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Paul Vienhage Date

The Quantum McKay Correspondence:
Classifying "Finite Subgroups" of a Quantum Group with Graphs

By

Paul Vienhage
Masters of Science

Mathematics

John Duncan
Advisor

Sharron Weiner
Committee Member

David Zureick-Brown
Committee Member

Accepted:

Lisa A. Tedesco, Ph.D.
Dean of the James T. Laney School of Graduate Studies

Date

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An abstract of
A thesis submitted to the Faculty of the
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Abstract

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The McKay Correspondence classifies finite subgroups of the rotation group of 3-space via graphs. In this paper we explore a quantum version of this correspondence. Specifically, we will cover the needed background on category theory, vertex operator algebras, and quantum groups to explain a powerful technique used by Kirillov and Ostrik to develop a quantum analog to the McKay correspondence.

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1 Background

1.1 Lie groups and Lie Algebras

The following section is a quick introduction to Lie groups and Lie algebras. For a more in depth and formal introduction see Kirillov [2008] or for a reference see [Knapp, 2013]. We start with definitions of Lie group and Lie algebra. A Lie group is a group that is also a differential manifold such that the group multiplication and inversion of elements are smooth operations, where smooth is defined as the function's derivatives being continuous everywhere in the domain for all orders of derivative. A simple example of a Lie group is $GL_2(\mathbb{R})$ which is all two by two real matrices with nonzero determinant. The theory of Lie groups is highly developed and encodes the information of continuous symmetry.

A Lie algebra is a vector space over a field with a bracket operator $\mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ which has the following properties.

1. Bilinearity: $[ax + by, z] = a[x, z] + b[y, z]$ and $[z, ax + by] = a[z, x] + b[z, y]$.
2. Alternativity: $[x, x] = 0$.
3. Jacobi Identity: $[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0$

The most immediate consequence of these conditions is that $[x, y] = -[y, x]$. With this observation we get good intuition of a simple example of a Lie Algebra.

Lemma 1. $\mathbb{R}^3$ *where* $[x, y] = x \times y$ *is the standard cross product is a Lie algebra.*

Proof. We start with the proof of bilinearity (applying standard results from vector calculus):

$$(c\mathbf{u} + \mathbf{v}) \times \mathbf{w} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ cu_i + v_i & cu_j + v_j & cu_k + v_k \\ w_i & w_j & w_k \end{vmatrix} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ cu_i & cu_j & cu_k \\ w_i & w_j & w_k \end{vmatrix} + \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ v_i & v_j & v_k \\ w_i & w_j & w_k \end{vmatrix}$$

$$= c \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ u_i & u_j & u_k \\ w_i & w_j & w_k \end{vmatrix} + \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ v_i & v_j & v_k \\ w_i & w_j & w_k \end{vmatrix} = c\,(\mathbf{u} \times \mathbf{w}) + \mathbf{v} \times \mathbf{w}$$

Next we conclude that alternativity holds because any vector in $\mathbb{R}^3$ is parallel to itself so $\mathbf{x} \times \mathbf{x} = \mathbf{x} \cdot \mathbf{x} \sin(0) = 0$. To show the Jacobi identity we use Lagrange's formula:

$$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \cdot \mathbf{c})\,\mathbf{b} - (\mathbf{a} \cdot \mathbf{b})\,\mathbf{c}$$

Then it is clear that:

$$\mathbf{a}\times(\mathbf{b}\times\mathbf{c})+\mathbf{b}\times(\mathbf{c}\times\mathbf{a})+\mathbf{c}\times(\mathbf{a}\times\mathbf{b}) = (\mathbf{a}\cdot\mathbf{c})\mathbf{b} - (\mathbf{a}\cdot\mathbf{b})\mathbf{c}+(\mathbf{b}\cdot\mathbf{a})\mathbf{c} - (\mathbf{b}\cdot\mathbf{c})\mathbf{a}+(\mathbf{c}\cdot\mathbf{b})\mathbf{a}-(\mathbf{c}\cdot\mathbf{a})\mathbf{b} = 0$$

□

We speak about many properties of Lie algebras in the same way as rings or associative algebras, with subalgebra, ideal, homomorphism, factor algebra and others defined in a way which attempts to encode similar information as for rings. The dimension of a Lie algebra is its dimension as a vector space. Lie groups are in correspondence with Lie algebras via a functor that maps the category of Lie groups to the category of finite dimensional real Lie algebras.

A simple Lie algebra is a non-abelian Lie algebra whose only ideals are 0 and itself. Note this definition excludes a one dimensional Lie algebra from being simple as they are necessarily abelian. A semisimple Lie algebra is a direct sum of simple Lie Algebras. The Killing form is a symmetric bilinear form defined by $B(x, y) = trace(ad(x)ad(y))$ where $ad(x)$ is the matrix form of the adjoint endomorphism of $\mathfrak{g}$ given by $ad(x)y = [x, y]$. A Lie algebra is semisimple if and only if the Killing form is non-degenerate.

We can classify semisimple Lie algebras according to their root systems. To explain this note that a Cartan subalgebra is a subalgebra of a Lie algebra which is nilpotent and self normalizing. The roots of a semisimple Lie algebra $\mathfrak{g}$ with Cartan subalgebra $\mathfrak{h}$ are those elements $\alpha \in \mathfrak{h}^*$ (where $\mathfrak{h}^*$ is the vector space dual to $\mathfrak{h}$) such that there is an $X \in G$ with the property $[H, X] = \alpha(H)X$ for all $H \in \mathfrak{h}$. The roots are the non-zero weights of the adjoint representation of $\mathfrak{g}$, a concept we will define in the next section. A root system is a set of vectors $\Phi \in V = \mathbb{R}^n$ with the following properties:

1. The roots span V.
2. The only scalar multiples of a root $\alpha \in \Phi$ that belong to Φ are α itself and $-\alpha$.
3. For every root $\alpha \in \Phi$, the set Φ is closed under reflection through the hyperplane perpendicular to α .
4. If α and β are roots in Φ , then the projection of β onto the line through α is an integer or half-integer multiple of α.

We define an inner product as a map $\langle \cdot, \cdot \rangle : V \to F$ with the following conditons:

1. $\langle x, y \rangle = \overline{\langle x, y \rangle}$
2. $\langle ax + by, z \rangle = a\langle x, z \rangle + b\langle y, z \rangle$
3. $\langle x, x \rangle \geq 0$ and $\langle x, x \rangle = 0$ if and only if $x = 0$.

We can define $\Delta \subset \Phi$ to be a set of simple roots if its elements span V and cannot be formed as the linear combination of other roots. Then it is clear that each root can be expressed as the linear combination of simple roots with integer coefficients. For a root system Φ and a choice of simple roots Δ we can define a valuation map which makes Φ into a poset. Define $ht : \Phi \to \mathbb{Z}$ by:

$$ht(\alpha) = \sum_{i=1}^{n} k_i \text{ where } \alpha = \sum_{\alpha_i \in \Delta} k_i \alpha_i$$

We define the maximal root (with respect to some set of simple roots Δ) as the max under the ordering given by ht. For this max root we define the Coxeter number as:

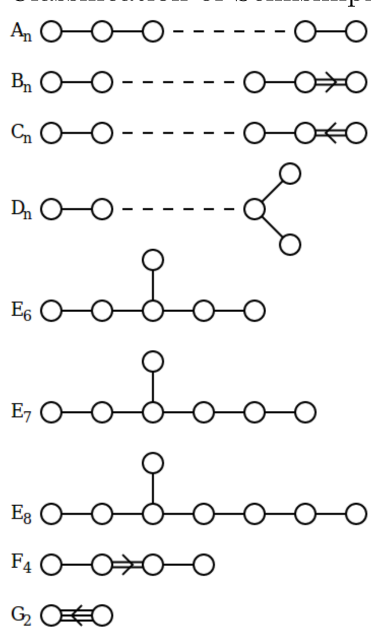
$$h = \sum_{i=1}^{n} k_i \text{ where } \alpha_j \text{ is the max root and } \alpha_j = \sum_{\alpha_i \in \Delta} k_i \alpha_i$$

We can define an "inverse" of the root system the dual roots by giving the dual of each root by $\alpha^\vee = \frac{2}{\langle \alpha, \alpha \rangle} \alpha$. We have a dual root system $\Phi^\vee$, dual simple roots $\Delta^\vee$, and a dual Coxeter number $h^\vee$ given in nearly identical ways to the originals.

The roots defined by our Lie algebra $\mathfrak{g}$ have inner product given by the Killing form. This inner product makes them a root system. Since the root system definition is highly restrictive all possible ones are well understood. Then it is a combinatorial exercise to classify all possible root systems via Dynkin Diagrams. This is well covered in Bosshardt [2012].

The following are all possible simple Lie algebras over complex numbers. $A_n : \mathfrak{sl}_{n+1}$ the special linear Lie algebra, the algebra given by $n+1$ by $n+1$ matrices with zero trace with bracket the usual commutator $[X, Y] = XY - YX$. $B_n : \mathfrak{so}_{2n-1}$ the special orthogonal Lie algebra for odd dimensions, skew-symmetric $n \times n$ matrices with the bracket given by the commutator. $C_n : \mathfrak{sp}_{2n}$ the symplectic Lie algebra, defined as the Lie algebra of matrices of size $2n$ that satisfy $M^T \Omega M = \Omega$ with Ω being the skew diagonal block matrix with I_n in the first row and $-I_n$ in the second. $D_n : \mathfrak{so}_{2n}$ the even dimensional special orthogonal Lie algebra. Then there are five exceptional Lie Algebras: E_6, E_7, E_8, F_4, and G_2. For descriptions of these Lie algebras see Cartan [1894]. The root systems of these Lie algebras are given by the Dynkin Diagrams in Figure 1.

Figure 1: The Classification of Semisimple Lie Algebras



These have Coxeter and dual Coxeter numbers given in table below:

Dynkin Diagram	Coxeter Number	Dual Coxeter Number
A_n	n+1	n+1
B_n	2n	2n-1
C_n	2n	n+1
D_n	2n-2	2n-2
E_6	12	12
E_7	18	18
E_8	30	30
F_4	12	9
G_2	6	4

For any root system we define the Dynkin diagram with a node at each root and a number of edges depending on the angle between the edges. Namely, we have no edges if the roots are perpendicular, a single undirected edge if they are 120 degrees, a double directed edge if they are 135 degrees apart, and a triple directed edge if they are 150 degrees apart. Note that we direct the edge from the longer to shorter root. In the case of semisimple Lie algebras this means the nodes of the Dynkin diagram are the roots of the Lie algebra and the number of edges is given by the value of the Killing form on the pairs of roots.

1.2 Representation Theory

A representation ϕ is defined as a homomorphism from a group G into $GL_n(K)$, where K is some field and n is called the dimension of the representation. An irreducible representation ϕ is one such that the vector space that the representation acts on has no proper non-trivial invariant subspaces. More specifically, given a representation $\varphi : G \to GL_n(K)$ if there does not exist a non-trivial proper subspace $W \subset V$ such that for all $g \in G$, $\varphi(g)W \subset W$ then it is irreducible. Irreducibility is closely related to the concept of simplicity of a representation. We define the direct sum of two representations (φ, V) and (ς, W), denoted $(\varphi \oplus \varsigma, V \oplus W)$ as $\varphi(g) \oplus \varsigma(g)$ for each element of G.

