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On Pisier type problems

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An abstract of
A dissertation submitted to the Faculty of the
James T. Laney School of Graduate Studies of Emory University
in partial fulfillment of the requirements for the degree of
Doctor of Philosophy
in Mathematics
2023

Abstract

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A subset $A \subseteq \mathbb{Z}$ of integers is *free* if for every two distinct subsets $B, B' \subseteq A$ we have

$$\sum_{b \in B} b \neq \sum_{b' \in B'} b'.$$

Pisier asked if for every subset $A \subseteq \mathbb{Z}$ of integers the following two statement are equivalent:

(i) A is a union of finitely many free sets.

(ii) There exists $\varepsilon > 0$ such that every finite subset $B \subseteq A$ contains a free subset $C \subseteq B$ with $|C| \geqslant \varepsilon |B|$.

In a more general framework, the Pisier question can be seen as the problem of determining if statements (i) and (ii) are equivalent for subsets of a given structure with prescribed property. We study the problem for several structures including B_h-sets, arithmetic progressions, independent sets in hypergraphs and configurations in the euclidean space.

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Contents

List of Figures

Notation

We use standard graph-theoretic notation throughout. We denote the vertex set and edge set of a graph or a hypergraph G by $V(G)$ and G (or $E(G)$), respectively. We will denote by $e(G) = |E(G)|$ the number of edges in G. For $v \in G$, we denote by $N_G(v)$ the neighbourhood of v and by $\deg_G(v) = |N_G(v)|$ its degree in G. For a subset $X \subseteq V(G)$ we denote the induced subgraph of G on this subset by $G[X]$. A *k-graph* or *k-uniform hypergraph* is a hypergraph with all edges of size k.

We use standard set-theoretic notation throughout as well. For a natural number N we set $[N] = \{1, \ldots, N\}$. Given a set X and a nonnegative integer k, we write $X^{(k)} = \{e \subseteq X : |e| = k\}$ for the set of all k-subsets of X. Unless stated otherwise, the elements of a set X will be always indexed in increasing order. That is, if we write $X = \{x_1, \ldots, x_k\}$, then we mean that $x_1 < \ldots < x_k$.

For functions $f = f(n)$ and $g = g(n)$, we write $f = O(g)$ to mean that there is a constant $C > 0$ such that $|f| \leqslant C|g|$; $f = \Omega(g)$ to mean that there is a constant $c > 0$ such that $|f| \geqslant c|g|$; $f = \Theta(g)$ to mean that $f = O(g)$ and $f = \Omega(g)$; and $f = o(g)$ to mean that $f/g \to 0$ as $n \to \infty$.

Chapter 1

Introduction

This dissertation consider the relation between Ramsey statements and density statements in various combinatorial problems. Ramsey theory refers to a large body of deep results in mathematics whose underlying philosophy is captured succinctly by the statement that "Every large system contains a large well-organized subsystem". This is an area in which a great variety of techniques from many branches of mathematics are used and whose results are important not only to combinatorics but also to logic, analysis, number theory, and geometry. A well known example of a Ramsey statement is the following celebrated result by Ramsey [39].

Theorem 1.0.1 ([39]). *For any integers $n, k, r \geqslant 1$, there exists integer N with the property that for any r-coloring of $[N]^{(k)}$ there exists a set $X \subseteq [N]$ of size n such that $X^{(k)}$ is monochromatic.*

The least number N satisfying the property of Theorem 1.0.1 is denoted by $R^{(k)}(n, r)$. In a more general way, a Ramsey statement usually can be described as a statement that no matter how we color a certain structure with finitely many colors, there exists a color class with a prescribed property.

On the other hand, density statements are more closely related to the area of extremal combinatorics. Extremal combinatorics studies how large (or small) an object that lies in a particular discrete mathematical system and satisfies a certain condition can be. A classical example is Mantel's theorem which states that every triangle free graph on n vertices has at most $n^2/4$ edges. This is indeed a density statement and can be rewritten as follows:

For every $\varepsilon > 0$, there exist n_0 such that any $(1/2+\varepsilon)$-proportion of the edges of K_n contains a triangle for $n \geqslant n_0$.

This statement can be considered as the density analogue of the Ramsey statement on Theorem 1.0.1 for $n = 3$. An interesting, perhaps natural question, is whether there is a relation between these two statements. The next question introduced by Pisier is the main motivation of this thesis.

In 1983 Pisier [37] formulated the following problem in the context of harmonic analysis. A set of integers $X = \{x_i\}_{i\in I} \subseteq \mathbb{Z}$ is called *free* if for any two distinct finite sets of indices $J, J' \subseteq I$ we have

$$\sum_{j\in J} x_j \neq \sum_{j'\in J'} x_{j'}. \tag{1.1}$$

Pisier was interested in a condition that guarantees that a set X is a union of a finite family of free sets. In this context, he asked if the following two statements are equivalent for every set $X \subseteq \mathbb{Z}$:

(i) X is the union of finitely many free sets.

(ii) There exists $\varepsilon > 0$ such that every finite subset $Y \subseteq X$ contains a free subset $Z \subseteq Y$ with $|Z| \geqslant \varepsilon|Y|$.

In a combinatorial sense, statement (i) can be written as the negation of a Ramsey statement:

¬ (i) Any finite coloring of X contains a monochromatic set that is not free.

Similarly, statement (ii) can be interpreted as the negation of a density statement:

¬ (ii) For every $\varepsilon > 0$, there exists a finite subset $Y \subseteq X$ such that any $Z \subseteq Y$ with $|Z| \geqslant \varepsilon|Y|$ is not free.

Hence, Pisier intrinsically asks if the Ramsey statement and the density statement for the property of not being a free set are equivalent. Clearly, by the pigeonhole principle, statement (i) implies statement (ii), i.e., the density statement implies the Ramsey one. However, the converse implication is still a open problem. For more about the history and related problems see [11, 14, 4]. In this thesis we will use this question as a general framework and study whether these two statements are equivalent for several properties in combinatorics.

1.1 Arithmetic progressions and B_h-sets

Given an integer $k \geqslant 1$, an arithmetic progression of length k (or AP_k) is a set of integers of the form

$$\{a, a+d, \ldots, a+(k-1)d\}$$

for integers $a \in \mathbb{Z}$ and $d > 0$. The theorem of van der Waerden is one of the earliest results in Ramsey theory. It asserts that every finite coloring of the integers yields a monochromatic arithmetic progression of any length. More precisely, for positive integers k and r we say that a set of integers $X \subseteq \mathbb{N}$ has the *van der Waerden property* $\mathrm{vdW}(k, r)$ if any r-coloring of X contains a monochromatic AP_k. With this notation, van der Waerden's theorem can be stated as follows:

Theorem 1.1.1 ([47]). *For integers $k \geqslant 3$ and $r \geqslant 2$, there exists an integer $W := W(k, r)$ such that for any $N \geqslant W$ the set of integers $[N]$ has the property $\mathrm{vdW}(k, r)$.*

Answering a long standing conjecture of Erdős and Turán [13], Szemerédi proved the following celebrated result.

Theorem 1.1.2 ([45]). *For an integer $k \geqslant 3$ and $\delta \in (0, 1]$, there exists an integer $N_0 := N_0(k, \delta)$ such that for $N \geqslant N_0$ the following holds. Every subset $A \subseteq [N]$ with $|A| \geqslant \delta N$ contains an arithmetic progression of length k.*

Basically, Szemerédi theorem states that any positive proportion of $\mathbb{N}$ contains an arithmetic progression of length k. Theorem 1.1.2 stimulated a lot of research and today several proofs, using tools of a variety of areas of mathematics, are known [21, 20, 22, 23, 33, 43, 46].

Similarly as with the van der Waerden property $\mathrm{vdW}(k, r)$, one can define a property related to Theorem 1.1.2. For an integer $k \geqslant 3$ and $\delta > 0$, we say that a finite set of integers $X \subseteq \mathbb{N}$ has the *Szemerédi property* $\mathrm{Sz}(k, \delta)$ if any subset $Y \subseteq X$ of size $|Y| \geqslant \delta |X|$ contains an AP_k. With this notation, Theorem 1.1.2 states that $[N]$ has the property $\mathrm{Sz}(k, \delta)$ for $N \geqslant N_0$.

A simple argument shows that the property $\mathrm{Sz}(k, \delta)$ implies property $\mathrm{vdW}(k, r)$ for $\delta \geqslant 1/r$. That is, Szemerédi theorem implies van der Waerden's theorem. Motivated by the problem of Pisier the following question was considered in [11, 3]:

Question 1.1.3. *Is it true that for any $k \geqslant 3$, there is $\delta > 0$ and set of integers X such that*

(i) X has property $\mathrm{vdW}(k, r)$ for every $r \geqslant 1$,

(ii) Every finite $Y \subseteq X$ fails to have property $\mathrm{Sz}(k, \delta)$?

A negative answer to Question 1.1.3 would imply that properties $\mathrm{vdW}(k, r)$ and $\mathrm{Sz}(k, \delta)$ are equivalent. This would in particular provide a surprising new proof of Szemerédi theorem by van der Waerden's theorem. For this reason, the authors in [11] conjectured that Question 1.1.3 has a positive answer. In this thesis, we confirm their conjecture.

Theorem 1.1.4. *For every $k \geqslant 3$ and $0 < \mu < \frac{k-1}{k}$ there is a set of integers $X := X(k, \mu) \subseteq \mathbb{N}$ such that*

(i) *For every* $r \geqslant 1$, *any* r*-coloring of* X *contains a monochromatic* AP_k,

(ii) *Every finite subset* $Y \subseteq X$ *contains a subset* $Z \subseteq Y$, $|Z| \geqslant \mu|Y|$ *with no* AP_k.

We note that Theorem 1.1.4 does not hold for $\mu > \frac{k-1}{k}$. Indeed, any set $X \subseteq \mathbb{N}$ satisfying condition (i) must contain an AP_k. By taking $Y \subseteq X$ to be an AP_k, we have that $|Y| = k$. Therefore, the only $Z \subseteq Y$ with $|Z| \geqslant \mu|Y|$ is Y itself. Hence, Y fails to have property $\mathrm{Sz}(k, \mu)$.

A similar result can be obtained for B_h-sets as well. For $h \geqslant 1$, we say that a set of integers $X = \{x_i\}_{i \in I}$ is a B_h*-set* if

$$\sum_{j \in J} x_j \neq \sum_{j' \in J'} x_{j'}$$

for $J \neq J'$, $|J| = |J'| = h$, i.e., if all the h-sums of X are distinct.

Note that a B_2-set is also called a Sidon set. The density statement and consequently the Ramsey statement were proved by Erdős and Turán [12], who showed that for every $\varepsilon > 0$, there exists $N_0 := N_0(\varepsilon)$ such that for every $N \geqslant N_0$ any ε-proportion of $[N]$ contains $\{a, b, c, d\}$ such that $a + b = c + d$ (They actually proved a much stronger bound on ε). Motivated by the Pisier problem, Alon and Erdős [1] asked if the following two statements for B_h-sets are equivalent:

(1) X is the union of finitely many B_h-sets.

(2) There exists $\varepsilon > 0$ such that every finite subset $Y \subseteq X$ contains a B_h subset $Z \subseteq Y$ with $|Z| \geqslant \varepsilon|Y|$.

As in the original Pisier problem, the implication (1) $\Rightarrow$ (2) holds. So it remains to determine whether the implication (2) $\Rightarrow$ (1) is true. The next result shows that it is not the case.

Theorem 1.1.5. *For every* $h \geqslant 1$ *there exists* $\varepsilon > 0$ *and a set of positive integers* X *with the following two properties:*

(i) X is not a union of finitely many B_h-sets.

(ii) Every finite subset $Y \subseteq X$ contains a B_h-set Z with $|Z| \geqslant \varepsilon |Y|$ elements.

1.2 Euclidean configurations

We will find it convenient to present our discussion in the framework of $\mathbb{R}^\infty$, by which we understand a subspace of ℓ^2 consisting of infinite sequences of real numbers with finite support, i.e., all but finitely many entries are zero and with $\mathbb{R}^\infty$ equipped by the usual euclidean metric. In other words, we can view $\mathbb{R}^\infty$ as the infinite union $\mathbb{R}^\infty = \bigcup_{d=1}^{\infty} \mathbb{R}^d$, where we understand that the copies of $\mathbb{R}^d$ are being included in one another.

For two configurations of points $A, B \subseteq \mathbb{R}^\infty$ we will write

$$A \to (B)_r$$

to denote the fact that any r-coloring of A yields a monochromatic copy of B. By a (*congruent*) *copy* of B, we mean a subconfiguration $B' \subseteq A$ that is isometric to B, i.e., that exists a bijective map $\varphi : B \to B'$ such that

$$||b_1 - b_2|| = ||\varphi(b_1) - \varphi(b_2)||$$

for every $b_1, b_2 \in B$. Given two configurations A, B we say that B is contained in A, and denote $B \subseteq A$, if there exists a copy A' of A such that $B \subseteq A'$ (in the set theoretical sense).

A finite configuration S is said to be *Ramsey* if $\mathbb{R}^\infty \to (S)_r$ for every integer $r \geqslant 1$. The concept was introduced in [10] by Erdős, Graham, Montgomery, Rothschild, Spencer and Strauss, who proved that the vertex set of every brick (rectangular parallelepiped) of arbitrary finite dimension is Ramsey. The list of Ramsey configurations was extended by a few more configurations in [15, 18, 28, 29]. On the other hand, the authors of [10] also proved that any Ramsey set is spherical, i.e., all points of S lie on some finite dimensional sphere. They asked if the opposite implication is also true: If any spherical set is Ramsey. In [26] Ron Graham offered \$1000 dollars for deciding if this implication holds as well. Based on the evidence coming from known Ramsey configurations Leader, Russel and Walters [30] proposed an alternative conjecture. Calling a finite set *transitive* if its symmetry group is transitive, i.e., if all points play the same role, their conjecture states that Ramsey sets are precisely the transitive sets together with their subsets.

While the progress on these conjectures was very small, some alternative concepts were considered in [24, 25, 32, 15, 19, 5]. In this thesis we will introduce another concept based on Pisier's problem. A *d-dimensional simplex* S is a configuration consisting of $d+1$ affinely independent points in $\mathbb{R}^\infty$. In [15] it was proved that all simplices are Ramsey. One interesting feature of their proof is that they actually show the following stronger statement.

Theorem 1.2.1 ([15]). *Let $S \subseteq \mathbb{R}^\infty$ be a d-dimensional simplex and $0 < \mu < 1$ a real number. Then there exists finite configuration $Y \subseteq \mathbb{R}^\infty$ such that any subconfiguration $Z \subseteq Y$ of size $|Z| \geqslant \mu|Y|$ contains a copy of S.*

In other words, Theorem 1.2.1 not only finds a configuration Y such that $Y \to (S)_r$, but also with the extra property that any subset of positive density contains a copy of S. One of the goals of this part of the thesis is to show an alternative construction of the fact that simplices are Ramsey where our set Y does not have the density property. The following definition is central for our exposition.

Definition 1.2.2. A finite configuration $X \subseteq \mathbb{R}^\infty$ is called *P-Ramsey* if there exists a configuration $Y \subseteq \mathbb{R}^\infty$ and a real number $\mu > 0$ such that the following holds:

(i) $Y \to (X)_r$ holds for every integer $r \geqslant 1$.

(ii) Every finite subconfiguration $Y' \subseteq Y$ contains a configuration $Z \subseteq Y'$ with $|Z| \geqslant \mu|Y'|$ such that Z is X-free

Note that statement (ii) of the P-Ramsey definition is in contrast with the density statement introduced in Theorem 1.2.1, since it says that every finite subconfiguration contains a large set without a copy of X.

Clearly, if X is P-Ramsey, then X is Ramsey. However, the converse is not so clear. In this thesis, we start the study of P-Ramsey configurations by showing the following two results.

Theorem 1.2.3. *All simplices are P-Ramsey.*

We say that a configuration $B \subseteq \mathbb{R}^\infty$ is a d-dimensional brick if there exists positive real numbers $a_1, \ldots, a_d \in \mathbb{R}$ such that B is congruent to the set

$$\{(x_1, \ldots, x_d) : \; x_i = 0 \text{ or } x_i = a_i, \; 1 \leqslant i \leqslant d\} \,.$$

Theorem 1.2.4. *All bricks are P-Ramsey.*

1.3 Organization

This thesis is organized as follows. In Chapter 2 we study variants of the Pisier problem for independent sets in hypergraphs. This variants will be important later in the proofs of Theorems 1.1.4, 1.2.3 and 1.2.4. Chapter 3 is devoted to the proof of the Pisier type problem for B_h-sets. The proof is based on a finitary set version of the problem (see Theorem 3.1.1). Moreover, we also prove in this section a partial one sided version of Pisier original problem (see Theorem 3.2.1). In Chapter 4 we prove Theorem 1.1.4 regarding arithmetic progressions. The proof is based on the partite construction by Rödl and Nesetril. Finally, Chapter 5 is devoted to the Pisier type problems of Euclidean configurations and in particular contains the proofs of Theorems 1.2.4 and 1.2.3.

Chapter 2

Independent sets on hypergraphs

In this chapter we consider the Pisier type problem for k-uniform hypergraphs. Viewing our sets as vertex sets from a hypergraph and replacing the notion of being free by being an independent set of vertices leads to the following question. For what values of μ is there a k-graph H with the properties:

(1) The chromatic number $\chi(H)$ is infinite,

(2) Every finite subset $Y \subseteq V(H)$ contains an independent set $Z \subseteq Y$ with $|Z| \geqslant \mu|Y|$ vertices?

That is, for what values of μ does the converse implication of the Pisier problem fail? We say that a hypergraph H satisfying statement (2) has the *μ-property*. By taking Y as the vertex set of an edge, one can clearly note that there is no nontrivial H satisfying the μ-property for $\mu > \frac{k-1}{k}$. On the other hand we will show that such hypergraphs exist for each $\mu < \frac{k-1}{k}$.

