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An abstract of
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Abstract

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The classical Christoffel's problem gives rise naturally to an elliptic partial differential equation $\Delta h + nh = \Phi$ on the n-dimensional unit sphere S^n, where under certain conditions h may represent the support function of a non-degenerate convex body in $\mathbb{R}^{n+1}$ and Φ/n the mean radius of curvature prescribed on S^n. We construct a closed form of the generalized Green's function for the differential operator $\Delta + n$ on the hypersphere by reducing the original equation to an ordinary differential equation and choosing the undetermined constants in the correct way. We compare the results with existing expressions for the generalized Green's function in the literature and investigate an incorrect claim about the choice of constants.

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Contents

1 Introduction

1.1 Christoffel's Problem

Let $F \subset \mathbb{R}^{n+1}$ be an orientable C^2 hypersurface embedded in $(n+1)$-dimensional Euclidean space. Since F is orientable, it is possible to choose a continuous normal field on F and calculate the mean radius of curvature $\Phi\colon F \to \mathbb{R}$ as a continuous function on F. A natural question is whether or not the above process is somehow reversible. When does a function describe the mean radius of curvature of some hypersurface? This is the question at the heart of Christoffel's problem.

Posed in 1865 by E. B. Christoffel, the classical Christoffel's problem is usually phrased in the language of convex bodies. Christoffel sought necessary and sufficient conditions on a function $\Phi\colon S^n \to \mathbb{R}$ such that $\Phi(u)$ is the mean radius of curvature of a convex surface at the point where u is the outer unit normal to the surface. Firey solved the classical Christoffel's problem in 1967 and the so-called generalized Christoffel's problem in 1968 [Fir67, Fir68]. The problem was also solved independently by Berg in 1969 [Ber69].

Firey's answer to Christoffel's problem involves solving an elliptic partial differential equation using the generalized Green's function and asking conditions on the solution such that it describes the support function of a convex body. Unfortunately, the conditions given by Firey and Berg would be difficult to verify in practice, in part because the representations for the Green's function are not easy to manipulate analytically. It would be beneficial to have a closed form of the generalized Green's function, and construction of such an expression is the primary goal of this paper.

1.2 Content of this Paper

In Section (2), we review basic facts about convex bodies in $\mathbb{R}^{n+1}$ and the Laplace operator on the n-dimensional unit sphere S^n. In Section (3), we construct the generalized Green's function for the differential operator $\mathcal{L} = \Delta + n$ on S^n by determining constants for the solution of the ordinary differential equation (12) in the correct way. In the process, we attempt to make transparent how our expressions satisfy the properties of the Green's function. Section (4) compares the expressions derived here to the expression for the generalized Green's function given in [Szm07] and discusses a claim cited in [Oli11] relating to choice of constants for the Green's function. In Section (5), we summarize the results and present possible directions for further study. Finally, Section (6) contains justification for the equations used in the paper.

2 Preliminaries

2.1 Convex Bodies

We consider Christoffel's problem in $(n+1)$-dimensional Euclidean space with the usual inner product, denoted $\langle \cdot, \cdot \rangle$. Let S^n be the n-dimensional unit sphere canonically embedded in $\mathbb{R}^{n+1}$ and σ_n denote the surface area of S^n, with explicit formula

$$\sigma_n = \frac{2\pi^{(n+1)/2}}{\Gamma((n+1)/2)}. \tag{1}$$

Remark. The following facts about convex bodies can be recovered from [Fir67] and [Sch93].

2.1.1 Definitions

Definition (Convex body)**.** A convex body $K \subset \mathbb{R}^{n+1}$ is a compact, convex subset of $(n+1)$-dimensional Euclidean space.

For our purposes, we always assume that the convex body has nonempty interior. A convex surface is defined to be the boundary of a convex body, and in the nondegenerate case, a convex surface is topologically equivalent to S^n.

In order to characterize the boundary of a convex body K, we define the notion of a support plane to K.

Definition (Support plane)**.** The support plane $H_u \subset \mathbb{R}^{n+1}$ to K with outer normal $u \in S^n$ is the n-dimensional hyperplane such that K is entirely contained on one side of H_u, K and H have nonempty intersection, and u points into the side of H_u which does not contain K.

The positions of supporting planes to the convex body K can be encoded into a function known as the support function.

Definition (Support function)**.** Let $K \subset \mathbb{R}^{n+1}$ be a convex body. The support function

$h_K\colon \mathbb{R}^{n+1} \to \mathbb{R}$ of K is defined as

$$h_K(u) = \sup_{x \in K} \langle x, u \rangle.$$

When $u \in S^n$, the support function describes the signed distance from the origin to the support plane H_u of K. Since K is compact and h_K is a continuous function, the sup above is actually a max. This represents that for each $u \in S^n$ the support plane H_u with outer normal u has nonempty intersection with K.

2.1.2 Properties of the Support Function

For all $u, v \in \mathbb{R}^{n+1}$, the support function h_K satisfies the following properties:

1. $h_K(\lambda u) = \lambda h_K(u)$ for all $\lambda \geq 0$, i.e. positive homogeneity of degree 1 and $h_K(0) = 0$

2. $h_K(u + v) \leq h_K(u) + h_K(v)$, i.e. subadditivity

These two properties require that the support function is convex. If the subadditivity is strict, then the support function is strictly convex. Furthermore, any function $h\colon \mathbb{R}^{n+1} \to \mathbb{R}$ which satisfies the above properties is the support function of some (possibly degenerate) convex body.

Given a support function h_K, we can recover its corresponding convex body K as

$$K = \{x \in \mathbb{R}^{n+1} \mid \langle x, u \rangle \leq h_K(u) \text{ for all } u \in S^n\}.$$

Geometrically, the above process is that of taking the intersection over all $u \in S^n$ of the "inward" sides of support planes H_u with outer normal u, where the inward side is the side into which u does not point. It follows that every convex body K is uniquely determined by its support function h_K.

2.1.3 Relation to Christoffel's Problem

Lastly, if K is a non-degenerate convex body in $\mathbb{R}^{n+1}$ with support function $h_K \in C^3(\mathbb{R}^{n+1})$ and strictly convex, then the Gauss map $N\colon \partial K \to S^n$ is invertible. For $u \in S^n$, the support function h_K satisfies

$$\nabla h_K(u) = N^{-1}(u)$$

where ∇ denotes the usual gradient in $\mathbb{R}^{n+1}$ [Fir68]. Furthermore, the restriction of h_K to S^n satisfies

$$\Delta_{S^n} h_K + n h_K = \Phi. \tag{2}$$

Here, Δ_{S^n} denotes the Laplacian on S^n and $\Phi(u)/n$ is the mean radius of curvature of ∂K at the point $N^{-1}(u)$.

Remark. If a function $f\colon \mathbb{R}^{n+1} \to \mathbb{R}$ is positive homogeneous of degree one, then

$$\Delta_{\mathbb{R}^{n+1}} f(u) = \Delta_{S^n} f(u) + n f(u)$$

when $u \in S^n$ (see p. 7 of [Fir67]). Firey solves the partial differential equation using the expression on the left, while we address the expression on the right.

Equation (2) is the foundation for solving the classical Christoffel's problem. Before solving the equation, we must recall some facts about the Laplacian on the hypersphere.

2.2 The Laplacian on the Unit Sphere

We examine $C^2(S^n)$ as a normed vector subspace of $L^2(S^n)$ endowed with the usual inner product

$$\langle f, g\rangle_{L^2} = \int_{S^n} fg \, d\sigma.$$

Recall that C^2 is not complete with respect to the L^2 norm.

Remark. The following facts about the Laplacian and hyperspherical harmonics can be recovered from [Szm07] and many other reliable sources.

2.2.1 Eigenvalues and Eigenfunctions

Consider the Laplace operator $\Delta\colon C^2(S^n) \to C(S^n)$ on the hypersphere. It is known that the operator Δ has a point spectrum consisting of the eigenvalues

$$\lambda_l = -l(l+n-l) \text{ for } l \geq 0.$$

Let d_l denote the dimension of the corresponding eigenspace, with

$$d_l = \frac{(2l+n-1)(l+n-2)!}{l!(n-1)!} \text{ for } l \geq 1$$

and $d_0 = 1$. Let $\{Y_{l,m}\}$ denote the set of normalized eigenfunctions of the Laplacian, where the first index l represents the corresponding eigenvalue $-l(l+n-1)$ and the second index is to distinguish between the d_l linearly independent eigenfunctions in each eigenspace. The functions $Y_{l,m}$ are the hyperspherical harmonics, satisfying the equation

$$\Delta Y_{l,m} = -l(l+n-1)Y_{l,m} \text{ for } 0 \leq l, 1 \leq m \leq d_l.$$

It is well known that the set $\{Y_{l,m}\}$ of hyperspherical harmonics comprises an orthonormal basis for $L_2(S^n)$.

