Pólya's Theorem with Zeros

By

Mari Castle
Doctor of Philosophy

Department of Mathematics and Computer Science

Victoria Powers, Ph.D.
Advisor

Eric Brussel, Ph.D.
Committee Member

Raman Parimala, Ph.D.
Committee Member

Accepted

Date

Pólya's Theorem with Zeros

By

Mari Castle

Advisor: Victoria Powers, Ph.D.

An Abstract of
a dissertation submitted to the Faculty of the Graduate School
of Emory University in partial fulfillment
of the requirements of the degree of
Doctor of Philosophy

Department of Mathematics and Computer Science

2008

Abstract

Let $\mathbb{R}[X] = \mathbb{R}[X_1, \ldots, X_n]$ and let Δ_n denote the standard n-simplex $\{(x_1, \ldots, x_n) \in \mathbb{R}^n \mid x_i \geq 0, \sum_i x_i = 1\}$. Pólya's Theorem says that if a form (homogeneous polynomial) $p \in \mathbb{R}[X]$ is positive on Δ_n, then for sufficiently large $N \in \mathbb{N}$, the coefficients of $(X_1 + \cdots + X_n)^N p$ are positive. In 2001, Powers and Reznick established an explicit bound for the N in Pólya's Theorem. The bound depends only on information about p, namely the degree and the size of the coefficients of p, and the minimum value of p on the simplex.

This thesis is part of an ongoing project, started by Powers and Reznick in 2006, to understand exactly when Pólya's Theorem holds if the condition "positive on Δ_n" is relaxed to "nonnegative on Δ_n", and to give bounds in this case. In this thesis, we will show that if a form p satisfies a relaxed version of Pólya's Theorem, then the set of zeros of p is a union of faces of the simplex. We characterize forms which satisfy a relaxed version of Pólya's Theorem and have zeros on vertices. Finally, we give a sufficient condition for forms with zero set a union of two-dimensional faces of the simplex to satisfy a relaxed version of Pólya's Theorem, with a bound.

Pólya's Theorem with Zeros

By

Mari Castle

Advisor: Victoria Powers, Ph.D.

A dissertation submitted to the Faculty of the Graduate School
of Emory University in partial fulfillment
of the requirements of the degree of
Doctor of Philosophy

Department of Mathematics and Computer Science

2008

Acknowledgments

Without the support of my committee members, friends, and family, this dissertation would not have been possible. I would like to express my deepest gratitude to my advisor, Dr. Vicki Powers. Her thoughtful guidance, endless patience, and enduring friendship provided me with a positive environment in which to do research. I am fortunate to have so many fond memories and honored to have worked with her.

I would also like to thank my committee members, Dr. Eric Brussel and Dr. Raman Parimala, who have generously given their time and expertise to better my work. I thank them for their contribution and their good-natured support. I would like to thank my many professors, the administrative staff, and the numerous friends I have made in the Mathematics Department at Emory. My time here has been truly enjoyable.

Many people and events have put me on this path, and to all I owe a great deal. I am profoundly grateful to Dr. Joe DeMaio, whose instruction and counsel made this end result not only possible, but attainable.

I am indebted to my friends and family, whose unlimited love and support bolstered me throughout this process. I am grateful to my parents, Juan and Marisela Fernandez, for instilling in me values and ideals I hold dear. I would especially like to thank my husband and friend, Jerry, for understanding and indulging me during this time. I will always treasure his ability to make me laugh, even during the most stressful times of my research. Special thanks go to my daughter, Nicole, for accepting the limits placed on our time together during the writing of this thesis. She is my inspiration and deserves much of the credit for motivating me through my research. To her, I dedicate this thesis.

Contents

Chapter 1

Introduction

Throughout this thesis we work in the real polynomial ring in n variables. Fix a positive integer n, let $\mathbb{R}[X] := \mathbb{R}[X_1, \ldots, X_n]$, and let $\mathbb{R}^+[X]$ denote the polynomials in $\mathbb{R}[X]$ with nonnegative coefficients. A *form* is a homogeneous polynomial. We let Δ_n denote the standard simplex,

$$\Delta_n = \{(x_1, \ldots, x_n) \in \mathbb{R}^n \mid x_i \geq 0, \sum_i x_i = 1\}.$$

Pólya's Theorem. *If a form $p \in \mathbb{R}[X]$ is positive on Δ_n, then for sufficiently large $N \in \mathbb{N}$, the coefficients of $(X_1 + \ldots + X_n)^N p$ are positive.*

Pólyas theorem appeared in 1928 [9] (in German) and is also in *Inequalities* by Hardy, Littlewood, and Pólya [7] (in English). In Hardy, Littlewood, and Pólya's words: "The theorem gives a systematic process for deciding whether a given form F is strictly positive for positive x. We multiply repeatedly by $\sum x_i$ and, if the form is positive, we shall sooner or later obtain a form with positive coefficients."[7]

In 2001, Powers and Reznick [10] established an explicit bound for the N in Pólya's Theorem in terms of the degree and the size of the coefficients of the given form, and the minimum value of the form on the simplex.

Before giving this result, we establish some notation. For $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^N$, X^α denotes the monomial $X_1^{\alpha_1} \cdots X_n^{\alpha_n}$. Let $|\alpha|$ denote $\sum \alpha_i$ and if $|\alpha| = d$, define $c(\alpha) := \frac{d!}{\alpha_1! \ldots \alpha_n!}$. Given $p \in \mathbb{R}[X]$, a form of degree d, say

$$p(X) = \sum_{|\alpha|=d} a_\alpha X^\alpha,$$

let $L(p)$ be the maximum of $|a_\alpha / c(\alpha)|$.

Theorem 1 (Powers, Reznick)**.** *Suppose $p(X) \in \mathbb{R}[X]$ is homogeneous of degree d with $p(X) > 0$ on Δ_n. Let λ be the minimum of $p(x)$ for $x \in \Delta_n$. Then for*

$$N > \frac{d(d-1)}{2} \frac{L(p)}{\lambda} - d$$

the coefficients of $(X_1 + ... + X_n)^N p$ are positive.

We describe a few applications of Pólya's Theorem and this bound. Pólya's Theorem has been used in the study of copositive programming. Let $\mathbb{S}^n$ denote the $n \times n$ symmetric matrices over $\mathbb{R}$ and define the copositive cone

$$C_n = \{M \in \mathbb{S}^n \mid Y^T M Y \geq 0 \text{ for all } Y \in \mathbb{R}^n_+\}.$$

Copositive programming is optimization over C_n. By Pólya's Theorem, the truncated cones

$$C_n^r := \{M \in \mathbb{S}^n \mid \left(\textstyle\sum_i x_i\right)^r X^T M X\}$$

have non-negative coefficients and will converge to C_n. Using linear programming, membership in C_n^r can be determined numerically. De Klerk and Pasechnik [3] use this fact, along with the bound for Pólya's Theorem from Theorem 1, to give results on approximating the stability number of a graph.

Handelman [5, 6] has studied a related question, namely, for which pairs (q, f) of polynomials does there exist $N \in \mathbb{N}$ so that $q^N f$ has nonnegative coefficients? (See also de Angelis and Tuncel [2].) Pólya's Theorem and the generalizations described in this thesis (without the bound) can be deduced from Handelman's work.

More recently, Schweighofer [13] used Pólya's Theorem to give an algorithmic proof of Schmüdgen's Positivstellensatz, which says that if the basic closed semialgebraic set $K = \{g_1 \geq 0, \ldots, g_k \geq 0\}$ is compact and $f > 0$ on K, then f can be written as a finite sum of products of the g_i's and squares in $\mathbb{R}[X]$. This can be used to give an algorithm for optimization of polynomials on compact semialgebraic sets; see [15] for details. Using the bound from Theorem 1, Schweighofer obtained complexity bounds for Schmüdgen's Positivstellensatz [16].

In 2006, Powers and Reznick [11] extended their results to non-negative polynomials allowed to have a certain type of zero at vertices of Δ_n.

Definition 1. Let $p(X) \in \mathbb{R}[X]$ be homogeneous of degree d and suppose $p(X) \geq 0$ on Δ_n. Write $v_1, \ldots, v_n$ for the vertices of Δ_n, i.e., $v_1 = (1, 0, \ldots, 0), \ldots, v_n = (0, \ldots, 0, 1)$. Then p has a *simple zero* at the unit vertex v_i if the coefficient of X_i^d in p is zero, but the coefficient of $X_i^{d-1}X_j$ is non-zero (and necessarily positive) for each $j \neq i$.

In [11], Powers and Reznick show that if a form p is positive on Δ_n except for simple zeros at v_i's, then $(X_1 + ... + X_n)^N p \in \mathbb{R}^+[X]$, for some $N \in \mathbb{N}$. The bound on N in this case depends on the size of the coefficients of p, the minimum of p away from the zeros, and some other constants determined by the coefficients of p.