The content of this chapter was obtained in joint work with Nešetřil, Reiher and Rödl and contains fragments of the manuscript of the following papers [36, 40, 42].

2.1 μ-fractional property

In this section we will prove a slightly stronger version of the problem described above.

Definition 2.1.1. We say that a weight vector $\mathbf{w} = (\mathbf{w}(i))_{i\in I}$ is *stochastical* if $\mathbf{w}(i) \in [0,1]$ for every $i \in I$ and $\sum_{i\in I} \mathbf{w}(i) = 1$. Let H be a k-graph. For given $\mu > 0$, we say that H has the *μ-fractional property* if for every finite subset $Y \subseteq V(H)$ and every stochastic weight vector $\mathbf{w} = (\mathbf{w}(y))_{y\in Y}$, there exists an independent set $Z \subseteq Y$ with

$$\sum_{z\in Z} \mathbf{w}(z) \geqslant \mu \sum_{y\in Y} \mathbf{w}(y) = \mu.$$

By taking $\mathbf{w}(y) = \frac{1}{|Y|}$ for every $y \in Y$, one can see that the μ-fractional property implies the μ-property. The next theorem shows the existence of k-graphs H with the μ-fractional property and infinite chromatic number. In particular, this answers the problem introduced in the beginning of the chapter.

Theorem 2.1.2. *For every $\mu < \frac{k-1}{k}$, there exists an infinite k-graph H with the following two properties:*

(i) The chromatic number $\chi(H)$ is infinite.

(ii) H has the μ-fractional property.

The proof of Theorem 2.1.2 is a corollary of the following finitary form of the statement. For integers k, N and $\mu \leqslant \frac{k-1}{k}$, set $\varepsilon = \frac{k-1}{k} - \mu$ and $\ell = \left\lceil \frac{2(k-1)^2}{\varepsilon k} \right\rceil$. Let $H := H(k, N, \mu)$ be the k-graph with vertex set $V(H) = [N]^{(\ell)}$ and edge set described as follows: A k-tuple $\{x_1, \ldots, x_k\} \in V(H)^{(k)}$ is an edge if and only if there exists a set $A = \{a_1, \ldots, a_{k+\ell-1}\} \in [N]^{(k+\ell-1)}$ such that

$$x_i = \{a_i, \ldots, a_{i+\ell-1}\},$$

for $1 \leqslant i \leqslant k$. That is, H is the shift k-graph on the ℓ-tuples of $[N]$.

Theorem 2.1.3. *For every $r \geqslant 2$, $k \geqslant 3$ and $\mu < \frac{k-1}{k}$, there exists an integer $N_0 := N_0(r, k, \mu)$ such that the k-graph $H := H(k, N, \mu)$ satisfies the following for $N \geqslant N_0$:*

(i) $\chi(H) > r$.

(ii) H has the μ-fractional property.

We start by proving the infinite version.

Proof of Theorem 2.1.2. For every integer $r \geqslant 1$, let $N_r := N_0(r, k, \mu)$ be the integer obtained by Theorem 2.1.3. Take H as the disjoint union of $H(k, N_r, \mu)$ for $r \geqslant 1$. Clearly, the k-graph H satisfies statements (i) and (ii) of Theorem 2.1.2 □

Now, we provide a proof of the finite version.

Proof of Theorem 2.1.3. Set $\varepsilon = \frac{k-1}{k} - \mu$ and $\ell = \left\lceil \frac{2(k-1)^2}{\varepsilon k} \right\rceil$. Let $N_0(r, k, \mu) = R^{(\ell)}(k + \ell - 1, r)$. We claim that $H := H(k, N, \mu)$ satisfies the statement of Theorem 2.1.3 for $N \geqslant N_0$.

Statement (i) follows from Ramsey theorem (Theorem 1.0.1).. Indeed, for any r-coloring of $[N]^{(\ell)}$, there exists a set $X \subseteq [N]$ of size $k + \ell - 1$ such that $X^{(\ell)}$ is monochromatic. In particular, this implies that H has an edge with all its vertices monochromatic. Hence, $\chi(H) > r$.

In order to address statement (ii), let $Y \subseteq V(H) = [N]^{(\ell)}$ be a subset of vertices and $\mathbf{w} = (\mathbf{w}(y))_{y \in Y}$ a stochastic weight vector. We will show by induction on the cardinality of Y that there is an independent set $Z \subseteq Y$ with $\sum_{z \in Z} \mathbf{w}(z) > \frac{k-1}{k} - \varepsilon$. For $|Y| = k$, the statement follows immediately from the fact that there exists independent set $Z \subseteq Y$ of size $|Y| - 1$ with

$$\sum_{z \in Z} \mathbf{w}(z) \geqslant \frac{|Y| - 1}{|Y|} > \frac{k-1}{k} - \varepsilon.$$

Assume now that $|Y| > k$. For an integer $c \in [N]$, we define

$$S(c) = \{y \in Y : \ c \in y\}$$

to be the set of vertices of Y that contain c. Similarly, let

$$S'(c) = \big\{y = \{b_1, \ldots, b_\ell\} : \ c \in \{b_k, \ldots, b_{\ell-(k-1)}\}\big\}$$

as the set of vertices of Y such that c is neither one of the first or last $k-1$ elements of Y.

We claim that $H[S(c)]$ is a k-partite k-graph for every $c \in [N]$. To see that consider the partition $S(c) = V_0 \cup \ldots \cup V_{k-1}$ where

$$V_j = \{y = \{b_1, \ldots, b_\ell\} \in S(c) : \ c = b_i \text{ and } i \equiv j \pmod k\}$$

for $0 \leqslant j \leqslant k-1$. That is, V_j are the vertices of $S(c)$ where c is in a position congruent to $j \pmod k$. Note that if $e = \{y_1, \ldots, y_k\}$ is an edge in $H[S(c)]$, then $|e \cap V_j| = 1$ for every $0 \leqslant i \leqslant k-1$. Hence, $H[S(c)]$ is k-partite.

By double counting the weights over all the pairs (c, y) where $y = \{b_1, \ldots, b_\ell\}$ and $c \in \{b_k, \ldots, b_{\ell-(k-1)}\}$, we obtain that

$$\sum_{c\in[N]} \sum_{y\in S'(c)} \mathbf{w}(y) = (\ell - 2(k-1)) \sum_{y \in Y} \mathbf{w}(y) = \ell - 2(k-1). \tag{2.1}$$

Similarly, by double counting the weights over all the pairs (c, y) with $c \in Y$, we have

$$\sum_{c\in[N]} \sum_{y\in S(c)} \mathbf{w}(y) = \ell \sum_{y \in Y} \mathbf{w}(y) = \ell. \tag{2.2}$$

Hence, comparing (2.1) and (2.2) yields that there exists $c_0 \in [N]$ such that

$$\sum_{y \in S'(c_0)} \mathbf{w}(y) \geqslant \frac{\ell - 2(k-1)}{\ell} \sum_{y \in S(c_0)} \mathbf{w}(y). \tag{2.3}$$

Since $S'(c_0) \subseteq S(c_0)$ we have that $H[S'(c_0)]$ is a k-partite graph and consequently by inequality (2.3) we have that there exists independent set $I_1 \subseteq S'(c_0)$ satisfying

$$\begin{aligned}\sum_{y \in I_1} \mathbf{w}(y) \geqslant \frac{k-1}{k} \sum_{y \in S'(c_0)} \mathbf{w}(y) &\geqslant \frac{k-1}{k} \left(\frac{\ell - 2(k-1)}{\ell} \right) \sum_{y \in S(c_0)} \mathbf{w}(y) \\ &\geqslant \left(\frac{k-1}{k} - \varepsilon \right) \sum_{y \in S(c_0)} \mathbf{w}(y). \end{aligned} \tag{2.4}$$

Furthermore, applying the inductive assumption to the set $Y - S(c_0)$ with stochastic weight vector $\mathbf{w}' = (\mathbf{w}(z))_{z \in Y - S(c_0)}$ given by $\mathbf{w}'(z) = \mathbf{w}(z)/(\sum_{y \in Y - S(c_0)} \mathbf{w}(y))$ gives us an independent set $I_2 \subseteq Y - S(c_0)$ with

$$\sum_{y \in I_2} \mathbf{w}(y) \geqslant \left(\frac{k-1}{k} - \varepsilon \right) \sum_{y \in Y - S(c_0)} \mathbf{w}(y). \tag{2.5}$$

We claim that if $e \in H$ is such that $e \cap S'(c_0) \neq \emptyset$, then $e \subseteq S(c_0)$. Indeed, let $e = \{y_1, \ldots, y_k\} \in H$ with

$$y_i = \{a_i, \ldots, a_{i+\ell-1}\}$$

for $1 \leqslant i \leqslant k$.

If $e \cap S'(c_0) \neq \emptyset$, then there exists a vertex $y_j = \{a_j, \ldots, a_{j+\ell-1}\}$ such that $c_0 \in \{a_{j+k}, \ldots, a_{j+\ell-k}\}$. However, because $1 \leqslant i \leqslant k$, we have $i < j + k < j + \ell - k < i + \ell - 1$ and consequently $c_0 \in \{a_{j+k}, \ldots, a_{j+\ell-k}\} \subseteq y_i$ for every $1 \leqslant i \leqslant k$ and hence $e \subseteq S(c_0)$.

Since $I_2 \subseteq Y - S(c_0)$, we obtain that there is no edge intersecting both I_1 and I_2.

This implies that $I_1 \cup I_2$ is and independent set and by inequalities (2.4) and (2.5) we have that

$$\sum_{y \in I_1 \cup I_2} \mathbf{w}(y) \geqslant \frac{k-1}{k} - \varepsilon.$$

□

2.2 A version for simple graphs

We observe that for $k \geqslant 3$, the shift k-graph $H(k, N, \mu)$ contains pairs of edges intersecting in more than one vertex, i.e., the hypergraph $H(k, N, \mu)$ is not simple. However, for the purposes of Chapter 4 we will need a simple hypergraph with the properties of Theorem 2.1.3.

Such a graph can be obtained by a standard application of the probabilistic method combined with Theorem 2.1.3 and the following observation:

Claim 2.2.1. *Let H be a k-graph with the μ-fractional property. If $\tilde{H} \subseteq H$ is a subgraph, then $\tilde{H}$ has the μ-fractional property. That is, the μ-fractional property is hereditary.*

Proof. We may assume that $V(\tilde{H}) = V(H)$ by adding some isolated vertices. Let $Y \subseteq V(H)$ be a finite subset of vertices and $\mathbf{w} = (\mathbf{w}(y))_{y \in Y}$ a stochastic weight vector. Since H has the μ-fractional property, there exists an independent set $Z \subseteq Y$ in H such that $\sum_{z \in Z} \mathbf{w}(z) \geqslant \mu$. The proof now follows because Z is also an independent set in $\tilde{H}$. □

Next we will show the following strengthening of Theorem 2.1.3.

Theorem 2.2.2. *For every $r \geqslant 2$, $k \geqslant 3$ and $\mu < \frac{k-1}{k}$ there exists an integer $M := M(r, k, \mu)$ and a simple k-graph $G \subseteq H(k, M, \mu)$ satisfying the properties:*

(i) $\chi(G) > r$.

(ii) G has the μ*-fractional property.*

Proof. Let $N_0 := N_0(r, k, \mu)$ be the integer obtained by Theorem 2.1.3 and let M be a sufficiently large integer such that

$$M \geqslant N_0^{3(k+\ell-1)}. \tag{2.6}$$

Consider the random subgraph $H_p \subseteq H(k, M, \mu)$ obtained by selecting each edge of $H(k, M, \mu)$ independently at random with probability $p = M^{3/2-k}$. Note that the k-graph $H(k, M, \mu)$ has $\binom{M}{\ell}$ vertices and $\binom{M}{k+\ell-1}$ edges. Moreover, because each edge is intersected by at most M^{k-2} edges in at least two vertices, the number of pairs intersecting in at least two vertices can be bounded by

$$\binom{M}{k+\ell-1}\binom{k}{2}M^{k-2} \leqslant M^{2k+\ell-3}. \tag{2.7}$$

Since the number of edges of H_p follows a binomial distribution, by the Chernoff bounds

$$|E(H_p)| = (1 + o(1))p\binom{M}{k+\ell-1} = \Theta(M^{\ell+1/2}) \tag{2.8}$$

holds with probability $1 - o(1)$.

Moreover, if X_p is the random variable counting the number of pairs of edges intersecting in at least two vertices, then by (2.7) we have

$$\mathbb{E}(X_p) \leqslant p^2 M^{2k+\ell-3} = M^\ell.$$

Hence, by Markov's inequality,

$$\mathbb{P}(X_p > 2M^\ell) \leqslant \frac{1}{2}. \tag{2.9}$$

Note that each subset $A \subseteq [M]$ of size N_0 induces a subgraph isomorphic to $H(k, N_0, \mu)$. By Theorem 2.1.3 any r-coloring of the vertices of $H(k, N_0, \mu)$ contains a monochromatic edge. Since there are $\binom{M}{N_0}$ subsets of size N_0 in $[M]$ and every edge is contained in at most $\binom{M-(k+\ell-1)}{N_0-(k+\ell-1)}$ of those induced graphs, we obtain that any r-coloring of $H(k, M, \mu)$ contains at least

$$\frac{\binom{M}{N_0}}{\binom{M-(k+\ell-1)}{N_0-(k+\ell-1)}} = (1+o(1))\left(\frac{M}{N_0}\right)^{k+\ell-1} \tag{2.10}$$

monochromatic edges.

Given an r-coloring $\varphi : V(H(k, M, \mu)) \to [r]$, let Y_φ be the random variable counting the number of monochromatic edges in H_p. Note that Y_p follows a binomial distribution. Relations (2.6) and (2.10) give us that

$$\mathbb{E}(Y_\varphi) \geqslant (1+o(1))p\left(\frac{M}{N_0}\right)^{k+\ell-1} = (1+o(1))\frac{M^{\ell+1/2}}{N_0^{k+\ell-1}} \geqslant (1+o(1))M^{\ell+1/6}.$$

Therefore, by the Chernoff bounds

$$\mathbb{P}\left(Y_\varphi \leqslant \frac{1}{2}M^{\ell+1/6}\right) \leqslant e^{-cM^{\ell+1/6}}$$

for some constant $c > 0$.

Let E be the event that $Y_\varphi \geqslant \frac{1}{2}M^{\ell+1/6}$ for every r-coloring φ. A union bound argument gives that

$$\mathbb{P}(\neg E) \leqslant r^{\binom{M}{\ell}} e^{-cM^{\ell+1/6}} = o(1). \tag{2.11}$$

Combining (2.8), (2.9) and (2.11) yields that

$$\mathbb{P}(E \wedge \{X_p \leqslant 2M^\ell\} \wedge \{|E(H_p)| = \Theta(M^{\ell+1/2})\}) \geqslant \frac{1}{2} - o(1).$$

Therefore, with positive probability, the k-graph H satisfies the event E and has at most $2M^\ell \ll \frac{1}{2}M^{\ell+1/6}$ pairs of edges intersecting in at least two vertices. Let $G \subseteq H_p$ be the hypergraph obtained from H_p by deleting all edges in those pairs. The resulting hypergraph G is simple and yet any r-coloring of $[M]^\ell$ yields at least $\frac{1}{2}M^{\ell+1/6} - 2M^\ell > 0$ monochromatic edges. Thus, $\chi(G) > r$, which proves property (i). Property (ii) follows from Claim 2.2.1 applied to G and $H(k, M, \mu)$. □

2.3 Independent sets of shift graphs

Note that the proof of Theorem 2.1.3 uses the fact that $\ell \gg k$. For some of the applications in Chapter 5 we will require a version of our Theorem for shift graphs on the pairs of $[N]$.

The shift graph $\mathrm{Sh}(2, \mathbb{N})$ is the graph with vertex set $V(\mathrm{Sh}(2, \mathbb{N})) = \mathbb{N}^{(2)}$, i.e., the pairs of natural numbers, and edge set

$$E(\mathrm{Sh}(2, \mathbb{N})) = \{\{\{x, y\}, \{y, z\}\} : \ x < y < z\}.$$

Claim 2.3.1. *The shift graph* $\mathrm{Sh}(2, \mathbb{N})$ *has the* $\frac{1}{4}$*-fractional property.*

Proof. Let $X \subseteq \mathbb{N}^{(2)}$ be a finite subset of vertices of $\mathrm{Sh}(2, \mathbb{N})$ and let $\mathbf{w} : X \to [0, 1]$ be a stochastic weight vector. Consider a random coloring $c : \mathbb{N} \to \{0, 1\}$, where each integer n is colored independently with probability

$$\mathbb{P}(c(n) = 0) = \frac{1}{2}.$$

Let $X_{0,1}$ be the random set defined by

$$X_{0,1} = \{\{x, y\} \in X : \, x < y \text{ and } c(x) = 0, \, c(y) = 1\}.$$

That is, $X_{0,1}$ are the ordered pairs of X such that the first integer is of color 0 and the last one of color 1. One can see that $X_{0,1}$ is an independent set in $\mathrm{Sh}(2, \mathbb{N})$. Moreover, by letting

$$Z_{0,1} = \sum_{x \in X_{0,1}} \mathbf{w}(x).$$

we have that

$$\mathbb{E}(Z_{0,1}) = \sum_{\{x,y\} \in X} \mathbb{P}\big(\{c(x) = 0\} \wedge \{c(y) = 1\}\big)\mathbf{w}(\{x, y\}) = \sum_{\{x,y\} \in X} \frac{1}{4}\mathbf{w}(\{x, y\}) = \frac{1}{4}.$$

Thus, by the first moment, with positive probability there is a coloring c such that $X_{0,1}$ is an independent set satisfying the statement of the claim. □

We remark that the constant $\mu = 1/4$ in Claim 2.3.1 is the best possible. This was proved in a joint paper with Arman and Rödl [2].

Chapter 3

Pisier type problem for B_h-sets

The content of this chapter was obtained in joint work with Nešetřil and Rödl and is based on [36].

