedron $\mathcal{O}_0\subseteq\mathcal{O}$*,*

$$|\mathcal{O}_0^{ind}(\mathcal{H})| = (1\pm\vartheta)\alpha^{\mathcal{O}_0}(1-\alpha)^{8-\mathcal{O}_0} d_{12}^4 d_{23}^4 d_{13}^4 m^6.$$

Note that $\mathcal{O}$ denotes the 3-partite 3-uniform octahedron $K_{2,2,2}^{(3)}$ on fixed vertex set $\{\sigma_{11},\sigma_{12},\sigma_{21},\sigma_{22},\sigma_{31},\sigma_{32}\}$ and fixed 3-partition $\{\sigma_{11},\sigma_{12}\}\cup\{\sigma_{21},\sigma_{22}\}$

$\cup\{\sigma_{31},\sigma_{32}\}$.

Proof of Theorem 3.10, Part (2)**.** As before let us start with stating the hierarchy of constants

$$0<\frac{1}{m}\le\frac{1}{m_0}\ll\varepsilon\ll\min\{\delta_2,d_0\}\le\delta_2\ll\alpha,\delta_1.$$

Let the 3-partite graph G and the 3-partite 3-uniform hypergraph $\mathcal{H}$ be given as in Theorem 3.10. Suppose that $\mathcal{H}$ is (α,δ_2)-minimal w.r.t. G. In order to show that $\mathcal{H}$ is also (α,δ_1)-quasirandom w.r.t. G we need to show that

$$\sum_{v_{11},v_{12}\in V_1}\sum_{v_{21},v_{22}\in V_2}\sum_{v_{31},v_{32}\in V_3} h_{v_{11}v_{12}v_{21}v_{22}v_{31}v_{32}}\le\delta_1 d_{12}^4 d_{23}^4 d_{13}^4 m^6, \tag{3.10}$$

where we recall that the function is defined as $h:V_1\times V_2\times V_3\to[-1,1]$ with $h(v_1,v_2,v_3)=\mathcal{H}(v_1,v_2,v_3)-\alpha$ for $\{v_1,v_2,v_3\}\in K_3(G)$ and 0 otherwise. Note that it suffices to consider only the terms corresponding to distinct choices of vertices, since the remaining terms contribute only $O(m^5)$. Moreover it suffices to regard only the terms $v_{11},\dots,v_{32}$ for which $\{v_{ai},v_{bj}\}\in G^{ab}$ for all $1\le a<b\le 3$ and $1\le i,j\le 2$.

Let us introduce some notation. Write

$$\begin{aligned}(V)_6^+=\Big\{\vec{v}=(v_{11},v_{12},v_{21},v_{22},v_{31},v_{32})\Big|\,&v_{ij}\in V_i, 1\le i\le 3, 1\le j\le 2 \text{ s.t.}\\ &(v_{a1},v_{a2})\in V_a^2=V_a\times V_a \text{ and } v_{a1}\ne v_{a2} \text{ for each } 1\le a\le 3;\\ &\{v_{ai},v_{bj}\}\in G^{ab}, \text{ for each } 1\le a<b\le 3 \text{ and } 1\le i,j\le 2\Big\}.\end{aligned}$$

For $\vec{v}\in(V)_6^+$, we shall also write $h_{\vec{v}}=h_{v_{11}v_{12}v_{21}v_{22}v_{31}v_{32}}$. Using this notation we then have

$$\sum_{v_{11}\ne v_{12}}\sum_{v_{21}\ne v_{22}}\sum_{v_{31}\ne v_{32}} h_{v_{11}v_{12}v_{21}v_{22}v_{31}v_{32}}=\sum_{\vec{v}\in(V)_6^+}h_{\vec{v}},$$

and therefore

$$\sum_{v_{11},v_{12}\in V_1}\sum_{v_{21},v_{22}\in V_2}\sum_{v_{31},v_{32}\in V_3} h_{v_{11}v_{12}v_{21}v_{22}v_{31}v_{32}}=O(m^5)+\sum_{\vec{v}\in(V)_6^+}h_{\vec{v}}. \tag{3.11}$$

We now investigate $\sum_{\vec{v}\in(V)_6^+} h_{\vec{v}}$.

Recall that $\mathcal{O}$ denotes the 3-partite 3-uniform octahedron $K^{(3)}_{2,2,2}$ on fixed vertex set $\{\sigma_{11}, \sigma_{12}, \sigma_{21}, \sigma_{22}, \sigma_{31}, \sigma_{32}\}$ and fixed 3-partition $\{\sigma_{11}, \sigma_{12}\} \cup \{\sigma_{21}, \sigma_{22}\} \cup \{\sigma_{31}, \sigma_{32}\}$. Note that $\mathcal{O}$ has $2^8 = 256$ labelled and spanning subhypergraphs which we call suboctahedrons. Now, fix a suboctahedron $\mathcal{O}_0 \subseteq \mathcal{O}$ and also fix a $\vec{v} \in (V)_6^+$. Note that $\vec{v} = (v_{11}, v_{12}, v_{21}, v_{22}, v_{31}, v_{32})$ spans an induced copy $\mathcal{O}_0'$ of $\mathcal{O}_0$ in $\mathcal{H}$ if for each $1 \le h, i, j \le 2$, $\{v_{1h}, v_{2i}, v_{3j}\} \in \mathcal{H}$ if, and only if $\{\sigma_{1h}, \sigma_{2i}, \sigma_{3j}\} \in \mathcal{O}_0$. Let

$$\mathcal{O}_0^{ind} = \left\{\vec{v} \in (V)_6^+ \middle| \vec{v} \text{ spans an induced copy } \mathcal{O}_0' \text{ of } \mathcal{O}_0 \text{ in } \mathcal{H}\right\}.$$

Therefore, (3.11) becomes

$$\begin{aligned}\sum_{\vec{v}\in(V)_6^+} h_{\vec{v}} &= \sum_{\mathcal{O}_0\subseteq\mathcal{O}} \sum_{\vec{v}\in\mathcal{O}_0^{ind}(\mathcal{H})} h_{\vec{v}} \\ &= \sum_{\mathcal{O}_0\subseteq\mathcal{O}} -\alpha^{8-|\mathcal{O}_0|}(1-\alpha)^{|\mathcal{O}_0|}\left|\mathcal{O}_0^{ind}(\mathcal{H})\right|,\end{aligned}$$

where the last equality is obtained by the definition of the function h. In order to show (3.10) we need to estimate $|\mathcal{O}_0^{ind}(\mathcal{H})|$ for every fixed $\mathcal{O}_0 \subseteq \mathcal{O}$. At this point, we apply Proposition 3.17 to each $\mathcal{O}_0 \subseteq \mathcal{O}$ with $\vartheta = \delta_1$.

$$\begin{aligned}\sum_{\vec{v}\in(V)_6^+} h_{\vec{v}} &= \sum_{\mathcal{O}_0\subseteq\mathcal{O}} -\alpha^{8-|\mathcal{O}_0|}(1-\alpha)^{|\mathcal{O}_0|}\left|\mathcal{O}_0^{ind}(\mathcal{H})\right| \\ &\le \sum_{\mathcal{O}_0\subseteq\mathcal{O}} -\alpha^{8-|\mathcal{O}_0|}(1-\alpha)^{|\mathcal{O}_0|}(1\pm\delta_1)\alpha^{|\mathcal{O}_0|}(1-\alpha)^{8-|\mathcal{O}_0|}d_{12}^4 d_{23}^4 d_{13}^4 m^6 \\ &= (\alpha(1-\alpha))^8 d_{12}^4 d_{23}^4 d_{13}^4 m^6 \sum_{\mathcal{O}_0\subseteq\mathcal{O}} (-1)^{8-|\mathcal{O}_0|}(1\pm(-1)^{|\mathcal{O}_0|}\delta_1) \\ &\le 2^8 \left(\frac{1}{4}\right)^8 \delta_1 d_{12}^4 d_{23}^4 d_{13}^4 m^6 \\ &= 2^{-8}\delta_1 d_{12}^4 d_{23}^4 d_{13}^4 m^6,\end{aligned}$$

where the last inequality is obtained by using the fact that $\alpha(1-\alpha) \le 1/4$. Returning to (3.11), we then have

$$\sum_{v_{11},v_{12}\in V_1} \sum_{v_{21},v_{22}\in V_2} \sum_{v_{31},v_{32}\in V_3} h_{v_{11}v_{12}v_{21}v_{22}v_{31}v_{32}} = O(m^5) + 2^{-8}\delta_1 d_{12}^4 d_{23}^4 d_{13}^4 m^6$$
$$\le \delta_1 d_{12}^4 d_{23}^4 d_{13}^4 m^6.$$

Therefore, $\mathcal{H}$ is also (α, δ_1)-quasirandom w.r.t. G.

■

3.2.4.1 Proof of Proposition 3.17

We finish the proof of Theorem 3.10 by showing that Proposition 3.17 holds. As mentioned before, we can deduce it from Theorem 3.12. The proof is very similar to the proof of Theorem 3.10, Part (1).

Proof of Proposition 3.17. As before, we first state the hierarchy of the constants:

$$0 < \frac{1}{m} \le \frac{1}{m_0} \ll \varepsilon \ll \min\{\delta, d_0\} \le \delta \ll \mu \ll \alpha, \vartheta < 1,$$

where μ is an auxiliary constant defined in the context. Let the 3-partite graph G and the 3-partite 3-uniform hypergraph $\mathcal{H}$ be given as in Proposition 3.17. We may assume that the number of vertices in each vertex partition $m = 2n$. Indeed, if m is odd we delete an arbitrary vertex from each V_i, $1 \le i \le 3$ to obtain a 3-partite graph G' with corresponding 3-partite hypergraph $\mathcal{H}'$. Note that the bipartite graphs of G' are $(d_{ij} \pm o(1), \varepsilon \pm o(1))$-regular and $\mathcal{H}'$ has relative density $\alpha \pm o(1)$ w.r.t. G' and $|K^{(3)}_{2,2,2}(\mathcal{H}')| \le |K^{(3)}_{2,2,2}(\mathcal{H})|$. Then, $|\mathcal{O}_0^{ind}(\mathcal{H})| - |\mathcal{O}_0^{ind}(\mathcal{H}')| = O(m^5)$.

Fix a suboctahedron $\mathcal{O}_0 \subseteq \mathcal{O}$. We need to calculate $|\mathcal{O}_0^{ind}(\mathcal{H})|$. We now consider the family $\mathbf{\Pi}$ of all subpartitions $V_1 = V_{11} \cup V_{12}$, $V_2 = V_{21} \cup V_{22}$, and $V_3 = V_{31} \cup V_{32}$, where $|V_{11}| \dots |V_{32}| = n$. For an arbitrary, fixed element $\Pi \in \mathbf{\Pi}$, i.e. $\Pi = (V_{11}, \dots, V_{32})$ we know that the graph $G[V_{ai}, V_{bj}]$, $1 \le a < b \le 3$ and $1 \le i, j \le 2$ is $(d_{ab}, 2\varepsilon)$-regular. Applying Fact 3.14 to $\mathbf{\Pi}$

we know that all but $o(\binom{m}{n}^3)$ elements $\Pi \in \mathbf{\Pi}$ satisfy that $\mathcal{H}[V_{1h}, V_{2i}, V_{3j}]$ is $(\alpha \pm o(1), 2\delta)$-minimal w.r.t. $G[V_{1h}, V_{2i}, V_{3j}]$ for each $1 \leq h, i, j \leq 2$. We call a partition $\Pi \in \mathbf{\Pi}$ 'good' if the above property is satisfied and we denote the set of all good partitions $\Pi \in \mathbf{\Pi}$ by $\mathbf{\Pi}^+$.

For a fixed $\Pi \in \mathbf{\Pi}^+$ we define a 6-partite graph $P = P_\Pi$ and a 6-partite hypergraph $\mathcal{J} = \mathcal{J}_{\Pi,\mathcal{O}_0}$ with vertex partition $V_{11} \cup \cdots \cup V_{32}$. Further we will show that the assumptions for Theorem 3.12 are satisfied (by introducing a claim) and apply it in order to estimate $|\mathcal{O}_0^{ind}(\mathcal{H})|$. First, let us define the graph P. For $1 \leq a \leq b \leq 3$ and $1 \leq i, j \leq 2$ and $V_{ai} \neq V_{bj}$ we set

$$P[V_{ai}, V_{bj}] = \begin{cases} G[V_{ai}, V_{bj}] \text{ if } a \neq b \\ K[V_{ai}, V_{bj}] \text{ otherwise.} \end{cases}$$

$K[V_{ai}, V_{bj}]$ denotes the complete bipartite graph with vertex bipartition $V_{ai} \cup V_{bj}$. As explained above, for $a \neq b$ the graph $P[V_{ai}, V_{bj}]$ is $(d_{ij}, 2\varepsilon)$-regular and for $a = b$ it is trivially $(1, 0)$-regular. Now, let us define the corresponding hypergraph $\mathcal{J} = \mathcal{J}_{\Pi,\mathcal{O}_0}$. For $1 \leq h, i, j \leq 2$ we set

$$\mathcal{J}[V_{1h}, V_{2i}, V_{3j}] = \begin{cases} \mathcal{H}[V_{1h}, V_{2i}, V_{3j}] \text{ if } \{\sigma_{1h}, \sigma_{2i}, \sigma_{3j}\} \in \mathcal{O}_0, \\ K_3(P[V_{1h}, V_{2i}, V_{3j}]) \backslash \mathcal{H}[V_{1h}, V_{2i}, V_{3j}]) \text{ otherwise .} \end{cases}$$

For all remaining $1 \leq a \leq b \leq c \leq 3$ and $1 \leq h, i, j \leq 2$ where V_{ah}, V_{bi}, V_{cj} are distinct, define

$$\mathcal{J}[V_{ah}, V_{bi}, V_{cj}] = K_3(P[V_{ah}, V_{bi}, V_{cj}]).$$

Note that every copy of $K_6^{(3)}$ in $\mathcal{J}$ corresponds to a copy of $\mathcal{O}_0' \in \mathcal{O}_0^{ind}(\mathcal{H})$. Next, we would like to apply Theorem 3.12 to P and $\mathcal{J}$ but the conditions of this theorem are not quite satisfied. More precisely, if $\{\sigma_{1h}, \sigma_{2i}, \sigma_{3j} \notin \mathcal{O}_0$, we do not know that $\mathcal{J}[V_{1h}, V_{2i}, V_{3j}] = K_3(P[V_{1h}, V_{2i}, V_{3j}]) \backslash \mathcal{H}[V_{1h}, V_{2i}, V_{3j}]$ is $((1-(\alpha \pm 1)), f(\delta))$-minimal w.r.t. $P[V_{1h}, V_{2i}, V_{3j}]$, for any $f(\delta) \to 0$ as $\delta \to 0$. However, Proposition 3.17 will imply that, in general, the complement of an (α, δ)-minimal hypergraph is $((1 - \alpha), f(\delta))$-minimal. With the following claim we are able to overcome the detail above. (We shall delay the proof of the claim momentarily in order to finish the proof of the proposition.)