In 2005, Schweighofer [14] gave a "localized" version of Pólya's Theorem that gives a condition which implies the conclusion of Pólyas theorem (with "positive coefficients" replaced by "nonnegative coefficients"). The idea is to find a representation of f, which depends on $x \in \Delta_n$, and which implies the conclusion of Pólya's Theorem for coefficients corresponding to X^α, where $\frac{\alpha}{|\alpha|}$ is contained in a neighborhood around x.

Proposition 1 (Schweighofer). *Let $f \in \mathbb{R}[X]$. Suppose that for every $x \in \Delta_n$ there are $m \in \mathbb{N}$, forms $g_1, ..., g_m$ and $h_1, ..., h_m \in \mathbb{R}^+[X]$ such that*
1) $f = g_1h_1 + ... + g_mh_m$, and
2) $g_i(x) > 0$ for all i.
Then there exists $N \in \mathbb{N}$ such that $(X_1 + ... + X_n)^N f \in \mathbb{R}^+[X]$.

This thesis is part of an ongoing project, begun in [11], to understand exactly when Pólya's Theorem holds if the condition "positive on Δ_n" is relaxed to "nonnegative on Δ_n", and to give bounds in this case. The author, along with Powers and Reznick, began work on this project in [1]; most of this work is contained in Chapter 3 and 4. In this work, we give a computational version of Proposition 1, replacing neighborhoods of x with closed subsets of Δ_n, along with a bound on N. We then obtain Proposition 1 as a corollary. Using this computational version of Proposition 1, we characterize forms that are positive on Δ_n, apart from zeros at v_i's, and satisfy the conclusion of Pólyas Theorem (with "positive coefficients" replaced by "nonnegative coefficients"). This is a generalization of the main result from [11].

In this thesis, we continue work begun in [1]. We establish possible locations for zeros of forms that satisfy a relaxed version of Pólya's Theorem. We include the work from [1] mentioned above. We then extend previous results to forms with zeros on two-dimensional faces, including a bound in this case. Finally, we include examples of forms that illustrate the work included in this thesis. Following is an outline of the thesis.

In Chapter 2, we give some preliminary notation and results. We show that if a form p satisfies Pólya's Theorem, then the zero set of p must be a union of faces of the simplex. We also give a necessary, but not sufficient, condition for p to satisfy Pólya's Theorem.

Chapters 3 and 4 contain work from [1]. Given p positive apart from zeros on the vertices Δ_n, we characterize those for which there is an N so that the coefficients of $(X_1 + ... + X_n)^N p$ are nonnegative, and give a bound on N.

The proof uses our computational version of Proposition 1 to get a bound on a "corner piece" of Δ_n and then noting p is positive on what remains of Δ_n, we can apply the generalization of Theorem 1 to get a bound here. We then obtain the main result from from [11] as a corollary.

In Chapter 5, we look at forms positive on Δ_n apart from zeros on two-dimensional faces. We consider subsets of Δ_n containing one-dimensional faces, two-dimension faces, and the rest of Δ_n. In each case, we find an appropriate representation of p, apply our computational version of Proposition 1, and establish a bound.

Very recently, we learned of related work by Hoi-Nam Mok and Wing-Keung To [8]. The main theorem in [8] is a sufficient condition for a form non-negative on the simplex to satisfy Pólya's Theorem, with a bound. This implies the main result in Chapter 5, however our bound is different. The proof in [8] is different from our proof.

Chapter 2

Preliminaries

Let $P_{n,d}(\Delta_n)$ denote the set of degree d forms in n variables which are non-negative on Δ_n and let $Po(n,d)$ be the degree d forms in n variables for which there exists an $N \in \mathbb{N}$ such that $(X_1 + ... + X_n)^N p \in \mathbb{R}^+[X]$. In other words, $Po(n,d)$ are the forms which satisfy the conclusion of Pólyas theorem, with "positive coefficients" replaced by "nonnegative coefficients."

For $I \subseteq \{1,...,n\}$, let $F(I)$ denote the *face* of Δ_n containing the vertices $\{v_i \mid i \in I\}$, i.e.,

$$F(I) = \{(u_1, u_2, \ldots, u_n) \in \Delta_n \mid u_j = 0 \text{ for } j \notin I\}.$$

Note that $F(\emptyset) = \Delta_n$ and for $i \in \{1,...,n\}$, $v_i = F(\{i\})$. The *relative interior* of the face $F(I)$ is the set $\{(u_1, \ldots, u_n) \in F(I) \mid u_i > 0 \text{ for } i \in I\}$. For $f(x) \in \mathbb{R}[X]$, we denote by $Z(f)$ the zeros of f, i.e., $Z(f) = \{u \in \mathbb{R}^n \mid f(u) = 0\}$.

Given $f = \sum a_\alpha X^\alpha \in \mathbb{R}[X]$ let

$$\Lambda^+(f) := \{\alpha \in \mathbb{N}^n \mid a_\alpha > 0\},$$
$$\Lambda^-(f) := \{\beta \in \mathbb{N}^n \mid a_\beta < 0\}$$

Let $\alpha = (\alpha_1, \ldots, \alpha_n)$ and $\beta = (\beta_1, \ldots, \beta_n)$ be n-tuples in $\mathbb{N}^n$, and let $I \subseteq \{1, \ldots, n\}$. Then we write $\beta \preceq_I \alpha$ if $\beta_i \leq \alpha_i$ for all $i \in I$, and $\beta \prec_I \alpha$ if

$\beta \preceq_I \alpha$ and there exists some $j \in I$ such that $\beta_j < \alpha_j$. Note if $I = \{i\}$, so that $F(I)$ is a vertex, then for a form $\in \mathbb{R}[X]$, if $\alpha \neq \beta$, $\beta \preceq_I \alpha$ implies $\beta \prec_I \alpha$.

In this section, we start with some observations about the possible location of zeros for a form $p \in Po(n, d)$. These results can be found without proof in [11].

Proposition 2. *Suppose $p \in Po(n, d)$. If $p(u) = 0$ for u a point in the relative interior of a face of Δ_n, then p vanishes everywhere on the face.*

Proof. For ease of exposition, we assume the face is $F(\{1, \dots, k\})$, where $1 \leq k \leq n$. Then $u = (u_1, \dots, u_k, 0, ..., 0)$ where each $u_i > 0$. By assumption, there is an $N \in \mathbb{N}$ such that $(X_1 + \cdots + X_n)^N p \in \mathbb{R}^+[X]$. Let $q = (X_1 + \cdots + X_n)^N p$, then $q(u) = 0$.

It is easy to see we can write $p = p_1 + p_2$ where $p_1 \in \mathbb{R}[X_1, \dots, X_k]$, and every monomial of p_2 contains at least one of $\{X_{k+1}, \dots, X_n\}$, or $p_2 \equiv 0$. Note that $p_2(u) = 0$. Write $(X_1 + \cdots + X_n)^N p_1 = \sum b_\gamma X_1^{\gamma_1} \dots X_k^{\gamma_k}$, where $b_\gamma \geq 0$ for all γ. Then

$$q = (X_1 + \cdots + X_n)^N p_1 + (X_1 + \cdots + X_n)^N p_2,$$

hence $q(u) = \sum b_\gamma u_1^{\gamma_1} \dots u_k^{\gamma_k}$. Since $b_\gamma \geq 0$ and $u_1^{\gamma_1}, \dots, u_k^{\gamma_k} > 0$, $q(u) = 0$ implies $b_\gamma = 0$ for all γ. Hence $p_1 \equiv 0$, which gives $p(w) = 0$ for all w of the form $(w_1, \dots, w_k, 0, ..., 0)$, i.e, all points on the face. Thus p vanishes everywhere on the face. □

Corollary 1. *If $p(u) = 0$ for u a point in the interior of Δ_n, then $p \equiv 0$.*

Corollary 2. *The set $Z(p) \cap \Delta_n$ is a union of faces of Δ_n.*

The preceding lemma and corollaries show we need only focus our attention on zeros on faces of the simplex. However, the location of the zeros does not determine if $p \in Po(n, d)$, as shown by the following example from [11].

Example 1. The following forms are non-negative on Δ_3 with zeros only at vertices:

$$f = xz^3 + yz^3 + x^2y^2 - xyz^2,$$

$$g = x^2y + y^2z + z^2x - xyz.$$

We will show $f \notin Po(3,3)$, but $g \in Po(3,3)$. We claim that the coefficient of $x^{N+1}yz^2$ in $(x+y+z)^N f$ is always negative. There is no contribution from the coefficient of $(x+y+z)^N xz^3$ or $(x+y+z)^N yz^3$ because the power of z is too large and there is no contribution from $(x+y+z)^N x^2y^2$ because the power of y is too large. Hence the only contribution comes from $(x+y+z)^N(-xyz^2)$ and thus the coefficient will always be -1. On the other hand, it is easy to compute that $(x+y+z)^3 g$ has only positive coefficients. Thus the location of the zeros of $p \in P_{n,d}(\Delta_n)$ is not enough to determine whether p is in $Po(n,d)$ or not.