The second eigenvalue of the Laplacian is $\lambda_1 = -n$ and has geometric multiplicity $d_1 = n+1$. It can be shown that the corresponding eigenspace consists of the functions $h_A = \langle u, A\rangle$ for any $A \in \mathbb{R}^{n+1}$, i.e. Euclidean inner products of $u \in S^n$ with the point A [Oli11]. From the perspective of convex bodies, these are support function of points.

2.2.2 The Shifted Laplacian

Define the shifted Laplacian $\mathcal{L}\colon C^2(S^n) \to C(S^n)$ as

$$\mathcal{L} = \Delta + n.$$

It is easy to see that the hyperspherical harmonics $Y_{l,m}$ are also the eigenfunctions of the operator $\mathcal{L}$, while the corresponding eigenvalues are shifted to

$$\lambda_n' = n - l(l + n - 1).$$

Since $\lambda_1 = -n$ was an eigenvalue of the original Laplacian, the $l = 1$ eigenvalue of $\mathcal{L}$ is a null eigenvalue. Its null space spanned by the functions $\{Y_{1,m}\}$.

2.2.3 Invertibility on a Subspace

There is hope for inverting the operator $\mathcal{L}$ provided we mod out the null space. Consider $\mathcal{L}$ restricted to $C^2(S^n) \setminus \mathrm{null}(\mathcal{L})$. Let $\Phi \in \mathrm{Range}(\mathcal{L})$ be L_2-orthogonal to the null space of the operator $\mathcal{L}$. Then the modified problem $\mathcal{L}h = \Phi$ restricted from the null space has a unique solution $h \in C^2(S^n) \setminus \mathrm{null}(\mathcal{L})$, and the general solution $\bar{h} \in C^2(S^n)$ to (2) can be recovered as

$$\bar{h} = h + \sum_{m=1}^{n+1} a_m Y_{1,m} = h + h_A,$$

where h_A is the support function of an arbitrary point $A \in \mathbb{R}^{n+1}$. Geometrically, this means that if the solution h is the support function of a convex body K, then K is unique up to translation [Fir67].

We can recover the solution by constructing the generalized Green's function for $\mathcal{L}$, which is the topic of the next section.

3 The Generalized Green's Function for a Shifted Laplacian

3.1 Basic Facts

Remark. In this section, we employ many of the definitions and identities presented in [Szm07]. In particular, this definition of the generalized Green's function is from [Szm07].

3.1.1 Definition

The generalized Green's function $G\colon S^n \times S^n \to (-\infty, \infty]$ for the operator $\mathcal{L}$ is a solution of the partial differential equation

$$\mathcal{L}_x G(x,x') = \delta(x,x') - \sum_{m=1}^{n+1} Y_{1,m}(x) Y_{1,m}(x') \tag{3}$$

such that $G(x,x')$ is also L^2-orthogonal to the null space of the shifted Laplacian $\mathcal{L}$, i.e.

$$\int_{S^n} G(x,x') Y_{1,m} = 0 \text{ for } m = 1, ..., n+1. \tag{4}$$

Here, $\delta(x,x')$ denotes the Dirac delta distribution centered at x' on the unit sphere.

By properties of spherical harmonics [Szm07], we have the identity

$$\sum_{m=1}^{n+1} Y_{1,m}(x) Y_{1,m}(x') = \frac{n+1}{(n-1)\sigma_n} C_1^{((n-1)/2)} \langle x, x' \rangle = \frac{n+1}{\sigma_n} \langle x, x' \rangle, \tag{5}$$

where $C_\lambda^{(\alpha)}(z)$ is the Gegenbauer function. Substituting (5) into (3), we conclude that $\mathcal{L}_x G(x,x')$ is given by

$$\Delta_x G(x,x') - nG(x,x') = \delta(x,x') - \frac{n+1}{\sigma_n} \langle x, x' \rangle. \tag{6}$$

3.1.2 Heuristics

Before delving into the details, we provide a non-rigorous but perhaps illuminating explanation for the action of the generalized Green's function. Suppose $h \in C^2(S^n) \setminus \mathrm{null}(\mathcal{L})$ and $\mathcal{L}h = \Phi$. For every $x' \in S^n$, the Green's function recovers $h(x')$ as follows:

$$\begin{aligned}\int_{S^n} G(x,x')\Phi(x)d_x\sigma &= \int_{S^n} G(x,x')\mathcal{L}h(x)d_x\sigma = \int_{S^n} \mathcal{L}G(x,x')h(x)d_x\sigma \\ &= \int_{S^n} \left(\delta(x,x') - \frac{n+1}{\sigma_n}\langle x,x'\rangle\right) h(x)d_x\sigma \\ &= h(x').\end{aligned} \tag{7}$$

The second equality is due to Green's second identity on S^n, and the final equality is because the function $h(x)$ was assumed to be L^2-orthogonal to the Euclidean inner product $\langle x, x'\rangle$. In reality, the integrals above are improper and should be treated with care.

3.1.3 Symmetry

A natural choice for the Green's function on the hypersphere is to write $G(x, x')$ as a function of $\langle x, x'\rangle = \cos\theta$, where θ is the angle between the unit vectors x and x'. Let

$$G(x,x') = (g \circ \gamma)(x,x'), \tag{8}$$

where $g(\theta)\colon [0,\pi] \to (-\infty,\infty]$ is a function of the angle between x and x' and $\gamma(x,x') = \arccos(\langle x,x\rangle)$. Under these assumptions, the generalized Green's function is symmetric in the two variables x and x', as desired.

3.1.4 Singularity

From (6), we observe that the generalized Green's function is finite-valued for $x \neq x'$ and singular whenever $x = x'$, as per the definition of Dirac delta. The order of the singularity is determined by the dimension n and must be chosen such that the value $h(x')$ is recovered exactly.

Remark. We say that the function $g(\theta)$ is singular of order n at $\theta = 0$ if $g(\theta) \sim C/\theta^n$ as $\theta \to 0^+$ from above. We say that $g(\theta)$ has logarithmic singularity if $g(\theta) \sim C \log\theta$ as $\theta \to 0^+$.

We now clarify the singularity of $G(x, x')$ and provide a rigorous argument for the assertion made in (7).

Fix $x' \in S^n$ and choose for convenience a coordinate system in which x' is the North pole. Let θ denote the angle descending from the North pole. For arbitrary $\epsilon \in (0, \pi)$, let

$$\Omega_\epsilon = \{x \in S^n \mid \arccos(\langle x, x' \rangle) \geq \epsilon\}$$

denote the subset of S^n with angle greater than ϵ radians from the North pole. Intuitively, Ω_ϵ is the set S^n with a small hole containing x' on top. The boundary $\partial\Omega_\epsilon$ of Ω_ϵ is an $(n-1)$-dimensional hypersphere with radius $\sin^{n-1}\epsilon$, surface area $(\sin^{n-1}\epsilon)\sigma_{n-1}$, and unit normal

$$\nu(x) = \frac{e_1 - x\cos\epsilon}{\sin\epsilon}.$$

As noted in [Oli11], the derivative $\partial/\partial\nu = -\partial/\partial\theta$ because of the choice of coordinate system.

Remark. The above construction can also be found in [Fir67].

As before, suppose $\mathcal{L}h = \Phi$ is L^2-orthogonal to the null space of $\mathcal{L}$. Because $G(x, x')$ is singular at $x = x'$, the integral $\int_{S^n} G(x, x')\Phi \, d_x\sigma$ is really an improper integral; the approach is to integrate over Ω_ϵ and let ϵ approach zero from above.