3.1 A local version of the Pisier problems for sets

In this section we introduce a version of the Pisier problem for finite sets that will be useful in the proof of Theorem 1.1.5. Let $\mathcal{A} = \{A_i\}_{i\in I}$ be a system of finite sets on the ground set X. We say that $\mathcal{A}$ is *h-independent* if for any indices $J, J' \subseteq I$ with $|J| = |J'| = h$,

$$\biguplus_{j\in J} A_j \neq \biguplus_{j'\in J'} A_{j'},$$

where $\biguplus$ stands for the multiset union operation, i.e., every element is counted according to its multiplicity in the operation. For instance, $\{1,2\} \uplus \{2,3\} = \{1,2,2,3\}$. One can see h-independent sets as the analogue of B_h-sets in the context of sets equipped with the multiset union operation.

In this context, statements (1) and (2) of the Pisier problem can be rewritten as

(1) $\mathcal{A}$ is the union of finitely many h-independent set systems.

(2) There exists $\varepsilon > 0$ such that every finite set system $\mathcal{A}' \subseteq \mathcal{A}$ contains a h-independent subset $\mathcal{A}''$ with $|\mathcal{A}''| \geqslant \varepsilon|\mathcal{A}'|$ elements.

The next result shows that statement (2) does not imply statement (1) and consequently these statements are not equivalent.

Theorem 3.1.1. *For every $h \geqslant 1$, there exists $\varepsilon > 0$ and a set system $\mathcal{A}$ on the ground set $\mathbb{N}$ with the following two properties:*

(i) $\mathcal{A}$ is not the union of finitely many h-independent sets.

(ii) Every finite subsystem $\mathcal{A}' \subseteq \mathcal{A}$ contains an h-independent set $\mathcal{A}'' \subseteq \mathcal{A}'$ with $|\mathcal{A}''| \geqslant \varepsilon|\mathcal{A}'|$ elements.

To prove Theorem 3.1.1 we will use the following result from [35]. A *partial Steiner (k, ℓ)-system* G is a k-uniform hypergraph (shortly k-graph) with the property that every ℓ-element subset of the vertex set of G is in at most one edge. For this problem all Steiner systems will be ordered, i.e., the vertex set of the graph has a linear order. We will say that F is a subgraph of G if there is an order preserving injective mapping $\varphi : V(F) \to V(G)$ which is a homomorphism. Let $\mathcal{S}_<(k, \ell)$ be the class of all ordered partial Steiner (k, ℓ)-systems. The next result shows that the class of ordered partial Steiner systems have the Ramsey property.

Theorem 3.1.2 ([35], Theorem 6.2)**.** *The class $\mathcal{S}_<(k, \ell)$ of all ordered partial Steiner (k, ℓ)-systems has the edge Ramsey property, i.e., for every $F \in \mathcal{S}_<(k, \ell)$ and for any integer r there exists $G \in \mathcal{S}_<(k, \ell)$ with the property that any r-coloring of the edges of G yields a monochromatic copy of F.*

Proof of Theorem 3.1.1. Let k be an even number and G a k-uniform graph with vertex set $V(G) \subseteq \mathbb{N}$. On a set $\mathbb{N} \times [k/2]$ we will construct a set system $\mathcal{A}_G$ as follows: For an edge $e = \{x_1, \ldots, x_k\}$, with $x_1 < \ldots < x_k$, define the set $A_e \subseteq \mathbb{N} \times [k/2]$ given by

$$A_e = \bigcup_{i=1}^{k/2} [x_{2i-1}, x_{2i}) \times \{i\},$$

where $[a, b) \times \{i\} = \{(a, i), (a+1, i), \ldots, (b-1, i)\}$ denotes the interval of integers between a and b, with b not included, in the i-th copy of $\mathbb{N}$. With this in mind, we define the set system $\mathcal{A}_G$ on the ground set $\mathbb{N} \times [k/2]$ as

$$\mathcal{A}_G = \{A_e :\ e \in G\}.$$

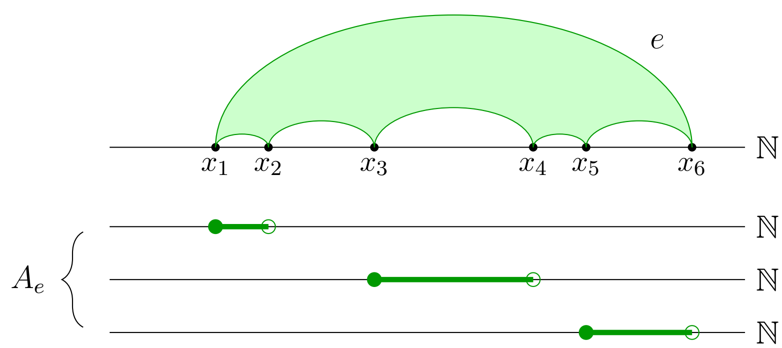

Figure 3.1: An edge e and its corresponding set A_e

We say that a graph G is *h-independent* if the associated set system $\mathcal{A}_G$ is h-independent, i.e., if there is no subgraph $F = \{f_1, \ldots, f_{2g}\} \subseteq G$ such that

$$\biguplus_{r=1}^{g} A_{f_r} = \biguplus_{s=g+1}^{2g} A_{f_s}$$

for $1 \leqslant g \leqslant h$. The following lemma shows that every non h-independent finite ordered k-partite k-graph has at least two edges with large intersection.

Lemma 3.1.3. *Let $k > h$ be integers with k even. Let H be a finite k-partite k-graph with vertex set V satisfying the following properties:*

(i) H is not h-independent.

(ii) There exists partition $V = V_1 \cup \ldots \cup V_k$ such that for every edge $e = \{x_1, \ldots, x_k\} \in H$ with $x_1 < \ldots < x_k$, we have $x_i \in V_i$.

Then there exist distinct edges $e, f \in H$ such that $|e \cap f| \geqslant k/h$.

Proof. Since H is not h-independent, there exists subgraph $F = \{f_1, \ldots, f_{2g}\} \subseteq H$ such that

$$\biguplus_{r=1}^{g} A_{f_r} = \biguplus_{s=g+1}^{2g} A_{f_s} \tag{3.1}$$

for some $1 \leqslant g \leqslant h$. Let $F' = \{f_1, \ldots, f_g\}$ and $F'' = \{f_{g+1}, \ldots, f_{2g}\}$. We claim that for every $x \in V$, we have $\deg_{F'}(x) = \deg_{F''}(x)$.

For $(a, i) \in \mathbb{N} \times [k/2]$ and subgraph $E \subseteq H$, let

$$\mu_E(a, i) = |\{e \in E : (a, i) \in A_e\}|,$$

i.e., $\mu_E(a, i)$ is the multiplicity of (a, i) in $\biguplus_{e \in E} A_e$. The relation (3.1) gives us that

$$\mu_{F'}(a, i) = \mu_{F''}(a, i) \tag{3.2}$$

for every $(a, i) \in \mathbb{N} \times [k/2]$.

Fix $i \in [k/2]$. We will prove that $\deg_{F'}(x) = \deg_{F''}(x)$ for every $x \in V_{2i-1} \cup V_{2i}$. Let x be the minimal integer in $V_{2i-1} \cup V_{2i}$ such that the statement is false. Suppose that $x \in V_{2i-1}$. Let $A \subseteq V_{2i-1}$, $B \subseteq V_{2i}$ be defined as

$$A = \{a \in V_{2i-1} : a < x\},$$
$$B = \{b \in V_{2i} : b < x\}.$$

That is, A and B are the subsets of V_{2i-1} and V_{2i} with elements smaller than x. If $e = \{x_1, \ldots, x_k\} \in E$ is an edge such that $(x, i) \in A_e$, then $x \in [x_{2i-1}, x_{2i})$. This implies that $x_{2i-1} \in A \cup \{x\}$ and $x_{2i} \notin B$. Hence,

$$\mu_E(x,i) = \sum_{a \in A, y \notin B} \deg_E(\{a,y\}) + \deg_E(x) = \sum_{a \in A} \deg_E(a) - \sum_{b \in B} \deg_E(b) + \deg_E(x). \tag{3.3}$$

By the minimality of x, we have that $\deg_{F'}(y) = \deg_{F''}(y)$ for all $y \in A \cup B$. Therefore, (3.2) and (3.3) gives us that $\deg_{F'}(x) = \deg_{F''}(x)$, which is a contradiction. If $x \in V_{2i}$, then one can show similarly that

$$\mu_E(x,i) = \sum_{a \in A} \deg_E(a) - \sum_{b \in B} \deg_E(b) - \deg_E(x)$$

and the result follows in the same way, which concludes the proof of the claim.

To finish the proof of Lemma 3.1.3 note that by the claim,

$$\begin{aligned} \sum_{f' \in F', f'' \in F''} |f' \cap f''| &= \sum_{i=1}^{k} \sum_{x \in V_i} \deg_{F'}(x) \deg_{F''}(x) \\ &= \sum_{i=1}^{k} \sum_{x \in V_i} \deg^2_{F'}(x) \geqslant \sum_{i=1}^{k} g = kg. \end{aligned}$$

Hence, by averaging, there exist $e \in F'$ and $f \in F''$ such that

$$|e \cap f| \geqslant \frac{kg}{g^2} = \frac{k}{g} \geqslant \frac{k}{h}.$$

□

The next lemma shows that for $\ell \leqslant k/h$ there exists a partial Steiner (k,ℓ)-system violating the h-independence condition.

Lemma 3.1.4. *For $h \geqslant 2$, there exists an even integer k and a partial Steiner (k,ℓ)-*

system $F = \{f_1, \ldots, f_{2h}\}$ *with* $\ell \leqslant k/h$ *such that*

$$\biguplus_{r=1}^{h} A_{f_r} = \biguplus_{s=h+1}^{2h} A_{f_s}.$$

Proof. We will construct a k-graph F satisfying the statement for $k = 2(h!)^2$ and $2h(h!)^2$ vertices. The construction depends on the parity and size of h.

Case 1: $h = 2t \geqslant 4$.

Let S_h be the set of permutations $\sigma : [h] \to [h]$. Write $S_h = \{\sigma_1, \ldots, \sigma_{h!}\}$. For a pair $(i, j) \in [h!]^2$, let $F_{ij} = C_{ij}^{(1)} \cup \ldots \cup C_{ij}^{(t)}$ be a labeled 2-graph consisting of $h/2 = t$ four cycles. For each $1 \leqslant q \leqslant t$, we label the $C_{ij}^{(q)}$ as follows: Let $V(C_{ij}^{(q)}) = \{x_1, x_2, x_3, x_4\}$ with $x_1 < x_2 < x_3 < x_4$ and label the edges of the cycle as in Figure 3.2.

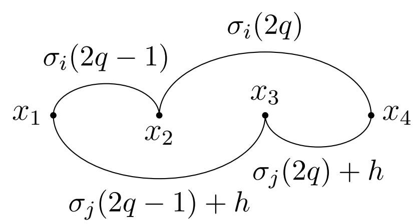

Figure 3.2: A four cycle $C_{ij}^{(q)}$

We order the vertices of all $C_{ij}^{(q)}$ such that $\max V(C_{ij}^{(q)}) < \min V(C_{i'j'}^{(q')})$ if and only if $(i, j, q) <_{\text{lex}} (i', j', q')$ in the lexicographical ordering. This in particular gives us a total ordering of $\bigcup_{1 \leqslant i,j \leqslant h!} V(F_{ij})$. For a fixed F_{ij}, each one of its $4t = 2h$ edges is labeled by precisely one of the labels from $[2h]$. Set $F_{ij} = \{f_{ij}^1, \ldots, f_{ij}^{2h}\}$, where f_{ij}^s is the edge of F_{ij} labeled with s.

We finally define the k-graph F as the graph with vertex set $V(F) = \bigcup_{1 \leqslant i,j, \leqslant h!} V(F_{ij})$, where the ordering of $V(F)$ respects the total ordering of $V(F_{ij})$ described above, and edge set given by

$$F = \left\{ f_s := \bigcup_{1 \leqslant i,j, \leqslant h!} f_{ij}^s : \; 1 \leqslant s \leqslant 2h \right\}$$

That is, the graph F consists of $2h$ edges of size $k = 2(h!)^2$ where the edge f_s of F is the union of all the pairs labeled with s.

We claim that F is a partial Steiner (k,ℓ)-system with $\ell = h(h-2)!h! + 1 \leqslant 2(h-1)!h! = k/h$ for $h > 2$. Let f_r and f_s be two edges of F such that $1 \leqslant r,s \leqslant h$. Then f_r and f_s only intersects in the cycles $C_{ij}^{(q)}$ such that

$$\{\sigma_i(2q-1), \sigma_i(2q)\} = \{r,s\} \tag{3.4}$$

For each $1 \leqslant q \leqslant t$, there are $2(h-2)!$ choices of σ_i satisfying (3.4). Consequently there are $2t(h-2)!h!$ choices of q and $\sigma_i, \sigma_j \in S_h$ such that f_r and f_s intersects in $C_{ij}^{(q)}$. Since f_r and f_s intersects in at most one vertex for each $C_{ij}^{(q)}$ we obtain that

$$|f_r \cap f_s| = 2t(h-2)!h! = h(h-2)!h!.$$

A similar computation shows that for $1 \leqslant r \leqslant h$ and $h+1 \leqslant s \leqslant 2h$

$$|f_r \cap f_s| = h((h-1)!)^2,$$

and for $h+1 \leqslant r,s, \leqslant 2h$

$$|f_r \cap f_s| = h(h-2)!h!.$$

Since $h(h-2)!h! > h((h-1)!)^2$ for $h \geqslant 2$, we obtain that F is a partial Steiner (k,ℓ)-system for $\ell = h(h-2)!h! + 1$.

It remains to show that $\biguplus_{r=1}^{h} A_{f_r} = \biguplus_{s=h+1}^{2h} A_{f_s}$. Since $k/2 = (h!)^2$, there exists an order preserving bijection $\varphi : [h!]^2 \to [k/2]$, where $[h!]^2$ is ordered lexicographically. Note that

$$A_{f_r} \cap (\mathbb{N} \times \{\varphi(i,j)\}) = \big[\min V(f_{ij}^r), \max(V(f_{ij}^r)\big) \times \{\varphi(i,j)\}$$

for every $1 \leqslant r \leqslant 2h$. Therefore,

$$\begin{aligned}\biguplus_{r=1}^{h} A_{f_r} &= \bigcup_{1\leqslant i,j\leqslant h!} \biguplus_{r=1}^{h} \big[\min V(f_{ij}^r), \max(V(f_{ij}^r)\big) \times \{\varphi(i,j)\} \\ &= \bigcup_{1\leqslant i,j\leqslant h!} \bigcup_{q=1}^{t} \Big[\min V(C_{ij}^{(q)}), \max(V(C_{ij}^{(q)})\Big) \times \{\varphi(i,j)\} \\ &= \bigcup_{1\leqslant i,j\leqslant h!} \biguplus_{s=h+1}^{2h} \big[\min V(f_{ij}^s), \max(V(f_{ij}^s)\big) \times \{\varphi(i,j)\} = \biguplus_{s=1}^{h} A_{f_s}\end{aligned}$$

since the pairs f_{ij}^r and f_{ij}^s for $1 \leqslant r \leqslant h$ and $h+1 \leqslant s \leqslant 2h$ cover precisely once the entire interval of each cycle $C_{ij}^{(q)}$ from $1 \leqslant q \leqslant t$.

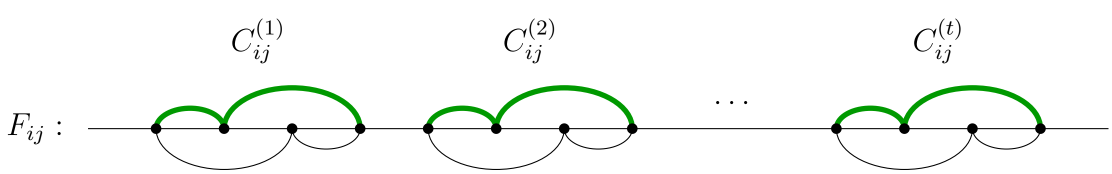

Figure 3.3: The pairs f_{ij}^r for $1 \leqslant r \leqslant h$

Case 2: $h = 2t+1 \geqslant 3$

The constructions is very similar to the previous case. For a pair $(i,j) \in [h!]^2$, let $F_{ij} = \bigcup_{q=1}^{t+1} C_{ij}^{(q)}$ be a labeled multigraph consisting of t four cycles and a 2-cycle $C_{ij}^{(t+1)}$. For each $1 \leqslant q \leqslant t$, we label the four cycle $C_{ij}^{(q)}$ exactly as in Case 1 (see Figure 3.2). We define $C_{ij}^{(t+1)}$ as the multigraph with two vertices and two edges labeled as in Figure 3.4.

As in Case 1, we label the vertices of $C_{ij}^{(q)}$ such that $\max V(C_{ij}^{(q)}) < \min V(C_{i'j'}^{(q')})$ if and only if $(i,j,q) <_{\text{lex}} (i',j',q')$. Moreover, F_{ij} is a multigraph with $2h$ edges labeled

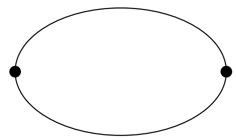

Figure 3.4: The 2-cycle $C_{ij}^{(t+1)}$

in an one-to-one correspondence with $[2h]$. Write $F_{ij} = \{f_{ij}^1, \ldots, f_{ij}^{2h}\}$, where f_{ij}^s is the edge of F_{ij} with label s.