Claim 3.18 *With* $\Pi \in \mathbf{\Pi}^+$, P *and* $\mathcal{J}$ *defined as above,*

$$|K_6^{(3)}(\mathcal{J})| = (1 \pm \vartheta)\alpha^{|\mathcal{O}_0|}(1-\alpha)^{8-|\mathcal{O}_0|}d_{12}^4 d_{23}^4 d_{13}^4 n^6.$$

We now double count the number of pairs $(\mathcal{O}_0', \Pi)$, where $\mathcal{O}_0' \in \mathcal{O}_0^{ind}(\mathcal{H})$, $\Pi \in \mathbf{\Pi}$ and Π splits $\mathcal{O}_0'$, i.e. if $\mathcal{O}_0' = (v_{11}, \ldots v_{32})$ and $\Pi = (V_{11}, \ldots, V_{32})$, then $\mathcal{O}_0' \in V_{11} \times \cdots \times V_{32}$. We get,

$$\begin{aligned}
|\mathcal{O}_0^{ind}(\mathcal{H})| \times \binom{m-2}{n-1}^3 &= |(\mathcal{O}_0', \Pi)| \\
&= \sum_{\Pi \in \mathbf{\Pi}} |K_6^{(3)}(\mathcal{J}_{\Pi, \mathcal{O}_0})| \\
&= \sum_{\Pi \in \mathbf{\Pi}^+} |K_6^{(3)}(\mathcal{J}_{\Pi, \mathcal{O}_0})| + \sum_{\Pi \in \mathbf{\Pi} \setminus \mathbf{\Pi}^+} |K_6^{(3)}(\mathcal{J}_{\Pi, \mathcal{O}_0})|.
\end{aligned}$$

Applying Fact 3.15 to the latter sum,

$$\begin{aligned}
\binom{m-2}{n-1}^{-3} \sum_{\Pi \in \mathbf{\Pi}^+} |K_6^{(3)}(\mathcal{J}_{\Pi, \mathcal{O}_0})| &\leq |\mathcal{O}_0^{ind}(\mathcal{H})| \\
&\leq \binom{m-2}{n-1}^{-3} \Big[\sum_{\Pi \in \mathbf{\Pi}^+} |K_6^{(3)}(\mathcal{J}_{\Pi, \mathcal{O}_0})| + n^6 \times o(\binom{m}{n}^3) \Big],
\end{aligned}$$

consequently,

$$|\mathcal{O}_0^{ind}(\mathcal{H})| = \binom{m-2}{n-1}^{-3} \sum_{\Pi \in \mathbf{\Pi}^+} |K_6^{(3)}(\mathcal{J}_{\Pi, \mathcal{O}_0})| \pm o(n^6). \tag{3.12}$$

Using Claim 3.18 to estimate $|K_6^{(3)}(\mathcal{J}_{\Pi, \mathcal{O}_0})|$ for each $\Pi \in \mathbf{\Pi}^+$ we get,

$$\begin{aligned}
\sum_{\Pi \in \mathbf{\Pi}^+} |K_6^{(3)}(\mathcal{J}_{\Pi, \mathcal{O}_0})| &\geq |\mathbf{\Pi}^+| \alpha^{|\mathcal{O}_0|}(1-\alpha)^{8-|\mathcal{O}_0|} d_{12}^4 d_{23}^4 d_{13}^4 n^6 (1 - \frac{\vartheta}{2}) \\
\sum_{\Pi \in \mathbf{\Pi}^+} |K_6^{(3)}(\mathcal{J}_{\Pi, \mathcal{O}_0})| &\leq |\mathbf{\Pi}| \alpha^{|\mathcal{O}_0|}(1-\alpha)^{8-|\mathcal{O}_0|} d_{12}^4 d_{23}^4 d_{13}^4 n^6 (1 + \frac{\vartheta}{2}).
\end{aligned}$$

Applying that $(1-o(1))|\mathbf{\Pi}| \leq |\mathbf{\Pi}^+| \leq |\mathbf{\Pi}|$ we receive

$$\begin{aligned}\sum_{\Pi\in\mathbf{\Pi}^+} |K_6^{(3)}(\mathcal{J}_{\Pi,\mathcal{O}_0})| &= |\mathbf{\Pi}|\alpha^{|\mathcal{O}_0|}(1-\alpha)^{8-|\mathcal{O}_0|}d_{12}^4 d_{23}^4 d_{13}^4 n^6(1\pm\frac{\vartheta}{2}\pm o(1))\\ &= \binom{m}{n}^3 \alpha^{|\mathcal{O}_0|}(1-\alpha)^{8-|\mathcal{O}_0|}d_{12}^4 d_{23}^4 d_{13}^4 n^6(1\pm\frac{3\vartheta}{4}).\end{aligned}$$

Combining the above equality with (3.12), we then conclude

$$\begin{aligned}|\mathcal{O}_0^{ind}(\mathcal{H})| &= \alpha^{|\mathcal{O}_0|}(1-\alpha)^{8-|\mathcal{O}_0|}d_{12}^4 d_{23}^4 d_{13}^4 (2n)^6(1\pm\frac{3\vartheta}{4}\pm o(1))\pm o(n^6)\\ &= \alpha^{|\mathcal{O}_0|}(1-\alpha)^{8-|\mathcal{O}_0|}d_{12}^4 d_{23}^4 d_{13}^4 m^6(1\pm\vartheta),\end{aligned}$$

as promised.

■

We now complete the proof of Proposition 3.17 (and so we also finish the proof of Theorem 3.10) by showing Claim 3.18.

Proof of Claim 3.18. For every edge $O_{hij} = \{\sigma_{1h}, \sigma_{2i}, \sigma_{3j}\} \in \mathcal{O}\backslash\mathcal{O}_0$, $1 \leq h, i, j \leq 2$, we define the 6-partite 3-uniform hypergraphs

$$\begin{aligned}\tilde{\mathcal{J}}_{O_{hij}} &= \mathcal{J}\cup\mathcal{H}[V_{1h}, V_{2i}, V_{3j}]\ (= K_3(P[V_{1h}, V_{2i}, V_{3j}]),\\ \tilde{\mathcal{J}} &= \bigcup_{O\in\mathcal{O}\backslash\mathcal{O}_0} \tilde{\mathcal{J}}_O,\ \text{and}\\ \hat{\mathcal{J}}_{O_{hij}} &= \tilde{\mathcal{J}}\backslash\mathcal{J}[V_{1h}, V_{2i}, V_{3j}]\ (= \mathcal{H}[V_{1h}, V_{2i}, V_{3j}]).\end{aligned}$$

Observe that by the inclusion exclusion principle we have

$$\begin{aligned}|K_6^{(3)}(\mathcal{J})| &= |K_6^{(3)}(\tilde{\mathcal{J}})| - \Big|\bigcup_{O\in\mathcal{O}\backslash\mathcal{O}'} K_6^{(3)}(\hat{\mathcal{J}}_O)\Big|\\ &= |K_6^{(3)}(\tilde{\mathcal{J}})| - \sum_{\emptyset\neq\mathcal{O}_0'\subseteq\mathcal{O}\backslash\mathcal{O}_0} (-1)^{|\mathcal{O}_0'|}\Big|\bigcap_{O\in\mathcal{O}_0'} K_6^{(3)}(\hat{\mathcal{J}}_O)\Big|\end{aligned}$$

Note that

$$\bigcap_{O\in\mathcal{O}_0'} K_6^{(3)}(\hat{\mathcal{J}}_O) = K_6^{(3)}\Big(\bigcap_{O\in\mathcal{O}_0'} \hat{\mathcal{J}}_O\Big).$$

Letting $\bigcap_{O\in\emptyset}\hat{\mathcal{J}}_O=\tilde{\mathcal{J}}$ the above equation reduces to

$$|K_6^{(3)}(\mathcal{J})|=\sum_{\mathcal{O}_0'\subseteq\mathcal{O}\backslash\mathcal{O}_0}(-1)^{|\mathcal{O}_0'|}\Big|K_6^{(3)}\Big(\bigcap_{O\in\mathcal{O}_0'}\hat{\mathcal{J}}_O\Big)\Big|. \tag{3.13}$$

For fixed $\mathcal{O}_0'\subseteq\mathcal{O}\backslash\mathcal{O}_0$ and fixed $1\le a\le b\le c\le 3$ and $1\le h,i,j\le 2$ it follows from the definition of $\hat{\mathcal{J}}$ that

$$\bigcap_{O\in\mathcal{O}_0'}\hat{\mathcal{J}}_O[V_{ah},V_{bi},V_{cj}]=\begin{cases}\mathcal{H}[V_{ah},V_{bi},V_{cj}] \text{ if } \{\sigma_{ah},\sigma_{bi},\sigma_{cj}\}\in\mathcal{O}_0\cup\mathcal{O}_0'\\ K_3(P[V_{ah},V_{bi},V_{cj}]) \text{ otherwise.}\end{cases}$$

Therefore,

$$\bigcap_{O\in\mathcal{O}_0'}\hat{\mathcal{J}}_O[V_{ah},V_{bi},V_{cj}] \text{ is } \begin{cases}(\alpha\pm o(1),2\delta)\text{-minimal if } \{\sigma_{ah},\sigma_{bi},\sigma_{cj}\}\in\mathcal{O}_0\cup\mathcal{O}_0'\\ (1,2\delta)\text{-minimal otherwise}\end{cases}$$

w.r.t. $P[V_{ah},V_{bi},V_{cj}]$. Moreover, we also know for $1\le a\le b\le 3$ and $1\le i,j\le 2$ that

$$P[V_{ai},V_{bj}] \text{ is } \begin{cases}(d_{ab},2\varepsilon)\text{- regular if } a\neq b,\\ (1,2\varepsilon)\text{- regular otherwise.}\end{cases}$$

Since all the conditions for Theorem 3.12 are satisfied for each term in (3.13) we have

$$\begin{aligned}|K_6^{(3)}(\mathcal{J})|&=\sum_{\mathcal{O}_0'\subseteq\mathcal{O}\backslash\mathcal{O}_0}(-1)^{|\mathcal{O}_0'|}\Big|K_6^{(3)}\Big(\bigcap_{O\in\mathcal{O}_0'}(\hat{\mathcal{J}}_O)\Big)\Big|\\ &=\sum_{\mathcal{O}_0'\subseteq\mathcal{O}\backslash\mathcal{O}_0}(-1)^{|\mathcal{O}_0'|}\Big((1\pm\mu)(\alpha\pm o(1))^{|\mathcal{O}_0+\mathcal{O}_0'|}d_{12}^4d_{23}^4d_{13}^4n^6\Big)\\ &=(1\pm 2\mu)\alpha^{|\mathcal{O}_0|}d_{12}^4d_{23}^4d_{13}^4n^6\sum_{\mathcal{O}_0'\subseteq\mathcal{O}\backslash\mathcal{O}_0}(-\alpha)^{|\mathcal{O}_0'|}\\ &=(1\pm 2\mu)\alpha^{|\mathcal{O}_0|}(1-\alpha)^{|\mathcal{O}|-|\mathcal{O}_0|}d_{12}^4d_{23}^4d_{13}^4n^6.\end{aligned}$$

With $|\mathcal{O}|=8$ and $2\mu<\vartheta/2$, the proof of Claim 3.18 is complete.

■

Chapter 4

Algorithmic Quasirandom Lemma

4.1 Statement of the Algorithmic Quasirandom Lemma

In this section we state our second main result the algorithmic quasirandom lemma for 3-uniform hypergraphs. It is based on Gowers' Quasirandom Lemma (see Theorem in [14]). His proof involved probabilistic arguments which we here derandomize to obtain an 'efficient' algorithm.

Theorem 4.1 (Algorithmic Quasirandom Lemma) *For all* γ, $\delta > 0$ *and functions* $\varepsilon : (0,1] \to (0,1]$ *there exist positive integers* P_0 *and* N_0 *so that the following holds. For every* 3*-uniform hypergraph* $\mathcal{H}$ *on vertex set* $V = V(\mathcal{H})$*, where* $|V| = N > N_0$*, one can construct in time* $O(N^6)$*:*

(i) a vertex partition $V = V_1 \cup \cdots \cup V_t$ *with* $|V_1| \leq \cdots \leq |V_t| \leq |V_1| + 1$*, and*

(ii) a pair-partition of $\binom{V}{2}$ *given by, for each* $1 \leq i < j \leq t$, $K[V_i, V_j] = G_1^{ij} \cup \cdots \cup G_{\ell_{ij}}^{ij}$*, with a total number of parts* $\sum_{1 \leq i < j \leq t} \ell_{ij} \leq P_0$ *and with the following property:*
All but γN^3 *triples* $\{v_i, v_j, v_k\} \in \binom{V}{3}$ *satisfy that whenever* $\{v_i, v_j, v_k\} \in K_3(G_a^{ij} \cup G_b^{jk} \cup G_c^{ik}) = K_3(G_{abc}^{ijk})$*, for some* $1 \leq i < j < k \leq t$ *and* $(a, b, c) \in [\ell_{ij}] \times [\ell_{jk}] \times [\ell_{ik}]$*, then*

(a) G_a^{ij}, G_b^{jk}, *and* G_c^{ik} *are, respectively,* $(d_{ija}, \varepsilon(d_{ija}))$, $(d_{jkb}, \varepsilon(d_{jkb}))$, $(d_{ikc}, \varepsilon(d_{ikc}))$*-quasirandom, with respective densities* d_{ija}, d_{jkb}, *and* d_{ikc}*;*

(b) $\mathcal{H}_{abc}^{ijk} = \mathcal{H} \cap K_3(G_{abc}^{ijk})$ *is* δ*-quasirandom w.r.t.* G_{abc}^{ijk}*.*

The proof of Theorem 4.1 is based on Algorithm 2.7 and on the upcoming Algorithm 4.3. The latter algorithm will consider a 3-partite graph $G = G^{XY} \cup G^{YZ} \cup G^{XZ}$ with vertex 3-partition $X \cup Y \cup Z$ where all bipartite graphs are quasirandom and a hypergraph $\mathcal{H}$ defined on $\mathcal{H} \subseteq K_3(G)$ which is not quasirandom w.r.t. G. In order to introduce Algorithm 4.3 we need some more definitions. In particular, we need to define the *index*-function which plays a pivotal role in the proof of Theorem 4.3. Let us recall that for a hypergraph $\mathcal{H}$ and vertices x, y, and z we let $\mathcal{H}(x, y, z)$ be the indication function for the edges of $\mathcal{H}$, i.e.

$$\mathcal{H}(x, y, z) = \begin{cases} 1 & \text{if } (x, y, z) \text{ is an edge in } \mathcal{H} \\ 0 & \text{otherwise.} \end{cases}$$

Also recall that $\alpha = d(\mathcal{H}|G) = |\mathcal{H}|/|K_3(G)|$ denotes the relative density of $\mathcal{H}$ w.r.t. G. Furthermore, consider the partition $G^{XY} = G_1^{XY} \cup \cdots \cup G_{p_{XY}}^{X,Y}$, $G^{YZ} = G_1^{YZ} \cup \cdots \cup G_{p_{YZ}}^{YZ}$, and $G^{XZ} = G_1^{XZ} \cup \cdots \cup G_{p_{XZ}}^{XZ}$ of the bipartite graphs of G. For each triangle with vertices $x \in X$, $y \in Y$ and $z \in Z$ in G define its *triad* to be the triple (i, j, k) such that $xy \in G_i^{XY}$, $yz \in G_j^{YZ}$, and $xz \in G_k^{XZ}$. The induced partition of $K_3(G)$ is the partition into at most $p = p_{XZ} p_{YZ} p_{XZ}$ cells according to which triad they belong to, i.e. $K_3(G) = \bigcup_{i=1}^{p} \Delta_i$. Note that a typical cell (i.e. one Δ_{ijk}; $1 \leq i \leq p_{XY}$, $1 \leq j \leq p_{YZ}$, $1 \leq k \leq p_{XZ}$) is of the form $\Delta_{ijk} = K_3(G_i^{XY} \cup G_j^{YZ} \cup G_k^{X,Z})$. For simplicity of notation we denote the number of triangles in G by T. Then, we can define the *index* of the tripartite 3-uniform hypergraph $\mathcal{H}$ with respect to the above partition.