We can apply the second Green's identity to obtain

$$\begin{aligned} \int_{\Omega_\epsilon} G(x, x')[\Delta_x h(x) + nh(x)] d_x\sigma - \int_{\Omega_\epsilon} [\Delta_x G(x, x') + nG(x, x')] h(x) d_x\sigma \\ = \int_{\partial\Omega_\epsilon} \left[G(x, x') \frac{\partial h(x)}{\partial\nu_x} - h(x) \frac{\partial G(x, x')}{\partial\nu_x} \right] d_x\sigma_\epsilon. \end{aligned} \tag{9}$$

Already, the second term of (9) converges to zero because on this domain $\mathcal{L}_x G(x, x') = -(n+1)\langle x, x' \rangle / \sigma_n$ and $h(x)$ is orthogonal to $\langle x, x' \rangle$.

The singularity of G at $x = x'$ must be such that the third term of (9) goes to zero as well and the final term recovers $h(x')$. Indeed, the third term approaches zero as ϵ becomes small provided that $g(\theta)$ has singularity of order strictly less than $n-1$, since

$$\int_{\partial\Omega_\epsilon} G(x,x')\frac{\partial h(x)}{\partial \nu_x} d_x\sigma_\epsilon \le \sigma_{n-1} \max_{x\in\partial\Omega_\epsilon} |\nabla h(x)| g(\epsilon) \sin^{n-1}\epsilon.$$

The final term in (9) converges to $Ch(x')$ for some constant C provided that $g(\theta)$ has singularity of logarithmic order for $n = 2$ and order $n-2$ for $n \ge 3$. Write

$$-\int_{\partial\Omega_\epsilon} h(x)\frac{\partial G(x,x')}{\partial \nu_x} d_x\sigma_\epsilon = \sigma_{n-1} M_h(\epsilon) \frac{dg(\theta)}{d\theta}\Big|_{\theta=\epsilon} \sin^{n-1}\epsilon \tag{10}$$

where $M_h(\epsilon)$ is the average of $h(x)$ taken over the boundary of Ω_ϵ, i.e.

$$M_h(\epsilon) = \frac{\int_{\partial\Omega_\epsilon} h(x) d_x\sigma_\epsilon}{\sigma_{n-1}\sin^{n-1}\epsilon}. \tag{11}$$

Suppose $n = 2$. If $g(\theta)$ has logarithmic singularity, then its derivative has singularity of order 1. Now suppose $n \ge 3$. If $g(\theta)$ is singular of order $n-2$, then its derivative has singularity of order $n-1$. Under these conditions, the limit of (10) as $\epsilon \to 0+$ exists and is given by the product of limits.

$$\sigma_{n-1} \lim_{\epsilon\to 0^+} \frac{dg(\theta)}{d\theta}\Big|_{\theta=\epsilon} M_h(\epsilon)\sin^{n-1}\epsilon = Ch(x').$$

Ideally, we want the Green's function such that $C = 1$ and $h(x')$ is recovered exactly.

3.2 Reduction to an Ordinary Differential Equation

Remark. The omitted details of this section can be found in [Oli11].

At last, it is time to construct the generalized Green's function. Under the symmetry conditions we proposed in (8), i.e. $G(x,x') = (g\circ\gamma)(x,x')$, the partial differential equation

(6) reduces nicely to an ordinary differential equation

$$g''(\theta) + (n-1)\cot\theta g'(\theta) + ng(\theta) = -\frac{n+1}{\sigma_n}\cos\theta \tag{12}$$

on the interval $\theta \in (0, \pi)$.

We do not solve the equation (12) here. It is sufficient to report the general solution deduced from the arguments in [Oli11]:

$$\begin{aligned} g_n(\theta) &= \cos\theta(\hat{g_n} + C_2) \\ \hat{g_n}(\theta) &= C_1 Q_{n-1} - (I_{n-1} - \log|\cos(\theta)|)/\sigma_n. \end{aligned} \tag{13}$$

The functions above are defined as

$$I_p = \int \frac{S_p d\theta}{\cos^2\theta \sin^p\theta} \tag{14}$$

$$S_p = \int \sin^p\theta d\theta \tag{15}$$

$$Q_p = \int \frac{d\theta}{\cos^2\theta \sin^p\theta}. \tag{16}$$

Also, $\cos\theta$ and $Q_{n-1}(\theta)$ are the solutions of the homogeneous equation on the interval $(0, \pi)$.

Observe that the $\log|\cos(\theta)|$ term in (13) is singular at $\theta = \pi/2$ and must somehow be eliminated. Moreover, the function Q_{n-1} is singular at $\theta = 0, \pi/2$, and π and must be scaled appropriately to the middle term I_{n-1} as to ensure boundedness at $\theta = \pi/2$ and π. The constant C_1 is responsible for this scaling.

In contrast, the constant C_2 does not affect the smoothness $g(\theta)$ and is only responsible for the orthogonality condition (32). In fact, the function $G(x, x')$ will recover the solution $h(x')$ regardless of the choice of C_2.

Remark. For later use, define the functions

$$M_p = \int \frac{d\theta}{\sin^p \theta} \tag{17}$$

$$L_q = \int \frac{d\theta}{\cos\theta \sin^q \theta} \tag{18}$$

. Also, define the double factorial

$$m!! = \begin{cases} (m)(m-2)\cdots 3\cdot 1 & m \text{ odd}, m > 0 \\ (m)(m-2)\cdots 2 & m \text{ even}, m > 0\ \cdot \\ 1 & \text{else} \end{cases} \tag{19}$$

We remark that this paper follows the convention that if the upper index if a sum is less than the lower index, then the sum is taken to be 0.

In the next sections, we expand each of the integrals in (13) and show how to choose the constants C_1 and C_2 to meet these requirements.

3.3 The Case of n Even

Let $n > 0$ be an even integer. Recall from (13) that we want expressions for Q_{n-1}, S_{n-1}, and I_{n-1}.

3.3.1 Closed Form for Q_{n-1}

Integrating by parts, we find that

$$Q_p = \frac{1}{\cos\theta \sin^{p+1}\theta} + (p+1)M_{p+2},\ p > 0. \tag{20}$$

We have a recursive formula for M_q given by

$$M_p = -\frac{\cos\theta}{(p-1)\sin^{p-1}\theta} + \frac{p-2}{p-1}M_{p-2},\ p > 1 \tag{21}$$

Using the recurrence (21), the function M_p can be written

$$M_p = -\sum_{k=1}^{(p-1)/2} \frac{(p-2)!!}{(p-1)!!}\frac{(2k-2)!!}{(2k-1)!!}\frac{\cos\theta}{\sin^{2k}\theta} + \frac{(p-2)!!}{(p-1)!!}\log\tan(\theta/2),\ p \text{ odd} \tag{22}$$

Substitute (22) back into (20). Since $\log\tan(\theta/2) = \log(1-\cos\theta) - \log(\sin\theta)$, let us write Q_{n-1} as

$$\begin{aligned} Q_{n-1} =& \frac{1}{\cos\theta\sin^{n-2}\theta} - \sum_{k=1}^{n/2-1}\frac{(n-1)!!}{(n-2)!!}\frac{(2k-2)!!}{(2k-1)!!}\frac{\cos\theta}{\sin^{2k}\theta} \\ &+ \frac{(n-1)!!}{(n-2)!!}(\log(1-\cos\theta) - \log(\sin\theta)) \end{aligned} \tag{23}$$

It is undesirable that the term $\log(\sin\theta)$ is singular at $\theta = \pi$, so eventually the constant C_1 will be chosen such that $\log(\sin\theta)$ cancels out with a logarithmic term in I_{n-1}.