Let $p \in \mathbb{R}[X]$. Then write $p = p^+ - p^-$ where

$$p^+ = \sum_{\alpha \in \Lambda^+(p)} a_\alpha X^\alpha \quad \text{and} \quad p^- = \sum_{\beta \in \Lambda^-(p)} b_\beta X^\beta,$$

with $a_\alpha, b_\beta \in \mathbb{R}^+$. Note $p^+, p^- \in \mathbb{R}^+[X]$.

Hence, for any $N \in \mathbb{N}$,

$$\begin{aligned}
(X_1 + \cdots + X_n)^N p =& (X_1 + \cdots + X_n)^N (p^+ - p^-) \\
=& (X_1 + \cdots + X_n)^N p^+ - (X_1 + \cdots + X_n)^N p^- \\
=& (X_1 + \cdots + X_n)^N \sum_{\alpha \in \Lambda^+} a_\alpha X^\alpha - (X_1 + \cdots + X_n)^N \sum_{\beta \in \Lambda^-} b_\beta X^\beta \\
=& \sum_{|\gamma| = N+d} A_\gamma X^\gamma - \sum_{|\gamma| = N+d} B_\gamma X^\gamma,
\end{aligned}$$

where, from calculations given in [10], we have

$$A_\gamma = \sum_{\alpha \in \Lambda^+(p), \alpha \preceq \gamma} \frac{N!}{(\gamma_1 - \alpha_1)! \cdots (\gamma_n - \alpha_n)!} \cdot a_\alpha \tag{2.1}$$

$$B_\gamma = \sum_{\beta \in \Lambda^-(p), \beta \preceq \gamma} \frac{N!}{(\gamma_1 - \beta_1)! \cdots (\gamma_n - \beta_n)!} \cdot b_\beta \tag{2.2}$$

Definition 2. For $I \subseteq \{1, \ldots, n\}$, let $\overline{I}$ denote $\{1, \ldots, n\} \setminus I$.

Proposition 3. *Let $p \in P_{n,d}(\Delta_n)$ and suppose $p \in Po(n,d)$. Let $I \subseteq \{1, \ldots, n\}$ and suppose $Z(p)$ contains $F(I)$. Let $\Lambda^+ = \Lambda^+(p)$ and $\Lambda^- = \Lambda^-(p)$. Then for every $\beta \in \Lambda^-$ there exists an $\alpha \in \Lambda^+$ so that $\alpha \preceq_{\overline{I}} \beta$.*

Proof. Since $p \in Po(n,d)$, there exists $N \in \mathbb{N}$ such that $(X_1 + \cdots + X_n)^N p \in \mathbb{R}^+[X]$. Suppose our assumption does not hold, i.e., there is a $\beta = (\beta_1, \ldots, \beta_n) \in \Lambda^-$ such that for all $\alpha = (\alpha_1, \ldots, \alpha_n) \in \Lambda^+$, $\alpha \npreceq_{\overline{I}} \beta$, i.e., $\beta \prec_{\overline{I}} \alpha$. Then, for each $\alpha \in \Lambda^+$, there is some $j \in \{1, \ldots, n\}$, $j \notin I$, so that $\alpha_j > \beta_j$.

Fix $i \in I$ and for each positive integer $N \geq 1$, define $\gamma = (\gamma_1, \ldots, \gamma_n)$ as follows:

$$\gamma_j := \begin{cases} \beta_j, & \text{if } j \neq i \\ N + \beta_i, & \text{if } j = i \end{cases} \tag{2.3}$$

Clearly, $|\gamma| = N + d$. For every $\alpha \in \Lambda^+$, since $\alpha_j > \beta_j = \gamma_j$ for some $j \neq i$, $\alpha \npreceq \gamma$ for any $\alpha \in \Lambda^+$. Hence, by (2.1) $A_\gamma = 0$. Additionally, from (2.3), we have $\gamma_j \geq \beta_j$ for any $j \in \{1, \ldots, n\}$. Hence, (2.3) and (2.2) imply $B_\gamma > 0$. Thus, for every positive integer $N \geq 1$ we have constructed a γ with $|\gamma| = N + d$ so that the coefficient of X^γ in $(X_1 + \cdots + X_n)^N p$ is negative, contradicting $p \in Po(n,d)$. $\square$

Proposition 3 gives a necessary but not sufficient condition for $p \in Po(n,d)$. This is demonstrated in the following example.

Example 2. Consider the following form p:

$$\begin{aligned} p(x,y,z,w) &= x^4 + y^4 + x^2(w-z)^2 \\ &= x^4 + y^4 + x^2z^2 + x^2w^2 - 2x^2zw \end{aligned}$$

Clearly, since p is a sum of squares, this form is nonnegative on Δ_n, hence $p \in P_{4,4}(\Delta_4)$. Also, $Z(p) \cap \Delta_4 = \{x = y = 0\} = F(\{3,4\})$.

We have the following:

$$\begin{aligned} &\Lambda^+(f) = \{(4,0,0,0),(0,4,0,0),(2,0,2,0),(2,0,0,2)\}. \\ &\Lambda^-(f) = \{(2,0,1,1)\}. \end{aligned}$$

It is easy to see p satisfies the conditions of Proposition 3, since

$$\begin{aligned} &(2,0,2,0) \prec_{\{1,2,3\}} (2,0,1,1) \\ &(2,0,0,2) \prec_{\{1,2,4\}} (2,0,1,1) \\ &(2,0,2,0) \preceq_{\{1,2\}} (2,0,1,1) \end{aligned}$$

We will show that p is not in $Po(4,4)$. Consider the $x^2z^{N+1}w^{N+1}$ term in $(x+y+z+w)^{2N}p$. Then for $\gamma = (2,0,N+1,N+1)$, from (2.1) we have

$$A_\gamma = \frac{2(2N+4)!}{0!0!(N-1)!(N+1)!}.$$

Likewise, from (2.2) we have

$$B_\gamma = \frac{2(2N+4)!}{0!0!N!N!}.$$

Thus $B_\gamma > A_\gamma$ which implies p is not in $Po(4,4)$.

Chapter 3

A localized Pólya's Theorem

The work in this chapter, which is from [1], is a computational version of Proposition 1. Proposition 1 says that given $f(X) \in \mathbb{R}[X]$, not necessarily homogeneous, if we can find certain types of representations of f, which depend on $x \in \Delta_n$, then there is an $N \in \mathbb{N}$ so that the coefficient of X^α in $(\sum X_i)^N f$ is nonnegative whenever $\frac{\alpha}{|\alpha|}$ is contained in a neighborhood around x. Taking a finite subcover from these neighborhoods yields a global N. Our version of this result replaces neighborhoods with finitely many closed subsets of Δ_n covering Δ_n, which allows us to give an explicit bound for the exponent N needed.

We first give a localized version of Theorem 1.

Lemma 1. *Suppose $S \subseteq \Delta_n$ is nonempty and closed, and $p \in \mathbb{R}[X]$ is homogeneous of degree d such that $p(x) > 0$ for all $x \in S$. Let λ be the minimum of p on S. Then for*

$$N > \frac{d(d-1)}{2}\frac{L(p)}{\lambda} - d$$

and $\beta \in \mathbb{N}^n$ such that $\frac{\beta}{|\beta|} \in S$, the coefficient of X^β in $(X_1 + ... + X_n)^N p$ is nonnegative.

Proof. The proof, which we give for completeness, is identical to the proof of Theorem 1 in [10]. We start with the technique of Pólya's proof of his theorem. For a positive number t, a non-negative integer m, and $x \in \mathbb{R}$, define

$$(x)_t^m := x(x-t)\cdots(x-(m-1)t) = \prod_{i=0}^{m-1}(x-it)\,.$$

Note for later reference that

$$(ty)_t^d = \prod_{i=0}^{d-1}\big(ty-(i-1)t\big) = t^d (y)_1^d, \tag{3.1}$$

and if $m > n$ are both integers, then $(n)_1^m = 0$, since one of the factors in the definition is zero. It follows immediately that in the special case that $x = k/M$ and $t = 1/M$, where M is a positive integer, we have

$$\left(\frac{k}{M}\right)_{1/M}^m = \frac{1}{M^m}\prod_{i=0}^{m-1}(k-i) = \begin{cases} \frac{1}{M^m}\frac{k!}{(k-m)!} = \frac{m!}{M^m}\binom{k}{m}, & \text{if } m \le k; \\ 0, & \text{otherwise.} \end{cases} \tag{3.2}$$

We fix $p = \sum a_\alpha X^\alpha$ and suppose that $p > 0$ on $S \subseteq \Delta_n$. We assume throughout that $d = \deg p > 1$; the $d = 1$ case is trivial. Following Pólya, we make the explicit computation:

$$(X_1 + \cdots + X_n)^N p(X_1, \ldots, X_n) =$$
$$\sum_{|\beta|=N} \frac{N!}{\beta_1!\cdots\beta_n!} X_1^{\beta_1}\cdots X_n^{\beta_n} \times \sum_{|\alpha|=d} a_\alpha X_1^{\alpha_1}\cdots X_n^{\alpha_n}.$$