Definition 4.2 (index of $\mathcal{H}$) *Let $\mathcal{H}$ and the partition Δ_i be as above. Then the index of $\mathcal{H}$ with respect to Δ_i is defined to be*

$$ind(\mathcal{H}, (\Delta_i)_{i=1}^p) = \frac{1}{T}\sum_{i=1}^{p} |\Delta_i| \left(|\Delta_i|^{-1} \sum_{(x,y,z)\in\Delta_i} \mathcal{H}(x,y,z) \right)^2 .$$

We are now able to state the main algorithm for the proof of Theorem 4.1.

Algorithm 4.3

Input: $1 > \alpha > 0$, $d_0, \delta, \varepsilon > 0$, *graph* G, *hypergraph* $\mathcal{H}$ *satisfying:*

1. $G = G^{XY} \cup G^{YZ} \cup G^{XZ}$ *has tripartition* $V(G) = X \cup Y \cup Z$, $m = |X| \leq |Y| \leq |Z| \leq m + 1$;
2. G^{XY}, G^{YZ}, G^{XZ} *are* ε*-quasirandom with respective densities* d_{XY}, d_{YZ}, $d_{XZ} \geq d_0$;
3. $\varepsilon = \varepsilon(d_0)$ *is sufficiently small,*
4. $\mathcal{H} \subseteq K_3(G)$ *where* $\alpha = d(\mathcal{H}|G)$, *but* $\mathcal{H}$ *is* <u>*not*</u> (α, δ)*-quasirandom w.r.t.* G.

Output:

$$\begin{aligned} G^{XY} &= G_1^{XY} \cup \cdots \cup G_{p_{XY}}^{XY}, \\ G^{YZ} &= G_1^{YZ} \cup \cdots \cup G_{p_{YZ}}^{YZ}, \\ G^{XZ} &= G_1^{XZ} \cup \cdots \cup G_{p_{XZ}}^{XZ}, \\ \mathbf{\Delta} &= (\Delta_i)_{i=1}^p \text{ so that} \end{aligned}$$

I. $ind(\mathcal{H}, \mathbf{\Delta}) \geq \alpha^2 + \delta^2/2^{10}$;

II. $p \leq 27^{1+d_0^{-12}}$.

Complexity: $O(m^5)$.

We present Agorithm 4.3 in Section 4.3.

4.2 Proof of Theorem 4.1

In this section we prove the algorithmic quasirandom lemma. The proof is based on iteratively applying Algorithms 2.7 and 4.3 and is similar to Szemeredi's original proof of the regularity lemma. As in Szemeredi's original proof the *index* function plays a pivotal role. Therefore, we will first prove some properties of this function which we will need later. Let us start with generalizing the definition of the *index*-function from (indicator function of) hypergraphs to arbitrary functions.

Definition 4.4 (index of f) *Let U be a set of size T and let $f : U \to [-1,1]$ be a function. Let further $\Delta_1, \ldots, \Delta_p$ be a partition of U. Then the index of f with respect to the partition $\Delta_1, \ldots, \Delta_p$ is defined to be*

$$ind(f, (\Delta_i)_{i=1}^p) = \frac{1}{T} \sum_{i=1}^{p} |\Delta_i| \left(|\Delta_i|^{-1} \sum_{x \in \Delta_i} f(x) \right)^2 .$$

We now state a technical lemma that we will need in the proof of Algorithm 4.1. For simplicity, from now on we denote the 2-norm of a function f, $||f||_2$ by $||f||$.

Lemma 4.5 *Let U be a set of size T and let $f : U \to [-1,1]$. Furtheromore, let $\Delta_1, \ldots, \Delta_p$ be a partition of U and let $g : U \to [-1,1]$ be a function that is constant on each Δ_i. Then,*

$$ind(f, (\Delta_i)_{i=1}^p) \geq \left(\frac{\langle f, g \rangle}{\sqrt{T}||g||} \right)^2 .$$

Proof. Since g is by assumption constant on each set Δ_i, let a_i be the value taken by g on Δ_i. Then, by the Cauchy-Schwartz Inequality and multiplying by $\sqrt{|\Delta_i|}/\sqrt{|\Delta_i|}$,

$$\begin{aligned}\langle f, g\rangle &= \sum_{x\in U} f(x)g(x)\\ &= \sum_{i=1}^{p} a_i \sum_{x\in\Delta_i} f(x)\\ &\overset{CS}{\leq} \left(\sum_{i=1}^{p} |\Delta_i| a_i^2\right)^{1/2} \left(\sum_{i=1}^{p} |\Delta_i|^{-1} \left(\sum_{x\in\Delta_i} f(x)\right)^2\right)^{1/2}\\ &= ||g|| \left(\sum_{i=1}^{p} |\Delta_i| \left(|\Delta_i|^{-1} \sum_{x\in\Delta_i} f(x)\right)^2\right)^{1/2}\\ &= ||g|| \left(T \cdot ind(f, (\Delta_i)_{i=1}^{p})\right)^{1/2}.\end{aligned}$$

Hence, the conclusion of the lemma follows.

■

For $G = G^{XY} \cup G^{YZ} \cup G^{XZ}$ a 3-partite graph with 3-partition $V(G) = X \cup Y \cup Z$, where $m = |X| \leq |Y| \leq |Z| \leq m+1$ and $\mathcal{H} \subseteq K_3(G)$ we would like to find a relation between Definition 4.4 and Definition 4.2 where our function of interest will be:

$$h(x, y, z) = \mathcal{H}(x, y, z) - \alpha$$

where $\alpha = d(\mathcal{H}|G) = |\mathcal{H}|/K_3(G)$ and $\mathcal{H}(x, y, z)$ is the characteristic function of the hyperedges in $\mathcal{H}$. The underlying partition will be the same as in Definition 4.2, i.e. let

$$\begin{aligned}G^{XY} &= G_1^{XY} \cup \cdots \cup G_{p_{XY}}^{XY},\\ G^{YZ} &= G_1^{YZ} \cup \cdots \cup G_{p_{YZ}}^{YZ}\\ G^{XZ} &= G_1^{XZ} \cup \cdots \cup G_{p_{XZ}}^{XZ}\end{aligned}$$

be arbitrary partitions into p_{XY}, p_{YZ} and p_{XZ} parts, respectively. These partitions induce a partition of $K_3(G)$ into at most $p = p_{XY}p_{YZ}p_{XZ}$ classes defined as follows: for $1 \le i \le p_{XY}$, $1 \le j \le p_{YZ}$ and $1 \le k \le p_{XZ}$, let

$$\Delta_{ijk} = K_3\big(G_i^{XY} \cup G_j^{YZ} \cup G_k^{XZ}\big).$$

The definition of the function h allows us now to relate the index of h to the index of $\mathcal{H}$ as follows.

Lemma 4.6 *Let the graph G, hypergraph $\mathcal{H}$, the function h, the relative density α, and the partition Δ_i be defined as above. Then the following holds:*

$$ind(h, (\Delta_i)_{i=1}^p) = ind(\mathcal{H}, (\Delta_i)_{i=1}^p) - \alpha^2.$$

Proof. Recall that we denote by T the number of triangles in G and p denotes the number of classes of the partition of $K_3(G)$.

$$\begin{aligned}
ind(h, (\Delta_i)_{i=1}^p) &= \frac{1}{T}\sum_{i=1}^{p} |\Delta_i| \left(|\Delta_i|^{-1} \sum_{(x,y,z)\in\Delta_i} h(x,y,z) \right)^2 \\
&= \frac{1}{T}\sum_{i=1}^{p} |\Delta_i| \left(|\Delta_i|^{-1} \sum_{(x,y,z)\in\Delta_i} \mathcal{H}(x,y,z) - \alpha \right)^2 \\
&= \frac{1}{T}\sum_{i=1}^{p} |\Delta_i| \left(\left(|\Delta_i|^{-1} \sum_{(x,y,z)\in\Delta_i} \mathcal{H}(x,y,z) \right) - \alpha \right)^2 \\
&= \frac{1}{T}\sum_{i=1}^{p} |\Delta_i| \left(|\Delta_i|^{-1} \sum_{(x,y,z)\in\Delta_i} \mathcal{H}(x,y,z) \right)^2 \\
&\qquad + \sum_{i=1}^{p} \frac{|\Delta_i|}{T} \left(\alpha^2 - 2\alpha \sum_{(x,y,z)\in\Delta_i} \frac{\mathcal{H}(x,y,z)}{|\Delta_i|} \right) \\
&= ind(\mathcal{H}, (\Delta_i)_{i=1}^p) + \alpha^2 - 2\alpha \sum_{i=1}^{p} \frac{1}{T} \sum_{(xyz)\in\Delta_i} \mathcal{H}(x,y,z) \\
&= ind(\mathcal{H}, (\Delta_i)_{i=1}^p) - \alpha^2.
\end{aligned}$$

■

Let us recall the familiar fact that if a partition $\mathbf{\Delta'} = (\Delta'_1, \ldots, \Delta'_q)$ of $K_3(G)$ *refines* $\mathbf{\Delta} = (\Delta_1, \ldots, \Delta_p)$, then $ind(\mathcal{H}, \mathbf{\Delta'}) \geq ind(\mathcal{H}, \mathbf{\Delta})$. Indeed, if $\mathbf{\Delta'}$ refines $\mathbf{\Delta}$, then we may write $\mathbf{\Delta} = (\Delta'_{ij} : 1 \leq i \leq p, 1 \leq j \leq q_i)$, for some integers $q_1, \ldots, q_p \geq 1$, where for each $1 \leq i \leq p$, $\Delta_i = \Delta'_{i1} \cup \cdots \cup \Delta'_{iq_i}$. Then for a fixed $1 \leq i \leq p$, the Cauchy-Schwarz inequality gives

$$\begin{aligned}
\sum_{j=1}^{q_i} \frac{|\mathcal{H} \cap \Delta'_{ij}|^2}{|\Delta'_{ij}|} &= \sum_{j=1}^{q_i} |\Delta'_{ij}| \Big(\frac{|\mathcal{H} \cap \Delta'_{ij}|}{|\Delta'_{ij}|} \Big)^2 \\
&\geq \frac{\left(\sum_{j=1}^{q_i} |\Delta'_{ij}| \times \frac{|\mathcal{H} \cap \Delta'_{ij}|}{|\Delta'_{ij}|} \right)^2}{\sum_{j=1}^{q_i} |\Delta'_{ij}|} \\
&= \frac{|\mathcal{H} \cap \Delta_i|}{|\Delta_i|}.
\end{aligned}$$

Therefore, we get

$$\begin{aligned}
ind(\mathcal{H}, \mathbf{\Delta'}) &= \frac{1}{T} \sum_{i=1}^{p} \sum_{j=1}^{q_i} \frac{|\mathcal{H} \cap \Delta'_{ij}|^2}{|\Delta'_{ij}|} \\
&\geq \frac{1}{T} \sum_{i=1}^{p} \frac{|\mathcal{H} \cap \Delta_i|^2}{|\Delta_i|} \\
&= ind(\mathcal{H}, \mathbf{\Delta}).
\end{aligned}$$

We collected all tools in order to prove Theorem 4.1.

Proof of Theorem 4.1. Let the constants γ, $\delta > 0$ be given as well as the function $\varepsilon : (0,1] \to (0,1]$. We may assume w.l.o.g. that $\varepsilon(x) \leq x$ and $\varepsilon(x)$ is small enough to enable an application of Algorithm 4.3 with $d_0 = x$. We shall not explicitly define the constant $P_0 = P_0(\gamma, \delta, \varepsilon)$, but we will describe it within the proof. Also, we shall choose the constant $N_0 = N_0(\gamma, \delta, \varepsilon, P_0)$ sufficiently large whenever needed. Furthermore, let $\mathcal{H}$ be a 3-uniform hypergraph on the vertex set $V = V(\mathcal{H})$ of size $|V| = N > N_0$. We now describe how to construct in $O(N^6)$ time our desired partition $\mathbf{\Pi}$ of V and $\binom{V}{2}$. We will show it by induction over the number of iterations.

4.2.1 Iteration I:

We will start by constructing a partition $\mathbf{\Pi}^{(1)}$. Therefore, we set $t_1 = \lceil 2/\gamma \rceil$ and let

$$V(\mathcal{H}) = U_1 \cup \cdots \cup U_{t_1}, \quad |U_1| \leq \cdots \leq |U_{t_1}| \leq |U_1| + 1,$$

be an arbitrary vertex partition. Furthermore, we set $l_1 = 1$ and let the complete bipartite graphs $K^{ij} = K[U_i, U_j]$, $1 \leq i < j \leq t_1$, be their own pair-partition. Now, we define $\mathbf{\Pi}$ to be the above described family of partitions constructible in linear time. We have to check that $\mathbf{\Pi}^{(1)}$ satisfies the conclusion of Theorem 4.1. Note that all but

$$\begin{aligned} t_1 \binom{\lceil N/t_1 \rceil}{2} N &= \frac{N^3}{2t_1} + O(N^2) \\ &\leq \frac{\gamma}{4} N^3 + O(N^2) \\ &\leq \frac{\gamma}{2} N^3 \end{aligned}$$

triples $\{x, y, z\} \in \binom{V}{3}$ cross the vertex partition $U_1 \cup \cdots \cup U_{t_1}$. Therefore, for some $1 \leq i < j < k \leq t_1$ the triple $\{x, y, z\} \in K_3(K^{ijk})$ where $K^{ijk} = K^{ij} \cup K^{jk} \cup K^{ik}$ is a triad of $\mathbf{\Pi}^{(1)}$. Observe that for every triad K^{ijk}, by construction each constituent bipartite graph K^{ij}, K^{jk}, and K^{ik} is $(1, 0)$-quasirandom. It remains to check if $\mathcal{H}^{ijk} = \mathcal{H} \cap \Delta(K^{ijk})$ is δ-quasirandom w.r.t. K^{ijk}. Therefore, we count the number of (crossing) triples $\{x, y, z\} \in \binom{V}{3}$ belonging to triads K^{ijk} for which $\mathcal{H}^{ijk}$ is not δ-quasirandom, i.e. for which

$$\sum_{u_i, u_i' \in U_i} \sum_{u_j, u_j' \in U_j} \sum_{u_k, u_k' \in U_k} h_{u_i, u_i', u_j, u_j', u_k, u_k'} \geq \delta (d_{ij} d_{jk} d_{ik})^4 |U_i|^2 |U_j|^2 |U_k|^2,$$

with densities $d_{ij} = d_{jk} = d_{ik} = 1$. This can be done in $O(N^6)$. We denote the 'bad' triads K^{ijk}, i.e., those which are not δ-quasirandom, by the indexing set $I_{bad} \subseteq \binom{[t_1]}{3}$, and compute, in time $O(N^3)$ the sum $\sum_{\{i,j,k\} \in I_{bad}} |K_3(K^{ijk})|$. If this sum is less than $(\gamma/2)N^3$, we are done, and

$\mathbf{\Pi} = \mathbf{\Pi}^{(1)}$ is the partition we seek. Otherwise,

$$\sum_{\{i,j,k\} \in I_{bad}} |K_3(K^{ijk})| \geq \frac{\gamma}{2} N^3 \quad \Rightarrow \quad |I_{bad}| \geq \frac{\gamma}{4} t_1^3. \tag{4.1}$$

In this case we refine $\mathbf{\Pi}^{(1)}$ to construct a new partition $\mathbf{\Pi}^{(2)}$.