3.3.2 Closed Form for S_{n-l}

Next, we look for an expression for the function S_{n-1}. Using the recurrence relation

$$S_p = -\frac{\sin^{p-1}\theta\cos\theta}{p} + \frac{p-1}{p}S_{p-2},\ p > 0, \tag{24}$$

it is possible to show that

$$S_p = -\sum_{k=0}^{(p-1)/2}\frac{(p-1)!!}{p!!}\frac{(2k-1)!!}{(2k)!!}\cos\theta\sin^{2k}\theta,\ p \text{ odd}. \tag{25}$$

3.3.3 Closed Form for I_{n-1}

Recall from (14) that

$$I_p = \int \frac{S_p d\theta}{\cos^2\theta \sin^p\theta}.$$

Substituting (25) into the above equation, the integral I_p only requires computing the integral for each term, i.e. L_q for odd q. Integration by parts gives the recursion formula

$$L_q = -\frac{1}{(q-1)\sin^{q-1}\theta} + L_{q-2}, q > 1. \tag{26}$$

By applying the recursion (26) over and over, we arrive at an expression for L_q, namely

$$L_q = \log|\tan\theta| - \sum_{k=1}^{(q-1)/2} \frac{1}{2k\sin^{2k}\theta}, q \text{ odd}. \tag{27}$$

Substitute the expression (27) back into (14) to obtain

$$I_{n-1} = -\sum_{k=0}^{n/2-1} \frac{(n-2)!!}{(n-1)!!}\frac{(2k-1)!!}{(2k)!!} L_{n-2k-1}. \tag{28}$$

3.3.4 Choice of C_1

Now, it is time to cancel the logarithmic terms. It is possible to show by induction that $\log|\tan\theta|$ terms hidden in (27) and (28) reduce nicely.

$$-\sum_{k=0}^{n/2-1} \frac{(n-2)!!}{(n-1)!!}\frac{(2k-1)!!}{(2k)!!} \log|\tan\theta| = \log|\cos\theta| - \log\sin\theta. \tag{29}$$

Thus, we have cancellation of the $\log|\cos\theta|$ terms in (13) for free!

Next, we choose C_1 such that the $-\log\sin\theta$ term in $C_1 Q_{n-1}$ (23) cancels the $\log\sin\theta$ term in $-I_{n-1}/\sigma_n$ (28). Inspecting the coefficient of $-\log\sin\theta$ in (23), one finds that the constant C_1 should be

$$C_1 = \frac{1}{\sigma_n}\frac{(n-2)!!}{(n-1)!!}. \tag{30}$$

3.3.5 Closed Form for $\hat{g_n}(\theta)$

For n even, the function $\hat{g_n}(\theta)$ in (13) can be represented by

$$\begin{aligned}\hat{g_n}(\theta) =& \frac{1}{\sigma_n}\frac{(n-2)!!}{(n-1)!!}\frac{1}{\cos\theta\sin^n\theta} - \frac{1}{\sigma_n}\sum_{k=1}^{n/2}\frac{(2k-2)!!}{(2k-1)!!}\frac{\cos\theta}{\sin^{2k}\theta} \\ &+ \frac{1}{\sigma_n}\log(1-\cos\theta) - \frac{1}{\sigma_n}\sum_{k=0}^{n/2-1}\frac{(n-2)!!}{(n-1)!!}\frac{(2k-1)!!}{(2k)!!}\sum_{l=1}^{n/2-k-1}\frac{1}{2l\sin^{2l}\theta}.\end{aligned} \tag{31}$$

3.3.6 Choice of C_2

We choose C_2 to fulfill the orthogonality condition (32).

As before, choose coordinates such that x' is the North pole. Suppose $\langle A, x\rangle$ is a function in the null space of the operator $\mathcal{L}$. Let $(x_1, ..., x_{n+1})$ and $(A_1, ..., A_{n+1})$ represent the components of x and A in the standard basis for Euclidean space. The following steps are equivalent.

$$\begin{aligned}\int_{S^n} G(x,x')\langle A, x\rangle d_x\sigma &= \int_0^\pi [\cos\theta(\hat{g_n}(\theta) + C_2)] \int_{\partial\Omega_\theta} \langle A, x\rangle d_x\sigma_\epsilon \, d\theta \\ &= \int_0^\pi [\cos\theta(\hat{g_n}(\theta) + C_2)] \int_{\partial\Omega_\theta} \sum_{k=1}^{n+1} A_k x_k \, d_x\sigma_\epsilon \, d\theta \\ &= A_{n+1}\sigma_{n-1}\int_0^\pi [\cos\theta(\hat{g_n} + C_2)]\cos\theta\sin^{n-1}\theta d\theta\end{aligned} \tag{32}$$

Therefore, we must choose C_2 such that the numerator in (32) is zero, i.e.

$$C_2 = -\frac{\int_0^\pi \hat{g_n}(\theta)\cos^2\theta\sin^{n-1}\theta d\theta}{\int_0^\pi \cos^2\theta\sin^{n-1}\theta d\theta}. \tag{33}$$

We do not compute this integral explicitly here; this is a direction for possible future work.

Remark. Recall that the choice of C_2 does not affect the smoothness of $g_n(\theta)$ nor the order of the singularity at $\theta = 0$. Since we ask the forcing term Φ in (9) to be L^2-orthogonal to the null space of $\mathcal{L}$, the function $g_n(\theta)$ will recover the solution of $\mathcal{L}h = \Phi$ regardless of C_2.

3.3.7 The Generalized Green's Function for $n = 2$

For the case $n = 2$, the explicit computation (33) is simple enough. Since the surface area of the 2-sphere is 4π, we have that

$$\hat{g}_2 = \frac{1}{4\pi}\left[\frac{1}{\cos\theta} + \log(1-\cos\theta)\right]. \tag{34}$$

Evaluating the definite integrals in (33), it can be shown that $C_2 = (4/3 - \log 2)/4\pi$.

The generalized Green's function in two dimensions is

$$g_2(\theta) = \frac{1}{4\pi}\left[1 + \cos\theta\left(\log\frac{1-\cos\theta}{2} + \frac{4}{3}\right)\right]. \tag{35}$$

As expected, the singularity is logarithmic. In addition, the equation agrees with the expression given in [Szm06].

3.4 The Case of n Odd

Suppose $n > 1$ is an odd integer. As in the previous section, we derive closed form expressions for Q_{n-1}, S_{n-1}, and I_{n-1} and show how to choose the constants C_1 and C_2.

3.4.1 Closed Form for Q_{n-1}

From the formula recursive formula (21) for M_q,

$$M_p = -\sum_{k=1}^{p/2} \frac{(p-2)!!}{(p-1)!!}\frac{(2k-3)!!}{(2k-2)!!}\frac{\cos\theta}{\sin^{2k-1}\theta},\ p \text{ even}. \tag{36}$$

After substituting (36) into the formula (20) for Q_{n-1}, it follows that

$$Q_{n-1} = \frac{1}{\cos\theta \sin^{n-2}\theta} - \sum_{k=1}^{(n-1)/2} \frac{(n-1)!!}{(n-2)!!} \frac{(2k-3)!!}{(2k-2)!!} \frac{\cos\theta}{\sin^{2k-1}\theta}. \tag{37}$$

This formula for Q_{n-1} differs from the formula (23) for even n in that there are no logarithmic terms to cancel.

3.4.2 Closed Form for S_{n-1}

Using the recurrence (24) for S_p, we obtain the formula

$$S_p = -\sum_{k=1}^{p/2-1} \frac{(p-1)!!}{p!!} \frac{(2k)!!}{(2k+1)!!} \cos\theta \sin^{2k+1}\theta + \frac{(p-1)!!}{p!!}(\theta - \sin\theta\cos\theta),\ p \text{ even}. \tag{38}$$

3.4.3 Closed Form for I_{n-1}

Because θ is present in the last term of (38), computing I_p may be more difficult in this case. As before, it is necessary to know the equations for L_q with q odd given by (18) and (27). More importantly, we must know the integral of $\theta/(\cos^2\theta \sin^p\theta)$, given by

$$\int \frac{\theta d\theta}{\cos^2\theta \sin^p\theta} = \frac{\theta}{\cos\theta \sin^{p-1}\theta} - L_{p-1} + p \int \frac{\theta d\theta}{\sin^p\theta}. \tag{39}$$

The last term in (39) can be integrated as

$$\int \frac{\theta d\theta}{\sin^p\theta} = \theta M_p - \int M_p d\theta. \tag{40}$$

We can compute an expression for $\int M_p d\theta$ by integrating (36) termwise and adjusting the indices:

$$\int M_p d\theta = \sum_{k=1}^{p/2-1} \frac{(p-2)!!}{(p-1)!!} \frac{(2k-1)!!}{(2k)!!} \frac{1}{2k\sin^{2k}\theta} - \frac{(p-2)!!}{(p-1)!!} \log|\sin\theta|,\ p \text{ even}. \tag{41}$$

Combining the various expressions from before, (28) becomes

$$I_p = -\sum_{k=1}^{p/2-1} \frac{(p-1)!!}{p!!} \frac{(2k)!!}{(2k+1)!!} L_{p-2k-1} + \frac{(p-1)!!}{p!!} \left[\frac{\theta}{\cos\theta \sin^{p-1}\theta} - 2L_{p-1} + p\left(\theta M_p - \int M_p d\theta\right)\right], \ p \text{ even}. \tag{42}$$

3.4.4 Cancellation of log Terms

For smoothness when $\theta \neq 0$, we expect that the $\log|\cos\theta|$ terms in $\hat{g_n}(\theta)$ cancel as in the previous section, and similarly for the $\log\sin\theta$ terms. Indeed, the log terms taken from the L_q above reduce nicely:

$$-\sum_{k=1}^{p/2-1} \frac{(p-1)!!}{p!!} \frac{(2k)!!}{(2k+1)!!} \log|\tan\theta| - \frac{2(p-1)!!}{p!!} \log|\tan\theta| = -\log|\tan\theta|. \tag{43}$$

The $\log|\sin\theta|$ term in (42) coming from $\int M_p d\theta$ also reduces nicely once coefficients are multiplied, resulting in the desired cancellations.

