Zero Cycles of Degree One on Principal Homogeneous Spaces

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Abstract

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Let k be a field and let G be a connected linear algebraic group over k. Let X be a principal homogeneous space under G over k. Jean-Pierre Serre has asked the following: "If X admits a zero cycle of degree one, does X have a k-rational point?" We give a positive answer to the question in two settings:

1. The field k is of characteristic different from 2 and the group G is simply connected or adjoint and of classical type.

2. The field k is perfect and of virtual cohomological dimension at most 2 and the simply connected group associated to G satisfies a Hasse principle over k.

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Contents

List of Tables

Chapter 1

Introduction

Galois cohomology is a powerful tool for studying a wide variety of interesting questions in pure mathematics. Perhaps the best known open problem in Galois cohomology is Serre's Conjecture II posed in 1962.

Serre's Conjecture II: Every principal homogeneous space under a semisimple simply connected linear algebraic group defined over a perfect field of cohomological dimension at most 2 has a rational point.

In the same year, Serre also posed the following question:

Serre's Question: Does a principal homogeneous space under a connected linear algebraic group defined over a field which admits a zero cycle of degree one have a rational point? (cf. Chapter 6)

Thanks to results of Chernousov [7] and Gille [16, Section III.2] a positive answer to Serre's question for simply connected groups of type E_8 would give a proof of Conjecture II in that case. This remark, given that both the question and the conjecture have been open for nearly five decades gives some indication of how difficult they are to solve. Nonetheless, some progress has been made in special cases. See for example [9, pg 41-56] for a fairly current exposition on the status of Conjecture II. A positive answer to Serre's question when the group is the orthogonal group of a quadratic form or the projective linear group comes as a straightforward consequence of classical results (cf. Section 6.3). Sansuc used the Hasse principle for number fields (cf. Section 5.1) to show that the answer to Serre's question is yes over a number field. Bayer and Lenstra (cf. Section 6.3) have proven a positive answer to Serre's question when the group is the group of isometries of an algebra with involution.

Inspired by the techniques used by Sansuc and Bayer-Lenstra we prove a positive answer to Serre's question in 2 settings:

1. The field is of characteristic different from 2 and the group is semisimple, simply connected or adjoint and of classical type.

2. The field is perfect and of virtual cohomological dimension at most 2 and the simply connected group associated to the group is of classical type, type F_4 or type G_2.

As a consequence of the results on similitudes used in the proof of the first result, we find that if (A, σ) and (A', σ') are central simple algebras with involution of the first kind over k which become isomorphic over an odd degree extension of k, then they are isomorphic over k (cf. Prop 7.11.)

The main tools for the first result are a norm principle for algebraic groups due to Gille and Merkurjev (cf. Section 4.5) and the previously mentioned result of Bayer and Lenstra. The main tool for the second result is a Hasse principle over perfect fields of virtual cohomological dimension at most 2 due to Bayer and Parimala (cf. Section 5.2).

Chapter 2

Galois Cohomology

The main sources for this section are [17] and [35].

2.1 Finite Group Cohomology

2.1.1 Defining the Cohomology Sets

Let R be a ring. An R-module P is said to be *projective* if for every surjective R-module homomorphism $h : M \to N$, the map $\mathrm{Hom}_R(P, M) \to \mathrm{Hom}_R(P, N)$ given by $f \to h \circ f$ is surjective.

Example 2.1. *Any free R-module is projective.*

A *projective resolution* of an R-module B is an exact sequence

$$\cdots \longrightarrow P_2 \xrightarrow{p_2} P_1 \xrightarrow{p_1} P_0 \xrightarrow{p_0} B \longrightarrow 0$$

where each P_i is a projective R-module.

Proposition 2.2. *Any R-module B has a projective resolution.*

Proof. We can construct such a resolution inductively. Define P_0 to be the free R-module composed of a direct sum of copies of R indexed by the elements of B. For b in the index set B, let 1_b denote the identity element in the b-th component. Define

$p_0(1_b) = b$. Then p_o extends to a surjective homomorphism from P_0 to B. Fix an index j and assume we have an exact sequence of projective modules

$$P_{j-1} \xrightarrow{p_{j-1}} \cdots \xrightarrow{p_0} B \longrightarrow 0$$

Define P_j to be the free R-module composed of a direct sum of copies of R indexed by $\ker(p_{j-1})$. As in the 0-th case, the map which sends $1_a \to a$ for each $a \in \ker(p_{j-1})$ induces a homomorphism $p_j : P_j \to P_{j-1}$ whose image is precisely $\ker(p_{j-1})$. □

Let Λ be a finite group. Consider the group ring $\mathbb{Z}[\Lambda]$ and let G be a left $\mathbb{Z}[\Lambda]$-module. For simplicity, we will call such a module a Λ-module. Take P_* a projective resolution of $\mathbb{Z}$ viewed as a $\mathbb{Z}[\Lambda]$-module. Let $\mathrm{Hom}_\Lambda(P_j, G)$ denote the group of Λ-module homomorphisms from P_j to G. Then for each $j \geq 0$, we have a map $d_j : \mathrm{Hom}_\Lambda(P_j, G) \to \mathrm{Hom}_\Lambda(P_{j+1}, G)$ given by $d_j(f) = f \circ p_{j+1}$. Observing the convention that for all $j < 0$, $P_j = \{0\}$ and d_j is the zero map, we obtain a complex:

$$\cdots \longrightarrow \mathrm{Hom}_\Lambda(P_{j-1}, G) \xrightarrow{d_{j-1}} \mathrm{Hom}_\Lambda(P_j, G) \xrightarrow{d_j} \mathrm{Hom}_\Lambda(P_{j+1}, G) \xrightarrow{d_{j+1}} \cdots$$

For $i \geq 0$ we define the *i-th cohomology group of Λ with values in G*, written $H^i(\Lambda, G)$, to be $\ker(d_i)/\operatorname{im}(d_{i-1})$.

Proposition 2.3. *The isomorphism class of $H^i(\Lambda, G)$ is independent of the choice of projective resolution.*

Proof. This is the second statement in Proposition 3.1.9 in [17]. □

Lemma 2.4. $H^0(\Lambda, G) = G^\Lambda$ *the set of elements in G invariant under the action of Λ.*

Proof. Since $\mathrm{Hom}(*, G)$ is a contravariant left-exact functor, and

$$P_1 \to P_0 \to \mathbb{Z} \to 0$$

is an exact sequence, the following sequence is exact

$$0 \longrightarrow \mathrm{Hom}_\Lambda(\mathbb{Z}, G) \longrightarrow \mathrm{Hom}_\Lambda(P_0, G) \xrightarrow{d_0} \mathrm{Hom}_\Lambda(P_1, G)$$

In particular, $\ker(d_0) \cong \mathrm{Hom}_\Lambda(\mathbb{Z}, G)$. The map $f \to f(1)$ gives an isomorphism $\mathrm{Hom}(\mathbb{Z}, G) \cong G^\Lambda$. Since we have assumed that $\mathrm{im}(d_{-1}) = 0$, we conclude that $H^0(\Lambda, G) \cong G^\Lambda$. □

The elements of $\ker(d_i)$ are called *i-cocycles* and those in $\mathrm{im}(d_i)$ are called *i-coboundaries*.

Proposition 2.5. *The set of 1-cocycles is precisely the set of maps $f : \Lambda \to G$ such that for all $s, t \in \Lambda$*

$$f(st) = f(s) + {}^s(f(t))$$

A 1-cocyle f is a 1-coboundary if there exists an element $b \in G$ such that for all $s \in \Lambda$,

$$g(s) = {}^s b - b$$

Proof. This is §3 in Chapter VII of [33]. □

Suppose that G is a non-abelian group with Λ-action. In this setting we can define $H^i(\Lambda, G)$ for $i \leq 1$. Define $H^0(\Lambda, G)$ to be G^Λ. For any Γ-group G, the set of *1-cocycles* of Λ with values in G, denoted $Z^1(\Lambda, G)$ is the set of maps $f : \Lambda \to G$ such that for all $s, t \in \Lambda$,

$$f(st) = f(s) \cdot {}^s(f(t))$$

We define an equivalence relation $\sim$ on $Z^1(\Lambda, G)$ as follows: we write $f \sim g$ if there is an element $b \in G$ such that for all $s \in \Lambda$,

$$g(s) = b^{-1} \cdot f(s) \cdot {}^s b$$

The set of equivalence classes of $Z^1(\Lambda, G)$ under $\sim$ is denoted $H^1(\Lambda, G)$. By 2.5 the set of elements in $H^1(\Lambda, G)$ coincides with the definition in the abelian case. However, while for G abelian $H^1(\Lambda, G)$ is a group, for G non-abelian $H^1(\Lambda, G)$ is a pointed set. The distinguished element in the pointed set $H^1(\Lambda, G)$ is given by the equivalence class of the map $\Lambda \to G$ given by $s \to 1_G$ for all $s \in \Lambda$. We will interchangeably denote this distinguished element in $H^1(\Lambda, G)$ by *point* or 1.

2.1.2 Functoriality

Proposition 2.6. *Let Λ be a finite group.*

1. *Let G and G' be groups with Λ-action. Each Λ-morphism $f : G \to G'$ gives a canonical map*

$$H^i(\Lambda, G) \xrightarrow{f} H^i(\Lambda, G')$$

where if G, G' are not abelian we set $i \leq 1$.

2. *Consider an exact sequence of groups with Λ-action*

$$1 \longrightarrow G_1 \xrightarrow{f_1} G_2 \xrightarrow{f_2} G_3 \longrightarrow 1$$

   (a) *There exits a connecting map $\delta_0 : G_3^\Lambda \to H^1(\Lambda, G_1)$ such that the following sequence is exact*

$$1 \longrightarrow G_1^\Lambda \xrightarrow{f_1} G_2^\Lambda \xrightarrow{f_2} G_3^\Lambda \xrightarrow{\delta_0} H^1(\Lambda, G_1) \xrightarrow{f_1} H^1(\Lambda, G_2) \xrightarrow{f_2} H^1(\Lambda, G_3)$$

   (b) *If G_1 is a central subgroup of G_2, then there is a connecting map $\delta_1 : H^1(\Lambda, G_3) \to H^2(\Lambda, G_1)$ such that the following sequence is exact*

$$1 \longrightarrow G_1^\Lambda \xrightarrow{f_1} G_2^\Lambda \xrightarrow{f_2} G_3^\Lambda \xrightarrow{\delta_0} H^1(\Lambda, G_1) \xrightarrow{f_1} H^1(\Lambda, G_2)$$

$$\xrightarrow{f_2} H^1(\Lambda, G_3) \xrightarrow{\delta_1} H^2(\Lambda, G_1)$$

   (c) *If G_1, G_2 and G_3 are abelian groups then there is a long exact sequence of abelian groups*

$$\cdots \longrightarrow H^{i-1}(\Lambda, G_3) \xrightarrow{\delta_{i-1}} H^i(\Lambda, G_1) \xrightarrow{f_1} H^i(\Lambda, G_2)$$

$$\xrightarrow{f_2} H^i(\Lambda, G_3) \xrightarrow{\delta_i} H^{i+1}(\Lambda, G_1) \longrightarrow \cdots$$

Proof. 1. This is part 2 of Proposition 3.1.9 in [17].

2. (a) This is Proposition 1 on pg 125 of [33].

   (b) This is Proposition 2 on pg 125 of [33].

(c) This is part 3 of Proposition 3.1.9 in [17].

□

Let Λ be a finite group and let G be a group with Λ-action. If Λ' is a subgroup of Λ, there is a restriction map $\mathrm{res} : H^1(\Lambda, G) \to H^1(\Lambda', G)$. If Λ' is a subgroup of finite index and G is abelian, there is a corestriction map $\mathrm{cor} : H^1(\Lambda', G) \to H^1(\Lambda, G)$ (cf Construction 3.3.5, 3.3.6 in [17]).

Proposition 2.7. *Let Λ' be a subgroup of Λ and assume G is an abelian group with Λ-action. The composite map*

$$H^i(\Lambda, G) \xrightarrow{\mathrm{res}} H^i(\Lambda', G) \xrightarrow{\mathrm{cor}} H^i(\Lambda, G)$$

is multiplication by the index of Λ' in Λ.

Proof. This result is Proposition 3.3.7 in [17]. □

Corollary 2.8. *Let G be an abelian group with Λ-action. Then $H^i(\Lambda, G)$ is a torsion group for all i.*

Proof. Consider $\Lambda' = \{0\}$. Since res is the trivial map, the composite map

$$H^i(\Lambda, G) \xrightarrow{\mathrm{res}} H^i(\Lambda', G) \xrightarrow{\mathrm{cor}} H^i(\Lambda, G)$$

is the trivial map. By 2.7, $\mathrm{cor} \circ \mathrm{res}$ is multiplication by the cardinality of Λ. We conclude that every element of $H^i(\Lambda, G)$ has order dividing the cardinality of Λ. □

Corollary 2.9. *Let Λ' be a subgroup of Λ of index n and assume G is an abelian group with Λ-action. If $\gcd(m, n) = 1$, then the restriction map $\mathrm{res} : H^1(\Lambda, G) \to H^1(\Lambda', G)$ is injective on the m-torsion part of $H^1(\Lambda, G)$.*

Proof. Choose λ an m-torsion element in the kernel of res. Then $point = \mathrm{cor}(\mathrm{res}(\lambda)) = n \cdot \lambda$. In particular, m divides n. Since $\gcd(m, n) = 1$ we conclude that $m = 1$ and $\lambda = point$. □

2.2 Profinite Group Cohomology

2.2.1 Definitions

If E/k is a Galois field extension of finite degree, the group $\mathrm{Gal}(E/k)$ is finite and the techniques in the previous section allow us to define $H^i(\mathrm{Gal}(E/k), G)$ for any group G with $\mathrm{Gal}(E/k)$-action. However, we also wish to consider the cohomology theory of Galois groups of field extensions which are not of finite degree. In what follows, we will demonstrate that an infinite Galois group is a subgroup of a product of finite Galois groups.

The first notion we will need to make this characterization explicit is the notion of an inverse system. An *inverse system of sets* consists of:

1. A set $\mathcal{I}$ with partial ordering $\leq$ such that for all $i, j \in \mathcal{I}$, there exists $k \in \mathcal{I}$ such that $i \leq k$ and $j \leq k$.

2. A set G_i corresponding to each $i \in \mathcal{I}$.

3. A map $\rho_{ij} : G_j \to G_i$ whenever $i \leq j$ such that for any $i \in \mathcal{I}$, ρ_{ii} is the identity and for any $i \leq j \leq k$, $\rho_{ik} = \rho_{ij} \circ \rho_{jk}$.

Let (G_i, ρ_{ij}) be an inverse system of sets. The *inverse limit* of (G_i, ρ_{ij}):

$$\varprojlim_i G_i = \{(g_i) \in \prod_{i \in I} G_i : \forall i \leq j,\ \rho_{ij}(g_j) = g_i\}$$

Let k be a field and let L be a Galois extension of k. Consider the set of all intermediate fieds $k \subset E_i \subset L$ which are finite and Galois over k. Let $\mathcal{I}$ be the index set of the E_i. Define a partial ordering on $\mathcal{I}$ by writing $i \leq j$ whenever $E_i \subset E_j$ and take ρ_{ij} to be the natural restriction map $\mathrm{Gal}(E_j/k) \to \mathrm{Gal}(E_i/k)$.

Proposition 2.10. *1.* $(\mathrm{Gal}(E_i/k), \rho_{ij})$ *is an inverse system.*

2. $\mathrm{Gal}(L/k) = \varprojlim_i \mathrm{Gal}(E_i/k)$.

Proof. This is Proposition 4.1.3 in [17]. □

In particular, $\mathrm{Gal}(L/k)$ is an inverse limit of finite groups. Any group which is an inverse limit of finite groups is called a *profinite group*.

To define the cohomology sets of $\mathrm{Gal}(L/k)$ we also need the notion of a direct limit. A *direct system of sets* consists of:

1. A set $\mathcal{I}$ with partial ordering $\leq$ such that for all $i, j \in \mathcal{I}$, there exists $k \in \mathcal{I}$ such that $i \leq k$ and $j \leq k$.

2. A set G_i corresponding to each $i \in \mathcal{I}$.

3. A map $\nu_{ij} : G_i \to G_j$ whenever $i \leq j$ and such that for any $i \in \mathcal{I}$, ν_{ii} is the identity and for any $i \leq j \leq k$, $\nu_{ik} = \nu_{jk} \circ \nu_{ij}$.

Let (G_i, ν_{ij}) be a direct system and let $\bar{G}$ be the disjoint union of the G_i. For $g_i \in G_i$ and $g_j \in G_j$ write $g_i \sim g_j$ whenever there exists and index $k \in \mathcal{I}$ such that $i \leq k$, $j \leq k$ and $\nu_{ik}(g_i) = \nu_{jk}(g_j)$. This gives an equivalence relation on $\bar{G}$. The set of equivalence classes of $\bar{G}$ under $\sim$ is called the *direct limit* of (G_i, ρ_{ij}) and is denoted $\varinjlim_i G_i$.

Let L be a Galois field extension of k and let $\{E_i\}$ be the set of all intermediate Galois extensions $k \subset E_i \subset L$. For all $i \leq j$ the restriction map $\rho_{ij} : \mathrm{Gal}(E_j/k) \to \mathrm{Gal}(E_i/k)$ induces a map $\nu_{ij} : H^l(\mathrm{Gal}(E_i/k), G^{\mathrm{Gal}(L/E_i)}) \to H^l(\mathrm{Gal}(E_j/k), G^{\mathrm{Gal}(L/E_j)})$. Since $(\mathrm{Gal}(E_i/k), \rho_{ij})$ is an inverse system, $(H^l(\mathrm{Gal}(E_i/k), G^{\mathrm{Gal}(L/E_i)}), \nu_{ij})$ is a direct system.

We may regard a group G with $\mathrm{Gal}(L/k)$-action as a topological space with the discrete topology. Take the discrete topology on $\mathrm{Gal}(E_i/k)$, the product topology on $\prod_i \mathrm{Gal}(E_i/k)$ and the subspace topology on $\mathrm{Gal}(L/k)$. The action of $\mathrm{Gal}(L/k)$ on G is said to be *continuous* if the stabilizer of each $g \in G$ is open in $\mathrm{Gal}(L/k)$ or equivalently, if $G = \cup G^U$ where U varies over the open subgroups of $\mathrm{Gal}(L/k)$. For G a group with continuous $\mathrm{Gal}(L/k)$ action, we define

$$H^l(\mathrm{Gal}(L/k), G) := \varinjlim_i H^l(\mathrm{Gal}(E_i/k), G^{\mathrm{Gal}(L/E_i)})$$

More generally, for any profinite group $\Gamma = \varprojlim_i \Gamma/U_i$ and discrete group G with continuous Γ-action, we can define $H^l(\Gamma, G) = \varinjlim_i H^l(\Gamma/U_i, G^{U_i})$. That $H^l(\Gamma, G)$ is functorial in Γ and G follows from the finite case by taking limits.