A representation is called decomposable it can be expressed as the direct sum of other representations. A representation φ is called simple if it is irreducible and indecomposable. These definitions are a way of formally talking about how representations are built from "smaller" representations. We should be careful though as there are indecomposable reducible representations, which contain sub-representations (restrictions of the representation which are also representations) but are not a direct sum of them. However we do know that these conditions are equivalent on finite or compact groups and semisimple Lie groups. Specifically if G is a finite group or a connected compact Lie group over a field of characteristic zero then its finite dimensional representations are the sum of irreducible representations. Thus if G is indecomposable it is the sum of a single irreducible representation and thus simple. Next we will define the tensor product of representations of a group G: given a pair of representations of G (φ, V) and (ϱ, W) we define $(\varphi \otimes \varrho, V \otimes W)$ element wise on $g \in G$ as $\varphi \otimes \varrho(g) = \varphi(g) \otimes \varrho(g)$.

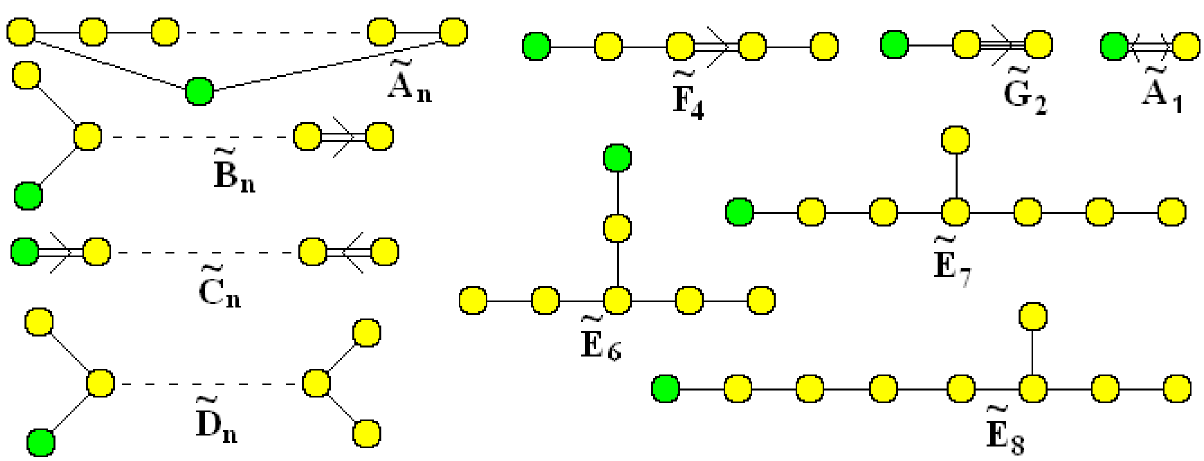
We next extend the previous definitions of representations to Lie algebras. Consider some Lie algebra $\mathfrak{g}$ and some vector space V. A representation of a Lie algebra is a homomorphism $\phi : \mathfrak{g} \to \mathfrak{gl}_n(K)$, where $\mathfrak{gl}_n(K)$ is the algebra of n by n matrices over K with the bracket given by the commutator. In other words this is a mapping $\phi([X,Y]) = \phi(X)\phi(Y) - \phi(Y)\phi(X)$. Note that the mapping and vector space are called a $\mathfrak{g}$-module. Given a pair of representations of a Lie algebra (φ, V) and (ϱ, W) we define their tensor product as $(\varphi \otimes \varrho, V \otimes W)$ with $\varphi \otimes \varrho$ given elementwise on $X \in \mathfrak{g}$ by $\varphi \otimes \varrho(X) = \varphi(X) \otimes I + I \otimes \varrho(X)$. Most other properties of representations on groups hold for Lie algebras with some simple alterations.

An important construction in the representation theory of Lie algebras is the universal enveloping algebra, which is an associative algebra whose representations are the same as those of the Lie algebra you used to construct it. It is denoted $U(\mathfrak{g})$ and its construction is given in 1.6. For further information on representation theory see Fulton and Harris [2013].

1.3 The McKay Correspondence

The McKay Correspondence is a relationship between finite subgroups of $SL(2, \mathbb{C})$ and a family of Dynkin diagrams called the affine simply laced Dynkin diagrams. The affine Dynkin diagrams are those classifying affine Lie algebras, they resemble standard Dynkin diagrams with an additional node. The affine Lie algebras are infinite dimensional extensions of the semisimple Lie Algebras, for a complete definition and introduction see Wray [2008]. Figure 2 demonstrates the classification of affine Lie algebras. Note that the green nodes are those which are added to the standard Dynkin diagrams.

Figure 2: The Classification of Affine Lie Algebras



We define simply laced Dynkin diagrams as those which have no multiple edges. These diagrams are important to the study of geometric representation theory which has produced many powerful results about representation theory. Another important object in geometric representation theory is the McKay quiver. A quiver is a directed graph which allows loops and multiple edges. The McKay graph is a quiver whose properties are defined based on the properties of a representation. We construct it for a representation of a group (ϕ, V) by creating a node for each irreducible representation of G. Then we add $k_{i,j}$ edges from node V_i to node V_j if V_j occurs $k_{i,j}$ times in the decomposition of $V_i \otimes V$ into irreducible sub-representations. For each finite subgroup H of $SL(2, \mathbb{C})$ we have a representation which is the canonical embedding $H \to SL(2, \mathbb{C})$, this will have some quiver. The McKay Correspondence says that the McKay quivers arising from subgroups $H \subset SL(2, \mathbb{C})$ in this way are the simply laced affine Dynkin diagrams.

Theorem 2 (McKay 1980)**.** *: Let G be a non-trivial finite subgroup of $SL(2, \mathbb{C})$ then the McKay quiver of the canonical representation $\phi : G \to SL(2, \mathbb{C})$ is a simply laced affine Dynkin diagram. For explicit classification see the following table from Sun [2010].*

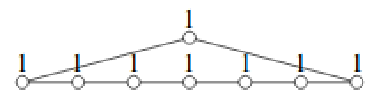
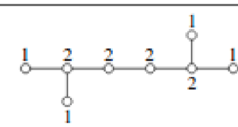
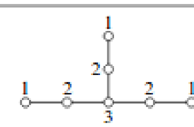
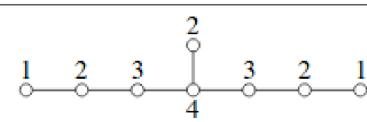
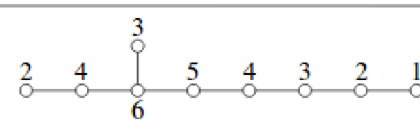
Finite subgroup of $SU(2)$		Affine simply laced Dynkin diagram	
$\mathbb{Z}/n\mathbb{Z}$	$\langle x \mid x^n = 1 \rangle$	$\widetilde{A}_{n-1}$	
$\mathbb{B}D_{2n}$	$\langle x, y, z \mid x^2 = y^2 = y^n = xyz \rangle$	$\widetilde{D}_{n-2}$	
$\mathbb{BT}$	$\langle x, y, z \mid x^2 = y^3 = z^3 = xyz \rangle$	$\widetilde{E}_6$	
$\mathbb{BO}$	$\langle x, y, z \mid x^2 = y^3 = z^4 = xyz \rangle$	$\widetilde{E}_7$	
$\mathbb{BD}$	$\langle x, y, z \mid x^2 = y^3 = z^5 = xyz \rangle$	$\widetilde{E}_8$	

Figure 3: The Explicit form of the McKay Correspondence

1.4 Category Theory

We now diverge into an entirely different topic: Category Theory. At the basic level Category Theory is a study of composable mappings between objects. It is phrased in such generality that given a category we can make into a meta-level category and in this meta-level category we then can impose structure which allows us to make conclusions about the mappings in lower level structures. This is the primary motivation of category theory, by studying the natural ways that objects can map to other objects we can make conclusions about mappings in many radically different fields.

A category $\mathcal{C}$ is a pair of classes $(\mathrm{obj}(\mathcal{C}), \mathrm{hom}(\mathcal{C}))$. The elements of $\mathrm{obj}(\mathcal{C})$ are called objects. The elements of $\mathrm{hom}(\mathcal{C})$ are directed arrows from some object $a \in \mathcal{C}$ to $b \in \mathcal{C}$, the collection of all morphisms from a to b is $hom(a, b)$. Finally a category has a binary operation $\circ : hom(b, c) \times hom(a, b) \to hom(a, c)$ which is associative and has some identity $id_x : x \to x$ for all $x \in \mathcal{O}$.

A functor is a mapping between categories that respects objects and morphisms. More specifically, $F : \mathcal{C} \to \mathcal{D}$ is a functor if for all $X \in \mathcal{C}$ there is a $F(X) \in \mathcal{D}$ and for all $f : X \to Y \in Mor(\mathcal{C})$ there is $F(f) : F(X) \to F(Y) \in \mathcal{D}$ where $F(id_X) = id_{F(X)}$ and $F(fg) = F(f)F(g)$. An example of a category is $Vec(k)$ whose objects are any vector space over the field k, and whose morphisms are linear transformations of vector spaces. $Vec_f(k)$ is the same limited to finite vector spaces. The categories used in this paper will be more structured and will generally be braided monoidal abelian categories, which we will now define and describe.

First we need to understand some abstractions of familiar concepts. Given two objects in a category X_1, X_2 we define a product as a triple $(X_1 \times X_2, \pi_1, \pi_2)$ where $X_1 \times X_2$ is an object in our category which we denote by X and $\pi_1 : X \to X_1$ $\pi_2 : X \to X_2$ are morphisms which satisfy the following universal property. Given a pair of morphisms $f_1 : Y \to X_1$ $f_2 : Y \to X_2$ there exists an $f : Y \to X$ such that $\pi_1 \circ f = f_1$ and $\pi_2 \circ f = f_2$. Note that f is called the product of the morphisms f_1 and f_2. We can extend this definition to any set of objects $\{X_i\}_{i \in I}$ in the natural way. This is a generalization of products of groups, sets and topological spaces.

Next given two objects in a category X_1, X_2 we define a coproduct as a triple $(X_1 \coprod X_2, i_1, i_2)$ where $X_1 \coprod X_2$ is an object in our category which we denote by X and $i_1 : X_1 \to X$ $i_2 : X_2 \to X$ are morphisms which satisfy the following universal property: Given a pair of morphisms $f_1 : X \to X_1$ $f_2 : X \to X_2$ there exists an $f : X \to Y$ such that $f \circ i_1 = f_1$ and $f \circ i_2 = f_2$. Note that f is called the coproduct of the morphisms f_1 and f_2. We can extend this definition to any set of objects $\{X_i\}_{i \in I}$ in the natural way. The coproduct is a generalization of the disjoint union of sets and the free product of groups.