We define F as the k-graph with vertex set $V(F) = \bigcup_{1 \leqslant i,j \leqslant h!} V(F_{ij})$ and edges

$$F = \left\{ f_s := \bigcup_{1 \leqslant i,j \leqslant h!} f_{ij}^s : 1 \leqslant s \leqslant 2h \right\}.$$

A similar argument as in Case 1 shows that $\biguplus_{r=1}^{h} A_{f_r} = \biguplus_{s=h+1}^{2h} A_{f_s}$. Furthermore, a careful analysis shows that

$$|f_r \cap f_s| = (h-1)!h!$$

for $1 \leqslant r, s \leqslant h$ or $h+1 \leqslant r, s \leqslant 2h$ and

$$|f_r \cap f_s| = (h+1)((h-1)!)^2.$$

Thus, F is a partial Steiner (k, ℓ)-system with $\ell = (h+1)((h-1)!)^2 + 1 \leqslant 2(h-1)!h! = k/h$ for $h > 1$.

Case 3: $h = 2$.

Let $F = \{f_1, f_2, f_3, f_4\}$ be the 8-uniform hypergraph on 16 vertices described in Figure 3.5, where, for each $1 \leqslant s \leqslant 4$, the edge f_s is the union of all the pairs labeled with s. Let the vertices of F be ordered from left to right exactly as shown in Figure 3.5.

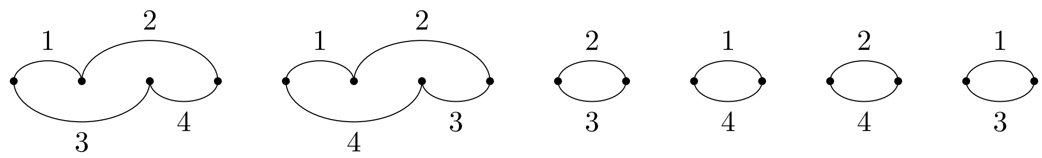

Figure 3.5: The graph F for $h = 2$

Following a similar argument as in Case 1, one can show that $A_{f_1} \uplus A_{f_2} = A_{f_3} \uplus A_{f_4}$. Moreover, one can also check that $|f_i \cap f_j| \geqslant 3$ for every $1 \leqslant i < j \leqslant 4$. Hence, F is a partial Steiner (k, ℓ)-system with $\ell = 4 = 8/2 = k/h$. $\square$

Since there is a bijection between $\mathbb{N} \times [k/2]$ and $\mathbb{N}$, to prove Theorem 3.1.1 we just need to show that there exists $\varepsilon > 0$ and a k-graph G such that $\mathcal{A}_G$ satisfies properties (i) and (ii) of the statement, i.e., a k-graph G such that

(i) Any finite coloring of G contains a monochromatic subgraph F that is not h-independent.

(ii) Every finite subgraph $G' \subseteq G$ contains an h-independent subgraph $G'' \subseteq G'$ with $e(G'') \geqslant \varepsilon e(G')$.

Let F be the partial Steiner (k, ℓ)-system obtained by Lemma 3.1.4. Given an integer r, by Theorem 3.1.2, there exists a partial Steiner (k, ℓ)-system G_r such that any r-coloring of the edges of G_r contains a monochromatic copy of F. Let $G = \bigcup_{r=1}^{\infty} G_r$ be the union of disjoint copies of G_r for $r \geqslant 1$. Order the vertex set of G such that $V(G) \subseteq \mathbb{N}$ and $\max V(G_r) < \min V(G_s)$ for $r < s$. We claim that G satisfies properties (i) and (ii).

For $r \geqslant 1$, consider an arbitrary r-coloring $c : G \to [r]$ of the edges of G. In particular, $c_{|G_r}$ is an r-coloring of $G_r \subseteq G$ and by Theorem 3.1.2, there exists a monochromatic copy of F. By Lemma 3.1.4, the graph F is not h-independent, which proves statement (i).

For statement (ii), let $G' \subseteq G$ be a finite subgraph of G. We are going to show that there exists a subgraph $H \subseteq G'$ with $e(H) \geqslant e(G')/k^k$ such that the vertex set of

H can be partitioned into $V(H) = V_1 \cup \ldots \cup V_k$ satisfying the following: for every edge $e = \{x_1, \ldots, x_k\} \in H$ with $x_1 < \ldots < x_k$, we have $x_i \in V_i$. Indeed, consider a random partition $V(G') = V_1 \cup \ldots \cup V_k$ such that every x is chosen to be in V_i independently with probability $1/k$. Thus, if $e = \{x_1, \ldots, x_k\} \in G'$, then $\mathbb{P}\left(\bigwedge_{i=1}^{k}\{x_i \in V_i\}\right) = 1/k^k$.

Let H be the graph consisting of all the transversal edges $e = \{x_1, \ldots, x_k\} \in G$ with $x_i \in V_i$ for $1 \leqslant i \leqslant k$. Then

$$\mathbb{E}(e(H)) = \sum_{\{x_1,\ldots,x_k\}\in G} \mathbb{P}\left(\bigwedge_{i=1}^{k}\{x_i \in V_i\}\right) = \frac{e(G')}{k^k},$$

which by Markov inequality implies that with positive probability one can obtain H with $e(H) \geqslant e(G')/k^k$. We claim that such H is h-independent. Suppose to the contrary that is not. Then by Lemma 3.1.3, there exists edges $e, f \in H$ such that $|e \cap f| \geqslant k/h$. However, by Lemma 3.1.4, the graph $H \subseteq G$ is a partial Steiner (k,ℓ)-system with $\ell \leqslant k/h$, which is a contradiction. Therefore, statement (ii) holds by taking $\varepsilon = 1/k^k$ and $G'' = H$. □

3.2 Proof of Theorem 1.1.5

In this section we prove Theorem 1.1.5 and also make partial progress on the original Pisier problem by answering in the negative a one sided version of the problem.

Proof of Theorem 1.1.5. Let $\mathcal{A} = \{A_i\}_{i\in I}$ be the set system on the ground set $\mathbb{N}$ obtained by Theorem 3.1.1. Let $X = \{x_i\}_{i\in I} \subseteq \mathbb{N}$ be the set of integers defined by

$$x_i = \sum_{j\in A_i} (h+1)^j.$$

Then for two set of indices $J, J' \subseteq I$ of size h, we have $\sum_{j\in J} x_j = \sum_{j'\in J'} x_{j'}$ if and only if $\biguplus_{j\in J} A_j = \biguplus_{j'\in J'} A_{j'}$. This implies that a subset $X' = \{x_{i'}\}_{i'\in I'} \subseteq X$ is a B_h-set if and only if the correspondent subfamily $\mathcal{A}' = \{A_{i'}\}_{i'\in I'} \subseteq \mathcal{A}$ is h-independent.

Hence, X satisfies statements (i) and (ii) of Theorem 1.1.5. $\square$

For an integer $h \geqslant 1$, we say that a set X is *h-free* if equation (1.1) holds for any distinct subset of indices $J, J' \subseteq I$ with $|J| \leqslant h$ (the size of J' may be arbitrary). We are going to prove the following:

Theorem 3.2.1. *For every $h \geqslant 1$ there exists $\varepsilon > 0$ and a set of positive integers X with the following two properties:*

(i) X is not a union of finitely many h-free sets.

(ii) Every finite subset $Y \subseteq X$ contains an h-free set Z with $|Z| \geqslant \varepsilon|Y|$ elements.

Proof. Let $A = \{a_i\}_{i \in I} \subseteq \mathbb{N}$ be the set of integers and $\varepsilon > 0$ the constant obtained from Theorem 1.1.5 satisfying statements (i) and (ii). Since A cannot be written as a finite union of B_h-sets, by a standard compactness argument ([6], Theorem 1) one can obtain for every $r \geqslant 1$ a finite set $A_r \subseteq A$ satisfying the following two properties:

(i) A_r is not an union of at most r B_h-sets.

(ii) Every subset $B \subseteq A_r$ contains a B_h-set $C \subseteq B$ with $|C| \geqslant \varepsilon|B|$.

We construct a sequence of finite sets $\{W_j\}_{j=0}^{\infty}$ satisfying the following: Let $X_r = \bigcup_{j=0}^{r} W_j$.

(i) X_r is not a union of at most r h-free sets.

(ii) Every subset $Y \subseteq X_r$ contains an h-free set $Z \subseteq Y$ with $|Z| \geqslant \varepsilon|Y|$.

Theorem 3.2.1 follows by taking $X = \bigcup_{j=0}^{\infty} W_j$.

Let $W_0 = \{0\}$. Suppose that we already constructed $W_0, \ldots, W_{r-1}$ and $X_{r-1} = \bigcup_{j=0}^{r-1} W_j$ satisfies statements (i) and (ii). We choose n_r and m_r to satisfy

$$n_r > \sum_{x \in X_{r-1}} x \quad \text{and} \quad m_r > n_r \left(1 + \sum_{a \in A_r} a\right). \tag{3.5}$$

Define $W_r = \{n_r a + m_r : a \in A_r\}$ and $X_r = \bigcup_{j=0}^{r} W_j = W_r \cup X_{r-1}$. It remains to prove that X_r satsifies properties (i) and (ii).

Property (i) follows by the fact that an ℓ-coloring of X_r, for $\ell \leqslant r$, is in particular an ℓ-coloring of W_r. Since there is a bijective linear map from A_r to W_r, we obtain that the ℓ-coloring in W_r corresponds to an ℓ-coloring in A_r. By construction, this coloring must contain a monochromatic equation

$$\sum_{b\in B} b = \sum_{b'\in B'} b$$

for $B, B' \subseteq A_r$ with $|B| = |B'| = h$. Then the equation

$$\sum_{b\in B}(n_r b + m_r) = \sum_{b'\in B'}(n_r b' + m_r)$$

is monochromatic in W_r, which implies that one of the colors classes is not h-free.

In order to prove Property (ii), consider an arbitray subset $Y \subseteq X_r$. Write $Y = Y' \cup Y''$, where $Y' = Y \cap X_{r-1}$ and $Y'' = Y \cap W_r$. By our induction hypothesis, there exists h-free set $Z' \subseteq Y'$ with $|Z'| \geqslant \varepsilon|Y'|$. Let $f : A_r \to W_r$ be the bijective linear map given by $f(a) = n_r a + m_r$. By property (ii) of A_r, there exists a B_h-set $C \subseteq f^{-1}(Y'') \subseteq A_r$ with $|C| \geqslant \varepsilon|f^{-1}(Y'')| = \varepsilon|Y''|$. Take $Z'' = f(C)$. We claim that $Z = Z' \cup Z''$ is h-free.

Suppose that $\sum_{p\in P} p = \sum_{q\in Q} q$ for some $P, Q \subseteq Z$. We want to show that $|P|, |Q| > h$. Let $P = P' \cup P''$ and $Q = Q' \cup Q''$ be partitions of the sets such that $P' = P \cap Z'$, $P'' = P \cap Z''$, $Q' = Q \cap Z'$ and $Q'' = Q \cap Z''$. A computation shows that

$$\begin{aligned}\left|\sum_{p\in P''} p - \sum_{q\in Q''} q\right| &= \left|\sum_{a\in f^{-1}(P'')} (n_r a + m_r) - \sum_{b\in f^{-1}(Q'')} (n_r b + m_r)\right| \\ &= \left|(|P''| - |Q''|)m_r + n_r\left(\sum_{a\in f^{-1}(P'')} a - \sum_{b\in f^{-1}(Q'')} b\right)\right| \qquad (3.6)\end{aligned}$$

Suppose that $|P''| \neq |Q''|$, then our choice of n_r and m_r in (3.5) and equation (3.6) gives us that

$$\left|\sum_{p\in P''} p - \sum_{q\in Q''} q\right| \geqslant m_r - \left|n_r\left(\sum_{a\in f^{-1}(P'')} a - \sum_{b\in f^{-1}(Q'')} b\right)\right| \geqslant m_r - n_r\left(\sum_{a\in A_r} a\right) > n_r.$$

Hence, by (3.5) and the fact that $P', Q' \subseteq X_{r-1}$,

$$0 = \left|\sum_{p\in P} p - \sum_{q\in Q} q\right| \geqslant \left|\sum_{p\in P''} p - \sum_{q\in Q''} q\right| - \left|\sum_{p\in P'} p - \sum_{q\in Q'} q\right| > n_r - \sum_{x\in X_r} x > 0,$$

which is a contradiction. Therefore, $|P''| = |Q''|$. We also claim that $\sum_{a\in f^{-1}(P'')} a = \sum_{b\in f^{-1}(Q'')} b$. Indeed, suppose to the contrary that $\sum_{a\in f^{-1}(P'')} a \neq \sum_{b\in f^{-1}(Q'')} b$. Then, by (3.5) and (3.6) we have

$$\left|\sum_{p\in P''} p - \sum_{q\in Q''} q\right| = \left|n_r\left(\sum_{a\in f^{-1}(P'')} a - \sum_{b\in f^{-1}(Q'')} b\right)\right| \geqslant n_r$$

and we reach a contradiction similarly as in the proof of $|P''| = |Q''|$. To finish the proof, note that $C = f^{-1}(Z'')$ is a B_h-set. Hence, $|P''| = |Q''| > h$ and consquently Z is h-free □

Chapter 4

Pisier type problems for arithmetic progressions

The content of this chapter was obtained in joint work with Christian Reiher and Vojtech Rödl and is based on [40].

4.1 A modification of Hales–Jewett theorem

We will now describe a modifictation of the Hales–Jewett theorem that is going to be used in the proof of Theorem 1.1.4. Given an alphabet $A = \{a_1, \dots, a_q\}$, we say that an n-tuple $\mathbf{u} = (\mathbf{u}(1), \dots, \mathbf{u}(n)) \in A^n$ is a *word* of length n in the combinatorial cube A^n. A collection of q words $L = \{\mathbf{u}_1, \dots, \mathbf{u}_q\}$ of length n with $\mathbf{u}_i = (\mathbf{u}_i(1), \dots, \mathbf{u}_i(n))$ is a *combinatorial line* if there exists a partition $[n] = M_L \cup F_L$ with $M_L \neq \emptyset$ and a sequence $\{b_s\}_{s \in F_L}$ of elements of A such that

$$\mathbf{u}_i(s) = \begin{cases} a_i, & \text{if } s \in M_L, \\ b_s, & \text{if } s \in F_L, \end{cases}$$

for $1 \leqslant i \leqslant q$ and $1 \leqslant s \leqslant n$. We will usually refer to M_L as the *moving indices* of the combinatorial line L, since for each word in L they correspond to a different letter of the alphabet. The set F_L is the *fixed indices* of L, because they are constant in every word of the combinatorial line.

Hales and Jewett [27] proved the following celebrated Ramsey result about combinatorial lines.

Theorem 4.1.1 ([27])**.** *Given integers $q, r \geqslant 1$, there exists an integer $N_0 := N_0(q, r)$ such that the following holds for $N \geqslant N_0$. For any alphabet A of size q and any r-coloring of the combinatorial cube A^N, there exists a monochromatic combinatorial line $L \subseteq A^N$.*

Let $\mathcal{L}(A^N)$ be the set of all combinatorial lines of A^N. One can view $\mathcal{L}(A^N)$ as the q-uniform hypergraph with vertex set A^N and combinatorial lines as edges. With this interpretation, Theorem 4.1.1 says that $\chi(\mathcal{L}(A^N)) > r$ for any $N \geqslant N_0(q, r)$.

Given a hypergraph H, a cycle of length ℓ in H consists of ℓ distinct edges $e_1, \dots, e_\ell$ and ℓ distinct vertices $x_1, \dots, x_\ell$ such that $x_i \in e_i \cap e_{i+1}$ for $1 \leqslant i \leqslant \ell$, where the indices are taken modulo ℓ. The girth $g(H)$ of a hypergraph H is the length of the shortest cycle in H. A famous result by Erdős, Hajnal and Lovász [8, 9, 31] states that for any integers k, g, r, there exists a k-graph H with chromatic number $\chi(H) > r$ and girth $g(H) > g$. We will use the following similar variation for the Hales–Jewett theorem established in [41].

Theorem 4.1.2 ([41])**.** *Let q, r, g be positive integers and A an alphabet of size q. Then there exists a integer $N := N(q, r, g)$ and a subgraph $H \subseteq \mathcal{L}(A^N)$ such that $\chi(H) > r$ and $g(H) > g$.*

In other words, Theorem 4.1.2 says that there exists a subset of combinatorial lines such that the hypergraph formed by them has high chromatic number and high girth. For simplicity, in the remaining of the paper, we will denote the graph obtained by Theorem 4.1.2 as $\mathcal{L}_g(A^N)$ instead of H.

4.2 The partite construction

Our proof of Theorem 1.1.4 will be based on a variant of the partite amalgamation construction (see [34, 16]). Partite amalgamation is a construction which allows us to alter one Ramsey type statement into another one. We will use Theorem 2.1.3 and 4.1.2 to prove the following finite form of Theorem 1.1.4.

Theorem 4.2.1. *For every $k \geqslant 3$, $r \geqslant 1$ and $0 < \mu < \frac{k-1}{k}$ there is a finite set of integers $X := X(k, r, \mu) \subseteq \mathbb{N}$ satisfying the two following properties:*

(i) Every r-coloring of X contains a monochromatic AP_k.

(ii) Every $Y \subseteq X$ contains an AP_k-free subset $Z \subseteq Y$ with $|Z| \geqslant \mu|Y|$.

Before proving Theorem 4.2.1, which occupies the remainder of this chapter, we show that Theorem 1.1.4 follows as a corollary of Theorem 4.2.1.

Proof of Theorem 1.1.4. For every $r \geqslant 1$, let $X_r := X(k, r, \mu)$ be the set obtained by Theorem 4.2.1 with parameters k, r and μ. Let $\{x_r\}_{r\geqslant 1}$ be a sequence of integers and $\{W_r\}_{r\geqslant 1}$ be a sequence of sets $W_r \subseteq \mathbb{N}$ defined as follows: For $r = 1$, set $x_1 = 0$ and $W_1 = X_1$. For $r > 1$, let

$$x_r = 2(\max W_{r-1} + \max X_r)$$

and $W_r = X_r + x_r = \{x + x_r : x \in X_r\}$. It is easy to check that $\max W_r < \min W_{r+1}$ for every $r \geqslant 1$. Set

$$X = \bigcup_{r\geqslant 1} W_r.$$

We claim that X satisfies the properties of Theorem 1.1.4.

