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Ian Wagner Date

Topics in the analytic theory of L-functions and harmonic Maass forms

By

Ian Wagner

Doctor of Philosophy

Mathematics

Ken Ono

Advisor

John Duncan

Committee Member

David Borthwick

Committee Member

Accepted:

Lisa A. Tedesco, Ph.D.

Dean of the James T. Laney School of Graduate Studies

Date

Topics in the analytic theory of L-functions and harmonic Maass forms

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Ian Wagner

B.A., Emory University, 2015

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Abstract

Topics in the analytic theory of L-functions and harmonic Maass forms

By Ian Wagner

This thesis presents several new results in the theory of L-functions, modular forms, and harmonic Maass forms. In particular, we prove results on the p-adic properties of modular and Maass forms, about the hyperbolicity of doubly infinite families of polynomials related to the partition function and general L-functions, and study Schwartz functions which tie together the field of modular forms and problems like sphere packing and energy optimization.

We prove a general congruence result for mixed weight modular forms using facts about direct products of Galois representations. As an application we prove explicit congruences for the conjugacy growth series of wreath products of finite groups and finitary permutations groups. We then start to answer a question of Mazur's about the existence of an eigencurve for harmonic Maass forms. We begin to answer Mazur's question by constructing two infinite familes of harmonic Maass Hecke eigenforms, and then assemble these forms to produce p-adic Hecke eigenlines.

In work with Larson, we make a result of Griffin, Ono, Rolen, and Zagier effective by showing that $p(n)$ satisfies the degree $3, 4$, and 5 Túran inequalities for all $n \geq 94, 206$, and 381 respectively. We also show that $p(n)$ satisfies the degree d Túran inequality for all $n \geq (3d)^{24d}(50d)^{3d^2}$.

Griffin, Ono, Rolen, and Zagier recently showed that for any degree d

all but at most finitely many of the Riemann zeta Jensen polynomials are hyperbolic. We extended this result to any suitable L-function. In order to prove this result, we also obtain improved estimates for the central derivatives of these L-functions.

Recently, Viazovska explicitly constructed special functions using modular forms which led to the resolution of the sphere packing problem in dimensions 8 and 24. Together with Rolen, we study possible generalizations of Viazovska's work which can be used to attack sphere packing problems in other dimensions and other related problems. We construct a number of infinite families of Schwartz functions using modular forms, which are eigenfunctions of the Fourier transform.

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Contents

Chapter 1

Introduction

We will begin this section with a general background on classical modular forms and the more recent field of harmonic Maass forms. For a more thorough introduction to these subjects see [44] and [3]. We will then give a more focused introduction to the main topics of this thesis and conclude each of these subsections with a statement of the main results.

1.1 Modular forms, harmonic Maass forms, and L-functions

Modular forms are connected to many important areas of mathematics including arithmetic geometry, combinatorics, and representation theory. Most famously, the modularity theorem ties modular forms to elliptic curves through the theory of Galois representations. This connection was used by Wiles in the 1990s to prove Fermat's last theorem.

Modular forms are holomorphic functions on the upper half-plane which have certain growth conditions and transform nicely under action by elements of $SL_2(\mathbb{Z})$ or one of its subgroups. Each modular form can be identified with its Fourier expansion at infinity. The Fourier coefficients of these expansions often have rich arithmetic meaning.

Harmonic Maass forms are natural generalizations of modular forms. They are functions on the upper half-plane which also satisfy modular transformation properties. However, harmonic Maass forms are not generally holomorphic; instead they are annihilated by the hyperbolic Laplacian operator. Zwegers famously showed that Ramanujan's mock theta functions are the holomorphic parts of harmonic Maass forms [66]. In many cases their Fourier coefficients also encode valuable arithmetic information.

1.1.1 Classical modular forms

We will begin with a review of classical modular forms. Denote the *upper half-plane* by $\mathbb{H} := \{z = x + iy \in \mathbb{C} : y > 0\}$. The *modular group*, denoted by $SL_2(\mathbb{Z})$, is the group of 2×2 integer matrices with determinant one. It is generated by the two elements

$$T := \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad S := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

An element $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})$ acts on a point $z \in \mathbb{H}$ by the Möbius transformation

$$\gamma z := \frac{az+b}{cz+d}.$$

We also define two *level* N *congruence subgroups* by

$$\Gamma_0(N) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}) : c \equiv 0 \pmod{N} \right\}$$
$$\Gamma_1(N) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}) : a \equiv d \equiv 1 \pmod{N},\ c \equiv 0 \pmod{N} \right\}$$

for any $N \in \mathbb{N}$. The action of a congruence subgroup on $\mathbb{H}$ extends to an action on $\mathbb{Q} \cup \{i\infty\}$. A *cusp* of a congruence subgroup Γ is an equivalence class of $\mathbb{Q} \cup \{i\infty\}$ under the action of Γ. For each integer k and each $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})$ define the *slash operator* on smooth functions $f : \mathbb{H} \to \mathbb{C}$ by

$$f|_k\gamma(z) := (cz+d)^{-k} f(\gamma z).$$

Let k be an integer in the following definition.

Definition 1.1.1. Suppose $f : \mathbb{H} \to \mathbb{C}$ is holomorphic and χ is a Dirichlet character modulo N. Then $f(z)$ is a *holomorphic modular form of weight* k *on* $\Gamma_0(N)$ *with character* χ if

1. $f|_k\gamma(z) = \chi(d)f(z)$ for all $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N)$.

2. $f(z)$ has at most polynomial growth as $z \to i\infty$, and analogous conditions hold at the other cusps of $\Gamma_0(N)$.

In particular, notice that the action of S implies that there are no nonzero odd weight modular forms on $SL_2(\mathbb{Z})$ with trivial character. We denote the space of all such functions by $M_k(\Gamma_0(N), \chi)$. We will suppress the character in this notation whenever it is trivial. It is a basic fact that $M_k(\Gamma_0(N), \chi)$ forms a finite dimensional complex vector space for each k and N. If $f(z)$ vanishes at every cusp of $\Gamma_0(N)$ then it is called a *cusp form*. We denote this subspace by $S_k(\Gamma_0(N), \chi)$. Because $T \in \Gamma_0(N)$, each modular form can be identified with its Fourier expansion at infinity

$$f(z) = \sum_{n\geq 0} a(n)q^n$$

where throughout $q := e^{2\pi iz}$. Another important space is the space of *weakly holomorphic modular forms* which is denoted by $M_k^!(\Gamma_0(N), \chi)$. Weakly holomorphic modular forms are allowed to have poles at the cusps and the vector space of such forms is infinite dimensional for any given k and N.

Modular forms can often be constructed by averaging a suitable function over the action of the modular group. The first examples of this idea are the

Eisenstein series

$$E_k(z) := \frac{1}{2\zeta(k)} \sum_{(0,0)\neq(n,m)\in\mathbb{Z}^2} \frac{1}{(nz+m)^k} \in M_k(SL_2(\mathbb{Z})), \tag{1.1}$$

where $\zeta(s)$ is the Riemann zeta-function. For $k > 2$, $E_k(z)$ is absolutely convergent and one can easily check the action of S and T to see it is modular on $SL_2(\mathbb{Z})$. For even $k > 2$, its Fourier expansion is given by

$$E_k(z) = 1 - \frac{2k}{B_k} \sum_{n\geq 1} \sigma_{k-1}(n) q^n,$$

where B_k is the kth Bernoulli number and $\sigma_{k-1}(n)$ is the sum of divisors function given by

$$\sigma_{k-1}(n) = \sum_{d|n} d^{k-1}.$$

For $k = 2$, the Eisenstein series is no longer absolutely convergent. One can still define the weight 2 Eisenstein series by its Fourier expansion:

$$E_2(z) = 1 - 24 \sum_{n\geq 1} \sigma_1(n) q^n.$$

$E_2(z)$ is still periodic, but has a slightly more complicated transformation under S, given by

$$z^{-2} E_2\left(-\frac{1}{z}\right) = E_2(z) + \frac{6}{\pi i z}.$$

The weight 2 Eisenstein series is the first example of a *quasi-modular form.* We will discuss one way to correct the modularity of $E_2(z)$ by giving up holomorphicity in Section 1.1.2. One can also sieve out some coefficients of $E_2(z)$ in order to construct a modular form on another group. For example,

$$F_2(z) := \sum_{n\geq 0} \sigma_1(2n+1)q^{2n+1} \in M_2(\Gamma_0(4)). \tag{1.2}$$

The first example of a cusp form is the *Delta function*

$$\Delta(z) := q\prod_{n\geq 1}(1-q^n)^{24} \in S_{12}(SL_2(\mathbb{Z})).$$

The Delta function is non-vanishing on the upper half-plane and so it can be used as a building block for the first example of a weakly holomorphic modular form. The modular *j-invariant* is given by

$$j(z) := \frac{E_4^3}{\Delta}(z) \in M_0^!(SL_2(\mathbb{Z})).$$

The j-invariant connects many different areas of mathematics; it parameterizes isomorphism classes of elliptic curves over $\mathbb{C}$ and its Fourier coefficients give the dimensions of the representations of the Monster group by the theory of moonshine.

There are natural linear operators which act on spaces of modular forms called *Hecke operators*. For each prime $p \nmid N$, the integer weight Hecke operator $T(p,k,\chi)$ preserves the space $M_k(\Gamma_0(N),\chi)$ (resp. $S_k(\Gamma_0(N),\chi)$) and acts on $f(z) = \sum_{n\geq 0} a(n)q^n \in M_k(\Gamma_0(N), \chi)$ by

$$f(z)|T(p,k,\chi) := \sum_{n\geq 0} \left(a(pn) + \chi(p)p^{k-1}a(n/p)\right) q^n, \tag{1.3}$$

where $a(n/p) = 0$ if $p \nmid n$. More generally, if m is a positive integer, then the action of $T(m,k,\chi)$ is given by

$$f(z)|T(m,k,\chi) := \sum_{n\geq 0} \left(\sum_{d|\gcd(m,n)} \chi(d)d^{k-1}a(mn/d^2) \right) q^n.$$

A modular form is called a Hecke eigenform if it is an eigenfunction $f(z)|T(m) = \alpha_m f(z)$ for each $m > 1$. For example, the Delta function and all of the modular Eisenstien series are Hecke eigenforms due to the fact that they reside in one dimensional spaces.

We will also review the theory of half-integral weight modular forms developed by Shimura [56]. Define

$$\epsilon_d := \begin{cases} 1 & \text{if } d \equiv 1 \pmod 4 \\ i & \text{if } d \equiv 3 \pmod 4, \end{cases}$$

and let $(\frac{c}{d})$ denote the *Kronecker character*. We will also choose the branch of the square root with argument in $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right]$.

Definition 1.1.2. Suppose $f : \mathbb{H} \to \mathbb{C}$ is holomorphic, χ is a Dirichlet character modulo $4N$, and $k \in \frac{1}{2} + \mathbb{Z}$. Then $f(z)$ is a *holomorphic modular form of weight k on $\Gamma_0(4N)$ with character χ* if

1. $f(\gamma z) = \chi(d)(\frac{c}{d})^{2k}\epsilon_d^{-2k}(cz+d)^k f(z)$ for all $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(4N)$.

2. $f(z)$ is holomorphic at the cusps of $\Gamma_0(4N)$.

Just as in the integer weight case, for any given k and N all such functions form a finite dimensional complex vector space. There are also analogous spaces of half-integral weight cusp forms and half-integral weight weakly holomorphic forms.

The first example of a half-integral weight modular form is given by *Jacobi's theta function*:

$$\theta(z) := \sum_{n\in\mathbb{Z}} q^{n^2} \in M_{\frac{1}{2}}(\Gamma_0(4)).$$

The coefficients of half-integral weight modular forms also encode arithmetic information. For example, it is well known that the coefficients of $\theta(z)^3$ are related to the Hurwitz class numbers, which we define in Section 1.1.2. We will also recall the Cohen-Eisenstein series, which are the half-integral weight analogues of the Eisenstein series. In order to do this we need to introduce a few new objects. Let

$$L(s,\chi) := \sum_{n\geq 1} \frac{\chi(n)}{n^s} \tag{1.4}$$

be the L-function associated to the Dirichlet character χ and let

$$\chi_D = \left(\frac{D}{\bullet}\right)$$

be the character associated to a fundamental discriminant D. For $N \geq 0$, write $(-1)^r N = v^2 D$, where D is a fundamental discriminant, and define the coefficients $H(r,N)$ by

$$H(r,N) := \begin{cases} \zeta(1-2r) & \text{if } N = 0 \\ L(1-r,\chi_D)\sum_{a|v}\mu(a)\chi_D(a)a^{r-1}\sigma_{2r-1}(v/a) & \text{if } N > 0, \end{cases}$$

where μ is the Möbius function. For an integer $r \geq 2$, the weight $r + \frac{1}{2}$ *Cohen-Eisenstein series* is given by

$$H_{r+\frac{1}{2}}(z) := \sum_{N\geq 0} H(r,N)q^N \in M_{r+\frac{1}{2}}(\Gamma_0(4)). \tag{1.5}$$

Remark. These series are analogues to the integer weight Eisenstein series because we can use the same averaging idea as in equation (1.1) to define

$$E_{r+\frac{1}{2}}(z) := \sum_{\substack{n>0 \text{ odd} \\ m}} \left(\frac{m}{n}\right)\left(\frac{-4}{n}\right)^{-r-\frac{1}{2}} (nz+m)^{-r-\frac{1}{2}}$$

and

$$F_{r+\frac{1}{2}}(z) := E_{r+\frac{1}{2}}\left(-\frac{1}{4z}\right) z^{-r-\frac{1}{2}}.$$

These are the two Eisenstein series corresponding to the regular cusps of $\Gamma_0(4)$. This means the Eisenstein series is non-vanishing at its associated cusp and vanishes at all other cusps. Using these forms we can write

$$H_{r+\frac{1}{2}}(z) = 2^{-2r-1}\zeta(1-2r)\left((1+i^{2r+1})E_{r+\frac{1}{2}}(z) + i^{2r+1}F_{r+\frac{1}{2}}(z)\right).$$

The Cohen-Eisenstein series can also be explicitly written in terms of the Jacobi theta function and $F_2(z)$ defined in equation (1.2). The first few series are given by

$$\begin{aligned} H_{\frac{5}{2}}(z) &= \frac{1}{120}\left(\theta(z)^5 - 20\theta(z)F_2(z)\right), \\ H_{\frac{7}{2}}(z) &= -\frac{1}{252}\left(\theta(z)^7 - 14\theta(z)^3F_2(z)\right), \\ H_{\frac{9}{2}}(z) &= \frac{1}{240}\left(\theta(z)^9 - 16\theta(z)^5F_2(z) + 16\theta(z)F_2(z)^2\right). \end{aligned}$$

Just like $E_2(z)$ was not quite a modular form, one can also define a Fourier series for $H_{\frac{3}{2}}(z)$ that is not quite modular. We will discuss how to complete this series as well in Section 1.1.2.

If $f(z) = \sum_{n\geq 0} a(n)q^n \in M_{r+\frac{1}{2}}(\Gamma_0(4N), \chi)$, then for each prime $p \nmid 4N$ the half-integral weight Hecke operator $T(p^2, r, \chi)$ preserves the space $M_{r+\frac{1}{2}}(\Gamma_0(4N), \chi)$ and acts by

$$f(z)|T(p^2, r, \chi) := \sum_{n\geq 0}\left(a(p^2n) + \chi^*(p)\left(\frac{n}{p}\right)p^{r-1}a(n) + \chi^*(p^2)p^{2r-1}a(n/p^2)\right)q^n, \tag{1.6}$$

where $\chi^*(n) := \left(\frac{(-1)^r}{n}\right)\chi(n)$. The Cohen-Eisenstein series are examples of half-integral weight Hecke eigenforms.

1.1.2 Harmonic Maass forms

In his famous deathbed letter Ramanujan wrote to Hardy about a strange family of q-series which he called *mock theta functions*. One such example is one of Ramanujan's fifth order mock theta functions:

$$f_0(q) := \sum_{n\geq 0} \frac{q^{n^2}}{(1+q)(1+q^2)\cdots(1+q^n)}. \tag{1.7}$$

Ramanujan gave an inconsistent definition for these mock theta functions and grouped them into categories that he never actually defined. Ramanujan's mock theta functions were not modular, but behaved somewhat like modular forms near roots of unity. Since the time of his letter it was big mystery to explain Ramanujan's mock theta functions and determine how they fit into the theory of modular forms.

Harmonic Maass forms are mild generalizations of the modular forms discussed in the previous section which were originally popularized by two papers [5, 66]. In his thesis Zwegers fit Ramanujan's mock theta functions into a beautiful framework of indefinite theta series. He showed that the mock theta functions were the holomorphic parts of special harmonic Maass forms coming from indefinite quadratic forms. Around the same time Bruinier and Funke defined the current definition of a harmonic Maass form while studying geometric theta lifts. Many of the foundational results in the subject can be found in their paper.

Each harmonic Maass form has a holomorphic and nonholomorphic part. The holomorphic part is often called a *mock modular form* and the nonholomorphic part has a modular cusp form associated to it called its *shadow*. We shall define its connection shortly. To define a harmonic Maass form we relax the condition that the function be holomorphic and replace it with demanding that the function satisfy a certain differential equation. To be precise, define the *weight k hyperbolic Laplacian operator* on $\mathbb{H}$ by

$$\Delta_k := -y^2\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)+iky\left(\frac{\partial}{\partial x}+i\frac{\partial}{\partial y}\right)=-4y^2\frac{\partial}{\partial z}\frac{\partial}{\partial \overline{z}}+2iky\frac{\partial}{\partial \overline{z}}. \quad (1.8)$$

We can now give the definition of a harmonic Maass form.

Definition 1.1.3. Suppose $f:\mathbb{H}\to\mathbb{C}$ is a smooth function, $k\in\frac{1}{2}\mathbb{Z}$, and χ is a Dirichlet character modulo N. Then f is a *harmonic Maass form of weight k on $\Gamma_0(N)$ with character χ* if

1. $$f(\gamma z)=\begin{cases}\chi(d)(cz+d)^k f(z) & \text{if } k\in\mathbb{Z}\\ \chi(d)\left(\frac{c}{d}\right)^{2k}\epsilon_d^{-2k}(cz+d)^k f(z) & \text{if } k\in\frac{1}{2}+\mathbb{Z},\end{cases}$$

   for all $\gamma=\begin{pmatrix}a & b\\ c & d\end{pmatrix}\in\Gamma_0(N)$.

2. $\Delta_k(f)=0$.

3. There exists a polynomial $P_f(q)\in\mathbb{C}[q^{-1}]$ such that $f(z)-P_f(q)=O\left(e^{-\varepsilon y}\right)$ as $y\to\infty$ for some $\varepsilon>0$, and analogous conditions hold at the other cusps of $\Gamma_0(N)$.

We denote the vector space of such forms by $H_k(\Gamma_0(N),\chi)$. If the third condition above is replaced with $f(z)=O\left(e^{\varepsilon y}\right)$ as $y\to\infty$ for some $\varepsilon>0$, then we call the form a *harmonic Maass form with manageable growth* and denote the vector space by $H_k^!(\Gamma_0(N), \chi)$. Both of these spaces are infinite dimensional. The differential equation defined by the hyperbolic Laplacian determines the shape of the Fourier expansion of a harmonic Maass form. In order to describe this expansion we must introduce one special function. The *incomplete gamma function* is defined by

$$\Gamma(s, z) := \int_z^\infty e^{-t} t^{s-1} dt,$$

for $\mathrm{Re}(s) > 0$ and $z \in \mathbb{C}$. (or any $s \in \mathbb{C}$ and $z \in \mathbb{H}$).

Lemma 1.1.4. *Let* $k \in \frac{1}{2}\mathbb{Z} \setminus \{1\}$ *and* $f(z) \in H_k(\Gamma_0(N))$. *Then* f *has the Fourier expansion*

$$\begin{aligned} f(z) &= f^+(z) + f^-(z) \\ &= \sum_{n \gg -\infty} c_f^+(n) q^n + c_f^-(0) y^{1-k} + \sum_{\substack{n \gg -\infty \\ n \neq 0}} c_f^-(n) \Gamma(1-k, 4\pi n y) q^{-n}. \end{aligned}$$

Here $f^+(z)$ is the holomorphic part of $f(z)$, often called a *mock modular form*, and $f^-(z)$ is the nonholomorphic part.

Define the weight k differential operator ξ_k by

$$\xi_k := 2iy^k \overline{\frac{\partial}{\partial \overline{z}}}.$$

The following result due to Bruinier and Funke [5] ties the spaces of harmonic Maass forms and modular forms to each other.

Lemma 1.1.5 (Bruinier-Funke). *Assuming the notation above, the following are true.*

1. *The map* $\xi_{2-k} : H_{2-k}(\Gamma_0(N) \twoheadrightarrow S_k(\Gamma_0(N))$ *is surjective and if* $f(z) \in H_{2-k}(\Gamma_0(N))$, *then*

$$\xi_{2-k}(f(z)) = \xi_{2-k}(f^-(z)) = -(4\pi)^{k-1} \sum_{n \geq 1} \overline{c_f^-(n)} n^{k-1} q^n.$$

2. *The map* $\xi_{2-k} : H^!_{2-k}(\Gamma_0(N) \twoheadrightarrow M^!_k(\Gamma_0(N))$ *is surjective and if* $f(z) \in H^!_{2-k}(\Gamma_0(N))$, *then*

$$\xi_{2-k}(f(z)) = \xi_{2-k}(f^-(z)) = (k-1)\overline{c_f^-(0)} - (4\pi)^{k-1} \sum_{n \geq 1} \overline{c_f^-(n)} n^{1-k} q^n.$$

For any $f(z) \in H_{2-k}(\Gamma_0(N))$, the associated cusp form $\xi_{2-k}(f(z)) \in S_k(\Gamma_0(N))$ is called the *shadow* of $f(z)$.

The averaging idea used to construct the Eisenstein series in Section 1.1.1 can be used to construct harmonic Maass forms as well. Let

$$\Gamma_\infty := \pm \left\{ \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix} \mid n \in \mathbb{Z} \right\}$$

and $\phi(z)$ be a translation invariant function. Then the *weight k level N Poincaré series for* $\phi(z)$ is given by

$$\mathbb{P}(\phi; z) := \sum_{\gamma \in \Gamma_\infty \backslash \Gamma_0(N)} \phi|_k \gamma(z).$$

One example of such a function is the completed weight 2 Eisenstein series. If we let

$$E_2^*(z) := E_2(z) - \frac{3}{\pi y},$$

then $E_2^*(z) \in H_2^!(SL_2(\mathbb{Z}))$. A wealth of examples of harmonic Maass forms come from Zwegers' work on indefinite theta functions. Given a quadratic form Q of type $(r-1,1)$, define its associated bilinear form by $B(X,Y) := Q(X+Y) - Q(X) - Q(Y)$. Then the *theta function associated to* Q with characteristic $a \in \mathbb{R}^r$ and $b \in \mathbb{R}^r$ is the series

$$\Theta_{a,b}(z) := \sum_{n \in a + \mathbb{Z}^r} \rho(n;z) e^{2\pi i B(n,b)} q^{Q(n)},$$

where $\rho(n;z)$ is a special function that ensures the series converges. These theta series are generally vector-valued harmonic Maass forms of weight $\frac{r}{2}$. For example, if we let $Q(j,k) = \frac{1}{2}(5j^2 - 2k^2)$, $a = \begin{pmatrix} \frac{1}{10} \\ 0 \end{pmatrix}$, and $b = \begin{pmatrix} 0 \\ \frac{1}{4} \end{pmatrix}$, then

$$\Theta_{a,b}^+(z) = 2q^{\frac{1}{40}} \left(\sum_{\substack{n+j \geq 0 \\ n-j \geq 0}} - \sum_{\substack{n+j<0 \\ n-j<0}} \right) (-1)^j q^{\frac{5n^2}{2} + \frac{n}{2} - j^2},$$

which is related to Ramanujan's fifth order mock theta function given in equation (1.7).

Remark. The complete definition of an indefinite theta function is slightly more complex and the full scope of Zwegers' work is much deeper than what is presented here. For a more complete overview see [66] or [3].

The first half-integral weight harmonic Maass form to be well understood was Zagier's weight $\frac{3}{2}$ Eisenstein series. For non-zero integers D, let $h(D)$ be the class number of $\mathbb{Q}(\sqrt{D})$. $h(D)$ counts the number of binary quadratic forms with discriminant D up to matrix equivalence. For a positive integer N, let $H(N)$ denote the *Hurwitz class number*, which counts the number of binary quadratic forms with discriminant $-N$ inversely weighted by the size of their automorphism group (which is generally 2). We mentioned above that the coefficients of $\theta(z)^3$ are related to the Hurwitz class numbers, so it seems like $\sum H(N)q^N$ should be a weight $\frac{3}{2}$ modular form. This is not quite true, but in [64] Zagier showed that the Hurwitz class number generating function can be completed to a weight $\frac{3}{2}$ harmonic Maass form. To be precise, Zagier showed that if

$$\mathcal{H}(z) := -\frac{1}{12} + \sum_{N\geq 1} H(N)q^N + \frac{1}{8\pi\sqrt{y}} + \frac{1}{4\sqrt{\pi}} \sum_{N\geq 1} \Gamma\left(-\frac{1}{2}, 4\pi N^2 y\right) q^{-N^2},$$

then $\mathcal{H}(z) \in H^!_{\frac{3}{2}}(\Gamma_0(4))$. Further, the shadow of $\mathcal{H}(z)$ is $\xi_{\frac{3}{2}}(\mathcal{H}(z)) = -\frac{1}{16\pi}\theta(z)$. This series beautifully fits into the family of Cohen-Eisenstien series. In fact, using the notation in Section 1.1.1 and the Dirichlet class number formula, one can write $\sum_{N\geq 0} H(1,N)q^N = -\frac{1}{12} + \sum_{n\geq 1} H(N)q^N$, and so the Fourier series of the weight $\frac{3}{2}$ Cohen-Eisenstein series is the holomorphic part of Zagier's weight $\frac{3}{2}$ Eisenstein series:

$$\mathcal{H}^{+}(z) = H_{\frac{3}{2}}(z).$$

1.1.3 *L*-functions

Dirichlet series and L-functions have classical roots going back to Euler and have become an integral part of modern number theory. Fundamental problems such as the Riemann Hypothesis, the Birch and Swinnerton-Dyer Conjecture, and the Langlands program are formulated in terms of L-functions. For an arithmetic function $a(n)$ we define its Dirichlet series by

$$L(a,s) := \sum_{n\geq 1} a(n) n^{-s}.$$

Depending on the arithmetic function $a(n)$, the associated Dirichlet series will usually converge absolutely in a small domain. If the function $a(n)$ is multiplicative, then its Dirichlet series has an *Euler product*:

$$L(a,s) = \prod_{p} \left(1 + a(p)p^{-s} + a(p^2)p^{-2s} + \dots\right),$$

where the product is over primes. The most famous example of a Dirichlet series is the *Riemann zeta function*

$$\zeta(s) := \sum_{n\geq 1} n^{-s} = \prod_{p} (1 - p^{-s})^{-1} \qquad \mathrm{Re}(s) > 1. \tag{1.9}$$

The Riemann zeta function is just the first case in many infinite families of L-functions. For example, if χ is a Dirichlet character modulo N, then we define the *Dirichlet L-function* as

$$L(\chi, s) := \sum_{n \geq 1} \chi(n) n^{-s} \qquad \mathrm{Re}(s) > 1. \tag{1.10}$$

If K is a number field and $\mathcal{O}_K$ is its ring of integers, then the *Dedekind zeta function for K* is given by

$$\zeta_K(s) := \sum_{\substack{\mathfrak{a} \subset \mathcal{O}_K \\ \text{Integral}}} N(\mathfrak{a})^{-s} \qquad \mathrm{Re}(s) > 1, \tag{1.11}$$

where $N(\mathfrak{a}) = [\mathcal{O}_K : \mathfrak{a}]$ reduces to the simple norm of the ideal $\mathfrak{a}$. Notice that if χ is the trivial character or $K = \mathbb{Q}$, then we return back to the Riemann zeta function. A wealth of interesting arithmetic functions come in the form of coefficients of modular forms. Let $f = \sum_{n \geq 1} a(n) q^n \in S_k(\Gamma_0(N))$ be an even weight newform with real coefficients and normalized so that $a(1) = 1$. The *modular L-function* associated to f is given by

$$L(f, s) := \sum_{n \geq 1} a(n) n^{-s} \qquad \mathrm{Re}(s) > 1 + \frac{k}{2}. \tag{1.12}$$

For special choices of arithmetic function the associated Dirichlet series can be analytically continued. Using the integral representation of the gamma function we can write the *completed Riemann zeta function* as

$$\Lambda(s) := \pi^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}\right)\zeta(s) = \int_0^\infty \sum_{n=1}^{\infty} e^{-\pi n^2 y} y^{\frac{s}{2}} \frac{dy}{y}.$$

We can notice that the theta function shows up in this integral in order to write

$$\Lambda(s) = \frac{1}{2}\int_0^\infty \left(\theta(iy) - 1\right) y^{\frac{s}{2}} \frac{dy}{y}.$$

By using the transformation property

$$\theta\left(-\frac{1}{z}\right) = (-iz)^{\frac{1}{2}}\theta(z)$$

one can show that $\Lambda(s)$ admits an analytic continuation to $\mathbb{C} \setminus \{0, 1\}$, has simple poles at $s = 0$ and $s = 1$ with residues -1 and 1 respectively, and satisfies the functional equation

$$\Lambda(s) = \Lambda(1 - s).$$

The following fundamental theorem says that this kind of behavior is not unique, but behavior may vary within families. For example, Dedekind zeta functions always have simple poles like the Riemann zeta funciton, while Dirichlet L-functions may not.

Theorem 1.1.6 (Fundamental Theorem)**.** *The Dirichlet L-functions, Dedekind zeta functions, and modular L-functions, $L(s)$, listed above have completions*

$\Lambda(s)$, have a meromorphic continuation to $\mathbb{C}$, and satisfy a functional equation

$$\Lambda(s) = \epsilon\Lambda(k - s)$$

for some $k \in \mathbb{Z}$ and $\epsilon \in \{\pm\}$.

For Dirichlet L-functions and Dedekind zeta functions the functional equation comes from the modular behavior of a certain theta function like in the Riemann zeta case. The functional equation for a modular L-function comes from the behavior of the modular form itself.

The completed zeta function clearly has zeros at each negative even integer. Each completed L-function has similar trivial zeros. One of the fundamental problems in all of number theory is to understand the non-trivial zeros of an L-function. The generalized Riemann hypothesis (GRH) is the statement that all of the non-trivial zeros should lie on the central line $\mathrm{Re}(s) = \frac{k}{2}$. More details on L-functions are given in Section 3.2.