For $|\beta| = N + d$, denote the coefficient of $X_1^{\beta_1}\ldots X_n^{\beta_n}$ in $(\sum X_i)^N p(X)$ by A_β. Then

$$\begin{aligned} A_\beta &= \sum_{|\alpha|=d,\ \alpha\preceq\beta} \frac{N!}{(\beta_1-\alpha_1)!\cdots(\beta_n-\alpha_n)!}\cdot a_\alpha \\ &= \frac{N!(N+d)^d}{\beta_1!\cdots\beta_n!} \sum_{|\alpha|=d,\ \alpha\preceq\beta} a_\alpha \prod_{\ell=1}^{n} \frac{\beta_\ell!}{(\beta_\ell-\alpha_\ell)!(N+d)^{\alpha_\ell}}. \end{aligned}$$

We now express A_β using the $(x)_t^m$ notation and (3.2):

$$A_\beta = \frac{N!(N+d)^d}{\beta_1! \cdots \beta_n!} \sum_{|\alpha|=d} a_\alpha \left(\frac{\beta_1}{N+d}\right)_{(N+d)^{-1}}^{\alpha_1} \cdots \left(\frac{\beta_n}{N+d}\right)_{(N+d)^{-1}}^{\alpha_n} \tag{3.3}$$

If $\alpha \not\leq \beta$, then the extra terms added in (3.3) are just 0. Still following Pólya, define

$$p_t(X_1, \ldots, X_n) := \sum_{|\alpha|=d} a_\alpha (X_1)_t^{\alpha_1} \cdots (X_n)_t^{\alpha_n}.$$

Clearly, $p_t \to p$ uniformly on Δ_n as $t \to 0$, hence for t sufficiently small, p_t is also positive on S. In view of the foregoing, this means that for N sufficiently large, and all $\frac{\beta}{|\beta|} = \left(\frac{\beta_1}{N+d}, \ldots, \frac{\beta_n}{N+d}\right) \in S$

$$A_\beta = \frac{N!(N+d)^d}{\beta_1! \cdots \beta_n!} p_{(N+d)^{-1}}\left(\tfrac{\beta_1}{N+d}, \ldots, \tfrac{\beta_n}{N+d}\right) > 0. \tag{3.4}$$

We now extend Pólya's work. Drop the constant factor in (3.4) and set $t = \frac{1}{N+d}$, $y_k = \frac{\beta_k}{N+d}$, and keep in mind that $\sum_k y_k = 1$. We have

$$p_t(y_1, \ldots, y_n) = p(y_1, \ldots, y_n) - \sum_{|\alpha|=d} a_\alpha \left(y_1^{\alpha_1} \cdots y_n^{\alpha_n} - (y_1)_t^{\alpha_1} \cdots (y_n)_t^{\alpha_n}\right).$$

If $(y_1, \ldots, y_n) \in S$, $p(y_1, \ldots, y_n) \geq \lambda$, hence

$$p_t(y_1, \ldots, y_n) \geq \lambda - L \sum_{|\alpha|=d} \frac{d!}{\alpha_1! \cdots \alpha_n!} \left|y_1^{\alpha_1} \cdots y_n^{\alpha_n} - (y_1)_t^{\alpha_1} \cdots (y_n)_t^{\alpha_n}\right| \tag{3.5}$$

If $\alpha_k > \beta_k$, then $(y_k)_t^{\alpha_k} = 0$, so $y_k^{\alpha_k} \geq (y_k)_t^{\alpha_k} \geq 0$ for all k; hence we may drop the absolute value in (3.5)

By the Multinomial Theorem,

$$\sum_{|\alpha|=d} \frac{d!}{\alpha_1! \cdots \alpha_n!} y_1^{\alpha_1} \cdots y_n^{\alpha_n} = (y_1 + \cdots + y_n)^d = 1.$$

By the iterated Vandermonde-Chu identity [10],

$$\sum_{|\alpha|=d} \frac{d!}{\alpha_1! \cdots \alpha_n!} (y_1)_t^{\alpha_1} \cdots (y_n)_t^{\alpha_n} = (y_1 + \cdots + y_n)_t^d = \prod_{k=0}^{d-1} (1 - kt). \tag{3.6}$$

By (3.5), we are done if we can show that

$$\lambda - L\big(1-(1-t)\cdots(1-(d-1)t)\big) > 0. \tag{3.7}$$

Suppose now that

$$t = \frac{1}{N+d} < \frac{2}{d(d-1)}\frac{\lambda}{L}.$$

It is easy to prove by induction that if $0 \le w_j \le 1$, then $\prod(1-w_j) \ge 1-\sum w_j$. Since $\lambda \le p(1,0,\dots,0) \le L$ and $d \ge 2$, we have $t < \frac{1}{d-1}$, hence

$$(1-(1-t)\cdots(1-(d-1)t)) < t(1+2+\cdots+(d-1)) = t\frac{(d-1)d}{2} < \frac{\lambda}{L},$$

and we are done. □

We want to apply Lemma 1 in the case where we have a representation of $p \in \mathbb{R}[X]$ of the type in Proposition 1 for a closed subset S of Δ_n. In other words, we want to write $p = g_1h_1 + \cdots + g_mh_m$ where $h_i \in \mathbb{R}[X]^+$ and $g_i(x) > 0$ for all $x \in S$, then apply Lemma 1 to the g_i. Our result will hold for a possibly smaller subset $T \subseteq S$ due to the fact that the exponents of $(\sum X_i)^N p$ are not the same as the exponents of $(\sum X_i)^N g_j$. For our specific application in Chapter 4, we will be able to take $T = S$.

Proposition 4. *Given $p \in \mathbb{R}[X]$ (not necessarily homogeneous) and a nonempty closed set $S \subseteq \Delta_n$ and suppose there exist homogeneous $g_1,\dots,g_m \in \mathbb{R}[X]$, and $h_1,\dots,h_m \in \mathbb{R}[X]^+$ with*

1. *$p = g_1h_1 + \cdots + g_mh_m$, and*
2. *$g_i(x) > 0$ for all $x \in S$.*

Suppose further that T is a nonempty closed subset of S and there exists $B \in \mathbb{N}$ with the following property: Whenever $\alpha, \beta, \gamma \in \mathbb{N}^n$ satisfy $\frac{\alpha}{|\alpha|} \in T$, $\beta + \gamma = \alpha$, $\gamma \in \operatorname{supp}(h_i)$ for some i, and $|\beta| \ge B$, then $\frac{\beta}{|\beta|} \in S$. Then there *exists $N \in \mathbb{N}$ such that for all $\alpha \in \mathbb{N}^n$ with $\frac{\alpha}{|\alpha|} \in T$, the coefficient of X^α in $(X_1 + ... + X_n)^N p$ is nonnegative.*

More precisely, for each i, let $k(i)$ be the bound from Lemma 1 for g_i on S, i.e.,

$$k(i) = \frac{d_i(d_i-1)}{2}\frac{L(g_i)}{\lambda_i} - d_i,$$

where λ_i is the minimum of g_i on S and $d_i = \deg g_i$. Then we can take

$$N \geq \max\{k(g_1), \ldots, k(g_m), B\}.$$

Proof. Given $\alpha \in \mathbb{N}^n$ with $\frac{\alpha}{|\alpha|} \in T$. Clearly, it suffices to show that for each $1 \leq j \leq m$, the coefficient of X^α in $(X_1 + ... + X_n)^N g_j h_j$ is nonnegative. Suppose $\beta, \gamma \in \mathbb{N}^n$ are such that $\beta + \gamma = \alpha$ and the coefficients of X^β in $(X_1 + \cdots + X_n)^N g_j$ and X^γ in h_j are non-zero. Since $h_j \in \mathbb{R}[X]^+$, the coefficient of X^γ in h_j is positive. Then since we have $|\beta| > N \geq B$ and $\alpha = \beta + \gamma$ for $\gamma \in \text{supp}(h_j)$, $\frac{\beta}{|\beta|} \in S$ by our assumption. Hence by the choice of $k(j)$ and Lemma 1, it follows that the coefficient of X^β in $(X_1 + \cdots + X_n)^N g_j$ is nonnegative and we are done. $\square$

We now obtain Proposition 1 as a corollary:

Corollary 3. *Let $f \in \mathbb{R}[X]$. Suppose that for every $x \in \Delta_n$ there are $m \in \mathbb{N}$, homogeneous $g_1, \ldots, g_m \in \mathbb{R}[X]$, and $h_1, \ldots, h_m \in \mathbb{R}[X]^+$ such that*

1) $f = g_1 h_1 + \cdots + g_m h_m$, and

2) $g_i(x) > 0$ for $i = 1, \ldots, m$.

Then there exists $N \in \mathbb{N}$ such that the coefficients of $(X_1 + \cdots + X_n)^N f$ are nonnegative.