Property (i) follows from the fact that W_r is a linear transformation of X_r and consequently preserves AP_k. This in particular implies that any r-coloring of W_r contains a monochromatic AP_k. Hence, because $X = \bigcup_{r\geqslant 1} W_r$, we obtain that any r-coloring of X contains a monochromatic AP_k for $r \geqslant 1$.

To check property (ii), first note that if $A \subseteq X$ is an AP_k, then $A \subseteq W_r$ for some $r \geqslant 1$. This is due to our choice of the quickly increasing sequence $\{x_r\}_{r\geqslant 1}$. Let $Y \subseteq X$ be a finite subset. Then there exists an integer t such that $Y \subseteq \bigcup_{r=1}^{t} W_r$. Write $Y_r = W_r \cap Y \subseteq W_r$. Since W_r is a linear transformation of X_r, by Theorem 4.2.1 there exists an AP_k free set $Z_r \subseteq Y_r$ with $|Z_r| \geqslant \mu|Y_r|$ for $1 \leqslant r \leqslant t$. Set $Z = \bigcup_{r=1}^{t} Z_r$. We claim that $Z \subseteq Y$ is AP_k-free with $|Z| \geqslant \mu|Y|$.

Suppose that $A \subseteq Z$ is an AP_k. Since $A \subseteq X$, there exists integer $r \geqslant 1$ such that $A \subseteq W_r$. Hence, $A \subseteq Z_r = Z \cap W_r$, which contradicts the fact that Z_r is AP_k-free. Finally,

$$|Z| = \sum_{r=1}^{t} |Z_r| \geqslant \sum_{r=1}^{t} \mu|Y_r| = \mu|Y|.$$

□

4.2.1 Construction of $X(k, r, \mu)$

We devote the rest of the section for the construction of the sets $X(k, r, \mu)$. Given $k \geqslant 3$, $r \geqslant 1$ and $0 < \mu < \frac{k-1}{k}$, let G be the simple k-graph obtained by Theorem 2.2.2 such that:

(i) $\chi(G) > r$.

(ii) G has the μ-fractional property.

Suppose that $V(G) = [n]$ and $m = |E(G)|$.

Our plan is to construct the set $X(k, r, \mu)$ by partite construction. This will be done inductively, by successively constructing the set of integers $P_0, P_1 \ldots, P_n$. We start with a set P_0 satisfying property (ii) of Theorem 4.2.1. For $1 \leqslant i \leqslant n$, the set P_i will be constructed by amalgamating several copies of P_{i-1}. The amalgamation will be done by using the modified version of Hales–Jewett given in Theorem 4.1.2 and in such a way that the new set P_i still satisfies property (ii), while it has new Ramsey properties. Finally, we set $X(k, r, \mu) = P_n$, which will have both properties (i) and (ii) of Theorem 4.2.1. Now we go into more details.

Construction of P_0

We start with the description of P_0. Let $E(G) = \{e_1, \ldots, e_m\}$ be an ordering of the edges of G, where $e_j = \{x_{1j}, \ldots, x_{kj}\}$ for $1 \leqslant j \leqslant m$. For $1 \leqslant i \leqslant k$ and $1 \leqslant j \leqslant m$, set

$$a_{ij} = i(2k)^j.$$

We construct for every vertex $t \in [n]$, the set of integers

$$P_{0,t} = \{a_{ij} : \; x_{ij} = t\}.$$

That is, $P_{0,t}$ is the set of integers corresponding to the vertex t, where for each edge containing t we have a unique integer a_{ij} depending on the edge e_j and the position of t in e_j. Clearly, $P_{0,t}$ is a set of size $\deg_G(t)$. Finally, we set

$$P_0 = \bigcup_{t=1}^{n} P_{0,t} = \{a_{ij} : \; 1 \leqslant i \leqslant k, \; 1 \leqslant j \leqslant m\}.$$

Note by construction that for every $1 \leqslant j \leqslant m$, the set $\{a_{ij}\}_{i=1}^{k}$ is an arithmetic progression. Moreover, these are the only arithmetic progressions of length k in P_0.

Indeed, let $D = \{d_1, \ldots, d_k\}$ be an AP_k in P_0, where $d_s = i_s(2k)^{j_s}$ for $1 \leqslant s \leqslant k$. Since $d_s + d_{s+2} = 2d_{s+1}$ for $1 \leqslant s \leqslant k-2$, we obtain that $i_s(2k)^{j_s} + i_{s+2}(2k)^{j_{s+2}} = 2i_{s+1}(2k)^{j_{s+1}}$. This implies that $j_s = j_{s+1} = j_{s+2}$. Hence, $D = \{i(2k)^j\}_{i=1}^k = \{a_{ij}\}_{i=1}^k$ for some $1 \leqslant j \leqslant m$.

Graphically, the set of integers P_0 can be seen as in Figure 4.1. On the vertical projection we have our k-graph G with labeled edges $\{e_1, \ldots, e_m\}$. For each edge e_j we have a corresponding AP_k given by the set $\{a_{ij}\}_{i=1}^k$. The sets $P_{0,t}$ corresponds to the horizontal dashed line in the picture. We usually refer to those as *musical lines*. Furthermore, if we think of P_0 as a k-graph with P_0 as the vertex set and edges being arithmetic progressions of length k, then P_0 is a matching.

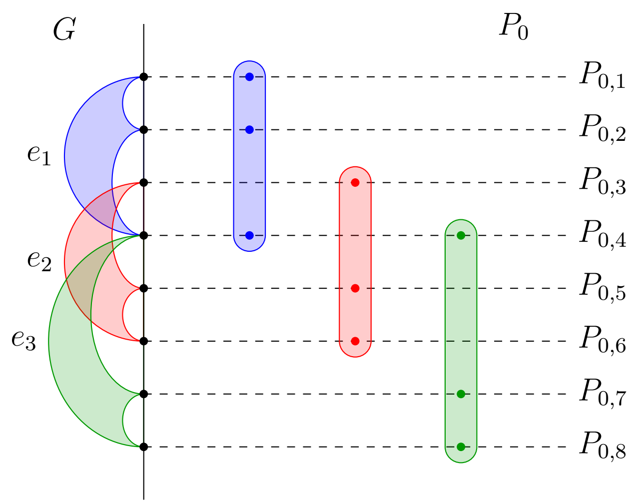

Figure 4.1: A visual representation of P_0

Construction of P_i

Next we will describe how to form P_i for $i \geqslant 1$. Suppose that we already constructed the set of integers $P_{i-1} = \bigcup_{t=1}^n P_{i-1,t}$, where $P_{i-1,t}$ is the musical line of P_{i-1} for the vertex $t \in [n]$.

Consider the alphabet $A = P_{i-1,i}$ of size $q = |P_{i-1,i}|$. By Theorem 4.1.2, there exists an integer $N := N(q, r, 3)$ and a set of combinatorial lines $\mathcal{L}_3 := \mathcal{L}_3(A^N) =$

$\mathcal{L}_3(P_{i-1,i}^N)$ such that $\mathcal{L}_3$ has girth greater than 3 and chromatic number greater than r. We will construct P_i from an auxiliary set of vectors $V_i \subseteq P_{i-1}^N$.

For a fixed combinatorial line $L \in \mathcal{L}_3$, let F_L and M_L be the set of fixed and moving indices of L and let $\{b_s\}_{s\in F_L}$ be the elements of $P_{i-1,i}$ corresponding to the fixed indices. For $a \in P_{i-1}$, we define the N-dimensional vector $\mathbf{v}_{a,L} \in P_{i-1}^N$ by

$$\mathbf{v}_{a,L}(s) = \begin{cases} b_s, & \text{if } s \in F_L \\ a, & \text{if } s \in M_L \end{cases} \tag{4.1}$$

For $t \in [n]$, let

$$V_{i,t}(L) = \{\mathbf{v}_{a,L} : a \in P_{i-1,t}\} \subseteq P_{i-1}^N. \tag{4.2}$$

Note in particular that by (4.1),

$$V_{i,i}(L) = L \tag{4.3}$$

That is, the set of vectors $V_{i,i}(L)$ is just the combinatorial line L in the Hales–Jewett cube $P_{i-1,i}^N$ itself. Let

$$V_i(L) = \bigcup_{t\in[n]} V_{i,t}(L). \tag{4.4}$$

In order to define P_i, we first consider the family of vectors $V_i = \bigcup_{t=1}^n V_{i,t}$, where

$$V_{i,t} = \bigcup_{L\in\mathcal{L}_3} V_{i,t}(L) \tag{4.5}$$

for $t \in [n]$ and set $T = 2\max P_{i-1}$. Now consider the linear mapping $\psi : P_{i-1}^N \to \mathbb{N}$

given by

$$\psi(a_1, \dots, a_N) = \sum_{j=1}^{N} a_j T^j \tag{4.6}$$

Finally, define

$$P_i = \psi(V_i) = \{\psi(\mathbf{u}) : \mathbf{u} \in V_i\}.$$

Similarly, we can define $P_i(L) = \psi(V_i(L))$ and $P_{i,t} = \psi(V_{i,t})$ for $t \in [n]$ and $L \in \mathcal{L}_3$.

Before we proceed, we would like to make the connection between the construction of P_i and the partite construction a little bit more transparent. We say that two sets of integers X and Y are *equivalent* (or X is a copy of Y), and write $X \cong Y$, if there exists a bijection $\varphi : X \to Y$ and $\alpha, \beta \in \mathbb{R}$ such that $\varphi(x) = \alpha x + \beta$ for $x \in X$.

Since φ is a bijective linear mapping, the arithmetic progressions of X are preserved under the mapping φ. Therefore, if X has the properties $\mathrm{vdW}(k, r)$ or $\mathrm{Sz}(k, \delta)$, then Y also has the property $\mathrm{vdW}(k, r)$ or $\mathrm{Sz}(k, \delta)$ as well, justifying the notation $X \cong Y$.

The concept is interesting for us because of the following. Given a combinatorial line $L \in \mathcal{L}_3$, we claim that $P_i(L) \cong P_{i-1}$. In view of (4.1), (4.2) and (4.4) we have that

$$V_i(L) = \bigcup_{t=1}^{n} \{\mathbf{v}_{a,L} : a \in P_{i-1,t}\}$$

and hence by (4.6)

$$P_i(L) = \psi(V_i(L)) = \{\psi(\mathbf{v}_{a,L}) : a \in P_{i-1}\} = \left\{ \sum_{s \in M_L} aT^s + \sum_{s \in F_L} b_s T^s : a \in P_{i-1} \right\}$$
$$= \{\alpha a + \beta : a \in P_{i-1}\} \cong P_{i-1}$$

where $\alpha = \sum_{s \in M_L} T^s$ and $\beta = \sum_{s \in F_L} b_s T^s$ are constants not depending on $a \in P_{i-1}$. In particular, this claim implies that $P_i = \bigcup_{L \in \mathcal{L}_3} P_i(L)$ is a union of $|\mathcal{L}_3|$ copies of P_{i-1}. A similar computation also shows that $P_{i,i}(L) = \psi(V_{i,i}(L)) = \psi(L) \cong P_{i-1,i}$. Moreover, given two combinatorial lines $L, L' \in \mathcal{L}_3$, we have by (4.1), (4.2), (4.3) and (4.4) that

$$V_i(L) \cap V_i(L') = V_{i,i}(L) \cap V_{i,i}(L') = L \cap L' \tag{4.7}$$

and consequently $P_i(L)$ and $P_i(L')$ only intersect at $P_{i,i}$.

Thus, one can interpret the construction of P_i as follows. First, we construct the musical line $P_{i,i}$ by creating a Ramsey system $\{P_{i,i}(L)\}_{L \in \mathcal{L}_3}$ with the property that any r-coloring of $P_{i,i}$ contains a monochromatic $P_{i,i}(L) \cong P_{i-1,i}$. Second, for each combinatorial line $L \in \mathcal{L}_3$ we construct a disjoint copy of P_{i-1} with musical line $P_{i-1,i}$ being precisely $P_{i,i}(L)$. The union of all those copies is exactly P_i.

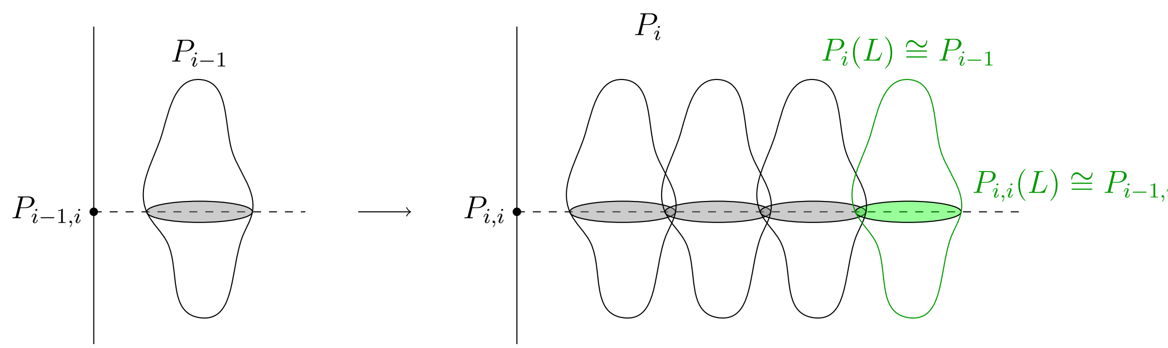

Figure 4.2: A visual representation of the construction of P_i

4.3 A property of the construction

Before we prove that our set of integers $X(k, r, \mu)$ satisfies the statement of Theorem 4.2.1, we will show the following structural property of the construction in Section 4.2. For $0 \leqslant i \leqslant n$, let $\pi : P_i \to [n]$ be the projection map defined by $\pi(a) = t$ if and only if $a \in P_{i,t}$. That is, the map π identifies in which musical line the integer a is located.

Lemma 4.3.1. *Let $P_0, \ldots, P_n$ be the sets of integers constructed in Section 4.2 and let G be the simple k-graph obtained by Theorem 2.2.2 used in the construction. Then the following holds:*

(a) For $1 \leqslant i \leqslant n$, if $A \subseteq P_i$ is an AP_k, then $A \subseteq P_i(L)$ for some combinatorial line $L \in \mathcal{L}_3 \subseteq P_{i-1,i}^N$.

(b) For $0 \leqslant i \leqslant n$, if $A \subseteq P_i$ is a non-trivial AP_k, i.e., not all the elements are equal, then $\pi(A) \in E(G)$.

(c) For $0 \leqslant i \leqslant n$, if $A, B \subseteq P_i$ are AP_k, then $|A \cap B| \leqslant 1$.

Proof. We proceed by induction on i. For $i = 0$, statements (b) and (c) as it can be seen in Figure 4.1. Now suppose that $1 \leqslant i \leqslant n$. We want to prove that P_i has properties (a), (b) and (c).

Note that property (a) implies properties (b) and (c). Indeed, if $A \subseteq P_i$ is an AP_k, then by property (a) we have that $A \subseteq P_i(L) \cong P_{i-1}$ for some $L \in \mathcal{L}_3$. Hence, by induction hypothesis $\pi(A) \in E(G)$, which proves property (b). Similarly, if $A, B \subseteq P_i$ are AP_k, then by property (a) we obtain that $A \subseteq P_i(L)$ and $B \subseteq P_i(L')$ for $L, L' \in \mathcal{L}_3$. If $L = L'$, then $A, B \subseteq P_i(L) \cong P_{i-1}$ and by the induction hypothesis we have $|A \cap B| \leqslant 1$. Otherwise, $A \cap B \subseteq P_i(L) \cap P_i(L')$. Since combinatorial lines intersect in at most one point, we have by (4.7) that

$$|A \cap B| \leqslant |P_i(L) \cap P_i(L')| = |V_i(L) \cap V_i(L')| = |L \cap L'| \leqslant 1,$$

which proves property (c).

Thus, it remains to show that P_i satisfies property (a). To simplify the argument, instead of working with the set of integers P_i, we are going to prove the statement for the set of vectors V_i introduced in Section 4.2. Our choice of bijective linear mapping $\psi : V_i \to P_i$ gives that an arithmetic progression $A = \{a_1, \ldots, a_k\} \subseteq P_i$ corresponds to a set of vectors $U = \{\mathbf{u}_1, \ldots, \mathbf{u}_k\} \subseteq V_i$ such that U is an AP_k in every coordinate. That is, if $\mathbf{u}_j = (\mathbf{u}_j(1), \ldots, \mathbf{u}_j(N))$ with $\mathbf{u}_j(s) \in P_{i-1}$, then the set $\{\mathbf{u}_j(s)\}_{j=1}^k$ is a (not necessarily non-trivial) AP_k for every $1 \leqslant s \leqslant N$.

Therefore, property (a) is equivalent to showing that if $U = \{\mathbf{u}_1, \ldots, \mathbf{u}_k\}$ is a collection of vectors that is an AP_k in every coordinate, then $U \subseteq V_i(L)$ for some $L \in \mathcal{L}_3$. Suppose that $\mathbf{u}_j \in V_i(L_j)$ for $1 \leqslant j \leqslant k$ and $L_1, \ldots, L_k \in \mathcal{L}_3$. Our goal is to prove that $L_1 = \ldots = L_k$. For each combinatorial line L_j, let F_{L_j} and M_{L_j} be its fixed and moving indices.

By the definition of $V_i(L_j)$ (see (4.1) and (4.4)), for each $\mathbf{u}_j = (\mathbf{u}_j(1), \ldots, \mathbf{u}_j(N)) \in V_i(L_j)$, there exists $c_j \in P_{i-1}$ such that $\mathbf{u}_j(s) = c_j$ for every $s \in M_{L_j}$. That is, $\mathbf{u}_j = \mathbf{v}_{c_j, L_j}$, where $\mathbf{v}_{c_j, L_j}$ is the vector defined in (4.1). Since $\mathbf{u}_j(s) \in P_{i-1,i}$ for $s \in F_{L_j}$, we obtain that the coordinates of $\mathbf{u}_j$ belong to the set of integers $P_{i-1,i} \cup \{c_j\}$ for $1 \leqslant j \leqslant k$ (note that is possible that $c_j \in P_{i-1,i}$). Therefore, the coordinate values of the entire set of vectors U belong to $P_{i-1,i} \cup \{c_1, \ldots, c_k\}$.