1.2 Congruences and p-adic modular forms

As noted above, modular forms often encode information about interesting combinatorial objects. One famous is $p(n)$, the number of partitions of n, which occurs as the n-th coefficient of a weight $-\frac{1}{2}$ modular form. In order to study the coefficients of modular forms we investigate their behavior under the action by the Hecke operators. The fact that certain Hecke operators annihilate specific modular forms modulo ℓ leads to the famous Ramanujan congruences [49]

$$p(5n+4) \equiv 0 \pmod 5, \quad p(7n+5) \equiv 0 \pmod 7, \text{ and } p(11n+6) \equiv 0 \pmod{11}.$$

These congruences are the first glimpse of the much deeper theory of modular Galois representations, which played a key role in the proof of Fermat's last theorem. Work of Eichler, Shimura, Deligne, and Serre [21,22] guarantees that for every integer weight newform and every prime ℓ, there exists an associated 2-dimensional ℓ-adic Galois representation that encodes the coefficients of the newform. A peculiar application of this theory by Serre, using the Chebotarev Density Theorem, implies that 100% of the coefficients of every newform are multiples of m for every m. Inspired by the congruence properties of modular forms Serre [55] introduced the notion of a p-adic modular form as a formal power series that has a sequence of modular forms p-adically converging to it. The theory of p-adic analytic families established by Serre has important consequences, in particular to the special values of p-adic L-functions. This section introduces the main results related to p-adic modular forms.

1.2.1 Conjugacy growth series for wreath product finitary symmetric groups

In [2], Bacher and de la Harpe developed the theory of conjugacy growth series. This theory uses the minimum word length statistics and the conjugacy classes of a group to produce the conjugacy growth series. In particular, this theory ties together infinite permutation groups with finite support and the usual number theoretic partition function.

Let G be a group and S a set that generates G. Then for each $g \in G$ define the *word length*, $\ell_{G,S}(g)$, to be the smallest nonnegative integer n for which there are $s_1, s_2, ..., s_n \in S \cup S^{-1}$ such that $g = s_1 s_2 \cdots s_n$. Define the *conjugacy length*, $\kappa_{G,S}(g)$, as the smallest integer n for which there exists h in the conjugacy class of g such that $\ell_{G,S}(h) = n$. For $n \in \mathbb{N}$ define $\gamma_{G,S}(n) \in \mathbb{N} \cup \{0\} \cup \{\infty\}$ as the number of conjugacy classes of G which contain elements g with $\kappa_{G,S}(g) = n$. Whenever $\gamma_{G,S}(n)$ is finite for all n, we can define the *conjugacy growth series*:

$$C_{G,S}(q) = \sum_{n=0}^{\infty} \gamma_{G,S}(n) q^n = \sum_{g \in Conj(G)} q^{\kappa_{G,S}(g)} \in \mathbb{N}[[q]],$$

where the second sum is taken over representatives of conjugacy classes of G.

We call $\mathrm{Sym}(\mathbb{N})$ the *finitary symmetric group* of $\mathbb{N}$. It is the group of permutations of $\mathbb{N}$ with finite support. Let the *finitary alternating group* of $\mathbb{N}$, $\mathrm{Alt}(\mathbb{N})$, be the subgroup of $\mathrm{Sym}(\mathbb{N})$ of permutations with even signature. Define the two generating sets of $\mathrm{Sym}(\mathbb{N})$, $S_{\mathbb{N}}^{Cox} = \{(i, i+1) : i \in \mathbb{N}\}$ and $T_{\mathbb{N}} = \{(x, y) : x, y \in \mathbb{N} \text{ are distinct}\}$. Let $S \subset \mathrm{Sym}(\mathbb{N})$ be a generating set such that $S_{\mathbb{N}}^{Cox} \subset S \subset T_{\mathbb{N}}$. Then Bacher and de la Harpe prove that (see Proposition 1 of [2])

$$C_{\mathrm{Sym}(\mathbb{N}),\mathrm{S}}(q) = \sum_{n=0}^{\infty} p(n) q^n = \prod_{n=1}^{\infty} \frac{1}{1-q^n},$$

where $p(n)$ counts the number of partitions of n. Similarly we can define two generating sets of $\mathrm{Alt}(\mathbb{N})$, $S^A_{\mathbb{N}} = \{(i, i+1, i+2) : i \in \mathbb{N}\}$ and $T^A_{\mathbb{N}} = \bigcup_{g \in \mathrm{Alt}(\mathbb{N})} g S^A_{\mathbb{N}} g^{-1}$, where $T^A_{\mathbb{N}}$ is the subset of all 3-cycles.

Let $S^{'} \subset \mathrm{Alt}(\mathbb{N})$ be a generating set such that $S^A_{\mathbb{N}} \subset S^{'} \subset T^A_{\mathbb{N}}$. Then Bacher and de la Harpe also prove that (see Proposition 11 of [2])

$$C_{\mathrm{Alt}(\mathbb{N}),\mathrm{S}'}(q) = \sum_{n=0}^{\infty} p(n) q^n \sum_{m=0}^{\infty} p_e(m) q^m = \frac{1}{2} \prod_{n=1}^{\infty} \frac{1}{(1-q^n)^2} + \frac{1}{2} \prod_{n=1}^{\infty} \frac{1}{1-q^{2n}},$$

where $p_e(m)$ denotes the number of partitions of m into an even number of parts.

In Section 2.1.1 we will generate conjugacy growth series, $C_{W'_M, S'_*}(q)$, that are powers of the series above. If M is a positive integer, let

$$\sum_{n=0}^{\infty} \gamma_{W'_M, S'_*}(n) q^n = \left(\frac{1}{2} \prod_{n=1}^{\infty} \frac{1}{(1-q^n)^2} + \frac{1}{2} \prod_{n=1}^{\infty} \frac{1}{1-q^{2n}} \right)^M. \tag{1.13}$$

Proposition 2.1.1 shows that these generating functions are the conjugacy growth series for wreath products of $\mathrm{Alt}(\mathbb{N})$.

Let H be a group and let $H^{(\mathbb{N})}$ be the group of functions from $\mathbb{N}$ to H. Then $W := H \wr_{\mathbb{N}} \mathrm{Sym}(\mathbb{N}) = H^{(\mathbb{N})} \rtimes \mathrm{Sym}(\mathbb{N})$ is called a *permutation wreath product*. $\mathrm{Sym}(\mathbb{N})$ has a natural action on $H^{(\mathbb{N})}$; $\sigma \in \mathrm{Sym}(\mathbb{N})$ acts on $\phi \in H^{(\mathbb{N})}$ by $\sigma(\phi) = \phi \circ \sigma^{-1}$. One can also think of $H^{(\mathbb{N})}$ as $|\mathbb{N}|$ copies of H, and so an element of $H^{(\mathbb{N})}$ can be thought of as $|\mathbb{N}|$ elements of H indexed by $\mathbb{N}$. In particular, $\mathrm{Sym}(\mathbb{N})$ acts naturally on these indices. For $\sigma, \tau \in \mathrm{Sym}(\mathbb{N})$ and $\phi, \psi \in H^{(\mathbb{N})}$, the multiplication in the wreath product is given by $(\phi, \sigma)(\psi, \tau) = (\phi\sigma(\psi), \sigma\tau)$.

The alternating wreath product, $W' := H \wr_{\mathbb{N}} \mathrm{Alt}(\mathbb{N})$, can be defined analogously.

In view of equation (1.13) and its interpretation in terms of conjugacy growth series, it is natural to study the congruence properties of the coefficients of these functions in the spirit of the earlier work of Bacher and de la Harpe and Cotron, Dicks, and Fleming in [2] and [17]. In [17] Cotron, Dicks, and Fleming only discuss congruences for $C_{W'_M,S'_*}(q)$ for $M = 1$ and for powers of the primes 5 and 7. For example, they proved that

$$\gamma_{W'_1,S'_*}(2 \cdot 5^4 n + 1198) \equiv 0 \pmod 5$$

and

$$\gamma_{W'_1,S'_*}(2 \cdot 7^6 n + 225494) \equiv 0 \pmod{49}.$$

It is natural to ask for a more complete description of congruences for all of the general wreath products. This also motivates the study of sums of mixed weight modular forms in general.

Cotron, Dicks, and Fleming use the theory of modular forms to obtain their results. Therefore, one expects to use modular forms to study the conjugacy growth series given in equation (1.13). However, a difficulty arises; these functions are *mixed weight modular forms*, finite sums of modular forms with different weights. Therefore, we must first obtain a general theorem about congruences for coefficients of mixed weight modular forms.

Theorem 1.2.1. *Let K be an algebraic number field with ring of integers*

$\mathcal{O}_K$. *Suppose* $f_i(z) = \sum_{n=0}^{\infty} a_i(n)q^n \in M^!_{k_i}(\Gamma_0(N_i), \chi_i) \cap \mathcal{O}_K((q))$, $g_j(z) = \sum_{m=0}^{\infty} b_j(m)q^m \in M^!_{\lambda_j+\frac{1}{2}}(\Gamma_0(M_j), \chi_j) \cap \mathcal{O}_K((q))$ *where* $4 \mid M_j$ *for every* j, *and let*

$$F(z) = \sum A(n)q^n = \sum_{i=1}^{u} f_i(z) + \sum_{j=1}^{v} g_j(z).$$

Let N *be minimal such that* $N_i \mid N$ *and* $M_j \mid N$ *for every* i *and* j, *and let* p *be prime such that* $(N, p) = 1$. *If* r *is a sufficiently large integer, then for each positive integer* j, *a positive proportion of primes* $Q \equiv -1 \pmod{Np^j}$ *have the property that*

$$A(Q^{4t+3}p^r n) \equiv 0 \pmod{p^j},$$

where $(Qp, n) = 1$ *and* t *is a nonnegative integer.*

Remark. If there are any half integral weight forms in the sum of forms above, then we will have $4|N$. In this case, it is clear that p must be an odd prime.

Applying Thereom 1.2.1 to the conjugacy growth series leads to the following theorem.

Theorem 1.2.2. *Suppose* $p \geq 5$ *is prime and let* j *be a positive integer. If* r *is a sufficiently large integer, then for a positive proportion of primes* $Q \equiv -1 \pmod{576p^j}$, *we have that*

$$\gamma_{W'_M, S'_*}\left(\frac{Q^{4t+3}p^r n + M}{12}\right) \equiv 0 \pmod{p^j}$$

for all n *coprime to* Qp, *and for all nonnegative integers* t.

1.2.2 Harmonic Maass form eigencurves

In [53] Serre introduced the notion of a p-adic modular form and showed the power of studying a p-adic analytic family of modular eigenforms. Work of Hida [29] and Coleman [13] expanded on Serre's initial definition of p-adic modular form to introduce overconvergent modular forms and offered more examples and applications. Coleman, in particular, defined the slope of an eigenform as the p-adic valuation of its U_p eigenvalue and proved that overconvergent modular forms with small slope are classical modular forms. In [14] Coleman and Mazur organized all of these results by constructing a geometric object called the "eigencurve." The *eigencurve* is a rigid-analytic curve whose points correspond to normalized finite slope p-adic overconvergent modular eigenforms.

Using Kummer's congruences, Serre was able to give the first examples of p-adic modular forms. Let v_p be the p-adic valuation on $\mathbb{Q}_p$. If $f = \sum a(n)q^n \in \mathbb{Q}[[q]]$ is a formal power series in q then define $v_p(f) := \inf_n v_p(a(n))$. We then say that f is a *p-adic modular form* if there exists a sequence of classical modular forms f_i of weights k_i such that $v_p(f - f_i) \to \infty$ as $i \to \infty$. The weight of a p-adic modular form is given by the limits of weights of the classical (holomorphic) modular forms in $X := \mathbb{Z}_p \times \mathbb{Z}/(p-1)\mathbb{Z}$. For a more in-depth discussion of weights see [53].

Using the Eisenstein series Serre constructed the p-adic Eisenstein series.

Define

$$\sigma_k^{(p)} := \sum_{\substack{d|n \\ \gcd(d,p)=1}} d^k,$$

and let $\zeta^{(p)}(s)$ be the *p-adic zeta function* (see [37]). We now have that

$$G_{2k}^{(p)}(z) = \frac{1}{2}\zeta^{(p)}(1-2k) + \sum_{n=1}^{\infty} \sigma_{2k-1}^{(p)}(n) q^n$$

is a p-adic Eisenstein series of weight $2k$. Clearly there is a sequence $2k_i$ of positive even integers that tends to $2k$ p-adically and $\sigma_{2k_i-1}(n)$ tends to $\sigma_{2k-1}^{(p)}(n)$ p-adically. The p-adic Eisenstein series are also classical modular forms on $\Gamma_0(p)$ and can be written as

$$G_{2k}^{(p)}(z) = G_{2k}(z) - p^{2k-1} G_{2k}(pz).$$

This form is a p-stabilization of $G_{2k}(z)$ so that $G_{2k}^{(p)}(z)$ is an eigenform for the U_p operator with eigenvalue coprime to p. The p-adic Eisenstein series satisfy incredible congruences; we have that $G_{k_1}^{(p)}(z) \equiv G_{k_2}^{(p)}(z) \pmod{p^a}$ whenever $k_1 \equiv k_2 \pmod{(p-1)p^{a-1}}$ and k_1 and k_2 are not divisible by $p-1$. For example, $6 \equiv 10 \pmod 4$ and $6, 10 \not\equiv 0 \pmod 4$, so we have

$$G_6^{(5)}(z) = \frac{781}{126} + q + 33q^2 + 244q^3 + 1057q^4 + q^5 + \cdots,$$

and

$$G_{10}^{(5)}(z) = \frac{488281}{66} + q + 513q^2 + 19684q^3 + 262657q^4 + q^5 + \cdots$$

are congruent modulo 5. The congruences can be explained using Kummer's congruences and Euler's theorem.

Mazur recently raised the question of whether or not an eigencurve-like object exists in the world of harmonic Maass forms. Harmonic Maass forms are traditionally built using methods which rarely lead to forms which are eigenforms (for background see [3]). Namely, the most well known constructions involve Poincaré series, indefinite theta functions, and Ramanujan's mock theta functions. These methods do not generally offer Hecke eigenforms. To this end, the first goal is to construct canonical families of harmonic Hecke eigenforms, out of which one hopes to be able to construct an eigencurve.

Here we construct two families, one integer weight and one half-integer weight, of harmonic Maass forms which are eigenforms for the Hecke operators (see Section 2.2.1 for the definition of the relevant Hecke operators). A natural place to look for a suitable family of harmonic Maass forms is the pullback under the ξ-operator of the classical Eisenstein series that Serre used. The pullback, however, is infinite dimensional. For example, the ξ-operator annihilates all weakly holomorphic modular forms. Therefore, the problem is to construct forms that are the pullback under the ξ-operator, and are also Hecke eigenforms and have p-adic properties. Our first family will be a pullback of the classical Eisenstein series that saitsfies these properties. For $k > 0$,

define

$$G(z,-2k) := \frac{(2k)!\,\zeta(2k+1)}{(2\pi)^{2k}} + \frac{(-1)^{k+1}y^{1+2k}2^{1+2k}\pi\zeta(-2k-1)}{2k+1}$$
$$+ (-1)^k(2\pi)^{-2k}(2k)!\sum_{n=1}^{\infty}\frac{\sigma_{2k+1}(n)}{n^{2k+1}}q^n$$
$$+ (-1)^k(2\pi)^{-2k}\sum_{n=1}^{\infty}\frac{\sigma_{2k+1}(n)}{n^{2k+1}}\Gamma(1+2k,4\pi ny)q^{-n}.$$

For half-integral weights, our family of forms will be a pullback of the Cohen-Eisenstein series under the ξ-operator. Define

$$T_r^{\chi}(v) := \sum_{a|v}\mu(a)\chi(a)a^{r-1}\sigma_{2r-1}(v/a),$$

where r is an integer. Set $(-1)^r N = Dv^2$ with D the discriminant of $\mathbb{Q}(\sqrt{D})$ and let $\chi_D = \left(\frac{D}{\cdot}\right)$ be the associated character as before. Let

$$c_r(N) = \begin{cases} i^{2r+1}L(1+r,\chi_D)\frac{1}{v^{2r+1}}T_{r+1}^{\chi_D}(v) & N>0 \\ i^{2r-1}\zeta(1+2r) + \frac{2^{2r+4}i\pi^{2r+1}y^{r+\frac{1}{2}}\zeta(-1-2r)}{(2r-3)\Gamma(2r+1)} & N=0 \\ \pi^{3/2}\frac{L(-r,\chi_D)T_{r+1}^{\chi_D}(v)}{N^{r+\frac{1}{2}}}\frac{\Gamma\left(\frac{r+a}{2}\right)}{\Gamma\left(\frac{r+1+a}{2}\right)\Gamma\left(r+\frac{1}{2}\right)}\Gamma\left(r+\frac{1}{2},-4\pi Ny\right) & N<0, \end{cases} \tag{1.14}$$

where $a = 0$ if r is odd and $a = 1$ if r is even. Then, for $r \geq 1$, define

$$\mathcal{H}\left(z,-r+\frac{1}{2}\right) := \sum_{N\in\mathbb{Z}} c_r(N)q^N. \tag{1.15}$$

Remark. The coefficients for $N > 0$ and $N < 0$ of $\mathcal{H}\left(z,-r+\frac{1}{2}\right)$ alternate between L-functions for real and imaginary quadratic fields as r changes parity. The L-functions for real quadratic fields are known to encode information about the torsion groups of K-groups for real quadratic fields. Therefore, the functions $\mathcal{H}\left(z, -r+\frac{1}{2}\right)$ create a grid that encodes this information for $K_n(\mathbb{Q}(\sqrt{D}))$ as both n and D vary.

We now have the following theorem.

Theorem 1.2.3. *Assuming the notation above, the following are true.*

1. *For positive integers k, we have that $G(z, -2k)$ is a weight $-2k$ harmonic Maass form on $SL_2(\mathbb{Z})$. Furthermore, $G(z, -2k)$ has eigenvalue $1+\frac{1}{p^{2k+1}}$ under the Hecke operator $T(p)$.*

2. *For positive integers r, we have that $\mathcal{H}\left(z, -r+\frac{1}{2}\right)$ is a weight $-r+\frac{1}{2}$ harmonic Maass form on $\Gamma_0(4)$. Furthermore, $\mathcal{H}\left(z, -r+\frac{1}{2}\right)$ has eigenvalue $1+\frac{1}{p^{2r+1}}$ under the Hecke operator $T(p^2)$.*

Remark. The proof of Theorem 1.2.3 will show that these forms can be viewed as two parameter functions in z and w where w is the weight of the form. Specializing w to $-2k$ for the integer weight case and to $-r+\frac{1}{2}$ in the half-integral weight case produces two families of harmonic Maass Hecke eigenforms which define lines on two Hecke eigencurves.

Remark. Just as the weight 2 Eisenstein series is not a modular form, the weight 0 form here is not a harmonic Maass form. However, we will see that there is a weight 0 p-adic harmonic Maass form in the same way that there is a weight 2 p-adic Eisenstein series.

Remark. Integer weight non-holomorphic Eisenstein series have been studied before. For example, in [65] Zagier considers the form

$$\widetilde{G}(z,s) = \frac{1}{2} \sum_{\substack{(m,n)\in\mathbb{Z}^2 \\ (m,n)\neq(0,0)}} \frac{y^s}{|nz+m|^{2s}},$$

which transforms as a weight 0 modular form with respect to z is an eigenform of Δ_0 with eigenvalue $s(1-s)$. This form plays an important role in the Rankin-Selberg method [50], [52]. Zagier shows that it has a meromorphic continuation so that $\widetilde{G}^*(z,s) = \pi^{-s}\Gamma(s)\widetilde{G}(z,s)$ satisfies $\widetilde{G}^*(z,s) = \widetilde{G}^*(z,1-s)$. The Maass lowering operator $L = -2iy^2\frac{\partial}{\partial \bar{z}}$ takes a function that transforms like a modular form of weight k to a function that transforms like a modular form of weight $k-2$. Furthermore, if f is an eigenform for Δ_k with eigenvalue λ, then $L(f)$ is an eigenform for Δ_{k-2} with eigenvalue $\lambda - k + 2$ (Chapter 5 of [3]). In particular, we can see that

$$L^k(\widetilde{G}(z,s)) = \frac{\Gamma(s+k)}{2\Gamma(s)} \sum_{\substack{(m,n)\in\mathbb{Z}^2 \\ (m,n)\neq(0,0)}} \frac{y^{s+k}(nz+m)^{2k}}{|nz+m|^{2s}}$$

is an eigenform of Δ_{-2k} with eigenvalue $-(s+k)(s-k-1)$. Evaluating at $s = k+1$ makes the form harmonic and gives the same forms as the ones in Theorem 1.2.3 (1).

Remark. The case of $r = 0$ has been constructed by Rhoades and Waldherr in [51] using a slightly different method. Their result can be recovered using the same method as in this paper and then sieving to suitably modify the

Fourier expansion. The work of Rhoades and Waldherr follows up on work of Duke and Imamoḡlu [25] and Duke, Imamoḡlu, and Tóth [26]. In [25] Duke and Imamoḡlu use the Kronecker limit formula to construct a function which has values of L-functions at $s = 1$ for its Fourier coefficients. This function was the first example and the motivation for the work in [26] where Duke, Imamoḡlu, and Tóth construct forms of weight $\frac{1}{2}$ on $\Gamma_0(4)$ whose Fourier coefficients are given in terms of cycle integrals of the modular j-function.

Remark. The forms in part 1 of Theorem 1.2.3 behave nicely under the flipping operator (see [3]), which essentially switches the holomorphic and non-holomorphic parts of a harmonic Maass form. Similar functions are studied by Bringmann, Kane, and Rhoades in [4].

Serre used the classical Eisenstein series to build p-adic modular forms. In a similar way we can use these harmonic Maass forms to build p-adic harmonic Maass forms.

Definition 1.2.4. A **weight k p-adic harmonic Maass form** is a formal power series

$$f(z) = \sum_{n \gg -\infty} c_f^+(n) q^n + c_f^-(0) y^{1-k} + \sum_{0 \neq n \ll \infty} c_f^-(n) \Gamma\left(1-k, -4\pi n y\right) q^n,$$

where $\Gamma(1-k, -4\pi n y)$ is taken as a formal symbol and where the coefficients $c_f^{\pm}(n)$ are in $\mathbb{C}_p$, such that there exists a series of harmonic Maass forms $f_i(z)$, of weights k_i, such that the following properties are satisfied:

1. $\lim_{i\to\infty} n^{1-k_i} c_{f_i}^{\pm}(n) = n^{1-k} c_f^{\pm}(n)$ for $n \neq 0$.

2. $\lim_{i\to\infty} c^{\pm}_{f_i}(0) = c^{\pm}_f(0)$.

Remark. Here $\lim_{i\to\infty} n^{1-k_i}c^{\pm}_{f_i}(n) = n^{1-k}c^{\pm}_f(n)$ means $v_p(n^{1-k_i}c^{\pm}_{f_i}(n) - n^{1-k}c^{\pm}_f(n))$ tends to ∞ and we have that k is the limit of the k_i in X.

We will need a few definitions before describing our p-adic harmonic Maass forms. Let $L_p(s,\chi)$ be the *p-adic L-function* (see [32]) and define

$$T_r^{\chi,(p)}(v) := \sum_{\substack{a\mid v\\ \gcd(a,p)=1}} \mu(a)\chi(a)a^{r-1}\sigma^{(p)}_{2r-1}(v/a).$$

Also define the usual *p-adic Gamma function* (see [40]) by

$$\Gamma^{(p)}(n) := (-1)^n \prod_{\substack{0<j<n\\ p\nmid j}} j \qquad \text{if } n\in\mathbb{Z},$$

and

$$\Gamma^{(p)}(x) := \lim_{n\to x}\Gamma^{(p)}(n) \qquad \text{if } x\in\mathbb{Z}_p.$$

For any $x\in\mathbb{Z}_p$ we have $v_p(\Gamma^{(p)}(x)) = 1$. In the following formulas we define $\pi := \Gamma^{(p)}\left(\frac{1}{2}\right)^2$ so that $v_p(\pi) = 1$. We now have the following theorem.

Theorem 1.2.5. *Suppose p is prime and let $\Gamma(\cdot,\cdot)$ be a formal symbol. Then the following are true.*

1. *For each $k\in X$, we have that*

$$G^{(p)}(z,-2k) := \frac{\Gamma^{(p)}(2k+1)\zeta^{(p)}(2k+1)}{(2\pi)^{2k}} + \frac{(-1)^{k+1}y^{1+2k}2^{1+2k}\pi\zeta^{(p)}(-2k-1)}{2k+1}$$

$$+(-1)^k(2\pi)^{-2k}\Gamma^{(p)}(2k+1)\sum_{n=1}^{\infty}\frac{\sigma_{2k+1}^{(p)}(n)}{n^{2k+1}}q^n$$
$$+(-1)^k(2\pi)^{-2k}\sum_{n=1}^{\infty}\frac{\sigma_{2k+1}^{(p)}(n)}{n^{2k+1}}\Gamma(1+2k,4\pi ny)q^{-n}$$

is a weight $-2k$ p-adic harmonic Maass form.

2. *For each $-r+\frac{1}{2}\in X$, let*

$$c_r^{(p)}(N) := \begin{cases} i^{2r+1}L_p(1+r,\chi_D)\frac{1}{v^{2r+1}}T_{r+1}^{\chi_D,(p)}(v) & N>0 \\ i^{2r-1}\zeta^{(p)}(1+2r)+\frac{2^{2r+4}i\pi^{2r+1}y^{r+\frac{1}{2}}\zeta^{(p)}(-1-2r)}{(2r-3)\Gamma^{(p)}(2r+1)} & N=0 \\ \pi^{3/2}\frac{L_p(-r,\chi_D)T_{r+1}^{\chi_D,(p)}(v)}{N^{r+\frac{1}{2}}}\frac{\Gamma^{(p)}\left(\frac{r+a}{2}\right)}{\Gamma^{(p)}\left(\frac{r+1+a}{2}\right)\Gamma^{(p)}\left(r+\frac{1}{2}\right)}\Gamma\left(r+\frac{1}{2},-4\pi Ny\right) & N<0. \end{cases}$$

Then $\mathcal{H}^{(p)}\left(z,-r+\frac{1}{2}\right)=\sum_{N\in\mathbb{Z}}c_r^{(p)}q^N$ is a weight $-r+\frac{1}{2}$ p-adic harmonic Maass form.

Remark. Note that these forms enjoy congruences similar to the ones for p-adic modular forms because of the generalized Bernoulli number congruences. Congruences for the holomorphic parts rely on the existence of a p-adic regulator for L-functions.

Remark. When k is an integer, the forms $G^{(p)}(z,-2k)$ satisfy

$$G^{(p)}(z,-2k)=G(z,-2k)-G(pz,-2k),$$

which is the analogue of the equation above that the classical p-adic Eisenstein series satisfy. This implies that $G^{(p)}(z,-2k)$ is a standard harmonic Maass form on $\Gamma_0(p)$. We do not know a similar formula for the half-integral weight forms.

Remark. Suppose p is a prime and consider an infinite sequence of even integers which p-adically go to zero (i.e. $\{2p^t\}_{t=1}^{\infty}$). By the proof of Theorem 1.2.5, taking the p-adic limit of a series of forms with these weights defines a p-adic harmonic Maass form of weight 0. As noted above, this is the analogue to the quasimodular form E_2 which is not quite a modular form, but Serre showed leads to a weight 2 p-adic modular form. In fact, the weight 0 p-adic harmonic Maass form constructed here is the preimage of Serre's weight 2 p-adic Eisenstein series under the ξ_0-operator.

Remark. Theorem 1.2.5 implies that the Cohen-Eisenstein series are p-adic modular forms in the sense of Serre. This fact was proven by Koblitz in [35].

Not much is known about harmonic Maass Hecke eigenforms except for the forms constructed here. The fact that Hecke operators increase the order of singularities at cusps poses a major roadblock in the study of harmonic Maass Hecke eigenforms. The forms constructed here stand out because this issue doesn't arise. It is an open question of Mazur to describe what the general structure of a "mock eigencurve" could be. For example, are there other branches of the mock eigencurve that connect together other harmonic Maass eigenforms?

1.3 Distributions and Jensen polynomials

Some of the most natural and difficult problems in number theory involve prime numbers. The first important result in the subject was the Prime Number Theorem which states that the number of primes up to X grows asymptotically like $X/\log X$. This result has been refined by Dirichlet's Theorem and the Chebotarev Density Theorem which, describe the proportion of primes in a given arithmetic progression and the proportion of primes that have a certain splitting behavior in a given Galois extension. These results have their origin in the study of the Riemann zeta-function $\zeta(s)$ and the Riemann Hypothesis, which conjectures that all of the non-trivial zeros of $\zeta(s)$ have real part $\frac{1}{2}$. Dyson, Montgomery, and Odlyzko [34, 39, 43] further conjectured that these non-trivial zeros are distributed like the eigenvalues of random Hermitian matrices. This prediction is now known as the *Gaussian Unitary Ensemble* (GUE) random matrix model. The Jensen polynomials were originally defined in order to study the zeros of $\zeta(s)$. This section introduces results related to Jensen polynomials and the distributions of zeros of various L-functions.

1.3.1 Hyperbolicity of the partition Jensen polynomials

Given a function $a : \mathbb{N} \to \mathbb{R}$ and positive integers d and n, the associated *Jensen polynomial of degree d and shift n* is defined by

$$J_a^{d,n}(X) := \sum_{j=0}^{d} \binom{d}{j} a(n+j) X^j. \tag{1.16}$$

A polynomial is said to be *hyperbolic* if all of its zeros are real. Given an entire real function $\varphi(x)$ with Taylor expansion $\varphi(x) = \sum_{n\geq 0} \frac{\alpha(n)x^n}{n!}$, it is a theorem of Jensen [33] that $\varphi(x)$ is in the Laguerre-Pólya class if and only if all of the associated Jensen polynomials $J_\alpha^{d,0}(X)$ are hyperbolic.

In this paper, we study the hyperbolicity of Jensen polynomials $J_p^{d,n}(X)$ associated to the partition function $p(n)$. Chen, Jia, and Wang conjectured that for each positive integer d, $J_p^{d,n}(X)$ is eventually hyperbolic [7]. For example, hyperbolicity of $J_p^{2,n}(X)$ is equivalent to $p(n+2)p(n) \leq p(n+1)^2$, a condition known as log concavity. Desalvo and Pak proved that this condition holds for all $n \geq 25$ in [23].

Recent results of Griffin-Ono-Rolen-Zagier [27] show that Jensen polynomials for a large family of functions, including those associated to $\xi(s)$ and the partition function, are eventually hyperbolic. A more exact statement of their theorem will be given in the next section. Their proof relates the polynomials $J_p^{n,d}(X)$ to the *Hermite polynomials* $H_d(X)$, defined by the generating function

$$e^{tX-t^2} = \sum_{d=0}^{\infty} H_d(X) \cdot \frac{t^d}{d!} = 1 + X \cdot t + (X^2 - 2) \cdot \frac{t^2}{2} + (X^3 - 6X) \cdot \frac{t^3}{6} + \dots .$$

More precisely, if

$$c := \frac{2}{3}\pi^2, \qquad w(n) := \frac{1}{\sqrt{c(n - \frac{1}{24})}}, \qquad \delta(n) := \frac{cw(n)^{\frac{3}{2}}}{\sqrt{2}},$$

the authors prove that

$$\lim_{n\to\infty} \frac{2^d}{p(n)\delta(n)^d} \cdot J_p^{d,n}\left(\delta(n)X - e^{-cw(n)/2}\right) = H_d(X). \tag{1.17}$$

Since the Hermite polynomials have distinct real roots, it follows that the polynomial on the left-hand side above, and hence $J_p^{d,n}(X)$, is eventually hyperbolic. In other words, for each d there exists some N such that for all $n \geq N$, the polynomial $J_p^{d,n}(X)$ is hyperbolic. Define $N(d)$ to be the minimal such N. For example, the results of Nicolas and Desalvo and Pak show $N(2) = 25$. We determine the following further values of $N(d)$.