Another abstraction we consider are the constant and coconstant mappings. Given a morphism $f : X \to Y$ we call f a constant mapping if for all objects W in $\mathcal{C}$ and all pairs of morphisms $g, h : W \to X$ we have that $f \circ g = f \circ h$. We call f coconstant if for all objects Z and pairs of morphisms $g, h : Y \to Z$ we have that $g \circ f = h \circ f$. A mapping f which is constant and coconstant is called a zero mapping. A category with zero morphisms is one where, for every two objects A and B in $\mathcal{C}$, there is a fixed morphism $0_{A,B} : A \to B$ and for all objects $X, Y, Z \in \mathcal{C}$ and all morphisms $f : Y \to Z$, $g : X \to Y$ we have that Figure 4 commutes.

$$\begin{tikzcd}
X \arrow[r, "0_{XY}"] \arrow[d, "g"'] \arrow[dr, "0_{XZ}"] & Y \arrow[d, "f"] \\
Y \arrow[r, "0_{YZ}"'] & Z
\end{tikzcd}$$

Figure 4: A commuting digram in a category with zero morphisms

In a category with zero morphisms we can define generalizations of kernel and cokernel. Given some morphism $f : X \to Y$ in our category we define the kernel as the pair (K, k) where K is an object in our category and $k : K \to X$ is a morphism which satisfies that:

1. We have that $f \circ k = 0_{K,Y}$.
2. Given (K', k') where K' is an object and $k' : K' \to X$ is a morphism with $f \circ k' = 0_{K',Y}$ then we have a morphism $\mu : K' \to K$ where $k \circ \mu = k'$.

The kernel is a generalization of the kernel of group homomorphisms (canonical embedding is our k and the subgroup is our K). Next we define the cokernel of a mapping $f : X \to Y$ as the pair (Q, q) where Q is an object and $q : Y \to Q$ is a morphism with the following properties:

1. We have that $q \circ f = 0_{K,Y}$.
2. Given (Q', q') where Q' is an object and $q' : Y \to Q'$ is a morphism with $q' \circ f = 0_{Q',Y}$ then we have a morphism $\mu : Q' \to Q$ where $q \circ \mu = q'$.

The cokernel's simplest form is in the category of abelian groups where for a group homomorphism $\phi : G \to H$ the cokernel is the quotient group $H/im(\phi)$. Using these generalizations we can define an abelian category $\mathcal{C}$ as one which has the following four properties:

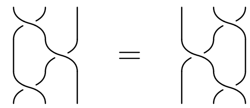
1. There exists a $0 \in \mathcal{O}$, such that for all $X \in \mathcal{C}$ there exists exactly one morphism $f : X \to 0$ and exactly one morphism $g : 0 \to X$.
2. $\mathcal{C}$ has all possible bi-products, ie that for any collection of objects X_i there is an object $X_1 \oplus X_2 \ldots \oplus X_n$ and morphisms $p_k : X_1 \oplus X_2 \ldots \oplus X_n \to X_k$ and $i_k : X_k \to X_1 \oplus X_2 \ldots \oplus X_n$ such that $(X_1 \oplus X_2 \ldots \oplus X_n, p_k)$ is a product and $(X_1 \oplus X_2 \ldots \oplus X_n, i_k)$ is a coproduct.
3. For every morphism $f \in hom(\mathcal{C})$ there must exist a kernel and cokernal in $\mathcal{C}$.
4. Every monomorphism (morphisms which left cancel under compositions) should be the kernel of its cokernel, and every epimorphism (morphisms which right cancel under compositions) is the cokernel of its kernel.

Monoidal Categories add a tensor product $\otimes$ which is a bifunctor $\mathcal{C} \times \mathcal{C} \to \mathcal{C}$ that has a unit object I such that for all $X \in \mathcal{C}$, $I \otimes X \cong X \otimes I \cong X$. Note that we often denote I as $\mathbf{1}$ and we say a monoidal category $\mathcal{C}$ is over complex numbers if $I = \mathbb{C}$. A braided tensor category is one that has a tensor product as described above, and it has a collection of morphisms $\sigma_{X_1,X_2} : X_1 \otimes X_2 \to X_2 \otimes X_1$ where X_1, X_2 are expressions formed from brackets, the unit object, and any objects from $\mathcal{O}$. Each $\sigma_{V,W}$ is uniquely identified with an element of the braid group, which we now explain.

The braid group has several realizations; one is the motion of a collection of points in a disk over time, but it is perhaps easier to visualize as braids of strands of string where the operation is concatenation and topologically equivalent strings are identified. Symbolically the braid group on n-strands is given by the following generators and relations:

$$B_n \doteq \langle \sigma_1, \sigma_2, \cdots \sigma_{n-1} \mid \sigma_{i-1}\sigma_i\sigma_{i-1} = \sigma_i\sigma_{i-1}\sigma_i \text{ and } \sigma_i\sigma_j = \sigma_j\sigma_i \text{ if } |i-j| \geq 2 \rangle.$$

We give an example of a pair of equivalent braids of B_3 below which demonstrates the braid relation. The equivalence of these braids gives the Yang-Baxter equation in braided tensor categories.



Now we explain duality for categories. The notion of duality in a category is motivated by the concept of duality in vector spaces. Let $V \in \mathcal{O}$ of some tensor category. A right dual of V is an object V^* and morphisms $e_V : V^* \otimes V \to \mathbf{1}$ and $i_V : \mathbf{1} \to V \otimes V^*$ such that:

1. The mapping $V \to V \otimes V^* \otimes V \to V$ by $i_v \otimes id_V \circ id_V \otimes e_V$ is the identity
2. The mapping $V^* \to V^* \otimes V \otimes V^* \to V^*$ by $i_V \otimes id_{V^*} \circ id_{V^*} \otimes e_V$ is equal to id_V

These two conditions are called the rigidity axioms. Also e_V is called the called counit (or evaluation map), and i_V is called the unit (or coevaluation map). We define a left dual with ${}^*V$ and morphisms which place the dual on the left (reversing the above definitions). Given a definition of dual object we can define the dual of a mapping. Given two objects V, W which have right duals (V^*, i_V, e_V) and (W^*, i_W, e_W) and a morphism $f : V \to W$ we define a mapping $f^* : W^* \to V^*$ as

$$f^* : W^* \xrightarrow{id_{W^*} \otimes e_v} W^* \otimes V \otimes V^* \xrightarrow{id_{W^*} \otimes f \otimes id_{V^*}} W^* \otimes W \otimes V^* \xrightarrow{i_W} V^*$$

A rigid monodial category is one which has all left and right duals.

A rigid braided tensor category is called a ribbon category if it has a functorial isomorphism $\delta_V : V \to V^{**}$ which satisfies the following conditions.

1. $\delta_{V \otimes W} = \delta_V \otimes \delta_W$
2. $\delta_{\mathbf{1}} = id$
3. $\delta_{V^*} = (\delta_V^*)^{-1}$

In other literature ribbon categories are sometimes called balanced rigid braided tensor categories. The representation category of a quantum group at a root of unity is a ribbon category.

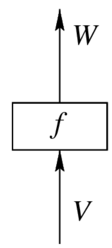
In any rigid braided tensor category $\mathcal{C}$ we have a functorial isomorphism $\psi_V : V^{**} \to V$ defined by $V^{**} \xrightarrow{i \otimes id} V \otimes V^* \otimes V^{**} \xrightarrow{id \otimes \sigma^{-1}} V \otimes V^{**} \otimes V^* \xrightarrow{id \otimes e} V$. This map allows the construction of twists in a category. A twist is a functorial isomorphism $\theta_V = \psi_V \delta_V : V \to V$. These satisfy the balancing axioms:

1. $\theta_{V \otimes W} = \sigma_{V,W} \sigma_{W,V} (\theta_V \otimes \theta_W)$
2. $\theta_1 = id$
3. $\theta_{V^*} = (\theta_V)^*$

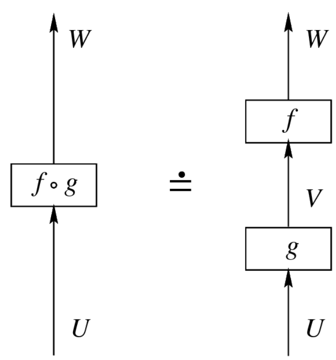
We define the trace of an endomorphism f of an object V in a ribbon category $\mathcal{C}$ by $\mathbf{1} \xrightarrow{i_v} V \otimes V^* \xrightarrow{f \otimes id} V \otimes V^* \xrightarrow{\delta_V \otimes id} V^{**} \otimes V^* \xrightarrow{e_{V^*}} \mathbf{1}$. The trace is denoted $tr(f)$ and the dimension of some object V is $tr(id_V)$. The name ribbon is partially inspired by the graphical calculus detailed in Bakalov and Kirillov [2001] which we explain in the next section.

1.5 Categorical Graphical Calculus

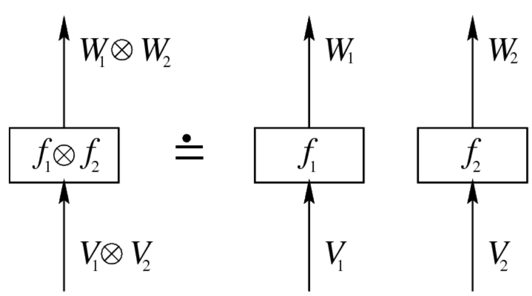
The following directly mirrors the introduction to this content in Bakalov and Kirillov [2001] which is where the diagrams are sourced from. We denote the morphisms of objects from bottom to top with the object on the bottom as the source and the top as the target. For example $f : V \to W$ is written as:



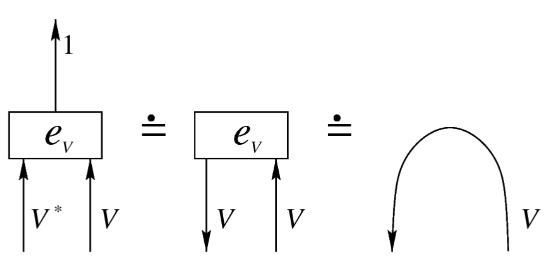
We denote a composition of morphisms by stacking such diagrams.