We claim that for $k \geqslant 4$ there exists at most one non-trivial AP_k in $P_{i-1,i} \cup \{c_1, \ldots, c_k\}$. This comes from the fact that if A is a non-trivial AP_k in $P_{i-1,i} \cup \{c_1, \ldots, c_k\} \subseteq P_{i-1}$, then by property (b) of the induction hypothesis we have that $\pi(A) \in E(G)$. In particular, this implies that $|\pi(A)| = k$ and consequently $|A \cap \{c_1, \ldots, c_k\}| \geqslant k-1$. Now if there were another non-trivial arithmetic progressions B in $P_{i-1,i} \cup \{c_1, \ldots, c_k\}$, then by the same argument $|B \cap \{c_1, \ldots, c_k\}| \geqslant k-1$. Hence, $|A \cap B| \geqslant k-2 > 1$, which contradicts property (c) of our induction hypothesis since $A, B \subseteq P_{i-1}$.

We remark that the claim made in the previous paragraph does not hold for $k = 3$. Unfortunately, in this case we can have more than one non-trivial AP_k in the set $P_{i-1,i} \cup \{c_1, \ldots, c_k\}$ and a special treatment will be required for this case. We split the proof now according to the number of non-trivial arithmetic progressions in the set $P_{i-1,i} \cup \{c_1, \ldots, c_k\}$.

<u>Case 1:</u> The set $P_{i-1,i} \cup \{c_1, \ldots, c_k\}$ has only trivial arithmetic progressions of length k.

By our assumption on U, the set $\{\mathbf{u}_j(s)\}_{j=1}^{k}$ is an AP_k in $P_{i-1,i} \cup \{c_1, \ldots, c_k\}$ for $1 \leqslant s \leqslant N$. Since there is no non-trivial AP_k in $P_{i-1,i} \cup \{c_1, \ldots, c_k\}$, we obtain that $\mathbf{u}_1(s) = \ldots = \mathbf{u}_k(s) \in P_{i-1,i}$. Hence, $\mathbf{u}_1 = \ldots = \mathbf{u}_k$, which implies that U consists of a single element and consequently there exists $L \in \mathcal{L}_3$ such that $U \subseteq V_i(L)$.

<u>Case 2:</u> The set $P_{i-1,i} \cup \{c_1, \ldots, c_k\}$ has exactly one non-trivial AP_k.

Let A be the non-trivial AP_k. By property (b) of the induction hypothesis, we have that $|\pi(A)| = k$. This in particular, implies that $|A \cap P_{i-1,i}| \leqslant 1$ and consequently at least $k-1$ values of $\{c_1, \ldots, c_k\}$ are not in $P_{i-1,i}$. Suppose without loss of generality that $c_1, \ldots, c_{k-1} \notin P_{i-1,i}$ and $A = \{c_1, \ldots, c_{k-1}, a\}$, where $a \in P_{i-1,i} \cup \{c_k\}$.

We claim that $M_{L_1} = \ldots = M_{L_{k-1}}$. Let $s \in M_{L_1}$. Since $\{\mathbf{u}_j(s)\}_{j=1}^{k}$ is an AP_k and $\mathbf{u}_1(s) = c_1$, we obtain that either $\{\mathbf{u}_j(s)\}_{j=1}^{k}$ is a trivial arithmetic progression with $\mathbf{u}_j(s) = c_1$ for $1 \leqslant j \leqslant k$ or $\{\mathbf{u}_j(s)\} = A = \{c_1, \ldots, c_{k-1}, a\}$. Note that $A = \{c_1, \ldots, c_{k-1}, a\}$ is a non-trivial AP_k and consequently $c_1, \ldots, c_{k-1}, a$ are all distinct integers. Hence, from the fact that $\mathbf{u}_j(s) \in P_{i-1,i} \cup \{c_j\}$ we obtain that $\mathbf{u}_j(s) \neq c_1$ for $2 \leqslant j \leqslant k-1$. This implies that

$$\mathbf{u}_j(s) = \begin{cases} c_j, & \text{if } 1 \leqslant j \leqslant k-1, \\ a, & \text{if } j = k. \end{cases} \tag{4.8}$$

Thus, for $1 \leqslant j \leqslant k-1$, we have that $s \in M_{L_j}$, which yields that $M_{L_1} \subseteq M_{L_j}$ for $2 \leqslant j \leqslant k-1$. By repeating the argument for $s \in M_{L_j}$ for $2 \leqslant j \leqslant k-1$, we obtain that $M_{L_1} = \ldots = M_{L_{k-1}}$.

Since $F_{L_j} = [N] \backslash M_{L_j}$, the last paragraph also implies that $F_{L_1} = \ldots = F_{L_{k-1}} = F$.

For $1 \leqslant j \leqslant k-1$, let $\{b_s^{(j)}\}_{s \in F_{L_j}}$ be the sequence of integers in $P_{i-1,i}$ corresponding to the fixed indices of L_j. Let $s \in F$. By definition,

$$\mathbf{u}_j(s) = b_s^{(j)}$$

for $1 \leqslant j \leqslant k-1$. The set $\{\mathbf{u}_j(s)\}_{j=1}^k$ is an AP_k with at least $k-1$ terms belonging to $P_{i-1,i}$. Hence, it is a trivial AP_k. This implies that

$$\mathbf{u}_j(s) = b_s^{(1)} \tag{4.9}$$

for every $s \in F$ and $1 \leqslant j \leqslant k$. Therefore, by (4.8), (4.9) and the definition of (4.1) we have that

$$\mathbf{u}_j = \begin{cases} \mathbf{v}_{c_j, L_1}, & \text{if } 1 \leqslant j \leqslant k-1, \\ \mathbf{v}_{a, L_1}, & \text{if } j = k \end{cases}$$

and consequently $U = \{\mathbf{u}_1, \ldots, \mathbf{u}_k\} \subseteq V_i(L_1)$.

Case 3: $k = 3$ and $P_{i-1,i} \cup \{c_1, c_2, c_3\}$ has at least two non-trivial arithmetic progressions.

We will prove that there is no such vector set U in this case. We first show that $P_{i-1,i} \cup \{c_1, c_2, c_3\}$ has exactly two non-trivial arithmetic progressions of length 3. By property (b) if $A \subseteq P_{i-1,i} \cup \{c_1, c_2, c_3\}$ is an AP_3, then $\pi(A) \in E(G)$ and consequently it must contain at least two elements of $\{c_1, c_2, c_3\}$.

Suppose that $A = \{c_1, c_2, c_3\}$. By property (c), if $B \subseteq P_{i-1,i} \cup \{c_1, c_2, c_3\}$ is another non-trivial AP_3, then $|A \cap B| \leqslant 1$. This implies that B must contain at least two elements of $P_{i-1,i}$. Hence, $\pi(B) \notin E(G)$, which contradicts property (c) of P_{i-1}. Therefore, $|A \cap \{c_1, c_2, c_3\}| = 2$ and by Property (c), there must be at most three distinct non-trivial arithmetic progression in $P_{i-1,i}$. Suppose by contradiction that we have three non-trivial arithmetic progressions and let $A = \{c_1, c_2, x\}$, $B = \{c_1, c_3, y\}$ and $C = \{c_2, c_3, z\}$ be them, where $x, y, z \in P_{i-1,i}$. In this case $\pi(c_1)$, $\pi(c_2)$, $\pi(c_3)$ and $\pi(x) = \pi(y) = \pi(z) = i$ are all distinct vertices of $[n]$. However, this implies that $|\pi(A) \cap \pi(B)| = 2$, which contradicts property (b), since G is a simple 3-graph.

Next assume without loss of generality that $A = \{c_1, c_2, x\}$ and $B = \{c_1, c_3, y\}$ are the only two non-trivial AP$_3$'s in $P_{i-1,i} \cup \{c_1, c_2, c_3\}$, where $c_1, c_2, c_3 \notin P_{i-1,i}$ and $x, y \in P_{i-1,i}$. Also suppose by contradiction that there exists a set of vectors $U = \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\} \subseteq V_i$ that is an AP$_3$ in every coordinate of $[N]$.

Claim 4.3.2. *$M_{L_2} \cap M_{L_3} = \emptyset$ and $M_{L_1} = M_{L_2} \cup M_{L_3}$*

Proof. Let $s \in M_{L_1}$. Then $\mathbf{u}_1(s) = c_1$. We claim that both $\mathbf{u}_2(s)$ and $\mathbf{u}_3(s)$ are different from c_1. Indeed if $\mathbf{u}_2(s) = c_1 \notin P_{i-1,i}$, then necessarily $s \in M_{L_2}$ and hence $c_1 = c_2 = \mathbf{u}_2(s)$. This however contradicts that $A = \{c_1, c_2, x\}$ is a non-trivial AP_3. Consequently we infer that $\mathbf{u}_2(s) \neq c_1$ and similarly (now using $B = \{c_1, c_3, y\}$) we observe that $\mathbf{u}_3(s) \neq c_1$. Since $\{\mathbf{u}_1(s), \mathbf{u}_2(s), \mathbf{u}_3(s)\}$ is an AP$_3$, $\mathbf{u}_1(s) = c_1$, $\mathbf{u}_2(s) \neq c_1$ and $\mathbf{u}_3(s) \neq c_1$, we obtain that either

$$\mathbf{u}_2(s) = c_2,\ \mathbf{u}_3(s) = x \quad \text{or} \quad \mathbf{u}_2(s) = y,\ \mathbf{u}_3(s) = c_3. \tag{4.10}$$

This implies that either $s \in M_{L_2}$ or $s \in M_{L_3}$ and consequently $M_{L_1} \subseteq M_{L_2} \cup M_{L_3}$.

Now suppose that $s \in M_{L_2}$. By the same argument, $\{\mathbf{u}_1(s), \mathbf{u}_2(s), \mathbf{u}_3(s)\}$ is a non-trivial AP_3 with $\mathbf{u}_2(s) = c_2$. Hence,

$$\mathbf{u}_1(s) = c_1 \text{ and } \mathbf{u}_3(s) = x$$

and therefore $s \in M_{L_1}$ and $s \notin M_{L_3}$. This implies that $M_{L_2} \subseteq M_{L_1}$ and $M_{L_2} \cap M_{L_3} = \emptyset$. Analogously, we have that $M_{L_3} \subseteq M_{L_1}$ and then $M_{L_1} = M_{L_2} \cup M_{L_3}$. □

Claim 4.3.2 gives us a partition of the set of indices $[N] = F_{L_1} \cup M_{L_2} \cup M_{L_3}$

and a neat description of the set of vectors $U = \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ and combinatorial lines $L_1, L_2, L_3 \in \mathcal{L}_3$. Let $\{b_s^{(1)}\}_{s\in F_{L_1}}$ be the sequence of integers in $P_{i-1,i}$ corresponding to the fixed indices F_{L_1} of L_1. Then L_2 has fixed indices $F_{L_2} = F_{L_1} \cup M_{L_3}$ and corresponding sequence $\{b_s^{(2)}\}_{s\in F_{L_2}}$ of integers in $P_{i-1,i}$ given by

$$b_s^{(2)} = \begin{cases} b_s^{(1)}, & \text{if } s \in F_{L_1}, \\ y, & \text{if } s \in M_{L_3}. \end{cases}$$

This is because if for some $s \in F_{L_1}$ the relation $b_s^{(2)} \neq b_s^{(1)}$ holds, then for such s the set $\{\mathbf{u}_1(s), \mathbf{u}_2(s), \mathbf{u}_3(s)\} \subseteq P_{i-1,i}$ would form a non-trivial AP_3 contradicting property (b) of the induction hypothesis. Similarly in view of (4.10) and the fact that the only AP_3 in $P_{i-1} \cup \{c_1, c_2, c_3\}$ containing c_1 and c_3 is $B = \{c_1, c_3, y\}$ we infer that $b_s^{(2)} = y$ for $s \in M_{L_3}$.

Similarly, we conclude that the line L_3 has fixed indices $F_{L_3} = F_{L_1} \cup M_{L_2}$ and corresponding sequence $\{b_s^{(3)}\}_{s\in F_{L_3}}$ given by

$$b_s^{(3)} = \begin{cases} b_s^{(1)}, & \text{if } s \in F_{L_1}, \\ x, & \text{if } s \in M_{L_2}. \end{cases}$$

Moreover, we have that

$$\mathbf{u}_1(s) = \mathbf{v}_{c_1,L_1}(s) = \begin{cases} b_s^{(1)}, & \text{if } s \in F_{L_1}, \\ c_1, & \text{if } s \in M_{L_2} \cup M_{L_3}; \end{cases}$$

$$\mathbf{u}_2(s) = \mathbf{v}_{c_2,L_2}(s) = \begin{cases} b_s^{(1)}, & \text{if } s \in F_{L_1}, \\ c_2, & \text{if } s \in M_{L_2}, \\ y, & \text{if } s \in M_{L_3}; \end{cases}$$

$$\mathbf{u}_3(s) = \mathbf{v}_{c_3,L_3}(s) = \begin{cases} b_s^{(1)}, & \text{if } s \in F_{L_1}, \\ x, & \text{if } s \in M_{L_2}, \\ c_3, & \text{if } s \in M_{L_3}. \end{cases}$$

Now note that $\mathbf{v}_{x,L_1} = \mathbf{v}_{x,L_3}$, $\mathbf{v}_{y,L_1} = \mathbf{v}_{y,L_2}$ and $\mathbf{v}_{x,L_2} = \mathbf{v}_{y,L_3}$. Since $\mathbf{v}_{x,L_1}, \mathbf{v}_{y,L_1} \in L_1$; $\mathbf{v}_{x,L_2}, \mathbf{v}_{y,L_2} \in L_2$ and $\mathbf{v}_{x,L_3}, \mathbf{v}_{y,L_3} \in L_3$, we have that $L_1 \cap L_3 \neq \emptyset$, $L_1 \cap L_2 \neq \emptyset$ and $L_2 \cap L_3 \neq \emptyset$, respectively. Hence, $\{L_1, L_2, L_3\}$ forms a 3-cycle on $\mathcal{L}_3$, which contradicts the fact that $g(\mathcal{L}_3) > 3$. □

4.4 Proof of Theorem 4.2.1

We are now ready to prove that the set of integers $X(k, r, \mu) = P_n$ satisfies statements (i) and (ii) of Theorem 4.2.1. First, we will show that our set satisfies the van der Waerden property.

Proposition 4.4.1. *Any r-coloring of $X(k, r, \mu)$ contains a monochromatic* AP_k.

Proposition 4.4.1 will be established by the following standard backwards induction on the partite construction.

Claim 4.4.2. *Let $P_0, \ldots, P_n$ be the set of integers constructed in Section 4.2 and G be the simple k-graph on n vertices obtained by Theorem 2.2.2 used in the construction.*

Then the following holds for $0 \leqslant i \leqslant n$. Every r-coloring of P_n contains a copy of P_i such that $P_{i,t}$ is monochromatic for $i+1 \leqslant t \leqslant n$.

Proof. We will proceed by backwards induction on $0 \leqslant i \leqslant n$. The statement is vacuously true for $i = n$. Suppose that we proved Claim 4.4.2 for i. Now we want to verify the claim for $i-1$. By the induction hypothesis, any r-coloring of P_n contains a copy of P_i such that $P_{i,t}$ is monochromatic for $i+1 \leqslant t \leqslant n$. Recall that by our construction

$$P_i = \bigcup_{L \in \mathcal{L}_3} P_i(L),$$

where each $P_i(L)$ is a copy of P_{i-1}. Restricting to the i-th musical line, we have that

$$P_{i,i} = \bigcup_{L \in \mathcal{L}_3} P_{i,i}(L),$$

where each $P_{i,i}(L)$ corresponds by a bijective linear map to the set of vectors $V_{i,i}(L) = L$ and therefore $P_{i,i}$ corresponds to $V_{i,i} = \mathcal{L}_3$. By Theorem 4.1.2, for any r-coloring of $\mathcal{L}_3$, there exists a monochromatic line $L \in \mathcal{L}_3$. Hence, for any r-coloring of $P_{i,i}$, there exists a combinatorial line $L \in \mathcal{L}_3$ such that $P_{i,i}(L)$ is monochromatic. Take $P_i(L)$ as our copy of P_{i-1}. By the induction hypothesis we have that $P_{i-1,t}$ is monochromatic for $i+1 \leqslant t \leqslant n$, while $P_{i-1,i}$ is monochromatic since it is equal to $P_{i,i}(L)$. □

Proof of Proposition 4.4.1. By Claim 4.4.2, for any r-coloring of $X(k,r,\mu)$ there exists a copy of P_0 such that $P_{0,t}$ is monochromatic for $1 \leqslant t \leqslant n$. Define the r-coloring $c : [n] \to [r]$ on the vertices of the auxiliary graph G by letting $c(t)$ be the color of the monochromatic set $P_{0,t}$. By Theorem 2.2.2, the k-graph G satisfies $\chi(G) > r$. Hence, there exists a monochromatic edge $e \in E(G)$ with respect to the coloring c. Due to the construction of P_0 (see Figure 4.1), there exists an arithmetic progression $A \subseteq P_0$ such that $\pi(A) = e$. Therefore, A is a monochromatic AP_k with the same color as

e. $\square$

Now we verify that any finite subset of $X(k, r, \mu)$ does not have the Szemerédi property.