Theorem 1.3.1. *Let $N(d)$ be defined as above. Then $N(3) = 94, N(4) = 206$, and $N(5) = 381$.*

Remark. During the preparation of this paper, the authors were notified that Chen, Jia, and Wang [7] independently proved $N(3) = 94$ using different methods.

The proof of Theorem 1.3.1 relies on obtaining functions that closely approximate the ratios $p(n+j)/p(n)$ and bounding the error of these approximations for large n. For $d = 3, 4, 5$, direct computation gives rise to good bounds, allowing us to reduce Theorem 1.3.1 to checking a reasonably small finite number of cases. As an illustration of these techniques, we also prove a recent conjecture of Chen which involves an inequality of polynomials in ratios of close partition numbers.

Theorem 1.3.2 (Conjecture 6.13 in [6])**.** *Let* $u_n = p(n+1)p(n-1)/p(n)^2$. *Then for all* $n \geq 2$*, we have*

$$4(1-u_n)(1-u_{n+1}) < \left(1 + \frac{\pi}{\sqrt{24}n^{3/2}}\right)(1-u_n u_{n+1})^2.$$

For arbitrary d, similar techniques, along with the convergence of $J_p^{d,n}(X)$ to the Hermite polynomials $H_d(X)$ after change of variable, gives rise to an upper bound for $N(d)$. However, without the benefit of direct computation we rely on rather rough estimates for the errors mentioned above. This yields the following.

Theorem 1.3.3. *For every positive integer* d*, we have* $N(d) \leq (3d)^{24d}(50d)^{3d^2}$.

1.3.2 The Jensen-Pólya program for various L-functions

By extending notes of Jensen, Pólya [47] proved that the Riemann hypothesis (RH) is equivalent to the hyperbolicity of the Jensen polynomials for Riemann's Xi-function. The Riemann Xi-function is the entire function that shifts the zeros of the Riemann zeta-function, $\zeta(s)$, from the line with real part $\frac{1}{2}$ to the real line. It is given by

$$\Xi(z) := \frac{1}{2}\left(-z^2 - \frac{1}{4}\right)\pi^{\frac{iz}{2}-\frac{1}{4}}\Gamma\left(-\frac{iz}{2}+\frac{1}{4}\right)\zeta\left(-iz+\frac{1}{2}\right),$$

where $\Gamma(s)$ is the gamma function. We can consider a change of variable and define the coefficients $\gamma(n)$ by the Taylor expansion of this new function:

$$\Xi_1(x) = \frac{1}{8} \cdot \Xi\left(\frac{i}{2}\sqrt{x}\right) =: \sum_{n\geq 0} \frac{\gamma(n)}{n!} \cdot x^n. \tag{1.18}$$

Pólya originally proved that RH is equivalent to Ξ_1 having an infinite product expansion of the form $\Xi_1(x) = ce^{\sigma x} \prod_{n\geq 1}\left(1 + \frac{x}{x_n}\right)$, where c is a constant, $\sigma \geq 0$, $x_n \in \mathbb{R}^+$, and $\sum x_n^{-1} < \infty$. This condition can be encoded by the hyperbolicity of Jensen polynomials.

RH is equivalent to the hyperbolicity of $J_\gamma^{d,n}(X)$ for all d and n and where γ is given in equation (1.18) as the Taylor coefficients of $\Xi_1(x)$ [19,24,47]. Due to the difficulty of proving RH, research before [27] focused on establishing hyperbolicity for all shifts n for small d. Work of Csordas, Norfolk, and Varga and Dimitrov and Lucas [18, 24] shows that $J_\gamma^{d,n}(X)$ is hyperbolic for all n when $d \leq 3$. In [27], Griffin, Ono, Rolen, and Zagier prove that for any $d \geq 1$, $J_\gamma^{d,n}(X)$ is hyperbolic with at most finitely exceptions n. They prove this by showing there is a family of sequences, called *Hermite-Jensen sequences*, and then showing that for a fixed d the Jensen polynomial of degree d for a sequence, a, in this family converges to the d-th Hermite polynomial as $n \to \infty$. The Hermite polynomials are known to have real distinct roots so $J_a^{d,n}(X)$ must also eventually have real distinct roots.

Definition 1.3.4. A real sequence $a : \mathbb{N} \to \mathbb{R}$ is **Hermite-Jensen** if there exists sequences of positive real numbers $\{A(n)\}$ and $\{\delta(n)\}$ with $\delta(n)$ tending to zero, which satisfy

$$\log\left(\frac{a(n+j)}{a(n)}\right) = A(n)j - \delta(n)^2 j^2 + o(\delta(n)^d) \qquad \text{as} \;\; n \to \infty \tag{1.19}$$

for some $d \geq 1$ and all $0 \leq j \leq d$.

Remark. In [27] the authors give a more general statement about the asymptotic behavior needed for the Jensen polynomials of a sequence to converge to other families of polynomials.

In order to show that the Taylor coefficients of Riemann's Xi-function are Hermite-Jensen, an arbitrary precision asymptotic formula for the derivatives $\Xi^{(2n)}(0)$ was found in [27]. To extend the results in [27] we show that any *good* Dirichlet series is Hermite-Jensen.

Definition 1.3.5. A Dirichlet series $L(s) = \sum_{n\geq 1} a(n)n^{-s}$ is **good** if the following hold.

1. $L(s)$ has a completed form, $\Lambda(s)$, that has an integral representation of the form
$$\Lambda(s) = N^{\frac{s}{2}} \int_0^\infty \left[f(t) - f(\infty)\right] t^s \frac{dt}{t},$$
where the function $f(t)$ has the form
$$f(t) = \alpha(0) + \sum_{n \geq n_0} \alpha(n) e^{-\pi n t},$$
where $f(\infty) = \alpha(0)$.

2. The function $f(t)$ satisfies

$$f\left(\frac{1}{Nt}\right) = \epsilon N^{\frac{k}{2}} t^k f(t),$$

where $\epsilon \in \{\pm 1\}$ which gives rise to an analytic continuation and a functional equation $\Lambda(s) = \epsilon\Lambda(k-s)$.

3. The coefficients of $\Lambda(s)$ are real.

For a good Dirichlet series $L(s)$, we define

$$\Xi(z) := \begin{cases} \left(-z^2 - \frac{k^2}{4}\right)\Lambda\left(\frac{k}{2} - iz\right) & \text{if } \Lambda(s) \text{ has a pole at } s = k \\ \Lambda\left(\frac{k}{2} - iz\right) & \text{otherwise.} \end{cases} \tag{1.20}$$

If $\Lambda(s) = \Lambda(k-s)$, then we define

$$\Xi_1(x) := \Xi(i\sqrt{x}) =: \sum_{n\geq 0} \frac{\gamma(n)}{n!} x^n, \tag{1.21}$$

where

$$\gamma(n) = (-1)^n \frac{n!}{(2n)!} \cdot \Xi^{(2n)}(0).$$

If $\Lambda(s) = -\Lambda(k-s)$, then define

$$\Xi_1(x) := \frac{\Xi(i\sqrt{x})}{\sqrt{x}} =: \sum_{n\geq 0} \frac{\gamma(n)}{n!} x^n, \tag{1.22}$$

where

$$\gamma(n) = i^{2n+1} \frac{n!}{(2n+1)!} \cdot \Xi^{(2n+1)}(0).$$

Theorem 1.3.6. *Suppose that $L(s)$ is a good Dirichlet series. Then $J_\gamma^{d,n}(X)$ is hyperbolic with at most finitely many exceptions n for each fixed $d \geq 1$.*

Remark. This offers evidence for the generalized Riemann Hypothesis (GRH).

Remark. Notice that all of the information of $\gamma(n)$ lies in the derivatives of $\Xi(z)$ at $z = 0$, or equivalently the derivatives of $\Lambda(s)$ at $s = \frac{k}{2}$. In order to prove Theorem 1.3.6 we will prove an asymptotic formula with arbitrary precision for these deriavtives.

Remark. All good L-series satisfy the *Gaussian Unitary Ensemble* (GUE) random matrix prediction in derivative aspect. Dyson, Montgomery, and Odlyzko [34, 39, 43] conjectured that the non-trivial zeros of the Riemann zeta function and other suitable L-functions are distributed like the eigenvalues of random Hermitian matrices. These eigenvalues and the roots of the suitably normalized Hermite polynomials, $H_d(X)$, as $d \to \infty$ both satisfy Wigner's Semicircular Law (Chapter 3 of [1]). The roots of $J_\gamma^{d,0}(X)$, as $d \to \infty$, approximate the zeros of $\Lambda\left(\frac{k}{2} - iz\right)$ [47] so these roots are also expected to satisfy Wigner's Semicircular Law. The derivatives of the completed L-function are also predicted to satisfy GUE and higher derivatives correspond to n growing in $J_\gamma^{d,n}(X)$ so it is natural to study $J_\gamma^{d,n}(X)$ as $n \to \infty$. For a good L-function the $J_\gamma^{d,n}(X)$ converge to the Hermite polynomials which satisfy GUE in degree aspect. This is what is meant by the statement that good L-functions satisfy

GUE in derivative aspect.

The following corollaries give some examples of Hermite-Jensen Dirichlet series.

Corollary 1.3.7. *Dirichlet L-functions for real primitive self-dual characters are good.*

Corollary 1.3.8. *Let $f \in S_{2k}^{new}(\Gamma_0(N))$ be a weight $2k$ modular newform on $\Gamma_0(N)$, then the modular L-function associated to f is good.*

Corollary 1.3.9. *The Dedekind zeta-function for a number field is good.*

In each of these cases we prove an arbitrary precision asymptotic formula for the derivatives of the completed L-series at its central value. We do this to show that these L-series are Hermite-Jensen, but these results are also of independent interest. The statements and proofs of these formulas will be given in Section 3.2.1.

1.4 Schwartz functions

In this section we will give a brief introduction to the recent advances in sphere packing. The sphere packing problem started in 1611 when Kepler asked for the best way to stack cannonballs in a crate. This is the dimension 3 case, but more generally one can ask what proportion of $\mathbb{R}^d$ can be covered with congruent balls. To be more precise, if $\mathcal{P}$ is a packing, then the *finite density* of $\mathcal{P}$ is

$$\Delta_{\mathcal{P}}(r) := \frac{\mathrm{Vol}(\mathcal{P} \cap B_d(0,r))}{\mathrm{Vol}(B_d(0,r))}.$$

The *density* of $\mathcal{P}$ is then $\Delta_{\mathcal{P}} := \limsup_{r\to\infty} \Delta_{\mathcal{P}}(r)$ and the *sphere packing constant* is

$$\Delta_d := \sup_{\mathcal{P}\subset\mathbb{R}^d} \Delta_{\mathcal{P}}. \tag{1.23}$$

The sphere packing problem is then to determine Δ_d for each dimension d. Clearly we have $\Delta_1 = 1$, and in 1892 Thue [60] showed that $\Delta_2 \approx 0.9068$ by proving the hexagonal packing corresponding to the A_2 lattice is optimal for $d = 2$. It wasn't until 1998 when Kepler's original question was answered; Hales showed that $\Delta_3 = \frac{\pi}{\sqrt{18}} \approx 0.7405$. Recently a breakthrough was made by Cohn and Elkies that showed solving the sphere packing problem in dimensions 8 and 24 was within reach. The *Fourier transform* of an L^1 function $f : \mathbb{R}^d \to \mathbb{C}$ is defined by

$$\mathcal{F}(f)(y) = \widehat{f}(y) := \int_{\mathbb{R}^d} f(x) e^{-2\pi i\langle x,y\rangle} dx \qquad y \in \mathbb{R}^d, \tag{1.24}$$

where $\langle x, y\rangle$ is the standard scalar product in $\mathbb{R}^d$. For example, the Fourier transform of the Gaussian is

$$\mathcal{F}\left(e^{-\alpha x^2}\right) = \left(\frac{\pi}{\alpha}\right)^{\frac{d}{2}} e^{-\frac{\pi^2 y^2}{\alpha}}.$$

Given a lattice Λ with shortest nonzero vector of length r_0, the density of the corresponding lattice packing is

$$\Delta_d := \frac{\pi^{d/2}}{\Gamma\left(\frac{d}{2}+1\right)} \left(\frac{r_0}{2}\right)^d \frac{1}{|\Lambda|}.$$

It is common to define the *center density sphere packing* as

$$\delta_d := \left(\frac{r_0}{2}\right)^d \frac{1}{|\Lambda|}.$$

A function $f(x)$ is a *Schwartz function* if f and all of its derivatives decay to zero faster than any inverse power of x. In the following theorem we assume the lattice Λ is self-dual. This kind of result is known as a *linear programming bound.*

Theorem 1.4.1 (Cohn, Elkies [9]). *Suppose $f : \mathbb{R}^d \to \mathbb{R}$ is a Schwartz function satisfying the following two conditions:*

1. *$f(x) \leq 0$ for all $|x| \geq r_0$.*
2. *$\widehat{f}(x) \geq 0$ for all $x \in \mathbb{R}^d$.*

Then

$$\Delta_d \leq \frac{\pi^{d/2}}{\Gamma\left(\frac{d}{2}+1\right)} \left(\frac{r_0}{2}\right)^d \frac{f(0)}{\widehat{f}(0)},$$

or equivalently, the center density of sphere packings in $\mathbb{R}^d$ is bounded above by $\left(\frac{r_0}{2}\right)^d \frac{f(0)}{\widehat{f}(0)}$.

Cohn and Elkies constructed functions for $4 \leq d \leq 36$ which, when combined with their theorem, led to the best known upper bounds for sphere packing in those dimensions. In particular, they showed that the upper bound in dimensions 8 and 24 was extremely close to the known lower bound, which provided evidence that there existed functions which would resolve the sphere packing problem in those dimensions. In 2017 Viazovska explicitly constructed such a function for $d = 8$ using special modular forms and quasi-modular forms.

Theorem 1.4.2 (Viazovska [62]). *There exists a radial Schwartz function* $f : \mathbb{R}^8 \to \mathbb{R}$ *which satisfies*

- $f(x) \leq 0$ *for all* $|x| \geq \sqrt{2}$,
- $\widehat{f}(x) \geq 0$ *for all* $x \in \mathbb{R}^8$,
- $\widehat{f}(0) = f(0) = 1$.

Therefore the E_8 *lattice packing is the optimal packing in* 8 *dimensions.*

Her methods were quickly modified by Cohn, Kumar, Miller, Radchenko, and Viazovska to resolve the sphere packing problem for $d = 24$.

Theorem 1.4.3 (Cohn, Kumar, Miller, Radchenko, Viazovska [11]). *There exists a radial Schwartz function* $f : \mathbb{R}^{24} \to \mathbb{R}$ *which satisfies*

- $f(x) \leq 0$ *for all* $|x| \geq 2$,
- $\widehat{f}(x) \geq 0$ *for all* $x \in \mathbb{R}^{24}$,
- $\widehat{f}(0) = f(0) = 1$.

Therefore the Leech lattice packing, Λ_{24}*, is the optimal packing in* 24 *dimensions.*

The main ideas behind the proofs of these theorems was to split the problem of constructing f into constructing a function f_+ which is a $+1$ eigenfunction for the Fourier transform and f_- which is a -1 eigenfunction for the Fourier transform. Letting f be a linear combination of these two functions allows control over the necessary inequalities. The Poisson Summation Formula also tells us that in order for the function f to resolve the sphere packing problem in a given dimension it also needs to have zeros of specific orders at specific points. To be precise, if r_0 is the shortest vector length in a lattice packing, then $f(x)$ must have double zeros at all lattice points $|x|> r_0$ and a simple zero when $|x|= r_0$. The details of these constructions will be given in Section 4.1.

For other dimensions not much is known for sphere packing. There are conjectures for optimal packings in small dimensions, but in general little is known. The best known lower bound is due to Venkatesh [61] and gives

$$\Delta_d \geq \frac{e^{-\gamma}}{2} \log\log d \cdot 2^{-d},$$

but is only true for a sparse sequence of dimensions. The best known upper bound has not been improved since 1978 when Kabatiansky and Levenshtein [36] proved

$$\Delta_d \leq 2^{-0.599d}.$$

There are problems related to sphere packing for which Viazovska's construction may be useful. In particular, sphere packing is just a special case of an energy optimization problem. Given a discrete closed subset $\mathcal{C} \subset \mathbb{R}^d$ and a potential function $p : (0, \infty) \to \mathbb{R}$, we say $\mathcal{C}$ has *p-energy*

$$E_p(\mathcal{C}) := \liminf_{r\infty} \frac{1}{|\mathcal{C} \cap B_d(0,r)|} \sum_{\substack{x,y \in \mathcal{C} \cap B_d(0,r) \\ x \neq y}} p(|x-y|),$$

provided the limit exists. We say that $\mathcal{C} \subset \mathbb{R}^d$ is *universally optimal* if it minimizes p-energy whenever $p : (0, \infty) \to \mathbb{R}$ is a completely monotonic function of squared distance. The following theorem is the analogous linear programming bound in this setting.

Theorem 1.4.4 (Cohn, Kumar [10])**.** *Let $p : (0, \infty) \to \mathbb{R}$ be any potential function and $f : \mathbb{R}^d \to \mathbb{R}$ be a Schwartz function satisfying*

- *$f(x) \leq p(|x|)$ for all $x \in \mathbb{R}^d \setminus \{0\}$,*
- *$\widehat{f}(x) \geq 0$ for all $x \in \mathbb{R}^d$.*

Then every density ρ discrete subset of $\mathbb{R}^d$ has lower p-energy at least

$$\rho \widehat{f}(0) - f(0).$$

Cohn and Kumar used this theorem to show that for $d = 1$, $\mathbb{Z}$ is universally optimal and made the following conjecture.

Conjecture (Cohn, Kumar)**.** *$A_2, E_8,$ and Λ_{24} are universally optimal in $2, 8,$ and 24 dimensions respectively.*

There is hope that the techniques developed to solve the sphere packing problems in dimensions 8 and 24 can be used to attack this conjecture. Here we construct Schwartz functions using quasi-modular and modular forms which behave well under the Fourier transform.

Theorem 1.4.5. *For each dimension d there exists a radial Schwartz function $f : \mathbb{R}^d \to \mathbb{R}$ and an $n \in \mathbb{N}$ such that*

- $f(x) = f_+(x) + f_-(x)$ *and* $\widehat{f}(x) = (-i)^{-\frac{d}{2}}(f_+(x) - f_-(x))$,
- $f(\sqrt{2m}) \neq 0$ *for* $0 \leq m < n$,
- $f(\sqrt{2n}) = 0$ *and* $f'(\sqrt{2n}) \neq 0$,
- $f(\sqrt{2m}) = f'(\sqrt{2m}) = 0$ *for* $m > n$.

In particular, for each $d \equiv 0 \pmod 8$, *let* $n_+ = \left\lfloor \frac{d}{16} + \frac{1}{2} \right\rfloor$ *and* $n_- = \left\lfloor \frac{d}{16} + 1 \right\rfloor$. *Then there exists radial Schwartz functions* $f_\pm : \mathbb{R}^d \to \mathbb{R}$ *which satisfies*

- $\widehat{f_\pm}(x) = \pm f_\pm(x)$.
- $f_\pm(\sqrt{2m}) \neq 0$ *for* $0 \leq m < n_\pm$.
- $f_\pm(\sqrt{2n_\pm}) = 0$ *and* $f'_\pm(\sqrt{2n_\pm}) \neq 0$.
- $f_\pm(\sqrt{2m}) = f'_\pm(\sqrt{2m}) = 0$ *for* $m > n_\pm$.

Remark. For any given d, it is straightforward to prove inequalities like those in Theorem 1.4.2 and Theorem 1.4.3.

Chapter 2

Congruences and *p*-adic modular forms

This chapter is devoted to the proofs of Theorem 1.2.1, Theorem 1.2.2, Theorem 1.2.3, and Theorem 1.2.5. Each section will contain some preliminary results needed for the proof of the main theorems.

2.1 Conjugacy growth series for wreath product finitary symmetric groups

To prove Theorems 1.2.1 and 1.2.2 we will make use of the theory of modular forms. The relevant generating functions for Theorem 1.2.2 turn out to be mixed weight modular forms. We will use the work of Treneer in [59] on weakly holomorphic modular forms and a famous theorem of Serre (see [44]).

We will show that Theorem 1.2.1 follows from a proposition of Ono and Skinner in [46] which allows us to use the theory of Galois representations attached to modular forms for a finite set of modular forms simultaneously. Section 2.1.2 will cover basic facts about The U and V operators on modular forms, the modularity of eta-quotients, and will give important propositions of Treneer, Serre, Ono, and others which are vital to our proofs. Theorem 1.2.1 will be proved in Section 2.1.3 and Theorem 1.2.2 will be proved in Section 2.1.4. The section will conclude with a short example in Section 2.1.5.

2.1.1 Conjugacy growth series for the finitary alternating wreath product

In Section 1.2.1 it was mentioned that there are other subgroups of the finitary symmetric group that produce interesting conjugacy growth series. Recall the wreath product $W = H \wr_{\mathbb{N}} \mathrm{Sym}(\mathbb{N})$.

For $a \in H \setminus \{1\}$ and $m \in \mathbb{N}$, let $\phi_m^a \in W$ be the permutation that maps $(h, m) \in H \times \mathbb{N}$ to (ah, m) and fixes (h, n) if $n \neq m$. Note that $(\phi_m^a)_{a \in H \setminus \{1\}, m \in \mathbb{N}}$ generates the subgroup $H^{(\mathbb{N})}$ and that ϕ_m^a and ϕ_k^b are conjugate in W if and only if a and b are conjugate in H. For $m \in \mathbb{N}$, let $H_m = \{\phi_m^a : a \in H \setminus \{1\}\}$ and let $T_H = \bigcup_{m \in \mathbb{N}} H_m$ be a subset of $H^{(\mathbb{N})}$. Recall that $T_{\mathbb{N}}$ is the subset of all transpositions in $\mathrm{Sym}(\mathbb{N})$. We consider subsets $S_H \subset T_H$ and $S_{\mathbb{N}} \subset T_{\mathbb{N}}$ and define S_* to be the disjoint union $S_H \sqcup S_{\mathbb{N}}$. If $S_H = \{\phi_{m_1}^{a_1}, ..., \phi_{m_r}^{a_r}\}$ where $\{a_1, ..., a_r\}$ generate H, then S_* generates W. Define $S_*^{'}$ analogously using subsets of T_H^A and $T_{\mathbb{N}}^A$; then $S_*^{'}$ generates $W' = H \wr_{\mathbb{N}} \mathrm{Alt}(\mathbb{N})$. This leads to the following proposition.

Proposition 2.1.1. *Let H be a finite group; denote by M the number of conjugacy classes of H. If $W_M^{'} = H \wr_{\mathbb{N}} \mathrm{Alt}(\mathbb{N})$ and $S_*^{'}$ is a generating set satisfying (PCwr) in [2], then*

$$C_{W_M^{'}, S_*^{'}}(q) = \left(\frac{1}{2} \prod_{n=1}^{\infty} \frac{1}{(1-q^n)^2} + \frac{1}{2} \prod_{n=1}^{\infty} \frac{1}{1-q^{2n}} \right)^M .$$

Proof. For each $w' = (\phi, \sigma) \in W_M^{'} = H \wr_{\mathbb{N}} \mathrm{Alt}(\mathbb{N})$ we can split σ into the product of an even number of cycles of even length, σ_e, and the product of cycles of odd length, σ_o, so that $w' = (\phi, \sigma_e \sigma_o)$. We can associate each conjugacy class in $W_M^{'}$ to an H_*-indexed family of partitions. Using the same notation as in [2] we associate the conjugacy classes in H to the family of partitions

$$\left(\lambda^{(1)}, \nu^{(1)}; (\mu^{(\eta)}, \gamma^{(\eta)})_{\eta \in H_* \setminus 1} \right),$$

where $\nu^{(1)}$ and $\gamma^{(\eta)}$ each have an even number of positive parts, in the following way.

Let $\mathbb{N}^{(w)}$ be the finite subset of $\mathbb{N}$ that is the union of the supports of ϕ and σ and let σ be the product of the disjoint cycles $c_1, ..., c_k$ where $c_i = (x_1^{(i)}, x_2^{(i)}, ..., x_{v_i}^{(i)})$ with $x_j^{(i)} \in \mathbb{N}^{(w)}$ and $v_i = \mathrm{length}(c_i)$. We include cycles of length 1 for each $n \in \mathbb{N}$ such that $n \in \sup(\phi)$ and $n \notin \sup(\sigma)$ so that $\mathbb{N}^{(w)} = \bigsqcup_{1 \leq i \leq k} \sup(c_i)$. Define $\eta_*^w(c_i) \in H_*$ to be the conjugacy class of $\phi(x_{v_i}^{(i)})\phi(x_{vi-1}^{(i)}) \cdots \phi(x_1^{(i)}) \in H$. For $\eta \in H_*$ and $\ell \geq 1$, let $m_\ell^{w,\eta}$ denote the number of cycles c in $\{c_1, ..., c_k\}$ that are of length ℓ and such that $\eta_*^w(c) = \eta$.

Let $\mu^{w,\eta} \vdash n^{w,\eta}$ be the partition with $m_\ell^{w,\eta}$ parts equal to ℓ, for all $\ell \geq 1$. Note that $\sum_{\eta \in H_*, \ell \geq 1} n^{w,\eta} = \sum_{\eta \in H_*, \ell \geq 1} \ell m_\ell^{w,\eta} = |\mathbb{N}^{(w)}|$. Also observe that the partition $\mu^{w,1}$ does not have parts of size 1 because if $v_i = 1$ then $\eta_*^w(c_i) \neq 1$. Using the same notation as above, let $\lambda^{w,1}$ be the partition with $m_\ell^{w,1}$ parts equal to $\ell - 1$. Because we are working in $\mathrm{Alt}(\mathbb{N})$, we can write $\sigma = \sigma_e \sigma_o$; and so this method actually splits to map to two partitions, one of which has an even number of parts. Define the *type* of w as the family $\left(\lambda^{(1)}, \nu^{(1)}; (\mu^{(\eta)}, \gamma^{(\eta)})_{\eta \in H_* \setminus 1}\right)$. Then two elements in W_M' are conjugate if and only if they have the same type. Thus, each H_*-indexed family of partitions, $\left(\lambda^{(1)}, \nu^{(1)}; (\mu^{(\eta)}, \gamma^{(\eta)})_{\eta \in H_* \setminus 1}\right)$, is the type of one conjugacy class in W_M'.

Consider an H_*-indexed family of partitions $\left(\lambda^{(1)}, \nu^{(1)}; (\mu^{(\eta)}, \gamma^{(\eta)})_{\eta \in H_* \setminus 1}\right)$ and the corresponding conjugacy class in W_M'. Let $u^{(1)}, v^{(1)}, u^{(\eta)}, v^{(\eta)}$ be the sums of the parts of $\lambda^{(1)}, \nu^{(1)}, \mu^{(\eta)}, \gamma^{(\eta)}$ and let $k^{(1)}, t^{(1)}, k^{(\eta)}, t^{(\eta)}$ be the be the number of parts of $\lambda^{(1)}, \nu^{(1)}, \mu^{(\eta)}, \gamma^{(\eta)}$ respectively.

Choose a representative $w' = (\phi, \sigma)$ of this conjugacy class such that

$$\sigma = \prod_{i=1}^{k} c_i = \prod_{i=1}^{k} (x_1^{(i)}, x_2^{(i)}, ..., x_{\mu_i}^{(i)})$$

and

$$\phi\left(x_j^{(i)}\right) = 1 \in H \text{ for all } j \in \{1, ..., \mu_i\} \qquad \text{when } \eta_*^w(c_i) = 1$$

$$\phi\left(x_j^{(i)}\right) = \begin{cases} 1 \text{ for all } j \in \{1, ..., \mu_i - 1\} \\ h \neq 1 \text{ for } j = \mu_i \end{cases} \qquad \text{when } \eta_*^w(c_i) \neq 1.$$

Observe that

$$k = k^{(1)} + t^{(1)} + \sum_{\eta \in H_* \backslash 1} (k^{(\eta)} + t^{(\eta)})$$
$$|\mathbb{N}^{(w')}| = u^{(1)} + k^{(1)} + v^{(1)} + t^{(1)} + \sum_{\eta \in H_* \backslash 1} (u^{(\eta)} + v^{(\eta)}).$$

Hence, the contribution to $C_{W'_M, S_*}(q)$ from $\left(\lambda^{(1)}, \nu^{(1)}; (\mu^{(\eta)}, \gamma^{(\eta)})_{\eta \in H_* \backslash 1}\right)$ is

$$\left(q^{u^{(1)}} q^{v^{(1)}} \prod_{\eta \in H_* \backslash 1} q^{u^{(\eta)}} q^{v^{(\eta)}}\right).$$

It follows that

$$\begin{aligned}
C_{W'_M, S_*}(q) &= \left[\left(\prod_{u_1=1}^{\infty} \frac{1}{1 - q^{u_1}}\right)\left(\frac{1}{2}\prod_{v_1}^{\infty} \frac{1}{1 - q^{v_1}} + \frac{1}{2}\prod_{v_1}^{\infty} \frac{1}{1 + q^{v_1}}\right)\right] \\
&\times \prod_{\eta \in H_* \backslash 1} \left[\left(\prod_{u_\eta=1}^{\infty} \frac{1}{1 - q^{u_\eta}}\right)\left(\frac{1}{2}\prod_{v_\eta}^{\infty} \frac{1}{1 - q^{v_\eta}} + \frac{1}{2}\prod_{v_\eta}^{\infty} \frac{1}{1 + q^{v_\eta}}\right)\right] \\
&= \left[\left(\frac{1}{2}\prod_{n_1=1}^{\infty} \frac{1}{1 - q^{2n_1}} + \frac{1}{2}\prod_{n_1=1}^{\infty} \frac{1}{(1 - q^{n_1})^2}\right)\right] \times \prod_{\eta \in H_* \backslash 1} \left[\left(\frac{1}{2}\prod_{n_\eta=1}^{\infty} \frac{1}{1 - q^{2n_\eta}} + \frac{1}{2}\prod_{n_\eta=1}^{\infty} \frac{1}{(1 - q^{n_\eta})^2}\right)\right] \\
&= \left(\frac{1}{2}\prod_{k=1}^{\infty} \frac{1}{1 - q^{2k}} + \frac{1}{2}\prod_{k=1}^{\infty} \frac{1}{(1 - q^k)^2}\right)^{|H_*|} \\
&= \left(\frac{1}{2}\frac{q^{1/12}}{\eta(z)^2} + \frac{1}{2}\frac{q^{1/12}}{\eta(2z)}\right)^{|H_*|}.
\end{aligned}$$

The equality between the first and second line is given in the appendix of [2]. The switch of variables to n_1 and n_η keeps track that the number of products is indexed by the conjugacy classes. □

2.1.2 Preliminaries

We will now recall some operators on integer weight modular forms. If $f(z) = \sum_{n\geq n_0} a(n)q^n$, the U-operator, U_t, on $f(z)$ is defined by

$$f(z)|U_t = \sum_{n\geq n_0} a(tn)q^n.$$

Similarly, the V-operator, V_t, is defined by

$$f(z)|V_t = \sum_{n\geq n_0} a(n)q^{tn}.$$

The following facts can be found in [44, p. 28]. Suppose $f(z) \in M_k(\Gamma_0(N), \chi)$, where k is an integer.