3.4.5 Choice of C_1

The constant C_1 has not yet been determined. We want to choose C_1 such that the limit in (10)

$$\sigma_{n-1} \lim_{\theta\to 0} \frac{dg_n(\theta)}{d\theta} \sin^{n-1}\theta \to 1, \tag{44}$$

as $\epsilon \to 0^+$. Because we want to differentiate, it is useful to consider the original expression (13). The derivative with respect to θ is given by

$$\frac{dg_n(\theta)}{d\theta} = \frac{C_1}{\cos\theta\sin^{n-1}\theta} - C_1 \sin\theta Q_{n-1} - \frac{1}{\sigma_n}\frac{S_{n-1}}{\cos\theta\sin^{n-1}\theta} + \frac{\sin\theta}{\sigma_n} I_{n-1} - \frac{1}{\sigma_n}\sin\theta(1+\log|\cos\theta|) + C_2(\cos\theta - \sin\theta). \tag{45}$$

Multiplying (45) by $\sigma_{n-1}\sin^{n-1}\theta$ causes every term to vanish under the limit except for the first term, converging to $C_1\sigma_{n-1}$. Taking

$$C_1 = 1/\sigma_{n-1}, \tag{46}$$

we conclude that the limit (44) converges to 1, as desired.

3.4.6 Closed Form for $\hat{g}_n(\theta)$

Combining all the previous expression, it is possible to write down an expression for $\hat{g_n}(\theta)$:

$$\begin{aligned}\hat{g_n}(\theta) = &\left(\frac{1}{\sigma_{n-1}} - \frac{\theta}{\sigma_n}\frac{(n-2)!!}{(n-1)!!}\right)\frac{1}{\cos\theta\sin^{n-2}\theta} \\ &+\left(\frac{\theta}{\sigma_n} - \frac{1}{\sigma_{n-1}}\frac{(n-1)!!}{(n-2)!!}\right)\sum_{k=1}^{(n-1)/2}\frac{(2k-3)!!}{(2k-2)!!}\frac{\cos\theta}{\sin^{2k-1}\theta} \\ &-\frac{1}{\sigma_n}\frac{(n-2)!!}{(n-1)!!}\left[\sum_{k=1}^{(n-1)/2-1}\frac{(2k)!!}{(2k+1)!!}\sum_{l=1}^{(n-1)/2-k-1}\frac{1}{2l\sin^{2l}\theta}\right. \\ &\left.+2\sum_{k=1}^{(n-1)/2-1}\frac{1}{2k\sin^{2k}\theta}\right]+\frac{1}{\sigma_n}\sum_{k=1}^{(n-1)/2-1}\frac{(2k-1)!!}{(2k-2)!!}\frac{1}{2k\sin^{2k}\theta}\end{aligned} \tag{47}$$

Remark. The constant C_2 is determined by the equation (33), as in the previous section.

3.4.7 The Generalized Green's Function for $n = 3$

An explicit formula for $g_3(\theta)$ is relatively easy to compute. Substituting $n = 3$ into the expression (47), we find that

$$\hat{g}_3(\theta) = \frac{(1-2\cos^2\theta)(\pi-\theta)}{4\pi^2\cos\theta\sin\theta}. \tag{48}$$

With the help of a computer algebra system, we determine the constant $C_2 = 1/8\pi^2$.

The generalized Green's function in three dimensions is

$$g_3(\theta) = \frac{(1 - 2\cos^2\theta)(\pi - \theta)}{4\pi^2 \sin\theta} + \frac{\cos\theta}{8\pi^2}. \tag{49}$$

Note that the singularity is of order one, as desired. The expression (49) agrees with the expression given in [Szm07].

4 Comparison of Closed Form Expressions

4.1 Reduction From Spectral Expansion [Szm07]

In [Szm07], Szmytkowski constructs a closed form of the (generalized) Green's function for a more general operator

$$\mathcal{H}_{S^n}^{(\lambda)} = \Delta + \lambda(\lambda + n - 1)$$

on the hypersphere, with λ any complex number. The operator $\mathcal{H}_{S^n}^{(\lambda)}$ is known as the Hemholtz operator on the hypersphere, for which our operator $\mathcal{L} = \Delta + n$ is the special case $\lambda = 1$. However, his expressions are in terms of higher functions. It would be informative to reduce and compare them with the expressions discovered in this paper.

It is known that the generalized Green's function for the Hemholtz operator with a null eigenvalue can be expanded in the basis of hyperspherical harmonics. Szmytkowski was able to manipulate the spectral expansion to construct a closed form for the generalized Green's function in terms of Gegenbauer polynomials $C_k^{(\alpha)}(z)$.

4.1.1 Closed Form for n Even

Suppose $n = 2m$ for $m \geq 1$. As recovered from [Szm07], the generalized Green's function for the operator $\mathcal{L}$ is

$$\begin{aligned} G(x, x') = &\frac{1}{\sigma_n}\left[\langle x, x'\rangle \left(\log \frac{1 - \langle x, x'\rangle}{2} + \sum_{l=m+1}^{n} \frac{2}{l} + \frac{1}{n+1}\right) + \frac{2}{n}\right] \\ &- \frac{(m-1)!}{(n-1)!!\sigma_n} \sum_{k=1}^{m-1} \frac{(n-2k-3)!!}{k(m-k-1)!} \frac{C_{k+1}^{(m-k-1/2)}(\langle x, x'\rangle)}{(1 - \langle x, x'\rangle)^k}. \end{aligned} \tag{50}$$

The main advantage of the expression (50) derived in Szmytkowski's paper is that the regularity of the Green's function is made transparent, since the terms with singularity are functions of $1 - \langle x, x'\rangle$. The singularity is of correct order because $(1 - \cos\theta) \sim (\sin^2\theta)/2$ as $x \to x'$. For $n = 2$, the summation drops out and the logarithmic singularity is recovered.

Because (50) must also satisfy the ordinary differential equation (12) and there is a unique way to choose the constants C_1 and C_2, some algebraic manipulations should transform (50) into the equation for $g_n(\theta)$ recovered in this paper for the case of even dimension.

4.1.2 Closed Form for n Odd

Now suppose $n = 2m+1$ for $m \geq 1$. The generalized Green's function for $\mathcal{L}$ is given by

$$\begin{aligned} G(x,x') =& \frac{1}{(m+1)\sigma_n} \sum_{k=0}^{m} \frac{(-1)^k}{2^k k!} C_k^{(m-k+1)}(\langle x,x'\rangle) X_k(-\langle x,x'\rangle) \\ &- \frac{\langle x,x'\rangle}{2(m+1)\sigma_n}, \end{aligned} \tag{51}$$

where the singularity is hiding in the expression

$$X_k(x) = \frac{d^k}{dx^k}\left(\sqrt{1-x^2}\arccos x\right) \text{ for } x \in \mathbb{R}. \tag{52}$$

In this case, the order of the singularity in (51) is somewhat obscured, whereas it is more explicit in the formula (47) derived here. This is an advantage of the expression derived in this paper over the general expression found in [Szm07]. Again, because $G(x,x')$ must satisfy the equation (12), some algebraic manipulations should show that the expression (51) is equal to the expression for $g_n(\theta)$ found in this paper.