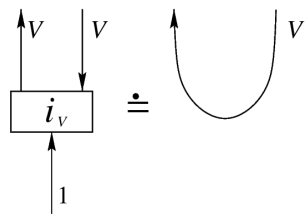
Then we denote the tensor product by placing the arrows next to each other, for $f_1 : V_1 \to W_2$ and $f_2 : V_2 \to W_2$ then we denote $f_1 \otimes f_2 = V_1 \otimes V_2 \to W_1 \otimes W_2$ by:



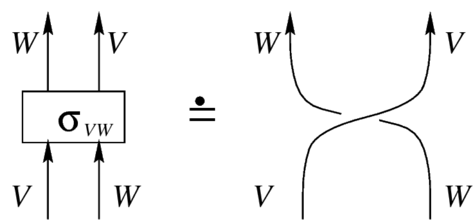
The empty diagram represents the identity morphism as we can remove or place an identity arrow at any point. Next we turn to duals, the evaluation map $e_V : V^* \otimes V \to \mathbf{1}$:



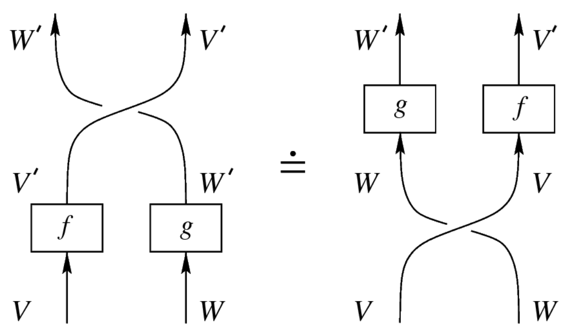
The other dual map, coevaluation $i_V : \mathbf{1} \to V \otimes V^*$ is denoted:



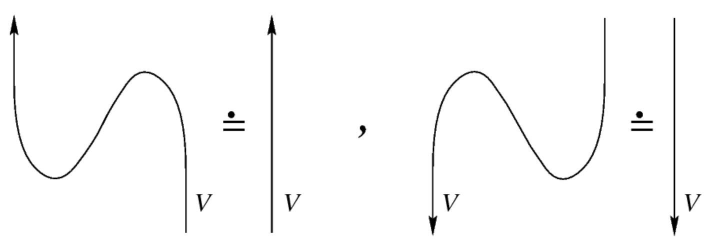
The braiding is very similar to the way braid groups are normally denoted. With $\sigma_{V,W}$: $V \otimes W \to W \otimes V$ as the following and the inverse with the strands crossed opposite to the following:



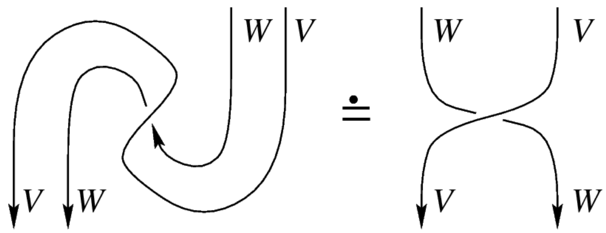
We present two examples, the funtorality of twists presented visually:



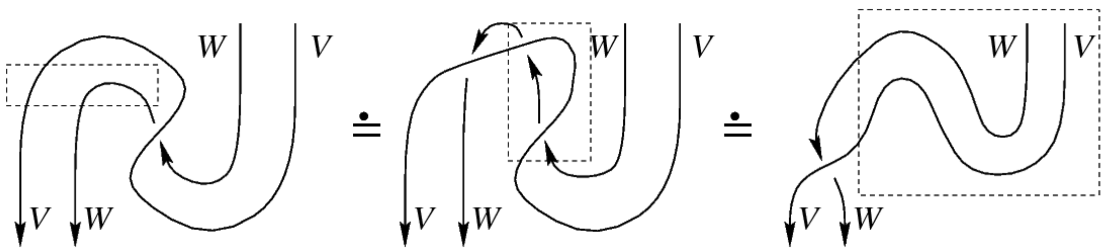
And the rigidity axioms expressed visually:



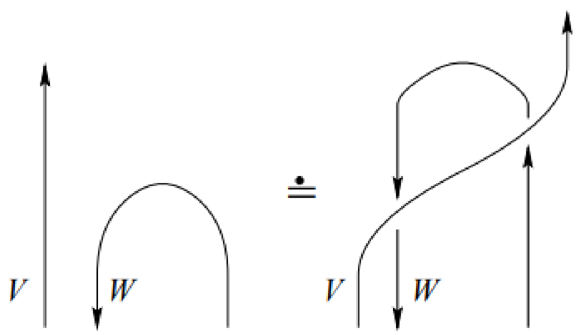
We have seen these presented earlier in the background, so to demonstrate the usefulness of this graphical calculus we present a proof via the graphical calculations and then one from axioms.



The graphical proof is:



Followed by an application of rigidity. The first step uses that:

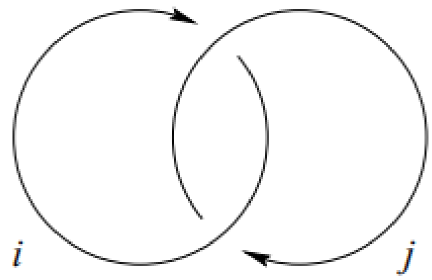


This graphically expresses that $id_V \otimes e_W : V \otimes (W \otimes W^*) \to V \otimes \mathbf{1}$ is equivalent to $\sigma_{V,W\otimes W^*} \circ e_W \otimes id_V \circ \sigma_{V,\mathbf{1}} : V \otimes (W \otimes W^*) \to (W \otimes W^*) \otimes V \to \mathbf{1} \otimes V \to V \otimes \mathbf{1}$. This follows from the axioms of the ribbon category. Next we present the proof that $(\sigma_{V,W})^* = \sigma_{V^*,W^*}$ from axioms.

Proof. $(\sigma_{V,W})^* = (e_{V^*} \otimes e_{W^*}) \circ (\sigma_{V,W}) \circ (\varphi_V \otimes id_W \otimes id_{V^*} \otimes id_{W^*}) \circ (id_{V_*} \otimes e_W \otimes id_{V^*} \otimes id_{W^*}) = (e_V \otimes e_W) \circ (\sigma_{V^*,W^*} \otimes id_V \otimes id_W) \circ (\phi_V \otimes id_{W^*} \otimes id_V \otimes id_W) \circ (\sigma_{V^*,W\otimes W^*}) \circ (e_W \otimes id_V \otimes \sigma_{V^*,1}) \circ \sigma_{V,W} = \sigma_{V^*,W^*} \circ (\eta_V \otimes id_V) \otimes (\eta_W \otimes id_W) \circ \alpha_{V^*,V,V^*} n \otimes \alpha_{W^*,W,W^*} \circ (id_{V^*} \otimes \varepsilon_{V^*}) \otimes (id_{W^*} \otimes \varepsilon_{W^*}) = \sigma_{V^*,W^*}$ □

This proof is nearly incomprehensible but can be seen by tracing each element of the visual proof and they are morally identical. This example demonstrates how the graphical calculus helps to understand morphisms.

A modular tensor category $\mathcal{C}$ is a semisimple ribbon category which has finitely many isomorphism classes of simple objects and which has that the matrix $\tilde{s} = (s_{i,j})_{i,j\in I}$ is invertible with $s_{i,j}$ defined by the diagram below:

$\tilde{s}_{ij} =$ 

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1.6 Quantum Groups

We now introduce the topic of quantum groups. A quantum group in the context of this paper will be defined as a Hopf algebra which is a deformation of a semi-simple Lie algebra. The universal enveloping algebra of a semi-simple Lie algebra is an associative algebra whose representation category is equivalent to that of the Lie algebra. Formally we define it as the quotient of $T(\mathfrak{g}) = k \oplus \mathfrak{g} \oplus (\mathfrak{g} \otimes \mathfrak{g}) \oplus \cdots$ (the free tensor algebra on $\mathfrak{g}$) by the two sided ideal of $T(\mathfrak{g})$ generated by elements of the form $a \otimes b - b \otimes a - [a, b]$. We denote this algebra by $U(\mathfrak{g})$. This construction has the universal property that if we consider the canonical embedding $h : \mathfrak{g} \to U(\mathfrak{g})$ (defined by mapping h^* into the tensor algebra and composing with the quotient map) and a map to an algebra A, $\phi : \mathfrak{g} \to A$ then there exists a unique algebra homomorphism $\hat{\phi} : U(\mathfrak{g}) \to A$ such that $\phi = \hat{\phi} \circ h$. This universal property shows that $U(\mathfrak{g})$ has a representation for every representation of $\mathfrak{g}$. The Poincaré–Birkhoff–Witt theorem gives a basis for $U(\mathfrak{g})$ and also has a coordinate free form see Birkhoff [1937].

To motivate the idea of the deformation of a universal enveloping algebra we give an example of the deformation of matrix groups. The matrix $T = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$ over some K-algebra (an algebra with left distributivity, right distributivity, and compatibility with scalars over a field K) is called a quantum matrix if the following relationships hold.

$$BA = qAB,\ DC = qCD,\ CA = qAC,\ DB = qBD,\ BC = CB$$

$$DA - AD = (q - q^{-1})BC,\ AC - qBD = 1$$

These relations allow T to function as a non-commutative transformation of the coordinates of a 2-d plane, which may be how coordinates behave in the quantum realm of physics. We note that as $q \to 1$ then these relationships approach the conditions on matrices over commutative elements. If we let $\mathcal{Q} = \{\{A, B, C, D\} \mid \text{the equations above hold for } q, A, B, C, \text{ and } D\}$ then we can defined a comultiplication $\Delta : \mathcal{Q} \to \mathcal{Q} \times \mathcal{Q}$, under which $(\mathcal{Q}, \Delta)$ is a pseudo-matrix grou. Note that $\otimes$ is a product and has that $(P \otimes Q)(R \otimes S) = PR \otimes QS$.

$$\Delta_{1,2}(T) = T_1 \otimes T_2 = \begin{bmatrix} A_1 & B_1 \\ C_1 & D_1 \end{bmatrix} \otimes \begin{bmatrix} A_2 & B_2 \\ C_2 & D_2 \end{bmatrix} = \begin{bmatrix} A_1 \otimes A_2 + B_1 \otimes C_2 & A_1 \otimes B_2 + B_1 \otimes D_2 \\ C_1 \otimes A_2 + D_1 \otimes C_2 & C_1 \otimes B_2 + D_1 \otimes D_2 \end{bmatrix}$$

This pseudo-matrix group is also called a quantum group, most notable in Takeuchi [2002] our source of these definitions, and is denoted $SL_q(2)$ because in the limit $q \to 1$ this algebra is $SL(2)$. To summarize how one could arrive at the motivation of the quantum group, a person might think that in the description of super small systems (ie quantum systems) the transformations of the coordinates may not be commutative. By building matrices with elements that don't commute but have deformed commutative relationships and defining the proper algebraic structure we get a deformed q-analogue of $SL(2)$. Since we have $SL_q(2)$ the natural question is: is there then an infinitesimal algebraic object related to $SL_q(2)$ in the same way that $SL(2)$ relates to $\mathfrak{sl}(2)$? The answer is yes and it is the quantum group that will be used throughout this paper.