1. If t is a positive integer, then

$$f(z)|V_t \in M_k(\Gamma_0(Nt), \chi).$$

2. If $t \mid N$, then

$$f(z)|U_t \in M_k(\Gamma_0(N), \chi).$$

Furthermore, if $f(z)$ is a cusp form, then so are $f(z)|U_t$ and $f(z)|V_t$.

We will now recall the analogous operators on half-integral weight modular forms. If $g(z) = \sum_{n\geq n_0} b(n)q^n$, the U-operator, U_t, on $g(z)$ is defined by

$$g(z)|U_t = \sum_{n\geq n_0} b(tn)q^n.$$

Similarly, the V-operator, V_t, is defined by

$$g(z)|V_t = \sum_{n \geq n_0} b(n)q^{tn}.$$

The following can be found in [44, p. 50]. Suppose $g(z) \in M_{\lambda+\frac{1}{2}}(\Gamma_0(4N), \chi)$.

1. If t is a positive integer, then

$$g(z)|V_t \in M_{\lambda+\frac{1}{2}}\left(\Gamma_0(4Nt), \left(\frac{4t}{\bullet}\right)\chi\right).$$

2. If $t \mid N$, then

$$g(z)|U_t \in M_{\lambda+\frac{1}{2}}\left(\Gamma_0(4N), \left(\frac{4t}{\bullet}\right)\chi\right).$$

Furthermore, if $g(z)$ is a cusp form, then so are $g(z)|U_t$ and $g(z)|V_t$.

Dedekind's eta-function is a weight $\frac{1}{2}$ modular form defined as

$$\eta(z) := q^{1/24}\prod_{n=1}^{\infty}(1-q^n).$$

An *eta-quotient* is a function $f(z)$ of the form

$$f(z) = \prod_{\delta|N} \eta(\delta z)^{r_\delta},$$

where $N \geq 1$ and r_δ is an integer. We have the following useful proposition for eta-quotients.

Proposition 2.1.2 ([44, p. 18]). *If* $f(z) = \prod_{\delta|N} \eta(\delta z)^{r_\delta}$ *is an eta-quotient* *with weight* $k = \frac{1}{2}\sum_{\delta|N} r_\delta$ *and the additional properties that*

$$\sum_{\delta|N} \delta r_\delta \equiv 0 \pmod{24}$$

and

$$\sum_{\delta|N} \frac{N}{\delta} r_\delta \equiv 0 \pmod{24},$$

then $f(z)$ *satisfies*

$$f\left(\frac{az+b}{cz+d}\right) = \chi(d)(cz+d)^k f(z)$$

for every $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N)$. *Here* $\chi(d) := \left(\frac{(-1)^k s}{d}\right)$ *where* $s := \prod_{\delta|N} \delta^{r_\delta}$.

In order to study congruence properties, we turn to a result from Serre on the action of the Hecke operator on integral weight modular forms.

Proposition 2.1.3 (Serre, [55])**.** *Suppose that* $f(z) = \sum_{n=1}^{\infty} a(n)q^n \in S_k(\Gamma_0(N), \chi)$ *has coefficients in* $\mathcal{O}_K$*, the ring of algebraic integers in the number field* K*, and* M *is a positive integer. Furthermore, suppose* $k > 1$*. Then a positive proportion of the primes* $p \equiv -1 \pmod{MN}$ *have the property that*

$$f(z)|T(p, k, \chi) \equiv 0 \pmod{M}.$$

There is an analogous proposition for half-integral weight modular forms due to Ono which is proved using Proposition 2.1.3 and Shimura's correspondence between half-integral weight modular forms and even integer weight modular forms.

Proposition 2.1.4 (Ono, [44, p. 56]). *Suppose that* $g(z) = \sum_{m=1}^{\infty} b(m)q^m \in S_{\lambda+\frac{1}{2}}(\Gamma_0(4N), \chi)$ *has coefficients in* $\mathcal{O}_K$*, the ring of algebraic integers in the number field* K*, and* M *is a positive integer. Furthermore, suppose* $\lambda > 1$*. Then a positive proportion of the primes* $\ell \equiv -1 \pmod{4MN}$ *have the property that*

$$g(z)|T(\ell^2, \lambda, \chi) \equiv 0 \pmod{M}.$$

It is natural to ask for a generalization of Propositions 2.1.3 and 2.1.4 where a Hecke operator for a prime p could simultaneously annihilate a finite set of modular forms. In order to tackle this problem we will now turn our attention to modular Galois representations. Let $\overline{\mathbb{Q}}$ be an algebraic closure of $\mathbb{Q}$, and for each rational prime ℓ, let $\overline{\mathbb{Q}}_\ell$ be an algebraic closure of $\mathbb{Q}_\ell$. Fix an embedding of $\overline{\mathbb{Q}}$ into $\overline{\mathbb{Q}}_\ell$. This fixes a choice of decomposition group $D_\ell = \{\sigma \in \mathrm{Gal(K/\mathbb{Q})} : œ(\mathfrak{p}_{\ell,\mathrm{K}}) = \mathfrak{p}_{\ell,\mathrm{K}}\}$. Specifically, if K is any finite extension of $\mathbb{Q}$ and $\mathcal{O}_K$ is the ring of integers of K, then for each ℓ this fixes a choice of prime ideal $\mathfrak{p}_{\ell,K}$ of $\mathcal{O}_K$ dividing ℓ. Let $\mathbb{F}_{\ell,K} = \mathcal{O}_K/\mathfrak{p}_{\ell,K}$ be the residue field of $\mathfrak{p}_{\ell,K}$ and let $|\cdot|_\ell$ be an extension to $\overline{\mathbb{Q}}_\ell$ of the usual ℓ-adic absolute value on $\mathbb{Q}_\ell$.

Theorem 2.1.5 ([44, p. 42]). *Let* $f(z) = \sum_{n=1}^{\infty} a(n)q^n \in S_k(\Gamma_0(N), \chi)$ *be a newform, and let* K_f *be the field extension obtained by adjoining all of the* $a(n)$ *and values of* χ *to* $\mathbb{Q}$*. If* K *is any finite extension of* $\mathbb{Q}$ *containing* K_f

and ℓ is any prime, then due to work of Eichler, Shimura, Deligne, and Serre there is a continuous semisimple representation

$$\rho_{f,\ell} : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{GL}_2(\mathbb{F}_{\ell,\mathrm{K}})$$

for which the following are true:

1. *$\rho_{f,\ell}$ is unramified at all primes $p \nmid N\ell$.*

2. $\mathrm{Tr}(\rho_{f,\ell}(\mathrm{Frob}_p)) \equiv a(p) \pmod{\mathfrak{p}_{\ell,K}}$ *for all primes $p \nmid N\ell$.*

3. $\det(\rho_{f,\ell}(\mathrm{Frob}_p)) \equiv \chi(p)p^{k-1} \pmod{\mathfrak{p}_{\ell,K}}$ *for all primes $p \nmid N\ell$.*

4. $\det(\rho_{f,\ell}(c)) = -1$ *for any complex conjugation c.*

Remark. Let $D_{f,\ell} = G_f \cap D_\ell$ where G_f is the subgroup of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ stabilizing f, and let

$$\mathbb{F}_{f,\ell} = \mathbb{F}_{\ell,K}^{D_{f,\ell}} = \{a \in \mathbb{F}_{\ell,K} : \sigma(a) = a \quad \forall \sigma \in D_{f,\ell}\}.$$

If f does not have complex multiplication and ℓ is sufficiently large, then the image of $\rho_{f,\ell}$ contains a normal subgroup H_f conjugate to $SL_2(\mathbb{F}_{f,\ell})$. Essentially, this means that the image of $\rho_{f,\ell}$ is almost always 'as large as possible'.

Newforms are eigenforms for the Hecke operator $T(p,k,\chi)$ with eigenvalues given by the pth coefficients of the newform. The fact that the image of $\rho_{f,\ell}$ is large, along with an application of the Chebotarev Density Theorem, tells us we can choose the image of Frob_p to have a trace of zero a positive proportion of the time. This determines the pth coefficient and thus implies

Proposition 2.1.3 of Serre. The following lemma from [46] extends the idea of these representations having large image and allows us to apply it to sums of modular forms.

Lemma 2.1.6 (Ono-Skinner, [46]). *Let $f_1, f_2, ..., f_v$ be newforms without complex multiplication, and let $f_i(z) = \sum_{n=1}^{\infty} a_i(n)q^n$. Write $\rho_{f_i,\ell} = \rho_i$ and $\mathbb{F}_{f_i,\ell} = \mathbb{F}_i$. Then*

1. *the image of $\rho_1 \times \cdots \times \rho_v$ is conjugate to $SL_2(\mathbb{F}_1) \times \cdots \times SL_2(\mathbb{F}_v)$.*

2. *For each positive integer d and each $w \in \mathbb{F}_i$, a positive density of primes $p \equiv 1 \pmod{d}$ satisfies*

$$a_i(p) \equiv w \pmod{\mathfrak{p}_{\ell,K}}.$$

3. *For each pair of coprime positive integers r, d, a positive density of primes $p \equiv r \pmod{d}$ satisfies $|a_i(p)|_\ell = 1$.*

Part (1) of Lemma 2.1.6 specifically tells us that, with small adjustments, we can apply Propositions 2.1.3 and 2.1.4 to a finite set of modular forms simultaneously. This fact is crucial for the proof of Theorem 1.2.1.

A large portion of the proofs of Theorems 1.2.1 and 1.2.2 will apply work of Treneer in [59] to $C_{W'_M,S'_*}(q)$. The main result from [59] follows.

Proposition 2.1.7 (Treneer, [59]). *Suppose p is an odd prime, and that k and r are integers with k odd. Let N be a positive integer with $4 \mid N$ and $(N,p) = 1$, and let χ be a Dirichlet character modulo N. Let K be an algebraic number* *field with ring of integers* $\mathcal{O}_K$*, and suppose* $f(z) = \sum a(n)q^n \in M^!_{\frac{k}{2}}(\Gamma_0(N), \chi) \cap \mathcal{O}_K((q))$*. If* r *is sufficiently large, then for each positive integer* j*, a positive proportion of primes* $Q \equiv -1 \pmod{Np^j}$ *have the property that*

$$a(Q^3 p^r n) \equiv 0 \pmod{p^j}$$

for all n *coprime to* Qp*.*

Remark. A similar statement is true for integer weight forms which will be apparent in the proof. This should not be surprising due to the fact that Serre has shown that almost all coefficients of integer weight forms are $0 \pmod{m}$ for any integer $m > 1$.

2.1.3 Proof of Theorem 1.2.1

In order to find congruences for sums of mixed weight modular forms, we must examine where their coefficients overlap. The following lemma will describe this.

Lemma 2.1.8. *Let* $f_i(z) = \sum_{n=0}^{\infty} a_i(n)q^n \in M_{k_i}(\Gamma_0(N_i), \chi_i)$ *and* $g_j(z) = \sum_{m=0}^{\infty} b_j(m)q^m \in M_{\lambda_j+\frac{1}{2}}(\Gamma_0(4M_j), \chi_j)$*. If*

$$f_i(z)|T_{p_i} \equiv 0 \pmod{Q}$$

and

$$g_j(z)|T(\ell_j^2) \equiv 0 \pmod{Q}$$

with all p_i *and* ℓ_j *distinct, then*

$$a_1\left(\prod_i p_i^{2r_i+1}\prod_j \ell_j^{4s_j+3}n\right)\equiv\cdots\equiv a_u\left(\prod_i p_i^{2r_i+1}\prod_j \ell_j^{4s_j+3}n\right)$$
$$\equiv b_1\left(\prod_i p_i^{2r_i+1}\prod_j \ell_j^{4s_j+3}n\right)\equiv\cdots\equiv b_v\left(\prod_i p_i^{2r_i+1}\prod_j \ell_j^{4s_j+3}n\right)\equiv 0 \pmod{Q},$$

where $gcd(p_1\cdots p_u\ell_1\cdots\ell_v, n)=1$ *and* r_i *and* s_j *are nonnegative integers.*

Remark. If $p_i=\ell_j$ for some i and j, then remove the p_i for the congruence to hold. This is made clear in the following corollary and in the proof of Lemma 2.1.8.

Corollary. *Let* $f_i(z)$ *and* $g_j(z)$ *be given as above. If* $f_i(z)|T_p\equiv 0 \pmod{Q}$ *and* $g_j(z)|T(p^2)\equiv 0 \pmod{Q}$ *for the same prime* p*, then*

$$a_1\left(p^{4t+3}n\right)\equiv\cdots\equiv a_u\left(p^{4t+3}n\right)\equiv b_1\left(p^{4t+3}n\right)\equiv\cdots\equiv b_v\left(p^{4t+3}n\right)\equiv 0 \pmod{Q},$$

where $gcd(p,n)=1$ *and* t *is a nonnegative integer.*

Proof of Lemma 2.1.8. Assume that $f(z)|T(p)\equiv 0 \pmod{Q}$ and $g(z)|T(\ell^2)\equiv 0 \pmod{Q}$. Because $f(z)|T(p,k,\chi)\equiv 0 \pmod{Q}$, whenever $p\nmid n$ we have $a(pn)\equiv 0 \pmod{Q}$. If $p\mid n$ then we can replace n with p^2r with r and p coprime in $f(z)|T(p,k,\chi)=\sum_{n=0}^{\infty}(a(pn)+\chi(p)p^{k-1}a(n/p))q^n$ to arrive at

$$\sum_{r=0}^{\infty}(a(p^3r)+\chi(p)p^{k-1}a(pr))q^{p^2r}\equiv 0 \pmod{Q}.$$

Since we know $a(pr)\equiv 0 \pmod{Q}$ in this case then $a(p^3r)\equiv 0 \pmod{Q}$

must be true for p and r coprime. This process can be repeated to show $a(p^{2t+1}n) \equiv 0 \pmod{Q}$ for p and n coprime.

Using the same idea, because

$$g(z)|T(\ell^2, \lambda, \chi) = \sum_{m=0}^{\infty} (b(\ell^2 m) + \chi^*(\ell) \left(\frac{m}{\ell}\right) \ell^{\lambda-1} b(m) + \chi^*(\ell^2)\ell^{2\lambda-1} b(m/\ell^2))q^m \equiv 0 \pmod{Q},$$

then whenever $\ell \mid m$ and $\ell^2 \nmid m$ we have $b(\ell^2 m) \equiv 0 \pmod{Q}$. Replacing m with ℓs where $(\ell, s) = 1$ shows that $b(\ell^3 s) \equiv 0 \pmod{Q}$. If $\ell^2 \mid m$, then we can replace m with $\ell^5 s$ with ℓ and s coprime to get

$$\sum_{s=0}^{\infty} (b(\ell^7 s) + \chi^*(\ell^2)\ell^{2\lambda-1} b(\ell^3 s))q^{\ell^5 s} \equiv 0 \pmod{Q}.$$

We know that $b(\ell^3 s) \equiv 0 \pmod{Q}$ so $b(\ell^7 s) \equiv 0 \pmod{Q}$ must be true for ℓ and s coprime. This process can be repeated to show $b(\ell^{4t+3} m) \equiv 0 \pmod{Q}$ for ℓ and m coprime. These two observations combined lead to Lemma 2.1.8.

□

Theorem 1.2.1 partly follows from being able to use the above corollary for any set of modular forms. The congruence in Proposition 2.1.7 comes from Treneer being able to show that any weakly holomorphic modular form has coefficients that are congruent $\pmod{p^j}$ to the coefficients of a cusp form. These details are worked out below for the conjugacy growth series in Section 2.1.1. Proposition 2.1.7 then follows by applying Proposition 2.1.3 or 2.1.4 to that cusp form. In order to apply the above corollary, we need to be able to use Propositions 2.1.3 and 2.1.4 on a finite set of modular forms. Lemma 2.1.6 tells us that the images of the Galois representations of a finite set of modular forms will almost always be simultaneously 'as large as possible'. Due to this fact, we can let $g \in Im(\rho_1 \times \cdots \times \rho_v)$ be conjugate to

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \times \cdots \times \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

and, by the Chebotarev Density Theorem, a positive proportion of primes $Q \equiv -1 \pmod{Np^j}$ satisfy $(\rho_1 \times \cdots \times \rho_v)(\mathrm{Frob_Q}) = \mathrm{g}$. This allows us to apply Propositions 2.1.3 and 2.1.4 to a finite set of modular forms for the same prime. Theorem 1.2.1 then follows from being able to use techniques from [59] for a finite set of weakly holomorphic modular forms to show they are all congruent to cusp forms. We can then use Propositions 2.1.3 and 2.1.4 on that set of forms to show they are simultaneously annihilated by Hecke operators, so we can apply the above corollary.

2.1.4 Proof of Theorem 1.2.2

In this section we will explicitly work out the congruence properties of the conjugacy growth series from Section 2.1.1 following the work of Treneer in [59].

Recall that

$$\sum_{n=0}^{\infty} \gamma_{W'_M, S'_*}(n) q^n = \left(\frac{1}{2} \frac{q^{1/12}}{\eta(z)^2} + \frac{1}{2} \frac{q^{1/12}}{\eta(2z)} \right)^M .$$

Define

$$F_M(z) := \sum_{n=-M}^{\infty} b_M(n)q^n = \left(\frac{1}{2}\frac{1}{\eta(12z)^2} + \frac{1}{2}\frac{1}{\eta(24z)}\right)^M$$
$$= \frac{1}{2^M}\sum_{k=0}^{M}\binom{M}{k}\frac{1}{\eta(12z)^{2(M-k)}}\frac{1}{\eta(24z)^k}$$
$$= \frac{1}{2^M}\sum_{k=0}^{M}\binom{M}{k}F_{M,k}(z),$$

where $F_{M,k}(z) = \sum_{n=-M}^{\infty} a_{M,k}(n)q^n \in M_{\frac{k}{2}-M}(\Gamma_0(N_{M,k}))$.

Lemma 2.1.9. *Suppose p is an odd prime and r and $N_{M,k}$ are integers with $(N_{M,k}, p) = 1$. If r is sufficiently large, then for every positive integer j there exists an integer $\beta \geq j-1$ and a cusp form*

$$F_{M,k,p,j}(z) \in S_{\frac{k}{2}-M+\frac{p^\beta(p^2-1)}{2}}(\Gamma_0(N_{M,k}p^2))$$

such that

$$F_{M,k,p,j}(z) \equiv \sum_{p\nmid n} a_{M,k}(p^r n)q^n (\text{mod } p^j).$$

Proof of Lemma 2.1.9. This proof will follow the proof of Proposition 3.1 in [59]. The plan for this proof is to divide the cusps of $\Gamma_0(N_{M,k}p^2)$ into two groups. We will pick r large enough so that $F_{M,k}(z)|U_{p^r} = \sum a_{M,k}(p^r n)q^n$ is holomorphic at each cusp $\frac{a}{c}$ with $p^2 \mid c$. Then we will define

$$F_{M,k,r}(z) = F_{M,k}(z)|U_{p^r} - F_{M,k}(z)|U_{p^{r+1}}|V_p = \sum_{p\nmid n} a_{M,k}(p^r n)q^n$$

such that it vanishes at these cusps. Define the eta-quotients

$$F_p(z) = \begin{cases} \frac{\eta^{p^2}(z)}{\eta(p^2 z)} \in M_{\frac{p^2-1}{2}}(\Gamma_0(p)) & p \geq 5 \\ \frac{\eta^{27}(z)}{\eta^3(9z)} \in M_{12}(\Gamma_0(9)) & p = 3. \end{cases}$$

$F_p(z)$ vanishes at every cusp $\frac{a}{c}$ where $p^2 \nmid c$ and is $1 \pmod p$. By induction it also clear that $F_p(z)^{p^{s-1}} \equiv 1 \pmod{p^s}$ for any integer s. Our cusp form will end up being $F_{M,k,r}(z) \cdot F_p(z)^{p^\beta}$ for some integer β. First we must find an explicit description of the Fourier expansion of $F_{M,k}(z)|U_{p^r}$ at a cusp $\frac{a}{c}$ with $p^2 \nmid c$. Note that for the remainder of this section the slash operator will be used on forms that may have half-integral weights. We have not explicitly defined the half-integral weight slash operator, but it can be defined analogously to the integer weight version. More information can be found in [44].

Proposition 2.1.10. *Let* $\gamma = \begin{pmatrix} a & b \\ cp^2 & d \end{pmatrix} \in SL_2(\mathbb{Z})$ *with* $ac > 0$. *Then there exists an integer* $n_0 \geq -24M$ *and a sequence* $\{a_{M,k,0}(n)\}_{n \geq n_0}$ *such that for each* $r \geq 1$,

$$(F_{M,k}(z)|U_{p^r})|_{\frac{k}{2}-M}\gamma = \sum_{\substack{n=n_0 \\ n \equiv 0 \pmod{p^r}}}^{\infty} a_{M,k,0}(n) q_{24p^r}^n$$

where $q_{24p^r}^n = e^{\frac{2\pi i n z}{24 p^r}}$.

Proof of Proposition 2.1.10. First we note that for any matrix $A \in SL_2(\mathbb{Z})$, $F_{M,k}(z)|_{\frac{k}{2}-M} A = \sum_{n=n_0}^{\infty} a_{M,k,0}(n) q_{24}^n$ where $n_0 \geq -24M$. This can be seen by following Theorem 1 in [31].

$$\prod_{i=i}^{j}\left(\eta(t_i z)|_{\kappa/2}\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right)^{n_i}$$

transforms to

$$C\prod_{i=i}^{j}\left(\eta(z)|_{\kappa/2}\begin{pmatrix} \alpha_i & \beta_i \\ 0 & \delta_i \end{pmatrix}\right)^{n_i}$$

where $\begin{pmatrix} t_i & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a_i & b_i \\ c_i & d_i \end{pmatrix}\begin{pmatrix} \alpha_i & \beta_i \\ 0 & \delta_i \end{pmatrix}$ and C is a constant. We can see from this that $t_i a = \alpha_i a_i$ and $c = \alpha_i c_i$, so $\alpha_i = (t_i, c) \leq t_i$. Taking $t_1 = 12, n_1 = -2(M-k), t_2 = 24$, and $n_2 = -k$ we arrive at the conclusion that $F_{M,k}(z)|_{\frac{k}{2}-M}A = \sum_{n=n_0}^{\infty} a_{M,k,0}(n)q_{24}^n$ where $n_0 \geq -24M$. If we define $\sigma_{v,t} := \begin{pmatrix} 1 & v \\ 0 & t \end{pmatrix}$, then notice that

$$F_{M,k}(z)|U_t = t^{\frac{k-2M}{4}-1}\sum_{v=0}^{t-1} F_{M,k}(z)|_{\frac{k}{2}-M}\sigma_{v,t}.$$

For each $0 \leq v \leq p^r - 1$, choose an integer $s_v \equiv 0 \pmod 4$ such that

$$s_v N_{M,k} \equiv (a + vcp^2)^{-1}(b + vd) \pmod{p^r}$$

and set $w_v = s_v N_{M,k}$. Note that w_v runs through the residue classes of p^r as

v does. Also define

$$\alpha_v := \begin{pmatrix} a + vcp^2 & \frac{b+vd-aw_v-w_vvcp^2}{p^r} \\ cp^{r+2} & d - w_vcp^2 \end{pmatrix},$$

so that $\sigma_{v,p^r}\gamma = \alpha_v\sigma_{w_v,p^r}$. Putting this all together, we have

$$\begin{aligned}(F_{M,k}(z)|U_{p^r})\,|_{\frac{k}{2}-M}\gamma &= (p^r)^{\frac{k-2M}{4}-1}\sum_{v=0}^{p^r-1} F_{M,k}(z)|_{\frac{k}{2}-M}\sigma_{v,p^r}\gamma \\ &= (p^r)^{\frac{k-2M}{4}-1}\sum_{v=0}^{p^r-1} F_{M,k}(z)|_{\frac{k}{2}-M}\alpha_v\sigma_{w_v,p^r}.\end{aligned}$$

In Lemma 3.4 of [59] Treneer shows that $\alpha_v\alpha_0^{-1} \in \Gamma_1(N_{M,k})$. Beacuse $F_{M,k}(z)$ is invariant under action by $\Gamma_1(N_{M,k})$, we now have

$$\begin{aligned}(F_{M,k}(z)|U_{p^r})\,|_{\frac{k}{2}-M}\gamma &= (p^r)^{\frac{k-2M}{4}-1}\sum_{v=0}^{p^r-1} F_{M,k}(z)|_{\frac{k}{2}-M}\alpha_v\sigma_{w_v,p^r} \\ &= (p^r)^{\frac{k-2M}{4}-1}\sum_{v=0}^{p^r-1} F_{M,k}(z)|_{\frac{k}{2}-M}(\alpha_v\alpha_0^{-1})^{-1}\alpha_v\sigma_{w_v,p^r} \\ &= (p^r)^{\frac{k-2M}{4}-1}\sum_{v=0}^{p^r-1} F_{M,k}(z)|_{\frac{k}{2}-M}\alpha_0\sigma_{w_v,p^r}.\end{aligned}$$

Since $\alpha_0 \in SL_2(\mathbb{Z})$, we have

$$\begin{aligned}\sum_{v=0}^{p^r-1} F_{M,k}(z)|_{\frac{k}{2}-M}\alpha_0\sigma_{w_v,p^r} &= \sum_{v=0}^{p^r-1}\left(\sum_{n=n_0}^{\infty} a_{M,k,0}(n)q_{24}^n\right)|_{\frac{k}{2}-M}\sigma_{w_v,p^r} \\ &= \sum_{v=0}^{p^r-1} p^{\frac{-r(k-2M)}{4}}\sum_{n=n_0}^{\infty} a_{M,k,0}(n)e^{\frac{2\pi in(z+w_v)}{24p^r}}\end{aligned}$$

$$= p^{\frac{-r(k-2M)}{4}} \sum_{n=n_0}^{\infty} a_{M,k,0}(n) q_{24p^r}^n \sum_{v=0}^{p^r-1} e^{\frac{2\pi i n w_v}{24p^r}}.$$

The numbers $\frac{w_v}{24}$ run through the residue classes of p^r as v does, therefore

$$\sum_{v=0}^{p^r-1} e^{\frac{2\pi i n w_v}{24p^r}} = \sum_{v=0}^{p^r-1} e^{\frac{2\pi i n v}{p^r}} = \begin{cases} p^r & \text{if } n \equiv 0 \pmod{p^r} \\ 0 & \text{otherwise,} \end{cases}$$

which gives us

$$\sum_{v=0}^{p^r-1} F_{M,k}(z)|_{\frac{k}{2}-M} \alpha_0 \sigma_{w_v,p^r} = p^{r(1-\frac{k-2m}{4})} \sum_{\substack{n=n_0 \\ n\equiv 0 \pmod{p^r}}}^{\infty} a_{M,k,0}(n) q_{24p^r}^n.$$

Therefore,

$$(F_{M,k}(z)|U_{p^r})|_{\frac{k}{2}-M} \gamma = \sum_{\substack{n=n_0 \\ n\equiv 0 \pmod{p^r}}}^{\infty} a_{M,k,0}(n) q_{24p^r}^n.$$

□

Proposition 2.1.11. *Define*

$$F_{M,k,r}(z) := F_{M,k}(z)|U_{p^r} - F_{M,k}(z)|U_{p^{r+1}}|V_p \in M_{\frac{k}{2}-M}(\Gamma_0(N_{M,k}p^2)).$$

Then for r sufficiently large, $F_{M,k,r}(z)$ vanishes at each cusp $\frac{a}{cp^2}$ of $(\Gamma_0(N_{M,k}p^2)$ with $ac > 0$.