Proof. For $\epsilon > 0$ and $x \in \mathbb{R}^n$, let $B_\epsilon(x) = \{y \in \mathbb{R}^n \mid ||y - x|| < \epsilon\}$, where $||\cdot||$ denotes the standard Euclidean norm in $\mathbb{R}^n$. In other words, $B_\epsilon(x)$ is the open ball of radius ϵ about x. For each $x \in \Delta_n$, by continuity of the g_i's, there is $\epsilon_x > 0$ so that a representation of f as above exists with $g_i > 0$ on

$B_{2\epsilon_x}(x)$. By compactness, we can choose a finite number of $B_{\epsilon_x}(x)$'s covering Δ_n. Then it is enough to show that for each $x \in \Delta_n$ there is an $N_x \in \mathbb{N}$ such that the coefficients of X^α in $(X_1 + \cdots + X_n)^{N_x} f$ for $\frac{\alpha}{|\alpha|} \in B_{\epsilon_x}(x)$ are nonnegative. Taking the maximum of the N_x's corresponding to the finite subcover, we are done.

Fix $x \in \Delta_n$, let $M = \max\{\deg(h_i)\}$, and choose $B \geq 2M/\epsilon_x$. Now set $S = \overline{B_{2\epsilon_x}} \cap \Delta_n$ and $T = \overline{B_{\epsilon_x}(x)} \cap \Delta_n$. Then S and T are nonempty and closed and $T \subseteq S$. Hence we need only show that the following property holds: Whenever $\alpha, \beta, \gamma \in \mathbb{N}^n$ with $\frac{\alpha}{|\alpha|} \in T, \beta + \gamma = \alpha, \gamma \in \operatorname{supp}(h_i)$ for some i, and $|\beta| \geq B$, then $\frac{\beta}{|\beta|} \in S$. We have $|\beta| \geq N \geq B \geq 1$. Thus we have $\dfrac{2|\gamma|}{|\beta|} \leq \dfrac{2M}{|\beta|} \leq \epsilon_x$ for $\gamma \in \operatorname{supp}(h_i)$. This gives us

$$\begin{aligned}
\|x - \beta\| &\leq \|\beta - \frac{\alpha}{|\alpha|}\| + \|x - \frac{\alpha}{|\alpha|}\| \\
&\leq \|\frac{|\alpha|\beta - |\beta|\alpha}{|\alpha||\beta|}\| + \epsilon_x \\
&= \|\frac{|\alpha|\gamma - |\gamma|\alpha}{|\alpha||\beta|}\| + \epsilon_x \\
&\leq \|\frac{|\alpha|\gamma\| + |\gamma|\alpha\|}{|\alpha||\beta|}\| + \epsilon_x \\
&= \frac{2|\alpha||\gamma|}{|\alpha||\beta|} + \epsilon_x \\
&= \frac{2|\gamma|}{|\beta|} + \epsilon_x \\
&\leq 2\epsilon_x.
\end{aligned}$$

□

Chapter 4

Pólya's Theorem With Zeros on Vertices

Most of the work in this section can be found in [1]. Recall from Lemma 2 that if $p \in Po(n,d)$, then $Z(p)$ must be a union of faces of Δ_n. In this chapter, we apply Proposition 4 to give a quantitative version of Pólya's Theorem for forms which are positive on Δ_n apart from zeros on the vertices of Δ_n, i.e., one-dimensional faces. This generalizes the main result from [11].

We begin with some notation. For $r \in \mathbb{R}$, $0 < r < 1$, and $i \in \{1, ..., n\}$, we define $\Delta(i, r)$ to be

$$\{(x_1, ..., x_n) \in \Delta_n \mid \sum_{j \neq i} x_j \leq r\} = \{(x_1, ..., x_n) \in \Delta_n \mid x_i \geq 1 - r\}.$$

In other words, $\Delta_n(i, r)$ is the scaled simplex $r \cdot \Delta_n$ translated by $(1 - r)v_i$ and nestled in the v_i corner of Δ_n.

The idea of the proof is to find, for each vertex v_i where p has a zero, an r so that a representation of p of the type in Proposition 4 exists on $\Delta_n(i, r)$. Then we apply Lemma 1 to the closure of Δ_n minus the corner simplices.

Let $f \in \mathbb{R}[X]$, say $f = \sum a_\alpha X^\alpha$. Recall that the *support* of f, denoted $\text{supp}(f)$, is $\{\alpha \in \mathbb{N}^n \mid a_\alpha \neq 0\}$. We define the following measure on the size of the coefficients of f:

$$W(f) := \sum_{\alpha \in \mathrm{supp}(f)} |a_\alpha|.$$

Lemma 2. *Given homogeneous $f \in \mathbb{R}[X]$ such that*

$$f = cX_i^e + \phi(X)$$

for some $i \in \{1, \dots, n\}$, where $c > 0$ and the degree of ϕ in X_i is less than e. Let $W = W(f)$ and define

$$r = \frac{c}{c + 2W}, \quad s = \frac{c}{2}\left(\frac{2W}{c + 2W}\right)^e.$$

Then $f \geq s$ on $\Delta_n(i, r)$.

Proof. Given $x = (x_1, \dots, x_n) \in \Delta_n$ with $x_i \neq 0$. For $j \neq i$, let $y_j = \dfrac{x_j}{x_i}$, then, since $\deg f = e$, we have $f(x_1, ..., x_n) = x_i^e f(y_1, \dots, y_{i-1}, 1, y_{i+1}, \dots, y_n)$. Let r be as given and suppose $(x_1, \dots, x_n) \in \Delta_n(i, r)$. Then $x_j \leq r$ for $j \neq i$, and $x_i \geq 1 - r$, hence for each $j \neq i$ we have

$$y_j = \frac{x_j}{x_i} \leq \frac{r}{1 - r} = \left(\frac{c}{c + 2W}\right)\left(\frac{c + 2W}{2W}\right) = \frac{c}{2W}.$$

Since the degree of ϕ in X_i is less than e and f is homogeneous of degree e, every monomial in $\phi(X)$ contains at least one X_j with $j \neq i$, thus $\phi(X_1, \dots, X_{i-1}, 1, X_{i+1}, \dots, X_n)$ has no constant term. Since $\frac{c}{2W} < 1$, it follows that each monomial in $\phi(X)$ evaluated at $(y_1, \dots, y_{i-1}, 1, y_{i+1}, \dots, y_n)$ has absolute value less than $\frac{c}{2W}$. Thus

$$|\phi(y_1, \dots, y_{i-1}, 1, y_{i+1}, \dots, y_n)| \leq (\frac{c}{2W})(W) = \frac{c}{2},$$

and hence

$$\begin{aligned} f(x_1,\ldots,x_n) &= x_i^e\left(c+\phi(y_1,\ldots,y_{i-1},1,y_{i+1},\ldots,y_n)\right) \\ &\geq (1-r)^e\left(c-\frac{c}{2}\right) \\ &= \left(\frac{c+2W-c}{c+2W}\right)^e\frac{c}{2} \\ &\geq \frac{c}{2}\left(\frac{2W}{c+2W}\right)^e = s. \end{aligned}$$

□

We start with a result for a form $p \geq 0$ on Δ_n with the set of zeros on Δ_n consisting of one vertex.

Proposition 5. *Given $p \in P_{n,d}(\Delta_n)$ such that $Z(p) \cap \Delta_n = \{v_i\}$ for some $1 \leq i \leq n$, suppose p can be written as*

$$p(X) = \sum_{k=1}^{m} M_k\left(c_k X_i^{l_k} + \phi_k(X)\right) + q(X)$$

where for all k, M_k is a monomial in $\{X_1,\ldots,X_n\}\setminus\{X_i\}$, $c_k > 0$, the degree in X_i of ϕ_k is strictly less than l_k, and $q(X)$ is a polynomial with only non-negative coefficients. Let $W = W(p), c = \min\{c_k\}, d = \deg(p)$ and define

$$r = \frac{c}{c+2W}, \quad s = \frac{c}{2}\left(\frac{2W}{c+2W}\right)^d.$$

Then if

$$N > \frac{d(d-1)}{2}\frac{L(p)}{s},$$

the coefficient of X^θ in $(X_1+\cdots+X_n)^N p$ is nonnegative for $\frac{\theta}{|\theta|} \in \Delta_n(i,r)$.

Proof. For each k, set $g_k := c_k X_i^{l_k} + \phi_k(X)$ and apply Lemma 2 to the g_k's. Let r_k, s_k be the bounds for g_k from Lemma 2, i.e.,

$$r_k = \frac{c_k}{c_k+2W(g_k)}, \quad s = \frac{c_k}{2}\left(\frac{2W(g_k)}{c_k+2W(g_k)}\right)^{l_k}.$$

Then $g_k \geq s_k$ on $\Delta_n(i, r_k)$. Note that the coefficients of the g_k's are a subset of the coefficients of p, hence $W(g_k) \leq W(p)$. Thus we have $c \leq c_k \leq W(g_k) \leq W(p)$, and this together with $l_k \leq d$ implies $r \leq r_k$, and $s \leq s_k$. Since $r \leq r_k$, $\Delta_n(i, r) \subseteq \Delta_n(i, r_k)$ and it follows that, for all k, $g_k \geq s$ on $\Delta_n(i, r)$.