2.2.2 Virtual Cohomological Dimension

Let k be a field with separable closure k_s and let Γ_k denote $\mathrm{Gal}(k_s/k)$, the absolute Galois group of k. Let p be any prime number. The *p-cohomological dimension* of k is less than or equal to r (written $\mathrm{cd}_p(k) \leq r$) if $H^n(\Gamma_k, A) = 0$ for every p-primary torsion Γ_k-module A and $n > r$. The *cohomological dimension* of k is less than or equal to r (written $\mathrm{cd}(k) \leq r$), if $\mathrm{cd}_p(k) \leq r$ for all primes p. Finally, the *virtual cohomological dimension* of k, written $\mathrm{vcd}(k)$ is precisely the cohomological dimension of $k(\sqrt{-1})$.

Proposition 2.11. *If k is a field of positive characteristic then* $\mathrm{vcd}(k) = \mathrm{cd}(k)$*.*

Proof. This is Theorem 1.1 in [4] and follows from Proposition 14 in §3.3 of Chapter I in [35]. $\square$

2.3 Galois Cohomology of Algebraic Groups

2.3.1 Definitions

An *algebraic group* over k is an algebraic variety G over k with a group structure such that the multiplication $\mu : G \times G \to G$ and inverse $i : G \to G$ are morphisms defined over k.

Example 2.12. *The general linear group GL_n is the group with k-points $GL_n(k) = \{a \in M_n(k) : \det(a) \neq 0\}$.*

An algebraic group is said to be *linear* if it is isomorphic to a closed subgroup of GL_n for some n.

Example 2.13. *The multiplicative group G_m is the group with k-points $G_m(k)$ given by k^*.*

Example 2.14. *The group of n-th roots of unity μ_n has k-points $\mu_n(k) = \{a \in k^* : a^n = 1\}$.*

Example 2.15. *Let A be an algebra over k. Then $GL_1(A)$ has k-points $GL_1(A)(k) = A^*$.*

Example 2.16. *Let L be a finite field extension of k. For an algebraic group G over k we define the Weil restriction of G denoted $R_{L/k}(G)$ to be the group with k-points given by $R_{L/k}(G)(k) = G(L)$.*

If G is an algebraic group over k then $\Gamma_k = \mathrm{Gal}(k_s/k)$ acts on $G(k_s)$. Thus we may define $H^i(\Gamma_k, G(k_s))$ for $0 \leq i \leq 1$. If G is an abelian group we may also define $H^i(\Gamma_k, G(k_s))$ for $i \geq 2$. We will write $H^i(k, G)$ for $H^i(\Gamma_k, G(k_s))$.

2.3.2 Tensors

Let m and n be non-negative integers. An $m \times n$ *tensor* over k consists of a finite dimensional vector space V over k and an element ϕ in $V^{\otimes m} \otimes (V^*)^{\otimes n}$ where V^* is the dual space of V.

Example 2.17. *Let $q = \sum_{i,j=1}^{n} a_{ij} X_i X_j$ be a quadratic form of dimension n over k. Let V be the set of $n \times 1$ column vectors with entries in k. Writing q with symmetric coefficients determines a unique $n \times n$ matrix (a_{ij}) and a quadratic map $Q : V \to k$ defined by $Q(x) = x^t (a_{ij}) x$. In turn, the map $b : V \times V \to k$ given by $b(x, y) = [Q(x + y) - Q(x) - Q(y)]/2$ is a symmetric bilinear form over k. In particular, (V, b) is a 0×2 tensor over k.*

A vector space isomorphism $f : V \to V'$ induces an isomorphism $V^{\otimes m} \to (V')^{\otimes m}$ which we denote by $f^{\otimes m}$. Composition with f^{-1} gives a vector space isomorphism $V^* \to (V')^*$ which we denote by f^*. Two $m \times n$ tensors (V, ϕ) and (V', ϕ') are said to be *isomorphic* if there is a vector space isomorphism f from V to V' such that $f^{\otimes m} \otimes (f^*)^{\otimes n}$ maps ϕ to ϕ'. Let L be a finite field extension of k. A k-tensor (V', ϕ') is called an L/k *twisted form* of a k-tensor (V, ϕ) if $(V \otimes L, \phi \otimes L) \cong (V' \otimes L, \phi' \otimes L)$.

Theorem 2.18. *Let k be a field and let (V, ϕ) be a tensor over k. Let A be the algebraic group whose k-points $A(k) = \mathrm{Aut}_k(V, \phi)$. Let L be a finite Galois extension of k. Then there is a bijection between $H^1(\mathrm{Gal}(L/k), A(L))$ and the set of isomorphism classes of L/k twisted forms of (V, ϕ).*

Proof. This is Proposition 4 in §2 of Chapter X in [33]. □

Example 2.19. *The automorphism group of a quadratic form q is called its orthogonal group written $O(q)$. By Theorem 2.18 above, there is a bijection between $H^1(\mathrm{Gal}(L/k), O(q)(L))$ and the set of isomorphism classes of quadratic forms q' over k such that $q'_L \cong q_L$.*

2.3.3 Fundamental Results

In this section we review some fundamental results in Galois Cohomology of linear algebraic groups which we will need for the work that follows.

Proposition 2.20. *(Hilbert's Theorem 90) Let k be a field and let A be a separable and associative algebra over k. Then $H^1(k, GL_1(A)) = 1$.*

Proof. This is Theorem 29.2 in [23] and is due to Speiser. □

One corollary of Hilbert's Theorem 90 is the following classical formulation.

Corollary 2.21 (Classical Hilbert 90)**.** *If L/k is a cyclic Galois extension of fields with Galois group generated by θ, then any element $\alpha \in L^*$ with $N_{L/k}(\alpha) = 1$ is of the form $\mu^{-1}\theta(\mu)$ for some $\mu \in L^*$.*

Proof. Let $[L : k] = n$. Let $f : \mathrm{Gal}(L/k) \to GL_1(L)$ be the group homomorphism defined by $f(\theta^i) = \prod_{j=0}^{i-1} \theta^j(\alpha)$. For any $0 \leq s, t \leq n$,

$$\begin{aligned} f(\theta^s) \cdot {}^{\theta^s}(f(\theta^t)) &= \left(\prod_{j=0}^{s-1} \theta^j(\alpha)\right)\left(\prod_{j=0}^{t-1} \theta^{s+j}(\alpha)\right) \\ &= \prod_{j=0}^{s+t-1} \theta^j(\alpha) \\ &= f(\theta^{s+t}) \\ &= f(\theta^s\theta^t) \end{aligned}$$

In particular, f is a 1-cocycle of $\mathrm{Gal}(L/k)$ with values in $GL_1(L)$. Let f_0 denote the trivial 1-cocyle given by $f(\theta^i) = 1$ for all i. By 2.20, $f \sim f_0$ and thus there exists $\mu \in L^*$ such that $f(\theta) = \mu^{-1} f_0(\theta)\theta(\mu)$. Since $\alpha = f(\theta)$, we find $\alpha = \mu^{-1}\theta(\mu)$. $\square$

Lemma 2.22 (Kummer)**.** $H^1(k, \mu_n) \cong k^*/(k^*)^n$

Proof. Consider the following short exact sequence.

$$1 \longrightarrow \mu_n \longrightarrow G_m \xrightarrow{n} G_m \longrightarrow 1\ .$$

The corresponding long exact sequence in cohomology begins

$$1 \longrightarrow \mu_n \longrightarrow k^* \xrightarrow{n} k^* \longrightarrow H^1(k, \mu_n) \longrightarrow H^1(k, G_m)$$

But since by 2.20, $H^1(k, G_m) = 1$, the sequence becomes

$$1 \longrightarrow \mu_n \longrightarrow k^* \xrightarrow{n} k^* \longrightarrow H^1(k, \mu_n) \longrightarrow 1$$

from which the desired result is clear. $\square$

Lemma 2.23 (Shapiro's Lemma)**.** *Let G be an algebraic k-group and let L/k be a finite extension of groups. Then $H^i(k, R_{L/k}(G)) \cong H^i(L, G)$.*

Proof. This is Lemma 29.6 in [23]. $\square$

Chapter 3

Algebras with Involution

In this chapter we review some of the theory of central simple algebras and then the theory of algebras with involution. The main sources of its content are [17] and [23].

3.1 Central Simple Algebras

Let K be a field. A *central simple algebra* A over K is an associative algebra A such that:

1. the center of A is K,

2. the only two-sided ideals of A are $\{0\}$ and A.

Example 3.1. *The $n \times n$ matrix algebra: $A = M_n(K)$ is a central simple algebra over K.*

Example 3.2. *Let K be a field of characteristic different from 2. A quaternion algebra over K is an algebra of dimension 4 over K with generators $\{i, j\}$ satisfying the relations $i^2 = a$, $j^2 = b$ and $ij = -ji$ and denoted $A = \left(\frac{a,b}{K}\right)$. A quaternion algebra is a central simple algebra over K.*

A central simple algebra A is said to be *division* if each nonzero $a \in A$ has a two-sided multiplicative inverse. The following characterization is fundamental in the study of central simple algebras.

Proposition 3.3 (Wedderburn's Theorem)**.** *Let A be a central simple algebra over K. Then there is a positive integer n and a division algebra D such that $A \cong M_n(D)$. Furthermore, the division algebra D is uniquely determined up to isomorphism.*

Proof. This is Theorem 2.1.3 in [17]. □

Since a quaternion algebra is of dimension four over K, one obvious consequence of Wedderburn's theorem is that any quaternion algebra is either a division algebra or a matrix algebra. Wedderburn's theorem also allows us to define an important invariant associated to a central simple algebra, namely its index. Let A be a central simple algebra over K and write $A \cong M_n(D)$ for D a division algebra over K. The *index* of A is the square root of the dimension of D over K. A central simple algebra of index 1 is necessarily a matrix algebra.

Let A be any ring and fix a unit $b \in A$. The map $a \to bab^{-1}$ defines an automorphism of A. An automorphism of this form is said to be *inner*.

Theorem 3.4 (Skolem-Noether Theorem)**.** *Let A be a central simple algebra over K. Then any automorphism of A is inner.*

Proof. This is Theorem 2.7.2 in [17]. □

3.2 Splitting Fields

Let K be a field and let L be a finite field extension of K. It is straightforward to verify that if A is a finite dimensional algebra over K such that $A \otimes_K L$ is central simple over L, then A is central simple over K. Since the converse is also true, we have the following:

Proposition 3.5. *Let A be a finite dimensional algebra over K and let L be a finite field extension of K. Then $A \otimes L$ is a central simple algebra over L if and only if A is a central simple algebra over K.*

Proof. This is Lemma 2.2.2 in [17]. □

It is clear that the index of $A \otimes L$ over L is less than or equal to the index of A over K. Furthermore, by the proposition which follows, any central simple algebra becomes index 1 over a finite field extension of K. A field extension L of K such that $A \otimes L \cong M_n(L)$ is called a *splitting field* of A.

Proposition 3.6 (Existence of Separable Splitting Fields)**.** *Let K be a field and let A be a central simple algebra over K. Then there exists a finite separable field extension L of K such that $A \otimes L \cong M_n(L)$.*

Proof. This is Theorem 2.2.5 in [17]. □

Proposition 3.7. *Let A be a central simple algebra over K.*

1. *A has a splitting field of degree equal to the index of A.*

2. *Any splitting field of A has degree divisible by the index of A.*

Proof.
1. This is Proposition 4.5.4 in [17].

2. This is Theorem 4.8 on page 221 of [20].

□

Splitting fields allow us to extend notions like the determinant and the trace of a matrix to a more general central simple algebra. Recall, that for any matrix $B \in M_n(L)$ we can define the *characteristic polynomial of B*, $P_B(X) := \det(XI - B)$ where I is the $n \times n$ identity matrix.

Proposition 3.8. *Let A be a central simple algebra over K and let L be a splitting field of A. Let $i : A \to A \otimes L$ be the natural inclusion map and fix an isomorphism $\phi : A \otimes L \cong M_n(L)$. For any $a \in A$ the characteristic polynomial of $\phi(i(a))$ satisfies the following properties:*

1. *$P_{\phi(i(a))}(X)$ does not depend on the choice of isomorphism ϕ. In particular we may fix any isomorphism ϕ and define $P_{i(a)}(X)$ as $P_{\phi(i(a))}(X)$.*

2. *$P_{i(a)}(X)$ is independent of the choice of the splitting field L and has coefficients in K.*

Proof. 1. Choose ϕ and ϕ' two isomorphisms $A \otimes L \to M_n(L)$. Then $\phi' \circ \phi^{-1}$ gives an automorphism of $M_n(L)$. Then by 3.4, $\phi' \circ \phi^{-1}$ is conjugation by b for some $b \in GL_n(L)$. In particular, $\phi(i(a))$ and $\phi'(i(a))$ are similar matrices and similar matrices have the same characteristic polynomial.

2. This is Lemma 5.7 in [32].

□

For $a \in A$, take the characteristic polynomial of a: $P_{i(a)}(X) = \sum_{i=0}^{n} \alpha_i X^i$. The *reduced norm* of a: $\mathrm{Nrd}(a) = (-1)^n \alpha_0$ and the *reduced trace* of a: $\mathrm{Trd}(a) = -\alpha_{n-1}$.

Let L/K be a finite extension of fields and let $G_{L/K}$ denote the group of embeddings of L into $\bar{K}$. We define the field norm $N_{L/K} : L \to K$ by

$$N_{L/K}(a) = \prod_{\sigma \in G_{L/K}} \sigma(a)$$

Proposition 3.9. *Let A be a central simple algebra over K and let L be a finite field extension of K. Let $N_{L/K}$ denote the field norm and let* Nrd *denote the reduced norm. Then $N_{L/K}(\mathrm{Nrd}(A \otimes L)) \subseteq \mathrm{Nrd}(A)$.*

Proof. This is Corollary 2.3 in [4]. □

To conclude this section we give a classification of central simple algebras by a Galois cohomology set.

Proposition 3.10. *Let K be a field with finite Galois extension L. There is a bijection between $H^1(\mathrm{Gal}(L/K), PGL_n(L))$ and the set of isomorphism classes of central simple algebras over K of dimension n split by L.*

Proof. Write V for $M_n(K)$ viewed as a vector space over K. Multiplication in $M_n(K)$ gives an element $\phi \in \mathrm{Hom}_K(V \otimes_K V, V)$ defined by $\phi(v \otimes w) = vw$.

$$\begin{aligned}
\mathrm{Hom}_K(V \otimes_K V, V) &\cong \mathrm{Hom}_K(V \otimes_K V, K) \otimes_K V \\
&\cong \mathrm{Hom}_K(V, K) \otimes_K \mathrm{Hom}_K(V, K) \otimes_K V \\
&\cong V^{*\otimes 2} \otimes_K V \\
&\cong V \otimes_K V^{*\otimes 2}
\end{aligned}$$

In particular, $M_n(K)$ determines an element in $V \otimes_K V^{*2}$, that is, a tensor of type 1×2 over K. Consider $\mathrm{Int} : GL_n(K) \to \mathrm{Aut}_K(M_n(K))$ given by $\mathrm{Int}(b)a = bab^{-1}$ for all $a \in M_n(K)$. The kernel of Int is the set of scalar matrices and 3.4 gives that Int is onto. In particular $\mathrm{Aut}_K(M_n(K)) \cong PGL_n(K)$. By 3.5 above, the L/K twisted forms of $M_n(K)$ are precisely the central simple algebras of dimension n over K split by L. The desired result now follows from 2.18. □

3.3 The Brauer group

We define an equivalence relation $\sim$ on the set of central simple algebras over K as follows: Two central simple algebras A and B are said to be *Brauer equivalent* if there exist positive integers m and n such that $A \otimes_K M_m(K) \cong B \otimes_K M_n(K)$.

Proposition 3.11. *Let K be a field. Let S be the set of Brauer equivalence classes of central simple algebras over K. Then the set S with multiplication given by the tensor product is an abelian group. This group is called the Brauer group of K and is denoted* $\mathrm{Br}(K)$.

Proof. We proceed by verifying the standard group axioms.

Closure This is Theorem 3.2 in Chapter 8 of [32].

Abelian property Since $A \otimes B \cong B \otimes A$, then $A \otimes B \sim B \otimes A$. In particular, the tensor product is abelian on S.

Associativity It is a standard exercise in tensor products to verify that $(A \otimes B) \otimes C \cong A \otimes (B \otimes C)$.

Existence of identity For any positive integer n and central simple algebra A over K, $M_n(k) \otimes A \sim A$. In particular $[M_n(K)]$ is the identity element in $\mathrm{Br}(K)$.

Existence of inverse Let A^o denote the opposite algebra of A. Choose $x \otimes y \in A \otimes A^o$. The map which sends any a in A to xay gives a K-endomorphism of A. This extends to a homorphism $\phi : A \otimes A^o \to \mathrm{End}_K(A)$. Since $A \otimes A^o$

is simple, ϕ has trivial kernel. Since $\dim_K(A \otimes A^o) = \dim_K(\mathrm{End}_k(A))$, ϕ is surjective. Thus $A \otimes A^o \cong \mathrm{End}_K(A)$. Since $\mathrm{End}_K(A)$ can be identified with a matrix algebra, we conclude that the inverse of $[A]$ is $[A^o]$.

□

We conclude with a cohomological description of the Brauer group.

Lemma 3.12 (The Cohomological Brauer Group)**.** *$H^2(K, G_m)$ is isomorphic $\mathrm{Br}(K)$ and $H^2(K, \mu_n)$ is isomorphic to the n-torsion subgroup of $\mathrm{Br}(K)$.*

Proof. This is Theorem 4.4.7 and Corollary 4.4.9 in [17]. □

Since by 2.8, $H^2(K, G_m)$ is a torsion group, we find that $Br(K)$ is a torsion group. Let A be a central simple algebra over K. The order of $[A]$ in $Br(K)$ is called the *period* of A or the *exponent* of A.