Proof of Proposition 2.1.11. From Proposition 2.1.10, we know

$$(F_{M,k}(z)|U_{p^r})|_{\frac{k}{2}-M}\gamma = \sum_{\substack{n=n_0 \\ n\equiv 0 \pmod{p^r}}}^{\infty} a_{M,k,0}(n)q_{24p^r}^n$$

where $n_0 \geq -24M$. For r sufficiently large, $-p^r < -24M \leq n_0$. In the Fourier expansion if $a_{M,k,0}(n) \neq 0$, in order for $n \equiv 0 \pmod{p^r}$ to be true, $n \geq 0$ must be true. Therefore, we have

$$(F_{M,k}(z)|U_{p^r})|_{\frac{k}{2}-M}\gamma = \sum_{\substack{n\geq 0 \\ n\equiv 0 \pmod{p^r}}} a_{M,k,0}(n)q_{24p^r}^n,$$

which shows $F_{M,k}(z)|U_{p^r}$ is holomorphic at the cusp $\frac{a}{cp^2}$. We will handle the second term in a similar way as the first term in the proof of Proposition 2.1.10. Define $\tau_{v,t} = \begin{pmatrix} 1 & v/t \\ 0 & 1 \end{pmatrix}$, and note that

$$F_{M,k}(z)|U_t|V_t = t^{-1}\sum_{v=0}^{t-1} F_{M,k}(z)|_{\frac{k}{2}-M}\tau_{v,t}.$$

Using this, we see that

$$(F_{M,k}(z)|U_{p^r})|U_p|V_p|_{\frac{k}{2}-M}\gamma = p^{-1}\sum_{v=0}^{p-1}(F_{M,k}(z)|U_{p^r})|_{\frac{k}{2}-M}\tau_{v,p}\gamma.$$

For each $0 \leq v \leq p-1$, choose $s_v^{'} \equiv 0 \pmod 4$ such that $s_v^{'}N_{M,k} \equiv a^{-1}vd$

(mod p), and set $w_v^{'} = s_v^{'} N_{M,k}$. Define

$$\delta_v := \begin{pmatrix} 1 + aw_v^{'}cp + vw_v^{'}c^2p^2 & \frac{avd - a^2w_v^{'}}{p} - acvw_v^{'} - bvcp \\ w_v^{'}c^2p^3 & 1 - aw_v^{'}cp \end{pmatrix},$$

so that $\tau_{v,p}\gamma = \delta_v \gamma \tau_{w_v^{'},p}$. In [59] Treneer also shows that $\delta_v \in \Gamma_1(N_{M,k}p)$, so

$$(F_{M,k}(z)|U_{p^r})|_{\frac{k}{2}-M}\tau_{v,p}\gamma = (F_{M,k}(z)|U_{p^r})|_{\frac{k}{2}-M}\delta_v\gamma\tau_{w_v^{'},p} = (F_{M,k}(z)|U_{p^r})|_{\frac{k}{2}-M}\gamma\tau_{w_v^{'},p}.$$

Following the same method as in the proof of Proposition 2.1.10, we can write

$$\begin{aligned}
(F_{M,k}(z)|U_{p^r})|U_p|V_p|_{\frac{k}{2}-M}\gamma &= p^{-1}\sum_{v=0}^{p-1}(F_{M,k}(z)|U_{p^r})|_{\frac{k}{2}-M}\gamma\tau_{w_v^{'},p} \\
&= p^{-1}\sum_{v=0}^{p-1}\left(\sum_{\substack{n\geq 0 \\ n\equiv 0 \pmod{p^r}}} a_{M,k,0}(n)q_{24p^r}^n\right)|_{\frac{k}{2}-M}\tau_{w_v^{'},p} \\
&= p^{-1}\sum_{v=0}^{p-1}\sum_{\substack{n\geq 0 \\ n\equiv 0 \pmod{p^r}}} a_{M,k,0}(n)\exp\left(\frac{2\pi i n(z+\frac{w_v^{'}}{p})}{24p^r}\right) \\
&= \sum_{\substack{n\geq 0 \\ n\equiv 0 \pmod{p^r}}} a_{M,k,0}(n)q_{24p^r}^n\sum_{v=0}^{p-1}\exp\left(\frac{2\pi i n w_v^{'}}{24p^{r+1}}\right) \\
&= \sum_{\substack{n\geq 0 \\ n\equiv 0 \pmod{p^r}}} a_{M,k,0}(n)q_{24p^r}^n\sum_{v=0}^{p-1}\exp\left(\frac{2\pi i w_v^{'}}{24p}\left(\frac{n}{p^r}\right)\right).
\end{aligned}$$

The numbers $\frac{w_v'}{24}$ run through the residue classes modulo p as v does, so

$$\sum_{v=0}^{p-1} \exp\left(\frac{2\pi i w_v'}{24p}\left(\frac{n}{p^r}\right)\right) = \sum_{v=0}^{p-1} \exp\left(\frac{2\pi i v}{p}\left(\frac{n}{p^r}\right)\right) = \begin{cases} p & \text{if } p \mid \frac{n}{p^r} \\ 0 & \text{otherwise.} \end{cases}$$

Putting everything together gives us

$$(F_{M,k}(z)|U_{p^r})|U_p|V_p|_{\frac{k}{2}-M}\gamma = \sum_{\substack{n\geq 0 \\ n\equiv 0 \pmod{p^{r+1}}}} a_{M,k,0}(n)q_{24p^r}^n.$$

To finish the proof of Proposition 2.1.11, we have

$$F_{M,k,r}(z)|_{\frac{k}{2}-M}\gamma = \sum_{\substack{n\geq 0 \\ n\equiv 0 \pmod{p^r}}} a_{M,k,0}(n)q_{24p^r}^n - \sum_{\substack{n\geq 0 \\ n\equiv 0 \pmod{p^{r+1}}}} a_{M,k,0}(n)q_{24p^r}^n.$$

The constant terms (which may be 0) of each expansion cancel, so $F_{M,k,r}(z)$ vanishes at the cusp $\frac{a}{cp^2}$. □

Before discussing the cusp $\frac{a}{c}$ where $p^2 \nmid c$, notice that

$$F_{M,k,r}(z) = \sum_{n=1}^{\infty} a_{M,k}(p^r n)q^n - \sum_{n=1}^{\infty} a_{M,k}(p^{r+1}n)q^{pn} = \sum_{\substack{n=1 \\ p\nmid n}}^{\infty} a_{M,k}(p^r n)q^n.$$

Recall the eta-quotients

$$F_p(z) = \begin{cases} \frac{\eta^{p^2}(z)}{\eta(p^2 z)} \in M_{\frac{p^2-1}{2}}(\Gamma_0(p)) & p \geq 5 \\ \frac{\eta^{27}(z)}{\eta^3(9z)} \in M_{12}(\Gamma_0(9)) & p = 3, \end{cases}$$

and recall that $F_p(z)$ vanishes at every cusp $\frac{a}{c}$ where $p^2 \nmid c$. The forms $F_p(z)$ are $1 \pmod{p}$, and by induction it is easy to show $F_p(z)^{p^{s-1}} \equiv 1 \pmod{p^s}$ for any integer s. Let r be sufficiently large, and fix j. If $\beta \geq j-1$ is sufficiently large, then

$$F_{M,k,p,j}(z) := F_{M,k,r}(z) \cdot F_p(z)^{p^\beta} \equiv F_{M,k,r}(z) \pmod{p^j}$$

vanishes at all cusps $\frac{a}{c}$ of $\Gamma_0(N_{M,k}p^2)$ where $p^2 \nmid c$. By Proposition 2.1.11, $F_{M,k,r,p,j}(z)$ also vanishes at the cusps $\frac{a}{c}$ where $p^2 \mid c$, so

$$F_{M,k,p,j}(z) \in S_{\frac{k}{2}-M+\frac{p^\beta(p^2-1)}{2}}(\Gamma_0(N_{M,k}p^2)).$$

As seen above,

$$F_{M,k,p,j}(z) \equiv F_{M,k,r}(z) \equiv \sum_{\substack{n=1 \\ p\nmid n}}^{\infty} a_{M,k}(p^r n)q^n \pmod{p^j}.$$

This completes the proof of Lemma 2.1.9. □

Lemma 2.1.12. *1. If k is even, then $F_{M,k,p,j}(z)$ is an integer weight cusp form, so for a positive proportion of primes $Q \equiv -1 \pmod{N_{M,k}p^j}$, we have*

$$a_{M,k}(Q^{2t+1}p^r n) \equiv 0 \pmod{p^j}$$

for all nonegative integers t, and n coprime to Qp.

2. If k is odd, then $F_{M,k,p,j}(z)$ is a half-integral weight cusp form, so for a *positive proportion of primes* $Q \equiv -1 \pmod{N_{M,k}p^j}$*, we have*

$$a_{M,k}(Q^{4t+3}p^r n) \equiv 0 \pmod{p^j}$$

for all nonegative integers t*, and* n *coprime to* Qp*.*

Proof of Lemma 2.1.12. If k is even (resp. odd), then $F_{M,k,p,j}(z)$ is an integral (resp. half-integral) weight cusp form. Thus, by Proposition 2.1.3 (resp. Proposition 2.1.4), for a positive proportion of primes $Q \equiv -1 \pmod{N_{M,k}p^j}$, we have $F_{M,k,p,j}(z)|T_Q \equiv 0 \pmod{p^j}$ (resp. $F_{M,k,p,j}(z)|T(Q^2) \equiv 0 \pmod{p^j}$). If we let $F_{M,k,p,j}(z) = \sum_{n=1}^{\infty} c_{M,k}(n)q^n$, then by Lemma 2.1.8 we have $c_{M,k}(Q^{2t+1}n) \equiv 0 \pmod{p^j}$ (resp. $c_{M,k}(Q^{4t+3}n) \equiv 0 \pmod{p^j}$) for any nonnegative integer t and Q and n coprime. The rest of the proof follows from the fact that $c_{M,k}(n) \equiv a_{M,k}(p^r n) \pmod{p^j}$. □

We will now refer back to part (1) of Lemma 2.1.6. Using this and Chebotarev's Density Theorem, we are able to apply Proposition 2.1.3 or 2.1.4 simultaneously to each term in a sum of modular forms, which in turn allows us to apply Lemma 2.1.8 to our entire sum at the same time instead of piece by piece as in Lemma 2.1.11. As in Theorem 1.2.1, if we have a sum of modular forms f_i of mixed weights and level N_i, we can replace the level N in Proposition 2.1.3 or 2.1.4 with the smallest N' such that each N_i divides N'.

We will now complete the proof of Theorem 1.2.2. Recall that

$$\sum_{n=0}^{\infty} \gamma_{W'_M, S'_*}(n) q^n = \left(\frac{1}{2} \frac{q^{1/12}}{\eta(z)^2} + \frac{1}{2} \frac{q^{1/12}}{\eta(2z)} \right)^M$$

and

$$F_M(z) := \sum_{n=-M}^{\infty} b_M(n)q^n = \left(\frac{1}{2}\frac{1}{\eta(12z)^2} + \frac{1}{2}\frac{1}{\eta(24z)}\right)^M$$
$$= \frac{1}{2^M}\sum_{k=0}^{M}\binom{M}{k}\frac{1}{\eta(12z)^{2(M-k)}}\frac{1}{\eta(24z)^k}$$
$$= \frac{1}{2^M}\sum_{k=0}^{M}\binom{M}{k}F_{M,k}(z),$$

so $F_{M,k}(z) = \sum_{n=-M}^{\infty} a_{M,k}(n)q^n \in M_{\frac{k}{2}-M}(\Gamma_0(N_{M,k}))$. Note also that

$$\sum_{n=-M}^{\infty} b_M(n)q^n = \sum_{k=0}^{M}\sum_{n=-M}^{\infty}\binom{M}{k}a_{M,k}(n)q^n.$$

In this sum there will be a form of level 576 and all of the other forms will have level dividing 576. Clearly, the sum will be a mix of integer weight and half-integral weight modular forms. From Lemma 2.1.11 we know that for a positive proportion of primes $Q \equiv -1 \pmod{N_{M,k}p^j}$, we have

$$a_{M,k}(Q^{2t+1}p^r n) \equiv 0 \pmod{p^j}$$

for k even and

$$a_{M,k}(Q^{4t+3}p^r n) \equiv 0 \pmod{p^j}$$

for k odd, for all nonegative integers t, and n coprime to Qp. Theorem 2.1.5 and Lemma 2.1.6 together imply that Lemma 2.1.11 can be applied to each $F_{M,k}(z)$ simultaneously for a positive proportion of primes $Q \equiv -1 \pmod{576p^j}$, so

$a_{M,k}(Q^{4t+3}p^r n) \equiv 0 \pmod{p^j}$ for each $a_{M,k}(n)$. Since the congruence holds for each part of the sum, we also have

$$b_M(Q^{4t+3}p^r n) \equiv 0 \pmod{p^j}$$

for a positive proportion of primes $Q \equiv -1 \pmod{576p^j}$. Because $b_M(n) = \gamma_{W'_M,S'_*}\left(\frac{n+M}{12}\right)$, we have

$$\gamma_{W'_M,S'_*}\left(\frac{Q^{4t+3}p^r n + M}{12}\right) \equiv 0 \pmod{p^j}.$$

2.1.5 An example

We give the following example to demonstrate Theorem 1.2.2.

Example. We find that

$$\gamma_{W'_2,S'_*}\left(\frac{7n+2}{12}\right) \equiv 0 \pmod{7}$$

whenever $n \not\equiv 10 \pmod{24}$. Moreover, the above congruence is true when $n = 24t + 10$ and $t \equiv 2, 4, 5, \text{ or } 6 \pmod{7}$.

We will now give the details of this example. Define

$$\sum_{n=0}^{\infty} a_1(n)q^n = \prod_{n=1}^{\infty} \frac{1}{(1-q^n)^4},$$
$$\sum_{n=0}^{\infty} a_2(n)q^n = \prod_{n=1}^{\infty} \frac{1}{(1-q^n)^2} \prod_{n=1}^{\infty} \frac{1}{(1-q^{2n})^2},$$

and

$$\sum_{n=0}^{\infty} a_3(n)q^n = \prod_{n=1}^{\infty} \frac{1}{(1-q^{2n})^2}.$$

It is clear that

$$\sum_{n=0}^{\infty} \gamma_{W_2',S_*'}(n)q^n = \frac{1}{4}\sum_{n=0}^{\infty} \left(a_1(n)+a_2(n)+a_3(n)\right)q^n.$$

By adapting Theorem 6 from [45], we have

$$\sum a_1\left(\frac{pn+2}{12}\right)q^n \equiv \frac{\Delta^{\frac{p^2-1}{6}}(z)|U_p|V_{12}}{\eta^{4p}(12z)} \pmod{p},$$
$$\sum a_2\left(\frac{pn+2}{12}\right)q^n \equiv \frac{\Delta^{\frac{p^2-1}{12}}(z)\Delta^{\frac{p^2-1}{24}}(2z)|U_p|V_{12}}{\eta^{2p}(12z)\eta^{p}(24z)} \pmod{p},$$

and

$$\sum a_3\left(\frac{pn+2}{12}\right)q^n \equiv \frac{\Delta^{\frac{p^2-1}{12}}(2z)|U_p|V_{12}}{\eta^{2p}(24z)} \pmod{p}.$$

Using a theorem of Sturm in [57], one can verify with a finite computation that

$$\Delta^8(z)|U_7 \equiv 0 \pmod{7},$$
$$\Delta^4(z)\Delta^2(2z)|U_7 \equiv 0 \pmod{7},$$

and

$$\Delta^4(2z)|U_7 \equiv 3\Delta(2z) \pmod{7}.$$

From this it is clear that

$$\sum \gamma_{W_2', S_*'}\left(\frac{7n+2}{12}\right) q^n \equiv \frac{3\Delta(24z)}{\eta^{14}(24z)} \equiv 3\eta^{10}(24z) \pmod 7.$$

In [54], Serre showed that $\eta^{10}(z)$ is lacunary, so one should expect a lot of congruences. In [20], Dawsey and the author use this fact to prove the congruences necessary to complete the example.

2.2 Harmonic Maass form eigencurves

We will prove Theorem 1.2.3 and Theorem 1.2.5 in this section. Section 2.2.2 will be dedicated to the construction of the forms in Theorem 1.2.3 and proving that they are Hecke eigenforms. Section 2.2.3 will be used to discuss the p-adic properties of the forms in Theorem 1.2.5.

2.2.1 Hecke operators for harmonic Maass forms and results of Zagier

In this subsection we will introduce Hecke operators for integer and half-integral weight harmonic Maass forms. We will then present some results of Zagier which are essential to the proofs of Theorem 1.2.3.

Proposition 2.2.1 (Proposition 7.1 of [3])**.** *Suppose that* $f(z) \in H_\kappa^!(\Gamma_0(N), \chi)$ *with* $\kappa \in \frac{1}{2}\mathbb{Z}$*. Then the following are true.*

1. *For* $m \in \mathbb{N}$*, we have that* $f|T(m) \in H_\kappa^!(\Gamma_0(N), \chi)$*.*

2. if $\kappa \in \mathbb{Z}$, $\epsilon \in \{\pm\}$, then, unless $n = 0$ and $\epsilon = -$,

$$c^{\epsilon}_{f|T(p)}(n) = c^{\epsilon}_f(pn) + \chi(p)p^{\kappa-1}c^{\epsilon}_f\left(\frac{n}{p}\right).$$

Moreover,

$$c^{-}_{f|T(p)}(0) = (p^{\kappa-1} + \chi(p))c^{-}_f(0).$$

3. if $\kappa \in \frac{1}{2}\mathbb{Z} \setminus \mathbb{Z}$, then, with $\epsilon \in \{\pm\}$ ($n \neq 0$ for $\epsilon = -$), we have that

$$c^{\epsilon}_{f|T(p^2)}(n) = c^{\epsilon}_f(p^2 n) + \chi^*(p)\left(\frac{n}{p}\right)p^{\kappa-\frac{3}{2}}c^{\epsilon}_f(n) + \chi^*(p^2)p^{2\kappa-2}c^{\epsilon}_f\left(\frac{n}{p^2}\right),$$

where $\chi^*(n) := \left(\frac{(-1)^{\kappa-\frac{1}{2}}}{n}\right)\chi(n)$. If $n = 0$ and $\epsilon = -$, then we have that

$$c^{-}_{f|T(p^2)}(0) = (p^{-2+2\kappa} + \chi^*(p^2))c^{-}_f(0).$$

The ξ-operator allows for a connection between the Hecke operators for harmonic Maass forms and modular forms (see [3]). In particular, we have that

$$p^{d(1-\kappa)}\xi_\kappa(f|T(p^d, \kappa, \chi)) = \xi_\kappa(f)|T(p^d, 2-\kappa, \chi), \tag{2.1}$$

where

$$d := \begin{cases} 1 & \text{if } \kappa \in \mathbb{Z}, \\ 2 & \text{if } \kappa \in \frac{1}{2} + \mathbb{Z}. \end{cases}$$

Several results of Zagier will be applicable to the construction of our forms.

We will state them here.

Proposition 2.2.2 (Zagier, [64]). *For positive integers a and c, let*

$$\lambda(a,c) = \begin{cases} i^{\frac{1-c}{2}} \left(\frac{a}{c}\right) & \text{if } c \text{ is odd, } a \text{ even} \\ i^{\frac{a}{2}} \left(\frac{c}{a}\right) & \text{if } a \text{ is odd, } c \text{ even} \\ 0 & \text{otherwise.} \end{cases}$$

Define the Gauss sum $\gamma_c(n)$ by

$$\gamma_c(n) := \frac{1}{\sqrt{c}} \sum_{a=1}^{2c} \lambda(a,c) e^{-\pi i n \frac{a}{c}}.$$

Let n be a nonzero integer and define a Dirichlet series $E_n(s)$ by

$$E_n(s) := \frac{1}{2} \sum_{\substack{c=1 \\ c \text{ odd}}}^{\infty} \frac{\gamma_c(n)}{c^s} + \frac{1}{2} \sum_{\substack{c=2 \\ c \text{ even}}}^{\infty} \frac{\gamma_c(n)}{(c/2)^s},$$

(i.e. $E_n(s) = \sum a_m m^{-s}$ where $a_m = \frac{1}{2}(\gamma_m(n) + \gamma_{2m}(n))$ when m is odd, and $a_m = \frac{1}{2}\gamma_{2m}(n)$ when m is even). Let $K = \mathbb{Q}(\sqrt{n})$, D be the discriminant of K, $\chi_D = \left(\frac{D}{\cdot}\right)$ be the character of K, and $L(s, \chi_D) = \sum \frac{\chi_D(n)}{n^s}$ be the L-series of K (if n is a perfect square, then $\chi(m) = 1$ for any m and $L(s,\chi) = \zeta(s)$). Then if $n \equiv 2, 3 \pmod 4$, we have

$$E_n(s) = 0.$$

If $n \equiv 0, 1 \pmod 4$, *we have*

$$E_n(s) = \frac{L(s,\chi_D)}{\zeta(2s)} \sum_{\substack{a,c\geq 1 \\ ac|v}} \frac{\mu(a)\chi_D(a)}{c^{2s-1}a^s} = \frac{L(s,\chi_D)}{\zeta(2s)} \frac{T_s^{\chi_D}(v)}{v^{2s-1}},$$

where $n = v^2 D$ *and*

$$T_s^{\chi}(v) = \sum_{t|v} t^{2s-1} \sum_{a|t} \frac{\mu(a)\chi(a)}{a^s} = \sum_{a|v} \mu(a)\chi(a)a^{s-1}\sigma_{2s-1}(v/a).$$

Furthermore, we have

$$E_0(s) = \frac{\zeta(2s-1)}{\zeta(2s)}.$$

Remark. It is clear from Zagier's proof in [64] that $E_n(s)$ can be continued to a meromorphic function on the whole s-plane. It will also be beneficial to note that $T_s^{\chi}(v) = v^{2s-1}T_{1-s}^{\chi}(v)$.

It will be useful for us to define

$$E_n^{odd}(s) := \sum_{\substack{c=1 \\ c \text{ odd}}}^{\infty} \gamma_c(n)c^{-s}, \tag{2.2}$$

and

$$E_n^{even}(s) := \sum_{\substack{c=1 \\ c \text{ even}}}^{\infty} \gamma_c(n)(c/2)^{-s}, \tag{2.3}$$

so that

$$E_n(s) = \frac{1}{2}\left(E_n^{odd}(s) + E_n^{even}(s)\right).$$

2.2.2 Proof of Theorem 1.2.3

Here we prove Theorem 1.2.3. There are two cases to consider, the integer weight and half-integral weight cases. In the next subsection we consider the integer weight case.

Proof of Theorem 1.2.3 Part 1

We will construct the forms from Theorem 1.2.3 part 1 first. Let $z \in \mathbb{H}$ and $k \in \mathbb{Z}$. Define

$$\mathcal{G}(z, -2k, s) := \frac{1}{2} \sum_{n,m}{}' \frac{(mz+n)^{2k}}{|mz+n|^{2s}},$$

where the primed sum means the sum runs over all (n, m) except $(0, 0)$. $\mathcal{G}(z, -2k, s)$ has a meromorphic continuation to the whole s-plane. Let

$$f(z, -2k, s) := \sum_{n=-\infty}^{\infty} (z+n)^{2k} |z+n|^{-2s}.$$

Then we have

$$f(z, -2k, s) = \sum_{n=-\infty}^{\infty} h_{n,2k}(s, y) e^{2\pi i n z} = \sum_{n=-\infty}^{\infty} h_n(y, -2k, s) e^{2\pi i n x} e^{-2\pi n y},$$

where

$$h_n(y, -2k, s) = \int_{iy-\infty}^{iy+\infty} z^{2k} |z|^{-2s} e^{-2\pi i n z} dz.$$

After making the substitution $z = yt + iy$ we have

$$h_n(y, -2k, s) = y^{1+2k-2s} e^{2\pi ny} \int_{-\infty}^{\infty} (t+i)^{2k} (t^2+1)^{-s} e^{-2\pi inyt} dt.$$

For $n = 0$, we have $h_0(y, -2k, s) = y^{1+2k-2s} \int_{-\infty}^{\infty} (t+i)^{2k}(t^2+1)^{-s} dt$. Following Zagier, we choose our branch cut along the negative imaginary axis. Then using contour integration, we find that

$$h_0(y, -2k, s) = 2iy^{1+2k-2s} e^{k\pi i} \sin(\pi(s-2k)) \int_{-i}^{-i\infty} |t+i|^{2k-s} |t-i|^{-s} dt.$$

We substitute t for $-i(2u+1)$ to arrive at

$$h_0(y, -2k, s) = 2^{2+2k-2s} y^{1+2k-2s} e^{k\pi i} \sin(\pi(s-2k)) \int_0^{\infty} u^{2k-s} (u+1)^{-s} du.$$

We make one more substitution, $u = \frac{1-v}{v}$. Then, we have

$$\begin{aligned} h_0(y, -2k, s) &= 2^{2+2k-2s} y^{1+2k-2s} e^{k\pi i} \sin(\pi(s-2k)) \int_0^1 (1-v)^{2k-s} v^{2s-2k-2} dv \\ &= 2^{2+2k-2s} y^{1+2k-2s} e^{k\pi i} \pi \frac{\Gamma(2s-2k-1)}{\Gamma(s-2k)\Gamma(s)}. \end{aligned}$$

For $n > 0$, we define a path c_1 as a clockwise path around $-i$ from $-i\infty$ to $-i\infty$. Then we have

$$h_n(y, -2k, s) = y^{1+2k-2s} \int_{c_1} (v+i)^{2k} (v^2+1)^{-s} e^{-2\pi inyv} dv.$$

Substitute v for $t - i$ and define the path $c_2 = c_1 + i$, then we have

$$h_n(y, -2k, s) = y^{1+2k-2s} e^{-2\pi ny} \int_{c_2} t^{2k-s}(t-2i)^{-s} e^{-2\pi inyt} dt.$$

For $n < 0$, define the path c_3 as before to circle i clockwise from $i\infty$ to $i\infty$. Making the substitutions $v = t + i$ and $c_4 = c_3 - i$, we arrive at

$$h_n(y, -2k, s) = y^{1+2k-2s} e^{2\pi ny} \int_{c_4} t^{-s}(t+2i)^{2k-s} e^{-2\pi inyt} dt,$$

for $n < 0$. Notice that $h_n(my, -2k, s) = m^{1+2k-2s} h_{mn}(y, -2k, s)$, so we have

$$\begin{aligned}
\mathcal{G}(z, -2k, s) &= \frac{1}{2} \sum_{m=-\infty}^{\infty} f(mz, -2k, s) \\
&= \zeta(2s - 2k) + \sum_{m=1}^{\infty} f(mz, -2k, s) \\
&= \zeta(2s - 2k) + \sum_{m=1}^{\infty} \sum_{n=-\infty}^{\infty} m^{1+2k-2s} h_{mn}(y, -2k, s) e^{2\pi inmx}.
\end{aligned}$$

We want to now look at the limit as s goes to zero in order to obtain a negative weight Eisenstein series. However, it is clear that for any $n \in \mathbb{Z}$, $h_n(y, -2k, 0) = 0$. Thus, our $\mathcal{G}(z, -2k, 0)$ functions will also go to zero. In order to work around this we will look at the derivative of our Eisenstein series with respect to s. Define

$$G(z, -2k) := \lim_{s \to 0} \frac{d}{ds} \mathcal{G}(z, -2k, s). \tag{2.4}$$

We will now calculate the q-expansion of $G(z,-2k)$. For $n=0$, we have

$$\begin{aligned}\frac{d}{ds}h_0(y,-2k,s)|_{s=0} &= y^{1+2k}2^{2+2k}e^{k\pi i}\pi\frac{\Gamma(-2k-1)}{\Gamma(-2k)}\\ &= \frac{(-1)^{k+1}y^{1+2k}2^{2+2k}\pi}{2k+1}.\end{aligned}$$

For $n>0$, we have

$$\begin{aligned}\frac{d}{ds}h_n(y,-2k,s)|_{s=0} &= -y^{1+2k}e^{-2\pi ny}\int_{c_2} t^{2k}\log(t(t-2i))e^{-2\pi inyt}dt\\ &= -y^{1+2k}e^{-2\pi ny}(2\pi i)\int_{-i\infty}^{0} t^{2k}e^{-2\pi inyt}dt\\ &= (-1)^k y^{1+2k}e^{-2\pi ny}(2\pi)\int_0^{\infty} t^{2k}e^{-2\pi nyt}dt\\ &= (-1)^k(2\pi)^{-2k}n^{-2k-1}\Gamma(2k+1)e^{-2\pi ny}.\end{aligned}$$

The log term jumps by $2\pi i$ across the branch cut, while everything else is continuous. Similarly, for $n<0$ we have

$$\frac{d}{ds}h_n(y,-2k,s)|_{s=0}= (-1)^{k+1}(2\pi)^{-2k}n^{-2k-1}e^{-2\pi ny}\Gamma(1+2k,-4\pi ny),$$

Let $h_n'(y,-2k,0):=\frac{d}{ds}h_n(y,-2k,s)|_{s=0}$, then, from equation (2.4), we have

$$G(z,-2k) = 2\zeta'(-2k) + \sum_{n=-\infty}^{\infty} h_n'(y,-2k,0)\sigma_{2k+1}(n)e^{2\pi inx}.$$

Recall that $\sigma_{2k+1}(0) = \frac{1}{2}\zeta(-2k-1)$. Putting everything together leads to the construction of the forms in Theorem 1.2.3 part 1. A short calculation shows these forms are harmonic. In order to show that $G(z,-2k)$ is a Hecke eigenform, notice that its image under the ξ-operator is a nonzero multiple of the weight $2k+2$ Eisenstein series, $E_{2k+2}(z)$. E_{2k+2} is known to be an eigenform with eigenvalue $\sigma_{2k+1}(p) = 1 + p^{2k+1}$ under the Hecke operator $T(p)$. By equation (2.1) and inspection it is clear that $G(z,-2k)$ is then an eigenform with eigenvalue $1+\frac{1}{p^{2k+1}}$.

Proof of Theorem 1.2.3 part 2

Let $k = 2r-1$ with $r \geq 1$. We define the two Eisenstein series $F\left(z, -\frac{k}{2}, s\right)$ and $E\left(z, -\frac{k}{2}, s\right)$ by

$$F\left(z,-\frac{k}{2},s\right) = \sum_{\substack{n,m\in\mathbb{Z}\\ n>0\\ 4|m}} \left(\frac{m}{n}\right) \varepsilon_n^{-k} \frac{(mz+n)^{k/2}}{|mz+n|^{2s}}, \tag{2.5}$$

and

$$E\left(z,-\frac{k}{2},s\right) = \frac{(2z)^{k/2}}{|2z|^{2s}} F\left(\frac{-1}{4z}, -\frac{k}{2}, s\right),$$

where $\left(\frac{m}{n}\right)$ is the *Kronecker symbol* and

$$\varepsilon_n := \begin{cases} 1 & if\ n \equiv 1 \pmod 4 \\ i & if\ n \equiv 3 \pmod 4. \end{cases}$$

A linear combination of these forms will have a meromorphic continuation to the whole s-plane and evaluating at $s = 0$ will give our weight $-\frac{k}{2}$ form. We will abuse this fact by letting $s = 0$ in the assembly of the forms. We have

$$E\left(z, -\frac{k}{2}, s\right) = 2^{\frac{k}{2}-2s} \sum_{\substack{n,m\in\mathbb{Z} \\ n>0,odd}} \left(\frac{m}{n}\right) \varepsilon_n^{-k} \frac{(nz-m)^{k/2}}{|nz-m|^{2s}}.$$

From this we have

$$\begin{aligned} E\left(z, -\frac{k}{2}, s\right) &= 2^{\frac{k}{2}-2s} \sum_{n>0,odd} \varepsilon_n^{-k} n^{\frac{k}{2}-2s} \sum_{m \pmod n} \left(\frac{m}{n}\right) \sum_{h=-\infty}^{\infty} \frac{\left(z - \frac{m}{n} + h\right)^{\frac{k}{2}}}{|z - \frac{m}{n} + h|^{2s}} \\ &= 2^{\frac{k}{2}-2s} \sum_{n>0,odd} \varepsilon_n^{-k} n^{\frac{k}{2}-2s} \sum_{N=-\infty}^{\infty} \sum_{m \pmod n} \left(\frac{m}{n}\right) \alpha_N\left(y, -\frac{k}{2}, s\right) e^{-\frac{2\pi i N m}{n}} q^N \\ &= \sum_{N=-\infty}^{\infty} a(N) q^N, \end{aligned}$$

where

$$a(N) = 2^{\frac{k}{2}-2s} \alpha_N\left(y, -\frac{k}{2}, s\right) \sum_{n>0,odd} \varepsilon_n^{-k} n^{\frac{k}{2}-2s} \sum_{m \pmod n} \left(\frac{m}{n}\right) e^{-\frac{2\pi i N m}{n}},$$

and by the Poisson summation formula

$$\alpha_N\left(y, -\frac{k}{2}, s\right) = \int_{iy-\infty}^{iy+\infty} z^{\frac{k}{2}} |z|^{-2s} e^{-2\pi i N z} dz.$$

Making the substitution $z = yt + iy$ gives us

$$\alpha_N\left(y, -\frac{k}{2}, s\right) = y^{\frac{k}{2}+1-2s} e^{2\pi N y} \int_{-\infty}^{\infty} (t+i)^{\frac{k}{2}} (t^2+1)^{-s} e^{-2\pi i N y t} dt.$$

Following Zagier, we choose the branch cut along the negative imaginary axis. Using contour integration we have

$$\alpha_N\left(y,-\frac{k}{2},s\right)=2e^{\frac{k\pi i}{4}}\sin\left(\pi\left(\frac{k}{2}-s\right)\right)y^{\frac{k}{2}+1-2s}\int_0^\infty t^{\frac{k}{2}-s}(t+2)^{-s}e^{-2\pi Nyt}dt.$$