Proposition 4.4.3. *For every $Y \subseteq X(k, r, \mu)$, there exists an* AP_k*-free subset of integer $Z \subseteq Y$ of size $|Z| \geqslant \mu|Y|$.*

Proof. Let $Y \subseteq X(k, r, \mu) = P_n$. Consider the partition $Y = \bigcup_{i=1}^{n} Y_i$, where $Y_i = \{y \in Y : \pi(y) = i\} = Y \cap P_{n,i}$. We define the stochastic weight vector $\mathbf{w} : [n] \to [0, 1]$ on the vertices of G by

$$\mathbf{w}(i) = \frac{|Y_i|}{|Y|}.$$

Clearly, the vector $\mathbf{w}$ is stochastic since

$$\sum_{i\in[n]} \mathbf{w}(i) = \sum_{i\in[n]} \frac{|Y_i|}{|Y|} = 1.$$

By Theorem 2.2.2, there exists an independent set $I \subseteq [n]$ in G such that

$$\sum_{i\in I} \mathbf{w}(i) \geqslant \mu.$$

Let $Z = \bigcup_{i\in I} Y_i$. Thus,

$$|Z| = \sum_{i\in I} |Y_i| = |Y| \sum_{i\in I} \mathbf{w}(i) \geqslant \mu|Y|.$$

Moreover, if $A \subseteq Z$ is an AP_k, then by property (b) of Lemma 4.3.1 the projection $\pi(A)$ is an edge of G. However, $\pi(A) \subseteq I$, which contradicts the independence of I. Hence, Z is AP_k-free. $\square$

Chapter 5

Euclidean configurations

The content of this chapter was obtained in joint work with Vojtech Rödl and is based on [42].

5.1 Segments are P-Ramsey

We prove in this section that segments are P-Ramsey. In fact, we will prove a stronger statement. Recall that a weight vector $\mathbf{w} : X \to [0,1]$ is *stochastical* if $\sum_{x \in X} \mathbf{w}(x) = 1$.

Lemma 5.1.1. *Let A be a segment of length a and $\gamma > 0$ be a real number. Then there exists a countable configuration $Y_A \subseteq \mathbb{R}^\infty$ satisfying the following:*

(i) The set of squares of all distances of points in Y_A is

$$\left\{ a^2, \frac{a^2}{1+\gamma+\gamma^2}, \frac{(1+\gamma^2)a^2}{1+\gamma+\gamma^2}, \frac{\gamma^2 a^2}{1+\gamma+\gamma^2} \right\}$$

(ii) $Y_A \to (C)_r$ holds for every $r \geqslant 1$ and finite configuration $C \subseteq Y_A$.

(iii) For every finite subconfiguration $Y' \subseteq Y_A$ and stochastic weight vector $\mathbf{w} : Y' \to [0,1]$, there exists a configuration $Z \subseteq Y'$ with no segments of lenght a such that

$\sum_{z\in Z} \mathbf{w}(z) \geqslant \frac{1}{4}$.

(iv) Y_A *does not contain an equilateral triangle of sides of lenght* a.

Proof. Let $\{e_i\}_{i=1}^{\infty}$ be the standard basis of $\mathbb{R}^{\infty}$. We construct a configuration $Y_A = \{y_e\}_{e\in\mathbb{N}^{(2)}} \subseteq \mathbb{R}^{\infty}$ by associating to each pair $e = \{i, j\} \in \mathbb{N}^{(2)}$, $i < j$, the point

$$y_e = \beta e_i - \beta\gamma e_j,$$

where $\beta = \frac{a}{\sqrt{2(1+\gamma+\gamma^2)}}$. We claim that the configuration Y_A satisfies properties (i), (ii), (iii) and (iv) of Lemma 5.1.1.

Property (i) comes from the fact that given two pairs $e = \{i, j\}$, $e' = \{i', j'\} \in \mathbb{N}^{(2)}$ the square of the distance between y_e and $y_{e'}$ can assume the following values

$$||y_e - y_{e'}||^2 = \begin{cases} 2\beta^2\gamma^2, & \text{if } i = i' \\ 2\beta^2, & \text{if } j = j' \\ 2\beta^2(1+\gamma+\gamma^2), & \text{if } i = j' \text{ or } i' = j \\ 2\beta^2(1+\gamma^2), & \text{if } \{i,j\} \cap \{i',j'\} = \emptyset \end{cases}$$

By plugging $\beta = \frac{a}{\sqrt{2(1+\gamma+\gamma^2)}}$ we obtain the set of distances of the statement. Moreover, another important consequence of the computation is that $||y_e - y_{e'}|| = a$ if and only if $e \sim e'$ in $\mathrm{Sh}(2, \mathbb{N})$.

In order to prove (ii), consider a finite configuration $C \subseteq Y_A$. Naturally C can be written as $C = \{y_e\}_{e\in E}$ for some $E \subseteq \mathbb{N}^{(2)}$. Since E is finite, there exists an integer n such that $E \subseteq [n]^{(2)}$. An r-coloring of Y_A corresponds to an r-coloring of $\mathbb{N}^{(2)}$. By Ramsey theorem (Theorem 1.0.1) there exists a set $W \subseteq \mathbb{N}$ of size n such that $W^{(2)}$ is monochromatic. Hence, this configuration $C' = \{y_e\}_{e\in W^{(2)}}$ is monochromatic. This implies property (i), since C' contains a copy of C.

To check property (iii), let $Y' \subseteq Y_A$ be a finite subconfiguration of Y_A. By our construction, this corresponds to a finite set $X \subseteq V(\mathrm{Sh}(2,\mathbb{N}))$. Let $\mathbf{w}' : X \to [0,1]$ be the stochastic weight vector given by $\mathbf{w}'(x) = \mathbf{w}(y)$, where $y \in Y'$ is the corresponding point to $x \in X$. Claim 2.3.1 applied to the vector $\mathbf{w}'$ gives us an independent set $I \subseteq X$ in $\mathrm{Sh}(2,\mathbb{N})$ such that $\sum_{i\in I} \mathbf{w}'(i) \geqslant \frac{1}{4}$. This corresponds to a subconfiguration $Z \subseteq Y'$ with no segments of length a and such that

$$\sum_{z\in Z} \mathbf{w}(z) \geqslant \frac{1}{4}.$$

Finally, property (iv) follows from the fact that an equilateral triangle of sides of length a corresponds to a triangle in $\mathrm{Sh}(2,\mathbb{N})$ and $\mathrm{Sh}(2,\mathbb{N})$ is triangle free. □

5.2 Robust configurations

One of the main techniques developed in [10] to prove that a configuration is Ramsey is the product theorem

Theorem 5.2.1 ([10], Theorem 20)**.** *Let A and B be finite configurations which are Ramsey and $X, Y \subseteq \mathbb{R}^\infty$ be such that $X \to (A)_r$ and $Y \to (B)_r$ for every $r \geqslant 1$. Then $X \times Y \to (C)_r$ for $C \subseteq A \times B$ for every $r \geqslant 1$.*

Unfortunately, it is not clear if a similar statement holds for P-Ramsey configurations. The goal of this section is to develop a partial version of the product theorem that will enable us to prove Theorems 1.2.3 and 1.2.4.

Definition 5.2.2. We say that a countable configuration Y is *robust* if for every finite configuration C with $C \subseteq Y$ we have that $Y \to (C)_r$ for every $r \geqslant 1$.

Note for instance, that by property (i) of Lemma 5.1.1 we have the following.

Corollary 5.2.3. *Let Y_A be the configuration obtained by Lemma 5.1.1. Then Y_A is a robust configuration.*

The following is our main result in the section. Recall that by $B \subseteq A$ we understand that there exists a copy A' of A such that $B \subseteq A$.

Theorem 5.2.4. *Let B be a brick and Y be a robust configuration. If $F \subseteq B \times Y$ and $F \nsubseteq Y$, then F is P-Ramsey.*

Theorem 5.2.4 is a consequence of the following lemma.

Lemma 5.2.5. *Let Y be a robust configuration, A be a segment and F a finite configuration with $|F| > 1$. Then the following holds:*

(a) If $F \subseteq A \times Y$ and $F \nsubseteq Y$, then F is P-Ramsey.

(b) If $F \nsubseteq A \times Y$, then there exists a robust configuration $\tilde{Y}$ such that $A \times Y \subseteq \tilde{Y}$ and $F \nsubseteq \tilde{Y}$.

Proof. Let a be the length of the segment A, also let D_Y be the set of all distances in Y and let D_F be the set of all distances in F. Consider the field extension $L = \mathbb{Q}(a, D_Y, D_F)$ of $\mathbb{Q}$, where $\mathbb{Q}(a, D_Y, D_F)$ is the minimal field containing a, D_Y, D_F and $\mathbb{Q}$. Since $D_Y \cup D_F \cup \{a\}$ is countable, we have that L is a countable extension of $\mathbb{Q}$ and consequently $L \neq \mathbb{R}$. Let $\gamma \in \mathbb{R}$ be a transcedental number over L, i.e.,

$$\text{there is no polynomial } p \in L[x] \text{ such that } p(\gamma) = 0 \tag{5.1}$$

Let Y_A be the configuration obtained by Lemma 5.1.1 with parameters a and γ. By property (iii) the set of all square distances is given by

$$\left\{a^2, \frac{a^2}{1+\gamma+\gamma^2}, \frac{(1+\gamma^2)a^2}{1+\gamma+\gamma^2}, \frac{\gamma^2 a^2}{1+\gamma+\gamma^2}\right\}$$

Note that while $a^2 \in L$, due to the fact that γ is transcedental, the other three distances are not in L. Indeed, to illustrate, assume for example that $\frac{a^2}{1+\gamma+\gamma^2} \in L$.

Then there exists $b \in L$ such that

$$\frac{a^2}{1+\gamma+\gamma^2} = b.$$

This implies that γ is a root of the polynomial $p \in L[x]$ given by $p(x) = bx^2+bx+b-a^2$, which contradicts the assumption that γ is transcedental over L.

Before we address statements (a) and (b) of Lemma 5.2.5, we will prove the following claim. Let $\pi_A : Y_A \times Y \to Y_A$ and $\pi_Y : Y_A \times Y \to Y$ be the projection maps of $Y_A \times Y$ on Y_A and Y, respectively.

Claim 5.2.6. *Let $F \subseteq Y_A \times Y$. Then either $F \subseteq Y$ or $\pi_A(F)$ is a copy of A.*

Proof. If $\pi_A(F)$ is a single point, then $F \subseteq Y$ and there is nothing to do. Thus, we may assume that $|\pi_A(F)| \geqslant 2$. Let p, q be two points of F such that $p' = \pi_A(p)$ and $q' = \pi_A(q)$ are distinct. We claim that $||p' - q'|| = a$. Let $p'' = \pi_Y(p)$ and $q'' = \pi_Y(q)$. Since all distances from points of F and Y are in L, we have that $||p - q||^2, ||p'' - q''||^2 \in L$. Thus, by Pythagoras theorem we have

$$||p' - q'||^2 = ||p - q||^2 - ||p'' - q''||^2 \in L. \tag{5.2}$$

On the other hand, by Lemma 5.1.1 we have that

$$||p' - q'||^2 \in \left\{a^2, \frac{a^2}{1+\gamma+\gamma^2}, \frac{(1+\gamma^2)a^2}{1+\gamma+\gamma^2}, \frac{\gamma^2 a^2}{1+\gamma+\gamma^2}\right\}.$$

Due to our choice of γ, the value a^2 is the only one of the four values in the field L. Hence, due to (5.2) we have $||p' - q'|| = a$.

Suppose that $|\pi_A(F)| \geqslant 3$. Then by the previous paragraph, there is an equilateral triangle of sides of length a in Y_A, which contradicts property (iv) of Lemma 5.1.1. Therefore, $\pi_A(F)$ is a segment of length a. ☐

Now we prove statement (a) of Lemma 5.2.5. Let F be a finite configuration, $|F| > 1$, such that $F \subseteq A \times Y$ and $F \not\subseteq Y$ for a segment A and a robust configuration Y. By Corollary 5.2.3, the configuration Y_A is robust, where Y_A is defined with parameters a and γ satisfying (5.1). We claim that $Y_A \times Y$ testifies that F is P-Ramsey.

To check property (i) of Definition 1.2.2 we note that since $F \subseteq A \times Y$, then there exists a finite configuration C such that $F \subseteq A \times C$. Because Y is robust, then $Y \rightarrow (C)_r$ for every $r \geqslant 1$. Lemma 5.1.1 gives us that $Y_A \rightarrow (A)_r$ for every $r \geqslant 1$. Thus, by Theorem 5.2.1, we have that $Y_A \times Y \rightarrow (F)_r$ for every $r \geqslant 1$.

In order to prove property (ii) of Definition 1.2.2, let $V \subseteq Y_A \times Y$ be a finite subconfiguration. Since V is finite, there exists a finite subconfiguration $X \subseteq Y_A$ such that $V \subseteq X \times Y$. We partition V into $V = \bigcup_{x \in X} V_x$ where $V_x = \pi_A^{-1}(x)$ are the elements of V that projects to the point x on Y_A. Let $\mathbf{w} : X \rightarrow [0, 1]$ be the stochastic weight vector defined by

$$\mathbf{w}(x) = \frac{|V_x|}{|V|}.$$

By property (ii) of Lemma 5.1.1, there exists a subconfiguration $Z \subseteq X$ with no segments of length a such that

$$\sum_{z \in Z} \mathbf{w}(z) \geqslant \frac{1}{4}. \tag{5.3}$$

Consider the configuration $U = \bigcup_{z \in Z} V_z$. We claim that U does not contain a copy of F. Suppose to the contrary that $F \subseteq U$. Since $F \not\subseteq Y$, then by Claim 5.2.6 the projection $\pi_A(U)$ contains a segment of length a. However, $\pi_A(U) = Z$, which contains no segment of length a, yielding a contradiction. Moreover, by (5.3)

$$|U| = \sum_{z \in Z} |V_z| = \sum_{z \in Z} |V|\mathbf{w}(z) \geqslant \frac{1}{4}|V|,$$

which proves property (ii) of Definition 1.2.2 with $\mu = \frac{1}{4}$. Hence, F is P-Ramsey.

Now we prove statement (b) of Lemma 5.2.5. Suppose that $F \not\subseteq A \times Y$. We claim that $\tilde{Y} = Y_A \times Y$ is a robust configuration such that $F \not\subseteq Y_A \times Y$. We first show that $\tilde{Y}$ is robust. If $C \subseteq \tilde{Y}$ is a finite configuration, then there exist finite configurations $W_A \subseteq Y_A$ and $W \subseteq Y$ such that $C \subseteq W_A \times W$. Since Y_A and Y are robust, we have that $Y_A \to (W_A)_r$ and $Y \to (W)_r$ for every $r \geqslant 1$. By Theorem 5.2.1, we obtain that $\tilde{Y} = Y_A \times Y \to (C)_r$, which proves that $\tilde{Y}$ is robust.

Assume by contradiction that $F \subseteq Y_A \times Y$. Then by Claim 5.2.6, we either have that $F \subseteq Y$ or $\pi_A(F)$ is a copy of A. In both cases, we have that $F \subseteq A \times Y$, which contradicts the hypothesis. □

We are now able to prove Theorem 5.2.4.

Proof of Theorem 5.2.4. Let B be a d-dimensional brick and let Y be a given robust configuration. We will write $B = A_1 \times \ldots \times A_d$ where A_i is a segment. By the hypothesis of Theorem 5.2.4 we are also given F satisfying $F \subseteq B \times Y$ and $F \not\subseteq Y$. Our goal is to prove that F is P-Ramsey. For that we will repeteadly apply Lemma 5.2.5. We will construct a sequence $Y_0, \ldots, Y_\ell$ of robust configurations with the property that $F \not\subseteq Y_i$, for $0 \leqslant i \leqslant \ell$, as follows. Let $Y_0 = Y$. Suppose that we already constructed $Y_0, \ldots, Y_i$. If $F \subseteq A_{i+1} \times Y_i$, then we stop the process and set $\ell = i$. Otherwise, by statement (b) of Lemma 5.2.5, there exists a robust configuration $\tilde{Y}$ such that $A_{i+1} \times Y_i \subseteq \tilde{Y}$ and $F \not\subseteq \tilde{Y}$. Set $Y_{i+1} = \tilde{Y}$. A simple induction shows that for every $1 \leqslant i \leqslant \ell$

$$A_1 \times \ldots \times A_i \times Y \subseteq Y_i.$$

Since $F \subseteq B \times Y = A_1 \times \ldots \times A_d \times Y$, the process terminates before the d-th step of the construction, i.e., $\ell < d$. This implies that $F \subseteq A_{\ell+1} \times Y_\ell$ and $F \nsubseteq Y_\ell$ and by statement (a) of Lemma 5.2.5, we have that F is P-Ramsey. □

A corollary of Theorem 5.2.4 is that bricks are P-Ramsey. In fact, we prove the slighter stronger statement that in particular implies Theorem 1.2.4.

Corollary 5.2.7. *Let B be a brick and $F \subseteq B$ be a subconfiguration with $|F| > 1$. Then F is P-Ramsey.*

Proof. Suppose that B is d-dimensional brick and write $B = A_1 \times \ldots \times A_d$, where A_i is a segment of length a_i and $a_1 \geqslant \ldots \geqslant a_d$. Let $\gamma > 0$ be an arbitrary real number and let Y_{A_d} be the configuration obtained by Lemma 5.1.1 with parameters γ and a_d. Suppose that $F \subseteq Y_{A_d}$. By the minimality of the segment A_d, we have that the minimum distance between two points in F is at least a_d. Moreover, by property (iii) of Lemma 5.1.1, the diameter of Y_{A_d} is exactly a_d. Hence, any two points of F have distance a_d. If $|F| \geqslant 3$, then Y_{A_d} contains an equilateral triangle of sides a_d. This contradicts property (iv) of Lemma 5.1.1. Thus, F is a copy of the segment A_d and in this case F is P-Ramsey by property (ii) and (iii) of Lemma 5.1.1.

Now suppose that $F \nsubseteq Y_{A_d}$. Since $A_d \subseteq Y_{A_d}$, then $F \subseteq A_1 \times \ldots \times A_d \subseteq A_1 \times \ldots \times A_{d-1} \times Y_{A_d}$. Therefore, F satisfies the hypothesis of Theorem 5.2.4 and we obtain that F is P-Ramsey. □

5.3 Simplices are P-Ramsey

In this section we prove Theorem 1.2.3. The proof follows the ideas from [15, 32]. First, we will introduce the terminology and auxiliary results from those papers. The main idea will be to prove that any simplex S can be embedded in a product $B \times Y$, where B is a brick and Y is a robust configuration.