We begin by establishing some formal considerations. For $1 \leq i < j < k \leq t_1$, write Δ_{ijk} as the trivial partition of $K_3(K^{ijk})$ given by the single class $K_3(K^{ijk})$. Then by the definition of the index (see Definition 4.2), we have

$$ind(\mathcal{H}^{ijk}, \Delta_{ijk}) = \alpha_{ijk}^2, \quad \text{where} \quad \alpha_{ijk} = d(\mathcal{H}^{ijk} | K^{ijk}).$$

Now, we define the index of $\mathbf{\Pi}^{(1)}$.

$$\begin{aligned} ind(\mathcal{H}, \mathbf{\Pi}^{(1)}) &= t_1^{-3} \sum_{1 \leq i < j < k \leq t_1} ind(\mathcal{H}_{ijk}, \Delta_{ijk}) \\ &= t_1^{-3} \sum_{1 \leq i < j < k \leq t_1} \alpha_{ijk}^2 \end{aligned}$$

Our goal is to construct a partition $\mathbf{\Pi}^{(2)}$ for which $ind(\mathcal{H}, \mathbf{\Pi}^{(2)})$ (defined later) is non-trivially larger than $ind(\mathcal{H}, \mathbf{\Pi}^{(1)})$. First, we will apply Algorithm 4.3 for which we now prepare.

For fixed $\{i, j, k\} \in I_{bad}$ we know that $\alpha_{ijk} = d(\mathcal{H}_{ijk} | K^{ijk}) \neq 0, 1$, since otherwise $\mathcal{H}^{ijk}$ would be quasirandom w.r.t. K^{ijk}. By construction, we know that each of the K^{ij}, K^{jk}, and K^{ik} are $(1, 0)$-quasirandom, so in particular, they are also $(1, \varepsilon_1)$-quasirandom, where $\varepsilon_1 = \varepsilon(1)$. Since all the conditions for an application of Algorithm 4.3 are met for $\mathcal{H}^{ijk}$ and K^{ijk} we construct the following partitions in $O(N^5)$

$$\begin{aligned} K^{ij} &= G_1^{ij} \cup \cdots \cup G_{p_{ij}}^{ij}, \\ K^{ij} &= G_1^{jk} \cup \cdots \cup G_{p_{jk}}^{ij}, \\ K^{ij} &= G_1^{ik} \cup \cdots \cup G_{p_{ik}}^{ij}; \end{aligned}$$

for some integers p_{ij}, p_{jk}, and p_{ik}. By the conclusion of Algorithm 4.3 we also know that the resulting partition Δ'_{ijk} which consists of $27^{1+\frac{1}{d_0^{-12}}} =$

$27^2 = 729$ classes of the form $\Delta^{ijk}_{abc} = K_3(G^{ij}_a \cup G^{jk}_b \cup G^{ik}_c)$, for $(a, b, c) \in [p_{ij}] \times [p_{jk}] \times [p_{ik}]$, satisfies

$$ind(\mathcal{H}^{ijk}, \Delta'_{ijk}) \geq \alpha^2_{ijk} + \frac{\delta^2}{2^{10}}.$$

For $\{i, j, k\} \notin I_{bad}$, we let

$$\begin{aligned} G^{ij}_1 &= K^{ij} \\ G^{jk}_1 &= K^{jk} \\ G^{ik}_1 &= K^{ik} \\ \Delta'_{ijk} &= \Delta_{ijk}. \end{aligned}$$

Since in any case Δ'_{ijk} refines Δ_{ijk} we conclude

$$ind(\mathcal{H}^{ijk}, \Delta'_{ijk}) \geq \begin{cases} \alpha^2_{ijk} + \frac{\delta^2}{2^{10}} & \text{if } \{i, j, k\} \in I_{bad}, \\ \alpha^2_{ijk} & \text{else.} \end{cases}$$

We now further refine the partitions above to get a common refinement. Let $K^{ijk_1}, \ldots, K^{ijk_r}$, $\{i, j\} \cap \{k_1, \ldots, k_r\} \neq \emptyset$, be the bad triads which include K^{ij} as a bipartite subgraph. Let

$$\begin{aligned} K^{ij}_1 &= G^{ij}_{11} \cup \cdots \cup G^{ij}_{p_{ij1}1} \\ &, \ldots, \\ K^{ij}_r &= G^{ij}_{1r} \cup \cdots \cup G^{ij}_{p_{ijr}r} \end{aligned}$$

be the partitions constructed by Algorithm 4.3. Now, construct in $O(N^2)$ the unique minimal partition of K^{ij}

$$K^{ij} = \tilde{G}^{ij}_1 \cup \cdots \cup \tilde{G}^{ij}_{q_{ij}}, \quad q_{ij} \leq 2^{t_1 27^2};$$

i.e.

$$\text{for all } e_1, e_2 \in K^{ij}, e_1 \sim e_2 \Leftrightarrow e_1 \text{ and } e_2 \text{ lie in the same partition class of } K^{ij}_s \text{ for all } 1 \leq s \leq r.$$

Similarly, we also construct

$$K^{jk} = \tilde{G}_1^{jk} \cup \cdots \cup \tilde{G}_{q_{jk}}^{jk}, \quad q_{jk} \le 2^{t_1 27^2}$$
$$K^{ij} = \tilde{G}_1^{ik} \cup \cdots \cup \tilde{G}_{q_{ik}}^{ik}, \quad q_{ik} \le 2^{t_1 27^2}.$$

Let $\tilde{\Delta}_{ijk}$ be the resulting partition of $\Delta(K^{ijk})$ whose classes are of the form

$$\tilde{\Delta}_{abc}^{ijk} = \Delta(\tilde{G}_a^{ij} \cup \tilde{G}_b^{jk} \cup \tilde{G}_c^{ik}), \quad (a,b,c) \in [q_{ij}] \times [q_{jk}] \times [q_{ik}].$$

Then $\tilde{\Delta}_{ijk}$ refines the partition Δ'_{ijk}, and therefore

$$\begin{aligned} ind(\mathcal{H}^{ijk}, \tilde{\Delta}_{ijk}) &\ge ind(\mathcal{H}^{ijk}, \Delta'_{ijk}) \\ &\ge \begin{cases} \alpha_{ijk}^2 + \frac{\delta^2}{2^{10}} & \text{if } \{i,j,k\} \in I_{bad} \\ \alpha_{ijk}^2 & \text{otherwise.} \end{cases} \end{aligned}$$

We continue refining the above partitions by applying Algorithm 2.7 since none of the bipartite graphs $\tilde{G}_a^{ij}$, $1 \le i < j \le t_1$, $1 \le a \le q_{ij}$, are guaranteed to be η-quasirandom for some $\eta \in (0,1)$. We apply Algorithm 2.7 with the following constants $t_0 = t_1$ and $l_0 = 2^{t_1 27^2}$ and set $\varepsilon_2 = \varepsilon(\gamma/(10 l_0))$. Then Algorithm 2.7 constructs in time $O(N^{2.376})$ vertex partitions $U_i = U_{i1} \cup \cdots \cup U_{it}$, $1 \le i \le t_1$; where $1 \le t \le T_0 = T_0(t_0, l_0, \varepsilon_2)$. We also know that all but $\varepsilon_2 (t_1 t)^2 l_0$ of the bipartite graphs $\hat{G}_a^{ij}(i',j') = \tilde{G}_a^{ij}[U_{ii'}, U_{jj'}]$ are ε_2-quasirandom, $1 \le i < j \le t_1$, $1 \le a \le q_{ij}$, $1 \le i', j' \le t$. Then for a fixed $1 \le i < j < k \le t_1$, this provides a refinement $\hat{\Delta}_{ijk}$ of $\tilde{\Delta}_{ijk}$ whose classes are of the form

$$\begin{aligned} \hat{\Delta}_{abc}^{ijk}(i',j',k') &= K_3(\hat{G}_a^{ij}(i',j') \cup \hat{G}_b^{jk}(j',k') \cup \hat{G}_c^{ik}(i',k')) \\ &= K_3(\hat{G}_{abc}^{ijk}(i',j',k')); \end{aligned}$$

$1 \le i', j', k' \le t$, $1 \le a \le q_{ij}$, $1 \le b \le q_{jk}$, and $1 \le c \le q_{ik}$, and therefore

$$\begin{aligned} ind(\mathcal{H}^{ijk}, \hat{\Delta}_{ijk}) &\ge ind(\mathcal{H}^{ijk}, \tilde{\Delta}_{ijk}) \\ &\ge \begin{cases} \alpha_{ijk}^2 + \frac{\delta^2}{2^{10}} & \text{if } \{i,j,k\} \in I_{bad} \\ \alpha_{ijk}^2 & \text{otherwise.} \end{cases} \end{aligned} \tag{4.2}$$

We are now ready to define $\mathbf{\Pi}^{(2)}$. For simplicity we first re-index the vertex sets $U_{ii'}$, $1 \le i \le t_1$, $1 \le i' \le t$, to $U_i^{(2)}$, $1 \le i \le t_2 = t_1 t$. Correspondingly, re-index the bipartite graphs $\hat{G}_a^{ij}(i', j')$, $1 \le i < j \le t_1$, $1 \le i', j' \le t$, $1 \le a \le q_{ij} \le l_2 = l_0 = 2^{t_1 27^2}$, to G_a^{ij} where now $1 \le i < j \le t_2$ and $1 \le a \le q_{ij}$ (there is no ambiguity in the parameter q_{ij}). Using the terminology after re-indexing we have that all but $\varepsilon_2 t_2^2 l_2$ of the bipartite graphs G_a^{ij} of $\mathbf{\Pi}^{(2)}$ are ε_2-quasirandom and the triads are of the form $G_{abc}^{ijk} = G_a^{ij} \cup G_b^{jk} \cup G_c^{ik}$, $1 \le i < j < k \le t_2$, $(a, b, c) \in [q_{ij}] \times [q_{jk}] \times [q_{ik}]$. These triads provide triangle partitions $\Delta_{ijk}^{(2)}$, $1 \le i < j < k \le t_2$, whose classes are of the form

$$\Delta_{abc}^{ijk} = K_3(G_{abc}^{ijk})$$

$(a, b, c) \in [q_{ij}] \times [q_{jk}] \times [q_{ik}]$. By the definition of the index of $\mathbf{\Pi}^{(2)}$ is

$$ind(\mathcal{H}, \mathbf{\Pi}^{(2)}) = t_2^{-3} \sum_{1 \le i < j < k \le t_2} ind(\mathcal{H}^{ijk}, \Delta_{ijk}^{(2)}).$$

It remains to show that we non-trivially increased the index of $\mathbf{\Pi}^{(2)}$ compared to the index of $\mathbf{\Pi}^{(1)}$, i.e., we will show that

$$ind(\mathcal{H}, \mathbf{\Pi}^{(2)}) \ge ind(\mathcal{H}, \mathbf{\Pi}^{(1)}) + \frac{\gamma \delta^2}{2^{13}}.$$

Applying the definition of the index and the notation prior to re-indexing, $ind(\mathcal{H}, \mathbf{\Pi}^{(2)})$ may be expressed as

$$\begin{aligned} ind(\mathcal{H}, \mathbf{\Pi}^{(2)}) &= t_2^{-3} \sum_{1 \le i < j < k \le t_2} ind(\mathcal{H}^{ijk}, \Delta_{ijk}^{(2)}) \\ &= (t_1 t)^{-3} \sum_{i,j,k} \sum_{i',j',k'} |K_3(K_{i'j'k'}^{ijk})|^{-1} \\ &\quad \times \sum_{a,b,c} |\mathcal{H} \cap \Delta_{abc}^{ijk}(i', j', k')|^2 |\Delta_{abc}^{ijk}(i', j', k')|^{-1}; \end{aligned}$$

where $1 \le i < j < k \le t_1$, $1 \le i', j', k' \le t$, $(a, b, c) \in [q_{ij}] \times [q_{jk}] \times [q_{ik}]$, and where

$$\begin{aligned} K_{i'j'k'}^{ijk} &= K^{ijk}[U_{ii'}, U_{jj'}, U_{kk'}] \\ &= K^{ij}[U_{ii'}, U_{jj'}] \cup K^{jk}[U_{jj'}, U_{kk'}] \cup K^{ik}[U_{ii'}, U_{kk'}]. \end{aligned}$$

Using that $t^3|K_3(K^{ijk}_{i'j'k'})| = (1 \pm o(1))|K_3(K^{ijk})|$, where $o(1) \to 0$ as $N \to \infty$. Note that if the U_{ii}'s all have the same size we get $t^3|K_3(K^{ijk}_{i'j'k'})| = |K_3(K^{ijk})|$. Again, by the definition of the index we get

$$\begin{aligned}
ind(\mathcal{H}, \mathbf{\Pi}^{(2)}) &= (t_1 t)^{-3} \sum_{i,j,k} \sum_{i',j',k'} |K_3(K^{ijk}_{i'j'k'})|^{-1} \\
&\quad \times \sum_{a,b,c} |\mathcal{H} \cap \Delta^{ijk}_{abc}(i',j',k')|^2 |\Delta^{ijk}_{abc}(i',j',k')|^{-1} \\
&\geq (1 - o(1)) t_1^{-3} \sum_{i,j,k} |K_3(K^{ijk})|^{-1} \\
&\quad \times \sum_{i',j',k'} \sum_{a,b,c} |\mathcal{H} \cap \Delta^{ijk}_{abc}(i',j',k')|^2 |\Delta^{ijk}_{abc}(i',j',k')|^{-1} \\
&\geq (1 - o(1)) t_1^{-3} \sum_{1 \leq i < j < k \leq t_1} ind(\mathcal{H}^{ijk}, \hat{\Delta}_{ijk}) \\
&= (1 - o(1)) t_1^{-3} \\
&\quad \times \Big(\sum_{\{i,j,k\} \in I_{bad}} ind(\mathcal{H}^{ijk}, \hat{\Delta}_{ijk}) + \sum_{\{i,j,k\} \notin I_{bad}} ind(\mathcal{H}^{ijk}, \hat{\Delta}_{ijk}) \Big) \\
&\overset{(4.2)}{\geq} (1 - o(1)) t_1^{-3} \Big(\sum_{\{i,j,k\} \in I_{bad}} (\alpha^2_{ijk} + \frac{\delta^2}{2^{10}}) + \sum_{\{i,j,k\} \notin I_{bad}} \alpha^2_{ijk} \Big) \\
&\overset{(4.1)}{\geq} (1 - o(1)) \Big(\frac{\gamma \delta^2}{2^{12}} + t_1^{-3} \sum_{1 \leq i < j < k \leq t_1} \alpha^2_{ijk} \Big) \\
&= (1 - o(1)) \Big(\frac{\gamma \delta^2}{2^{12}} + ind(\mathcal{H}, \mathbf{\Pi}^{(1)}) \Big) \\
&\geq \frac{\gamma \delta^2}{2^{13}} + ind(\mathcal{H}, \mathbf{\Pi}^{(1)}),
\end{aligned}$$

as promised.

4.2.2 Iteration s:

Essentially, this proof is identical to Iteration I. We will again apply Algorithms 4.3 and 2.7 to further refine the current partition and non-trivially increase the index function. We will limit this proof to an outline.