Letting $s=0$ we arrive at

$$\begin{aligned}\alpha_N\left(y,-\frac{k}{2},s\right)&=2e^{\frac{k\pi i}{4}}\sin\left(\frac{\pi k}{2}\right)y^{\frac{k}{2}+1}\int_0^\infty t^{\frac{k}{2}}e^{-2\pi Nyt}dt\\&=2e^{\frac{k\pi i}{4}}\sin\left(\frac{\pi k}{2}\right)y^{\frac{k}{2}+1}(2\pi Ny)^{-\frac{k}{2}-1}\int_0^\infty t^{\frac{k}{2}}e^{-t}dt\\&=2e^{\frac{k\pi i}{4}}\sin\left(\frac{\pi k}{2}\right)(2\pi N)^{-\frac{k}{2}-1}\Gamma\left(\frac{k}{2}+1\right),\end{aligned}$$

for $N>0$. If we evaluate the similar integral for $N\leq 0$, because we do not cross a branch cut the integral is zero. It will be useful to evaluate the derivative. We have that

$$\begin{aligned}\frac{d}{ds}\alpha_N\left(y,-\frac{k}{2},s\right)\Big|_{s=0}&=-y^{\frac{k}{2}+1}e^{4\pi Ny}(2\pi i)\int_{i\infty}^0(t+2i)^{\frac{k}{2}}e^{-2\pi iNyt}dt\\&=-2y^{\frac{k}{2}+1}\pi i^{\frac{k}{2}}\int_2^\infty t^{\frac{k}{2}}e^{2\pi Nyt}dt\\&=i^{-\frac{k}{2}}(2\pi)^{-\frac{k}{2}}N^{-\frac{k}{2}-1}\Gamma\left(\frac{k}{2}+1,-4\pi Ny\right),\end{aligned}$$

for $N<0$, while

$$\frac{d}{ds}\alpha_0\left(y,-\frac{k}{2},s\right)\Big|_{s=0}=-\frac{2^{\frac{7}{2}-r}i^{\frac{k}{2}}y^{\frac{k}{2}+1}\pi}{2r-3}.$$

Similarly, for $F\left(z,-\frac{k}{2},s\right)$ we have

$$\begin{aligned}F\left(z,-\frac{k}{2},s\right) &= 1+\sum_{\substack{m>0\\4|m}} m^{\frac{k}{2}-2s}\sum_{n \pmod m}\left(\frac{m}{n}\right)\varepsilon_n^{-k}\sum_{h=-\infty}^{\infty}\frac{\left(z+\frac{n}{m}+h\right)^{\frac{k}{2}}}{|z+\frac{n}{m}+h|^{2s}}\\
&= 1+\sum_{\substack{m>0\\4|m}} m^{\frac{k}{2}-2s}\sum_{n \pmod m}\left(\frac{m}{n}\right)\varepsilon_n^{-k}\sum_{N=-\infty}^{\infty}\alpha_N\left(y,-\frac{k}{2},s\right)e^{\frac{2\pi i N n}{m}}q^N\\
&= 1+\sum_{N=-\infty}^{\infty} b(N)q^N,\end{aligned}$$

where

$$b(N)=\alpha_N\left(y,-\frac{k}{2},s\right)\sum_{\substack{m>0\\4|m}} m^{\frac{k}{2}-2s}\sum_{n \pmod m}\left(\frac{m}{n}\right)\varepsilon_n^{-k}e^{\frac{2\pi i N n}{m}}.$$

Using Proposition 2.2.2 and by manipulating the inner sums of $a(N)$ and $b(N)$, it is not hard to show that

$$\begin{aligned}a(N) &= 2^{\frac{k}{2}-2s}\alpha_N\left(y,-\frac{k}{2},s\right)\sum_{n>0,odd} n^{\frac{k}{2}-2s}\sum_{\substack{m=1\\m\ even}}^{2n}\lambda(m,n)e^{-\pi i(-1)^r N\frac{m}{n}}\\
&= 2^{\frac{k}{2}-2s}\alpha_N\left(y,-\frac{k}{2},s\right)\sum_{n>0,odd} n^{\frac{k}{2}+\frac{1}{2}-2s}\gamma_n((-1)^r N)\\
&= 2^{\frac{k}{2}+1-2s}\alpha_N\left(y,-\frac{k}{2},s\right)\frac{1}{2}E^{odd}_{(-1)^r N}\left(-\frac{k}{2}-\frac{1}{2}+2s\right),\end{aligned}$$

and

$$b(N)=(1+i^{2r+1})4^{\frac{k}{2}+\frac{1}{2}-2s}\alpha_N\left(y,-\frac{k}{2},s\right)\frac{1}{2}\sum_{\substack{m>0\\m\ even}}\frac{\gamma_m((-1)^r N)}{(m/2)^{-\frac{k}{2}-\frac{1}{2}+2s}}$$

$$= (1 + i^{2r+1}) 4^{\frac{k}{2}+\frac{1}{2}-2s} \alpha_N \left(y, -\frac{k}{2}, s\right) \frac{1}{2} E^{even}_{(-1)^r N} \left(-\frac{k}{2} - \frac{1}{2} + 2s\right).$$

We are now able to define our forms. Define

$$\mathcal{H}\left(z, -r + \frac{1}{2}\right) = \sum_{N=-\infty}^{\infty} c_r(N) q^N$$
$$:= \lim_{s \to 0} \zeta(1 + 2r - 4s) \left(i^{2r-1} F\left(z, -r + \frac{1}{2}, s\right) + 2^{r-\frac{1}{2}} (1 + i^{2r-1}) E\left(z, -r + \frac{1}{2}, s\right)\right). \tag{2.6}$$

The rest of the construction is using Proposition 2.2.2. Similar calculations can be found in [8] or [65]. Note that the functional equations for the zeta function and the L-function are used and that there is pole when evaluating the non-holomorphic coefficients. The image of $\mathcal{H}\left(z, -r + \frac{1}{2}\right)$ under the ξ-operator is a nonzero multiple of the weight $r + \frac{3}{2}$ Cohen-Eisenstein series. The weight $r + \frac{3}{2}$ Cohen-Eisenstein series is a Hecke eigenform with eigenvalue $1 + p^{2r+1}$ under the Hecke operator $T(p^2)$. Therefore, using equation (2.3), we can see that

$$\mathcal{H}\left(z, -r + \frac{1}{2}\right) \Big| T(p^2) - \left(1 + \frac{1}{p^{2r+1}}\right) \mathcal{H}\left(z, -r + \frac{1}{2}\right)$$

is a weight $-r + \frac{1}{2}$ holomorphic modular form in the Kohnen plus space (see [44]). This space is empty and so $\mathcal{H}\left(z, -r + \frac{1}{2}\right)$ must be a Hecke eigenform with eigenvalue $1 + \frac{1}{p^{2r+1}}$.

2.2.3 Proof of Theorem 1.2.5

In order to discuss p-adic harmonic Maass forms we will first need to recall some facts about Bernoulli numbers. Values of the Reimann zeta function at negative integers are tied to Bernoulli numbers. In fact we have $\zeta(1-2k) = -\frac{B_{2k}}{2k}$, and $\zeta(-2k) = 0$. In a similar way, there is a connection between generalized Bernoulli numbers and the values of L-functions at negative integers. The *generalized Bernoulli numbers* $B(n, \chi)$ are defined by the generating function

$$\sum_{n=0}^{\infty} B(n, \chi)\frac{t^n}{n!} = \sum_{a=1}^{m-1} \frac{\chi(a)te^{at}}{e^{mt}-1},$$

Where χ is a Dirichlet character modulo m. Generalized Bernoulli numbers are known to give the values of Dirichlet L-functions at non-positive integers. In fact, from [44] we know that if k is a positive integer and χ is a nontrivial Dirichlet character, then

$$L(1-k, \chi) = -\frac{B(k, \chi)}{k}.$$

This connection helps one define a p-adic L-function, $L_p(s, \chi)$. The p-adic L-function is analytic except for a pole at $s = 1$ with residue $\left(1 - \frac{1}{p}\right)$. For $n \geq 1$ we have that

$$L_p(1-n, \chi) = -(1 - \chi \cdot \omega^{-n}(p)p^{n-1})\frac{B(n, \chi \cdot \omega^{-n})}{n}, \tag{2.7}$$

where ω is the Teichmüller character. The Teichmüller character is a p-adic Dirichlet character of conductor p if p is odd and conductor 4 if $p = 2$. It is best to view it as a p-adic object. For more information see Chapter 5 of [63]. Kummer famously showed that if $n \equiv m \pmod{(p-1)p^a}$ and $(p-1) \nmid n, m$ for an odd prime p, then

$$(1 - p^{n-1})\frac{B_n}{n} \equiv (1 - p^{m-1})\frac{B_m}{m} \pmod{p^{a+1}}, \tag{2.8}$$

where a is a nonnegative integer. Similar congruences hold for generalized Bernoulli numbers as well. For example, if we let $\chi \neq 1$ be a primitive Dirichlet character with conductor not divisible by p, then if $n \equiv m \pmod{p^a}$ we have

$$(1-\chi\cdot\omega^{-n}(p)p^{n-1})\frac{B(n, \chi \cdot \omega^{-n})}{n} \equiv (1-\chi\cdot\omega^{-m}(p)p^{m-1})\frac{B(m, \chi \cdot \omega^{-m})}{m} \pmod{p^{a+1}}. \tag{2.9}$$

Notice that twisting by the appropriate power of the Teichmüller character removes the dependence on the residue class of n and m modulo $p-1$ here. The family of p-adic harmonic Maass forms coming from the integer weight forms in Theorem 1.2.3 are constructed in the exact same way as the p-adic Eisenstein series in [53]. Equation (2.8) shows that the constant term, the p-adic zeta function at a negative integer, will satisfy congruences. The other terms satisfy congruences due to Euler's theorem which generalizes Fermat's Little Theorem. The algebraic parts of these p-adic harmonic Maass forms enjoy similar congruences as their modular counterparts. In fact, the non-holomorphic parts are nearly identical to the p-adic Eisenstein series. The holomorphic parts behave not quite as nicely only because the p-adic zeta function at positive integers does not behave as nicely as at negative integers. However, it is still expected that it satisfies similar congruences modulo some p-adic regulator. For example, we have

$$G^{+,(5)}(z,-2) = -\frac{1}{2\pi^2}\left(\zeta^{(5)}(3) + q + \frac{9}{8}q^2 + \frac{28}{27}q^3 + \frac{73}{64}q^4 + \frac{1}{75}q^5 + \cdots\right),$$

while

$$G^{+,(5)}(z,-6) = -\frac{45}{4\pi^6}\left(\zeta^{(5)}(7) + q + \frac{129}{128}q^2 + \frac{2188}{2187}q^3 + \frac{16513}{16384}q^4 + \frac{1}{78125}q^5 + \cdots\right).$$

The family of p-adic harmonic Maass forms coming from the half-integral weight forms from Theorem 1.2.3 are defined using p-adic L-functions and the fact that $T_r^{\chi,(p)}(v)$ is the p-adic limit of $T_r^{\chi}(v)$. As in the previous case, the non-holomorphic parts satisfy nice congruences due to equation (2.9). The holomorphic parts are expected to satisfy congruences modulo a p-adic regulator.

Chapter 3

Distributions and Jensen polynomials

In this chapter we will prove the results related to Jensen polynomials and hyperbolicity presented in Section 1.3. Section 3.1.1 will contain some preliminary results for the partition Jensen polynomials. The proofs of Theorems 1.3.1 and 1.3.2 and Theorem 1.3.3 will be presented in Sections 3.1.2 and 3.1.3 respectively. Section 3.2.1 will contain a discussion of the asymptotics for the derivatives of a general good Dirichlet series at its central point. We will present and prove an asymptotic formula with arbitrary precision for these derivatives in this section. We will prove Theorem 1.3.6 in Section 3.2.2 and prove Corollary 1.3.7, Corollary 1.3.8, and Corollary 1.3.9 in Section 3.2.3, Section 3.2.4, and Section 3.2.5 respectively. These sections will also contain a short introduction to the type of Dirichlet series discussed there and and an asymptotic formula for the derivatives of that specific type of Dirichlet series.

3.1 Hyperbolicity of the partition Jensen polynomials

In this section we will be concerned with the Jensen polynomials associated to the partition function. We will present some necessary results and then prove Theorem 1.3.1, Theorem 1.3.2, and Theorem 1.3.3.

3.1.1 Hankel determinants and ratios of close partition numbers

The hyperbolicity of a polynomial $P(X) = a_d X^d + a_{d-1} X^{d-1} + \ldots + a_0$ is equivalent to certain polynomial conditions in the coefficients a_i, which we now describe. If $\lambda_1, \ldots, \lambda_d$ are the roots of $P(X)$, let $S_k = \lambda_1^k + \ldots + \lambda_d^k$ denote the sum of kth powers of the roots. The $m \times m$ *Hankel determinant associated to* $P(X)$ is defined by

$$\Delta_m(P(X)) := \begin{vmatrix} S_0 & S_1 & \cdots & S_{m-1} \\ S_1 & S_2 & \cdots & S_m \\ \vdots & \vdots & & \vdots \\ S_{m-1} & S_m & \cdots & S_{2m-2} \end{vmatrix} = \sum_{i_1 < \cdots < i_m} \prod_{a<b} (\lambda_{i_a} - \lambda_{i_b})^2. \qquad (3.1)$$

In addition, let

$$D_{d,m}(P(X)) = D_{d,m}(a_0, \ldots, a_d) := a_d^{2m-2} \cdot \Delta_m(P(X))$$

so that $D_{d,d}(a_0, \ldots, a_d)$ is the discriminant of $P(X)$ and $D_{d,m}(a_0, \ldots, a_d)$ is a homogeneous polynomial of degree $2m-2$ in the coefficients a_i. A theorem of Hermite [42] says the hyperbolicity of $P(X)$ is equivalent to the condition $D_{d,m}(P(X)) \geq 0$ for all $m = 2, \ldots, d$.

We will prove Theorems 1.3.1 and 1.3.3 by showing that

$$\mathcal{D}_{d,m}(n) := D_{d,m}\left(\frac{J_p^{d,n}(X)}{p(n)}\right) = D_{d,m}\left(1, \binom{d}{1}\frac{p(n+1)}{p(n)}, \binom{d}{2}\frac{p(n+2)}{p(n)}, \ldots, \frac{p(n+d)}{p(n)}\right) > 0$$

for each $m = 2, \ldots, d$ and all n greater than the claimed quantities. Note that $\mathcal{D}_{d,m}(n)$ approaches 0 in the limit as $n \to \infty$, since $\lim_{n\to\infty} J_p^{d,n}(X)/p(n) = (X+1)^d$. This fact is true because the partition ratios $\frac{p(n+j)}{p(n)} \to 1$ as $n \to \infty$ for any fixed j. A priori, this makes the sign of $\mathcal{D}_{d,m}(n)$ difficult to ascertain.

However, the results in [27] determine the rate at which $\mathcal{D}_{d,m}(n)$ approaches 0 and the coefficient of the leading term. More precisely, by the behavior of Δ_m under change of variable and equation (1.17), we know that

$$\lim_{n\to\infty} \frac{1}{\delta(n)^{m(m-1)}} \Delta_m\left(\frac{J_p^{d,n}(X)}{p(n)}\right) = \lim_{n\to\infty} \Delta_m\left(\frac{J_p^{d,n}(\delta(n)X - e^{-cw(n)/2})}{p(n)}\right) = \Delta_m(H_d(X)).$$

Equivalently in terms of $w = w(n) = 1/\sqrt{c(n-1/24)}$ and $\mathcal{D}_{d,m}(n)$, we have

$$\lim_{w\to 0} \frac{1}{w^{\frac{3}{2}m(m-1)}} \mathcal{D}_{d,m}(n) = \left(\frac{c}{\sqrt{2}}\right)^{m(m-1)} \Delta_m(H_d(X)). \tag{3.2}$$

Because the Hermite polynomials have distinct, real roots, the term on the right is a positive constant. Our strategy is to expand $\mathcal{D}_{d,m}(n)$ in powers of w around zero, up to $w^{\frac{3}{2}m(m-1)}$. Because the above limit exists, we are guaranteed that all lower powers of w cancel, and the coefficient of the $w^{\frac{3}{2}m(m-1)}$ term is the specified positive multiple of $\Delta_m(H_d(X))$. We then must find explicit bounds for the remaining terms that are tending to zero.

To do this, we need to study ratios of close partition numbers. In terms of w, the Hardy-Ramanunjan asymptotic formula for the partition numbers [28] takes the form

$$p(n) \sim F(w) := \frac{\pi^2}{6\sqrt{3}}(w^2 - w^3)e^{1/w}.$$

As observed in [27], $w(n+j) = \frac{w(n)}{\sqrt{1+cjw(n)^2}}$, so the function

$$R(j,w) := \frac{F\left(\frac{w}{\sqrt{1+cjw^2}}\right)}{F(w)} = \frac{e^{\frac{cjw}{1+\sqrt{1+cjw^2}}}(\sqrt{1+cjw^2}-w)}{(1-w)(1+cjw^2)^{3/2}} \tag{3.3}$$

closely approximates $p(n+j)/p(n)$.

To bound the error of this approximation, we use Lehmer's error bound for Rademacher's convergent series for the partition function, in which $F(w)$ is the leading term. In what follows, $A_k(n)$ is a Kloosterman sum. The only property we need is $|A_1(n)| = |A_2(n)| = 1$, so we do not define it here, instead referring the reader to [38].

Theorem 3.1.1 (Lehmer)**.** *Let* $w = w(n) = 1/\sqrt{c(n-1/24)}$*. For all* $n \geq 1$*, we have*

$$p(n) = \frac{\pi^2}{6\sqrt{3}} w^2 \sum_{k=1}^{N} \frac{A_k(n)}{\sqrt{k}} \left((1-w)e^{1/kw} + (1+w)e^{-1/kw}\right) + B(n,N) \quad (3.4)$$

where

$$|B(n,N)| < \frac{\pi^2 N^{-2/3}}{\sqrt{3}} \left(N^3 w^3 \sinh\left(\frac{1}{Nw}\right) + \frac{1}{6} - N^2 w^2\right) < \frac{\pi^2 N^{-2/3}}{\sqrt{3}} \left(N^3 w^3 \frac{e^{1/Nw}}{2} + \frac{1}{6}\right).$$

In order for us to state precisely how well $R(j,w)$ approximates $p(n+j)/p(n)$, let

$$L(w) := \frac{1+21w}{1-w} \cdot e^{-1/2w} + \frac{e^{-1/w}}{w^2 - w^3}.$$

Lemma 3.1.2. *For all* $n \geq 1$*, we have*

$$\left|\frac{p(n+j)}{p(n)} - R(j,w)\right| \leq R(j,w) \frac{2L(w)}{1-L(w)} \sim 2e^{-1/2w}.$$

Proof. Let $E(w(n)) = p(n) - F(w(n))$. The function $F(w)$ appears in the $k=1$ term of (3.4). Gathering the rest of that term, the $k=2$ term, and the Lehmer's bound on $|B(n,2)|$ we find

$$\begin{aligned} |E(w)| &\leq \frac{\pi^2}{6\sqrt{3}} \left((w^2+w^3)e^{-1/2w} + (w^2 - w^3 + 12 \cdot 2^{5/6} w^3)e^{1/2w} + 2^{-7/6}\right) \\ &\leq \frac{\pi^2}{6\sqrt{3}} \left((w^2 + 21w^3)e^{1/2w} + 1\right), \end{aligned}$$

where in the last line we have used that $w \le 1/\sqrt{c}$. Hence, $|E(w)/F(w)| \le L(w)$. Noting that the function $L(w)$ is increasing in w for $0 < w \le 1/\sqrt{c}$ it follows that

$$\left|\frac{p(n+j)}{p(n)} - \frac{F(w(n+j))}{F(w(n))}\right| = \frac{F(w(n+j))}{F(w(n))}\left|\frac{1+\frac{E(w(n+j))}{F(w(n+j))}}{1+\frac{E(w)}{F(w)}} - 1\right|$$
$$= R(j,w)\left|\frac{\frac{E(w(n+j))}{F(w(n+j))} - \frac{E(w(n))}{F(w(n))}}{1+\frac{E(w)}{F(w)}}\right| \le R(j,w)\frac{2L(w)}{1-L(w)}.$$

□

To study the behavior $p(n+j)/p(n)$ for large n, we want to study $R(j,w)$ near $w=0$. To this end, let $A_s(j,w)$ be the degree $s-1$ Taylor polynomial of $R(j,w)$. Applying Lemma 3.1.2 and Taylor's theorem, we immediately obtain the following.

Lemma 3.1.3. *Let $n \ge 1$ and suppose $w = 1/\sqrt{c(n-1/24)} \in [0,\epsilon]$ for some $0 < \epsilon \le 1/\sqrt{c}$. Then we have*

$$\frac{p(n+j)}{p(n)} = A_s(j,w) + E_s(j,w)w^s$$

where

$$|E_s(j,w)| \le \frac{1}{s!}\cdot \sup_{x\in[0,\epsilon]}\left|R^{(s)}(j,x)\right| + \sup_{x\in[0,\epsilon]}\left|R(j,x)\frac{2L(x)}{x^s(1-L(x))}\right|. \qquad (3.5)$$

3.1.2 Proof of Theorems 1.3.1 and 1.3.2

We now prove Theorem 1.3.1 by bounding the error terms that accumulate from approximating $p(n+j)/p(n)$ by the Taylor polynomials $A_s(j,w)$ in the polynomial expression for $\mathcal{D}_{d,m}(n)$. This allows us to reduce to checking finitely many cases.

Proof of Theorem 1.3.1. Using the Newton-Girard identities to write the power sums of the roots in terms of the elementary symmetric functions, one can generate symbolic expressions for the polynomials $D_{d,m}(a_0,\ldots,a_d)$ in terms of $a_0,\ldots,a_n$. To obtain $\mathcal{D}_{d,m}(n)$, we substitute

$$\binom{d}{j}(A_{10}(j,w)+E_jw^s)$$

in for a_j in these polynomials, introducing E_j as a variable. This gives rise to a polynomial expression in w whose coefficients are polynomials in E_j. It turns out that all coefficients of w^i for $i<k=\frac{3}{2}m(m-1)$ vanish in this expression. In addition, dividing through by w^k gives rise to an expression of the form

$$\mathcal{D}_{d,m}(w)=c_0+c_1w+c_2(E_1,\ldots,E_d)w^2+\ldots+c_{(2m-2)s-k}(E_1,\ldots,E_d)w^{(2m-2)s-k}$$

where c_0 and c_1 are positive constants.

We then use Mathematica to calculate the upper bound on $E_j=E_{10}(j,w)$

for $w \in [0, \epsilon]$ given in Lemma 3.1.3, where we choose

$$\epsilon = 0.021, 0.0163, 0.0081 \qquad \text{for } d = 3, 4, 5 \text{ respectively.}$$

From these, we can obtain a lower bound $-c_i' \leq c_i(E_1, \ldots, E_d)$ for each $i \geq 2$, giving rise to an expression of the form

$$\mathcal{D}_{d,m}(w) \geq c_0 + c_1 w - c_2' w^2 - \ldots - c_{(2m-2)s-k}' w^{(2m-2)s-k}.$$

Moreover, we can arrange for each of the c_i' above to be nonnegative so that the function on the right crosses zero at most once in the interval $[0, \epsilon]$. For our chosen values of ϵ, evaluating the right-hand side at $w = \epsilon$ is positive, so $\mathcal{D}_{d,m}(w) > 0$ for all $1 \leq m \leq d$ and $w \leq \epsilon$. Equivalently, $J_p^{d,n}(X)$ is hyperbolic for all $n \geq \frac{1}{c\epsilon^2} + \frac{1}{24}$. Using the values of ϵ listed above, this shows $J_p^{3,n}(X)$ is hyperbolic for all $n > 344$, $J_p^{4,n}(X)$ is hyperbolic for all $n > 572$ and $J_p^{5,n}$ is hyperbolic for all $n > 2316$. Checking the finite number of remaining possible counter examples directly now proves the theorem. Annotated Sage and Mathematica code to implement the full procedure described above appears in the appendix. □

Remark. With our chosen parameters, the total run time of this procedure is about 15 minutes. We note that by increasing the number of terms s that we take in the Taylor expansion of $R(j, w)$, the number of cases one needs to check directly can be brought down. However, this increases total run time, as checking more particular cases directly is faster than carrying out the more complex symbolic manipulations. For example, when $d = 5$, by increasing s to 16, one may increase ϵ to 0.013, corresponding to checking $n = 899$ cases directly, but this has a total run time of about an hour.

Remark. For $d \geq 6$ one would need to keep more than $s = 10$ terms in order to see the cancellation of lower order terms in w take place. The main obstruction of applying this method in higher degrees is tracking the increasing number of error terms in the increasingly complex symbolic expressions for $\mathcal{D}_{d,m}(n)$. A code for $d = 6$ with $s = 16$ did not finish within 36 hours when run on a laptop.

Taylor expanding $R(j, w)$ and symbolically keeping track of errors can be used to prove inequalities about other polynomial equations involving ratios of close partition numbers. We now prove Theorem 1.3.2 using this idea.

Proof. Setting $a_i = p(n+i)/p(n)$ we can rewrite equation (1.3.2) as

$$0 < \left(1 + \frac{\pi^4}{9}w^3\right)(a_1^2 - a_{-1}a_1a_2)^2 - 4a_1^2(1 - a_{-1}a_1)(a_1^2 - a_2).$$

We follow the same procedure and notation as in the proof of Theorem 1.3.1, taking $s = 6$ and $\epsilon = 0.013$. Substituting $a_i = A_6(i, w) + E_i w^6$ into the right-hand side above gives rise to a polynomial expression in w with coefficients that are polynomials in the E_i, where the first term is a positive constant times w^{10}. We then minimize all the coefficients as before, using the bounds on $|E_i|$

from Lemma 3.1.3. This leaves us with an expression of the form

$$w^{10}\left(\frac{25}{729}\pi^{12}-x(w)\right)\leq\left(1+\frac{\pi^4}{9}w^3\right)(a_1^2-a_{-1}a_1a_2)^2-4a_1^2(1-a_{-1}a_1)(a_1^2-a_2),$$

where $x(w)$ is a strictly increasing polynomial in w. Evaluating the left-hand side at $w=\epsilon$ yields a positive number, so the right-hand side is positive for all $w\in[0,\epsilon]$. Equivalently, the proposition holds for all $n>900$. Checking all $n\leq 900$ directly completes the proof. □

3.1.3 Bounds for general d

The polynomial $\mathcal{D}_{d,m}(n)$ we wish to study is homogeneous of degree $2m-2$ in the coefficients of $J_p^{d,n}(X)/p(n)$ and homogeneous of degree $m(m-1)$ in its roots. That is, it has the form

$$\mathcal{D}_{d,m}(n)=\sum_{i_1+\ldots+i_{2m-2}=m(m-1)}A_{i_1,\ldots,i_{2m-2}}\cdot\prod_{k=1}^{2m-2}\binom{d}{i_k}\frac{p(n+d-i_k)}{p(n)},\tag{3.6}$$

where the $A_{i_1,\ldots,i_{2m-2}}$ are constants. To bound errors when we expand in terms of w, we find bounds on the derivatives $R^{(s)}(j,w)$ for w in the interval $[0,\epsilon]$, where $\epsilon:=(3d)^{-12d}(50d)^{-\frac{3}{2}d^2}$, corresponding to our eventual bound on $N(d)$. For convenience, let $t=t(j):=cj$.

Lemma 3.1.4. *Assume that $w\in[0,\epsilon]$ with ϵ as above. Then*

$$\left|R^{(m)}(j,w)\right|\leq m!\binom{m+3}{3}e^{g(\epsilon)}(4e^{2t\epsilon}t)^m,\tag{3.7}$$

where $g(\epsilon) = \frac{t\epsilon}{1+\sqrt{1+t\epsilon^2}}$.

Proof. The idea of the proof is to use the product rule to split up $R(j,w)$ into four more manageable parts and use Faà di Bruno's formula for iterated applications of the chain rule to evaluate each part as needed. This formula says that for differentiable functions $f(x)$ and $g(x)$, we have

$$\frac{d^n}{dx^n} f\left(g(x)\right) = \sum_{m_1+2\cdot m_2+\cdots+n\cdot m_n=n} \frac{n!}{m_1!\cdots m_n!} f^{(m_1+m_2+\cdots+m_n)}\left(g(x)\right) \prod_{j=1}^{n} \left(\frac{g^{(j)}(x)}{j!}\right)^{m_j}. \tag{3.8}$$

Let

$$A = A(t,w) := e^{\frac{tw}{1+\sqrt{1+tw^2}}} \qquad B = B(t,w) := \sqrt{1+tw^2} - w,$$
$$C = C(t,w) := \frac{1}{1-w} \qquad D = D(t,w) := \frac{1}{(1+tw^2)^{3/2}},$$

so that

$$R^{(m)}(j,w) = \sum_{m_1+m_2+m_3+m_4=m} \frac{m!}{m_1!\cdots m_4!} \left(\frac{d^{m_1}A}{dw^{m_1}}\right)\cdot\left(\frac{d^{m_2}B}{dw^{m_2}}\right)\cdot\left(\frac{d^{m_3}C}{dw^{m_3}}\right)\cdot\left(\frac{d^{m_4}D}{dw^{m_4}}\right). \tag{3.9}$$

We will focus on A first. Let $f(w) = e^w$ and $g(w) = \frac{tw}{1+\sqrt{1+tw^2}}$. By equation (3.8), we have

$$\frac{d^nA}{dw^n} = \frac{d^n}{dw^n} f(g(w)) = \sum_{m_1+2\cdot m_2+\cdots+n\cdot m_n=n} \frac{n!}{m_1!\cdots m_n!} e^{g(w)} \prod_{i=1}^{n} \left(\frac{g^{(i)}(w)}{i!}\right)^{m_i}. \tag{3.10}$$

By the product rule, it is easy to see that

$$g^{(i)}(w) = tw\left(\frac{d}{dw}\right)^i \frac{1}{1+\sqrt{1+tw^2}} + it\left(\frac{d}{dw}\right)^{i-1} \frac{1}{1+\sqrt{1+tw^2}}. \tag{3.11}$$

Next, let $g_*(w) := \frac{1}{1+\sqrt{1+tw^2}}$ and let $\alpha(k) := \left(\frac{d}{dw}\right)^k \sqrt{1+tw^2}$. We use equation (3.8) again to show

$$g_*^{(i)}(w) = \sum_{r_1+\cdots+i\cdot r_i=i} \frac{i!}{r_1!\cdots r_i!} \frac{(-1)^{r_1+\cdots+r_i}(r_1+\cdots+r_i)!}{(1+\sqrt{1+tw^2})^{r_1+\cdots+r_i+1}} \prod_{k=1}^{i} \left(\frac{\alpha(k)}{k!}\right)^{r_k}. \tag{3.12}$$

Using equation (3.8) once more we have

$$\alpha(k) = \sum_{s_1+2s_2=k} \frac{k!}{s_1!\, s_2!} \binom{\frac{1}{2}}{s_1+s_2} \frac{(2tw)^{s_1} t^{s_2}}{(1+tw^2)^{s_1+s_2-\frac{1}{2}}} \leq k!\, e^{2tw} t^k.$$

We can plug this back into equation (3.12) to find that

$$g_*^{(i)}(w) \leq i!\,(e^{2tw}t)^i \sum_{r_1+\cdots+i\cdot r_i=i} \frac{(r_1+\cdots+r_i)!}{r_1!\cdots r_i!} \leq i!\,(2e^{2tw}t)^i,$$

where we used the fact that the sum is counting the number of ordered partitions of i. Next, we plug this into equation (3.11) and use the fact that $tw \leq 1$ to find $\left|g^{(i)}(w)\right| \leq i! \cdot 2(2\lambda t)^i$. Finally, we are able to plug into equation (3.10) to find that

$$\left|\frac{d^n A}{dw^n}\right| \leq n!\, e^{g(w)}(2e^{2tw}t)^n \cdot \sum_{m_1+\cdots+n\cdot m_n=n} \frac{2^{m_1+\cdots+m_n}}{m_1!\cdots m_n!} \leq n!\, e^{g(w)}(4e^{2tw}t)^n. \tag{3.13}$$

Next, it is easy to show that

$$\left|\frac{d^nB}{dw^n}\right| \leq |\alpha(n)| \leq n!\,(e^{2tw}t)^n, \tag{3.14}$$

and

$$\left|\frac{d^nC}{dw^n}\right| = \frac{n!}{(1-w)^{n+1}} \leq n!\,(e^{2tw}t)^n. \tag{3.15}$$

Lastly, we have

$$\left|\frac{d^nD}{dw^n}\right| \leq \sum_{r_1+\cdots+n\cdot r_n=n} \frac{n!}{r_1!\cdots r_n!}\frac{(\frac{3}{2})_{r_1+\cdots+r_n}}{(1+tw^2)^{\frac{3}{2}+r_1+\cdots+r_n}}\prod_{k=1}^{n}\left(\frac{|\alpha(k)|}{k!}\right)^{r_k} \leq n!\,(2e^{2tw}t)^n. \tag{3.16}$$

where $(x)_n := x(x+1)\cdots(x+n-1)$ is the rising factorial. Finally, we substitute the bounds in equations (3.13), (3.14), (3.15), and (3.16) back into equation (3.9) and use the fact that the sum over $m_1+\ldots+m_4=m$ contains $\binom{m+3}{3}$ terms. □

Given some $\underline{i} = (i_1,\ldots,i_{2m-2})$ with $i_1+\ldots+i_{2m-2} = m(m-1)$, let $T_{d,m}(\underline{i};w)$ be the degree $\frac{3}{2}m(m-1)$ Taylor polynomial of $\prod_{k=1}^{2m-2} R(d-i_k,w)$.