We now want to apply Proposition 4 to p with g_k as above, $h_k := M_k$, $S = T = \Delta_n(i, r)$, and $B = 1$. We must check that S, T, and B satisfy the conditions of Proposition 4. Assume $\alpha, \beta, \gamma \in \mathbb{N}^n$ with $\frac{\alpha}{|\alpha|} \in \Delta_n(i, r), \beta + \gamma = \alpha, \gamma \in \operatorname{supp}(M_k)$, and $|\beta| \geq 1$. Since $\frac{\alpha}{|\alpha|} \in \Delta_n(i, r)$, we have $\frac{\alpha_i}{|\alpha|} \geq 1 - r$. Also, $\gamma \in \operatorname{supp}(M_k)$ implies $\gamma_i = 0$, since M_k is a monomial in $\{X_1 \cdots X_n\} \setminus \{X_i\}$, and hence $\alpha_i = \beta_i$. Since $|\beta| \leq |\alpha|$, it follows that

$$\frac{\beta_i}{|\beta|} \geq \frac{\alpha_i}{|\alpha|} \geq 1 - r.$$

Hence $\frac{\alpha}{|\alpha|} \in \Delta_n(i, r)$ implies $\frac{\beta}{|\beta|} \in \Delta_n(i, r)$. Thus, by Proposition 4, for $\frac{\theta}{|\theta|} \in \Delta_n(i, r)$, the coefficient of X^θ in $(X_1 + \cdots + X_n)^N p$ is nonnegative. $\square$

Let $p \in \mathbb{R}[X]$ and write $p = \sum a_\alpha X^\alpha$, and set:

$$C_0(p) := \min_{\alpha \in \operatorname{supp}(p)} \{|a_\alpha|\}$$

$$C_1(p) := \max_{\alpha \in \operatorname{supp}(p)} \{|a_\alpha|\}$$

Let d be the degree of p. We define the following constants for p:

$$r(p) := \frac{C_0(p)}{C_1(p) + 2W(p)}, \quad s(p) := \frac{C_0(p)}{2} \left(\frac{2W(p)}{C_1(p) + 2W(p)} \right)^d.$$

Corollary 4. *Suppose $p \in P_{n,d}(\Delta_n)$ such that $Z(p) \cap \Delta_n = \{v_i\}$ for some $1 \leq i \leq n$. Suppose further that for every $\beta \in \Lambda^-(p)$ there is an $\alpha \in \Lambda^+(p)$ with $\alpha \preceq_{\overline{\{i\}}} \beta$. Then we can find r, s so that for*

$$N > \frac{d(d-1)}{2} \frac{L(p)}{s},$$

the coefficient of X^θ in $(X_1 + \cdots + X_n)^N p$ is nonnegative whenever $\frac{\theta}{|\theta|} \in \Delta_n(i,r)$ In particular, we can take $r = r(p)$ and $s = s(p)$.

Proof. Suppose in p we have the terms aX^α and $-bX^\beta$ with $\alpha \preceq_{\overline{\{i\}}} \beta$, where $\beta = (\beta_1, \ldots, \beta_n) \in \Lambda^-(p)$ and $\alpha = (\alpha_1, \ldots, \alpha_n) \in \Lambda^+(p)$. This means $\alpha_j \leq \beta_j$ for all $j \neq i$ and $\alpha_j < \beta_j$ for at least one j. Since $|\alpha| = |\beta|$, it follows that $\alpha_i > \beta_i$. Then

$$\begin{aligned}&aX_1^{\alpha_1} \cdots X_n^{\alpha_n} - bX_1^{\beta_1} \cdots X_n^{\beta_n}\\ =&X_1^{\alpha_1} \cdots X_{i-1}^{\alpha_{i-1}} X_{i+1}^{\alpha_{i+1}} \cdots X_n^{\alpha_n}(aX_i^{\alpha_i} - bX_1^{\beta_1-\alpha_1} \cdots X_i^{\beta_i} \cdots X_n^{\beta_n-\alpha_n}).\end{aligned}$$

Since we can do this for every $\beta \in \Lambda^-$, it is clear we can write p in the form

$$p(X) = \sum_{k=1}^{m} M_k \left(c_k X_i^{l_k} + \phi_k(X) \right) + q(X)$$

where for all k, M_k is a monomial in $\{X_1, \ldots, X_n\} \setminus \{X_i\}$, $c_k > 0$, the degree in X_i of ϕ_k is strictly less than l_k, and $q(X)$ is a polynomial with only non-negative coefficients. Hence we can apply Proposition 5. Let c be the constant from Proposition 5. Note that the c_k's are a subset of the coefficients of p, hence $C_0 \leq c$ and $C_1 \geq c$. It follows that

$$r(p) \leq \frac{c}{c + 2W(p)}, \quad s(p) \leq \frac{c}{2} \left(\frac{2W(p)}{c + 2W(p)} \right)^d$$

hence we are done by Proposition 5. $\square$

Remark 1. The constants r and s in Corollary 4 are slightly different from the ones found in [1]. This is because we want a "universal" r and s, i.e., an r and s that will work for all vertices simultaneously. This will be needed in Chapter 5.

Suppose $p \in P_{n,d}(\Delta_n)$ such that $Z(p) \cap \Delta_n \subseteq \{v_1, \ldots, v_n\}$. Our main result in this chapter is a characterization of when such p is in $Po(n,d)$.

The idea of the proof is to break up Δ_n into "corner pieces" $\Delta_n(i,r)$, which contain the zeros, and the rest of Δ_n. Then we apply Proposition 5 to the corner pieces and use Lemma 1 for the remaining piece of Δ_n.

Theorem 2. *Given p as above, then p is in $Po(n,d)$ if and only if the following holds: Given $i \in \{1,\dots,n\}$ with $v_i \in Z(p)$, then for every $\beta \in \Lambda^-(p)$ there is an $\alpha \in \Lambda^+(p)$ with $\alpha \preceq_{\overline{\{i\}}} \beta$.*

Moreover, if $p \in Po(n,d)$, then we can take

$$N > \max\left\{\frac{d(d-1)}{2}\frac{L(p)}{s}, \frac{d(d-1)}{2}\frac{L(p)}{\lambda}\right\}$$

where $r = r(p)$, $s = s(p)$, λ is the minimum of p on $\overline{\Delta_n \setminus \left(\bigcup_{v_i\in Z(p)} \Delta_n(i,r)\right)}$, and d is the degree of p.

Proof. If p is in $Po(n,d)$, the given condition holds by Proposition 3.

Now suppose that the assumption holds. Then by Corollary 4, if $\frac{\theta}{|\theta|} \in \Delta_n(i,r)$ for some i, then the coefficient of X^θ in $(X_1+\cdots+X_n)^N p$ is nonnegative. Note that at each vertex $v_i \in Z(p)$, the bound on N is the same. If $\frac{\theta}{|\theta|} \notin \Delta_n(i,r)$, for all $v_i \in Z(p)$, then $\frac{\theta}{|\theta|} \in \overline{\Delta_n \setminus \left(\bigcup_{v_i\in Z(p)} \Delta_n(i,r)\right)}$ and the coefficient of X^θ is nonnegative by Lemma 1. $\square$

We obtain as a corollary a new proof of the main result in [11]. As in [11], we say that a form p of degree d which is nonnegative on Δ_n has a *simple zero* at v_j if the coefficient of X_j^d in p is zero, but the coefficient of $X_j^{d-1}X_i$ is non-zero (and necessarily positive) for each $i \neq j$.

Corollary 5. *Suppose p is positive on Δ_n except for simple zeros at some v_j's. Then $p \in Po(n,d)$ and there is a bound for the exponent N as in Theorem 2.*

Proof. Given j so that $v_j \in Z(p)$, then p has no terms of the form aX_j^d and necessarily has terms of the form $aX_j^{d-1}X_i$ for all $i \neq j$. Suppose $\beta \in \Lambda^-(p)$

and fix $v_j \in Z(p)$. Pick $i \neq j$ so that $\beta_i \neq 0$ and let α be the exponent of the $X_j^{d-1}X_i$ term. Then $\alpha \in \Lambda^+(p)$ and $\alpha \preceq_{\overline{\{j\}}} \beta$. Hence, we are done by Theorem 2. $\square$

Example 3. This example is from [11], using the bound from Theorem 2. For $0 < \alpha < 1$, let

$$p(x, y, z) := x(y - z)^2 + y(x - z)^2 + z(x - y)^2 + \alpha xyz.$$

Then $p \in P_{3,3}(\Delta_3)$ with zeros at all three vertices.