Let $s \geq 2$ be an integer, and assume we have constructed, in time $O(N^6)$, the successively refining partitions $\mathbf{\Pi}^{(1)}, \ldots, \mathbf{\Pi}^{(s)}$, where $\mathbf{\Pi}^{(s)}$ consists of

(i) a vertex partition

$$V(\mathcal{H}) = U_1^{(s)} \cup \cdots \cup U_{t_s}^{(s)}, \text{ with } |U_1^{(s)}| \quad \leq \cdots \leq |U_{t_s}^{(s)}| \leq |U_1^{(s)}| + 1,$$

where t_s is independent of N, and

(ii) a pair partition given by, for each $1 \leq i < j \leq t_s$,

$$K^{ij}(s) = K[U_i^{(s)}, U_j^{(s)}] = G_1^{ij}(s) \cup \cdots \cup G_{\ell_{ij}^{(s)}}^{ij}(s),$$

where $\ell_{ij}^{(s)} \leq \ell_s$ for some integer ℓ_s independent of N,

and assume $\mathbf{\Pi}^{(s)}$ satisfies the following properties:

(i) all but $\varepsilon_s t_s^2 \ell_s$ bipartite graphs $G_a^{ij}(s)$, $1 \leq i < j \leq t_s$, $1 \leq a \leq \ell_{ij}^{(s)}$, are ε_s-quasirandom, where $\varepsilon_s = \varepsilon(\gamma/(10\ell_s))$,

(ii)

$$ind(\mathcal{H}, \mathbf{\Pi}^{(s)}) \geq ind(\mathcal{H}, \mathbf{\Pi}^{(s-1)}) + \gamma\delta^2/2^{13}$$

where for $1 \leq i < j < k \leq t_s$ and $(a, b, c) \in [\ell_{ij}^{(s)}] \times [\ell_{jk}^{(s)}] \times [\ell_{ik}^{(s)}]$

(a)

$$ind(\mathcal{H}, \mathbf{\Pi}^{(s)}) = t_s^{-3} \sum_{1 \leq i < j < k \leq t_s} ind(\mathcal{H}^{ijk}(s), \Delta_{ijk}^{(s)});$$

(b)

$$\begin{aligned} \mathcal{H}^{ijk}(s) &= \mathcal{H} \cap K_3(K^{ijk}(s)) \\ &= \mathcal{H} \cap K_3(K^{ij}(s) \cup K^{jk}(s) \cup K^{ik}(s)), \end{aligned}$$

(c) $\Delta_{ijk}^{(s)}$ is the partition of $K_3(K^{ijk}(s))$ whose classes are given by

$$\Delta_{abc}^{ijk}(s) = K_3(G_{abc}^{ijk}(s)) = K_3(G_a^{ij}(s) \cup G_b^{jk}(s) \cup G_c^{ik}(s)).$$

In what follows, s is assumed to be fixed, and so for simplicity, we shall abbreviate the symbols $U_i^{(s)}$, $K^{ij}(s)$, $G_a^{ij}(s)$, $l_{ij}^{(s)}$, $\mathcal{H}^{ijk}(s)$, $K^{ijk}(s)$ and $G_{abc}^{ijk}(s)$ to U_i, K^{ij}, G_a^{ij}, l_{ij}, $\mathcal{H}^{ijk}$, K^{ijk} and G_{abc}^{ijk}, respectively. (But the other references to s will stay.)

As in the first iteration, we first check whether $\mathbf{\Pi}^{(s)}$ already satisfies the conclusion of Theorem 4.1. By assumption we know, that all but $(\gamma/4)N^3$ crossing triples $\{x, y, z,\} \in \binom{V}{3}$ belong to triads G_{abc}^{ijk}, $1 \leq i < j < k \leq t_s$, $(a, b, c) \in [\ell_{ij}^{(s)}] \times [\ell_{jk}^{(s)}] \times [\ell_{ik}^{(s)}]$, for which the constituent bipartite graphs G_a^{ij}, G_b^{jk}, and G_c^{ik} are ε_s-quasirandom with respective densities d_{ija}, d_{jkb}, and $d_{ikc} > d_s = \gamma/(10\ell_s)$. Indeed, the 'bad' crossing triples contribute at most

$$\begin{aligned}(\varepsilon_s + d_s)t_s^2\ell_s\lceil N/t_s\rceil^2 N &\leq \frac{\gamma}{5}t_s^2\lceil N/t_s\rceil^2 N\\ &< \frac{\gamma}{4}N^3,\end{aligned}$$

where we used that $\varepsilon(\gamma/(10\ell_s)) \leq \gamma/(10\ell_s)$. We now determine in time $O(N^6)$ if the 'good' crossing triples, meaning triples belonging to triads for which the underlying bipartite graphs are ε_s-quasirandom and of density greater than $\gamma/(10l_s)$, also belong to triads G_{abc}^{ijk} for which $\mathcal{H}_{abc}^{ijk} = \mathcal{H} \cap K_3(G_{abc}^{ijk})$ is δ-quasirandom w.r.t. G_{abc}^{ijk}. To that end, let $I_{bad} = I_{bad}^{(s)} \subseteq \binom{[t_s]}{3} \times [\ell_s]^3$ be the indexing set for those triads G_{abc}^{ijk} of $\mathbf{\Pi}^{(s)}$ for which $G_a^{ij}, G_b^{jk} G_c^{ik}$ are ε_s-quasirandom with respective densities $d_{ija}, d_{jkb}, d_{ikc} > d_s$, but for which $\mathcal{H}_{abc}^{ijk}$ is not δ-quasirandom w.r.t. G_{abc}^{ijk}. We then compute in time $O(N^3)$ the sum $\sum_{I_{bad}} |K_3(G_{abc}^{ijk})|$. If this sum is less than $(\gamma/4)N^3$, we are done, and $\mathbf{\Pi} = \mathbf{\Pi}^{(s)}$ is the partition we seek. Otherwise,

$$\sum_{(\{i,j,k\},a,b,c)\in I_{bad}} \left|K_3(G_{abc}^{ijk})\right| \geq \frac{\gamma}{4}N^3$$

in which case we refine the partition $\mathbf{\Pi}^{(s)}$ (in $O(N^5)$) to receive a partition $\mathbf{\Pi}^{(s+1)}$ that has a non-trivially larger index than that of $\mathbf{\Pi}^{(s)}$.

Indeed, fix $(\{i,j,k\},a,b,c) \in I_{bad}$. Apply Algorithm 4.3 to $\mathcal{H}^{ijk}_{abc}$ and G^{ijk}_{abc} to construct partitions

$$\begin{aligned} G^{ij}_a &= G^{ij}_{a1} \cup \cdots \cup G^{ij}_{ap_{ija}}\,, \\ G^{jk}_b &= G^{jk}_{b1} \cup \cdots \cup G^{jk}_{bp_{jkb}}\,, \\ G^{ik}_c &= G^{ik}_{c1} \cup \cdots \cup G^{ik}_{cp_{ikc}}\,, \end{aligned}$$

so that the resulting partition $\mathbf{\Delta}^{ijk}_{abc}$ of $K_3(G^{ijk}_{abc})$ with classes of the form

$$K_3\big(G^{ijk}_{abc}(a',b',c')\big) = K_3\big(G^{ij}_{aa'} \cup G^{jk}_{bb'} \cup G^{ik}_{cc'}\big),$$

$(a',b',c') \in [p_{ija}] \times [p_{jkb}] \times [p_{ikc}]$ satisfies the following properties:

I.

$$ind(\mathcal{H}^{ijk}_{abc}, \mathbf{\Delta}^{ijk}_{abc}) \geq (\alpha^{ijk}_{abc})^2 + \delta^2/2^{10}, \quad \text{where} \quad \alpha^{ijk}_{abc} = d(\mathcal{H}^{ijk}_{abc}|G^{ijk}_{abc});$$

II.

$$p_{ija}p_{jkb}p_{ikc} \leq 27^{1+d_s^{-12}}.$$

As in the first iteration we further refine the partitions above to obtain, for each $1 \leq i < j \leq t_s$ and $1 \leq a \leq \ell_s$, a common refinement

$$G^{ij}_a = \tilde{G}^{ij}_{a1} \cup \cdots \cup \tilde{G}^{ij}_{aq_{ija}}, \quad q_{ija} \leq 2^{t_s \ell_s^2 27^{1+d_s^{-12}}} = Q_s.$$

These common refinements then yield partitions $\mathbf{\Delta}^{ijk}_{abc}$ of $K_3(G^{ijk}_{abc})$, $1 \leq i < j < k \leq t_s$, $(a,b,c) \in [\ell_{ij}] \times [\ell_{jk}] \times [\ell_{ik}]$, whose classes are of the form

$$\begin{aligned} \tilde{\Delta}^{ijk}_{abc}(a',b',c') &= K_3\big(\tilde{G}^{ijk}_{abc}(a',b',c')\big) \\ &= K_3\big(\tilde{G}^{ij}_{aa'} \cup \tilde{G}^{jk}_{bb'} \cup \tilde{G}^{ik}_{cc'}\big), \end{aligned}$$

where $(a',b',c') \in [q_{ija}] \times [q_{jkb}] \times [q_{ikc}]$. We continue refining the above partitions by applying Algorithm 2.7 to $G^{ij}_{aa'}$ $1 \leq i < j \leq t_s$, $a \in [\ell_{ij}]$, $a' \in [q_{ija}]$ since none of the bipartite graphs are guaranteed to be sufficiently quasi-random. We set $t_0 = t_s$, $\ell_0 = \ell_s Q_s$ and $\varepsilon_{s+1} = \varepsilon(\gamma/(10\ell_0))$. Algorithm 2.7

constructs, with the above choice of our constants, in $O(N^{2.376})$ vertex partitions $U_i = U_{i1} \cup \cdots \cup U_{it}$, $1 \le i \le t_s$, where $1 \le t \le T_0 = T_0(t_0, \ell_0, \varepsilon_{s+1})$. We also know by the output of Algorithm 2.7 that all but $\varepsilon_{s+1}(t_s t)^2 \ell_0$ bipartite graphs $\hat{G}^{ij}_{aa'}(i',j') = \tilde{G}^{ij}_{aa'}[U_{ii'}, U_{jj'}]$, $1 \le i < j \le t_s$, $a \in [l_{ij}]$, $a' \in [q_{ija}]$, $1 \le i', j' \le t$, are ε_{s+1}-quasirandom. Then for fixed $1 \le i < j < k \le t_s$, $(a,b,c) \in [\ell_{ij}] \times [\ell_{jk}] \times [\ell_{ik}]$, the application of Algorithm 2.7 provides a refinement $\hat{\mathbf{\Delta}}^{ijk}_{abc}$ of $\tilde{\mathbf{\Delta}}^{ijk}_{abc}$ whose classes are of the form

$$\begin{aligned}\hat{\Delta}^{ijk}_{abc}(a',b',c',i',j',k') &= K_3\big(\hat{G}^{ijk}_{abc}(a',b',c',i',j',k')\big)\\ &= K_3\big(\hat{G}^{ij}_{aa'}(i',j') \cup \hat{G}^{jk}_{bb'}(j',k') \cup \hat{G}^{ik}_{cc'}(i',k')\big)\end{aligned}$$

$(a',b',c') \in [q_{ija}] \times [q_{jkb}] \times [q_{ikc}]$, $1 \le i',j',k' \le t$. Then, for each $1 \le i < j < k \le t_s$, $(a,b,c) \in [\ell_{ij}] \times [\ell_{jk}] \times [\ell_{ik}]$ and the fact that the index function is nondecreasing w.r.t. refinements we get

$$\begin{aligned} ind(\mathcal{H}^{ijk}_{abc}, \hat{\mathbf{\Delta}}^{ijk}_{abc}) &\ge ind(\mathcal{H}^{ijk}_{abc}, \tilde{\mathbf{\Delta}}^{ijk}_{abc})\\ &\ge ind(\mathcal{H}^{ijk}_{abc}, \mathbf{\Delta}^{ijk}_{abc})\\ &\ge \begin{cases} (\alpha^{ijk}_{abc})^2 + \frac{\delta^2}{2^{10}} & \text{if } (\{i,j,k\},a,b,c) \in I_{bad},\\ (\alpha^{ijk}_{abc})^2 & \text{otherwise.}\end{cases}\end{aligned}$$

We are now ready to define the partition $\mathbf{\Pi}^{(s+1)}$. For simplicity, we first re-index the vertex sets $U_{ii'}$, $1 \le i \le t_s$, $1 \le i' \le t$, to $U^{(s+1)}_i$, $1 \le i \le t_{s+1} = t_s t$. Correspondingly, re-index the bipartite graphs $\hat{G}^{ij}_{aa'}(i',j')$, $1 \le i < j \le t_s$, $a \in [\ell_{ij}]$, $a' \in [q_{ija}]$, $1 \le i',j' \le t$, to G^{ij}_a, $1 \le i < j \le t_{s+1}$, $1 \le a \le \ell_{ij} \le \ell_{s+1} = \ell_0 = \ell_s Q_s$. Using the terminology after re-indexing we know that all but $\varepsilon_{s+1} t^2_{s+1} \ell_{s+1}$ bipartite graphs G^{ij}_a, $1 \le i < j \le t_{s+1}$, $1 \le a \le \ell_{ij}$, are ε_{s+1}-quasirandom. As a consequence of re-indexing, $\mathbf{\Pi}^{(s+1)}$ admits triangle partitions $\mathbf{\Delta}^{(s+1)}_{ijk}$, $1 \le i < j < k \le t_{s+1}$, whose classes are of the form

$$\Delta^{ijk}_{abc} = K_3(G^{ij}_a \cup G^{jk}_b \cup G^{ik}_c)$$

$(a,b,c) \in [\ell_{ij}] \times [\ell_{jk}] \times [\ell_{ik}]$. The index of $\mathbf{\Pi}^{(s+1)}$ is

$$ind(\mathcal{H}, \mathbf{\Pi}^{(s+1)}) = t^{-3}_{s+1} \sum_{1 \le i < j < k \le t_{s+1}} ind(\mathcal{H}^{ijk}, \mathbf{\Delta}^{(s+1)}_{ijk}).$$

It remains to show that

$$ind(\mathcal{H}, \mathbf{\Pi}^{(s+1)}) \geq ind(\mathcal{H}, \mathbf{\Pi}^{(s)}) + \gamma\delta^2/2^{13}.$$

Applying the definition of the index and the notation prior to re-indexing $ind(\mathcal{H}, \mathbf{\Pi}^{(s+1)})$ may be expressed as

$$\begin{aligned} ind(\mathcal{H}, \mathbf{\Pi}^{(s+1)}) = (t_s t)^{-3} \sum_{i,j,k} \sum_{i',j',k'} \left|K_3\big(K^{ijk}_{i'j'k'}\big)\right|^{-1} \\ \times \sum_{abc} \sum_{a'b'c'} \frac{\left|\mathcal{H} \cap \hat{\Delta}^{ijk}_{abc}(a',b',c',i',j',k')\right|^2}{\left|\hat{\Delta}^{ijk}_{abc}(a',b',c',i',j',k')\right|^{-1}}, \end{aligned}$$

where the sums run over $1 \leq i < j < k \leq t_s$, $1 \leq i', j', k' \leq t$, $(a,b,c) \in [\ell_{ij}] \times [\ell_{jk}] \times [\ell_{ik}]$, $(a',b',c') \in [q_{ija}] \times [q_{jkb}] \times [q_{ikc}]$, and where $K^{ijk}_{i'j'k'}$ is defined analogously as in the first iteration. Using the fact that $t^3|K_3(K^{ijk}_{i'j'k'})| = (1 \pm o(1))|K_3(K^{ijk})|$, where $o(1) \to 0$ as $N \to \infty$, we conclude that

$$\begin{aligned} ind(\mathcal{H}, \mathbf{\Pi}^{(s+1)}) \geq (1 - o(1)) t_s^{-3} \sum_{i,j,k} \sum_{a,b,c} \left|K_3\big(K^{ijk}\big)\right|^{-1} \\ \times \sum_{a'b'c'} \sum_{i',j',k'} \frac{\left|\mathcal{H} \cap \hat{\Delta}^{ijk}_{abc}(a',b',c',i',j,k')\right|^2}{\left|\hat{\Delta}^{ijk}_{abc}(a',b',c',i',j',k')\right|^{-1}}. \end{aligned}$$