We define a q-number in the context of quantum groups as:

$$[\![n]\!]_q = \frac{q^n - q^{-n}}{q - q^{-1}}$$

This q-number has the nice property that $[\![n]\!]_q \approx n$ as $q \to 1$. We define a generic T matrix parameterized by α, β, and λ and the equation $T = e_{q^{-2}}^{\lambda \mathcal{X}_-} e^{2\alpha \mathcal{X}_0} e_{q^2}^{\beta \mathcal{X}_+}$, where our matrices $\{\mathcal{X}_-, \mathcal{X}_0, \mathcal{X}_+\}$ obey the algebra defined by:

$$[\mathcal{X}_0, \mathcal{X}_\pm] = \pm \mathcal{X}_\pm \text{ and } [\mathcal{X}_+, \mathcal{X}_-] = \frac{q^{2\mathcal{X}_0} - q^{-2\mathcal{X}_0}}{q - q^{-1}} = [\![2\mathcal{X}_0]\!]$$

This algebra is equivalent to the structure of the "$\mathfrak{sl}_q(2)$" which is more formally the q deformation of the universal enveloping algebra of $\mathfrak{sl}(2)$ (see Takeuchi [2002] for a proof of the equivalence). That is to say that the algebra $U_q(\mathfrak{sl}(2))$ is generated by polynomials over $\{\mathcal{X}_-, \mathcal{X}_0, \mathcal{X}_+\}$. We note that the T parametrization above defines the elements of $SL_q(2)$ giving a relationship similar to the relationship between $SL(2)$ and $\mathfrak{sl}(2)$. We can define multpilcation, comultiplicaiton, unit, counit, and antipode which induce a Hopf algebra structure on the algebra $U_q(\mathfrak{sl}(2))$. We will not define them explicitly as it will repeat the general construction bellow.

The quantum group $U_q(\mathfrak{g})$ is defined for an arbitrary Lie algebra $\mathfrak{g}$, via the root system and a set of unital associative algebra relationships. Let $q \in C^*$ and $q \neq 1$ and λ element of the root lattice of $\mathfrak{g}$ (where the root lattice is the lattice generated by the root system of $\mathfrak{g}$). Then our algebra is generated by elements k_λ, e_i, and f_i (where i is the index of the roots α_i) which have the following rules:

$$k_0 = 1 \quad k_\lambda k_\mu = k_{\lambda+\mu} \quad k_\lambda e_i k_\lambda^{-1} = q^{(\lambda \alpha_i)} e_i \quad k_\lambda f_i k_\lambda^{-1} = q^{-(\lambda \alpha_i)} f_i \quad [e_i, f_j] = \delta_{i,j} \frac{k_i - k_i^{-1}}{q_i - q_i^{-1}}$$

$$k_i = k_{\alpha_i} \quad q_i = q^{\frac{1}{2}(\alpha_i \alpha_i)}$$

Let $A = (a_{i,j})$ be the Cartan matrix of the Lie algebra $\mathfrak{g}$. For all $i \neq j$ we have the q-Serre relations:

$$\sum_{n=0}^{1-a_{i,j}} (-1)^n \frac{[\![1 - a_{i,j}]\!]_{q_i}!}{[\![1 - a_{i,j} - n]\!]_{q_i}! \quad [\![n]\!]_{q_i}!} e_i^n e_j e_i^{1-a_{i,j}-n} = 0$$

$$\sum_{n=0}^{1-a_{i,j}} (-1)^n \frac{[\![1 - a_{i,j}]\!]_{q_i}!}{[\![1 - a_{i,j} - n]\!]_{q_i}! \quad [\![n]\!]_{q_i}!} f_i^n f_j f_i^{1-a_{i,j}-n} = 0$$

With these relationships our algebra defined by the above relationships we have that $U_q(\mathfrak{g})$ behaves more and more like $U(\mathfrak{g})$ as $q \to 1$. We establish that $U_q(\mathfrak{g})$ is a Hopf algebra by defining a coproduct, counit, and antipode (the other mapping are described in Majid [2002]).

$$\Delta(k_\lambda) = k_\lambda \otimes k_\lambda \quad \Delta(e_i) = 1 \otimes e_i + e_i \otimes k_i \quad \Delta(f_i) = k_i^{-1} \otimes f_i + f_i \otimes 1$$

$$\epsilon(k_\lambda) = 1 \quad \epsilon(e_i) = 0 \quad \epsilon(f_i) = 0$$

$$S(k_\lambda) = k_{-\lambda} \quad S(e_i) = -e_i k_i^{-1} \quad S(f_i) = -k_i f_i$$

There are alternate definitions of these functions under which $U_q(\mathfrak{g})$ is also a Hopf algebra, but we will work exclusively with this one in this paper.

We now turn to the representation theory of $U_q(\mathfrak{g})$ where q is a root of unity. This representation theory has been well documented in Bakalov and Kirillov [2001] and Klimyk and Schmüdgen [2012]. We will only present the representation theory in the case where our quantum group is $U_q(\mathfrak{sl}_2)$ where $q = e^{\frac{\pi i}{k}}$. The Weyl module is defined via the weight lattice which in the case of $\mathfrak{sl}_2$ is identical to $\mathbb{Z}$, for $\mathfrak{sl}_2$ we have the Weyl module n is:

$$V_n = \sum_{i=0}^{n} \mathbb{C} v_i \quad n \in \mathbb{N}$$

Where v_i is the highest weight vector and $v_i = f^{(i)} v_0$. Since q is a root of unity we can denote it $q = e^{\frac{i\pi}{k}}$. Then we have from Bakalov and Kirillov [2001] that the n Weyl module of $U_q(\mathfrak{sl}_2)$ is irreducible if and only if $n \leq k$ or $n = lk - 1$ for some $l \in \mathbb{N}$. The textbook Bakalov and Kirillov [2001] gives the general form of the category of representations of the quantum group of $\mathfrak{g}$ at the root of unity $q = e^{\frac{i\pi}{k}}$ denoted $\mathcal{C}(\mathfrak{g}, k)$ and a proof of the following theorem.

Theorem 3. *$\mathcal{C}(\mathfrak{g}, k)$ is a ribbon category over $\mathbb{C}$.*

This paper works primarily with the category theory properties of $\mathcal{C}(\mathfrak{sl}_2, k)$ so will not go into more detail, instead turning to background on vertex operator algebras.

1.7 Vertex Operator Algebras

A vertex algebra is a structure defined on a vector space V with an identity $1 \in V$, translation endomorphism $T : V \to V$, and a map $Y : V \otimes V \to V((z)))$ where $V((z))$ denotes Laurent series with coefficients in V. If the following axioms hold this collection of objects is a vertex algebra.

1. For any $u \in V : Y(1, z)u = u = uz^0$
2. $T(1) = 0$ and for all $u, v \in V : [T, Y(u, z)]v = TY(u, z)v - Y(u, z)Tv = \frac{d}{dz} Y(u, z)v$
3. For any u,v $\in V$ there is an $N \in \mathbb{Z}$ such that $(z-x)^N Y(u, z)Y(v, x) = (z-x)^N Y(v, x)Y(u, z)$

When we equip a vertex algebra with an element ω called the conformal element which has that $Y(\omega, z)$ is a Virasoro field. We define the spanning $L(z)$ via $Y(\omega, z) = \sum_{n\in\mathbb{Z}} \omega_n z^{-n-1} = L(z) = \sum_{n\in\mathbb{Z}} L_n z^{-n-2}$. The structure of the Virasoro algebra is that $[L_m, L_n] = (m-n)L_{m+n} + (\delta_{m+n,0}/12)(m^3 - m)c\, id_v$ where c is called the central charge of the vertex algebra. These axioms induce a quantum field theory and describe the particle behavior of some system. We often abbreviate vertex operator algebra as VOA. For example the monster VOA $V^\natural$ describes the particle interactions on an orbifold of a 24 dimensional torus. This construction of quantum field theory is equivalent to the conformal net one as there is a "dictionary" which translates between conformal nets and VOAs Carpi et al. [2015].

2 Kirillov and Ostrick's q-Analogue

The focus of this section will be the paper of Kirillov and Ostrick Ostrik [2001]. This paper gives a classification of "finite subgroups" in $U_q(\mathfrak{sl}_2)$ (where $q = e^{\frac{\pi i}{l}}$) via the properties of the category of representations of $U_q(\mathfrak{sl}_2)$. The classification is not new (having appeared in Michel et al. [1992]) but the method of classification is new. Specifically the paper proves that these "finite subgroups" are classified by the Dynkin diagrams of types A_n, D_{2n}, E_6, E_8 with Coxeter numbers equal to l (where l is the denominator of the exponential of our deformation via $q = e^{\frac{\pi i}{l}}$) the category theoretic properties of the representations of $U_q(\mathfrak{sl}_2)$.

The purpose of the following sections will be proving the following main theorem:

Theorem. *There is a correspondence between rigid $\mathcal{C}$-algebras with $\theta_A = id$ and Dynkin diagrams of types A_n, D_2n, E_6, E_8 with Coxeter numbers equal to l. Under this correspondence the simple objects of Rep(A) are represented by the vertices of the Dynkin diagram. Finally the matrix of multiplication by $F(V_1)$ in the Grothendieck ring of Rep(A) is $2 - C$ where C is the Cartan matrix of the Dynkin diagram.*

To do this we will first outline some basic category theory results on the category of representations of $U_q(\mathfrak{sl}_2)$, apply them to finite groups, then present a small review of vertex operator algebra results, and tie all of this together in a main proof which will establish the correspondence then check case by case which diagrams correspond to realizable unique algebras. We finish by presenting the calculations which assign representations to the vertices of the diagrams.

2.1 Preliminary results

As we have seen in the background the theory of q-deformations is built in parallel to the questions one finds in traditional Lie Theory. Often the nice notions of Lie Theory do not carry into its deformation, for example $U_q(\mathfrak{sl}_2)$ is not a group so does not have subgroups. Kirillov and Ostrick utilize a way of defining a "finite subgroup" for $U_q(\mathfrak{sl}_2)$ which has parallel properties to finite subgroups of $\mathfrak{sl}_2$. Given a semi simple abelian rigid balanced braided tensor category $\mathcal{C}$ over $\mathbb{C}$, we define a $\mathcal{C}$-Algebra A as an object $A \in \mathcal{C}$ with morphisms $\mu : A \otimes A \to A$ and $i_A : \mathbf{1} \to A$ that satisfy the following conditions:

1. $\mu \circ (\mu \otimes id) = \mu \circ (id \otimes \mu)$
2. $\mu \circ \sigma_{A,A} : A \otimes A \to A$ is equal to μ.
3. $\mu \circ (i_A \otimes A) : \mathbf{1} \otimes A \to A$ is equal to id_A
4. $\dim(Hom_{\mathcal{C}}(\mathbf{1}, A) = 1$

This definition of a $\mathcal{C}$-Algebra was not new but rather the paper allows more general use of it via properties of $Rep^0(A)$ and $Rep(A)$, where $Rep(A) = \{\, (V, \mu_V) \,\}$ with μ_V satisfying:

1. $\mu_V \circ (\mu \otimes \mu_V) = \mu_V \circ (id \otimes \mu_V) : A \otimes A \otimes A \to V$
2. $\mu_V(i_A \otimes id) = id : \mathbf{1} \otimes V \to V$

We denote sets of morphisms in $Rep(A)$ as Hom_A and define them as:

$$Hom_A((V, \mu_V), (W, \mu_W)) = \{\varphi \in Hom_{\mathcal{C}}(V, W) \mid \mu_W \circ (id \otimes \varphi) = \varphi \circ \mu_V\}$$

One of the primary differences between the work of Kirillov and Ostrick and previous work is that they do not assume that $\mathcal{C}$ has that $\sigma_{A,V}\sigma_{V,A} = id$. Rather they define a a full subcategory of $Rep(A)$ named $Rep^0(A)$ as those representations (V, μ_V) which have that $\mu_V \circ \sigma_{A,V}\sigma_{V,A} = \mu_V$. Such objects have been referred to as transparent or dyslexic by previous authors, such as in Bruguieres [2000]. Section One of Ostrik [2001] also establishes that $Rep(A)$ is monodial with unit object and the existence and properties of the following functors in $Rep(A)$.