3.4 Involutions

Let K be a field and let A be a central simple algebra over K. An *involution* σ on A is a map $\sigma : A \to A$ such that for all $a, b \in A$:

1. $\sigma(a + b) = \sigma(a) + \sigma(b)$
2. $\sigma(ab) = \sigma(b) \cdot \sigma(a)$
3. $\sigma^2(a) = a$

Example 3.13. *If $A = M_n(K)$ then the map which sends each matrix to its transpose is an involution on A.*

Example 3.14. *Let K be a field of characteristic different from 2. Let $A = (\frac{a,b}{K})$ be a quaternion algebra over K. The map which sends any element $a + bi + cj + dk \in A$ to $a - bi - cj - dk$ is an involution on A.*

Let K be a field and let A be a central simple algebra over K with an involution σ. It is clear that if a is central in A, then $\sigma(a)$ in central in A. In particular, $\sigma(K) \subset K$. However, an involution need not fix all the elements of K.

Lemma 3.15. *Let K^σ denote the set of elements in K fixed by σ. Then K^σ is a subfield of K and $[K : K^\sigma] \leq 2$.*

Proof. By the properties included in the definition of an involution, K^σ is closed under addition, multiplication and inverses. If $K^\sigma \neq K$ then σ is an automorphism of K of period 2. From whence it is clear that $[K : K^\sigma] = |\langle\sigma\rangle| = 2$. □

Let $k = K^\sigma$. If $k \neq K$ we call σ an *involution of the second kind* or an *involution of unitary type.* If $k = K$ we call σ an *involution of the first kind.* Let σ be an involution of the first kind on A. For any field extension E of k, $\sigma_E = \sigma \otimes id_E$ is an involution on $A \otimes E$. In particular, $\sigma_{\bar{k}}$ is an involution on $M_n(\bar{k})$. If $\sigma_{\bar{k}}$ is isomorphic to the transpose map we say that σ is an *orthogonal* involution. An involution of the first kind which is not orthogonal is called a *symplectic* involution.

Proposition 3.16 (Existence of Involutions)**.**

1. *Let A be a central simple algebra over a field K. A admits an involution of the first kind if and only if $A \otimes A$ is split.*

2. *Let K/k be a quadratic field extension and let A be a central simple algebra over K. Then there is an involution of the second kind of A with $K^\sigma = k$ if and only if the corestriction algebra $N_{K/k}(A)$ splits.*

Proof. This is Theorem 3.1 in [23]. □

3.5 Hermitian Forms

The main source for this section was [32].

Let A be a central simple algebra over K with an involution σ and let $K^\sigma = k$. Let ϵ be 1 or -1. An *ϵ-hermitian form* on a right A-module V is a map $h : V \times V \to A$ such that for all $v, w \in V$ and $a, b \in A$:

1. $h(va, wb) = \sigma(a)h(v, w)b$

2. $h(v, w) = \epsilon\sigma(h(w, v))$

3. h is bi-additive

We will often omit the factor ϵ and refer simply to a *hermitian form* (V, h). We will assume that hermitian forms satisfy a *non-degeneracy condition*, that is to say, for all $v \in V - \{0\}$ there is a $w \in V - \{0\}$ such that $h(v, w) \neq 0$.

The *orthogonal sum* of two ϵ-hermitian forms (V, h) and (V', h') over (A, σ) written $(V, h) \perp (V', h')$ is defined as $(V \times V', h \perp h')$ where $(h \perp h')((v, v'), (w, w')) = h(v, w) + h'(v', w')$. Two ϵ-hermitian forms (V, h) and (V', h') over (A, σ) are said to be *isomorphic* if there is a vector space isomorphism $f : V \to V'$ such that $h' \circ (f \times f) = h$. Let $\mathcal{H}^\epsilon(A, \sigma)$ denote the set of isomorphism classes of ϵ-hermitian forms over (A, σ). Then $\mathcal{H}^\epsilon(A, \sigma)$ is a commutative semigroup under $\perp$. We denote the Grothendieck group of $\mathcal{H}^\epsilon(A, \sigma)$ by $\mathcal{G}r^\epsilon(A, \sigma)$. An ϵ-hermitian form (V, h) is said to be *metabolic* if there exists a subspace W of V such that for all $w \in W$, the set of all $v \in V$ such that $h(w, v) = 0$ is precisely W. Let the set of ϵ-metabolic forms on (A, σ) be denoted by $\mathcal{M}^\epsilon(A, \sigma)$. Then $\mathcal{M}^\epsilon(A, \sigma)$ is a subgroup of $\mathcal{G}r^\epsilon(A, \sigma)$ and we define the *Witt group of ϵ-hermitian forms over* (A, σ) written $W^\epsilon(A, \sigma)$ as the quotient of $Gr^\epsilon(A, \sigma)$ by $\mathcal{M}^\epsilon(A, \sigma)$. In the case $A = K$ and $\epsilon = 1$, $W(A, \sigma) = W(k)$ the Witt group of quadratic forms over K. If (V, h) is an ϵ-hermitian form over (A, σ) and (M, b) is a quadratic form over K then $(M \otimes V, b \otimes h)$ defines an ϵ-hermitian form over A. Thus the Witt group of ϵ-hermitian forms over (A, σ), can be regarded as a left $W(K)$-module.

Let A be a simple algebra over k. Let L be a finite extension of k and let $s : L \to k$ be any k-linear map. Then s induces a map $A \otimes L \to A$ and in turn a group homomorphism $s_* : W^\epsilon(A_L, \sigma_L) \to W^\epsilon(A, \sigma)$. Let $r^* : W^\epsilon(A, \sigma) \to W^\epsilon(A_L, \sigma_L)$ denote the standard extension of scalars $(V, h) \to (V_L, h_L)$.

Proposition 3.17 (Frobenius Reciprocity)**.** *Let k be a field and let A be a simple algebra over k with k-linear involution σ. Let (V, h) be an ϵ-hermitian form over (A, σ). Let L be a finite field extension of k. Let M be a finite dimensional vector space over L and let $b : M \times M \to L$ be a quadratic form. Then as elements in*

$W^\epsilon(A,\sigma)$

$$s_*(b \otimes_L r^*(h)) = s_*(b) \otimes_k h$$

Proof. Choose $(m \otimes \lambda \otimes v, m' \otimes \lambda' \otimes v') \in M \otimes_L (L \otimes_k V)$.

$$\begin{aligned} s_*(b \otimes r^*(h))(m \otimes \lambda \otimes v, m' \otimes \lambda' \otimes v') &= s((b \otimes r^*(h))(m \otimes \lambda \otimes v, m' \otimes \lambda' \otimes v')) \\ &= s(b \otimes h)(m\lambda \otimes v, m'\lambda' \otimes v') \\ &= s(b(m\lambda, m'\lambda')h(v,v')) \\ &= s(b(m\lambda, m'\lambda'))h(v,v') \\ &= (s_*(b) \otimes h)(m \otimes \lambda \otimes v, m' \otimes \lambda' \otimes v') \end{aligned}$$

□

Let a be an algebraic element over k and let $k(a) = L$. Consider the k-linear map $s : k(a) \to k$ defined by $s(1) = 1$ and $s(a^j) = 0$ for all $1 \leq j < m$ where $m = [L : k]$. We refer to the induced homomorphism $s_* : W(A_L, \sigma_L) \to W(A, \sigma)$ as *Scharlau's transfer homomorphism.*

Proposition 3.18. *Let $L = k(a)$ be a simple field extension of k of odd degree. Let $s_* : W(L) \to W(k)$ be Scharlau's transfer homomorphism. Then*

$$s_*(\langle 1 \rangle) = \langle 1 \rangle$$

and

$$s_*(\langle a \rangle) = \left\langle N_{k(a)/k(a)} \right\rangle$$

Proof. This is Lemma 5.8 in Chapter 2 of [32]. □

Proposition 3.19. *If L is a finite field extension of k of odd degree then*

$$r^* : W^\epsilon(A,\sigma) \to W^\epsilon(A_L, \sigma_L)$$

is injective.

Proof. This proof is due to Bayer and Lenstra. (cf. Proposition 1.2 in [2]).

Let h be an ϵ-hermitian form over (A, σ) and take $\langle 1\rangle \otimes r^*(h) \in W^\epsilon(A_L, \sigma_L)$. Let s_* be Scharlau's transfer homomorphism. By 3.17, $s_*(\langle 1\rangle \otimes r^*(h)) = s_*(\langle 1\rangle) \otimes h$. Since $[L : k]$ is odd, by 3.18 $s_*(\langle 1\rangle) \otimes h = \langle 1\rangle \otimes h$. Thus $s_*(r^*(h)) = h$ and r^* is injective. □

We can define an involution associated to any ϵ-hermitian form. Let h be an ϵ-hermitian form over an algebra with involution (A, σ). We define the *adjoint involution* τ_h on $\mathrm{End}_A(V)$ to be the involution such that:

1. $\tau_h(a) = \sigma(a)$ for all $a \in K$, and
2. for all $v, w \in V$ and $f \in \mathrm{End}_k(V)$, $h(v, f(w)) = h(\tau_h(f)(v), w)$

Proposition 3.20. *Let k be a field. Let K be a quadratic field extension of k and let A be a central simple algebra over K. Let σ_1 be an involution of the first kind on A and let σ_2 be an involution of the second kind on A such that $K^{\sigma_2} = k$. The map $h \to \tau_h$ gives a bijective correspondence between:*

1. *ϵ-hermitian forms (V, h) over (A, σ_1) up to a factor in K^* and involutions of the first kind on $\mathrm{End}_A(V)$.*
2. *1-hermitian forms (V, h) over (A, σ_2) up to a factor in k^* and involutions of the second kind on $\mathrm{End}_A(V)$ whose restriction to K is σ_2.*

Proof. This is Theorem 4.2 in [23]. □

By 3.3, the result 3.20 admits the following corollary:

Corollary 3.21. *Let A be a central simple algebra over K with an involution σ. Then there is a division algebra D with an involution θ and an ϵ-hermitian form (V, h) over (D, θ) such that $(A, \sigma) = (End_D(V), \tau_h)$.*

3.6 Groups Associated to an Algebra with Involution

Let A be a central simple algebra over K with involution σ. The group of *similitudes* of (A, σ) is the algebraic group whose K-points is the set of all $a \in A$ such that $\sigma(a)a$ is in K^*. This element $\sigma(a)a$ is called the *multiplier* of a written $\mu(a)$. We denote the group of similitudes of (A, σ) by $GO(A, \sigma)$ if σ is of orthogonal type, $GSp(A, \sigma)$ if σ is of symplectic type and $GU(A, \sigma)$ if σ is of unitary type. Let the quotients of these groups by their centers be denoted by $PGO(A, \sigma)$, $PGSp(A, \sigma)$ and $PGU(A, \sigma)$ respectively, and let them be referred to as the group of *projective similitudes* of (A, σ) in each case. The group of similitudes with multiplier 1 is called the group of *isometries* of (A, σ) and is denoted $O(A, \sigma)$, $Sp(A, \sigma)$ and $U(A, \sigma)$ in the cases σ orthogonal, symplectic and unitary respectively. Let $SU(A, \sigma)$ be the elements in $U(A, \sigma)$ with trivial reduced norm.

For σ an orthogonal involution on a central simple K-algebra A of even degree, let $GO^+(A, \sigma)$ denote the set of elements a in $GO(A, \sigma)$ such that $\mathrm{Nrd}(a) = \mu(a)^{\deg(A)/2}$ and $PGO^+(A, \sigma)$ be the quotient of $GO^+(A, \sigma)$ by its center. Let $GO^-(A, \sigma)$ be the coset of $GO^+(A, \sigma)$ in $GO(A, \sigma)$ consisting of elements a such that $\mathrm{Nrd}(a) = -\mu(a)^{\deg(A)/2}$. We will call elements of $GO^+(A, \sigma)$ *proper similitudes* and those of $GO^-(A, \sigma)$ *improper similitudes*. The intersection of $GO^+(A, \sigma)$ with $O(A, \sigma)$ is denoted by $O^+(A, \sigma)$.

Let A be a central simple algebra of even degree with orthogonal involution σ. Construct the tensor algebra of A: $T(A) = \bigoplus_{m \geq 0} T^m(A)$ where $T^0(A) = k$ and for $m \geq 1$, $T^m(A) = A^{\otimes m}$. Let $\bar{\sigma}$ be the involution on $T(A)$ such that for all m, $\bar{\sigma}(a_1 \otimes \cdots \otimes a_m) = \sigma(a_1) \otimes \cdots \otimes \sigma(a_m)$. Let J_1 be the ideal in $T(A)$ generated by the elements of the form $s - \frac{1}{2}\mathrm{Trd}(s)$ for $s \in A$ such that $\sigma(s) = s$. Let J_2 be the ideal in $T(A)$ generated by all elements of the form $a \otimes b - \frac{1}{2}ab$ for $a, b \in A$ such that $axb = a\sigma(x)b$ for all $x \in A$. The *Clifford algebra* of (A, σ):

$$C(A, \sigma) = \frac{T(A)}{J_1 + J_2}$$

Let $T_+(A) = \bigoplus_{m\geq 1} T^m(A)$ with left $T(A)$-action denoted by $*$ and right $T(A)$-action denoted by $\cdot$. The *Clifford bimodule* of (A, σ):

$$B(A,\sigma) = \frac{T_+(A)}{J_1 * T_+(A) + T_+(A) \cdot J_1}$$

The natural inclusion of A in $T_+(A)$ induces an injection $b : A \to B(A, \sigma)$. The *Clifford group* of (A, σ) written $\Gamma(A, \sigma)$ is the group whose K-points is the set of all $c \in C(A, \sigma)$ such that $c^{-1} * b(A) \cdot c \in b(A)$. Define the *spinor norm* homomorphism, $sn : \Gamma(A, \sigma) \to G_m$ by sending c to $\bar{\sigma}(c)c$. Let $\mathrm{Spin}(A, \sigma)$ denote the kernel of the spinor norm.

Chapter 4

Linear Algebraic Groups

This chapter was informed by [5] and [23]. In it we expand on the introduction to algebraic groups given in Chapter 2. Recall that an *algebraic group* over a field k is an algebraic variety G over k with a group structure such that the multiplication $\mu : G \times G \to G$ and inverse $i : G \to G$ are morphisms defined over k. An algebraic group is said to be *linear* if it is isomorphic to a closed subgroup of GL_n for some n and *affine* if its underlying algebraic variety is affine. That is to say, as a variety $G = \mathrm{Spec}(A)$ for A a reduced, finitely generated algebra over k.

Proposition 4.1. *An algebraic group is linear if and only if it is affine.*

Proof. That any affine algebraic group is isomorphic to a closed subgroup of GL_n is Proposition 1.10 in [5]. The reverse implication is trivial. Since GL_n is given by the non-vanishing of the determinant, it is a closed subset of $M_n = \mathbb{A}^{n^2}$ in the Zariski topology. In particular, it is an affine variety and therefore any closed subset of GL_n is an affine variety. $\square$

We will be particularly interested in connected linear algebraic groups. An algebraic group is said to be *connected* if its underlying algebraic variety is connected. Of particular interest among connected linear algebraic groups are semisimple groups.

4.1 Semisimple Groups

A connected linear algebraic group G is called *semisimple* if it has no nonzero, connected, solvable, normal subgroups. A surjective morphism of algebraic groups with finite kernel is called an *isogeny* of algebraic groups. An isogeny is called *central* if its kernel is central. A semisimple group G is said to be *simply connected* if every central isogeny $G' \to G$ is an isomorphism and *adjoint* if every central isogeny $G \to G'$ is an isomorphism.

Proposition 4.2. *Let G be a semisimple k-group.*

1. *There is a unique simply connected group $\tilde{G}$ such that there is a central isogeny $\tilde{\pi} : \tilde{G} \to G$*

2. *There is a unique adjoint group $\bar{G}$ such that there is a central isogeny $\bar{\pi} : G \to \bar{G}$*

Proof. This is Theorem 26.7 in [23]. □

We refer to $\tilde{G}$ as the *simply connected cover* of G and the kernel of $\tilde{\pi}$ as the *fundamental group* of G.

A semisimple group G is said to be *absolutely simple* if G_{k_s} has no nonzero connected normal subgroups.

Proposition 4.3. *Any simply connected (resp. adjoint) semisimple group is a product of groups of the form $R_{E_j/k}G_j$ where each E_j is a finite separable field extension of k and each group G_j is a simply connected (resp. adjoint) absolutely simple group*

Proof. This is Theorem 26.8 in [23]. □

4.2 Classification of Absolutely Simple Semisimple Groups

Any absolutely simple group is of type A_n, B_n, C_n, D_n, E_6, E_7, E_8, F_4 or G_2 [23, §25-26]. An absolutely simple group which is of type A_n, B_n, C_n or D_n but not of type trialitarian D_4 is said to be a *classical group*. All other simple groups are called *exceptional groups*.

4.2.1 Classical Groups

Let k be a field of characteristic different from 2. An absolutely simple, simply connected, classical k-group G has one of the following forms [23, 26.A.], [3]:

1. Type ^1A_n

   $G = SL_1(A)$ for a central simple algebra A over k.

2. The unitary case (Type ^2A_n)

   $G = SU(A, \sigma)$ for A a central simple algebra of degree at least 2 over a field K and σ a unitary involution on A with $K^\sigma = k$.

3. The symplectic case (Type C_n)

   $G = Sp(A, \sigma)$ for A a central simple algebra over k of even degree and σ a symplectic involution on A.

4. The orthogonal case (Type B_n, D_n)

   $G = \mathrm{Spin}(A, \sigma)$ for A a central simple algebra over k of degree at least 3 and σ an orthogonal involution on A.

Let k be field of characteristic different from 2. An absolutely simple, adjoint, classical k-group G has one of the following forms: [23, 26.A.]:

1. Type ^1A_n

   $G = PGL_1(A)$ for A a central simple algebra over k.

2. The unitary case (Type ^2A_n)

   $G = PGU(A, \sigma)$ for A a central simple algebra of degree at least 2 over a field K and σ a unitary involution on A with $K^\sigma = k$.

3. The symplectic case (Type C_n)

   $G = PGSp(A, \sigma)$ for A a central simple algebra over k of even degree and σ a symplectic involution on A.

4. <u>The orthogonal case</u>

   (a) <u>Type B_n</u>

   $G = O^+(A, \sigma)$ for A a central simple algebra over k of odd degree at least 3 and σ an orthogonal involution on A.

   (b) <u>Type D_n</u>

   $G = PGO^+(A, \sigma)$ for A a central simple algebra over k of even degree at least 4 and σ an orthogonal involution on A.