For each $1 \leq i < j < k \leq t_s$ and $(a,b,c) \in [\ell_{ij}] \times [\ell_{jk}] \times [\ell_{ik}]$ and again by the definition of the index we know that

$$\left|K_3\big(G^{ijk}_{abc}\big)\right| \times ind(\mathcal{H}^{ijk}_{abc}, \hat{\mathbf{\Delta}}^{ijk}_{abc}) = \sum_{a'b'c'} \sum_{i',j',k'} \frac{\left|\mathcal{H} \cap \hat{\Delta}^{ijk}_{abc}(a',b',c',i',j,k')\right|^2}{\left|\hat{\Delta}^{ijk}_{abc}(a',b',c',i',j',k')\right|^{-1}}$$

and so

$$
\begin{aligned}
&ind(\mathcal{H}, \mathbf{\Pi}^{(s+1)}) \\
&\quad \geq (1-o(1))t_s^{-3} \sum_{i,j,k} \sum_{a,b,c} \big|K_3\big(K^{ijk}\big)\big|^{-1} \big|K_3\big(G_{abc}^{ijk}\big)\big| \times ind(\mathcal{H}_{abc}^{ijk}, \hat{\mathbf{\Delta}}_{abc}^{ijk}) \\
&\quad = (1-o(1))t_s^{-3} \Big[\sum_{(\{i,j,k\},a,b,c) \in I_{bad}} \big|K_3\big(K^{ijk}\big)\big|^{-1} \big|K_3\big(G_{abc}^{ijk}\big)\big| \\
&\qquad \times ind(\mathcal{H}_{abc}^{ijk}, \hat{\mathbf{\Delta}}_{abc}^{ijk}) \\
&\qquad + \sum_{(\{i,j,k\},a,b,c) \notin I_{bad}} \big|K_3\big(K^{ijk}\big)\big|^{-1} \big|K_3\big(G_{abc}^{ijk}\big)\big| \times ind(\mathcal{H}_{abc}^{ijk}, \hat{\mathbf{\Delta}}_{abc}^{ijk}) \Big] \\
&\quad \geq (1-o(1))t_s^{-3} \\
&\qquad \times \Big[\sum_{(\{i,j,k\},a,b,c) \in I_{bad}} \big|K_3\big(K^{ijk}\big)\big|^{-1} \big|K_3\big(G_{abc}^{ijk}\big)\big| \big((\alpha_{abc}^{ijk})^2 + \frac{\delta^2}{2^{10}}\big) \\
&\qquad + \sum_{(\{i,j,k\},a,b,c) \notin I_{bad}} \big|K_3\big(K^{ijk}\big)\big|^{-1} \big|K_3\big(G_{abc}^{ijk}\big)\big| (\alpha_{abc}^{ijk})^2 \Big] \\
&\quad = (1-o(1))t_s^{-3} \Big[\frac{\delta^2}{2^{10}} \sum_{(\{i,j,k\},a,b,c) \in I_{bad}} \big|K_3\big(K^{ijk}\big)\big|^{-1} \big|K_3\big(G_{abc}^{ijk}\big)\big| \\
&\qquad + \sum_{i,j,k} \big|K_3\big(K^{ijk}\big)\big|^{-1} \sum_{a,b,c} (\alpha_{abc}^{ijk})^2 \big|K_3\big(G_{abc}^{ijk}\big)\big| \Big].
\end{aligned}
$$

Now, for each $1 \leq i < j < k \leq t_s$, we know

$$
|K_3(K^{ijk})|^{-1} \sum_{a,b,c} (\alpha_{abc}^{ijk})^2 |K_3(G_{abc}^{ijk})| = ind(\mathcal{H}^{ijk}, \mathbf{\Delta}_{ijk}^{(s)})
$$

and subsequently

$$
t_s^{-3} \sum_{i,j,k} ind(\mathcal{H}^{ijk}, \mathbf{\Delta}_{ijk}^{(s)}) = ind(\mathcal{H}, \mathbf{\Pi}^{(s)}).
$$

Note that each $|K_3(K^{ijk})|$, $1 \leq i < j < k \leq t_s$, satisfies $|K_3(K^{ijk})| = (1-o(1))(N/t_s)^3$, where $o(1) \to 0$ as $N \to \infty$. We therefore infer

$$
\begin{aligned}
ind(\mathcal{H}, \mathbf{\Pi}^{(s+1)}) &\geq (1-o(1)) \big[\frac{\gamma\delta^2}{2^{12}} + ind(\mathcal{H}, \mathbf{\Pi}^{(s)}) \big] \\
&\geq ind(\mathcal{H}, \mathbf{\Pi}^{(s)}) + \frac{\gamma\delta^2}{2^{13}}
\end{aligned}
$$

as promised and this concludes Iteration s.

We now finish the proof of Theorem 4.3. Knowing that the index function is bounded by 1 and that for $s \geq 1$ we have

$$ind(\mathcal{H}, \mathbf{\Pi}^{(s)}) + \gamma\delta^2/2^{13} \leq ind(\mathcal{H}, \mathbf{\Pi}^{(s+1)}) \leq 1,$$

we can at most perform $2^{13}/(\gamma\delta^2)$ iterations before we arrive at a partition $\mathbf{\Pi}$ satisfying the conclusion of Theorem 4.3. Observe that $\mathbf{\Pi}$ can be constructed in time $O(N^6)$.

Finally, it is clear that P_0, the number of parts of $\mathbf{\Pi}$, is independent of N. Indeed, suppose the procedure above terminates in $\mathbf{\Pi} = \mathbf{\Pi}^{(s+1)}$ so that $P_0 \leq t_{s+1}^2 \ell_{s+1}$. The preceding partition $\mathbf{\Pi}^{(s)}$ had only $t_s^2 \ell_s$ parts, where t_s and ℓ_s are constant in N. Recall $\ell_{s+1} = \ell_s Q_s$, where

$$Q_s = 2^{t_s \ell_s^2 27^{1+d_s^{-12}}}, \quad d_s = \frac{\gamma}{10\ell_s}.$$

Hence $\ell_{s+1} = \ell_s Q_s$ is independent of N. Recall $t_{s+1} = t_s t$, where $t \leq T_0(t_s, \ell_{s+1}, \varepsilon_{s+1})$, $\varepsilon_{s+1} = \varepsilon(\gamma/10\ell_{s+1})$, is a constant. Hence, t_{s+1} is independent of N.

■

4.3 Proof of Algorithm 4.3

Here, we prove the correctness of Algorithm 4.3. We basically follow Gowers [14] ideas but derandomize his probabilistic arguments. The proof of Algorithm 4.3 consists of two algorithm which will be proven later. We just state them at the appropriate point.

Proof of Algorithm 4.3. Let G be a graph with vertex classes X, Y an Z and let the densities of the bipartite graphs be d_{XY}, d_{YZ}, and d_{XZ}.

Further, let $\mathcal{H}$ be a 3-partite, 3-uniform hypergraph with $\mathcal{H} \subseteq K_3(G)$. Since $\mathcal{H}$ is not δ-quasirandom relative to G we know by Definition 3.8

$$\sum_{xx'yy'zz'} h_{xx'yy'zz'} \geq \delta(d_{XY}d_{YZ}d_{XZ})^4(m^3)^2. \tag{4.3}$$

For all triples $(x, y, z) \in \Delta(G)$ let us define the function

$$\begin{aligned}
&H_{xyz} : X \times Y \times Z \to [-1, 1] \quad \text{by} \\
&H_{xyz}(x', y', z') = h_{xx'yy'zz'}/h(x', y', z') \\
&= h(x, y, z)h(x', y, z)h(x, y', z)h(x, y, z')h(x', y', z)h(x, y', z')h(x', y, z').
\end{aligned}$$

Additionally, let

$$\begin{aligned}
H : X \times Y \times Z \to \mathbb{R} \ \text{ be given by} \\
H(x', y', z') = \sum_{(x,y,z)} H_{xyz}(x', y', z').
\end{aligned}$$

We then have that

$$\begin{aligned}
\langle h, H\rangle &= \sum_{(x',y',z')} h(x', y', z')H(x', y', z') \\
&= \sum_{(x',y',z')} \sum_{(x,y,z)} h(x', y', z')H_{xyz}(x', y', z') \\
&= \sum_{xx'yy'zz'} h_{xx'yy'zz'} \\
&\geq \delta(d_{XY}d_{YZ}d_{XZ})^4 m^6
\end{aligned} \tag{4.4}$$

where the last inequality is obtained by applying (4.3). Since for fixed $(x, y, z) \in K_3(G)$ we have $H_{xyz}(x', y', z') \in [-1, 1]$ we can rewrite $H_{xyz}(x', y', z')$ as follows

$$\begin{aligned}
H_{xyz}(x', y', z') &= u_{xyz}(x', y')v_{xyz}(y', z')w_{xyz}(x', z') \ \text{ where} \\
u_{xyz}(x', y') &= h(x', y', z)h(x', y, z)h(x, y', z)h(x, y, z) \in [-1, 1] \\
v_{xyz}(y', z') &= h(x, y', z')h(x, y, z') \in [-1, 1] \\
w_{xyz}(x', z') &= h(x', y, z') \in [-1, 1].
\end{aligned} \tag{4.5}$$

Following Gowers [14], the first main step in the proof of Algorithm 4.3 will be to replace the $[-1,1]$-valued functions $H_{xyz} = u_{xyz}v_{xyz}w_{xyz}$, $(x,y,z) \in K_3(G)$, by $\{-1,0,1\}$-valued functions $\hat{H}_{xyz} = \hat{u}_{xyz}\hat{v}_{xyz}\hat{w}_{xyz}$, $(x,y,z) \in K_3(G)$. We do that constructively in the algorithm below.

Algorithm 4.7
Input: *$(x,y,z) \in K_3(G)$ and $[-1,1]$ valued functions $u_{xyz}(x',y')$, $v_{xyz}(y',z')$, $w_{xyz}(x',z')$, $(x,y,z) \in K_3(G)$, as above.*
Output: *$\{-1,0,1\}$-valued functions $\hat{u}_{xyz}(x',y')$, $\hat{v}_{xyz}(y',z')$, $\hat{w}_{xyz}(x',z')$, such that*

$$\hat{H}_{xyz} = \hat{u}_{xyz}\hat{v}_{xyz}\hat{w}_{xyz} \quad \text{and} \quad \hat{H} = \sum_{x,y,z \in K_3(G)} \hat{H}_{xyz} \quad \text{satisfy}$$

I. $\langle h, \hat{H}\rangle \geq \langle h, H\rangle$;

II. $||\hat{H}||^2 = \langle \hat{H}, \hat{H}\rangle \leq 16(d_{XY}d_{YZ}d_{XZ})^4|K_3(G)|^3$.

Complexity: $O(m^5)$.

We describe Algorithm 4.7 in Section 4.3.1. Next we want to make a small selection of r functions $R_0 = \{\hat{H}_{\alpha_1}, \ldots, \hat{H}_{\alpha_r}\}$ from $\{\hat{H}_{xyz} | (x,y,z) \in K_3(G)\}$ that still preserves the good properties of $\hat{H}$. Before we explain how to choose them explicitly let us follow Gowers' proof of Lemma 8.4 [see page 176, in [14]]. In particular, set

$$d = d_{XY}d_{YZ}d_{XZ} \quad \text{and} \quad r = \lceil d^{-4} \rceil$$

and for $R \in \binom{K_3(G)}{r}$, define

$$\hat{H}_R = \sum_{(x,y,z) \in R} \hat{H}_{xyz}.$$

We now want to seek a set $R_0 \in \binom{K_3(G)}{r}$ for which the following holds:

$$256rd^2\langle h, \hat{H}_{R_0}\rangle - \delta||\hat{H}_{R_0}||^2 \geq 64r^2\delta d^4|K_3(G)|. \tag{4.6}$$

In fact, for $R_{rand} \in \binom{K_3(G)}{r}$ uniformly chosen at random, we have:

$$\mathbb{E}(\langle h, \hat{H}_{R_{rand}} \rangle) = \frac{r}{|K_3(G)|} \langle h, \hat{H} \rangle \quad \text{and}$$
$$\mathbb{E}(||\hat{H}_{R_{rand}}||) = \frac{r^2}{|K_3(G)|^2} ||\hat{H}||^2.$$

Using Property I of Algorithm 4.7 and inequality (4.4) we get

$$\begin{aligned}
\mathbb{E}(\langle h, \hat{H}_{R_{rand}} \rangle) &= \frac{r}{|K_3(G)|} \langle h, \hat{H} \rangle \\
&\geq \frac{r}{|K_3(G)|} \langle h, H \rangle \\
&\geq \frac{1}{|K_3(G)|} r\delta d^4 m^6 \\
&\geq \frac{1}{2} r\delta d^3 m^3 \\
&\geq \frac{1}{3} r\delta d^2 |K_3(G)|,
\end{aligned}$$

where the two last inequalities are obtained by applying Theorem 2.8, i.e. by using $|K_3(G)| \leq (1.5)dm^3$. Property II of Algorithm 4.7 then ensures that

$$\begin{aligned}
\mathbb{E}(||\hat{H}_{R_{rand}}||) &= (1 \pm o(1)) \frac{r^2}{|K_3(G)|^2} ||\hat{H}||^2 \\
&\leq (16 + o(1)) r^2 d^4 |K_3(G)| \\
&\leq 17 r^2 d^4 |K_3(G)|.
\end{aligned}$$

Therefore, we summarize that

$$\begin{aligned}
256 r d^2 \mathbb{E}(\langle h, \hat{H}_{R_{rand}} \rangle) - \delta \mathbb{E}(||\hat{H}_{R_{rand}}||^2) &\geq 68 r^2 \delta d^4 |K_3(G)| \\
&\geq 64 r^2 \delta d^4 |K_3(G)|. \qquad (4.7)
\end{aligned}$$

We therefore have proven the existence of the set R_0. Observe that the set R_0 can certainly be determined in time $\binom{|K_3(G)|}{r}$ by regarding all possible choices. We will derandomize Gowers' probabilistic techniques so that we are able to find these r functions more efficiently, i.e. in $O(m^3)$ time. Let us formulate this in the next algorithm.

Algorithm 4.8
Input: $\hat{H}_{xyz}$, $(x, y, z) \in K_3(G)$, *as above.*
Output: $R_0 \in \binom{K_3(G)}{r}$ *for which* $256rd^2\langle h, \hat{H}_{R_0}\rangle - \delta||\hat{H}_{R_0}||^2 \geq 64r^2\delta d^4|K_3(G)|$.
Complexity: $O(m^3)$.