Theorem 4. *Define functors* $F : \mathcal{C} \to Rep(A)$, $G : Rep(A) \to \mathcal{C}$ *via* $F(V) = A \otimes V$, $\mu_{F(V)} = \mu \otimes id$ *and* $G(V, \mu_V) = V$ *Then:*

1. *Both F and G are exact and injective on morphisms.*
2. *F and G are adjoint: one has canonical functorial isomorphisms:*

$$Hom_A(F(V), X) = Hom_{\mathcal{C}}(V, G(X)), \quad V \in \mathcal{C}, \quad X \in Rep(A)$$

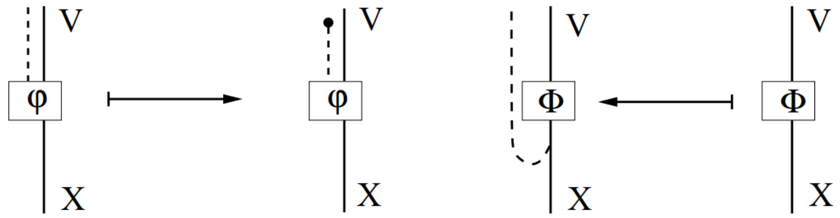
3. *F is a tensor functor:* $F(V \otimes W) = F(X) \otimes_A F(W), \quad F(\mathbf{1}) = A$
4. *There are canonical isomorphisms* $G(F(V)) = A \otimes V$ *and* $G(F(V) \otimes_A X) = V \otimes G(X)$.

Proof. (1) follows from the definition. We define the (2)'s maps using the following diagram and appeal to graphical calculus for a proof of the requirements. Next we define:

$$f = id_A \otimes id_V \otimes \imath_A \otimes id_W : A \otimes V \otimes W \to (A \otimes V) \otimes_A (A \otimes W)$$

$$g : (A \otimes V) \otimes_A (A \otimes W) \xrightarrow{R^{-1}_{A,V}} A \otimes_A A \otimes V \otimes V \xrightarrow{\mu} A \otimes V \otimes W$$

These are inverses and well defined showing that our functors are tensor. (4) follows directly from the defition of F, G and the previous properties.



□

Theorem 5. *Rep*(A) *has an an additive functor* $\boxtimes : \mathcal{C} \times Rep(A) \to Rep(A)$ *and isomorphisms* $(V_1 \otimes V_2) \boxtimes X \cong V_1 \boxtimes (V_2 \boxtimes X)$ *and* $\mathbf{1} \boxtimes X \cong X$.

Proof. We just take $V \boxtimes X \cong F(V) \otimes_A X$ and apply the previous theorem. □

Next we note that $Rep(A)$ is not a braided category using the inherited commutativity morphism because $R_{V,W} \circ (\mu_1 - \mu_2) \not\equiv 0$. This prevents the commutativity morphism from descending properly in $Rep(A)$. We however do have:

Theorem 6 (Pareigis [1995])**.** *: The category* $Rep^0(A)$ *is a braided tensor category with the commutativity isomorphism inherited from* $\mathcal{C}$.

Proof. This is only a sketch of the proof. We show that $X, Y \in Rep^0(A)$ and $X \otimes_A Y \in Rep^0(A)$. This is done by showing that the $X \otimes Y \to X \otimes_A Y$ is the canonical projection. Next we show that $R_{X,Y} : X \otimes Y \to Y \otimes X$ descends to isomorphism $R_{X,Y} : X \otimes_A Y \to Y \otimes_A X$ in $Rep^0(A)$. This is equivalent to showing $R_{X,Y}(I) \subset (I)$ where I is the kernel of a canonical isomorphism $X \otimes Y \to X \otimes_A Y$. This is done via the graphical calculus, but we don't include it here. □

Recalling the definition of rigidity from the background we give a definition of a rigid $\mathcal{C}$-Algebra as one which has that $e_A : A \otimes A \xrightarrow{\mu} A \xrightarrow{\eta_A} \mathbf{1}$ is a non-degenerate pairing and $\dim_{\mathcal{C}}(A) \neq 0$.If that holds then there is a unique $i_A : \mathbf{1} \to A \otimes A$ such that e_A, i_A follow the rigidity conditions. Independent of rigidity we also have e_A, i_A but also two maps $\imath_A : \mathbf{1} \to A$ and $\varepsilon_A : A \to \mathbf{1}$, with $\imath_A \circ \varepsilon_A = id_A$. We note that these maps are "compatible" with e_A, i_A in the natural way.

Lemma 7. *If* A *is a rigid* $\mathcal{C}$*-algebra with* $\theta_A = id$ *then* $\mu \circ e_A = dim(A)\,\imath_A$.

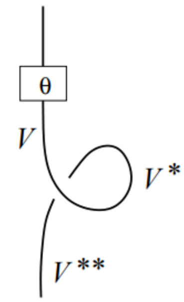
Proof. These are both maps $\mathbf{1} \to A$ then by the uniqueness of the unit map for the $\mathcal{C}$-algebra these must be proportional. □

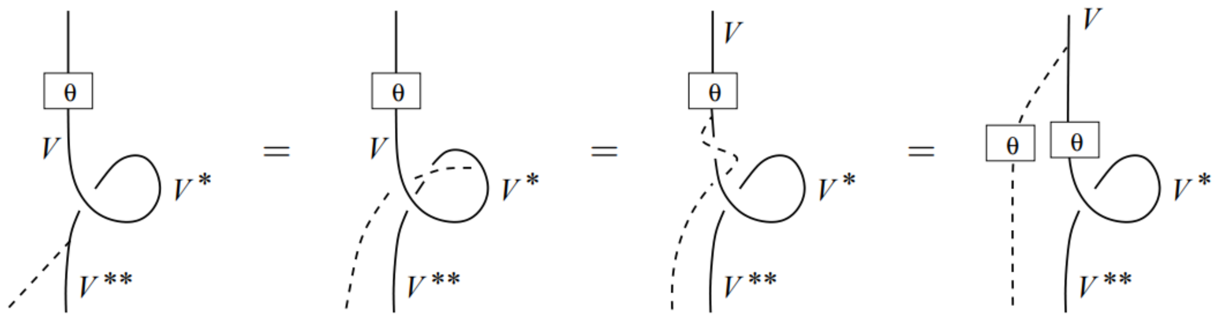
Theorem 8. *Let* $\mathcal{C}$ *be a rigid balanced braided category and* A*, a rigid* $\mathcal{C}$*-algebra,* $\theta_A = id$ *then*

1. $Rep^0(A) = \{V \in Rep(A) \mid \theta_A$ *is an* A*-morphism*$\}$
2. $Rep^0(A)$ *is a rigid balanced braided category with* θ *inherited from* $\mathcal{C}$
3. *For any* $V \in RepA$*, the morphism* $\delta_V : V \to V^{**}$ *is an* A*-morphism.*

Proof. First we note that $\theta_{A\otimes V} = R_{A,V} R_{V,A}\, \theta_V \otimes \theta_A$ which when $\theta_V = id$ implies that $Rep^0(A) = \{V \in Rep(A) \mid \theta_A \text{ is an } A\text{-morphism}\}$. Then applying the previous theorems and part (1) gives (2). Then (3) is given by the graphical calculus.

$$\delta_V^{-1} =$$





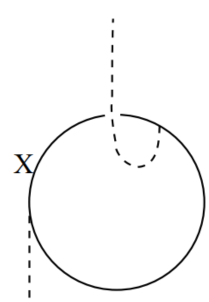
□

Theorem 9. *Let $\mathcal{C}$ be a rigid balanced braided category, and A a rigid $\mathcal{C}$-algebra such that $\theta_A = id_A$. Then for every $X, Y \in Rep(A)$,*

1. $dim_A(X \otimes_A Y) = dim_A(X) dim_A(Y)$
2. $dim_A(X) = \frac{dim_{\mathcal{C}}(X)}{dim_{\mathcal{C}}(A)}$
3. $dim_A(F(V)) = dim_{\mathcal{C}}(V)$

Proof. The first inequality follows in any rigid monoidal category with a natural monoidal transformation $V \to (V^*)^*$. We have that the dimension is defined by the following after an application of the rigidity axioms.

$$\frac{1}{\dim A}$$



$$= (\dim_A X)\, \mathrm{id}_A$$

Both sides of this diagram are A-automorphisms, so we compose them with $\imath_A, \epsilon_A$ and we get that $\frac{1}{\dim(A)}\dim(\mathcal{C}) = \dim_A(X)$. We apply this to $F(v) = V \otimes A$ and get that $\dim_A(F(V)) = \dim_{\mathcal{C}}(V)$. $\square$

We turn the simplicity of our defined objects. Notably we define a $\mathcal{C}$-algebra A as semi simple if $Rep(A)$ is semi simple and give a corollary to the previous theorem:

Corollary 10. *Let $\mathcal{C}$ be rigid and A, a semisimple $\mathcal{C}$-algebra with $\theta_A = id$. Then,*

1. *If $X \in Rep(A)$ is simple, then $X \in Rep^0(A)$ is simple if and only if, $\theta_X = c \cdot id$.*
2. *$Rep^0(A)$ is semisimple, with simple objects exactly those which are simple in $Rep(A)$ and satisfy 1*

Proof. This immediately follows from the previous theorem, the fact that the dimension of simple objects is 1, and that for simple objects $X \in Hom_A(X, X) = \mathbb{C}$ $\square$

A lemma from the literature will give us our final preliminary theorem.

Lemma 11 (Bruguieres [2000])**.** *: If A is rigid then every $X \in Rep(A)$ is a direct summand in $F(V)$ for some $V \in \mathcal{C}$*

Proof. We only give a sketch of the proof here. We define a subjective map from $A \otimes A \to A$ and its one sided inverse. Using these we can get $A \otimes_A X \cong X$ and thus that $(A \otimes A) \otimes_A X = A \otimes (A \otimes X) = A \otimes X = F(G(X))$. For the exact definitions of these we refer to Lemma 3.4 in Ostrik [2001]. $\square$

Theorem 12. *Let $\mathcal{C}$ be rigid and A, a $\mathcal{C}$-algebra be rigid. Then A is semi simple.*

Before the proof we state some properties of projective objects and the Ext^1 functor. An object in $\mathcal{C}$ is projective if it is universal to epimorphisms, ie the $Hom(O, -)$ functor preserves epimorphsims. Now the $Ext^1(O_1, O_2)$ functor describes the properties of sequence of mappings $O_1 \to V \to O_2$. Without diving deeply into the theory we suffice to say that if $Ext^1(O_1, O_2) = 0$ then $V = O_1 \oplus K$. Again glossing over details, if every $Ext^1(O_1, O_2) = 0$ then the category is semisimple (a sketch of the proof is that all objects are related by directed sums so the only possible decompositions are directed sums).