4.2.2 Groups of Type G_2

Let k be a field of characteristic different from 2. Let $A = \left(\frac{a,b}{k}\right)$ be a quaternion algebra over k with involution θ which sends any element $a + bi + cj + dk$ in A to $a - bi - cj - dk$. Let λ be any element of k^*. Let v be a symbol. Then $A \oplus vA$ is a vector space over k. Define multiplication on $A \oplus vA$ by

$$(x + vy) \cdot (a + vb) = xa + \lambda b\theta(y) + v(\theta(x)b + ay)$$

An algebra of the form $(A \oplus vA, \cdot)$ is called a *Cayley algebra* over k.

Proposition 4.4. *Let k be a field of characteristic different from 2. Let G be a simple group of type G_2 over k. Then $G = \mathrm{Aut}(C)$ for some Cayley algebra C over k.*

Proof. This is Theorem 26.19 in [23]. $\square$

The map which sends $a + vb \in A \oplus vA$ to $\theta(a) - vb$ gives an involution on $A \oplus A$ which we denote by θ. We can associate to any element x in a Cayley algebra C its norm $q_C(x) = x \cdot \theta(x)$.

Proposition 4.5. *For any Cayley algebra C, q_C is a 3-fold Pfister form over k. We call q_C the norm form of C.*

Proof. This is part of Proposition 33.18 in [23]. $\square$

Proposition 4.6. *Let k be a field of characteristic different from 2 and let C and C' be Cayley algebras over k with norm forms q_C and q_C' respectively. Then $C \cong C'$ if and only if $q_C \cong q_C'$*

Proof. This is Proposition 33.19 in [23]. □

4.2.3 Groups of Type F_4

This section was informed by [23] and [34, §9].

Let k be a field of characteristic different from 2. A *Jordan algebra* J over k is a finite dimensional commutative k-algebra such that multiplication in J satisfies the following condition: for all $x, y \in J$, $((x \cdot x) \cdot y) \cdot x = (x \cdot x) \cdot (y \cdot x)$.

Example 4.7. *Let B be any associative algebra over k and let B^+ be the elements of B with multiplication given by $x \cdot y = (xy + yx)/2$. Then B^+ is a Jordan algebra over k.*

A Jordan algebra J which is not isomorphic to a sub-algebra of an algebra of the form B^+ is said to be *exceptional*.

Example 4.8. *Let C be a Cayley algebra over k. For $\alpha \in k^3$, define*

$$\mathcal{H}_3(C, \alpha) = \{X \in M_3(C) : \alpha^{-1} \bar{X}^t \alpha = X\}$$

where $\bar{X}$ denotes conjugation on the entries of X. Then $\mathcal{H}_3(C, \alpha)$ is an exceptional Jordan algebra over k.

Proposition 4.9. *Let k be a field of characteristic different from 2 and let G be a simple group of type F_4. Then $G = \mathrm{Aut}(J)$ for some exceptional Jordan algebra J of dimension 27 over k.*

Proof. This is Theorem 26.18 in [23]. □

Proposition 4.10. *Any central simple exceptional Jordan algebra over k is a twisted form of an algebra of the form $\mathcal{H}_3(C, \alpha)$ for a Cayley algebra C over k.*

Proof. cf. Theorem 17 in [1] and pg 516 in [23]. □

An exceptional Jordan algebra of dimension 27 J which is of the form $\mathcal{H}_3(C, \alpha)$ is said to be *reduced*.

We can associate to a Jordan algebra J its trace form T_J which is a quadratic form of the form $\langle 1,1,1\rangle \perp b_{q_C} \otimes \langle -c,-d,cd\rangle$ where $c,d \in k^*$ and q_C is the norm form of the underlying Cayley algebra C [23, pg 516].

Proposition 4.11. *Let k be a field of characteristic different from 2. Let J, J' be reduced exceptional Jordan algebras of dimension 27 with trace forms $T_J, T_{J'}$ respectively. Then $J \cong J'$ if and only if $T_J \cong T_{J'}$.*

Proof. This is due to Springer and is pg 421 Theorem 1 in [36]. See also Theorem 5.8.1 in [37]. □

Proposition 4.12. *Let T_J be the trace form of an exceptional Jordan algebra J. The isomorphism class of T_J is determined by the isomorphism class of q_C and the isomorphism class of $q_C \otimes \langle 1,-c,-d,cd\rangle$. In particular, if J is a reduced Jordan algebra, then the isomorphism class of J is determined by the isomorphism class of a 3-fold Pfister form and a 5-fold Pfister form.*

Proof. This is Corollary 37.16 in [23] and Theorem 22.4 on page 50 of [14]. □

4.3 The Homological Torsion Primes

This section was informed by [34].

Let G be an absolutely simple algebraic group over k. The *homological torsion primes* of G is the set of prime numbers p satisfying one of the following conditions:

1. p divides the order of the automorphism group of the Dynkin graph of G
2. p divides the order of the fundamental group of G
3. p is a torsion prime of the root system of G

The set of homological torsion primes of a group G is denoted by $S(G)$. Table 4.1 shows the elements in $S(G)$ for each type of absolutely simple group G.

Group	$S(G)$
Type ${}^1A_{n-1}$	prime divisors of n
unitary case	2, prime divisors of n
symplectic case	2
orthogonal case	2
G_2	2
F_4	2,3
${}^{3,6}D_4,\ E_6,\ E_7$	2,3
E_8	2,3,5

Table 4.1: The Homological Torsion Primes

4.4 Unipotent Groups and Reductive Groups

An algebraic group is said to be *unipotent* if all its elements are unipotent.

Example 4.13. *The additive group G_a with k-points:*

$$G_a(k) = \left\{ \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} : x \in k \right\}$$

is a unipotent group over k.

Proposition 4.14. *Let k be a perfect field and let U be a connected, unipotent, linear algebraic group over k. Then $H^i(k, U)$ is trivial for $i \geq 1$.*

Proof. This is Proposition 6 on page 128 of [35]. □

A connected linear algebraic group is called *reductive* if it has no nontrivial, connected, unipotent, normal subgroups. Let G^u be the maximal connected unipotent normal subgroup in G and let G^{red} denote G/G^u.

Proposition 4.15. *Let k be a perfect field and let G be a linear algebraic group over k. Then the natural map*

$$H^1(k, G) \to H^1(k, G^{\mathrm{red}})$$

has trivial kernel.

Proof. The short exact sequence

$$1 \to G^u \to G \to G^{\text{red}} \to 1$$

induces the following exact sequence in Galois Cohomology

$$H^1(k, G^u) \to H^1(k, G) \to H^1(k, G^{\text{red}})$$

Since G^u is unipotent, the desired result follows from Proposition 4.14. □

This result will allow us to reduce questions about $H^1(k, G)$ for more general algebraic groups to the setting of reductive groups. One useful strategy for studying reductive groups is to utilize a special covering.

A *special covering* of a reductive group G is an isogeny

$$1 \to \mu \to G_0 \times S \to G \to 1$$

where G_0 is a simply connected semisimple algebraic k-group and S is a quasitrival k-torus. A torus T is said to be *quasitrivial* if it is a product of groups of the form $R_{E_i/k}G_m$ where $\{E_i\}_{1\leq i\leq r}$ is a family of finite field extensions of k.

Proposition 4.16. *Let G be a reductive group. Then, there exists an integer n and a quasitrival torus T such that $G^n \times T$ admits a special covering*

Proof. This is Lemme 1.10 in [31]. □

4.5 Norm Principles

In this section, we review the norm principles for algebraic groups due to Gille and Merkurjev. Its content was informed by [25]. We will need the notion of R-equivalence to state the results.

Let G be an algebraic variety. Two rational points x and y in $G(k)$ are said to be *strictly equivalent* if there is a rational map $f : \mathbb{A}^1_k \to G$ such that $f(0) = x$ and $f(1) = y$. Two rational points x and y are said to be *R-equivalent* if there is finite sequence of rational points $x = x_1, x_2, \ldots, x_n = y$ such that x_i is strictly equivalent to x_{i+1}. Let $RG(k)$ denote the elements of $G(k)$ which are R-equivalent to 1.

Theorem 4.17 (Merkurjev's Norm Principle)**.** *Let k be a perfect field. Let T be an algebraic k-torus and let G_1 and G be a connected reductive k-groups such that the following sequence is exact:*

$$1 \longrightarrow G_1 \longrightarrow G \xrightarrow{f} T \longrightarrow 1 \tag{4.17.1}$$

Then

$$N_{L/k}(f(RG(L)) \subseteq f(RG(k))$$

Proof. This is Theorem 3.9 in [25]. □

Notable consequences of 4.17 include the following two results:

Corollary 4.18 (Gille's Norm Principle)**.** *Let k be a perfect field. Consider a k-isogeny of semisimple algebraic groups*

$$1 \to \mu \to \tilde{G} \to G \to 1$$

defined over k. Let δ denote the first connecting map in Galois Cohomology $G(k) \to H^1(k, \mu)$. Then

$$N_{L/k}(\delta(RG(L))) \subseteq \delta(RG(k))$$

Proof. This is Lemma 3.11, Lemma 3.12 in [25]. □

Remark 4.19. The result 4.18 had been previously proven by Phillipe Gille. (cf. Théorème II.3.2 in [15]).

Corollary 4.20. *Let K be a field and let A be a central simple algebra over K with involution σ of the second kind. Let $k = K^\sigma$. For any $y \in K$, the following are equivalent:*

1. *$y = \mathrm{Nrd}(a)$ for some $a \in A^*$ such that $\sigma(a)a = 1$*
2. *$y = \mathrm{Nrd}(c\sigma(c)^{-1})$ for some $c \in A$*

Proof. This is Proposition 6.1 in [25]. □

We will be particularly interested in norm principles for G when the underlying variety of G is rational. An algebraic variety G is said to be *rational* if its function field $k(G)$ is a rational function field in finitely many variables over k.

Theorem 4.21. *If the underlying variety of a connected algebraic group G defined over a field k is rational then $RG(k) = G(k)$.*

Proof. See [10]. □

By Theorem 4.21, Theorem 4.17 also admits the following corollaries:

Corollary 4.22. *Let k be a perfect field. Let T be an algebraic k-torus and let G_1 and G be a connected reductive k-groups such that the following sequence is exact:*

$$1 \longrightarrow G_1 \longrightarrow G \xrightarrow{f} T \longrightarrow 1 \tag{4.22.1}$$

If G is a rational group then

$$N_{L/k}(f(G(L)) \subseteq f(G(k))$$

Corollary 4.23. *Let k be a field of characteristic 0. Consider a k-isogeny of semisimple algebraic groups*

$$1 \to \mu \to \tilde{G} \to G \to 1$$

defined over k. Let δ denote the first connecting map in Galois Cohomology $G(k) \to H^1(k, \mu)$. If G is a rational group then

$$N_{L/k}(\delta(G(L))) \subseteq \delta(G(k))$$

4.6 The Rost Invariant

The main source for this section is [14].

Let G be an algebraic group over k. Then G defines a functor from the category of field extensions of k to the category of groups given by $E \to G(E)$. Let H be a functor from the category of field extensions of k to the category of abelian groups. An *invariant of G with values in H* is a morphism of functors $H^1(*, G) \to H$. An invariant ϕ is said to be *normalized* if it vanishes on the point in $H^1(*, G)$. For G an absolutely simple simply connected group the *Rost invariant* R_G is an invariant of G with values in $H^3(*, \mathbb{Q}/\mathbb{Z}(2))$ which satisfies the following property:

Proposition 4.24. *The Rost invariant generates the group of all normalized invariants of G with values in $H^3(*, \mathbb{Q}/\mathbb{Z}(2))$*

Proof. This is Theorem 9.11 on page 129 in [14]. □

One can associate to each absolutely simple, simply connected algebraic group G an integer called the *Dynkin index* of G and denoted n_G. The list of possible values for the Dynkin index for each absolutely simple, simply connected group is shown in Table 4.2. In the case of a group of type A_n, A refers to the underlying central simple algebra.

Group	n_G
Type ${}^1A_n$	the exponent of A
unitary case	the exponent of A or twice the exponent of A
symplectic case	1 or 2
orthogonal case	2 or 4
${}^{3,6}D_4$	6 or 12
E_6	6 or 12
E_7	12
E_8	60
F_4	6
G_2	2

Table 4.2: The Dynkin Index

We will make use of the following result.

Proposition 4.25. *Let G be an absolutely simple, simply connected group over k and let $R_G : H^1(k, G) \to H^3(k, \mathbb{Q}/\mathbb{Z}(2))$ be the Rost invariant. Then $R_G(H^1(k, G)) \subseteq H^3(k, \mathbb{Z}/n_G\mathbb{Z}(2))$.*

Proof. This is Proposition 31.40 in [23]. ☐

Theorem 4.26. *Let k be a field and let G be a quasisplit, absolutely simple, simply connected group of type ${}^{3,6}D_4, E_6$ or E_7. Then the Rost invariant has trivial kernel.*

Proof. This result is due to Skip Garibaldi [12], [13]. See also [8] or [28]. ☐

One can associate to any exceptional Jordan algebra J an invariant g_3 with values in $H^3(k, \mathbb{Z}/3\mathbb{Z})$. See [30], [27] or §40 in [23]. We will refer to this invariant as the *Serre-Rost mod-3* invariant.

Theorem 4.27. *Let k be a field of characteristic different from 2 and 3. Then $g_3(J) = 0$ if and only if J is reduced.*

Proof. This result is due to Rost [30]. See also §3.3 in [27]. ☐

Chapter 5

The Hasse principle

5.1 The Hasse Principle over a Number Field

The main sources for this section are [29] and [11].

Let k be a number field. A *valuation* on k is a function $v : k \to \mathbb{R}$ such that for all $x, y \in k$:

1. $v(x) \geq 0$ and $v(0) = 0$ if and only if $x = 0$,
2. $v(xy) = v(x)v(y)$ and
3. $v(x + y) \leq v(x) + v(y)$.

Example 5.1. *Fix a prime p. Any $x \in \mathbb{Q} - \{0\}$ can be written in the form*

$$x = \frac{ap^n}{b}$$

where a, b and n are integers and a and b are coprime to p. The map $v : \mathbb{Q} \to \mathbb{R}$ defined by $v(x) = \left(\frac{1}{p}\right)^n$ if $x \neq 0$ and $v(0) = 0$ is a valuation on $\mathbb{Q}$ called the p-adic valuation.

Example 5.2. *The map $v : \mathbb{Q} \to \mathbb{R}$ given by $v(x) = 1$ for all $x \neq 0$ is a valuation on $\mathbb{Q}$ called the trivial valuation.*

Example 5.3. *The usual absolute value function is a valuation on $\mathbb{Q}$.*

Proposition 5.4 (The Completion of k at v)**.** *Let k be a number field and let v be a valuation on k. Then there is a complete field k_v with a valuation v' and a field homomorphism $j : k \to k_v$ such that:*

1. $v(a) = v'(j(a))$ *for all* $a \in k$,
2. $j(k)$ *is dense in* k_v *and*
3. k_v *satisfies the following universal property: for any complete field L with valuation w and homomorphism $i : k \to L$ satisfying $v(a) = w(i(a))$ for all $a \in k$, there is a unique homomorphism $i' : k_v \to L$ such that $i = i' \circ j$.*

Proof. This is Theorem 10 in §3 of Chapter II of [11]. □

We refer to the field k_v in Proposition 5.4 as the completion of k at v.

Example 5.5. *The completion of $\mathbb{Q}$ at the absolute value is $\mathbb{R}$.*

Example 5.6. *The completion of $\mathbb{Q}$ at the p-adic valuation is the field of p-adic numbers $\mathbb{Q}_p$.*

Let k be a field with valuation v. Let L be a field extension of k. A valuation w on L is called an *extension* of v to L if w restricted to k is v.

Proposition 5.7. *Let k be a field with valuation v. Let L be a finite Galois extension of k and let w be an extension of v to L. Then L_w is a Galois extension of k_v and we can identify $\mathrm{Gal}(L_w/k_v)$ with a subgroup of $\mathrm{Gal}(L/k)$.*

Proof. This is Theorem 21 in §1 of Chapter III in [11]. □

In particular, if L is a Galois extension of k which admits an extension of v, then there is a restriction map $H^1(\mathrm{Gal}(L/k), G) \to H^1(\mathrm{Gal}(L_w/k_v), G)$. Let V denote the set of valuations of k. We say that an algebraic group G satisfies a *Hasse principle* over k if the product of the restriction maps $H^1(k, G) \to \prod_{v \in V} H^1(k_v, G)$ is injective.

Proposition 5.8. *Let k be a number field and let G be a semisimple simply connected group over k. Let V_∞ denote the set of all valuations on k which restrict to the absolute value on $\mathbb{Q}$. Then for any $v \in V - V_\infty$, $H^1(k_v, G)$ is trivial.*

Proof. This is Theorem 6.4 in [29] and is due to Kneser. □

Proposition 5.9. *Let k be a number field and let G be a semisimple simply connected group over k. Then the canonical map $H^1(k, G) \to \prod_{v \in V_\infty} H^1(k_v, G)$ is injective.*

Proof. This is due to Kneser, Harder and Chernousov [18], [21], [6] and is Theorem 6.6 in [29]. □

Combining 5.8 and 5.9, any semisimple simply connected group satisfies a Hasse principle over a number field.

5.2 The Hasse Principle over a Field of Virtual Cohomological Dimension at most 2

Our main source for this section is [32].

An *ordering* v of a field k is given by a binary relation $\leq_v$ such that for all $a, b, c \in k$:

1. $a \leq_v a$,
2. if $a \leq_v b$ and $b \leq_\nu c$ then $a \leq_v c$,
3. if $a \leq_v b$ and $b \leq_v a$ then $a = b$,
4. either $a \leq_v b$ or $b \leq_v a$,
5. if $a \leq_v b$ then $a + c \leq_v b + c$ and
6. if $a \leq_v b$ and $0 \leq_v c$ then $ca \leq_v cb$

Let k be a field with an ordering v and let L be a field extension of k. An *extension* of v to L is an ordering v' on L such that v' restricted to k is v.

Lemma 5.10. *If L is a finite field extension of k of odd degree there is an extension of v to L.*

Proof. This is Theorem 1.10 in Chapter 3 of [32]. □

A field k is said to be *formally real* if -1 is not a sum of squares in k. A field k is called *real closed* if it is a formally real field and no proper algebraic extension is formally real.

Proposition 5.11 (The Real Closure of k at v)**.** *Let k be a field with an ordering v. Then k has an algebraic extension k_v which is real closed and ordered by an extension of v.*

Proof. This is Theorem 1.13 in Chapter 3 of [32]. □

The field k_v in Proposition 5.11 is called the *real closure* of k at v.