Again, we defer the proof of Algorithm 4.8 to Section 4.3.2 in favor of finishing the proof of Algorithm 4.3. Since $\delta||\hat{H}_{R_0}||^2 \geq 0$ and $64\delta d^4 r^2|K_3(G)| \geq 0$ the inequality in Algorithm 4.8 implies that

$$\langle h, \hat{H}_{R_0}\rangle \geq \frac{1}{4}\delta d^2 r|K_3(G)| \qquad \text{and} \tag{4.8}$$

$$256d^2 r\langle h, \hat{H}_{R_0}\rangle\delta^{-1} \geq ||\hat{H}_{R_0}||^2 \tag{4.9}$$

respectively. Therefore, $\hat{H}_{R_0}$ also satisfies

$$\begin{aligned} \frac{\langle h, \hat{H}_{R_0}\rangle^2}{||\hat{H}_{R_0}||^2} &\geq \frac{\langle h, \hat{H}_{R_0}\rangle}{256d^2 r\delta^{-1}} && \text{(by (4.9))} \\ &\geq \frac{\delta d^2 r|K_3(G)|}{2^{10}d^2 r\delta^{-1}} && \text{(by (4.8))} \\ &= \frac{\delta^2|K_3(G)|}{2^{10}}. && (4.10) \end{aligned}$$

To end the proof we want to apply Lemma 4.5. In order to apply this lemma we first need to specify $U^{Lemma\ 4.5}$, $f^{Lemma\ 4.5}$, and $g^{Lemma\ 4.5}$. Let $U^{Lemma\ 4.5} = K_3(G)$ and $f^{Lemma\ 4.5} = h$. Partition the bipartite graphs G^{XY}, G^{YZ}, and G^{XZ} into at most 3^r subgraphs G_i^{XY}, G_j^{YZ}, and G_k^{XZ} such that $\hat{u}$, $\hat{v}$, and $\hat{w}$ are constant on each G_i^{XY}, G_j^{YZ}, and G_k^{XZ} respectively. Let us define the $\Delta_i^{Lemma\ 4.5}$. For every $(x, y, z) \in K_3(G)$ define its triad to be the triple (i, j, k) such that $xy \in G_i^{XY}$, $yz \in G_j^{YZ}$, and $xz \in G_k^{XZ}$. Partition $K_3(G)$ into triples according to their triads. Observe that the function $\hat{H}$ is constant on each partition class. This partition consists of at most $p_{XY}p_{YZ}p_{XZ} \leq 3^{3r} = 27^r$ cells Δ_i, where the choice of r implies that $r \leq 1 + d_0^{-12}$, for $d_0 \leq \min\{d_{XY}, d_{YZ}, d_{XZ}\}$. Each cell is of the form

$K_3(G_i^{XY} \cup G_j^{YZ} \cup G_k^{XZ})$ and

$$\begin{aligned} \hat{H}_{R_0}(x', y', z') &= \sum_{(x,y,z)\in R_0} \hat{H}_{xyz}(x', y', z') \\ &= \sum_{(x,y,z)\in R_0} \hat{u}_{xyz}(x', y')\hat{v}_{xyz}(y', z')\hat{w}_{xyz}(x', z') \end{aligned}$$

is constant on each class of $\mathbf{\Delta}$. By the conclusion of Lemma 4.5 we know that $ind(f, (\Delta_i)_{i=1}^{3^{3r}}) \geq (\langle h, \hat{H}_{R_0}\rangle / \sqrt{|K_3(G)|}||\hat{H}_{R_0}||)^2$. Applying inequality (4.10) we get that $ind(f, (\Delta_i)_{i=1}^{3^{3r}}) \geq 2^{-10}\delta^2$ which implies by Lemma 4.6 that $ind(\mathcal{H}, (\Delta_i)_{i=1}^{3^{3r}}) \geq \alpha^2 + 2^{-10}\delta^2$.

■

4.3.1 Proof of Algorithm 4.7

Again, the proof of Algorithm 4.7 was already established by Gowers. Below we derandomize his argument and describe a deterministic algorithm.

Proof of Algorithm 4.7. We first define the promised functions $\hat{u}_{xyz}$, $\hat{v}_{xyz}$, $\hat{w}_{xyz}$, $(x, y, z) \in K_3(G)$, and then prove that the functions fulfill the desired properties. For what follows, fix $(x, y, z) \in \Delta(G)$. By the definition of u_{xyz}, v_{xyz}, and w_{xyz} we have

$$\begin{aligned} \langle h, H\rangle &= \sum_{(xx'yy'zz')} h(x', y', z')u_{xyz}(x', y')v_{xyz}(y', z')w_{xyz}(x', z') \\ &= \sum_{(xx'yy'z)} u_{xyz}(x', y') \sum_{z'} h(x', y', z')v_{xyz}(y', z')w_{xyz}(x', z'). \end{aligned} \tag{4.11}$$

For a fixed (x, x', y, y', z) consider the term

$$u_{xyz}(x', y') \sum_{z'} h(x', y', z')v_{xyz}(y', z')w_{xyz}(x', z')$$

and define $\hat{u}_{xyz} : X \times Y \to \{-1, 0, 1\}$ as follows

$$\hat{u}_{xyz}(x', y') = \begin{cases} 1 & \text{if } \sum_{z'} h(x', y', z') v_{xyz}(y', z') w_{xyz}(x', z') \geq 0 \text{ and} \\ & u_{xyz}(x', y') \neq 0 \\ 0 & \text{if } u_{xyz}(x', y') = 0 \\ -1 & \text{if } \sum_{z'} h(x', y', z') v_{xyz}(y', z') w_{xyz}(x', z') < 0 \text{ and} \\ & u_{xyz}(x', y') \neq 0. \end{cases}$$

Obviously, by the definition of $\hat{u}_{xyz}(x', y')$ we have

$$\hat{u}_{xyz}(x', y') \sum_{z'} h(x', y', z') v_{xyz}(y', z') w_{xyz}(x', z') \\ \geq u_{xyz}(x', y') \sum_{z'} h(x', y', z') v_{xyz}(y', z') w_{xyz}(x', z'). \quad (4.12)$$

Observe that $\hat{u}_{xyz}(x', y')$ does not depend on the values of $u_{x_1 y_1 z_1}(x'_1, y'_1)$ for $(x, y, z, x', y') \neq (x_1, y_1, z_1, x'_1, y'_1)$. This allows to successively repeat the above procedure for all five tuples (x, y, z, x', y') replacing $u_{xyz}(x', y')$ by a corresponding $\hat{u}_{xyz}(x', y')$ so that (4.12) holds. After this process is completed the functions $u_{xyz}(x', y') : X \times Y \to [-1, 1]$ are replaced by $\hat{u}_{xyz}(x', y') : X \times Y \to \{-1, 0, 1\}$ for all $xyz \in K_3(G)$. Also having replaced u_{xyz} by $\hat{u}_{xyz}$ in the definition of $H_{xyz}(x', y', z')$ (of (4.5)) for each $xyz \in K_3(G)$ we obtain $H^{new}_{xyz}(x', y', z')$ which in view of (4.11) and (4.12) satisfies

$$\langle h, H^{new} \rangle \geq \langle h, H \rangle.$$

Next, we repeat the process over all five tuples (x, y, z, y', z') with initial values of the functions $\hat{u}$, v, and w replacing v attaining values in $[-1, 1]$ by $\hat{v}$ with values in $\{-1, 0, 1\}$. Finally, starting with the functions $\hat{u}$, $\hat{v}$, and w we repeat the process now over all five tuples (x, y, z, x', z') to obtain $\hat{u}$, $\hat{v}$, and $\hat{w}$. Setting $\hat{H}_{xyz}(x', y', z') = \hat{u}_{xyz}(x', y') \hat{v}_{xyz}(y', z') \hat{w}_{xyz}(x', z')$ we observe that $\langle h, \hat{H} \rangle \geq \langle h, H \rangle$ holds (part(i) of Claim 1).

On the other hand, by the definition of $\hat{u}_{xyz}(x',y')$, $\hat{v}_{xyz}(y',z')$, and $\hat{w}_{xyz}(x',z')$ we know that

$$\hat{H}_{xyz}(x',y',z') = \hat{u}_{xyz}(x',y')\hat{v}_{xyz}(y',z')\hat{w}_{xyz}(x',z')$$

is only nonzero if (x,x',y,y',z,z') are vertices of an octahedron in the graph G. Let $G_{xx'yy'zz'} = 1$ if (x,x',y,y',z,z') are the vertices of an octahedron in G and 0 otherwise. Then we are able to find an upper bound on $||\hat{H}||^2$.

$$\begin{aligned} ||\hat{H}||^2 &= \sum_{(x',y',z')} \left(\sum_{(x,y,z)} \hat{H}_{xyz}(x',y',z') \right)^2 \\ &\leq \sum_{(x',y',z')} \left(\sum_{(x,y,z)} G_{xx'yy'zz'} \right)^2 \\ &= \sum_{(x',y',z')} \sum_{(x_1 y_1 z_1 x_2 y_2 z_2)} G_{x_1 x' y_1 y' z_1 z'} G_{x_2 x' y_2 y' z_2 z'} \end{aligned}$$

But the last sum just counts the number of copies of a certain graph $G(9,21)$ with 9 vertices and 21 edges in G (see picture below).

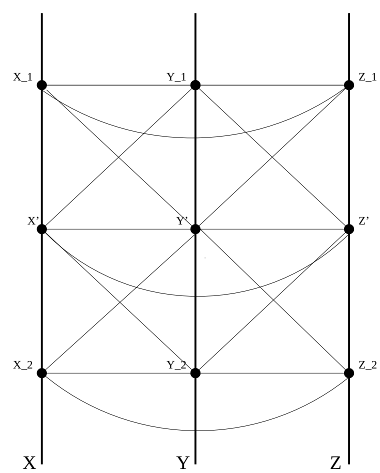

Figure 4.1: Graph G(9,21)

Since all conditions for the application of the graph counting lemma are satisfied, we know that we can find at most

$$\left[(d_{XY}d_{YZ}d_{XZ})^7 + 2^{21}\varepsilon^{1/4}\right] m^9 \overset{(2^{21}\varepsilon^{1/4}\leq d^7)}{\leq} (2d^7)m^9$$

copies of $G(9,21)$ in G. Combining the two inequalities above and the fact that $dm^3/2 \leq |K_3(G)| \leq 2dm^3$ we get that

$$||\hat{H}||^2 \leq 2d^7(m^3)^3 \leq 2d^4 8|K_3(G)|^3 = 16d^4|K_3(G)|^3$$

which implies Property II of Algorithm 4.7.

■

4.3.2 Proof of Algorithm 4.8

In this section we show the correctness of Algorithm 4.8 completing the proof of Algorithm 4.3.

Proof of Algorithm 4.8. For simplicity, we shall write $T = |K_3(G)|$ for the number of triangles in $G>$ Recall that we want to choose an r-tuple $R_0 \in \binom{K_3(G)}{r} = [T]^r$ $(r = \lceil d^{-4} \rceil)$ such that

$$256rd^2\langle h, \hat{H}_{R_0}\rangle - \delta||\hat{H}_{R_0}||^2 \geq 64r^2\delta d^4 T.$$

We shall choose the elements of R_0 one-by-one. Suppose that all the T functions $\hat{H}_{xyz}$ are labelled $\hat{H}_1, \hat{H}_2, \ldots, \hat{H}_T$. The only terms that are effected by a random choice in inequality 4.6 are $\langle h, \hat{H}_{\mathbb{R}}\rangle$ and $||\hat{H}_{R_{rand}}||^2$. We therefore introduce functions μ_k and σ_k that can be related to $\mathbb{E}(\langle h, \hat{H}_{R_{rand}}\rangle)$ and $\mathbb{E}(||\hat{H}_{R_{rand}}||^2)$ respectively. Let

$$\mu_0 = \mathbb{E}(\langle h, \hat{H}_{R_{rand}}\rangle) = \frac{1}{\binom{T}{r}} \sum_{R\in[T]^r} \langle h, \sum_{i\in R} \hat{H}_i\rangle = \frac{r}{T}\sum_{i=1}^{T}\langle h, \hat{H}_i\rangle.$$

Now, for all $1 \leq k \leq T$ we define

$$\mu_k = \langle h, \hat{H}_k \rangle + \frac{r-1}{T-1} \sum_{\substack{i=1 \\ i \neq k}}^{T} \langle h, \hat{H}_i \rangle.$$

Additionally, we know that

$$\begin{aligned} \sum_{k=1}^{T} \mu_k &= \sum_{k=1}^{T} \langle h, \hat{H}_k \rangle + \frac{r-1}{T-1}(T-1) \sum_{i=1}^{T} \langle h, \hat{H}_i \rangle \\ &= r \sum_{i=1}^{T} \langle h, \hat{H}_i \rangle = T\mu_0. \end{aligned} \tag{4.13}$$

Next we introduce an expression corresponding to $\mathbb{E}(||\hat{H}_{R_{rand}}||^2)$.

$$\begin{aligned} \sigma_0 = \mathbb{E}(||\hat{H}_{R_{rand}}||^2) &= \frac{1}{\binom{T}{r}} \sum_{R \in [T]^r} \langle \sum_{i \in R} \hat{H}_i, \sum_{i \in R} \hat{H}_i \rangle \\ &= \frac{r}{T} \sum_{i=1}^{T} ||\hat{H}_i||^2 + \frac{r(r-1)}{T(T-1)} \sum_{i \neq j} \langle \hat{H}_i, \hat{H}_j \rangle. \end{aligned}$$

For all $1 \leq k \leq T$ let

$$\begin{aligned} \sigma_k = ||\hat{H}_k||^2 + 2\langle \hat{H}_k, \frac{r-1}{T-1} \sum_{i \neq k} \hat{H}_i \rangle + \frac{r-1}{T-1} \sum_{i \neq k} ||\hat{H}_i||^2 \\ + \frac{(r-1)(r-2)}{(T-1)(T-2)} \sum_{\substack{i \neq j \\ i,j \neq k}} \langle \hat{H}_i, \hat{H}_j \rangle. \end{aligned}$$

As before, we determine

$$\begin{aligned}\sum_{k=1}^{T}\sigma_k &= \sum_{k=1}^{T}||\hat{H}_k||^2 + 2\sum_{k=1}^{T}\langle\hat{H}_k, \frac{r-1}{T-1}\sum_{i\neq k}\hat{H}_i\rangle + \frac{r-1}{T-1}(T-1)\sum_{i=1}^{T}||\hat{H}_i||^2 \\ &\quad + \frac{(r-1)(r-2)}{(T-1)(T-2)}(T-2)\sum_{i\neq j}\langle\hat{H}_i,\hat{H}_j\rangle \\ &= r\sum_{i=1}^{T}||\hat{H}_i||^2 + 2\frac{r-1}{T-1}\sum_{k\neq i}\langle\hat{H}_k,\hat{H}_i\rangle + \frac{(r-1)(r-2)}{T-1}\sum_{i\neq j}\langle\hat{H}_i,\hat{H}_j\rangle \\ &= r\sum_{i=1}^{T}||\hat{H}_i||^2 + \frac{r(r-1)}{T-1}\sum_{i\neq j}\langle\hat{H}_i,\hat{H}_j\rangle \\ &= T\sigma_0. \end{aligned} \tag{4.14}$$

Using the conclusions of (4.13) and (4.14) we have

$$\begin{aligned}\mathbb{E}(256d^2r\langle h,\hat{H}_{R_{rand}}\rangle - \delta||\hat{H}_{R_{rand}}||^2) &= 256d^2r\mu_0 - \delta\sigma_0 \\ &= 256d^2r\frac{1}{T}\sum_{k=1}^{T}\mu_k - \delta\frac{1}{T}\sum_{k=1}^{T}\sigma_k. \end{aligned} \tag{4.15}$$