To address the robust configuration consider the following definition. Let $\{e_i\}_{i\geqslant 1}$ be the standard basis of $\mathbb{R}^\infty$. Given an integer k, a vector $c = (c_1, \ldots, c_k) \in \mathbb{R}^k$ and a k-tuple $J = (j_1, \ldots, j_k) \in \mathbb{N}^{(k)}$, we define the point spread$(c, J) \in \mathbb{R}^\infty$ as

$$\text{spread}(c, J) = \sum_{\ell=1}^{k} c_\ell e_{j_\ell}$$

Given a subset of integers $A \subseteq \mathbb{N}$, one can then define the configuration Spread(c, A) as

$$\text{Spread}(c, A) = \{\text{spread}(c, J) : \ J \in A^{(k)}\}.$$

The reason why spread configurations are interesting for us is twofold. One is that those configurations approximate simplices very well. The second is that it fits well in the context of P-Ramseyeness (see Claim 5.3.3 below) The next result was proven in [32]. Given real number $\rho > 0$, we denote by $S_\rho(\mathbb{R}^\infty)$ the sphere of radius ρ in $\mathbb{R}^\infty$. For a linear subspace $Z \subseteq \mathbb{R}^\infty$, let $S_\rho(Z) = S_\rho(\mathbb{R}^\infty) \cap Z$.

Proposition 5.3.1 ([32])**.** *For every $\delta > 0$ and every integer m, there exist an integer n, k, a k-dimensional vector $c = (c_1, \ldots, c_k) \in \mathbb{R}^\infty$ with $||c|| = \rho$ and an m-dimensional subspace $Z \subseteq \mathbb{R}^\infty$ such that the following holds. For every $z \in Z$, there is a point $y \in$ Spread$(c, [n])$ such that $||z - y|| < \delta$.*

Since any d-dimensional simplex can be embedded in any d-dimensional vector space, we obtain the following corollary from Proposition 5.3.1.

Corollary 5.3.2. *For $\delta < \rho/2$ and a d-dimensional simplex $S = \{y_0, \ldots, y_d\}$ of circumradius $\rho(S) = \rho$, there exist integers n, k, a k-dimensional vector $c = (c_1, \ldots, c_k) \in \mathbb{R}^k$ with $||c|| = \rho$ and a d-dimensional simplex $S' = \{z_0, \ldots, z_d\} \subseteq$ Spread$(c, [n])$ such that $||y_i - z_i|| < \delta$ for $0 \leqslant i \leqslant d$.*

The second reason is that Spread configurations are robust.

Claim 5.3.3. Spread$(c, \mathbb{N})$ *is robust.*

Proof. Let $X \subseteq \text{Spread}(c, \mathbb{N})$ be a finite configuration. Then there exist N and $\mathcal{J} = \{J_1, \ldots, J_t\} \subseteq [N]^{(k)}$ such that $X = \{\text{spread}(c, J) : J \in \mathcal{J}\}$. Note that there exists a bijective map φ from Spread$(c, \mathbb{N})$ to $\mathbb{N}^{(k)}$ given by $\varphi(\text{spread}(c, J)) = J$. Thus, for any finite coloring of Spread$(c, \mathbb{N})$ there is a corresponding coloring of $\mathbb{N}^{(k)}$. By Ramsey's theorem, there exists $A \subseteq \mathbb{N}$ of size N such that $A^{(k)}$ is monochromatic. Therefore, the configuration Spread(c, A) is monochromatic. The result follows now since $X \subseteq \text{Spread}(c, A)$. $\square$

Another important result for our proof is the next characterization of configurations of points in an Euclidean space. Let $M = (m_{ij})_{0 \leqslant i,j \leqslant d}$ be a symmetric real matrix with zero entries on the main diagonal. We say that the matrix M is of *negative type* if

$$\sum_{0 \leqslant i < j \leqslant d} m_{ij} \lambda_i \lambda_j \leqslant 0 \tag{5.4}$$

holds for all choices of $\lambda_0, \ldots, \lambda_d$ with $\lambda_0 + \ldots + \lambda_d = 0$ and $\lambda_0^2 + \ldots + \lambda_d^2 = 1$.

Theorem 5.3.4 ([44]). *A finite configuration $X = \{x_0, \ldots, x_d\}$ with distances $d_{ij} = ||x_i - x_j||$ can be embedded in the Euclidean space $\mathbb{R}^d$ if and only if the matrix $M = (m_{ij})_{0 \leqslant i,j \leqslant d}$ given by $m_{ij} = d_{ij}^2$ is of negative type. Moreover, X is a d-dimensional simplex if and only if the inequality in (5.4) is strict for all choices of $\lambda_0, \ldots, \lambda_d$.*

As a consequence of Theorem 5.3.4, we can show that all almost regular simplices are realizable.

Corollary 5.3.5. *Let d be an integer and $\beta, \varepsilon > 0$ be real numbers such that $\varepsilon < \frac{\beta}{64d^2}$. For any symmetric matrix of distances $D = \{d_{ij}\}_{0 \leqslant i,j \leqslant d}$ satisfying*

$$\beta - \varepsilon \leqslant d_{ij} \leqslant \beta + \varepsilon, \tag{5.5}$$

there exists a d-dimensional simplex $S = \{x_0, \ldots, x_d\}$ *such that* $||x_i - x_j|| = d_{ij}$ *for every* $0 \leqslant i < j \leqslant d$.

Proof. Let $M = (m_{ij})_{0 \leqslant i,j \leqslant d}$ be the symmetric matrix with zero entries in the main diagonal given by $m_{ij} = d_{ij}^2$ for $i \neq j$. For real numbers $\lambda_0, \ldots, \lambda_d$ satisfying $\lambda_0 + \ldots + \lambda_d = 0$ and $\lambda_0^2 + \ldots + \lambda_d^2 = 1$ we have that

$$0 = \left(\sum_{i=0}^{d} \lambda_i\right)^2 = 1 + 2 \sum_{0 \leqslant i < j \leqslant d} \lambda_i \lambda_j.$$

Hence,

$$\sum_{0 \leqslant i < j \leqslant d} \lambda_i \lambda_j = -\frac{1}{2}. \tag{5.6}$$

Thus, by (5.5) and (5.6) we have

$$\begin{aligned} \left|\left| \sum_{0 \leqslant i < j \leqslant d} m_{ij} \lambda_i \lambda_j + \frac{\beta^2}{2} \right|\right| = \left|\left| \sum_{0 \leqslant i < j \leqslant d} (m_{ij} - \beta^2) \lambda_i \lambda_j \right|\right| &\leqslant \sum_{0 \leqslant i < j \leqslant d} ||m_{ij} - \beta^2|| \\ &\leqslant (d+1)^2 (2\varepsilon\beta + \varepsilon^2) < \frac{\beta^2}{4}. \end{aligned}$$

This implies that $\sum_{0 \leqslant i < j \leqslant d} m_{ij} \lambda_i \lambda_j < -\frac{\beta^2}{4}$ and M is strictly of negative type. Therefore, by Theorem 5.3.4 there exists a simplex $S = \{x_0, \ldots, x_d\}$ with $||x_i - x_j|| = d_{ij}$ for $0 \leqslant i < j \leqslant d$. □

Finally, the last auxiliary result shows that any almost regular simplex can be embedded in a brick.

Theorem 5.3.6 ([15, 32])**.** *For every* $\beta, d > 0$*, there exists a real number* $\eta := \eta(\beta, d)$ *such that the following holds. For any simplex* $S = \{w_0, \ldots, w_d\}$ *satisfying*

$$\beta - \eta \leqslant ||w_i - w_j|| \leqslant \beta + \eta$$

for $0 \leqslant i < j \leqslant d$*, there exists a* $\binom{d+1}{2}$*-dimensional brick* B *with* $S \subseteq B$*.*

We are now ready to prove Theorem 1.2.3.

Proof of Theorem 1.2.3. To prove that a simplex is P-Ramsey we will apply again Theorem 5.2.4, now combined with ideas from [15, 32]. We find it convenient to divide the proof in the next four steps.

Step 1: For a simplex $S = \{x_0, \ldots, x_d\}$ with circumradius $\rho(S) = \rho$, we will find a simplex $S_1 = \{y_0, \ldots, y_d\}$ a small positive real number β such that

$$||y_i - y_j||^2 = ||x_i - x_j||^2 - \beta$$

for all $0 \leqslant i \neq j \leqslant d$. This implies that the circumradius $\rho(S_1) = \rho' < \rho$.

Let $M = (m_{ij})_{0 \leqslant i,j \leqslant d}$ be the matrix given by $m_{ij} = ||x_i - x_j||^2$. Since S is a simplex, by Theorem 5.3.4 there exists $\gamma > 0$ such that

$$\sum_{0 \leqslant i < j \leqslant d} m_{ij} \lambda_i \lambda_j \leqslant -\gamma$$

for all choices of $\lambda_0, \ldots, \lambda_d$ with $\lambda_0 + \ldots + \lambda_d = 0$ and $\lambda_0^2 + \ldots + \lambda_d^2 = 1$. Set $\beta = \frac{\gamma}{8d^2}$ and let $M' = (m'_{ij})_{0 \leqslant i,j \leqslant d}$ be the matrix defined by $m'_{ij} = m_{ij} - \beta$ for $i \neq j$ and zero entries in the main diagonal. Since $\beta(d+1)^2 \leqslant 4\beta d^2 < \gamma/2$, then

$$\sum_{0 \leqslant i < j \leqslant d} m'_{ij} \lambda_i \lambda_j \leqslant -\beta \sum_{0 \leqslant i < j \leqslant d} \lambda_i \lambda_j - \gamma \leqslant \beta(d+1)^2 - \gamma < -\frac{\gamma}{2} < 0.$$

Consequently, M' is strictly negative, which implies that there exists a simplex $S_1 = \{y_0, \ldots, y_d\}$ such that

$$||y_i - y_j||^2 = m'_{ij} = ||x_i - x_j||^2 - \beta \tag{5.7}$$

for $0 \leqslant i < j \leqslant d$.

Step 2: For $\delta \ll \beta$, we find a k-dimensional vector $c = (c_1, \ldots, c_k) \in \mathbb{R}^k$ and $S_2 = \{z_0, \ldots, z_d\} \subseteq \text{Spread}(c, [n])$ with $||z_i - y_i|| < \delta$ for $0 \leqslant i \leqslant d$. Moreover,

$$||z_i - z_j||^2 = ||x_i - x_j||^2 - \beta \pm \varepsilon$$

where $\varepsilon := \varepsilon(\beta, d) \to 0$ as $\delta \to 0$.

Let $\eta := \eta(\beta, d) > 0$ be the positive real number given by Theorem 5.3.6, let $\varepsilon < \min\{\beta/64d^2, \eta\}$ and let $\delta := \delta(\eta, \rho)$ be sufficiently small. By Corollary 5.3.2, there exist integers n, k, a k-dimensional vector $c = (c_1, \ldots, c_k) \in \mathbb{R}^k$ with $||c|| = \rho'$ and a simplex $S_2 = \{z_0, \ldots, z_d\} \subseteq \text{Spread}(c, [n])$ such that $||y_i - z_i|| < \delta$ for $0 \leqslant i \leqslant d$. Thus, the triangle inequality gives us that

$$||y_i - y_j|| - 2\delta < ||z_i - z_j|| < ||y_i - y_j|| + 2\delta. \tag{5.8}$$

Hence, by combining (5.7) and (5.8)

$$||x_i - x_j||^2 - \beta + 4\delta^2 - 4\delta||y_i - y_j|| < ||z_i - z_j||^2 < ||x_i - x_j||^2 - \beta + 4\delta^2 + 4\delta||y_i - y_j||.$$

Since $||y_i - y_j|| < 2\rho'$ and $4\delta^2 + 8\delta\rho' < \varepsilon$ for sufficiently small δ, we have that

$$||x_i - x_j||^2 - \beta - \varepsilon < ||z_i - z_j||^2 < ||x_i - x_j||^2 - \beta + \varepsilon.$$

Step 3: We find an "almost" regular simplex $S_3 = \{w_0, \ldots, w_d\}$ satisfying

$$||w_i - w_j||^2 = ||x_i - x_j||^2 - ||z_i - z_j||^2 = \beta \pm \varepsilon$$

for all $0 \leqslant i \neq j \leqslant d$. Furthermore, there exists a brick B such that $S_3 \subseteq B$.

This is an easy consequence of our preliminary results. By our choice of ε, Corollary 5.3.5 guarantees that there exists a simplex $S_3 = \{w_0, \ldots, w_d\}$ such that

$$||w_i - w_j||^2 = ||x_i - x_j||^2 - ||z_i - z_j||^2.$$

Moreover, by Theorem 5.3.6, there exists a $\binom{d+1}{2}$-dimensional brick B such that $W \subseteq B$.

Step 4: We construct a simplex $F \cong S$ such that $F \subseteq B \times \text{Spread}(c, [n])$ and $F \nsubseteq \text{Spread}(c, [n])$ and apply Theorem 5.2.4.

Let $F = \{f_0, \ldots, f_d\}$ be the simplex defined by

$$f_i = w_i * z_i,$$

where the symbol $*$ stands for the usual concatenation, i.e., if $a = (a_1, \ldots, a_r)$ and $b = (b_1, \ldots, b_s)$, then $a * b = (a_1, \ldots, a_r, b_1, \ldots, b_s)$. Hence,

$$||f_i - f_j||^2 = ||w_i - w_j||^2 + ||z_i - z_j||^2 = ||x_i - x_j||^2$$

for every $0 \leqslant i, j \leqslant d$. Thus, the configuration F is a copy of S. Furthermore, by the construction of F we have that

$$F \subseteq W \times Z \subseteq B \times \text{Spread}(c, \mathbb{N}),$$

where B is a $\binom{d+1}{2}$-dimensional brick and $\text{Spread}(c, \mathbb{N})$ is a robust configuration (Claim 5.3.3). Since $\rho(\text{Spread}(c, \mathbb{N})) = \rho' < \rho = \rho(F)$, we obtain that $F \nsubseteq \text{Spread}(c, \mathbb{N})$ and by Theorem 5.2.4 the configuration $F \cong S$ is P-Ramsey. $\square$

Chapter 6

Concluding remarks

6.1 Pisier type problems for linear system of equations

Note that an arithmetic progression of length k can be written as a system of homogeneous linear equations

$$x_i - 2x_{i+1} + x_{i+2} = 0 \tag{6.1}$$

for $1 \leqslant i \leqslant k-2$. A solution $\mathbf{x} = \{x_i\}_{i=1}^k$ to this system in $\mathbb{N}$ is an AP_k. In this case, the van der Waerden theorem can be seen as the Ramsey statement that any r-coloring of $\mathbb{N}$ contains a solution to the linear system given in (6.1). Similarly, Szemerédi theorem is the density statement that any subset $X \subseteq \mathbb{N}$ with positive density contains a solution to the system in (6.1). Such concepts can be extended to any system of linear equations on the integers.

Given a matrix $A \in \mathbb{Z}^{m\times n}$ with integer coefficients, the system of homogeneous linear equations $A\mathbf{x} = 0$ is called *partition regular* if for any finite coloring of $\mathbb{N}$, there exists a monochromatic solution $\mathbf{x} = \{x_i\}_{i=1}^n$ to the system. Examples of partition regular systems include the system $x_1 + x_2 = x_3$ (Schur's theorem) and arithmetic progressions (van der Waerden's theorem). A full characterization of the systems A that are partition regular was proven by Rado [38, 7].

A similar concept was introduced in [17]. A linear system $A\mathbf{x} = 0$ is *density regular* if any subset $X \subseteq \mathbb{N}$ of positive density contains a non-trivial solution of the system. One can observe that density regular systems are partition regular. However, the opposite is not true, For instance, the equation $x_1 + x_2 = x_3$ is partition regular, but the odd numbers do not contain any solution of it.

It would be interesting to study for which systems of linear equations there exists a version of Theorem 1.1.4.

Question 6.1.1. *Given a system of linear equations $A\mathbf{x} = 0$ with $A \in \mathbb{Z}^{m\times n}$ are there integer set $X \subseteq \mathbb{N}$ and real number $\varepsilon > 0$ such that*

(i) Any finite coloring of X contains a monochromatic solution of $A\mathbf{x} = 0$,

(ii) For every finite $Y \subseteq X$, there exists a set $Z \subseteq Y$ with $|Z| \geqslant \varepsilon|Y|$ such that Z does not contain any non-trivial solution to $A\mathbf{x} = 0$?

We conjecture that such statements should be true for both partition regular and density regular systems.

6.2 Euclidean considerations

The list of known Ramsey configurations is quite limited. Apart from simplices and bricks, the most significant result is due to Křížž [28] who proved that regular polygons are Ramsey. Unfortunately, our method of robust configurations does not apply here. This leaves us with the following question:

Question 6.2.1. *Are regular polygons P-Ramsey?*

Differently from Theorem 1.2.1, the proof in [28] does not provide a density result for regular polygons. Another interesting question would be to determine if such a result exists.

Question 6.2.2. *Let F be a regular polygon. For every $\mu > 0$, is there a configuration $Y := Y(F, \mu)$ such that any set $Z \subseteq Y$ of size $|Z| \geqslant \mu|Y|$ contains a copy of F?*

Lastly, another direction of research would be to obtain sharp bounds for the real number μ in the P-Ramsey definition. It is not hard to show that for a configuration X with k points we cannot take $\mu > \frac{k-1}{k}$. However, our proofs of Theorem 1.2.3 and 1.2.4 only give $\mu = \frac{1}{4}$. It would be interesting to close the gap for simplices.

Question 6.2.3. *Let S be a d-dimensional simplex. What is the largest value of $\mu > 0$ such that there exists a configuration Y satisfying properties (i) and (ii) of Definition 1.2.2?*

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