Proposition 5.12. *Let k be a real closed field. Then $k(\sqrt{-1})$ is algebraically closed.*

Proof. This is Theorem 2.3 in Chapter 3 of [32]. □

Assume k is a perfect field. Then since k_v is an algebraic extension of k, it is a separable extension of k and we have the inclusion $k \subset k_v \subset k_s$. In particular, given an algebraic group G over k we can define a restriction map $H^1(k, G) \to H^1(k_v, G)$. We say that an algebraic group G satisfies a *Hasse principle* over a perfect field k if the map $H^1(k, G) \to \prod_v H^1(k_v, G)$ is injective where v varies over the orderings of k.

Proposition 5.13. *Let k be a perfect field. Assume* $\mathrm{vcd}(k) \leq 2$ *and let G be a simply connected group of classical type, type F_4 or type G_2. Then G satisfies a Hasse principle over k*

Proof. This is Theorem 10.1 in [4] and is due to Bayer and Parimala. □

It is not known whether a simply connected group of type trialitarian D_4, E_6 E_7 or E_8 satisfies a Hasse principle over a perfect field of virtual cohomological dimension at most 2.

Chapter 6

A question of Serre

6.1 Zero Cycles on Principal Homogeneous Spaces

This section was informed by [19] and [35].

Let G be a group. A right *group action* of G on a set X is a map $\phi : X \times G \to X$ denoted by $\phi(x, g) = x \cdot g$ and such that $x \cdot 1 = x$ and for all $g, h \in G$ and $x \in X$, $(x \cdot g) \cdot h = x \cdot gh$. The action of G on X is said to be *simply transitive* if for all $x, y \in X$ there is a unique $g \in G$ such that $x \cdot g = y$. Now let k be a field, let G be an algebraic group over k and let X be a variety over k. A right *algebraic group action* of G on X is a morphism of varieties $\phi : X \times G \to X$ denoted $\phi(x, g) = x \cdot g$ which induces a group homomorphism from G to $\operatorname{Aut} X$ defined over k. A k-variety X is said to be a *principal homogeneous space* under G over k if there is a right algebraic group action of G on X which induces a simply transitive group action of $G(\bar{k})$ on $X(\bar{k})$.

Example 6.1. *Let X be the set G with G-action given by right multiplication. Then X is a principal homogeneous space under G over k.*

Example 6.2. *Let $s \to \alpha_s$ be a 1-cocyle of $\Gamma_k = \mathrm{Gal}(k_s/k)$ with values in $G(k_s)$. Define the α-twisted action of Γ_k on G by ${}^{s*}g = \alpha_s{}^s g$ for all $g \in G$ and $s \in \mathrm{Gal}(k_s/k)$. Let G_α denote the set G with α-twisted action of Γ_k and G-action given by right multiplication. Then G_α is a principal homogeneous space under G over k.*

A *zero cycle* on a principal homogeneous space X under G over k is an element of the free abelian group on closed points of X. We may associate to any zero cycle $\sum_{i=1}^{m} n_i x_i$ its *degree* $\sum_{i=1}^{m} n_i[k(x_i) : k]$ where $k(x_i)$ is the residue field of x_i. A closed point with residue field k is called a *rational point*.

Jean-Pierre Serre [35, pg 192] has asked the following question:

Q: Let k be a field and let G be a connected linear algebraic group over k. Let X be a principal homogeneous space under G over k. If X admits a zero cycle of degree one, does X have a k-rational point?

6.2 The Kernel of the Restriction Map

We begin with a classification of principal homogeneous spaces by a cohomology set.

Proposition 6.3. *The map $\alpha \to G_\alpha$ induces a bijection between $H^1(k, G)$ and the set of isomorphism classes of principal homogeneous spaces under G over k. Under this bijection, the trivial class in $H^1(k, G)$ is associated to the principal homogeneous space under G over k with rational point.*

Proof. This is Proposition 33 in Chapter I of [35]. □

Let X be a principal homogeneous space under G over k and let $\sum n_i x_i$ be a zero cycle of degree one on X. By 6.3 we may associate to X an element $\lambda \in H^1(k, G)$. By construction of the residue field, x_i is a rational point of $X_{k(x_i)}$ over $k(x_i)$ for all i and $X_{k(x_i)}$ is associated to the trivial element in $H^1(k(x_i), G)$ and λ is in the kernel of the restriction map $H^1(k, G) \to H^1(k(x_i), G)$. If the zero cycle is of degree one, then the field extensions $k(x_i)$ are necessarily of coprime degree over k.

Guided by this insight, one may restate **Q** as follows.

Q: Let k be a field and let G be a connected, linear algebraic group defined over k. Let $\{L_i\}_{1 \le i \le m}$ be a collection of finite extensions of k with $\gcd([L_i : k]) = 1$. Does the canonical map

$$H^1(k, G) \to \prod_{i=1}^{m} H^1(L_i, G)$$

have trivial kernel?

In view of a standard argument (see for example [9, pg 47]) one can reduce to the setting where k is a field of characteristic 0.

6.3 Known Results

Serre's question is known to have positive answer in a number of settings. A positive answer in the case $G = PGL_n$ is classical. The argument proceeds as follows:

Lemma 6.4. *Let k be a field, let $\{L_i\}_{1 \le i \le m}$ be a set of finite field extensions of k with gcd($[L_i : k]$) $= 1$ and let $G = PGL_n$. Then the canonical map*

$$H^1(k, G) \to \prod_i H^1(L_i, G)$$

has trivial kernel.

Proof. By 3.10, there is a bijection between $H^1(k, G)$ and the set of isomorphism classes of k_s/k twisted forms of $M_n(k)$. By 3.6, any central simple algebra admits a separable splitting field. In particular, any central simple algebra is a k_s/k twisted form of $M_n(k)$. So choose λ in the kernel of the product of the restriction maps $H^1(k, G) \to \prod_i H^1(L_i, G)$. Associate to λ the isomorphism class of a central simple algebra A. By choice of λ, $A \otimes L_i$ is split for all L_i. By 3.7 this implies that the index of A divides $[L_i : k]$ for all i. Since the L_i are assumed to be of coprime degree, this implies that the index of A is 1. Thus A is split over k and λ is the point in $H^1(k, G)$. □

Proposition 6.5 (Springer's Theorem)**.** *Let k be a field and let q be a quadratic form over k with orthogonal group $G = O(q)$. Let L be a finite field extension of k of odd degree. Then the restriction map*

$$H^1(k, G) \to H^1(L, G)$$

has trivial kernel.

Proof. By 2.18 there is a bijection between $H^1(k, O(q))$ and the set of isomorphism classes of k_s/k twisted forms of q which is the set of quadratic forms of dimension n over k. Choose λ in the kernel of the product of the restriction maps $H^1(k, G) \to \prod_i H^1(L_i, G)$ and associate to λ a quadratic form q' over k. By choice of λ, $q'_L \cong q_L$. Since $[L : k]$ is odd, by Theorem 2.7 in Chapter VII of [24], $q' \cong q$ and thus λ is the point in $H^1(k, G)$. $\square$

Theorem 6.6. *Let A be a central simple algebra over a field K with an involution σ. Let $K^\sigma = k$ and let L be a finite field extension of k of odd degree. Let $G = \mathrm{Iso}(A, \sigma)$ the group of isometries of (A, σ). Then the restriction map*

$$H^1(k, G) \to H^1(L, G)$$

has trivial kernel.

Proof. This is due to Eva Bayer and H.W. Lenstra and is Theorem 2.1 in [2]. $\square$

Theorem 6.7. *Let k be a number field and let G be a connected linear algebraic group over k. Let $\{L\}_{1 \le i \le m}$ be a set of finite field extensions of k such that $\gcd[L_i : k] = 1$. Then the restriction map*

$$H^1(k, G) \to \prod_i H^1(L_i, G)$$

has trivial kernel.

Proof. This is Corollaire 4.8 in [31]. $\square$

Chapter 7

Results under Semisimple Groups

7.1 Absolutely Simple Simply Connected Groups of Classical Type

The main result of this section is the following:

Theorem 7.1. *Let k be a field of characteristic different from 2. Let G be an absolutely simple, simply connected, classical algebraic group over k. Let $\{L_i\}_{1\leq i\leq m}$ be a set of finite field extensions of k and let the greatest common divisor of the degrees of the extensions $[L_i : k]$ be d. If d is coprime to $S(G)$, then the canonical map*

$$H^1(k, G) \to \prod_{i=1}^{m} H^1(L_i, G)$$

has trivial kernel.

We will need the following lemma in the rest of this section.

Lemma 7.2. *Let K be a field and let A be a central simple algebra over K of index s. Let* Nrd *be the reduced norm. For every $\alpha \in K^*$, there exists $\beta \in A^*$ such that* $\mathrm{Nrd}(\beta) = \alpha^s$

Proof. By part 1 of 3.7, choose a splitting field E for A such that $[E : K] = s$. Since A_E is split, $\mathrm{Nrd} : A_E \to E$ is onto. In particular α is in $\mathrm{Nrd}(A_E)$. Since by 3.9 $N_{E/K}(\mathrm{Nrd}(A_E)) \subset \mathrm{Nrd}(A)$ and $N_{E/K}(\alpha) = \alpha^s$, it follows that α^s is in $\mathrm{Nrd}(A)$. $\square$

Proposition 7.3. *Let k be a field of characteristic different from 2. Let A be a central simple algebra over k with an involution σ of the first kind. Let* $\mathrm{Iso}(A,\sigma)$ *denote the group if isometries of (A,σ). If A is not split, then every element of* $\mathrm{Iso}(A,\sigma)(k)$ *has trivial reduced norm.*

Proof. This is Lemma 1 b in [22]. □

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Proposition 7.4. *Let k be a field of characteristic different from 2, let A be a central simple algebra of degree n over k and $G = SL_1(A)$. Let $\{L_i\}_{1\le i\le m}$ be a set of finite extensions of k let* $\gcd([L_i : k]) = d$. *If d is coprime to n, then the canonical map*

$$H^1(k,G) \to \prod_{i=1}^{m} H^1(L_i,G)$$

has trivial kernel

Proof. Consider the short exact sequence

$$1 \longrightarrow SL_1(A) \longrightarrow GL_1(A) \xrightarrow{\mathrm{Nrd}} G_m \longrightarrow 1 \tag{7.4.1}$$

which by 2.20 induces the following commutative diagram with exact rows.

$$\begin{array}{ccccccc} A^* & \xrightarrow{Nrd} & k^* & \xrightarrow{\delta} & H^1(k,SL_1(A)) & \longrightarrow & 1 \\ \downarrow & & \downarrow g & & \downarrow h & & \\ \prod A^*_{L_i} & \xrightarrow{Nrd} & \prod L_i^* & \xrightarrow{\delta} & \prod H^1(L_i,SL_1(A)) & \longrightarrow & 1 \end{array} \tag{7.4.2}$$

Choose $\lambda \in \ker(h)$. By the exactness of the top row of the diagram, choose $\lambda' \in k^*$ such that $\delta(\lambda') = \lambda$. Fix an index i. Since $\delta(g(\lambda'))$ = point, by exactness of the bottom row choose $(\lambda_i'') \in A^*_{L_i}$ such that $\mathrm{Nrd}(\lambda_i'') = g(\lambda')$. By 2.7, $N_{L_i/k}(g(\lambda')) = (\lambda')^{m_i}$ where $m_i = [L_i : k]$. By 3.9, $N_{L_i/k}(\mathrm{Nrd}(A^*_{L_i})) \subset \mathrm{Nrd}(A^*)$. In particular, $(\lambda')^{m_i}$ is in $\mathrm{Nrd}(A^*)$. Since $d = \sum m_i n_i$ for appropriate choice of integers n_i, $(\lambda')^d = \prod((\lambda')^{m_i})^{n_i}$ is in $\mathrm{Nrd}(A^*)$.

Let s be the index of A. Then by 7.2, $(\lambda')^s$ is in $\mathrm{Nrd}(A^*)$. Since s divides n and by assumption d and n are coprime, then d and s are coprime. So choose a and b such that $sa + db = 1$. Then $\lambda' = (\lambda')^{sa}(\lambda')^{db}$ is in $\mathrm{Nrd}(A^*)$ and by exactness of the top row $\lambda = \delta(\lambda')$ is the point in $H^1(k, SL_1(A))$. □

The Unitary case

Theorem 7.5. *Let k be a field of characteristic different from 2. Let A be a central simple algebra of degree n with center K and σ a unitary involution on A with $K^\sigma = k$. Suppose $deg_K(A) \geq 2$. Let $G = SU(A, \sigma)$. Let $\{L\}_{1 \leq i \leq m}$ be a set of finite field extensions of k with $\gcd([L_i : k]) = d$. If d is odd and coprime to n, then the canonical map*

$$H^1(k, G) \to \prod_{i=1}^{m} H^1(L_i, G)$$

has trivial kernel.

Proof. Consider the short exact sequence

$$1 \longrightarrow SU(A,\sigma) \longrightarrow U(A,\sigma) \xrightarrow{Nrd} R^1_{K/k}G_m \longrightarrow 1 \tag{7.5.1}$$

which induces the following commutative diagram in Galois Cohomology with exact rows.

$$\begin{tikzcd} U(A,\sigma)(k) \arrow[r, "Nrd"] \arrow[d] & R^1_{K/k}G_m(k) \arrow[r, "\delta"] \arrow[d, "f"] & H^1(k, SU(A,\sigma)) \arrow[r, "j"] \arrow[d, "g"] & H^1(k, U(A,\sigma)) \arrow[d, "h"] \\ \prod U(A,\sigma)(L_i) \arrow[r, "Nrd"] & \prod R^1_{K/k}G_m(L_i) \arrow[r, "\delta"] & \prod H^1(L_i, SU(A,\sigma)) \arrow[r, "j"] & \prod H^1(L_i, U(A,\sigma)) \end{tikzcd} \tag{7.5.2}$$

Choose $\lambda \in \ker(g)$. By assumption, there is an index i such that $[L_i : k]$ is odd. Fix that index i and let $L_i = L$. By 6.6, $H^1(k, U(A,\sigma)) \to H^1(L, U(A,\sigma))$ has trivial kernel. In particular, h has trivial kernel and λ is in $\ker(j)$. So choose $\lambda' \in R^1_{K/k}G_m(k)$ such that $\delta(\lambda') = \lambda$. Since $\delta(f(\lambda')) =$ point, exactness of the bottom row of the diagram gives $(\lambda''_i) \in \prod U(A,\sigma)(L_i)$ such that $\mathrm{Nrd}(\lambda''_i) = f(\lambda')$. Applying $N_{L_i/k}$ to both sides of this equality we find $N_{L_i/k}(\mathrm{Nrd}(\lambda''_i)) = N_{L_i/k}(f(\lambda'))$. Since $U(A,\sigma)$ is a rational group, 4.22 gives that for each i, $N_{L_i/k}(\mathrm{Nrd}(\lambda''_i))$ is in the image of $\mathrm{Nrd} : U(A,\sigma)(k) \to R^1_{K/k}G_m(k)$. By 2.7, for each i, $N_{L_i/k}(f(\lambda')) = (\lambda')^{m_i}$ for $m_i = [L_i : k]$. So for each i, $(\lambda')^{m_i}$ is in the image of $\mathrm{Nrd} : U(A,\sigma)(k) \to R^1_{K/k}G_m(k)$. Since $(\lambda')^d = \prod((\lambda')^{m_i})^{n_i}$ for appropriate choice of integers n_i, then $(\lambda')^d$ is in the image of $\mathrm{Nrd} : U(A,\sigma)(k) \to R^1_{K/k}G_m(k)$.

By 2.21 write $\lambda' = \mu^{-1}\bar{\mu}$ for $\mu \in K^*$ and $\bar{\mu}$ the image of μ under the nontrivial automorphism of K over k. Let s be the index of A and write $(\lambda')^s = (\mu^s)^{-1}\bar{\mu^s}$. By 7.2, $\mu^s = \mathrm{Nrd}(a)$ for some $a \in A^*$. Thus $(\lambda')^s = \mathrm{Nrd}(a^{-1}\sigma(a))$ and by 4.20, $(\lambda')^s$ is in the image of $\mathrm{Nrd} : U(A,\sigma)(k) \to R^1_{K/k}G_m(k)$.

Certainly, s divides n and since by assumption d is coprime to n, then d is coprime to s. In particular, there exist $v, w \in \mathbb{Z}$ such that $dv + sw = 1$. Therefore $\lambda' = (\lambda')^{dv}(\lambda')^{sw}$ is in the image of $\mathrm{Nrd} : U(A,\sigma)(k) \to R^1_{K/k}G_m(k)$ and by exactness of the top row of (7.5.2), $\lambda = \delta(\lambda') =$ point. $\square$

The Symplectic case

Proposition 7.6. *Let k be a field of characteristic different from 2, let A be a central simple algebra over k of even degree with a symplectic involution σ and let $G = Sp(A,\sigma)$. Let L be a finite extension of k of odd degree. Then the canonical map*

$$H^1(k,G) \to H^1(L,G)$$

has trivial kernel

Proof. Since G is the group of isometries of an algebra with involution, this is just a special case of 6.6 due to Bayer and Lenstra. $\square$

The Orthogonal case

Our proof in this case makes use of the following result.

Proposition 7.7. *Let k be a field of characteristic different from 2 and let A be a central simple algebra over k of degree ≥ 3 with orthogonal involution σ. Let $G = O^+(A,\sigma)$ and let L be a finite extension of k of odd degree. Then the canonical map*

$$H^1(k,G) \to H^1(L,G)$$

has trivial kernel.

Proof. We have the short exact sequence

$$1 \longrightarrow O^+(A,\sigma) \longrightarrow O(A,\sigma) \xrightarrow{Nrd} \mu_2 \longrightarrow 1 \tag{7.7.1}$$

In the case A is split, $O(A,\sigma) = O(q)$ the orthogonal group of a quadratic form q, $O^+(A,\sigma) = O^+(q)$ and the reduced norm is the determinant. Then 6.5 gives $H^1(k, O(q)) \to H^1(L, O(q))$ has trivial kernel. That $H^1(k, O^+(q)) \to H^1(k, O(q))$ has trivial kernel follows from the observation that the determinant map $O(q)(k) \to \mu_2$ is onto. Combining these two results, 7.7 holds.

So assume A is not split. Then 7.3 gives $O^+(A,\sigma)(k) = O(A,\sigma)(k)$. Since A admits an involution of the first kind and L/k is odd, A_L is not split and $O^+(A,\sigma)(L) = O(A,\sigma)(L)$.