This implies by (4.7) and (4.15) that there must exist an $\alpha_1 \in [T]$ such that

$$256d^2r\mu_{\alpha_1} - \delta\sigma_{\alpha_1} \geq 64\delta d^4r^2T.$$

After this step we already decided on one element of R_0. Note that the first element can be found in time $O(|T|) = O(m^3)$. We will now describe how to choose the remaining ones by using an inductive argument. Assume that we already chose the $s-1$ functions $E_{\alpha_1},\dots,E_{\alpha_{s-1}}$ in time $O(m^3)$. For simplicity we suppose that $\alpha_1 = 1,\dots,\alpha_{s-1} = s-1$. Furthermore from now on, let $W = [T]-[s-1] = \{s,\dots,T\}$. Then we are able to define

$$\begin{aligned}\mu_{[s-1]} &= \langle h, \sum_{i\in[s-1]}\hat{H}_i\rangle + \frac{1}{\binom{T-s+1}{r-s+1}}\sum_{R\in[W]^{r-s+1}}\langle h, \sum_{i\in R}\hat{H}_i\rangle \\ \sigma_{[s-1]} &= ||\sum_{i\in[s-1]}\hat{H}_i||^2 + 2\langle\sum_{i\in[s-1]}\hat{H}_i, \frac{1}{\binom{T-s+1}{r-s+1}}\sum_{R\in[W]^{r-s+1}}\sum_{i\in R}\hat{H}_i\rangle \\ &\quad + \frac{1}{\binom{T-s+1}{r-s+1}}\sum_{R\in[W]^{r-s+1}}\langle\sum_{i\in R}\hat{H}_i, \sum_{i\in R}\hat{H}_i\rangle. \end{aligned}$$

Simple calculations yield

$$\mu_{[s-1]} = \langle h, \sum_{i\in[s-1]} \hat{H}_i\rangle + \frac{r-s+1}{T-s+1}\sum_{i\in W}\langle h, \hat{H}_i\rangle$$

$$\sigma_{[s-1]} = ||\sum_{i\in[s-1]} \hat{H}_i||^2 + 2\langle \sum_{i\in[s-1]} \hat{H}_i, \frac{r-s+1}{T-s+1}\sum_{i\in W}\hat{H}_i\rangle + \frac{r-s+1}{T-s+1}\sum_{i\in W}||\hat{H}_i||^2 + \frac{(r-s+1)(r-s)}{(T-s+1)(T-s)}\sum_{\substack{i,j\in W\\ i\neq j}}\langle \hat{H}_i, \hat{H}_j\rangle.$$

Furthermore, we assume that the $s-1$ chosen functions satisfy

$$256d^2r\mu_{[s-1]} - \delta\sigma_{[s-1]} \geq 256d^2r\mu_{[s-2]} - \delta\sigma_{[s-2]} \geq \cdots \geq 64\delta d^4r^2T. \quad (4.16)$$

Observe that for $t=2$ inequality (4.16) results in $256d^2\mu_1 - \delta\sigma_1 \geq 256d^2r\mu_0 - \delta\sigma_0$ which we just proved. We will know explain how to choose the s-th ($2 < s \leq r$) function and prove that for $s \leq \alpha_s \leq T$

$$256d^2r\mu_{[s-1]\cup\{\alpha_s\}} - \delta\sigma_{[s-1]\cup\{\alpha_s\}} \geq 256d^2r\mu_{[s-1]} - \delta\sigma_{[s-1]}$$

is satisfied. For $s \leq k \leq T$ we set

$$\mu_{[s-1]\cup\{k\}} = \langle h, \hat{H}_k + \sum_{i\in[s-1]}\hat{H}_i\rangle + \frac{r-s}{T-s}\sum_{\substack{i\in W\\ i\neq k}}\langle h, \hat{H}_i\rangle \quad \text{and}$$

$$\sigma_{[s-1]\cup\{k\}} = \sum_{i\in[s-1]}||\hat{H}_i||^2 + ||\hat{H}_k||^2 + 2\langle \hat{H}_k + \sum_{i\in[s-1]}\hat{H}_i, \frac{r-s}{T-s}\sum_{\substack{j\in W\\ j\neq k}}\hat{H}_j\rangle + 2\langle \sum_{i\in[s-1]}\hat{H}_i, \hat{H}_k\rangle + 2\sum_{\substack{i,j\in[s-1]\\ i<j}}\langle \hat{H}_i, \hat{H}_j\rangle + \frac{r-s}{T-s}\sum_{\substack{i\in W\\ i\neq k}}||\hat{H}_i||^2 + \frac{(r-s)(r-s-1)}{(T-s)(T-s-1)}\sum_{\substack{i,j\in W\\ i,j\neq k}}\langle \hat{H}_i, \hat{H}_j\rangle.$$

Observe that

$$\begin{aligned}
\sum_{k\in W}\mu_{[s-1]\cup\{k\}} &= (T-s+1)\langle h, \sum_{i\in[s-1]}\hat{H}_i\rangle + \sum_{k\in W}\langle h, \hat{H}_k\rangle \\
&\quad + \frac{r-s}{T-s}(T-s)\sum_{i\in W}\langle h, \hat{H}_i\rangle \\
&= (T-s+1)\langle h, \sum_{i\in[s-1]}\hat{H}_i\rangle + (r-s+1)\sum_{i\in W}\langle h, \hat{H}_i\rangle \\
&= (T-s+1)\mu_{[s-1]}. \qquad (4.17)
\end{aligned}$$

A similar equality holds for $\sigma_{[s-1]\cup\{k\}}$. Indeed,

$$\begin{aligned}
\sum_{k\in W}\sigma_{[s-1]\cup\{k\}} &= (T-s+1)\sum_{i\in[s-1]}||\hat{H}_i||^2 + \sum_{k\in W}||\hat{H}_k||^2 \\
&\quad + 2\frac{r-s}{T-s}(T-s)\langle \sum_{i\in[s-1]}\hat{H}_i, \sum_{j\in W}\hat{H}_j\rangle \\
&\quad + 2\frac{r-s}{T-s}\sum_{\substack{k,j\in W\\ k\neq j}}\langle \hat{H}_k, \hat{H}_j\rangle + 2\langle \sum_{i\in[s-1]}\hat{H}_i, \sum_{k\in W}\hat{H}_k\rangle \\
&\quad + 2(T-s+1)\sum_{\substack{i,j\in[s-1]\\ i<j}}\langle \hat{H}_i, \hat{H}_j\rangle \\
&\quad + \frac{r-s}{T-s}(T-s)\sum_{i\in W}||\hat{H}_i||^2 \\
&\quad + \frac{(r-s)(r-s-1)}{(T-s)(T-s-1)}(T-s-1)\sum_{\substack{i,j\in W\\ i\neq j}}\langle \hat{H}_i, \hat{H}_j\rangle \\
&= (T-s+1)\sum_{i\in[s-1]}||\hat{H}_i||^2 + (r-s+1)\sum_{i\in W}||\hat{H}_i||^2 \\
&\quad + 2(r-s+1)\langle \sum_{i\in[s-1]}\hat{H}_i, \sum_{j\in W}\hat{H}_j\rangle \\
&\quad + 2(T-s+1)\sum_{\substack{i,j\in[s-1]\\ i<j}}\langle \hat{H}_i, \hat{H}_j\rangle
\end{aligned}$$

$$+\frac{(r-s+1)(r-s)}{T-s}\sum_{\substack{i,j\in W\\ i\neq j}}\langle \hat{H}_i,\hat{H}_j\rangle$$

$$=(T-s+1)\sigma_I. \tag{4.18}$$

Combining the results of the above equalities (4.17) and (4.18) we get

$$256d^2r\frac{1}{T-s+1}\sum_{k\in W}\mu_{[s-1]\cup\{k\}}-\delta\frac{1}{T-s+1}\sum_{k\in W}\sigma_{[s-1]\cup\{k\}}$$
$$=256d^2r\mu_{[s-1]}-\delta\sigma_{[s-1]}$$
$$\stackrel{(4.16)}{\geq} 64\delta d^4r^2T.$$

Therefore, there must exist $\alpha_s\in\{s,\dots,T\}$ such that

$$256d^2r\mu_{[s-1]\cup\{\alpha_s\}}-\delta\sigma_{[s-1]\cup\{\alpha_s\}}\geq 64\delta d^4r^2T.$$

After r-steps we end up with $\hat{H}_{R_0}$, the sum of the previous r chosen functions $\hat{H}_{\alpha_1},\dots,\hat{H}_{\alpha_r}$. Furthermore, we know that $\hat{H}_{R_{rand}}$ satisfies the inequality in Algorithm 4.8.

■

Bibliography

[1] N. Alon, R. A. Duke, H. Lefmann, V. Rödl, and R. Yuster. The algorithmic aspects of the regularity lemma (extended abstract). *Proceedings of the IEEE Symposium on Foundations of Computer Science*, pages 473–481, 1992.

[2] N. Alon, R. A. Duke, H. Lefmann, V. Rödl, and R. Yuster. The algorithmic aspects of the regularity lemma. *J. Algorithms*, 16(1):80–109, 1994.

[3] Christian Avart, Vojtěch Rödl, and Mathias Schacht. Every monotone 3-graph property is testable. *SIAM J. Discrete Math.*, 21(1):73–92 (electronic), 2007.

[4] F. R. K. Chung and R. L. Graham. Quasi-random hypergraphs. *Random Structures Algorithms*, 1(1):105–124, 1990.

[5] F. R. K. Chung and R. L. Graham. Quasi-random set systems. *J. Amer. Math. Soc.*, 4(1):151–196, 1991.

[6] F. R. K. Chung, R. L. Graham, and R. M. Wilson. Quasi-random graphs. *Combinatorica*, 9(4):345–362, 1989.

[7] Fan R. K. Chung. Quasi-random classes of hypergraphs. *Random Structures Algorithms*, 1(4):363–382, 1990.

[8] Oliver Cooley, Nikolaos Fountoulakis, Daniela Kühn, and Deryk Osthus. 3-uniform hypergraphs of bounded degree have linear Ramsey numbers. *J. Combin. Theory Ser. B*, 98(3):484–505, 2008.

[9] Oliver Cooley, Nikolaos Fountoulakis, Daniela Kühn, and Deryk Osthus. 3-uniform hypergraphs of bounded degree have linear Ramsey numbers. *J. Combin. Theory Ser. B*, submitted.

[10] Reinhard Diestel. *Graph theory*, volume 173 of *Graduate Texts in Mathematics*. Springer-Verlag, Berlin, third edition, 2005.

[11] Richard A. Duke, Hanno Lefmann, and Vojtěch Rödl. A fast approximation algorithm for computing the frequencies of subgraphs in a given graph. *SIAM J. Comput.*, 24(3):598–620, 1995.

[12] Peter Frankl and Vojtěch Rödl. Extremal problems on set systems. *Random Structures Algorithms*, 20(2):131–164, 2002.

[13] Alan Frieze and Ravi Kannan. The regularity lemma and approximation schemes for dense problems. In *37th Annual Symposium on Foundations of Computer Science (Burlington, VT, 1996)*, pages 12–20. IEEE Comput. Soc. Press, Los Alamitos, CA, 1996.

[14] W. T. Gowers. Quasirandomness, counting and regularity for 3-uniform hypergraphs. *Combin. Probab. Comput.*, 15(1-2):143–184, 2006.

[15] W.T. Gowers. Hypergraph regularity and the multidimensional szemerédi theorem. *submitted*.

[16] P. E. Haxell and V. Rödl. Integer and fractional packings in dense graphs. *Combinatorica*, 21(1):13–38, 2001.

[17] Penny Haxell, Brendan Nagle, and Vojtěch Rödl. An algorithmic version of the hypergraph method. *SIAM J. Comput.*, to appear.

[18] Svante Janson, Tomasz Łuczak, and Andrzej Rucinski. *Random graphs.* Wiley-Interscience Series in Discrete Mathematics and Optimization. Wiley-Interscience, New York, 2000.

[19] Y. Kohayakawa, V. Rödl, and L. Thoma. An optimal algorithm for checking regularity. *SIAM J. Comput.*, 32(5):1210–1235 (electronic), 2003.

[20] Yoshiharu Kohayakawa, Brendan Nagle, and Vojtěch Rödl. Hereditary properties of triple systems. *Combin. Probab. Comput.*, 12(2):155–189, 2003.

[21] Yoshiharu Kohayakawa, Vojtěch Rödl, and Jozef Skokan. Hypergraphs, quasi-randomness, and conditions for regularity. *J. Combin. Theory Ser. A*, 97(2):307–352, 2002.

[22] János Komlós, Ali Shokoufandeh, Miklós Simonovits, and Endre Szemerédi. The regularity lemma and its applications in graph theory. In *Theoretical aspects of computer science (Tehran, 2000)*, volume 2292 of *Lecture Notes in Comput. Sci.*, pages 84–112. Springer, Berlin, 2002.

[23] Brendan Nagle and Vojtěch Rödl. The asymptotic number of triple systems not containing a fixed one. *Discrete Math.*, 235(1-3):271–290, 2001. Combinatorics (Prague, 1998).

[24] Brendan Nagle, Vojtěch Rödl, and Mathias Schacht. The counting lemma for regular k-uniform hypergraphs. *Random Structures Algorithms*, 28(2):113–179, 2006.

[25] Brendan Nagle, Vojtěch Rödl, and Mathias Schacht. Extremal hypergraph problems and the regularity method. In *Topics in discrete mathematics*, volume 26 of *Algorithms Combin.*, pages 247–278. Springer, Berlin, 2006.

[26] V. Rödl, B. Nagle, J. Skokan, M. Schacht, and Y. Kohayakawa. The hypergraph regularity method and its applications. *Proc. Natl. Acad. Sci. USA*, 102(23):8109–8113 (electronic), 2005.

[27] V. Rödl, M. Schacht, M. H. Siggers, and N. Tokushige. Integer and fractional packings of hypergraphs. *J. Combin. Theory Ser. B*, 97(2):245–268, 2007.

[28] V. Rödl, M. Schacht, E. Tengan, and N. Tokushige. Density theorems and extremal hypergraph problems. *Israel J. Math.*, 152:371–380, 2006.

[29] Vojtech Rödl and Andrzej Ruciński. Ramsey properties of random hypergraphs. *J. Combin. Theory Ser. A*, 81(1):1–33, 1998.

[30] Vojtěch Rödl and Mathias Schacht. Regular partitions of hypergraphs: Counting lemmas. *Combinatorics, Probability and Computing*, to appear.

[31] Vojtěch Rödl and Mathias Schacht. Regular partitions of hypergraphs: Regularity lemmas. *Combinatorics, Probability and Computing*, to appear.

[32] Vojtěch Rödl and Jozef Skokan. Regularity lemma for k-uniform hypergraphs. *Random Structures Algorithms*, 25(1):1–42, 2004.

[33] Vojtěch Rödl and Jozef Skokan. Applications of the regularity lemma for uniform hypergraphs. *Random Structures Algorithms*, 28(2):180–194, 2006.

[34] Endre Szemerédi. Regular partitions of graphs. In *Problèmes combinatoires et théorie des graphes (Colloq. Internat. CNRS, Univ. Orsay, Orsay, 1976)*, volume 260 of *Colloq. Internat. CNRS*, pages 399–401. CNRS, Paris, 1978.

[35] Terence Tao. The Gaussian primes contain arbitrarily shaped constellations. *J. Anal. Math.*, 99:109–176, 2006.

[36] Terence Tao. A variant of the hypergraph removal lemma. *J. Combin. Theory Ser. A*, 113(7):1257–1280, 2006.

[37] Andrew Thomason. Pseudorandom graphs. In *Random graphs '85 (Poznań, 1985)*, volume 144 of *North-Holland Math. Stud.*, pages 307–331. North-Holland, Amsterdam, 1987.