Proof. We know from the exactness and adjointness of the functors F and G that every $F(X)$ is projective. Thus applying the lemma we have that every object is the direct sum of projective objects. Every sum of projective objects is projective. Now consider $Ext^1(O_1, O_2)$ since both objects are projective this is 0. Thus we have that A is semi simple. $\square$

2.2 Sanity Check: Representations of Finite Groups

In this section we make sure that all of the "abstract nonsense" we have just defined and delineated has recognizable forms in a simple case. Let G be a finite group and $\mathcal{F}(X)$ be the set of complex valued functions on set X. Then what we will show is that each $Rep(A)$ for A a $Rep(G)$-Algebra corresponds to a category $Rep(H)$ for $H \subset G$. Then we will show that each subgroup $H \subset G$ of finite index we have that $\mathcal{F}(G/H)$ is a semi simple *Rep*(G)-Algebra which is equivalent as a category to *Rep*(H). Together these results tell us that the the A's which are *Rep*(G)-Algebras precisely identify the finite subgroups of G, exactly our original goal. In Ostrik [2001] they present this for any object G which has a finite complex representation space, we present this proof in a limited form.

Theorem 13. *For $H \subset G$ of finite index $\mathcal{F}(G/H) = A$ is a semisimple $\mathcal{C}$-Algebra and Rep(A) is equivalent as a category to Rep(H).*

Proof. Each object of *Rep*(A) is a G module V with decomposition $V = \bigoplus_{x \in G/H} V_x$ and representation structure $gV_x = V_{gx}$. The tensor in A is given by $(V \otimes_A W)_x = V_x \otimes W_x$. Together these facts naturally induce the $\mathcal{C}$-Algebra Structure. Define functor $\chi : Rep(A) \to Rep(H)$ by $\oplus V_x \to V_1$ and $\varphi : Rep(H) \to Rep(A)$ by $E \to Ind_H^G(E)$. Then clearly $\chi(V \otimes W) = \chi(V) \otimes \chi(W)$ and by the properties of Ind $\varphi(E_1 \otimes E_2) = Ind_H^G(E_1 \otimes E_2) = Ind_H^G(E_1) \otimes Ind_H^G(E_2)$. Finally because $V = Ind(E) = \bigoplus_{x \in G/H} V_x$, we have that $\chi \circ \varphi = id$ so they are inverses. This tells us that the categories are equivalent. □

Corollary 14. *In the equivalence of the $\mathcal{C}$-Algebra Rep$(\mathcal{F}(G/H))$ and Rep(H) the $\mathcal{C}$-Algebra functors F and G are mapped to the functors Res and Ind respectively in Rep(H)*

Theorem 15. *For $\mathcal{C} = Rep(G)$ any rigid $\mathcal{C}$-algebra is of the form $\mathcal{F}(G/H)$ for some H of finite index.*

Proof. By definition, a $\mathcal{C}-$Algebra is a commutative associative algebra over $\mathbb{C}$ on which G acts via automorphisms. So if A is rigid then it is semisimple as an algebra, meaning that the largest nilpotent ideal is zero. We denote this as N then N is invariant under automorphism by G and thus an $\mathcal{C}$-Algebra ideal. The Lemma following this proof shows that N is zero.

Semisimple commutative associative algebras have a highly defined structure. Via the structure and the fact that A is a subcategory of *Rep*(G) we can conclude that A is the algebra of functions on some finite X where X is the set of primitive idempotents of A. When we decompose A as a G-module (under the permutation action) $\mathbb{C}$ appears as a G-module with multiplicity 1, from this we conclude that the action is transitive. This implies that $X = G/H$. □

Lemma 16. *Let $\mathcal{C}$ be a rigid and A a $\mathcal{C}$-algebra such that $\theta_A = id$ and $dim_{\mathcal{C}}(A) \neq 0$. Then A is a rigid $\mathcal{C}$-Algebra if and only if A is simple as an A-module.*

2.3 Results on Vertex Operator Algebras

This section is a photo of a photo, in that it is an an overview of overviews of results on VOAs described in Ostrik [2001] but proved elsewhere. This paper also gives the following results via appeal to references. First we restrict our investigation to the friendliest vertex operator algebras, those which have the following

1. For every simple $\mathscr{V}$-module M, its conformal dimension (the lowest eigenvalue of the transformation by L_0) is real and non-negative. With it being zero if and only if $M = \mathscr{V}$ in which case $\dim(\mathscr{V}_0) = 1$. This is a technical condition to ensure the vacuum vector is unique.

2. The category of representations of $\mathscr{V}$ is semi-simple, with finitely many simple objects, and all spaces of conformal blocks (i.e. intertwining operators between tensor products of representations) are finite dimensional. Finally $\mathscr{V}$ is simple as a $\mathscr{V}$-module

3. The category of $\mathcal{C}$ of $\mathscr{V}$-modules is a rigid braided tensor category.

Next we introduce a sample construction and an important theorem. First however we must define "level" in the context of affine Lie algebras. We can construct an Lie algebra as a central extension of a loop algebra of the corresponding simple Lie algebra. We note that the central extensions of the affine Lie groups induce a k which is the scalar that the central element acts by in the representation and it parameterizes the representations of an affine Lie algebra. We present the following construction of a VOA which has properties 1-3.

Construction 17. *Let $\mathfrak{g}$ be a simple Lie algebra, $\hat{\mathfrak{g}}$ the corresponding affine Lie algebra and k a level (with $k \neq -h^\vee$ where $h^\vee$ is the dual Coxeter number). Then let $L_{0,k}$ be the integrable $\hat{\mathfrak{g}}$ module of level k and highest weight 0 (corresponding to the vacuum vector). This is a VOA canonically, denoted $\mathscr{V}(\mathfrak{g}, k)$*

As an abeliean category $\mathscr{V}(\mathfrak{g}, k)$ is just the integral $\hat{\mathfrak{g}}$-modules of level k. It has also been shown to be modular Huang et al. [1999]. It was show in Finkelberg [1996] to be the "semisimple part" of $U_q(\mathfrak{g})$ with $q = e^{\pi i/m(k+h)}$ where h is the Coxeter dual number and m depends on the diagram of the algebra. We define an extension of $\mathscr{V}$ as $\mathscr{V} \subset \mathscr{V}_e$ such that $\mathscr{V}_e$ is finite length (where this is its length as a module, ie the maximum size of chains of intermediary modules) and preserves the vertex algebra structure. Then we have this theorem from Huang et al. [2015].

Theorem 18. *Let $\mathscr{V}$ be a VOA satisfying 1-3 above, and let $\mathcal{C}$ be the category of $\mathscr{V}$-modules. The the following definitions describe the same collection of objects.*

1. *An extension $\mathscr{V} \subset \mathscr{V}_e$, where $\mathscr{V}_e$ is also a VOA satisfying conditions 1-3.*

2. *A rigid $\mathcal{C}$-algebra with $\theta_A = 1$.*

Under this correspondence the category of $\mathscr{V}_e$-modules is identified with $Rep^0(A)$.

Given $\mathscr{V}(\mathfrak{g}, k)$ we will define an example of conformal embedding given an embedding $\mathfrak{g} \subset \mathfrak{g}'$ which is an embedding of Lie algebras. Then this induces an embedding $\hat{\mathfrak{g}} \subset \hat{\mathfrak{g}}'$ where $\hat{\mathfrak{g}}'$ has level k'. We naturally get an embedding of VOAs $\hat{\mathfrak{g}} \subset \mathscr{V}(\mathfrak{g}', k')$ which we call a conformal embedding if it preserves the Virasoro element. Note that not all conformal VOA embeddings are built in this way but we do get examples we will use later in this way.

First note that for $\mathcal{C}(\mathfrak{sl}_2, 10)$ the category of integrable modules over $\hat{\mathfrak{sl}_2}$ of level 10, that there is a conformal embedding into $\mathcal{C}(sp(4), 1)$. We can describe this embedding by observing that that the four dimensional representation of $\mathfrak{sl}_2$ has a an invariant non-degenerate skew-symmetric form. This form induces the required embedding. We observe that the decomposition of $\mathscr{V}(sp(4), 1)$ as a $\mathscr{V}(\mathfrak{sl}_2, 10)$ module is given by $\mathscr{V} = L_{0,10} \oplus L_{6,10}$.

Similarly when $k = 28$, $\mathscr{V}(\mathfrak{sl}_2, 28)$ has a conformal embedding into $\mathscr{V}(G_2, 1)$. Where $\mathscr{V}(G_2, 1)$ as a $\mathscr{V}(\mathfrak{sl}_2, 28)$-module is given by $\mathscr{V} = L_{0,28} \oplus L_{10,28} \oplus L_{18,28} \oplus L_{28,28}$. The textbook "Conformal Field Theory" covers these results in detail Francesco et al. [2012].

2.4 The Main Proof

The following proof is the main purpose of the paper. Since most of the results had been previously shown it is the proof structure itself that is of interest. We consider the case where $\mathcal{C}$ is the semisimple part of the representations of $U_q(\mathfrak{sl}_2)$ with $q = e^{\pi i/l}$ and $l \geq 2$. We refer to the background for an overview of the properties of $\mathcal{C}$ and to Bakalov and Kirillov [2001] for the full details. By definition $\mathcal{C}$ is semisimple and it is well know that it has $l - 2 = k$ simple objects $V_1, \cdots, V_k$ where each V_i is a standard $i + 1$ dimensional representation of $U_q(\mathfrak{sl}_2)$. Its Grothendieck ring K is generated by V_1 and quantum dimensions given by $\dim_{\mathcal{C}}(V_n) = [n+1]_q$. This category is modular and in Finkelberg [1996] they prove that it is equivalent to $\mathscr{V}(\mathfrak{sl}_2, k)$. Given this setup we have:

Theorem 19. *There is a correspondence between rigid $\mathcal{C}$-algebras with $\theta_A = id$ and Dynkin diagrams of types A_n, D_{2n}, E_6, E_8 with Coxeter numbers equal to l. Under this correspondence the simple objects of Rep(A) are represented by the vertices of the Dynkin diagram. Finally the matrix of multiplication by $F(V_1)$ is the Grothendieck ring of Rep(A) is $2 - C$ where C is the Cartan matrix of the Dynkin diagram.*

Let A be any rigid $\mathcal{C}$-algebra with $\theta_A = id$. Then by Theorem 4 we know that it is a monoidal category and it is a module category over $\mathcal{C}$. Thus the Grothendieck rings of *Rep*(A) and A are equivalent and are modules over the Grothendieck ring $K(\mathcal{C})$. We apply Theorem 11 and conclude *Rep*(A) is semisimple. We then can apply semisimplicity to show a number of technical properties of modules over $K(\mathcal{C})$.

The exact nature of these technical conditions is excluded here. It suffices to note that in Etingof and Khovanov [1994] they prove that any module which has these conditions must correspond to the finite Dynkin diagrams with loops and Coxeter number equal to l, where under this correspondence the vertices of the Dynkin diagram correspond to the elements of the distinguished basis of the module and the multiplication matrix is in the required form.