Then (7.7.1) induces the following diagram with exact rows and commuting rectangles.

$$\begin{tikzcd}
1 \arrow[r] & \mu_2 \arrow[r, "\delta"] \arrow[d, "h"] & H^1(k, O^+(A,\sigma)) \arrow[r, "i"] \arrow[d, "f"'] & H^1(k, O(A,\sigma)) \arrow[d, "g"] \\
1 \arrow[r] & \mu_2 \arrow[r, "\delta"] & H^1(L, O^+(A,\sigma)) \arrow[r, "i"] & H^1(L, O(A,\sigma))
\end{tikzcd} \tag{7.7.2}$$

Let $\lambda \in \ker(f)$. By the commutativity of the rightmost rectangle in (7.7.2), $g(i(\lambda)) =$ point. Then 6.6 gives $i(\lambda) =$ point. By the exactness of the top row, there exists $\lambda' \in \mu_2$ such that $\delta(\lambda') = \lambda$. Since the left rectangle in (7.7.2) commutes, $\delta(h(\lambda')) =$ point. Since h is the identity map and δ has trivial kernel, $\lambda' = 1$ and thus $\lambda = \delta(\lambda') =$ point. $\square$

Now we give the proof for absolutely simple, simply connected groups in the orthogonal case.

Theorem 7.8. *Let k be a field of characteristic different from 2 and let A be a central simple algebra over k of degree ≥ 4 with orthogonal involution σ. Let $G = Spin(A,\sigma)$ and let L be a finite extension of k of odd degree. Then the canonical map*

$$H^1(k, G) \to H^1(L, G)$$

has trivial kernel.

Proof. The short exact sequence

$$1 \longrightarrow \mu_2 \xrightarrow{i} Spin(A,\sigma) \xrightarrow{\eta} O^+(A,\sigma) \longrightarrow 1 \tag{7.8.1}$$

induces the following commutative diagram with exact rows.

$$\begin{tikzcd} O^+(A,\sigma)(k) \arrow[r, "\delta"] \arrow[d, "f"] & H^1(k,\mu_2) \arrow[r, "i"] \arrow[d, "g"] & H^1(k, Spin(A,\sigma)) \arrow[r, "\eta"] \arrow[d, "h"'] & H^1(k, O^+(A,\sigma)) \arrow[d, "j"] \\ O^+(A,\sigma)(L) \arrow[r, "\delta"] & H^1(L,\mu_2) \arrow[r, "i"] & H^1(L, Spin(A,\sigma)) \arrow[r, "\eta"] & H^1(L, O^+(A,\sigma)) \end{tikzcd} \tag{7.8.2}$$

Choose $\lambda \in \ker(h)$. By commutativity of the rightmost rectangle in (7.8.2) $j(\eta(\lambda)) =$ point. In particular, $\eta(\lambda) \in \ker(j)$ and by 7.7, $\eta(\lambda) =$ point. By exactness of the top row, we may choose $\lambda' \in H^1(k,\mu_2)$ such that $i(\lambda') = \lambda$. By the commutativity of the central rectangle in (7.8.2), $i(g(\lambda')) =$ point. So from exactness of the bottom row, we may choose $\lambda'' \in O^+(A,\sigma)(L)$ such that $\delta(\lambda'') = g(\lambda')$. Applying the norm map to both sides of this equality we find, $N_{L/k}(\delta(\lambda'')) = N_{L/k}(g(\lambda'))$. By 2.7 the latter is $(\lambda')^{[L:k]}$. Let $\tilde{\lambda}$ be a representative of λ' in $k^*/(k^*)^2$. Since $[L:k]$ is odd, $\tilde{\lambda}^{[L:k]} = \tilde{\lambda}$ in $k^*/(k^*)^2$. In turn $[(\lambda')^{[L:k]}] = [\lambda']$ in $H^1(k,\mu_2)$. Thus $N_{L/k}(\delta(\lambda'')) = \lambda'$. Since $O^+(A,\sigma)$ is rational, 4.23 gives

$$N_{L/k}(\mathrm{im}(O^+(A,\sigma)(L) \xrightarrow{\delta} H^1(L,\mu_2)) \subset \mathrm{im}(O^+(A,\sigma)(k) \xrightarrow{\delta} H^1(k,\mu_2))) .$$

In particular λ' is in the image of $O^+(A,\sigma)(k) \to H^1(k,\mu_2)$. But then by exactness of the top row, $\lambda = i(\lambda') =$ point. □

7.2 Absolutely Simple Adjoint Groups of Classical Type

We begin by recording some general results which we shall use in the proof of 7.13.

Proposition 7.9. *Let K be a field of characteristic different from 2. Let A be a central simple algebra over a field K with involution σ of any kind and $k = K^\sigma$. Let L be a finite extension of k of odd degree. Let G be the group of similitudes of (A,σ). Then the canonical map*

$$H^1(k,G) \to H^1(L,G)$$

has trivial kernel.

Proof. Let G_0 be the group of isometries of (A, σ). We have the exact sequence

$$1 \to G_0 \to G \to G_m \to 1$$

where the map $G \to G_m$ takes each similitude a to its multiplier $\sigma(a)a$. In view of 2.20, the sequence yields the following commutative diagram with exact rows.

$$\begin{tikzcd} k^* \arrow[r, "\delta"] \arrow[d] & H^1(k, G_0) \arrow[r, "i"] \arrow[d, "r^*"] & H^1(k, G) \arrow[r] \arrow[d, "g"] & 1 \\ L^* \arrow[r, "\delta"] & H^1(L, G_0) \arrow[r, "i"] & H^1(L, G) \arrow[r] & 1 \end{tikzcd} \tag{7.9.1}$$

Let $\psi \in \ker(g)$. By the exactness of the top row of (7.9.1), there exists $\langle x \rangle \in H^1(k, G_0)$ such that $i(\langle x \rangle) = \psi$. Here $\langle x \rangle$ is a rank one hermitian form over (A, σ). Since commutativity of the right rectangle gives $i(r^*(\langle x \rangle)) =$ point, exactness of the second row gives an $a \in L^*$ such that $r^*(\langle x \rangle) = \delta(a)$. We note that $\delta(a)$ is the isomorphism class of the rank one hermitian form $\langle a \rangle$ over $(A, \sigma)_L$.

Let $k(a)$ be the subfield of L generated by a over k. Since L is an odd degree extension of $k(a)$ and $\langle a \rangle_L \cong r^*(\langle x \rangle)_L$ then by 3.19 $\langle a \rangle_{k(a)} \cong r^*(\langle x \rangle)_{k(a)}$. Let $s : k(a) \to k$ be the k-linear map given by $s(1) = 1$ and $s(a^j) = 0$ for all $1 \leq j < m$ where $m = [k(a) : k]$ and let s_* be the induced transfer homomorphism. Write $\langle a \rangle$ as $\langle a \rangle \otimes \langle 1 \rangle_{k(a)}$ in $W(k(a)) \otimes W(A_{k(a)}, \sigma_{k(a)})$. Since $[k(a) : k]$ is odd, 3.17 and 3.18 give that $s_*(\langle a \rangle \otimes \langle 1 \rangle_{k(a)})$ is Witt equivalent to $\left\langle N_{k(a)/k}(a) \right\rangle \otimes \langle 1 \rangle$. On the other hand, $s_*(r^*(\langle x \rangle)) = s_*(\langle 1 \rangle \otimes \langle x \rangle)$ and since $[L : k]$ is odd, $s_*(\langle 1 \rangle \otimes \langle x \rangle) \cong \langle x \rangle$. So $\left\langle N_{k(a)/k}(a) \right\rangle$ is Witt equivalent to $\langle x \rangle$ and since the two forms have dimension one over (A, σ), by Witt's cancellation for hermitian forms, $\left\langle N_{k(a)/k}(a) \right\rangle \cong \langle x \rangle$. Then $\langle x \rangle = \delta(N_{k(a)/k}(a))$ and thus $\psi = i(\langle x \rangle) =$ point. □

The following is a straightforward corollary of 7.9.

Proposition 7.10. *Let A be a central simple algebra over a field k with an involution σ of the first kind. Let G be the group of projective similitudes of (A, σ). Let L be a finite extension of k of odd degree. Then the canonical map*

$$H^1(k, G) \to H^1(L, G)$$

has trivial kernel

Proof. Let G_0 be the group of similitudes $Sim(A,\sigma)$. Then we have the short exact sequence

$$1 \longrightarrow G_m \xrightarrow{i} G_0 \xrightarrow{\eta} G \longrightarrow 1 \tag{7.10.1}$$

which induces the following commutative diagram with exact rows.

$$\begin{tikzcd} 1 \arrow[r] & H^1(k,G_0) \arrow[r, "\eta"] \arrow[d, "f"] & H^1(k,G) \arrow[r, "\delta"] \arrow[d, "g"] & H^2(k,G_m) \arrow[d, "h"] \\ 1 \arrow[r] & H^1(L,G_0) \arrow[r, "\eta"] & H^1(L,G) \arrow[r, "\delta"] & H^2(L,G_m) \end{tikzcd} \tag{7.10.2}$$

The set $H^1(k,G)$ is in bijection with the isomorphism classes of central simple algebras of the same degree as A with involution of the same type as σ. Choose $(A',\sigma') \in \ker(g)$. By commutativity of (7.10.2), $h(\delta(A',\sigma')) = \text{point}$. Now $\delta(A',\sigma') = [A'][A]^{-1}$ [23, pg 405] which is 2-torsion in the Brauer group by 3.16. In particular, since $[L:k]$ is odd, h has trivial kernel on the image of δ and $\delta(A',\sigma') = \text{point}$. By exactness of the top row of the diagram, choose $\lambda' \in H^1(k,G_0)$ such that $\eta(\lambda') = (A',\sigma')$. By choice of λ', $\eta(f(\lambda')) = \text{point}$. Then by exactness of the bottom row $f(\lambda') = \text{point}$. But that f has trivial kernel was shown in 7.9. So $\lambda' = \text{point}$ and in turn (A',σ') is the point in $H^1(k,G)$. □

As a consequence of this result, we obtain a result on isomorphism of central simple algebras with involution of the first kind. Let (A,σ) and (A',σ') be two algebras with involution of the first kind over k. A *homomorphism* $f : (A,\sigma) \to (A',\sigma')$ is a k-algebra homomorphism $f : A \to A'$ such that $\sigma' \circ f = f \circ \sigma$.

Corollary 7.11. *Let k be a field of characteristic different from 2 and let L be a finite field extension of k of odd degree. Let A and A' be central simple algebras of degree n over k with involutions of the first kind σ and σ' respectively. If $(A_L,\sigma_L) \cong (A'_L,\sigma'_L)$ then $(A,\sigma) \cong (A',\sigma')$.*

Proof. Since $\mathrm{Aut}(A,\sigma) = PGO(A,\sigma)$, 2.18 gives that $H^1(L, PGO(A,\sigma))$ classifies the L/k-twisted forms of (A,σ). The desired result now follows from 7.10. □

Next we prove a norm principle for multipliers of similitudes.

Lemma 7.12. *Let A be a central simple K-algebra with k-linear involution σ. Let L be a finite extension of k of odd degree and let g be a similitude of $(A,\sigma)_L$ with multiplier $\mu(g)$. Then $N_{L/k}(\mu(g))$ is the multiplier of a similitude of (A,σ)*

Proof. Let g be a similitude of $(A,\sigma)_L$. Let $\mu(g) = \sigma(g)g$ be the multiplier of g. By definition, the hermitian form $\langle\mu(g)\rangle_L$ is isomorphic to $\langle 1\rangle_L$. In particular left multiplication by g gives an explicit isomorphism between the hermitian forms. We may identify $\langle\mu(g)\rangle_L$ with $\langle\mu(g)\rangle_L \otimes \langle 1\rangle_L$ in $W(L)\otimes W(A_L,\sigma_L)$. Since $[L : k(\mu(g))]$ is odd and $\langle\mu(g)\rangle_L \otimes \langle 1\rangle_L \cong \langle 1\rangle_L$ then by 3.19, $\langle\mu(g)\rangle_{k(\mu(g))} \otimes \langle 1\rangle_{k(\mu(g))} \cong \langle 1\rangle_{k(\mu(g))}$. Let s be Scharlau's transfer map from $k(\mu(g)) \to k$ and let s_* be the induced transfer homorphism. Then by 3.18 and 3.17, $s_*(\langle\mu(g)\rangle_{k(a)} \otimes \langle 1\rangle_{k(a)})$ is Witt equivalent to $\left\langle N_{k(\mu(g))/k}(\mu(g))\right\rangle \otimes \langle 1\rangle$. Since on the other hand $s_*(\langle 1\rangle_{k(\mu(g))}) = \langle 1\rangle$, then $\left\langle N_{k(\mu(g))/k}(\mu(g))\right\rangle \otimes \langle 1\rangle$ is Witt equivalent to 1. Since both are rank 1 hermitian forms, it follows from Witt's cancellation that they are in fact isomorphic which gives precisely that $N_{k(\mu(g))/k}(\mu(g))$ is the multiplier of a similitude of (A,σ). □

Having established these results we move on to the main result of this section.

Theorem 7.13. *Let k be a field of characteristic different from 2 and G an absolutely simple, adjoint, classical group over k. Let $\{L_i\}_{1\le i\le m}$ be a set of finite field extensions of k and let the greatest common divisor of the degrees of the extensions $[L_i : k]$ be d. If d is coprime to $S(G)$ the canonical map*

$$H^1(k,G) \to \prod_{i=1}^{m} H^1(L_i,G)$$

has trivial kernel.

Type ${}^1A_{n-1}$

Theorem 7.14. *Let k be a field of characteristic different from 2, A a central simple algebra of degree n over k and $G = PGL_1(A)$. Let $\{L_i\}_{1\le i\le m}$ be a set of finite field* *extensions of k and let* $\gcd([L_i : k]) = d$*. If d is coprime to n then the canonical map*

$$H^1(k, G) \to \prod_{i=1}^{m} H^1(L_i, G)$$

has trivial kernel.

Proof. Consider the short exact sequence

$$1 \longrightarrow G_m \longrightarrow GL_1(A) \longrightarrow PGL_1(A) \longrightarrow 1 \tag{7.14.1}$$

Since 2.20 gives $H^1(k, GL_1(A)) = 1$, the induced long exact sequences in Galois Cohomology produces the following commutative diagram with exact rows.

$$\begin{tikzcd} 1 \arrow[r] & H^1(k, PGL_1(A)) \arrow[r, "\delta"] \arrow[d, "f"] & H^2(k, G_m) \arrow[d, "g"'] \\ 1 \arrow[r] & \prod H^1(L_i, PGL_1(A)) \arrow[r, "\delta"] & \prod H^2(L_i, G_m) \end{tikzcd} \tag{7.14.2}$$

The pointed set $H^1(k, PGL_1(A))$ classifies isomorphism classes of central simple algebras of degree n over k and for $B \in H^1(k, PGL_1(A))$, $\delta(B) = [B][A]^{-1}$. Choose $B \in \ker(f)$. By commutativity of the diagram, $g(\delta(B)) =$ point in $\prod H^2(L_i, G_m)$.

Let A^o denote the opposite algebra of A and choose $B \otimes A^o$ a representative for the class $[B][A]^{-1}$ in $H^2(k, G_m)$. Let the exponent of $B \otimes A^o$ be s. Since by assumption $B \otimes A^o$ splits over each L_i, s divides each $[L_i : k]$. It follows that s divides d. Since the degree of $B \otimes A^o$ is n^2, s divides n^2.

Since by assumption n and d are coprime, $s = 1$, $B \otimes A^o$ is split and B is Brauer equivalent to A. Then since B and A are of the same degree, they are isomorphic and B is the point in $H^1(k, PGL_1(A))$. □

The unitary case

Theorem 7.15. *Let K be a field of characteristic different from 2. Let A be a central simple algebra of degree n over K with $n \geq 2$, Let σ be a unitary involution on A. Let k be the subfield of elements of K fixed by σ and $G = PGU(A, \sigma)$. Let $\{L_i\}_{1 \leq i \leq m}$ be a set of finite field extensions of k and let* $\gcd([L_i : k]) = d$*. If d is odd and coprime* *to n then the canonical map*

$$H^1(k,G) \to \prod_{i=1}^{m} H^1(L_i,G)$$

has trivial kernel.

Proof. We have the short exact sequence

$$1 \longrightarrow R_{K/k}G_m \longrightarrow GU(A,\sigma) \longrightarrow PGU(A,\sigma) \longrightarrow 1 \tag{7.15.1}$$

By 2.23, $H^1(k,R_{K/k}G_m) \cong H^1(K,G_m)$ and the latter is trivial. Therefore, (7.15.1) induces the following commutative diagram with exact rows.

$$\begin{tikzcd}
1 \arrow[r] & H^1(k,GU(A,\sigma)) \arrow[r, "\pi"] \arrow[d, "f"] & H^1(k,PGU(A,\sigma)) \arrow[r, "\delta"] \arrow[d, "g"] & H^2(k,R_{K/k}G_m) \arrow[d, "h"] \\
1 \arrow[r] & \prod H^1(L_i,GU(A,\sigma)) \arrow[r, "\pi"] & \prod H^1(L_i,PGU(A,\sigma)) \arrow[r] & \prod H^2(L_i,R_{K/k}G_m)
\end{tikzcd} \tag{7.15.2}$$

Now $H^1(k,PGU(A,\sigma))$ is the set of isomorphism classes of triples (A',σ',ϕ') where A' is a central simple algebra over a field K' which is a quadratic extension of k, the degree of A' over K' is n, σ' is a unitary involution on A' with $(K')^{\sigma'}=k$, and ϕ' is an isomorphism from K' to K [23, pg. 400].

Choose $(A',\sigma',\phi') \in \ker(g)$. Now $\delta(A',\sigma',\phi') = [A'\otimes_{K'}K][A]^{-1}$. Let A^o denote the opposite algebra of A and choose $(A'\otimes_{K'}K)\otimes_K A^o$ a representative of $[A'\otimes_{K'}K][A]^{-1}$ in $H^2(k,R_{K/k}G_m)$. Let the exponent of $A'\otimes_{K'}K)\otimes_K A^o$ be t.

By commutativity of the rightmost rectangle of (7.15.2), $h(\delta(A',\sigma',\phi')) =$ point. In particular, 2.7 gives that $m_i \cdot ((A\prime \otimes_{K'} K) \otimes_K A^o)$ is split for each $m_i = [L_i : k]$. Then t divides each m_i and in turn t divides d.