We start by computing the bound on the corners. In this case, we have $d = 3$, $C_0(p) = \alpha$, $C_1(p) = 2$, and $W(p) = 12 - \alpha$, hence the constants from Corollary 4 are

$$r(p) = \frac{\alpha}{26 - 2\alpha}, \quad s(p) = \frac{\alpha}{2}\left(\frac{24 - 2\alpha}{26 - 2\alpha}\right)^3, \quad L(p) = \max\left\{1, \frac{6 - \alpha}{6}\right\} = 1.$$

Thus

$$\frac{d(d-1)}{2}\frac{L(p)}{s} = \frac{6}{\alpha}\left(\frac{26 - 2\alpha}{24 - 2\alpha}\right)^3.$$

From calculations in [11], we have that the minimum of p on the interior of the closure of Δ_3 minus the three corners is $\frac{\alpha}{27}$. Thus,

$$\frac{d(d-1)}{2}\frac{L(p)}{\lambda} = \frac{81}{\alpha}.$$

Putting this together, if

$$N > \max\left\{\frac{6}{\alpha}\left(\frac{26 - 2\alpha}{24 - 2\alpha}\right)^3, \frac{81}{\alpha}\right\},$$

then $(X_1 + \cdots + X_n)^N p$ has nonnegative coefficients. As α approaches 0, this N behaves like $\frac{1}{\alpha}$. From [11] we have $(X_1 + \cdots + X_n)^N p$ has nonnegative coefficients for $N \geq \frac{18}{\alpha} - 3$, and this bound is sharp if $\frac{18}{\alpha} - 3 \in \mathbb{N}$. Hence, the computed bound has the same order of growth as the true bound, $\frac{18}{\alpha} - 3$.

Chapter 5

Zeros On A Two Dimensional Face

In this chapter, we look at forms $p \in P_{n,d}(\Delta_n)$ with $Z(p) \cap \Delta_n$ a union of two-dimensional faces. We show that in this case, if p satisfies the condition of Proposition 3 with "$\alpha \preceq_{\overline{I}} \beta$" replaced with "$\alpha \prec_{\overline{I}} \beta$," then $p \in Po(n, d)$.

Definition 3. For $r \in \mathbb{R}, 0 < r < 1$ and $I \subseteq \{1, ..., n\}$, let $\Delta(I, r)$ be the set $\{(x_1, \ldots, x_n) \in \Delta_n \mid \sum_{j \notin I} x_j \leq r\} = \{(x_1, \ldots, x_n) \in \Delta_n \mid \sum_{i \in I} x_i \geq 1 - r\}$.

We start with the case where $Z(p) \cap \Delta_n$ consists of one two-dimensional face. For ease of exposition, assume $Z(p) \cap \Delta_n = F(I)$, where $I = \{1, 2\}$. Given $r \in (0, \frac{1}{2})$ and $t \in (0, \frac{r}{2}]$, define

$$U(r, t) = \{(x_1, \ldots, x_n) \in \Delta_n \mid x_1, x_2 \geq r - t, \text{ and } \sum_{i=3}^{n} x_i < t\} \subseteq \Delta(I, t)$$

It is easy to see that $\Delta(I, t) \subseteq U(r, t) \cup \Delta(1, r) \cup \Delta(2, r)$.

To prove our main theorem, we are going to find r and t so that we can apply Proposition 4 to $\Delta(1, r)$, $\Delta(2, r)$, and $U(r, t)$, and then apply Lemma 1 to $\overline{\Delta_n \setminus \Delta(I, t)}$.

Given a form p, let d be the degree of p. Recall that we define

$$r(p) := \frac{C_0(p)}{C_1(p) + 2W(p)}, \quad s(p) := \frac{C_0(p)}{2}\left(\frac{2W(p)}{C_1(p) + 2W(p)}\right)^d,$$

and set

$$t(p) = r(p)\left(\frac{C_0(p)}{C_1(p) + 2W(p)}\right), \quad u(p) = \frac{C_0(p)}{2}\left(\frac{2C_0(p)W(p)}{(C_1(p) + 2W(p))^2}\right)^d.$$

Proposition 6. *Let $\tilde{X} = (X_3, \dots, X_n)$, so that for $\Gamma = (\gamma_3, \dots, \gamma_n) \in \mathbb{N}^{n-2}$, $\tilde{X}^\Gamma$ denotes $X_3^{\gamma_3} \cdots X_n^{\gamma_n}$. Suppose $p \in \mathbb{R}[X]$ is a form which can be written as*

$$\sum_{j=1}^{m} \tilde{X}^{\Gamma_j}\left(c_j X_1^{k_j} X_2^{l_j} + \phi_j(X)\right) + q(X)$$

where for all j, $c_j > 0$ and the degree in $X_1 X_2$ of ϕ_j is strictly less than $k_j + l_j$, and $q(X)$ is a form with only non-negative coefficients. Let $W = W(p)$, $r = r(p)$, $t = t(p)$, and $u = u(p)$. Then for $\theta \in \mathbb{N}^n$ with $\frac{\theta}{|\theta|} \in U(r,t)$ the coefficient of X^θ in $(X_1 + \cdots + X_n)^N p$ is non-negative for

$$N > \frac{d(d-1)}{2}\frac{L(p)}{u}.$$

Proof. For each j, set $g_j := c_j X_1^{k_j} X_2^{l_j} + \phi_j(X)$ and and $h_j := \tilde{X}^{\Gamma_j}$. We are going to apply Proposition 4 with $S = T = U(r,t)$, thus we need a lower bound for the g_j's on $U(r,t)$. Given $x = (x_1, \dots, x_n) \in U(r,t)$, then $x_1, x_2 \geq r - t$ and $x_i \leq t$ for $i = 3, \dots, n$. Fix $1 \leq j \leq n$ and let $X^\beta = X_1^{\beta_1} \dots X_n^{\beta_n}$ be a monomial in $\phi_j(X)$. Then

$$\frac{x_1^{\beta_1} x_2^{\beta_2} \dots x_n^{\beta_n}}{x_1^{k_j} x_2^{l_j}} = \frac{x_3^{\beta_3} \dots x_n^{\beta_n}}{x_1^{k_j - \beta_1} x_2^{l_j - \beta_2}} \leq \frac{t^{\beta_3 + \cdots + \beta_n}}{(r-t)^{(k_j + l_j) - (\beta_1 + \beta_2)}}.$$

Since g_j is a form of degree $k_j + l_j$, $(k_j + l_j) - (\beta_1 + \beta_2) = \beta_3 + \cdots + \beta_n$, hence

$$\frac{t^{\beta_3 + \cdots + \beta_n}}{(r-t)^{(k_j + l_j) - (\beta_1 + \beta_2)}} = \frac{t^{\beta_3 + \cdots + \beta_n}}{(r-t)^{\beta_3 + \cdots + \beta_n}} = \left(\frac{t}{r-t}\right)^{\beta_3 + \cdots + \beta_n}.$$

Since $\dfrac{t}{r-t} < 1$, we have $\left(\dfrac{t}{r-t}\right)^{\beta_3+\cdots+\beta_n} < \left(\dfrac{t}{r-t}\right)$. Hence for each term $M(X) = b_\beta X_1^{\beta_1} X_2^{\beta_2} \dots X_n^{\beta_n}$ in $\phi_j(X)$,

$$\left| \frac{M(x_1, x_2 \dots, x_n)}{x_1^{k_j} x_2^{l_j}} \right| \leq |b_\beta| \left(\frac{t}{r-t}\right).$$

Note that

$$r - t = r\left(1 - \frac{C_0}{C_1 + 2W}\right) = r\left(\frac{C_1 - C_0 + 2W}{C_1 + 2W}\right)$$

and thus,

$$\begin{aligned}
|\phi(x_1, \dots, x_n)/x_1^{k_j} x_2^{l_j}| &\leq \left(\frac{t}{r-t}\right) W \\
&= \left(\frac{C_0}{C_1 + 2W}\right)\left(\frac{C_1 + 2W}{C_1 - C_0 + 2W}\right) W \\
&= \left(\frac{C_0}{C_1 - C_0 + 2W}\right) W \\
&\leq \left(\frac{C_0}{2W}\right) W = \frac{C_0}{2}.
\end{aligned}$$

Since $x_1, x_2 \geq r - t$, and noting that $k_j + l_j \leq d$ and $C_1 \geq C_0$,

$$\begin{aligned}
x_1^{k_j} x_2^{l_j} &\geq (r-t)^{k_j + l_j} \\
&= \left[r\left(\frac{C_1 - C_0 + 2W}{C_1 + 2W}\right)\right]^{k_j + l_j} \\
&\geq \left[r\left(\frac{2W}{C_1 + 2W}\right)\right]^d \\
&= \left[\left(\frac{C_0}{C_1 + 2W}\right)\left(\frac{2W}{C_1 + 2W}\right)\right]^d \\
&= \left(\frac{2C_0 W}{(C_1 + 2W)^2}\right)^d,
\end{aligned}$$

Thus

$$\begin{aligned} g_j(x_1,\ldots,x_n) &= x_1^{k_j}x_2^{l_j}\left(c_j + \phi(x_1,\ldots,x_n)/x_1^{k_j}x_2^{l_j}\right) \\ &\geq \left(\frac{2C_0W}{(C_1+2W)^2}\right)^d \left(C_0 - \frac{C_0}{2}\right) = u. \end{aligned}$$