This result restricts our classification to several cases but does not tell us exactly which exist in this case. We note that the matrix of the tensor product is symmetric in the basis of the module restricting our classification to ADET type diagrams. Note that the T_n diagram is an A_n with a loop on one of the ends (these are called tadpole diagrams because of the shape). Next the following Lemma allows us to determine the the structure of the potential module.

Lemma 20. *If A is a rigid $\mathcal{C}$-algebra, then A corresponds to the end (vertex of degree 1) of the longest leg of the corresponding Dynkin diagram.*

Since this determines the vertex of A up to an automorphism of the Dynkin diagram, we can actually use it to determine the decomposition of A in $\mathcal{C}$. The determination of the A vertex allows us to determine the class of $F(V_1)$. Since F is a tensor functor and V_1 generates our Grothendieck ring we can determine the mapping F as a mapping of Grothendieck rings. Then using this determination we can calculate the inverse mapping $G : K(A) \to K$. Then we apply the correspondence of Dynkin diagrams to elements of $K(A)$ to map them to elements of K. These mappings give a decomposition of A in $\mathcal{C}$

Fortunately the authors of Ostrik [2001] worked these out by hand and presented them in the following table from their paper. We now must check exactly which of these possibilities are really $\mathcal{C}$-Algebras

Diagram	$k = h - 2$	A
A_n	$n - 1$	V_0
D_n	$2n - 4$	$V_0 + V_k$
T_n	$2n - 1$	$V_0 + V_k$
E_6	10	$V_0 + V_6$
E_7	16	$V_0 + V_8 + V_{16}$
E_8	28	$V_0 + V_{10} + V_{18} + V_{28}$

2.4.1 Case: A

Since our algebra A must decompose into V_0 it must be exactly $\mathbf{1}$. This obviously has the structure of a commutative associative algebra and tells us that our $Rep(A) = \mathcal{C}$.

2.4.2 Case: D

Theorem 21. *The object $A = \mathbf{1} \oplus V_k$ in $\mathcal{C}$ has a structure of a rigid $\mathcal{C}$-algebra if and only if $4|k$. In this case, the structure of an algebra is unique up to isomorphism and this algebra satisfies $\theta_A = id$*

Proof. Let μ be the multiplication $\mu : (\mathbf{1} \otimes V_k) \otimes (\mathbf{1} \otimes V_k) \to (\mathbf{1} \otimes V_k)$. Each component of this is uniquely determined by the unit axiom, except for $\mu_{V_k V_k} : V_k \otimes V_k \to \mathbf{1}$. Now note that $V_k \otimes V_n \cong V_{k-n}$ then $V_k \otimes V_k \cong \mathbf{1}$. Then by rigidity is not only non zero but also fixed up to a constant, which allows us to conclude that it is unique. Now if we fix some nonzero $\mu_{V_k V_k}$ the associativity and commutativity are equivalent to:

$$\mu_{V_k V_k} \circ (id \otimes \mu_{V_k V_k}) = (\mu_{V_k V_k} \otimes id) \circ \mu_{V_k V_k}$$

$$\mu_{V_k V_k} R_{V_k, V_k} = \mu_{V_k V_k}$$

□

These follow from Lemma 6.6 in Ostrik [2001]

Lemma 22. *For generic values of q, let $f : V_a \otimes V_a \to V_{2b}$ be a non-zero homomorphism. Then*

$$f \circ R_{V_a, V_a} = (-1)^{a-b} \theta_a^{-1} (\theta_{2b})^{1/2} f$$

Where by the definition of universal twist, $\theta_A = q^{a(a+2)/2}$ and $\theta_{2b}^{1/2} = q^{2b(2b+2)/4}$.

Finally to complete this section we appeal to a detailed and explicit construction of the category of representations preformed above in Ostrik [2001], which is omitted here because it contributes less to understanding and proof strategy than it does to length.

2.4.3 Case: T

There are no possible algebras with modules corresponding to T_n. All such A s must have a simple object composition $V_0 \oplus V_k$ and it was proved in Theorem 20 that there is a unique structure associated with this decomposition, namely the one associated with D_n. Thus if there was an algebra corresponding to T_n we would violate the uniqueness of D_n a contradiction to our results in 2.4.2.

2.4.4 Case: E_6

Applying the table we notice that $V_0 \oplus V_6$ has a natural realization in the embedding of $\mathscr{V}(sp(4), 1)$ in $\mathscr{V}(\mathfrak{sl}_2, 10)$. Thus we only need to prove that this is unique up to isomorphism is $\mathcal{C}$-algebra. The only non-trivial components of our multiplication map μ are μ' and μ''. By our preliminary results these are unique up to constant multiple in $\mathbb{C}$. Then for any other $e : V_6 \otimes V_6 \to \mathbf{1}$ and $f : V_6 \otimes V_6 \to V_6$ we know that our algebras will be compatible in the sense that: $\mu' = \alpha e$ and $\mu'' = \beta f$ for complex α, β. Then we define an isomorphism $\phi : V_6 \oplus \mathbf{1} \to V_6 \oplus V_6$ by $\phi|_{\mathbf{1}} = id$ and $\phi|_{V_6} = \alpha^{1/2} id$. This is clearly an isomorphism and it implies that $\alpha^{1/2} = \alpha$ ie that $\alpha = 1$. Then we note that $\mu|_{V_6 \otimes V_6} = e + \beta f$. Since we only consider associative algebras we have that

$$\beta^2(f \circ (id \otimes f) - f \circ (f \otimes id)) = e \otimes id - id \otimes e$$

$e \otimes id - id \otimes e$ is clearly not zero, thus we have that this quadratic form is non degenerate. Thus it either has zero solutions which gives a contradiction, or two solutions $\pm\beta_0$. These two solutions allow a natural isomorphism for any V_k which is that $\phi : V_k \otimes \mathbf{1} \to \mathbf{1} \otimes V_k$ given by $\phi|_{\mathbf{1}} = 1$ and $\phi|_{V_k} = -1$. This is an isomorphism of our distinct algebras thus there is only one isomorphism class corresponding to E_6

2.4.5 Case: E_7

This diagram cannot appear corresponding to a $K(A)$ for a commutative associative algebra A. First we reference the table to conclude that it must break down as $A = V_0 \oplus V_8 \oplus V_{16} = (1 \oplus V_k) \oplus V_8$. Which tells us that $A' = \mathbf{1} \oplus V_k$ is a subalgebra in A. The multiplication on A defines a structure of A'-module on V_8 and module morphism $V_8 \otimes V_8 \to (\mathbf{1} \oplus V_k)$ which by rigidity cannot be trivial. By the Lemma 21 this cannot be symmetric.

2.4.6 Case: E_8

Since we need an an A of the form $A = V_0 \oplus V_{10} \oplus V_{18} \oplus V_{28}$ we use the observation from the end of the section on VOAs which gives a conformal embedding $\mathscr{V}(\mathfrak{sl}_2, 28)$ into $\mathscr{V}(G_2, 1)$. This clearly gives the existence of such a $\mathcal{C}$-algebra. We will not fully reproduce the proof of uniqueness here, rather just give a sketch.

Consider the algebra generated $V_0 \oplus V_{10}$ then take an extension of this as a VOA. This is an extension of $\mathscr{V}(\mathfrak{sl}_2, 28)$. By considering the conformal dimensions it has defined this mapping $\mathfrak{sl}_2 \oplus L_{10}$. Using the properties of VOAs, literature results and a Lemma, this definition induces a mapping of $\mathfrak{sl}_2$ into G_2. Since the VOA extension is consistent along central charge our embedding of $\mathfrak{sl}_2$ into G_2 induces a conformal mapping $\mathscr{V}(\mathfrak{sl}_2, 28)$ into $\mathscr{V}(G_2, 1)$, which is know from the literature is unique.

2.4.7 Diagram Representation Composition

Assigning representations to each node of the Dynkin diagrams is one of the unique strengths of this method as compared to others. The calculations in the paper are done by using the method described by the proof of Theorem 18 then by using Corollary 9 if they have $\theta_A = id$. The following three figures are the examples calculated in Ostrik [2001] using the notation that a filled dot is in $Rep^0(A)$, an open one is in $Rep(A)$, with V_i being given by i so $i+m$ represents $V_i \oplus V_m$.

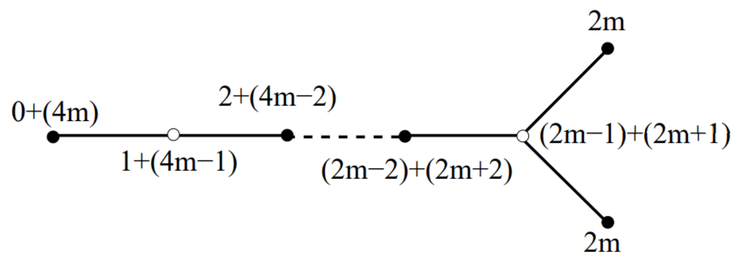


Figure 5: D_n with n even

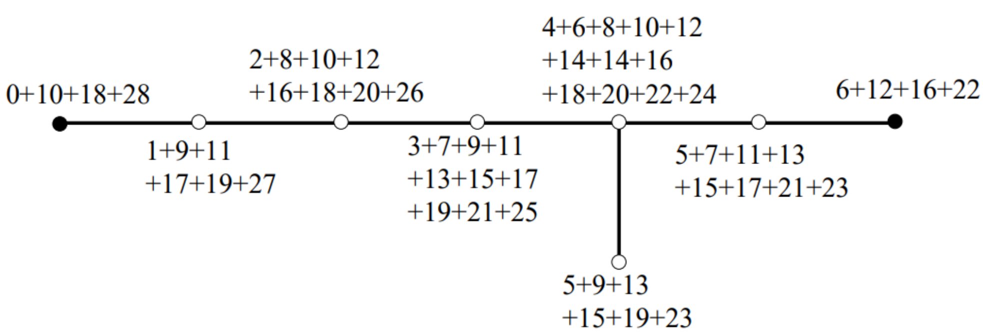


Figure 6: E_8

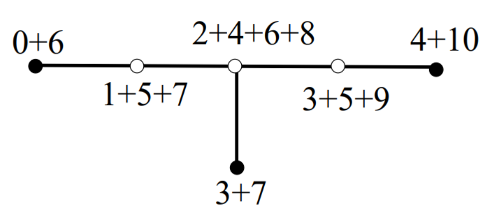


Figure 7: E_6

3 Conclusion

The previous proof is an efficient method of classifying the "finite subgroups" of a quantum group, especially because the definition of $Rep^0(A)$ allowed the identification and description of the representations corresponding to each vertex. The quantum McKay Correspondence had been established using Von Neumann algebras but this method did not identify specific representations with vertex operator algebras. Thus the method of Ostrik [2001] which uses category theory and vertex operator algebras to establish it is a notable improvement. This paper establishes the power of category theory to work with the representation theory of deformational algebras, and the author believes that there may be future applications of this to the deformational ("quantum") versions of finite groups.

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