On the other hand, t divides the degree of $(A'\otimes_{K'}K)\otimes_K A^o$ which is n^2. Since d and n^2 are by assumption coprime, we find that the exponent of t is 1. Thus $(A\prime\otimes_{K'}K)\otimes_K A^o$ is the point in $H^2(k,R_{K/k}G_m)$. Exactness of the top row of (7.15.2) gives a $\lambda \in H^1(k,GU(A,\sigma))$ such that $\pi(\lambda) = (A',\sigma',\phi')$. Commutativity of the left rectangle in (7.15.2) gives $\pi(f(\lambda)) =$ point. And thus by the exactness of the bottom row, $f(A',\sigma',\phi') =$ point. It follows from 7.9 that $\lambda =$ point and thus $(A',\sigma',\phi') =$ point. $\square$

The symplectic case

Proposition 7.16. *Let k be a field of characteristic different from 2. Let A a central simple algebra over k of even degree and let σ be a symplectic involution on A. Let $G = PGSp(A, \sigma)$ and let L be a finite field extension of k of odd degree. Then the canonical map*

$$H^1(k, G) \to H^1(L, G)$$

has trivial kernel.

Proof. This is just a special case of 7.10 above. □

The orthogonal case

The case in which G is of type B_n is a special case of 7.7. For the case in which G is of type D_n we will need the following result on the existence of improper similitudes.

Proposition 7.17. *Let k be a field of characteristic different from 2. Let A be a central simple algebra over k with orthogonal involution σ and discriminant δ. Let $g \in GO(A, \sigma)$ be a similitude of (A, σ) and let $\mu(g)$ denote the multiplier of g. Consider the quaternion algebra $\left(\frac{\delta, \mu(g)}{k}\right)$.*

1. *If g is a proper similitude then $\left(\frac{\delta, \mu(g)}{k}\right)$ splits.*

2. *If g is an improper similitude then $\left(\frac{\delta, \mu(g)}{k}\right)$ is Brauer-equivalent to A.*

Proof. This is Theorem 13.38 in [23] □

Lemma 7.18. *Let k be a field of characteristic different from 2 and A a central simple algebra over k of even degree at least 4 with an orthogonal involution σ. Let L be a finite field extension of k of odd degree. If A is not split, then $GO^-(A, \sigma)(k)$ nonempty if and only if $GO^-(A, \sigma)(L)$ nonempty.*

Proof. If $g \in A$ is an improper similitude of A over k, then certainly g_L is an improper similitude of A_L over L. Conversely, choose $g \in A_L$ an improper similitude of A_L over L and let $\sigma(g)g = \mu(g)$. Then by 7.17 A_L Brauer equivalent to the quaternion algebra

$(\delta, \mu(g))$ over L where δ is the discriminant of σ. From this we find $\mathrm{cor}(A_L)$ Brauer equivalent to $\mathrm{cor}((\delta, \mu(g)))$. Now $\mathrm{res} : H^2(k, \mu_2) \to H^2(L, \mu_2)$ certainly takes A to A_L and $\mathrm{cor}(\mathrm{res}(A)) = A$ since A is 2-torsion and $[L : k]$ is odd. On the other hand, $\mathrm{cor}((\delta, \mu(g))) = (\delta, N_{L/k}(\mu(g)))$. By 7.12 write $N_{L/k}(\mu(g))$ as $\mu(g')$ for g' a similitude of A over k. Thus A is Brauer equivalent to $(\delta, \mu(g'))$. If g' is a proper similitude then by 7.17 $(\delta, \mu(g'))$ splits. But then A splits and we arrive at a contradiction. So g' is an improper similitude of A over k. □

Proposition 7.19. *Let k be a field of characteristic different from 2 and A a central simple algebra over k of even degree at least 4 with an orthogonal involution σ. Let $G = GO^+(A, \sigma)$ and let L be a finite field extension of k of odd degree. Then the canonical map*

$$H^1(k, G) \to H^1(L, G)$$

has trivial kernel.

Proof. Consider the short exact sequence

$$1 \longrightarrow GO^+(A, \sigma) \xrightarrow{i} GO(A, \sigma) \xrightarrow{\eta} \mu_2 \longrightarrow 1 \tag{7.19.1}$$

where the map η takes $a \in GO(A, \sigma)$ to 1 if $\mathrm{Nrd}(a) = \mu(a)^{\deg(A)/2}$ and $\eta^{-1}(-1)$ is precisely $GO^-(A, \sigma)$.

In the case A is split, each hyperplane reflection gives an improper similitude. Thus $GO(A, \sigma)(k) \to \mu_2$ is onto and (7.19.1) induces the following commutative diagram with exact rows.

$$\begin{tikzcd} 1 \arrow[r] & H^1(k, GO^+(A, \sigma)) \arrow[r, "i"] \arrow[d, "g"] & H^1(k, GO(A, \sigma)) \arrow[d, "h"] \\ 1 \arrow[r] & H^1(L, GO^+(A, \sigma)) \arrow[r, "i"] & H^1(L, GO(A, \sigma)) \end{tikzcd} \tag{7.19.2}$$

Choose $\lambda \in \ker(g)$. Since the diagram (7.19.2) commutes and h has trivial kernel by 7.9, $i(\lambda)$ =point. Then exactness of the top row of (7.19.2) gives $\lambda =$ point.

In the case A is not split, we need only consider two scenarios. Firstly, suppose A and A_L both admit improper similitudes. Then $GO(A, \sigma)(k) \to \mu_2$ and

$GO(A,\sigma)(L) \to \mu_2$ are both onto and the proof proceeds exactly as in the split case. Otherwise, by 7.18 neither admits an improper similitude. That is $GO^+(A,\sigma)(k) = GO(A,\sigma)(k)$, $GO^+(A,\sigma)(L) = GO(A,\sigma)(L)$ and (7.19.1) induces the following commutative diagram with exact rows.

$$\begin{tikzcd} 1 \arrow[r] & \mu_2 \arrow[r, "\delta"] \arrow[d, "f"] & H^1(k, GO^+(A,\sigma)) \arrow[r, "i"] \arrow[d, "g"] & H^1(k, GO(A,\sigma)) \arrow[d, "h"] \\ 1 \arrow[r] & \mu_2 \arrow[r, "\delta"] & H^1(L, GO^+(A,\sigma)) \arrow[r, "i"] & H^1(L, GO(A,\sigma)) \end{tikzcd} \tag{7.19.3}$$

Choose $\lambda \in \ker(g)$. Commutativity of the rightmost rectangle in (7.19.3) gives $i(\lambda) \in \ker(h)$. But by 7.9, this gives $i(\lambda) =$ point. Then, by exactness of the top row of (7.19.3), there is an element $\lambda' \in \mu_2$ such that $\delta(\lambda') = \lambda$. Commutativity of the left rectangle in (7.19.3) gives $\delta(f(\lambda')) =$ point. From whence, since the bottom row of (7.19.3) is exact we find $f(\lambda') = 1$. But certainly f is the identity map. So in fact $\lambda' = 1$ and in turn, $\lambda = \delta(\lambda') =$ point. $\square$

We may now prove 7.13 for the absolutely simple group in the orthogonal case.

Theorem 7.20. *Let k be a field of characteristic different from 2 and A a central simple algebra over k of degree at least 4 with an orthogonal involution σ. Let $G = PGO^+(A,\sigma)$ and let L be a finite field extensions of k of odd degree. Then the canonical map*

$$H^1(k, G) \to H^1(L, G)$$

has trivial kernel.

Proof. Consider the short exact sequence

$$1 \longrightarrow G_m \longrightarrow GO^+(A,\sigma) \xrightarrow{\eta} PGO^+(A,\sigma) \longrightarrow 1 \tag{7.20.1}$$

By 2.20, this induces the following commutative diagram with exact rows.

$$\begin{tikzcd} 1 \arrow[r] & H^1(k, GO^+(A,\sigma)) \arrow[r, "\eta"] \arrow[d, "f"] & H^1(k, PGO^+(A,\sigma)) \arrow[r, "\delta"] \arrow[d, "g"] & H^2(k, G_m) \arrow[d, "h"] \\ 1 \arrow[r] & H^1(L, GO^+(A,\sigma)) \arrow[r, "\eta"] & H^1(L, PGO^+(A,\sigma)) \arrow[r] & H^2(L, G_m) \end{tikzcd} \tag{7.20.2}$$

$H^1(k, PGO^+(A,\sigma))$ classifies k-isomorphism classes of triples (A', σ', ϕ') where A' is a central simple algebra over k of the same degree as A, σ' is an orthogonal involution on A' and ϕ' is an isomorphism from the center of the Clifford algebra of A' to the center of the Clifford algebra of A. For any such triple (A', σ', ϕ'), $\delta(A', \sigma', \phi') = [A'][A]^{-1}$ which is 2-torsion in the Brauer group since both A and A' admit involutions of the first kind. Then, since $[L : k]$ is odd, h is injective on the image of δ in $H^2(k, G_m)$.

So choose $(A', \sigma', \phi') \in \ker(g)$. By commutativity of the rightmost rectangle in (7.20.2), $h(\delta(A', \sigma', \phi')) =$ point and thus $\delta(A', \sigma', \phi') =$ point. Then by the exactness of the top row of the diagram, there is a $\lambda' \in H^1(k, GO^+(A,\sigma))$ such that $\eta(\lambda') = (A', \sigma', \phi')$. By commutativity of the left rectangle of (7.20.2), $\eta(f(\lambda')) =$ point which by exactness of the bottom row, gives $f(\lambda') =$ point. Then by 7.19, $\lambda' =$ point and thus $(A', \sigma', \phi') = \eta(\lambda') =$ point. □

7.3 Exceptional Groups

The main result of this section is the following.

Theorem 7.21. *Let k be a field of characteristic different from 2. Let G be a quasisplit, absolutely simple exceptional algebraic group over k which is not of type E_8. Let $\{L_i\}_{1\le i\le m}$ be a set of finite field extensions of k such that* $\gcd([L_i : k]) = d$. *If d is coprime to $S(G)$, then the canonical map*

$$H^1(k, G) \to \prod_{i=1}^{m} H^1(L_i, G)$$

has trivial kernel.

G absolutely simple of type G_2

Proposition 7.22. *Let k be a field of characteristic different from 2. Let G be an absolutely simple group of type G_2 over k. Let L be any finite field extension of k of odd degree. Then the canonical map*

$$H^1(k, G) \to H^1(L, G)$$

has trivial kernel.

Proof. By 4.4, $G \cong \mathrm{Aut}(C)$ for C a Cayley algebra over k. Then by 2.18, $H^1(k, G)$ is in bijection with k_s/k-twisted forms of C. Since every Cayley algebra over k splits over k_s, $H^1(k, G)$ is in bijection with the isomorphism classes of Cayley algebras over k.

Choose $[C'] \in \ker(H^1(k, G) \to H^1(L, G))$ and let q_C denote the norm form of C. By choice of C', C'_L is isomorphic to C_L and thus by 4.6 $(q_C)_L \cong (q'_C)_L$. But since $[L : k]$ is odd, by 6.5 $q_C \cong q_{C'}$. Applying 4.4 again, we conclude that C' is isomorphic to C,and $[C']$ is the point in $H^1(k, G)$. □

G absolutely simple of type F_4

Proposition 7.23. *Let k be a field of characteristic different from 2 and 3. Let G be a split, absolutely simple, simply connected group of type F_4. Let $\{L_i\}_{1\leq i\leq m}$ be a set of finite extensions of k and let* $\gcd([L_i : k]) = d$. *If d is coprime to 2 and 3, then the canonical map*

$$H^1(k, G) \to \prod_{i=1}^{m} H^1(L_i, G)$$

has trivial kernel.

Proof. Consider the Serre-Rost invariant $g_3 : H^1(k, G) \to H^3(k, \mathbb{Z}/3\mathbb{Z})$. The following diagram commutes

$$\begin{tikzcd} H^1(k, G) \arrow[r, "g_3"] \arrow[d, "f"] & H^3(k, \mathbb{Z}/3\mathbb{Z}) \arrow[d, "g"] \\ \prod H^1(L_i, G) \arrow[r, "g_3"] & \prod H^3(L_i, \mathbb{Z}/3\mathbb{Z}) \end{tikzcd}$$

Choose J in $\ker(f)$. By commutativity of the diagram, $g_3(J)$ is in $\ker(g)$ and by our assumption on d, g has trivial kernel. Thus $g_3(J) = 0$ and by 4.27, J is reduced. We associate to J its trace form T_J. Since J is reduced, it is determined up to isomorphism by the isomorphism class of T_J which is in turn determined by the isometry classes of Pfister forms ϕ_3 and ϕ_5 by 4.12 . Since by assumption J is split over each L_i, ϕ_3 and ϕ_5 are hyperbolic over each L_i. Since at least one of the L_i is odd degree, then by 6.5, ϕ_3 and ϕ_5 are hyperbolic over k from whence we have J is split over k. Thus J is the point in $H^1(k, G)$. $\square$

G absolutely simple of type ${}^{3,6}D_4, E_6, E_7$

We begin by considering the simply connected case from which the general case will follow.

Proposition 7.24. *Let k be a field of characteristic different from 2 and 3 and let G be a quasisplit, absolutely simple, simply connected group of type ${}^{3,6}D_4, E_6, E_7$. Let $\{L_i\}_{1\leq i\leq m}$ be a set of finite field extensions of k and let* $\gcd([L_i : k]) = d$. *If d is coprime to 2 and 3 then the canonical map*

$$H^1(k, G) \to \prod_{i=1}^{m} H^1(L_i, G)$$

has trivial kernel.

Proof. The following diagram commutes.

$$\begin{tikzcd} H^1(k,G) \arrow[r, "R_G"] \arrow[d, "f"] & H^3(k, (\mathbb{Z}/n_G\mathbb{Z})(2)) \arrow[d, "g"] \\ \prod H^1(L_i, G) \arrow[r, "R_G"] & \prod H^3(L_i, (\mathbb{Z}/n_G\mathbb{Z})(2)) \end{tikzcd} \tag{7.24.1}$$

Choose $\lambda \in \ker(f)$. Let $R_G(\lambda) = \lambda'$. Since $S(G)$ contains the prime divisors of n_G, and we have assumed d coprime to $S(G)$, 2.7 gives that g has trivial kernel. So by commutativity of (7.24.1), λ is in $\ker(R_G)$. By 4.26, R_G has trivial kernel. So we conclude that $\lambda =$ point. $\square$

Proposition 7.25. *Let k be a field of characteristic different from 2 and 3 and let G be a quasisplit, absolutely simple group of type ${}^{3,6}D_4, E_6, E_7$. Let $\{L_i\}$ be a set of finite field extensions of k and let* $\gcd([L_i : k]) = d$. *If d is coprime to 2 and 3, then the canonical map*

$$H^1(k, G) \to \prod_{i=1}^{m} H^1(L_i, G)$$

has trivial kernel.

Proof. By 7.24 we may assume that G is not simply connected. Then we have a short exact sequence

$$1 \longrightarrow \mu \longrightarrow G^{sc} \xrightarrow{\pi} G \longrightarrow 1 \tag{7.25.1}$$

where G^{sc} is a simply connected cover of G and μ is its center. Since G is by assumption quasisplit, then G^{sc} is quasisplit. So let T be the maximal, quasitrivial torus in G^{sc}.

As $\mu \subset T \subset G^{sc}$, the map $H^1(k,\mu) \to H^1(k,G^{sc})$ induced by the inclusion of μ in G^{sc} factors through the map $H^1(k,\mu) \to H^1(k,T)$ induced by the inclusion of μ in T. But since T is quasitrivial, $H^1(k,T)$ is trivial, and thus the image of the map $H^1(k,\mu) \to H^1(k,G^{sc})$ is trivial. Given this result, (7.25.1) induces the following commutative diagram with exact rows.

$$\begin{tikzcd}
1 \arrow[r] & H^1(k,G^{sc}) \arrow[r, "\pi"] \arrow[d, "f"] & H^1(k,G) \arrow[r, "\delta"] \arrow[d, "g"] & H^2(k,\mu) \arrow[d, "h"] \\
1 \arrow[r] & \prod H^1(L_i,G^{sc}) \arrow[r, "\pi"] & \prod H^1(L_i,G) \arrow[r, "\delta"] & \prod H^2(L_i,\mu)
\end{tikzcd} \tag{7.25.2}$$

Choose $\lambda \in \ker(g)$. The prime divisors of the order of μ are contained in $S(G)$. Then since d is coprime to $S(G)$, d is coprime to the order of μ and 2.7 gives that h has trivial kernel. So by commutativity of the rightmost rectangle of (7.25.2), $\lambda \in \ker(\delta)$. By exactness of the top row of (7.25.2) choose $\lambda' \in H^1(k,G^{sc})$ such that $\pi(\lambda') = \lambda$. Commutativity of the left rectangle of (7.25.2) gives $f(\lambda') \in \ker(\pi)$ which is trivial by the exactness of the bottom row of (7.25.2). So $f(\lambda') =$ point, from whence by 7.24, λ' is the point in $H^1(k,G^{sc})$. It is then immediate that $\lambda = \pi(\lambda')$ is the the point in $H^1(k,G)$. □

Remark 7.26. One can avoid the extra restrictions on the characteristic k in the last 3 results by giving a proof in the flat cohomology sets $H^1_{\mathrm{fppf}}(*,*)$ as defined in [38]. Since G is by assumption smooth, $H^1_{\mathrm{fppf}}(k,G) = H^1(k,G)$.

7.4 Main Result

Theorem 7.27. *Let k be a field of characteristic different from 2. Let $\{L_i\}_{1\leq i\leq m}$ be a set of finite field extensions of k and let $\gcd([L_i : k]) = d$. Let G be a simply connected or adjoint semisimple algebraic k-group which does not contain a simple factor of type E_8 and such that every exceptional simple factor of type other than G_2 is quasisplit. If d is coprime to $S(G)$, then the canonical map*

$$H^1(k, G) \to \prod_{i=1}^{m} H^1(L_i, G)$$

has trivial kernel.

Proof. Write G as a product of groups of the form $R_{E_j/k}G_j$ where each G_j is an absolutely simple, simply connected or adjoint group and each E_j is a finite separable field extension of k. It is sufficient to consider a group of the form $R_{E/k}G$ for an absolutely simple group G and a finite separable field extension E of k. By 2.23, $H^1(k, R_{E/k}G) \cong H^1(E, G)$ and $\prod_i H^1(L_i, R_{E/k}G) \cong \prod_i H^1_{\text{et}}(L_i \otimes E, G)$ where the subscript et denotes the étale cohomology as in [26].