Lemma 3.1.5. *Suppose* $w \in [0,\epsilon]$. *Then*

$$\prod_{k=1}^{2m-2}\frac{p(n+d-i_k)}{p(n)} = T_{d,m}(\underline{i};w) + E_{d,m}(\underline{i};w)w^{\frac{3}{2}m(m-1)+1} \tag{3.17}$$

where

$$|E_{d,m}(\underline{i};w)| \leq e^2(3d)^{10d-10}(4cd)^{\frac{3}{2}d^2} + 8m\cdot 6^{2m} \leq 2e^2(3d)^{10d-10}(4cd)^{\frac{3}{2}d^2}.$$

Proof. By Lemma 3.1.2, we can write

$$\prod_{k=1}^{2m-2} \frac{p(n+d-i_k)}{p(n)} = \prod_{k=1}^{2m-2} R(d-i_k,w)(1+U_k(w)) = \prod_{k=1}^{2m-2} R(d-i_k,w) + U(w),$$

where

$$\begin{aligned} |U(w)| &\leq \prod_{k=1}^{2m-2} R(d-i_k,w)\left(\left(1+\frac{2L(w)}{1-L(w)}\right)^{2m-2} - 1\right) \\ &\leq 2^{2m-2}\cdot(2m-2)\cdot 3^{2m-2}\cdot\frac{2L(w)}{1-L(w)} \leq 8m\cdot 6^{2m}\cdot e^{-1/2w}. \end{aligned}$$

Let $s = \frac{3}{2}m(m-1)+1$. Note also that we can easily bound

$$\frac{e^{-1/2w}}{w^s} \leq \frac{e^{-1/2\epsilon}}{\epsilon^s} \leq \exp\left(\frac{3}{2}d^2\left(2d\log(3d)+\frac{3}{2}d^2\log(50d)\right) - \frac{1}{2}(3d)^{12d}(50d)^{\frac{3}{2}d^2}\right) < 1.$$

Meanwhile, from Lemma 3.1.4 and the product rule, we know that

$$\begin{aligned} \frac{1}{s!}\left|\frac{d^s}{dw^s}\prod_{k=1}^{2m-2} R(d-i_k,w)\right| &\leq e^{(2m-2)g(\epsilon)}(4e^{2cd\epsilon}cd)^s \\ &\quad \times \sum_{n_1+\cdots+n_{2m-2}=\frac{3}{2}m(m-1)+1} \binom{n_1+3}{3}\cdots\binom{n_{2m-2}+3}{3}. \end{aligned}$$

The largest term in the sum on the right hand side occurs if each n_i is equal, which is in turn bounded by replacing each n_i with $m \geq \frac{\frac{3}{2}m(m-1)+1}{2m-2}$. Counting the number of terms, we see that the sum is bounded above by

$$\binom{\frac{3}{2}m(m-1)+2m-2}{2m-3}\cdot\binom{m+3}{3}^{2m-2} \leq (2m^2)^{2m-2}\cdot\left(\frac{3}{2}m^3\right)^{2m-2} = (3m)^{10m-10}.$$

This shows that

$$\left|\prod_{k=1}^{2m-2} R(d-i_k,w) - T_{d,m}(\underline{i};w)\right| \leq e^{(2m-2)g(\epsilon)}(4e^{2d\epsilon}cd)^s(3m)^{10m-10}\cdot w^s$$

$$\leq e^2(3d)^{10d-10}(4cd)^{\frac{3}{2}d^2}\cdot w^s. \qquad \square$$

In order to finish bounding the monomials in equation (3.6) we need the following result. We include the extra factor out front because of how it enters in equation (3.2).

Lemma 3.1.6. *Suppose* $0 \leq m \leq d$ *and* $i_1 + \ldots + i_{2m-2} = m(m-1)$ *for positive integers* i_k*. Then we have*

$$\left|\left(\frac{\sqrt{2}}{c}\right)^{m(m-1)}\prod_{k=1}^{2m-2}\binom{d}{i_k}\right| \leq \left(e^{\frac{4e}{c^2}}\right)^{d^2}. \tag{3.18}$$

Proof. The product $\prod_{k=1}^{2m-2}\binom{d}{i_k}$ is maximized when all i_k are equal (i.e. $i_k = \frac{m}{2}$). Using standard bounds on binomial coefficients, we therefore have $\prod_{k=1}^{2m-2}\binom{d}{i_k} \leq \left(\frac{2ed}{m}\right)^{m(m-1)}$. For $0 \leq m \leq d$, the function $\left(\frac{2\sqrt{2}ed}{cm}\right)^{m^2}$ achieves its maximum at $m = \frac{2\sqrt{2}ed}{c}$. Thus

$$\left|\left(\frac{\sqrt{2}}{c}\right)^{m(m-1)}\prod_{k=1}^{2m-2}\binom{d}{i_k}\right| \leq \left|\left(\frac{2\sqrt{2}ed}{cm}\right)^{m^2}\right| \leq \left(e^{\frac{4e}{c^2}}\right)^{d^2}.$$

$\square$

We now have bounds on the errors of our approximations of each monomial in (3.6). We also must bound the number of such terms that appear in this equation for $\mathcal{D}_{d,m}(n)$.

Lemma 3.1.7. *Suppose* $n > (3d)^{24d}(50d)^{3d^2}$ *and let* $A_{i_1,\ldots,i_{2m-2}}$ *be as in equation* (3.6). *Then*

$$\sum_{i_1,\ldots,i_{2m-2}} |A_{i_1,\ldots,i_{2m-2}}| \leq m!\,(m-1)^m 2^{m^2-2} \leq d^{2d} \cdot 2^{d^2}. \tag{3.19}$$

Proof. By the Newton-Girard identities, the power sums S_k in the matrix in (3.1) can be written as a sum of at most

$$k \sum_{r_1+\cdots+k\cdot r_k=k} \frac{(r_1+\cdots+r_k-1)!}{r_1!\cdots r_k!} \leq k2^{k-1}$$

monomials in the coefficients of our polynomial. The determinant of the matrix in equation (3.1) is made up of a sum of at most $m!$ monomials of the form

$$\prod_{\ell=1}^{m} S_{i_\ell} \qquad \text{where } i_1+\cdots+i_m = m(m-1).$$

Plugging in the elementary symmetric functions for each S_{i_ℓ} in this product and expanding will express each of these "S-monomials" as a sum of at most

$$\prod_{\ell=1}^{m} i_l 2^{i_\ell-1} \leq (m-1)^m 2^{m(m-2)}$$

monomials in the coefficients. To obtain $\mathcal{D}_{d,m}(n)$ from this, we must multiply by $(\frac{p(n+d)}{p(n)})^{2m-2}$. Since n is so large, we easily have $p(n+d)/p(n) \leq 2$, for example by using Lemma 3.1.2 with $s = 1$. Multiplying together the factors discussed above gives the result. □

The last ingredient we need to prove Theorem 1.3.3 is a lower bound on the Hankel determinants of Hermite polynomials.

Lemma 3.1.8. *For each $m \le d$, we have $\Delta_m(H_d(X)) \ge 1$.*

Proof. We know $\Delta_m(H_d(X)) = \sum_{i_1<\cdots<i_m} \prod_{a<b}(\lambda_{i_a} - \lambda_{i_b})^2$ so by the inequality of the arithmetic and geometric mean

$$\begin{aligned}\Delta_m(H_d(X)) \ge \binom{d}{m} \prod_{i_1<\cdots<i_m} \left(\prod_{a<b}(\lambda_{i_a} - \lambda_{i_b})^2\right)^{\frac{1}{\binom{d}{m}}} &= \binom{d}{m} \left(\prod_{j<k}(\lambda_j - \lambda_k)^{2\binom{d-2}{m-2}}\right)^{\frac{1}{\binom{d}{m}}} \\ &= \binom{d}{m} \Delta_d(H_d(X))^{\frac{m(m-1)}{d(d-1)}}.\end{aligned}$$

By Theorem 6.71 of [58], and the fact that $a_d(H_d(X)) = 2^d$, we have

$$\Delta_d(H_d(X)) = \frac{\mathrm{Disc}(H_d(X))}{2^{2d(d-1)}} = 2^{-\frac{d(d-1)}{2}} \prod_{\nu=1}^{d} \nu^\nu \ge 1,$$

so the result follows. □

Proving Theorem 1.3.3 is now just a matter of collecting and bounding all of the higher order terms from expanding $\mathcal{D}_{d,m}(n)$ in terms of w.

Proof of Theorem 1.3.3. Suppose $n > (3d)^{24d}(50d)^{3d^2}$ so that $w(n) \in [0, \epsilon]$. By

(3.2), we have

$$\frac{\mathcal{D}_{d,m}(n)}{w^{\frac{3}{2}m(m-1)}} = \sum_{i_1,\ldots,i_{2m-2}=m(m-1)} \frac{A_{i_1,\ldots,i_{2m-2}}}{w^{\frac{3}{2}m(m-1)}} \cdot \prod_{k=1}^{2m-2} \binom{d}{i_k} \left(T_{d,m}(\underline{i};w) + E_{d,m}(w) w^{\frac{3}{2}m(m-1)+1}\right)$$
$$= \left(\frac{c}{\sqrt{2}}\right)^{m(m-1)} \Delta_m(H_d(X)) + w \cdot \mathcal{E}_{d,m}(w),$$

where by Lemmas 3.1.5, 3.1.6, and 3.1.7

$$\left(\frac{\sqrt{2}}{c}\right)^{m(m-1)} \cdot |\mathcal{E}_{d,m}(w)| \cdot w \le d^{2d} \cdot 2^{d^2} \cdot \left(e^{\frac{4e}{c^2}}\right)^{d^2} \cdot 2e^2(3d)^{10d-10}(4cd)^{\frac{3}{2}d^2} \cdot w$$
$$< (3d)^{12d}(50d)^{\frac{3}{2}d^2} \cdot w \le 1.$$

Since $\Delta_m(H_d(X)) \ge 1$, it follows that $\mathcal{D}_{d,m}(n) > 0$ and therefore, $J_p^{d,n}(X)$ is hyperbolic. □

3.2 The Jensen-Pólya program for various L-functions

In this section we will the hyperbolicity of the Jensen polynomials associated to various different L-functions. We will present and prove an arbitrary precision asymptotic formula for the derivatives of good L-functions at their central values and then use this to prove Theorem 1.3.6. The corollaries on specific cases of good L-functions will also be discussed.

3.2.1 Asymptotics for $\Xi^{(n)}(0)$.

Let $L(s) = \sum_{n\geq 1} a(n)n^{-s}$ be a good Dirichlet series. We thus know that $L(s)$ has a completed form

$$\begin{aligned}\Lambda(s) &= N^{\frac{s}{2}} \prod_{j=1}^{J} \Gamma_{\mathbb{R}}(s) \prod_{m=1}^{M} \Gamma_{\mathbb{C}}(s) \cdot L(s) \\ &= N^{\frac{s}{2}} \int_0^\infty \left[f(t) - f(\infty)\right] t^s \frac{dt}{t},\end{aligned}$$

where $\Gamma_{\mathbb{R}}(s) := \pi^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}\right)$, $\Gamma_{\mathbb{C}}(s) := 2(2\pi)^{-s}\Gamma(s)$, and $\Gamma(s) := \int_0^\infty e^{-t}t^{s-1}dt$ is the usual gamma function. Because of the transformation properties of $f(t)$, we split the integral at $\frac{1}{\sqrt{N}}$ to arrive at

$$\Lambda(s) = \frac{(\epsilon s - s + k)f(\infty)}{s(s-k)} + \int_{\frac{1}{\sqrt{N}}}^\infty (f(t) - f(\infty)) \left(\epsilon N^{\frac{k-s}{2}} t^{k-s} + N^{\frac{s}{2}} t^s\right) \frac{dt}{t}. \tag{3.20}$$

We have the following expression for the derivatives of $\Lambda(s)$:

$$\begin{aligned}\Lambda^{(n)}(s) = &\frac{[(-1)^{n+1}(k-s)^{n+1} - \epsilon s^{n+1}]f(\infty)n!}{s^{n+1}(k-s)^{n+1}} \\ &+ \int_{\frac{1}{\sqrt{N}}}^\infty (f(t) - f(\infty)) \left(N^{\frac{s}{2}} t^s + (-1)^n \epsilon N^{\frac{k-s}{2}} t^{k-s}\right) \left(\frac{1}{2}\log(N) + \log(t)\right)^n \frac{dt}{t}.\end{aligned}$$

At $s = \frac{k}{2}$ and $z = 0$ we have

$$\Lambda^{(n)}\left(\frac{k}{2}\right) = \frac{2^{n+1} f(\infty) n! \left((-1)^{n+1} - \epsilon\right)}{k^{n+1}} + F(n) \tag{3.21}$$

and

$$\Xi^{(n)}(0) = \begin{cases} (-i)^n \frac{8\binom{n}{2}F(n-2)-k^2F(n)}{4} & \text{if } \Lambda(s) \text{ has a pole at } s=k \\ (-i)^n F(n) & \text{otherwise,} \end{cases} \tag{3.22}$$

where

$$F(n) = \frac{1}{2^n}\int_{\frac{1}{\sqrt{N}}}^{\infty} (f(t)-f(\infty))N^{\frac{k}{4}}t^{\frac{k}{2}-1}(1+(-1)^n\epsilon)\left(\log(N)+2\log(t)\right)^n\,dt \tag{3.23}$$

for all $n \geq 0$. The large asymptotics of $\Lambda^{(n)}\left(\frac{k}{2}\right)$ and $\Xi^{(n)}(0)$ are obtained from the following theorem.

Theorem 3.2.1. *The function $F(n)$ defined in equation (3.23) is given to all orders in n by the asymptotic expansion*

$$F(n) \sim \frac{\alpha(n_0)\sqrt{2\pi}N^{\frac{k}{4}}(1+(-1)^n\epsilon)}{2^n}\frac{L^{n+1}}{\sqrt{\left(1+\frac{L}{2}\right)n-\left(\frac{k}{2}-1\right)L^2}} \tag{3.24}$$
$$\times\, e^{\frac{k}{4}(L-\log(N))-\frac{2n}{L}-\frac{k}{2}+1}\left(1+\frac{b_1}{n}+\frac{b_2}{n^2}+\cdots\right) \qquad (n\to\infty),$$

where $L = L(n) \approx 2\log\left(\frac{n\sqrt{N}}{\log(n\sqrt{N})}\right)$ is the unique positive solution of the equation $n = \frac{1}{2}\left(\pi n_0 e^{\frac{L-\log(N)}{2}} - \frac{k}{2}+1\right)L$ and each coefficient b_k belongs to $\mathbb{Q}(L)$, the first value being $b_1 = \frac{2(31L^4+189L^3+542L^2+744L+496)}{3(L+2)^3}$.

Proof of Theorem 3.2.1. We approximate the integrand in equation (3.23) by the function

$$g(t) = \alpha(n_0)e^{-\pi n_0 t}N^{\frac{k}{4}}(1 + (-1)^n\epsilon)t^{\frac{k}{2}-1}(\log(N) + 2\log(t))^n.$$

From now on we let n be fixed and will omit it from our notations. We have that $t\frac{d}{dt}(\log(g(t))) = \frac{2n}{\log(N)+2\log(t)} - \pi n_0 t + \frac{k}{2} - 1$, so $g(t)$ assumes its unique maximum at $t = a$ where a is the solution in $\left(\frac{1}{\sqrt{N}}, \infty\right)$ of

$$n = \frac{1}{2}\left(\pi n_0 a - \frac{k}{2} + 1\right)(\log(N) + 2\log(a)).$$

For convenience, we define $L = \log(N) + 2\log(a)$ so we have

$$n = \frac{1}{2}\left(\pi n_0 e^{\frac{L-\log(N)}{2}} - \frac{k}{2} + 1\right)L.$$

We can then use Lambert's W function to asymptotically solve this equation. Lambert's W function is defined as the solution to $z = W(z)e^{W(z)}$. It has the nice property that $Y = Xe^X$ if and only if $X = W(Y)$. If we take a branch cut to restrict W to be real valued, then we have that the principal branch has a Taylor series around 0 given by $W(x) = \sum_{n=1}^{\infty} \frac{(-n)^{n-1}}{n!}x^n$. For large x, $W(x)$ is asymptotic to $W(x) = \ln(x) - \ln\ln(x) + \frac{\ln\ln(x)(\ln\ln(x)-2)}{\ln^2(x)} + O\left(\left(\frac{\ln\ln(x)}{\ln(x)}\right)^3\right)$ [15]. Therefore, we have $L \approx 2\log\left(\frac{n\sqrt{N}}{\log(n\sqrt{N})}\right)$. We now follow [27] and apply the saddle point method. The Taylor expansion of $g(t)$ around $t = a$ is given by

$$\frac{g((1+\lambda)a)}{g(a)} = \left(1 + \frac{2\log(1+\lambda)}{\log(N) + 2\log(a)}\right)^n (1+\lambda)^{\frac{k}{2}-1}e^{-\pi n_0 \lambda a} = e^{-\frac{C\lambda^2}{2}}\left(1 + A_3\lambda^3 + A_4\lambda^4 + \cdots\right),$$

where $C = n\left(\frac{\varepsilon}{2}+\varepsilon^2\right)+\frac{k}{8}-\frac{1}{4}$, $\varepsilon = \frac{1}{\log(N)+2\log(a)} = L^{-1}$, and the A_i are polynomials of degree $\lfloor i/3 \rfloor$ in n with coefficients in $\mathbb{Q}[\varepsilon]$. This expansion was found by expanding $\log(g((1+\lambda)a)) - \log(g(a))$ in λ. The linear term vanises by choice of a and the quadratic term is $-\frac{C\lambda^2}{2}$. The coefficients of the higher powers of λ are all linear expressions in n with coefficients in $\mathbb{Q}[\varepsilon]$. Exponentiating this expansions gives our expression for $g((1+\lambda)a)/g(a)$. The important behavior is that the dominant term of each A_i comes primarily from the exponential of the cubic term of the logarithmic expansion. The first few A_i are

$$\begin{aligned}
A_3 &= 2n\left(\frac{\varepsilon}{3}+\varepsilon^2+\frac{4\varepsilon^3}{3}\right)+\frac{k}{6}-\frac{1}{3},\\
A_4 &= -n\left(\frac{\varepsilon}{2}+\frac{11\varepsilon^2}{6}+4\varepsilon^3+4\varepsilon^4\right)-\frac{k}{8}+\frac{1}{4},\\
A_5 &= n\left(\frac{2\varepsilon}{5}+\frac{5\varepsilon^2}{3}+\frac{14\varepsilon^3}{3}+8\varepsilon^4+\frac{32\varepsilon^5}{5}\right)+\frac{k}{10}-\frac{1}{5},\\
A_6 &= n^2\left(\frac{2\varepsilon^2}{9}+\frac{4\varepsilon^3}{3}+\frac{34\varepsilon^4}{9}+\frac{16\varepsilon^5}{3}+\frac{32\varepsilon^6}{9}\right)+\frac{k^2-7k+10}{36}\\
&\quad +n\left(\frac{(10k-50)\varepsilon}{90}+\frac{(30k-197)\varepsilon^2}{90}+\frac{(40k-530)\varepsilon^3}{90}-\frac{34\varepsilon^4}{3}-16\varepsilon^5-\frac{32\varepsilon^6}{3}\right).
\end{aligned}$$

We plug in $t = (1+\lambda)a$ to arrive at the asymptotic expansion

$$\begin{aligned}
\frac{1}{2^n}\int_{\frac{1}{\sqrt{N}}}^{\infty} g(t)dt &= \frac{ag(a)}{2^n}\int_{-1+\frac{1}{a\sqrt{N}}}^{\infty} e^{-\frac{C\lambda^2}{2}}\left(1+A_3\lambda^3+A_4\lambda^4+\cdots\right)d\lambda\\
&= \frac{ag(a)}{2^n}\sqrt{\frac{2\pi}{C}}\left(1+\frac{3A_4}{C^2}+\frac{15A_6}{C^3}+\cdots+\frac{(2i-1)!!\,A_{2i}}{C^i}+\cdots\right).
\end{aligned}$$

This expression and the one in Theorem 3.2.1 are interpreted as asymptotic expansions. These series do not converge for a fixed n, but we can truncate the approximation at $O(n^{-A})$ for some $A > 0$, and as $n \to +\infty$ this approximation becomes true to the given precision. We substitute the formulas for C and A_i in terms of n in order to obtain the statement in the theorem. We also replace $F(n)$ by the integral over $g(t)$ with only the A_{2i} with $i \leq 3k$ contributing to b_k. The same asymptotic formula will hold with this replacement because the ratio of $g(t)$ and the integrand of $F(n)$ is equal to $1 + O(n^{-K})$ for any $K > 0$ for t near a.

□

3.2.2 Proof of Theorem 1.3.6

Our goal is to show that $\{\gamma(n)\}$ satisfies the growth conditions of Definition 1.3.4. Recall from Section 1 that

$$\gamma(n) = \begin{cases} (-1)^n \frac{n!}{(2n)!} \Xi^{(2n)}(0) & \text{if } \epsilon = 1 \\ i^{2n+1} \frac{n!}{(2n+1)!} \Xi^{(2n+1)}(0) & \text{if } \epsilon = -1, \end{cases} \tag{3.25}$$

where $\gamma(n)$ are the Taylor coefficients of $\Xi_1(x)$. Therefore, if we have

$$\widehat{F}(n) = \frac{\alpha(n_0)\sqrt{2\pi} N^{\frac{k}{4}} (1 + (-1)^n \epsilon) L^{n+1}}{2^n \sqrt{\left(1 + \frac{L}{2}\right) n - \left(\frac{k}{2} - 1\right) L^2}} e^{\frac{k}{4}(L - \log(N)) - \frac{2n}{L} - \frac{k}{2} + 1} \left(1 + \frac{b_1}{n}\right) \tag{3.26}$$

and

$$\widehat{\Xi}^{(n)}(0) = \left\{ \begin{matrix} (-i)^n 2\binom{n}{2}\widehat{F}(n-2) & \text{if } \Lambda(s) \text{ has a pole} \\ (-i)^n\widehat{F}(n) & \text{otherwise} \end{matrix} \right\} = \Xi^{(n)}(0)\cdot\left(1+O\left(\frac{1}{n^{2-\varepsilon}}\right)\right)$$

as in the example above, then

$$\widehat{\gamma}(n) = \left\{ \begin{matrix} \frac{n!}{(2n-2)!}\widehat{F}(2n-2) & \text{if } \Lambda(s) \text{ has a pole and } \epsilon = 1 \\ \frac{n!}{(2n-1)!}\widehat{F}(2n-1) & \text{if } \Lambda(s) \text{ has a pole and } \epsilon = -1 \\ \frac{n!}{(2n)!}\widehat{F}(2n) & \text{if } \Lambda(s) \text{ does not have a pole and } \epsilon = 1 \\ \frac{n!}{(2n+1)!}\widehat{F}(2n+1) & \text{if } \Lambda(s) \text{ does not have a pole and } \epsilon = -1 \end{matrix} \right\} = \gamma(n)\cdot\left(1+O\left(\frac{1}{n^{2-\varepsilon}}\right)\right). \tag{3.27}$$

We will show that the $\gamma(n) = \frac{n!}{m!}\widehat{F}(m)\cdot\left(1+O\left(\frac{1}{n^{2-\varepsilon}}\right)\right)$ for $m = 2n-2, 2n-1, 2n,$ or $2n+1$ form a Hermite-Jensen sequence. Recall that

$$b_1 = \frac{2(31L^4 + 189L^3 + 542L^2 + 744L + 496)}{3(L+2)^3}.$$

Using Stirling's approximation $r! = \sqrt{2\pi r}\left(\frac{r}{e}\right)^r \cdot \left(1+\frac{1}{12r}\right)\cdot(1+O(1/r^2))$, we have

$$\gamma(n) = \frac{\alpha(n_0)N^{\frac{k}{4}}(1+(-1)^m\epsilon)e^{m-n}n^{n+\frac{1}{2}}\left(1+\frac{1}{12n}\right)L(m)^m}{2^m m^{m+\frac{1}{2}}\left(1+\frac{1}{12m}\right)}\cdot\sqrt{\frac{2\pi}{C(m)}} \tag{3.28}$$
$$\times \exp\left(\frac{k}{4}(L(m)-\log(N)) - \frac{2m}{L(m)} - \frac{k}{2} + 1\right)\left(1+\frac{b_1(m)}{m}\right)\cdot\left(1+O\left(\frac{1}{n^{2-\varepsilon}}\right)\right).$$

Recall that $L(m)$ and $b_1(m)$ are given in Theorem 3.2.1 and $C(m) = m\left(\frac{L(m)^{-1}}{2} + L(m)^{-2}\right) + \frac{k}{8} - \frac{1}{4}$. $L(m)$ can be viewed as a holomorphic and non-vanishing function for $\mathrm{Re}(m) > 0$, so we have a Taylor expansion in j for the ratio $L(m+2j)/L(m)$ given by

$$\mathcal{L}(j;m) := \frac{L(m+2j)}{L(m)} = 1 + \sum_{r\geq 1} \frac{\ell_r(m)}{r!} \cdot j^r \tag{3.29}$$

which converges when $|j| < \frac{m}{2}$, so we will assume this throughout the proof. If we let $J = \frac{\lambda m}{2}$ for some $-1 < \lambda < 1$, then the asymptotic $L(m) \approx \log\left(\frac{\sqrt{N}m}{\log(\sqrt{N}m)}\right)$ gives the limit

$$\lim_{m\to\infty} \mathcal{L}(J;m) = \lim_{m\to\infty} \frac{L(m(\lambda+1))}{L(m)} = 1.$$

This implies that $\ell_r(m) = o\left(\left(\frac{2}{m}\right)^r\right)$. If we expand

$$m + 2j = \frac{L(m)\cdot\mathcal{L}(j;m)}{2}\left(\pi n_0 e^{\frac{L(m)\cdot\mathcal{L}(j;m)-\log(N)}{2}} - \frac{k}{2} + 1\right)$$

in j then we find $\ell_1(m) = \frac{8}{4m(L/2+1)+L^2(k/2-1)} = \frac{2}{C\cdot L^2}$ and $\ell_2(m) = \frac{-(L/2+2)(m+kL/4-L/2)}{C^3\cdot L^5}$, where $L = L(m)$ and $C = C(m)$. We will also define

$$\mathcal{C}(j;m) := \frac{C(m+2j)}{C(m)} = 1 + \sum_{r\geq 1} \frac{c_r(m)}{r!} \cdot j^r,$$

and

$$\mathcal{B}(j;m) := \frac{1 + \frac{b_1(m+2j)}{m+2j}}{1 + \frac{b_1(m)}{m}} = 1 + \sum_{r\geq 1} \frac{\beta_r(m)}{r!} \cdot j^r.$$

We have the limits

$$\lim_{m\to\infty} \mathcal{C}(J;m) = 1+\lambda \quad \text{and} \quad \lim_{m\to\infty} \frac{m}{2}\left(\mathcal{B}(J;m)-1\right) = 0,$$

which imply $c_r(m) = o\left(\left(\frac{2}{m}\right)^r\right)$ and $\beta_r(m) = o\left(\left(\frac{2}{m}\right)^{r+1}\right)$. Using the expansion for $\mathcal{L}(j;m)$ and the expression for $\ell_1(m)$ we can find that $c_1(m) = \frac{L+2}{C\cdot L^2} - \frac{m(L+4)}{C^2\cdot L^4}$. Define $R_\gamma(j;m) := \frac{\widehat{\gamma}(n+j)}{\widehat{\gamma}(n)}$, then after some manipulations we have

$$R_\gamma(j;m) = \frac{e^j n^j L(m)^{2j}}{2^{2j} m^{2j}} \cdot \frac{\left(\frac{n+j}{n}\right)^{n+j+\frac{1}{2}}}{\left(\frac{m+2j}{m}\right)^{m+2j+\frac{1}{2}}} \frac{\left(1+\frac{1}{12(n+j)}\right)\left(1+\frac{1}{12m}\right)}{\left(1+\frac{1}{12n}\right)\left(1+\frac{1}{12(m+2j)}\right)} \cdot \frac{\mathcal{L}(j;m)^{m+2j}}{\sqrt{\mathcal{C}(j;m)}}$$
$$\times \exp\left(\frac{kL(m)}{4}(\mathcal{L}(j;m)-1) - \frac{2(m+2j)}{\mathcal{L}(j;m)L(m)} + \frac{2m}{L(m)}\right) \cdot \mathcal{B}(j;m).$$

By equation (3.28), for j fixed and as $m \to \infty$ (and thus $n \to \infty$), we have

$$\frac{\gamma(n+j)}{\gamma(n)} = R_\gamma(j;m) \cdot \left(1 + O\left(\frac{1}{n^{2-\varepsilon}}\right)\right). \tag{3.30}$$

Notice that the first factor in $R(j;m)$ is the jth power of $\frac{enL(m)^2}{2^2 m^2}$. This factor will essentially be $e^{A(n)}$. We will now look at the expansion

$$\log R(j;m) =: \sum_{r\geq 1} g_r(m) j^r.$$

We again let $J = \frac{m\lambda}{2}$ for $-1 < \lambda < 1$, then we have

$$\lim_{m\to\infty} \frac{2\log R(J;m)}{m} = \sum_{r\geq 1} g_r(m) \left(\frac{m}{2}\right)^{r-1} \lambda^r = -(\lambda+1)\log(\lambda+1),$$

which tells us that $g_r(m) = O\left(\left(\frac{m}{2}\right)^{1-r}\right)$. We can use our previous expansions and the formula for $R_\gamma(j; m)$ to find

$$g_1(m) = \log\left(\frac{nL^2}{4m^2}\right) + m \cdot \ell_1(m)\left(\frac{L+2}{L}\right) - \frac{4}{L} + \frac{k \cdot \ell_1(m) \cdot L}{4} - \frac{c_1(m)}{2} + O\left(\frac{1}{n^{2-\varepsilon}}\right),$$
$$g_2(m) = -\frac{2}{m} + (4\ell_1(m) + m \cdot \ell_2(m))\left(\frac{L+2}{2L}\right) - m \cdot \ell_1(m)^2\left(\frac{L+4}{2L}\right) + O\left(\frac{1}{n^{2-\varepsilon}}\right).$$

We can use the formulas for $\ell_1(m)$ and $\ell_2(m)$ to simplify these to

$$g_1(m) = \log\left(\frac{nL^2}{4m^2}\right) + \frac{(k-1)L-2}{2C \cdot L^2} + \frac{m(L+4)}{2C^2 \cdot L^4} + O\left(\frac{1}{n^{2-\varepsilon}}\right), \tag{3.31}$$
$$g_2(m) = -\frac{2}{m} + \frac{4}{C \cdot L^2} + O\left(\frac{1}{n^{2-\varepsilon}}\right). \tag{3.32}$$

We now let

$$A(n) = \log\left(\frac{nL^2}{4m^2}\right) + \frac{(k-1)L-2}{2C \cdot L^2} + \frac{m(L+4)}{2C^2 \cdot L^4} \tag{3.33}$$
$$\delta(n) = \sqrt{\frac{2}{m} - \frac{4}{C \cdot L^2}}. \tag{3.34}$$

These functions satisfy the conditions of Definition 1.3.4 for the sequence $\{\gamma(n)\}$. The fact that $\delta(n) \to 0$ follows from the asymptotics given above and the precision of $O\left(\frac{1}{n^{2-\varepsilon}}\right)$ satisfies the necessary growth conditions given in Definition 1.3.4.