4.2 Construction in [Oli11]

In [Oli11], the differential equation (12) is replaced by a variant, namely

$$g''(\theta) + (n-1)\cot\theta g'(\theta) + ng(\theta) = a\cos\theta. \tag{53}$$

for some constant a undetermined. The general solution to the differential equation is given by

$$
\begin{aligned}
g_{n,a}(\theta) &= \cos\theta(\hat{g_n} + C_2) \\
\hat{g_{n,a}}(\theta) &= C_1 Q_{n-1} + \frac{a}{n+1}\left(I_{n-1} - \log|\cos(\theta)|\right).
\end{aligned} \tag{54}
$$

The expression (13) is recovered for $a = -(n+1)/\sigma_n$.

4.2.1 A Note About Uniqueness

Suppose $g_{n,a}$ satisfies the differential equation equation (53) on $(0, \pi)$ for a and $g_{n,a'}$ satisfies the equation for a'. Define

$$
D(\theta) = a' g_{n,a}(\theta) - a g_{n,a'}(\theta). \tag{55}
$$

It is clear that $D(\theta)$ satisfies the homogeneous equation

$$
\mathcal{L}D(\theta) = 0
$$

on the interval $(0, \pi)$, which implies that

$$
D(\theta) = K_1 \cos\theta + K_2 Q_{n-1}(\theta)
$$

for some constants $K_1, K_2 \in \mathbb{R}$. Moreover, if both $g_{n,a}$ and $g_{n,a'}$ are continuous on $(0, \pi]$, then K_2 must be zero and $g_{n,a}$ must be a multiple of $g_{n,a'}$, up to a $\cos\theta$ term.

4.2.2 Choice of the Constant a

There is a claim reproduced in [Oli11], originating in another paper, that one should first set $C_1 = 1$ and then let

$$a = \begin{cases} -3 & n = 2 \\ -\frac{2^{m+2}}{\pi} & n = 2m+1 \text{ for } m \geq 1 \\ -\frac{1}{3(m/2)-2}\frac{(n+1)!!}{(m-1)!!} & n = 2m, m > 1 \text{ even} \\ -\frac{2}{3(m-1)}\frac{(n+1)!!}{(m-2)!!} & n = 2m, m > 1 \text{ odd.} \end{cases} \qquad (56)$$

The claim is that $g_{n,a}(\theta)$ with this choice of constants recovers the solution to the partial differential equation (2) such that the solution is scaled by some constant C which depends only on the dimension. We want to analyze and eventually rebut the above claim: The choice of a given above is not correct except in the case $n = 2$.

4.2.3 The Case of n Even

The claim should be easy to check for even $n > 2$ because the ratio $R = a/C_1$ must be such that the $\log|\sin\theta|$ terms in Q_{n-1} and I_{n-1} cancel. The desired ratio R is determined by C_1 chosen in (30) and $a = -(n+1)/\sigma_n$:

$$R = -\frac{(n+1)!!}{(n-2)!!}. \qquad (57)$$

Let us compare the correct ratio R and the ratio a/C_1 given in the claim. Recall that the claim assumes that $C_1 = 0$. We compare the left and right hand expressions of

$$(n-2)!! \text{ and } (3(m/2)-2)(m-1)!! \qquad m \text{ even}, \qquad (58)$$

$$(n-2)!! \text{ and } (3m-1)m!!/2 \qquad m > 1 \text{ odd}. \qquad (59)$$

It is clear that neither is an equality in general, e.g. take $m = 2$ and $m = 3$, respectively. In particular, the left and right expressions are never equal for the given values of m.

Lastly, for the case $n = 2$, the ratio $R = -3$ by substituting into (57).

4.2.4 The Case of n Odd

Suppose $n = 2m+1$ and $g_{n,a}$ is finite valued at $\theta = \pi$. We can examine the ratio $R = a/C_1$, as before. For our choice of $C_1 = 1/\sigma_{n-1}$ in (46) and $a = -(n+1)/\sigma_n$, we have

$$R = -\frac{(n+1)}{\pi}\frac{4^m m!m!}{(n-1)!} \neq -\frac{2^{m+2}}{\pi}. \tag{60}$$

We conclude that the choice of a reproduced in [Oli11] is not correct except in the case $n = 2$.

5 Conclusion

In this paper, we introduced a linear partial differential equation arising naturally in the geometric context of Christoffel's problem. With some perseverance, we constructed the generalized Green's function for the corresponding differential operator $\Delta + n$ on the n-dimension unit sphere by solving an ordinary differential equation and choosing the constants in the correct way. Expanding the integrals and justifying the choice of C_1 was the main original contribution of this paper. Following the construction, we compared the expressions developed in this paper to the existing expressions in [Szm07]. Moreover, we examined a curious expression cited in [Oli11] as to what the choice of constants should be for the generalized Green's function, and the expression was shown to be incorrect in general. Investigating this last point was the main motivation for this paper, and it serves as a reminder for the author that the mathematical literature is imperfect.

5.0.5 Further Work

There are some obvious directions for further investigation into the generalized Green's function derived in this paper. The most obvious is that we did not compute a satisfying expression for the constant C_2 chosen in (33). Knowing explicitly the constant C_2 is not vital for solving the partial differential equation, but it does restore uniqueness to the function $g_n(\theta)$ and allows expressions for $g_n(\theta)$ to be compared more easily.

Another possible direction is to manipulate the expressions (50) and (51) given in [Szm07] to collapse with the respective expressions derived in this paper, since the expressions are known to be equal a priori, barring mistakes. In the case of n even, the equation (50) should reduce with some computational persistence. For n odd, reducing the expression (51) could be more difficult because of its dependence on the function $X_k(x)$ defined in (52). Perhaps such manipulations would be best approached with a computer algebra system on hand.

Lastly, it could be interesting from a numerical standpoint to use the generalized Green's function to compute solutions to the equation (2) and visualize them for low dimensions.

The Green's function would have to be numerically optimized to maximize accuracy and minimize the number of operations, especially for large n. The issue of integrating $G(x, x')$ near the singularity would be another hurdle. How would a computational method based on the Green's function fair against standard finite element methods for solving partial differential equations? Perhaps these considerations could generate a whole research project by themselves.

5.0.6 Acknowledgments

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6 Appendix

6.1 Formulas From [Szm06] and [Szm07]

6.1.1 The Case $n = 2$

From Equation (4.1) in [Szm06], the generalized Green's function of the Hemholtz operator for $\lambda = 1$ in two dimensions is

$$\begin{aligned} G(x, x') &= \frac{1}{4\pi} P_1(\langle x, x' \rangle) \left[\log \frac{1 - \langle x, x' \rangle}{2} + \frac{4}{3} \right] + \frac{1}{4\pi} P_0(\langle x, x' \rangle) \\ &= \frac{1}{4\pi} \left[1 + \cos\theta \left(\log \frac{1 - \cos\theta}{2} + \frac{4}{3} \right) \right], \end{aligned}$$

where $P_k(z)$ is the kth Legendre polynomial, such that $P_0(z) = 1$ and $P_1(z) = z$. The above expression is exactly the expression we derived for $g_2(\theta)$ in (35).

6.1.2 The Case $n = 3$

From Equation (4.23) in [Szm07], the generalized Green's function of the Hemholtz operator for $\lambda = 1$ in three dimensions is

$$\begin{aligned} G(x, x') &= -\frac{1}{4\pi^2} T_2(\langle x, x' \rangle) \frac{\arccos(-\langle x, x' \rangle)}{\sqrt{1 - \langle x, x' \rangle^2}} + \frac{1}{16\pi^2} U_1(\langle x, x' \rangle) \\ &= \frac{(1 - 2\langle x, x' \rangle^2)}{4\pi^2} \frac{\arccos(-\langle x, x' \rangle)}{\sqrt{1 - \langle x, x' \rangle^2}} + \frac{\langle x, x' \rangle}{8\pi^2} \\ &= \frac{(1 - 2\cos^2\theta)(\pi - \theta)}{4\pi^2 \sin\theta} + \frac{\cos\theta}{8\pi^2}, \end{aligned}$$

where $T_2(z) = 2z^2 - 1$ and $U_1(z) = 2z$ are Chebyshev polynomials of the first and second kinds, respectively. Also, note that the functions $\arccos(-\cos\theta)$ and $(\pi - \theta)$ agree on the interval $[0, 1]$, so we use the latter.

The expression above is exactly the expression we derived for $g_3(\theta)$ in (49).