Since E is separable, for each index i, $E \otimes L_i \cong \prod_s L_{i,s}$ for $L_{i,s}$ finite extensions of E and therefore $H^1_{\text{et}}(L_i \otimes E, G) \cong \prod_s H^1(L_{i,s}, G)$. Let d' be the greatest common divisor of $\sum_{i,s}[L_{i,s} : k]$. Since for each i, $\sum_s [L_{i,s} : k] = [L_i : k]$, then d' divides d and thus d' is coprime to $S(G)$. Thus the map $H^1(E, G) \to \prod_i \prod_s H^1_{\text{et}}(L_{i,s}, G)$ has trivial kernel in view of 7.1, 7.13 and 7.21 above and thus the map $H^1(k, G) \to \prod_{i=1}^{m} H^1(L_i, G)$ has trivial kernel. $\square$

Chapter 8

Results over Virtual Cohomological Dimension 2 Fields

8.1 Preliminaries

Recall that given a connected linear algebraic group G, the *unipotent radical* of G denoted G^u is the maximal connected unipotent normal subgroup of G. It is clear that G/G^u is always a reductive group. We denote G/G^u by G^{red}. The commutator subgroup of G^{red} is denoted G^{ss}. We denote the simply connected cover of G^{ss} by G^{sc}. In the discussion which follows we will need the following lemmas.

Lemma 8.1. *Let k be a field and let G be a reductive group over k. Fix an integer n and a quasitrivial torus T such that $G^n \times T$ admits a special covering*

$$1 \to \mu \to G_0 \times S \to G^n \times T \to 1$$

Then G^{sc} satisfies a Hasse Principle over k if and only if G_0 satisfies a Hasse principle over k.

Proof. Taking commutator subgroups we have a short exact sequence

$$1 \to \tilde{\mu} \to [G_0 \times S, G_0 \times S] \to [G^n \times T, G^n \times T] \to 1$$

Since S and T are tori, $[G_0 \times S, G_0 \times S] \cong [G_0, G_0]$ and $[G^n \times T, G^n \times T] = [G^n, G^n]$. That G_0 is semisimple gives $[G_0, G_0] = G_0$. It is clear that $[G^n, G^n] = [G, G]^n$ which in turn is $(G^{ss})^n$ by definition of G^{ss}. Therefore, we have the following short exact sequence

$$1 \to \tilde{\mu} \to G_0 \to (G^{ss})^n \to 1$$

where $\tilde{\mu}$ is some finite group. In particular, G_0 is a simply connected cover of $(G^{ss})^n$. Since $(G^{sc})^n$ is certainly a simply connected cover of $(G^{ss})^n$, uniqueness of the simply connected cover of $(G^{ss})^n$ gives $(G^{sc})^n \cong G_0$. In particular, the simple factors of G^{sc} are the same as the simple factors of G_0 and G^{sc} satisfies the Hasse principle over k if and only if G_0 satisfies the Hasse principle over k. □

Lemma 8.2. *Let k be a real closed field and let G be a reductive group over k which admits a special covering*

$$1 \to \mu \to G_0 \times S \to G \to 1 \tag{8.2.1}$$

Let L be a finite étale k-algebra. Let δ be the first connecting map in Galois Cohomology and let $N_{L/k}$ denote the corestriction map $H^1(k \otimes L, \mu) \to H^1(k, \mu)$. Then

$$N_{L/k}(\mathrm{im}(G(k \otimes L) \xrightarrow{\delta_L} H^1(k \otimes L, \mu)) \subset \mathrm{im}(G(k) \xrightarrow{\delta} H^1(k, \mu))$$

Proof. Since k is perfect, L is separable and $k \otimes L$ is a product of finite extensions of k. By 5.12 the only field extensions of k are k itself and $k(\sqrt{-1})$. Thus there exists finite numbers r and s such that $k \otimes L$ is isomorphic to a product of r copies of k and s copies of $k(\sqrt{-1})$. Thus

$$H^1(k \otimes L, \mu) \cong \prod_{r \text{ copies}} H^1(k, \mu) \prod_{s \text{ copies}} H^1(k(\sqrt{-1}), \mu)$$

Since k is real closed, $k(\sqrt{-1})$ is algebraically closed, $H^1(k(\sqrt{-1}), \mu)$ is trivial and $H^1(k \otimes L, \mu)$ is just a product of r copies of $H^1(k, \mu)$. Therefore,

$$N_{L/k} : H^1(k \otimes L, \mu) \to H^1(k, \mu)$$

is just the product map

$$\prod_{r \text{ copies}} H^1(k, \mu) \to H^1(k, \mu)$$

That $k \otimes L$ is a product of r copies of k and s copies of $k(\sqrt{-1})$ also gives that

$$G(k \otimes L) \cong \prod_{r \text{ copies}} G(k) \prod_{s \text{ copies}} G(k(\sqrt{-1}))$$

Therefore, the connecting map

$$\prod_{r \text{ copies}} G(k) \prod_{s \text{ copies}} G(k(\sqrt{-1})) \xrightarrow{\delta} \prod_{r \text{ copies}} H^1(k,\mu) \prod_{s \text{ copies}} H^1(k(\sqrt{-1}),\mu)$$

is just the product of the connecting maps

$$G(k) \to H^1(k,\mu)$$

and

$$G(k(\sqrt{-1}) \to H^1(k(\sqrt{-1}),\mu)$$

the latter of which is necessarily the trivial map.

So choose

$$(x_1,\dots,x_r,y_1,\dots,y_s) \in G(k \otimes L)$$

Then

$$\begin{aligned} N_{L/k}(\delta(x_1,\dots,x_r,y_1,\dots,y_s)) &= N_{L/k}(\delta(x_1),\dots,\delta(x_r),\delta(y_1),\dots,\delta(y_s)) \\ &= \delta(x_1)\cdots\delta(x_r) \\ &= \delta(x_1\cdots x_r) \end{aligned}$$

Since the x_i were chosen to be in $G(k)$ for all i, then $x_1\cdots x_r \in G(k)$ and the desired result holds. □

Lemma 8.3. *Let k be a field and let V denote the set of orderings of k. Let G be a reductive group and L be a finite field extension of k of odd degree. The kernel of the canonical map $H^1(k,G) \to H^1(L,G)$ is contained in the kernel of the canonical map $H^1(k,G) \to \prod_{v\in\Omega} H^1(k_v,G)$.*

Proof. By 5.10 each ordering v of k extends to an ordering w of L. In particular each real closure k_v is L_w for some ordering w on L. Since the natural map $H^1(k,G) \to H^1(L_w,G)$ factors through the canonical map $H^1(k,G) \to H^1(L,G)$, the desired result is immediate. □

8.2 Main Result

We now return to the result which is the main goal of this chapter.

Theorem 8.4. *Let k be a perfect field of virtual cohomological dimension ≤ 2 and let G be a connected linear algebraic group over k. Let $\{L_i\}_{1\leq i\leq m}$ be a set of finite field extensions of k such that the greatest common divisor of the degrees of the extensions $[L_i : k]$ is 1. If G^{sc} satisfies a Hasse principle over k, then the canonical map*

$$H^1(k, G) \to \prod_{i=1}^{m} H^1(L_i, G)$$

has trivial kernel.

Proof. By 4.15, the natural map $H^1(k, G) \to H^1(k, G^{\text{red}})$ has trivial kernel. Thus to prove 8.4 it is sufficient to consider the case where G is a reductive group. Then fix an integer n and quasitrivial torus T such that $G^n \times T$ admits a special covering

$$1 \to \mu \to G_0 \times S \to G^n \times T \to 1$$

By functoriality, $H^1(k, G^n \times T) \cong H^1(k, G)^n \times H^1(k, T)$ and since T is quasitrivial, $H^1(k, T) = 1$. It follows that our result holds for G if and only if it holds for $G^n \times T$. Replacing G by $G = G^n \times T$ we assume that G admits a special covering

$$1 \to \mu \to G_0 \times S \to G \to 1$$

If k is a field of positive characteristic it is not formally real and thus has no orderings. Since by hypothesis G^{sc} satisfies a Hasse principle over k then $H^1(k, G^{sc}) = \{1\}$. In particular $H^1(k, G_0) = \{1\}$ and the special covering of G above induces the following commutative diagram with exact rows

$$\begin{tikzcd}
1 \arrow[r] & H^1(k,G) \arrow[r, "h"] \arrow[d, "q"] & H^2(k,\mu) \arrow[d, "r"] \\
1 \arrow[r] & \prod_i H^1(L_i,G) \arrow[r] & \prod_i H^2(L_i,\mu)
\end{tikzcd} \tag{8.4.1}$$

Choose $\lambda \in \ker(q)$. By commutativity of the diagram $h(\lambda) \in \ker(r)$ where r is the product of the restriction maps $H^1(k, G) \to H^1(L_i, G)$. By 3.12 and 2.9, r has trivial kernel. Thus $h(\lambda) = $ point. Then by exactness of the top row of the diagram, $\lambda =$ point.

Therefore, we may assume that the characteristic of k is zero. Fix an index i. The special covering of G above induces the following commutative diagram with exact rows where the vertical maps are the restriction maps.

$$\begin{tikzcd} H^1(k,\mu) \arrow[r, "f"] \arrow[d] & H^1(k,G_0) \arrow[r, "g"] \arrow[d, "p"] & H^1(k,G) \arrow[r, "h"] \arrow[d, "q"] & H^2(k,\mu) \arrow[d, "r"] \\ \prod H^1(L_i,\mu) \arrow[r, "f"] & \prod H^1(L_i,G_0) \arrow[r, "g"] & \prod H^1(L_i,G) \arrow[r] & \prod H^2(L_i,\mu) \end{tikzcd} \tag{8.4.2}$$

Let λ be in $\ker(q)$. Taking $\mathrm{cor}\circ\mathrm{res}$ we find that r has trivial kernel and thus by commutativity of (8.4.2), λ is in $\ker(h)$. By exactness of the top row, we choose $\lambda' \in H^1(k,G_0)$ such that $g(\lambda') = \lambda$. Write $p(\lambda') = (\lambda'_{L_i})$. Since $g(\lambda'_{L_i}) = $ point, by exactness of the bottom row of (8.4.2) choose $\eta_{L_i} \in H^1(L_i,\mu)$ such that $f(\eta_{L_i}) = \lambda'_{L_i}$.

For each ordering v of k, the special covering of G above also induces the following commutative diagram with exact rows.

$$\begin{tikzcd} H^1(k,\mu) \arrow[r, "f"] \arrow[d] & H^1(k,G_0) \arrow[r, "g"] \arrow[d, "p'"] & H^1(k,G) \arrow[r, "h"] \arrow[d, "q'"] & H^2(k,\mu) \arrow[d, "r'"] \\ \prod_{v\in\Omega} H^1(k_v,\mu) \arrow[r, "f"] & \prod_{v\in\Omega} H^1(k_v,G_0) \arrow[r, "g"] & \prod_{v\in\Omega} H^1(k_v,G) \arrow[r] & \prod_{v\in\Omega} H^2(k_v,\mu) \end{tikzcd} \tag{8.4.3}$$

By 8.3, λ is in the kernel of q'. Thus by commutativity of (8.4.3), $(\lambda'_v) = p'(\lambda')$ is in $\ker(g)$. Then by exactness of the bottom row of (8.4.3) choose $\alpha_v \in H^1(k_v,\mu)$ such that $f(\alpha_v) = \lambda'_v$. Let $(\alpha_v)_{L_i}$ denote the image of α_v under the canonical map $H^1(k_v,\mu) \to H^1(k_v \otimes L_i,\mu)$. Let $(\eta_{L_i})_v$ denote the image of η_{L_i} under the canonical map $H^1(L_i,\mu) \to H^1(k_v \otimes L_i,\mu)$.

By choice of α_v and η_{L_i}, $f((\alpha_v)_{L_i}) = (\lambda'_v)_{L_i} = (\lambda'_{L_i})_v = f((\eta_{L_i})_v)$. In particular, $f((\alpha_v)_{L_i}(\eta_{L_i})_v^{-1})$ is the point in $H^1(k_v \otimes L_i, G_0)$. We have a commutative diagram

$$\begin{tikzcd} G(k_v) \arrow[r, "\delta"] \arrow[d] & H^1(k_v,\mu) \arrow[r, "f"] \arrow[d] & H^1(k_v,G_0) \arrow[d] \\ \prod_i G(k_v\otimes L_i) \arrow[r, "\delta_{L_i}"] & \prod_i H^1(k_v\otimes L_i,\mu) \arrow[r, "f"] & \prod_i H^1(k_v\otimes L_i,G_0) \end{tikzcd} \tag{8.4.4}$$

Exactness of the bottom row of (8.4.4) gives that $(\alpha_v)_{L_i}(\eta_{L_i})_v^{-1}$ is in the image of δ_{L_i}. Choose m_i such that $\sum m_i[L_i : k] = 1$. Since δ_{L_i} is multiplicative, it follows that for each index i, $(\alpha_v)_{L_i}^{m_i}((\eta_{L_i})_v^{-1})^{m_i}$ is in the image of δ_{L_i}

By Lemma 8.2 above, there exists γ_v in $G(k_v)$ such that

$$\delta(\gamma_v) = \prod_i N_{L_i/k}((\alpha_v)_{L_i}^{m_i}((\eta_{L_i})_v^{-1})^{m_i})$$

Since by 2.7 $N_{L_i/k}((\alpha_v)_{L_i}^{m_i}) = \alpha_v^{m_i[L_i:k]}$. It follows that

$$\begin{aligned}\delta(\gamma_v) &= \prod_i N_{L_i/k}((\alpha_v)_{L_i}^{m_i}((\eta_{L_i})_v^{-1})^{m_i}) \\ &= \alpha_v^{\sum_i m_i[L_i:k]} \prod_i (N_{L_i/k}(\eta_{L_i})_v^{-1})^{m_i} \\ &= \alpha_v \prod_i (N_{L_i/k}(\eta_{L_i})_v^{-1})^{m_i}\end{aligned}$$

In turn

$$\delta(\gamma_v) \prod_i (N_{L_i/k}(\eta_{L_i})_v)^{m_i} = \alpha_v$$

Since f is well-defined on the cosets of $G(k_v)$ in $H^1(k_v, \mu)$ [25] and the top row of (8.4.4) is exact, it follows that

$$f\left(\prod_i (N_{L_i/k}(\eta_{L_i})_v)^{m_i}\right) = f(\alpha_v)$$

By choice of α_v the latter is λ'_v. Since G^{sc} satisfies a Hasse principle over k, Lemma 8.1 gives that G_0 satisfies a Hasse principle over k. In particular, the map $H^1(k, G_0) \to \prod_v H^1(k_v, G_0)$ is injective, and since $f(\prod_i (N_{L_i/k}(\eta_{L_i})^{m_i}))_v = \lambda'_v$ for all v, then

$$f\left(\prod_i (N_{L_i/k}(\eta_{L_i}))^{m_i}\right) = \lambda'$$

Taking g as in (8.4.2) above

$$g\left(f\left(\prod_i (N_{L_i/k}(\eta_{L_i}))^{m_i}\right)\right) = g(\lambda')$$

Then by exactness of the top row of (8.4.2), $\lambda = g(\lambda')$ =point. $\square$

Corollary 8.5. *Let k be a perfect field of virtual cohomological dimension ≤ 2 and let G be a connected linear algebraic group over k. Let $\{L_i\}_{1\leq i\leq m}$ be a set of finite field extensions of k such that the greatest common divisor of the degrees of the extensions $[L_i : k]$ is 1. If G^{sc} is of classical type, type F_4 or type G_2, then the canonical map*

$$H^1(k, G) \to \prod_{i=1}^{m} H^1(L_i, G)$$

has trivial kernel.

Proof. Apply 5.13 to 8.4. □

Chapter 9

Conclusion

This chapter gives a summary of the the major results contained in this work. We began with the following question due to Jean-Pierre Serre:

Q: Let k be a field and let G be a connected, linear algebraic group defined over k. Let $\{L_i\}_{1\leq i\leq m}$ be a collection of finite extensions of k with $\gcd([L_i : k]) = 1$. Does the canonical map

$$H^1(k, G) \to \prod_{i=1}^{m} H^1(L_i, G)$$

have trivial kernel?

Motivated by Bayer and Lenstra's positive result [2] for the group of isometries of an algebra with involution, we studied more general groups associated to algebras with involution. By utilizing their result, results on hermitian forms used to produce it and the Gille-Merkurjev Norm Principle [25], [15] we produced a positive answer to Serre's question for absolutely simple, simply connected and adjoint classical groups over fields of characteristic different from 2. In the case of simple split groups of type F_4, simple groups of type G_2 and quasisplit, simply connected or adjoint groups of type ${}^{3,6}D_4, E_6$ or E_7 we utilized results due Rost [30], Garibaldi [13] and Chernousov [8] on the Rost invariant to produce a positive answer to Serre's question. More precisely, we showed the following:

Theorem 9.1. *Let k be a field of characteristic different from 2. Let $\{L_i\}_{1\leq i\leq m}$ be a set of finite field extensions of k and let $\gcd([L_i : k]) = d$. Let G be a simply connected or adjoint semisimple algebraic k-group which does not contain a simple factor of type E_8 and such that every exceptional simple factor of type other than G_2 is quasisplit. If d is coprime to $S(G)$, then the canonical map*

$$H^1(k, G) \to \prod_{i=1}^{m} H^1(L_i, G)$$

has trivial kernel.

A positive answer to Serre's question for fields k and groups G as in 9.1 comes from considering the case $d = 1$.

Motivated by Sansuc's approach [31] to Serre's question over number fields we showed the following:

Theorem 9.2. *Let k be a perfect field of virtual cohomological dimension ≤ 2 and let G be a connected linear algebraic group over k. Let $\{L_i\}_{1\leq i\leq m}$ be a set of finite field extensions of k such that the greatest common divisor of the degrees of the extensions $[L_i : k]$ is 1. If G^{sc} satisfies a Hasse principle over k, then the canonical map*

$$H^1(k, G) \to \prod_{i=1}^{m} H^1(L_i, G)$$

has trivial kernel.

From whence, by the Bayer-Parimala Hasse principle [4] one obtains a positive answer to Serre's question for groups G over perfect fields of virtual cohomological dimension at most 2 with G^{sc} of classical type, type F_4 or type G_2.

Serre's question is as intriguing as it is challenging. As such, it is likely to continue to inspire interesting work in the future. We hope to continue to make contributions to its study as well as the study of related questions.

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