We now apply Proposition 4 to p with g_j, h_j as above, $S = T = U(r,t)$, and $B = 1$. We must check that S, T, and B satisfy the conditions of Proposition 4. Assume $\alpha, \beta, \gamma \in \mathbb{N}^n$ with $\frac{\alpha}{|\alpha|} \in U(r,t), \beta + \gamma = \alpha, \gamma \in \operatorname{supp}(h_j)$, and $|\beta| \geq 1$. Since $\frac{\alpha}{|\alpha|} \in U(r,t)$, we have $\frac{\alpha_1+\alpha_2}{|\alpha|} \geq 1-t$, and $\frac{\alpha_1}{|\alpha|} \geq r-t$, $\frac{\alpha_2}{|\alpha|} \geq r-t$. Also, $\gamma \in \operatorname{supp}(h_j)$ implies $\gamma_1 = \gamma_2 = 0$, since h_j is a monomial in $\{X_1 \cdots X_n\} \setminus \{X_1, X_2\}$, and hence $\alpha_1 = \beta_1$, $\alpha_2 = \beta_2$. Since $|\beta| \leq |\alpha|$, it follows that

$$\frac{\beta_1+\beta_2}{|\beta|} \geq \frac{\alpha_1+\alpha_2}{|\alpha|} \geq 1-t,$$

$$\frac{\beta_1}{|\beta|} \geq \frac{\alpha_1}{|\alpha|} \geq r-t, \quad \frac{\beta_2}{|\beta|} \geq \frac{\alpha_2}{|\alpha|} \geq r-t.$$

Hence $\frac{\alpha}{|\alpha|} \in U(r,t)$ implies $\frac{\beta}{|\beta|} \in U(r,t)$. Note that the bound from Proposition 4 for each g_j is

$$\frac{(k_j+l_j)((k_j+l_j)-1)}{2}\frac{L(g_j)}{\lambda_j} - (k_j+l_j)$$

where λ_j is the minimum of g_j on $U(r,t)$. Since $u \leq \lambda_j, k_j + l_j \leq d$, and the coefficients of each g_j are a subset of the coefficients of p, we can use the bound

$$\frac{d(d-1)}{2}\frac{L(p)}{u}$$

for each g_j in Lemma 1. Therefore we obtain the bound on N as given. □

Theorem 3. *Given $p \in P_{n,d}(\Delta_n)$ with $Z(p) \cap \Delta_n = F(I)$. Suppose for every $\beta \in \Lambda^-(p)$ there are $\alpha, \gamma, \delta \in \Lambda^+(p)$ such that $\alpha \prec_{\overline{\{1\}}} \beta$, $\gamma \prec_{\overline{\{2\}}} \beta$, $\delta \prec_{\overline{\{1,2\}}} \beta$. Then $p \in Po(n,d)$. In particular, if we let $r = r(p), s = s(p), t = t(p), u = u(p)$, and let λ be the minimum of p on $\overline{\Delta_n \setminus \Delta(I,t)}$, then whenever*

$$N > \max\left\{\frac{d(d-1)}{2}\frac{L(p)}{s}, \frac{d(d-1)}{2}\frac{L(p)}{u}, \frac{d(d-1)}{2}\frac{L(p)}{\lambda}\right\}$$

the coefficients of $(X_1 + ... + X_n)^N p$ are nonnegative.

Proof. We are going to apply Proposition 4 to appropriate closed subsets of the simplex. In particular, we use $\Delta(1,r)$, $\Delta(2,r)$, and $U(r,t)$, and recall that $\Delta(I,t) \subseteq U(r,t) \cup \Delta(1,r) \cup \Delta(2,r)$. We then will apply Lemma 1 to $\overline{\Delta_n \setminus \Delta(I,t)}$.

Let N be as in the statement. Given $\theta \in \mathbb{N}^n$ with $|\theta| = N + d$ consider the X^θ term in $(X_1 + \cdots + X_n)^N p$. If $\frac{\theta}{|\theta|} \in \Delta(1,r) \cup \Delta(2,r)$, the coefficient of X^θ is nonnegative, by Corollary 4.

Suppose $\frac{\theta}{|\theta|} \in U(r,t)$. By assumption, for any $\beta \in \Lambda^-$ there exists an $\delta \in \Lambda^+$ such that $\delta \prec_{\overline{\{1,2\}}} \beta$. As in the vertex case, we can write

$$p = \left(\sum_{j=1}^{m} \tilde{X}^{\Gamma_j}\left(c_j X_1^{k_j} X_2^{l_j} + \phi_j(X)\right)\right) + q(X),$$

where the degree in $X_1 X_2$ of ϕ_j is strictly less than $k_j + l_j$, and $q(X)$ is a polynomial with only non-negative coefficients. Then by Proposition 6, the coefficient of X^θ is nonnegative.

Finally, we note that $p > 0$ on $\overline{\Delta_n \setminus \Delta(I,t)}$ and hence we can apply Lemma 1 in the case where $\frac{\theta}{|\theta|} \in \overline{\Delta_n \setminus \Delta(I,t)}$. Therefore $p \in Po(n,d)$ with the bound on N as given.

□

Remark 2. Theorem 3 with a more complicated bound also follows from [8].

Example 4. Consider the following family of psd polynomials, defined in [12]: For $n \in \mathbb{N}$, and $0 < \epsilon < 1$, let

$$M_n(x_1, \dots, x_n) := x_1^2 \cdot \dots \cdot x_{n-1}^2 \left(\sum_{i=1}^{n-1} x_i^2 \right) + x_n^{2n} - (n - \epsilon) x_1^2 \cdot \dots \cdot x_n^2.$$

Since M_n is psd, it is nonnegative on Δ_n. It is easy to see $Z(M_n) = \bigcup_{1 \leq i \leq j \leq n-1} F(\{i, j\})$.

Let $\beta = (2, \dots, 2)$, then $\Lambda^-(M_n) = \{\beta\}$. Given $I = \{i, j\}$ with $1 \leq i \leq j \leq n - 1$, let α be the exponent of $X_1^2 \cdots X_{n-1}^2 \cdot X_i^2 \in \Lambda^+(M_n)$. Then clearly, $\alpha \preceq_{\overline{\{i,j\}}} \beta$ and $\alpha \prec_{\overline{\{i\}}} \beta$. Thus the conditions of Theorem 3 hold and $M_n \in Po(n, d)$.

Bibliography

[1] M. Castle, V. Powers, B. Reznick, *A Quantitative Pólya's Theorem with Zeros*, J. Symbolic Computation, to appear.

[2] V. de Angelis and S. Tuncel, *Handelman's theorem on polynomials with positive multiples*, in Codes, Systems, and Graphical Models (Minneapolis, MN, 1999), 439–445, IMA Vol. Math. Appl., 123, *Springer, New York,* 2001.

[3] E. de Klerk and D. Pasechnik, *Approximation of the stability number of a graph via copositive programming*, SIAM J. Optimization **1**2 (2002), 875-892.

[4] W. Habicht, *Über die zerlegung strikte definter formen in quadrate*, Comment Math. Helv., **12** (1940), 317-322.

[5] D. Handelman, *Deciding eventual positivity of polynomials*, Ergod. Th. & Dynam. Sys. **6** (1986), 57-79.

[6] D. Handelman, *Representing polynomials by positive linear functions on compact convex polyhedra*, Pac. J. Math. **132** (1988), 35–62.

[7] G.H. Hardy, J.E. Littlewood, and G. Pólya, *Inequalities.* Cambridge University Press, 2nd Edition, 1952.

[8] H.-N. Mok and W.-K. To, *Effective Pólya semi-positivity for non-negative polynomials on the simplex*, J. of Complexity, (2008).

[9] G. Pólya, *Über positive Darstellung von Polynomen Vierteljschr*, Naturforsch. Ges. Zürich **73** (1928 141–145, in *Collected Papers* **2** (1974), MIT Press, 309-313.

[10] V. Powers and B. Reznick, *A new bound for Pólya's Theorem with applications to polynomials positive on polyhedra*, J. Pure Appl. Alg. **1**64 (2001), 221-229.

[11] V. Powers and B. Reznick, *A quantitative Pólya's Theorem with corner zeros*, in Proceedings of the 2006 International Symposium on Symbolic and Algebraic Computations, J.-G. Dumas, editor, New York, NY, ACM Press, pp. 285–290.

[12] B. Reznick, *Forms Derived from the Arithmetic-Geometric Inequality*, Math. Ann. **283** (1989), 431-464.

[13] M. Schweighofer, *An algorithmic approach to Schmüdgen's Positivstellensatz*, J. Pure and Appl. Alg. **1**66 (2002), 307-319.

[14] M. Schweighofer, *Certificates for nonnegativity of polynomials with zeros on compact semialgebraic sets*, Manuscripta Math. **1**17 (2005), 407-428.

[15] M. Schweighofer, *Optimization of polynomials on compact semialgebraic sets*, SIAM J. Optimization, **15** (2005), 805-825.

[16] M. Schweighofer, *On the complexity of schmüdgen's positivstellensatz*, J. Complexity, **20** (2004), 529-543.