6.1.3 The General Case n Even

For $\lambda = 1$ and $n = 2m$ even, the expression (4.41) in [Szm07] becomes

$$\begin{aligned}G(x,x') =& \frac{1}{(n-1)\sigma_n} C_1^{(m-1/2)}(\langle x,x'\rangle) \log \frac{1-\langle x,x'\rangle}{2} \\ & - \frac{(m-1)!}{(n-1)!!\sigma_n} \sum_{k=1}^{m-1} \frac{(n-2k-3)!!}{k(m-k-1)!} \frac{C_{1+k}^{(m-k-1/2)}(\langle x,x'\rangle)}{(1-\langle x,x'\rangle)^k} \\ & + \frac{2}{n\sigma_n} C_0^{(m-1/2)}(\langle x,x'\rangle) \\ & + \frac{1}{(n-1)\sigma_n}[\Psi(n+2)+\Psi(n+1) \\ & - 2\Psi(m+1)] C_1^{(m-1/2)}(\langle x,x'\rangle).\end{aligned} \tag{61}$$

where C_k^α are the Gegenbauer functions, which are polynomials for $k \in \mathbb{N}$, as occurs here. The function $\Psi(z)$ is the digamma function

$$\Psi(z) = \frac{1}{\Gamma(z)} \frac{d\Gamma(z)}{dz}$$

satisfying

$$\Psi(k+1) = -\gamma + \sum_{l=1}^{k} \frac{1}{l}$$

where γ is the Euler-Mascheroni constant.

We can substitute $C_0^{(\alpha)} = 1$ and $C_1^{(\alpha)} = 2\alpha x$ into (61) and evaluate

$$\Psi(n+2) + \Psi(n+1) - 2\Psi(m+1) = \sum_{l=m+1}^{n} \frac{2}{l} + \frac{1}{n+1},$$

to arrive the expression

$$
\begin{aligned}
G(x,x') =& \frac{\langle x,x'\rangle}{\sigma_n}\log\frac{1-\langle x,x'\rangle}{2} \\
&-\frac{(m-1)!}{(n-1)!!\sigma_n}\sum_{k=1}^{m-1}\frac{(n-2k-3)!!}{k(m-k-1)!}\frac{C_{1+k}^{(m-k-1/2)}(\langle x,x'\rangle)}{(1-\langle x,x'\rangle)^k} \\
&+\frac{2}{n\sigma_n}+\frac{1}{\sigma_n}\left(\sum_{l=m+1}^{n}\frac{2}{l}+\frac{1}{n+1}\right)\langle x,x'\rangle,
\end{aligned} \tag{62}
$$

which rearranges to become

$$
\begin{aligned}
G(x,x') =& \frac{1}{\sigma_n}\left[\langle x,x'\rangle\left(\log\frac{1-\langle x,x'\rangle}{2}+\sum_{l=m+1}^{n}\frac{2}{l}+\frac{1}{n+1}\right)+\frac{2}{n}\right] \\
&-\frac{(m-1)!}{(n-1)!!\sigma_n}\sum_{k=1}^{m-1}\frac{(n-2k-3)!!}{k(m-k-1)!}\frac{1}{(1-\langle x,x'\rangle)^k}C_{k+1}^{(m-k-1/2)}(\langle x,x'\rangle).
\end{aligned}
$$

Remark. To keep the notation consistent in this paper, we have substituted $2n+2 \to n$ and $n+1 \to m$ in the original expression (4.41) in [Szm07].

6.1.4 The General Case n Odd

For $\lambda = 1$ and $n = 2m+1$ odd, the expression (4.34) in [Szm07] becomes

$$
\begin{aligned}
G(x,x') =& \frac{1}{(m+1)\sigma_n}\sum_{k=0}^{m}\frac{(-1)^k}{2^k k!}C_k^{(m-k+1)}(\langle x,x'\rangle)X_k(-\langle x,x'\rangle) \\
&-\frac{1}{4m(m+1)\sigma_n}C_1^{(m)}(\langle x,x'\rangle).
\end{aligned}
$$

Substituting $C_1^{(m)}(z) = 2mz$ into the expression above, we obtain the expression (51) in the paper.

6.2 Proof of Minor Claims

6.2.1 Equation (20)

$$u = \frac{1}{\sin^{p+1}\theta}, v' = \frac{\sin\theta}{\cos^2\theta}, u' = -\frac{(p+1)\cos\theta}{\sin^{p+2}\theta}, v = \frac{1}{\cos\theta}$$

$$\int \frac{d\theta}{\cos^2\theta \sin^p\theta} = \frac{1}{\cos\theta\sin^{p+1}\theta} + \int \frac{(p+1)d\theta}{\sin^{p+2}d\theta}$$

6.2.2 Equation (21)

$$\begin{aligned}
\frac{d}{d\theta}\left(\frac{\cos\theta}{\sin^{p-1}\theta}\right) &= -\frac{\sin\theta}{\sin^{p-1}\theta} - (p-1)\frac{\cos^2\theta}{\sin^p\theta} \\
&= -\frac{1}{\sin^{p-2}\theta} - (p-1)\frac{1}{\sin^p\theta} + (p-1)\frac{1}{\sin^{p-2}\theta} \\
&= (p-2)\frac{1}{\sin^{p-2}\theta} - (p-1)\frac{1}{\sin^p\theta} \\
\int \frac{1}{\sin^p\theta}d\theta &= -\frac{\cos\theta}{(p-1)\sin^{p-1}\theta} + \frac{p-2}{p-1}\int\frac{1}{\sin^{p-2}\theta}d\theta
\end{aligned}$$

6.2.3 Equation (22)

Let $p = 1$. Integrating directly, we find that $M_1 = \log\tan(\theta/2)$, so the equation (22) holds. Next, let $p \geq 3$ odd and suppose (22) holds for $p-2$. Applying the recursion (21), we have

$$\begin{aligned}
M_p &= -\frac{\cos\theta}{(p-1)\sin^{p-1}\theta} + \frac{p-2}{p-1}M_{p-2} \\
&\overset{(22)}{=} -\frac{\cos\theta}{(p-1)\sin^{p-1}\theta} - \frac{p-2}{p-1}\sum_{k=1}^{(p-3)/2}\frac{(p-4)!!}{(p-3)!!}\frac{(2k-2)!!}{(2k-1)!!}\frac{\cos\theta}{\sin^{2k}\theta} \\
&\quad + \frac{p-2}{p-1}\frac{(p-4)!!}{(p-3)!!}\log\tan(\theta/2) \\
&= -\sum_{k=1}^{(p-1)/2}\frac{(p-2)!!}{(p-1)!!}\frac{(2k-2)!!}{(2k-1)!!}\frac{\cos\theta}{\sin^{2k}\theta} + \frac{(p-2)!!}{(p-1)!!}\log\tan(\theta/2)
\end{aligned}$$

6.2.4 Equation (24)

$$\begin{aligned}
\frac{d}{d\theta}\left(\sin^{p-1}\theta\cos\theta\right) &= (p-1)\sin^{p-2}\theta\cos^2\theta - \sin^p\theta \\
&= (p-1)\sin^{p-2}\theta - (p-1)\sin^p\theta - \sin^p\theta \\
&= (p-1)\sin^{p-2}\theta - p\sin^p\theta \\
\int \sin^p\theta d\theta &= -\frac{\sin^{p-1}\theta\cos\theta}{p} + \frac{p-1}{p}\int \sin^{p-2}\theta d\theta
\end{aligned}$$

6.2.5 Equation (25)

Let $p = 1$. A simple integration shows that $S_1 = -\cos\theta$, and therefore the equation (25) holds. As in the previous section, let $p \geq 3$ odd and suppose (25) holds for $p-2$. By (24), we have

$$\begin{aligned}
S_p(\theta) &= -\frac{\sin^{p-1}\theta\cos\theta}{p} + \frac{p-1}{p}S_{p-2}(\theta) \\
&\overset{(25)}{=} -\frac{\sin^{p-1}\theta\cos\theta}{p} - \frac{p-1}{p}\sum_{k=0}^{(p-3)/2}\frac{(p-3)!!}{(p-2)!!}\frac{(2k-1)!!}{(2k)!!}\cos\theta\sin^{2k}\theta \\
&= -\sum_{k=0}^{(p-1)/2}\frac{(p-1)!!}{p!!}\frac{(2k-1)!!}{(2k)!!}\cos\theta\sin^{2k}\theta
\end{aligned}$$

6.2.6 Equation (26)

$$\frac{d}{d\theta}\left(\frac{1}{(q-1)\sin^{q-1}\theta}\right) = -\frac{\cos\theta}{\sin^q\theta} = -\frac{\cos^2\theta}{\cos\theta\sin^q\theta} = \frac{1}{\cos\theta\sin^{q-2}\theta} - \frac{1}{\cos\theta\sin^q\theta}$$

$$\int \frac{1}{\cos\theta\sin^q\theta}d\theta = -\frac{1}{(q-1)\sin^{q-1}\theta} + \int \frac{1}{\cos\theta\sin^{q-2}\theta}d\theta$$

6.2.7 Equation (27)

Let $q = 1$. A simple integration shows that $L_1 = \log|\tan\theta|$, and therefore the equation (27) holds. Let $q \geq 3$ odd and suppose (27) holds for $q-2$. Using the recursion (26), we have

$$\begin{aligned} L_q &= -\frac{1}{(q-1)\sin^{q-1}\theta} + L_{q-2} \\ &\overset{(27)}{=} -\frac{1}{(q-1)\sin^{q-1}\theta} - \sum_{k=1}^{(q-3)/2} \frac{1}{2k\sin^{2k}\theta} + \log|\tan\theta| \\ &= -\sum_{k=1}^{(q-1)/2} \frac{1}{2k\sin^{2k}\theta} + \log|\tan\theta| \end{aligned}$$

6.2.8 Equation (29)

If $n = 2$, the equation (29) is equal to $-\log|\tan\theta|$ trivially. Suppose $n \geq 4$ even and (29) is true for $n-2$.

$$\begin{aligned} -\sum_{k=0}^{n/2-1} \frac{(n-2)!!}{(n-1)!!}\frac{(2k-1)!!}{(2k)!!}\log|\tan\theta| &= -\frac{n-2}{n-1}\left[\sum_{k=0}^{n/2-2} \frac{(n-4)!!}{(n-3)!!}\frac{(2k-1)!!}{(2k)!!}\log|\tan\theta|\right] \\ &\quad - \frac{\log|\tan\theta|}{n-1} \\ &\overset{(29)}{=} -\left(\frac{n-2}{n-1} + \frac{1}{n-1}\right)\log|\tan\theta| \\ &= \log|\cos\theta| - \log\sin\theta \end{aligned}$$

6.2.9 Equation (36)

A direct integration shows that M_2 is equal to $-\cos\theta/\sin\theta$, so (36) holds for $p = 2$. Now suppose $p \geq 4$ even and (36) is true for $p-2$. By the recursion (21),

$$
\begin{aligned}
M_p &= -\frac{\cos\theta}{(p-1)\sin^{p-1}\theta} + \frac{p-2}{p-1}M_{p-2} \\
&\overset{(36)}{=} -\frac{\cos\theta}{(p-1)\sin^{p-1}\theta} - \frac{p-2}{p-1}\sum_{k=1}^{p/2-1}\frac{(p-4)!!}{(p-3)!!}\frac{(2k-3)!!}{(2k-2)!!}\frac{\cos\theta}{\sin^{2k-1}\theta} \\
&= -\sum_{k=1}^{p/2}\frac{(p-2)!!}{(p-1)!!}\frac{(2k-3)!!}{(2k-2)!!}\frac{\cos\theta}{\sin^{2k-1}\theta}
\end{aligned}
$$

6.2.10 Equation (38)

By integrating directly, one may show that $S_2 = (\theta - \sin\theta\cos\theta)/2$, so the equation (38) holds for $p = 2$. Let $p \geq 4$ even and assume (38) holds for $p-2$.

$$
\begin{aligned}
S_p(\theta) &= -\frac{\sin^{p-1}\theta\cos\theta}{p} + \frac{p-1}{p}S_{p-2}(\theta) \\
&\overset{(38)}{=} -\frac{\sin^{p-1}\theta\cos\theta}{p} - \frac{p-1}{p}\sum_{k=1}^{p/2-2}\frac{(p-3)!!}{(p-2)!!}\frac{(2k)!!}{(2k+1)!!}\cos\theta\sin^{2k+1}\theta \\
&\quad + \frac{p-1}{p}\frac{(p-3)!!}{(p-2)!!}(\theta - \sin\theta\cos\theta) \\
&= -\sum_{k=1}^{p/2-1}\frac{(p-1)!!}{p!!}\frac{(2k)!!}{(2k+1)!!}\cos\theta\sin^{2k+1}\theta + \frac{(p-1)!!}{p!!}(\theta - \sin\theta\cos\theta)
\end{aligned}
$$

6.2.11 Equation (39)

Integration by parts using

$$
u = \frac{\theta}{\sin^p\theta},\ v' = \frac{1}{\cos^2\theta},\ u' = \frac{1}{\sin^p\theta} - p\frac{\theta\cos\theta}{\sin^{p+1}\theta},\ v = \tan\theta
$$

gives the desired result

$$\begin{aligned}\int \frac{\theta d\theta}{cos^2\theta \sin^p \theta} &= \frac{\theta}{\cos\theta \sin^{p-1}\theta} - \left(\int \frac{d\theta}{\cos\theta \sin^{p-1}\theta} - p\int \frac{\theta d\theta}{\sin^p\theta}\right) \\ &= \frac{\theta}{\cos\theta \sin^{p-1}\theta} - L_{p-1} + p\int \frac{\theta d\theta}{\sin^p\theta}.\end{aligned}$$

6.2.12 Equation (40)

The equation follows from integration by parts with

$$u = \theta,\ v' = \frac{1}{\sin^p\theta},\ u' = 1,\ v = M_p(\theta).$$

6.2.13 Equation (43)

For $p = 2$, the claim is trivially true. Let $p \geq 4$ even and suppose the claim is true for $p - 2$.

$$\begin{aligned}&-\sum_{k=1}^{p/2-1} \frac{(p-1)!!}{p!!}\frac{(2k)!!}{(2k+1)!!}\log|\tan\theta| - \frac{2(p-1)!!}{p!!}\log|\tan\theta| \\ &\quad = -\frac{p-1}{p}\sum_{k=1}^{p/2-2}\frac{(p-3)!!}{(p-2)!!}\frac{(2k)!!}{(2k+1)!!}\log|\tan\theta| - \frac{\log|\tan\theta|}{p} \\ &\qquad - \frac{p-1}{p}\frac{2(p-3)!!}{(p-2)!!}\log|\tan\theta| \\ &\quad = -\frac{p-1}{p}\log|\tan\theta| - \frac{\log|\tan\theta|}{p} = -\log|\tan\theta|\end{aligned}$$

6.2.14 Equation (56)

For $n = 2m$ and $m > 1$, the original expression found in [Oli11] is

$$a = -(n+1)(n-1)\cdots\frac{n+1-(-1)^{m-1}}{2} \Big/ \sum_{j=1}^{m-1}\sqrt{2}^{1+(-1)^j}$$

We divide into cases for m even and odd.

If m is odd, then the product in the numerator must end at $n/2 = m$. Likewise, if m is even, the product must end at $(n+2)/2 = m+1$.

The summand in the denominator alternates between 1 and 2 so that for m even, the sum is $3(m-2)/2+1$, or $3(m/2)-2$. For m odd, the sum is simply $3(m-1)/2$.

The resulting expression for $n = 2m$ in (56) is

$$a = \begin{cases} -\frac{1}{(3(m/2)-2)}\frac{(n+1)!!}{(m-1)!!} & m > 1 \text{ even} \\ -\frac{2}{3(m-1)}\frac{(n+1)!!}{(m-2)!!} & m > 1 \text{ odd.} \end{cases}$$

6.2.15 Equation (60)

For $n = 2m+1$, we have the equation

$$\begin{aligned} R &= -(n+1)\frac{\sigma_{n-1}}{\sigma_n} \\ &\overset{(1)}{=} -(n+1)\frac{2\pi^{n/2}}{\Gamma(n/2)}\frac{\Gamma((n+1)/2)}{2\pi^{(n+1)/2}} \\ &\overset{(63)}{=} -\frac{(n+1)}{\pi}\frac{4^m m! m!}{(n-1)!} \end{aligned}$$

where we used the identity

$$\Gamma\left(\tfrac{1}{2}+k\right) = \frac{(2k)!}{4^k k!}\pi^{1/2} = \frac{(2k-1)!!}{2^k}\pi^{1/2} \tag{63}$$

for the gamma function at half-integers.

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