3.2.3 Dirichlet L-functions

Proof of Corollary 1.3.7

Let χ be a Dirichlet character modulo $N > 1$. Then we define the Dirichlet L-function as

$$L(\chi, s) := \sum_{n \geq 1} \frac{\chi(n)}{n^s} \tag{3.35}$$

for $\mathrm{Re}(s) > 1$. If we let χ be the trivial character, then our L-function is the Riemann zeta function. This case was handled in [27]. Next, recall the twisted theta function

$$\theta_\chi(z) = \chi(0) + 2 \sum_{n \geq 1} \chi(n) n^\nu e^{\pi i n^2 z} \tag{3.36}$$

where $\nu = 0$ if χ is even and $\nu = 1$ if χ is odd. The twisted theta function satisfies the functional equation

$$\theta_\chi(z) = \frac{\tau(\chi)}{i^\nu \sqrt{N} (-iNz)^{\frac{1}{2}+\nu}} \theta_{\bar{\chi}} \left(\frac{1}{N^2 z} \right) \tag{3.37}$$

where $\tau(\chi)$ is a Gauss sum and $\bar{\chi}$ is the dual character. We will focus on real primitive self-dual characters so we have $\chi = \bar{\chi}$ and $\tau(\chi) = i^\nu \sqrt{N}$. Define the completed Dirichlet L-function by

$$\begin{aligned} \Lambda(\chi, s) &:= \left(\frac{N}{\pi} \right)^{\frac{s+\nu}{2}} \Gamma\left(\frac{s+\nu}{2} \right) L(\chi, s) \\ &= \frac{1}{2} N^{\frac{s+\nu}{2}} \int_0^\infty \theta_\chi(iy) y^{\frac{s+\nu}{2}} \frac{dy}{y}. \end{aligned} \tag{3.38}$$

Using equation 3.37 and the fact that χ is a real primitive self-dual character, we have the following functional equation

$$\Lambda(\chi, s) = \Lambda(\chi, 1-s). \tag{3.39}$$

The completed Dirichlet L-function has the required integral representation, functional equation, and real coefficients so it is good.

Derivatives at central values and Dirichlet Jensen polynomials

We want to study the derivatives of $\Lambda(\chi, s)$ which are given by

$$\Lambda^{(n)}(\chi, s) = \frac{1}{2^{2n+1}} \int_{\frac{1}{N}}^{\infty} \theta_{\chi}(iy) \left((Ny)^{\frac{s+\nu}{2}} + (-1)^n (Ny)^{\frac{1-s+\nu}{2}} \right) \left(\log(N^2) + 2\log(y) \right)^n \frac{dy}{y}. \tag{3.40}$$

At the central value $s = \frac{1}{2}$ we have

$$\Lambda^{(n)}\left(\chi, \frac{1}{2}\right) = \frac{1}{2^{2n+1}} \int_{\frac{1}{N}}^{\infty} \theta_{\chi}(iy)(Ny)^{\frac{1}{4}+\frac{\nu}{2}}(1+(-1)^n)(\log(N^2)+2\log(y))^n \frac{dy}{y}. \tag{3.41}$$

Because the Dirichlet L-functions fit into our framework we have the following theorem which gives an arbitrary precision asymptotic formula for these derivatives.

Theorem 3.2.2. *Assume the notation above. The large n asymptotics for*

$\Lambda^{(n)}\left(\chi,\frac{1}{2}\right)$ and $\Xi^{(n)}(\chi,0)$ are given to all orders by the asymptotic expansion

$$F(n) \sim \frac{\sqrt{2\pi}N^{\frac{1}{4}+\frac{\nu}{2}}(1+(-1)^n)}{2^{2n}} \frac{L^{n+1}}{\sqrt{4n\left(1+\frac{L}{2}\right)-\left(\frac{3}{4}-\frac{\nu}{2}\right)L^2}} \tag{3.42}$$
$$\times e^{\left(\frac{1}{8}+\frac{\nu}{4}\right)(L-\log(N^2))-\frac{2n}{L}+\frac{3}{4}-\frac{\nu}{2}}\left(1+\frac{b_1}{n}+\frac{b_2}{n^2}+\cdots\right) \qquad (n\to\infty),$$

where $L = L(n) \approx 2\log\left(\frac{nN}{\log(nN)}\right)$ is the unique positive solution to $n = \frac{1}{2}\left(\pi e^{\frac{L-\log(N^2)}{2}}+\frac{3}{4}-\frac{\nu}{2}\right)L$ and each coefficient b_k belongs to $\mathbb{Q}(L)$, the first value being $b_1 = \frac{L^4+9L^3+32L^2+24L+16}{24(L+2)^3}$.

Example. *Let χ_4 be the odd Dirichlet character of modulus 4. Using the two-term approximation $\widehat{F}(n)$ given in equation (3.26) we give some approximations $\widehat{\gamma}_{\chi_4}(n)$ in the table below.*

n	$\widehat{\gamma}_{\chi_4}(n)$	$\gamma_{\chi_4}(n)$	$\gamma_{\chi_4}(n)/\widehat{\gamma}_{\chi_4}(n)$
10	$\approx 8.6123842782 \times 10^{-14}$	$\approx 8.5921206983 \times 10^{-14}$	≈ 0.997647158
100	$\approx 1.0054943805 \times 10^{-174}$	$\approx 1.0057597216 \times 10^{-174}$	≈ 0.9997361785
1000	$\approx 1.7838444188 \times 10^{-2350}$	$\approx 1.7838866878 \times 10^{-2350}$	≈ 0.9999763051
10000	$\approx 1.7271165350 \times 10^{-30650}$	$\approx 1.7271200653 \times 10^{-30650}$	≈ 0.9999979560
100000	$\approx 8.1291521235 \times 10^{-384416}$	$\approx 8.1291531304 \times 10^{-384416}$	≈ 0.9999998761

In the previous section we showed that the Dirichlet L-function $L(\chi,s)$ is good. Dirichlet L-functions have a pole at $s = 1$ if χ is principal so we define

$$\Xi(\chi, z) := \begin{cases} \left(-z^2 - \frac{1}{4}\right) \Lambda\left(\frac{1}{2} - iz\right) & \text{if } \chi \text{ is principal} \\ \Lambda\left(\chi, \frac{1}{2} - iz\right) & \text{else} \end{cases} \tag{3.43}$$

and

$$\Xi_1(\chi, x) := \Xi(\chi, i\sqrt{x}) = \sum_{n \geq 0} \frac{\gamma_\chi(n)}{n!} x^n \tag{3.44}$$

where

$$\gamma_\chi(n) = (-1)^n \frac{n!}{(2n)!} \cdot \Xi^{(2n)}(\chi, 0). \tag{3.45}$$

By Theorem 1.3.6 or by using the asymptotic expansion above we know that if $d \geq 1$, then $J_{\gamma_\chi}^{d,n}(X)$ is hyperbolic with at most finitely many exceptions n.

3.2.4 Modular L-functions

Proof of Corollary 1.3.8

Let $f \in S_k(\Gamma_0(N))$ be an even weight newform with real coefficients and write $f(z) = \sum_{n \geq 1} a(n) e^{2\pi i n z}$. Assume that f is normalized so that $a(1) = 1$. We focus newforms with trivial character. Define the L-function associated to f by

$$L(f, s) := \sum_{n \geq 1} \frac{a(n)}{n^s} \tag{3.46}$$

for $\mathrm{Re}(s) > 1 + \frac{k}{2}$. Define the completed modular L-function by

$$\Lambda(f, s) := N^{\frac{s}{2}} (2\pi)^{-s} \Gamma(s) L(f, s). \tag{3.47}$$

We have the transformation property

$$f\left(\frac{i}{Ny}\right) = i^k \epsilon_f N^{\frac{k}{2}} y^k f(iy), \tag{3.48}$$

which gives rise to the functional equation

$$\Lambda(f,s) = i^k \epsilon_f \Lambda(f, k-s), \tag{3.49}$$

where $\epsilon_f \in \{\pm 1\}$ is the eigenvalue of f under the Atkin-Lehner involution. The completed modular L-function $\Lambda(f,s)$ has the required integral representation, the modular properties of $f(z)$ gives a functional equation, and the coefficients are real so $L(f,s)$ is good.

Derivatives at central values and modular Jensen polynomials

Similarly to the Dirichlet L-function case, the nth derivative takes the form

$$\Lambda^{(n)}(f,s) = \frac{1}{2^n}\int_{\frac{1}{\sqrt{N}}}^{\infty} f(iy)\left(N^{\frac{s}{2}}y^s + (-1)^n i^k \epsilon_f N^{\frac{k-s}{2}} y^{k-s}\right)(\ln(N) + 2\ln(y))^n \, \frac{dy}{y}. \tag{3.50}$$

At the central value $s = \frac{k}{2}$ we have

$$\Lambda^{(n)}\left(f, \frac{k}{2}\right) = \frac{1}{2^n}\int_{\frac{1}{\sqrt{N}}}^{\infty} f(iy) N^{\frac{k}{4}} y^{\frac{k}{2}-1}(1 + (-1)^n i^k \epsilon_f)(\ln(N) + 2\ln(y))^n \, dy. \tag{3.51}$$

The following theorem gives an arbitrary precision asymptotic formula for these derivatives at central values.

Theorem 3.2.3. *Assume the notation above. Large n asymptotics for $\Lambda^{(n)}\left(f, \frac{k}{2}\right)$ and $\Xi^{(n)}(f,0)$ is given to all orders by the asymptotic expansion*

$$F(n) \sim \frac{\sqrt{2\pi} N^{\frac{k}{4}}(1+(-1)^n i^k \epsilon_f)}{2^{n+1}} \frac{L^{n+1}}{\sqrt{\left(1+\frac{L}{2}\right)n - \left(\frac{k}{2}-1\right)L^2}} \times e^{\frac{k}{4}(L-\log(N)) - \frac{2n}{L} - \frac{k}{2}+1}\left(1+\frac{b_1}{n}+\frac{b_2}{n^2}+\cdots\right) \qquad (n\to\infty), \tag{3.52}$$

where $L = L(n) \approx 2\log\left(\frac{n\sqrt{N}}{\log(n\sqrt{N})}\right)$ is the unique solution of the equation $n = \frac{1}{2}\left(\pi e^{\frac{L-\log(N)}{2}} - \frac{k}{2}+1\right)L$ and each coefficient b_k belongs to $\mathbb{Q}(L)$, the first value being $b_1 = \frac{L^4+9L^3+32L^2+24L+16}{24(L+2)^3}$.

We have showed that the modular L-function $L(f,s)$ and does not have a pole so define

$$\Xi(f,z) := \Lambda\left(f, \frac{k}{2} - iz\right). \tag{3.53}$$

Depending on the sign of the functional equation we define the Taylor coefficients by

$$\Xi_1(f,x) = \sum_{n\geq 0} \frac{\gamma_f(n)}{n!} x^n = \begin{cases} \Xi(i\sqrt{x}) & \text{if } i^k\epsilon_f = 1 \\ \frac{\Xi(i\sqrt{x})}{\sqrt{x}} & \text{if } i^k\epsilon_f = -1, \end{cases} \tag{3.54}$$

where

$$\gamma_f(n) = \begin{cases} (-1)^n \frac{n!}{(2n)!} \cdot \Xi^{(2n)}(0) & \text{if } i^k\epsilon_f = 1 \\ i^{2n+1} \frac{n!}{(2n+1)!} \cdot \Xi^{(2n+1)}(0) & \text{if } i^k\epsilon_f = -1. \end{cases} \tag{3.55}$$

By Theorem 1.3.6 or from the asymptotic expansion above we have that if

$d \geq 1$, then$J^{d,n}_{\gamma_f}(X)$ is hyperbolic with at most finitely many exceptions n.

3.2.5 Dedekind zeta-functions

Proof of Corollary 1.3.9

The Dedekind zeta-function case will require some setup and notation. We will mostly follow the notation in [41]. Let K be a number field of degree j and $\mathcal{O}_K$ its ring of integers. Denote the embeddings by $\sigma_1, \ldots, \sigma_{r_1}, \rho_1, \overline{\rho}_1, \ldots, \rho_{r_2}, \overline{\rho}_{r_2}$ where there are r_1 real embeddings and r_2 pairs of complex embeddings so that $r_1 + 2r_2 = j$. Denote the class group of K by $Cl(K)$. Let $\mathbf{C} = \prod_\tau \mathbb{C}$ and $\mathbf{R} = [\prod_\tau \mathbb{C}]^+ = \{z \in \mathbf{C} : z = \overline{z}\}$ be the Minkowski space of K where $\overline{z} = \overline{(z_\tau)} = (\overline{z}_{\overline{\tau}})$ is the usual complex conjugation and τ runs over the j embeddings. We define the trace and norm by

$$Tr(z) = \sum_\tau z_\tau \qquad N(z) = \prod_\tau z_\tau, \tag{3.56}$$

and have a Hermitian scalar product given by

$$\langle x, y \rangle = \sum_\tau x_\tau \overline{y}_\tau. \tag{3.57}$$

We will also require the spaces

$$\mathbf{R}_\pm = \left[\prod_\tau \mathbb{R}\right]^+ = \{x \in \mathbf{R} : x_\tau = x_{\overline{\tau}}\} \tag{3.58}$$

and

$$\mathbf{R}_+^* = \left[\prod_\tau \mathbb{R}_+^*\right]^+ = \{x \in \mathbf{R}_\pm : \forall\tau\ x_\tau > 0\} \tag{3.59}$$

in order to define the two homomorphisms

$$|\ | : \mathbf{R}^* \to \mathbf{R}_+^* \qquad x = (x_\tau) \mapsto |x| = (|x_\tau|) \tag{3.60}$$

$$\log : \mathbf{R}_+^* \xrightarrow{\sim} \mathbf{R}_\pm \qquad x = (x_\tau) \mapsto \log x = (\log x_\tau). \tag{3.61}$$

Let $\mathfrak{p} = \{\sigma, \overline{\sigma}\}$ denote a conjugacy class of embeddings (so $\mathfrak{p}$ has one or two elements depending on whether the embedding is real or complex) and observe that there is an isomorphism between $\mathbf{R}_+^*$ and $\prod_\mathfrak{p} \mathbb{R}_+^*$. We now have a Haar measure, which we denote by $\frac{dy}{y}$, that corresponds to the product measure $\prod_\mathfrak{p} \frac{dt}{t}$ where $\frac{dt}{t}$ is the usual Haar measure on $\mathbb{R}_+^*$. We can now define a suitable generalization of the gamma function by

$$\Gamma_K(s) = 2^{(1-2s)r_2}\Gamma(s)^{r_1}\Gamma(2s)^{r_2} = \int_{\mathbf{R}_+^*} N(e^{-y}y^s)\frac{dy}{y}. \tag{3.62}$$

The Dedekind zeta-function for K is given by

$$\zeta_K(s) = \sum_{\substack{\mathfrak{a}\subset\mathcal{O}_K \\ \text{integral}}} N(\mathfrak{a})^{-s} \tag{3.63}$$

for $\mathrm{Re}(s) > 1$, where $N(\mathfrak{a}) = [\mathcal{O}_K : \mathfrak{a}]$ is the norm of the ideal $\mathfrak{a}$. For each

$B \in Cl(K)$ we define the partial zeta function by

$$\zeta_B(s) = \sum_{\substack{\mathfrak{a} \in B \\ \text{integral}}} N(\mathfrak{a})^{-s}. \tag{3.64}$$

We therefore have

$$\zeta_K(s) = \sum_{B \in Cl(K)} \zeta_B(s). \tag{3.65}$$

We define the completed partial Dedekind zeta-function by

$$\begin{aligned} \Lambda(B, s) &= |d_K|^{\frac{s}{2}} \pi^{-\frac{js}{2}} \Gamma_K\left(\frac{s}{2}\right) \zeta_B(s) \\ &= \int_{\mathbf{R}_+^*} g(iy) N(y)^s \frac{dy}{y} \end{aligned} \tag{3.66}$$

where d_K is the discriminant of K and g is some theta function that we will not specify now. The image of the unit group $\mathcal{O}_K^*$ under the mapping $|\ |\colon \mathbf{R}^* \to \mathbf{R}_+^*$, which we will denote by $|\mathcal{O}_K^*|$, is contained in the norm-one hypersurface

$$S = \{x \in \mathbf{R}_+^* : N(x) = 1\}. \tag{3.67}$$

We obtain a direct decomposition $\mathbf{R}_+^* = S \times \mathbb{R}_+^*$ by writing

$$y = xt^{\frac{1}{j}}, \qquad x = \frac{y}{N(y)^{\frac{1}{j}}}, \qquad t = N(y)$$

for any $y \in \mathbf{R}_+^*$. We will need to choose a fundamental domain F for the action of the group

$$|\mathcal{O}_K^*|^2 = \{|\epsilon|^2 \colon \epsilon \in \mathcal{O}_K^*\}$$

on S. The log map $\log : \mathbf{R}_+^* \to \mathbf{R}_\pm$ takes S to the trace-zero space

$$H = \{x \in \mathbf{R}_\pm : Tr(x) = 0\}$$

and by Dirichlet's unit theorem the group $|\mathcal{O}_K^*|$ is taken to a complete lattice G in H. We may choose F to be the preimage of any fundamental mesh of the lattice $2G$. Now using this decomposition we have that

$$\Lambda(B, s) = \int_0^\infty (f(\mathfrak{a}, t) - f(\mathfrak{a}, \infty)) t^{\frac{s}{2}} \frac{dt}{t} \tag{3.68}$$

where B is the class of $\mathfrak{a}^{-1}$ and

$$f(\mathfrak{a}, t) = f_F(\mathfrak{a}, t) = \frac{1}{w_K} \int_F \theta(\mathfrak{a}, ixt^{\frac{1}{j}}) d^*x. \tag{3.69}$$

In the above equation w_K is the number of roots of unity in K, d^*x is the appropriate Haar measure such that $d^*x \times \frac{dt}{t} = \frac{dy}{y}$, and the theta function is defined by

$$\theta(\mathfrak{a}, z) = \sum_{a \in \mathfrak{a}} e^{\pi i d_{\mathfrak{a}}^{-\frac{1}{j}} \langle az, a \rangle} \tag{3.70}$$

where $d_{\mathfrak{a}} = |N(\mathfrak{a})|^2 |d_K|$ is the absolute value of the discriminant of $\mathfrak{a}$. Using the properties of the theta function it is not difficult to show

$$f_F\left(\mathfrak{a}, \frac{1}{t}\right) = t^{\frac{1}{2}} f_{F^{-1}}((\mathfrak{a}\mathfrak{d}_K)^{-1}, t) \tag{3.71}$$

and

$$f_F(\mathfrak{a}, \infty) = f(\infty) = \frac{2^{r_1+r_2-1}R(k)}{w_K}, \tag{3.72}$$

where F^{-1} is again a fundamental domain, $\mathfrak{d}_K$ is the different ideal of K, and $R(K)$ is the regulator of K. Note that $(\mathfrak{a}\mathfrak{d}_K)^{-1}$ is the dual lattice of $\mathfrak{a}$ and that $f(\infty)$ does not depend on the fundamental domain or ideal choice so we will supress notation whenever possible. We now define the completed Dedekind zeta-function by

$$\begin{aligned}\Lambda(K, s) &= \sum_{B \in Cl(K)} \Lambda(B, s) = |d_K|^{\frac{s}{2}} \pi^{-\frac{js}{2}} \Gamma_K\left(\frac{s}{2}\right) \zeta_K(s) \\ &= \frac{2^{r_1+r_2}R(K)h_K}{s(s-1)w_K} + \sum_{i=1}^{h_K} \int_1^\infty (f(\mathfrak{a}_i, t) - f(\infty)) \left(t^{\frac{s}{2}} + t^{\frac{1-s}{2}}\right) \frac{dt}{t},\end{aligned} \tag{3.73}$$

where h_K is the class number of K and if $B_i, 1 \leq i \leq h_k$ are ideal classes, then B_i is the class of $\mathfrak{a}_i^{-1}$. This shows that we have the functional equation

$$\Lambda(K, s) = \Lambda(K, 1-s). \tag{3.74}$$

The completed Dedekind zeta-function has suitable integral representation, functional equation, and real coefficients so $\zeta_K(s)$ is good.

Derivatives at central values and Dedekind Jensen polynomials

The nth derivative of the completed Dedekind zeta-function has the form

$$\Lambda^{(n)}(K,s) = \frac{2^{r_1+r_2}R(K)h_K \cdot n!}{w_K} \cdot \frac{(s-1)^{n+1} - s^{n+1}}{s^{n+1}(1-s)^{n+1}} + \sum_{i=1}^{h_K} \frac{1}{2^n} \int_1^\infty (f(\mathfrak{a}_i,t) - f(\infty)) \left(t^{\frac{s}{2}} + (-1)^n t^{\frac{1-s}{2}} \right) \log^n(t) \frac{dt}{t}. \tag{3.75}$$

At the central value $s = \frac{1}{2}$ we have

$$\Lambda^{(n)}\left(K, \frac{1}{2}\right) = \frac{2^{r_1+r_2+n+1}R(K)h_K((-1)^n-1)n!}{w_K} + \sum_{i=1}^{h_K} \frac{1}{2^n} \int_1^\infty (f(\mathfrak{a}_i,t) - f(\infty))\, t^{\frac{1}{4}} (1+(-1)^n) \log^n(t) \frac{dt}{t}. \tag{3.76}$$

In order to state the asymptotic expansion we need to find the first nonzero coefficient of each $f(\mathfrak{a}_i t)$. Let ϵ be a unit with norm 1, then the smallest nonzero exponent in $f(\mathfrak{a}_i, t)$ is given by

$$m_{\mathfrak{a}_i} = \min\{\langle a\epsilon, a\rangle : a \in \mathfrak{a}_i, a \neq 0\}.$$

Let

$$M_{\mathfrak{a}_i} = \#\{a \in \mathfrak{a}_i : \langle a\epsilon, a\rangle = m_{\mathfrak{a}_i}\},$$

then the expansion of $f(\mathfrak{a}_i, t)$ begins

$$f(\mathfrak{a}_i,t) = f(\infty) + \frac{2^{r_1+r_2-1}R(K)}{w_K} M_{\mathfrak{a}_i} e^{-\pi m_{\mathfrak{a}_i}\left(\frac{t}{d_{\mathfrak{a}_i}}\right)^{\frac{1}{j}}} + \cdots. \tag{3.77}$$

We will let $C_i = \frac{2^{r_1+r_2-1}R(K)}{w_K} M_{\mathfrak{a}_i}$ and

$$F_i(n) = \frac{1}{2^n} \int_i^{\infty} \left(f(\mathfrak{a}_i, t) - f(\infty)\right) t^{-\frac{3}{4}} (1 + (-1)^n) \log^n(t) dt \tag{3.78}$$

in order to simplify the next theorem.

Theorem 3.2.4. *Assume the notation above, then we have*

$$\Lambda^{(n)}\left(K, \frac{1}{2}\right) = \frac{2^{r_1+r_2+n+1} R(K) h_K ((-1)^n - 1) n!}{w_K} + \sum_{i=1}^{h_K} F_i(n) \tag{3.79}$$

and $F_i(n)$ is given to all orders by the asymptotic expansion

$$\begin{aligned} F_i(n) \sim \frac{C_i \sqrt{2\pi}(1+(-1)^n)}{2^n} & \frac{L_i^{n+1}}{\sqrt{n\left(1+\frac{L_i}{j}\right) - \frac{3}{4j} L_i^2}} \\ & \times e^{\frac{L_i}{4} - \frac{jn}{L_i} + \frac{3j}{4}} \left(1 + \frac{b_{i,1}}{n} + \frac{b_{i,2}}{n^2} + \cdots\right) \qquad (n \to \infty), \end{aligned} \tag{3.80}$$

where $L_i = L_i(n) \approx j \log\left(\frac{n}{\log(n)}\right)$ is the unique solution of the equation $n = \left(\frac{m_{\mathfrak{a}_i} d_{\mathfrak{a}_i}^{-\frac{1}{j}}}{j} \pi e^{\frac{L_i}{j}} + \frac{3}{4}\right) L_i$ and each coefficient $b_{i,k}$ belongs to $\mathbb{Q}(L_i)$.

We have shown that $\zeta_K(s)$ is good so define

$$\Xi(z) := \left(-z^2 - \frac{1}{4}\right) \Lambda\left(K, \frac{1}{2} - iz\right) \tag{3.81}$$

and

$$\Xi_1(x) := \Xi(i\sqrt{x}) = \sum_{n \geq 0} \frac{\gamma_K(n)}{n!} x^n \tag{3.82}$$

where the Taylor coefficients are given by

$$\gamma_k(n) = (-1)^n \frac{n!}{(2n)!} \cdot \Xi^{(2n)}(0). \tag{3.83}$$

The derivatives $\Xi^{(2n)}(0)$ are given by

$$\Xi^{(2n)}(0) = (-1)^n \sum_{i=1}^{h_K} \frac{8\binom{2n}{2} F_i(2n-2) - F_i(2n)}{4}$$

and so we can use the above asymptotic expansion above or Theorem 1.3.6 to show that if $d \geq 1$, then $J_{\gamma_K}^{d,n}(X)$ is hyperbolic with at most finitely many exceptions n.

Chapter 4

Schwartz functions

This section will contain generalizations of the constructions use by Viazovska and Cohn, Kumar, Miller, Radchennko, and Viazovska to solve the sphere packing problem in dimension 8 and 24. We will discuss constructions analogous to Viazovska's $+1$ eigenfunction in Section 4.1 and constructions analogous to Viazovska's -1 eigenfunction in Section 4.2. Together these sections prove the first part of Theorem 1.4.5. In Section 4.3 we will discuss sphere packing upper bounds via modular forms which proves the remainder of Theorem 1.4.5. Appendix B contains some other constructions of Schwartz functions which have nice Fourier transforms using modular forms.

4.1 Background on modular forms

Recall that we have the following structure for the following graded rings:

$$M(SL_2(\mathbb{Z})) = \bigoplus_{k\in\mathbb{Z}} M_k(SL_2(\mathbb{Z})) = \mathbb{C}[E_4, E_6],$$
$$M^!(\Gamma) = \bigoplus_{k\in\mathbb{Z}} M^!_k(\Gamma) = M(\Gamma)[\Delta^{-1}].$$

We also recall the *Jacobi theta functions*

$$\theta_2(z) = \sum_{n\in\mathbb{Z}} e^{\pi i\left(n+\frac{1}{2}\right)^2 z},$$
$$\theta_3(z) = \sum_{n\in\mathbb{Z}} e^{\pi i n^2 z},$$
$$\theta_4(z) = \sum_{n\in\mathbb{Z}} (-1)^n e^{\pi i n^2 z}.$$

Following the notation in [12] we define

$$\begin{aligned} U(z) &= \theta_3(z)^4 \\ V(z) &= \theta_2(z)^4 \\ W(z) &= \theta_4(z)^4. \end{aligned} \tag{4.1}$$

With this notation we can write the Jacobi identity as $U = V + W$ and we have the fact

$$M(\Gamma(2)) = \mathbb{C}[V, W]. \tag{4.2}$$

The modular forms U, V, and W transform under $SL_2(\mathbb{Z})$ as follows:

$$U|_2T = W, \quad V|_2T = -V, \quad W|_2T = U, \tag{4.3}$$

$$U|_2S = -U, \quad V|_2S = -W, \quad W|_2S = -V. \tag{4.4}$$

We will also require the modular function

$$\lambda(z) := \frac{V}{U}(z) \in M_0^!(\Gamma(2)). \tag{4.5}$$

The function $\lambda(s)$ is the Hauptmodul for $\Gamma(2)$. It takes the values $0, 1$, and ∞ at the cusps $i\infty, 0$, and -1 of $\Gamma(2)$ respectively, and it decreases from 1 to 0 as z goes from 0 to $i\infty$ along the imaginary axis. The function $\lambda(z)$ satisfies the transformation properties

$$\begin{aligned} (\lambda|_0S)(z) &= 1 - \lambda(z) \\ (\lambda|_0T)(z) &= -\frac{\lambda(z)}{1-\lambda(z)}. \end{aligned} \tag{4.6}$$

If we define $\lambda_S(z) := (\lambda|_0S)(z)$, then we also have

$$(\lambda_S|_0T)(z) = \frac{1}{\lambda_S(z)}. \tag{4.7}$$

We again follow [12] to define logarithms of λ and λ_S. Because λ and λ_S do not vanish on $\mathbb{H}$ we can define

$$\mathcal{L}(z) := \int_0^z \frac{\lambda'(w)}{\lambda(w)} dw \quad \text{and} \quad \mathcal{L}_s(z) := -\int_z^{i\infty} \frac{\lambda_S'(w)}{\lambda_S(w)} dw, \tag{4.8}$$

where the contours are chosen to approach 0 or $i\infty$ along vertical lines. These functions are essentially the regularized Eichler integrals of the weight 2 weakly holomorphic modular form $\frac{\lambda'(z)}{\lambda(z)}$ at the cusps 0 and $i\infty$. They therefore are the holomorphic parts of some weight 0 harmonic Maass form and will play the same role for constructing Schwartz functions on the "minus" side as E_2 plays on the "plus" side. For more information on these topics see [3]. These functions satisfy

$$\mathcal{L}(it) = \log(\lambda(it)) \quad \text{and} \quad \mathcal{L}_S(it) = \log(\lambda_S(it)) = \log(1 - \lambda(it))$$

for $t > 0$, and so are holomorphic functions for which $e^{\mathcal{L}} = \lambda$ and $e^{\mathcal{L}_S} = \lambda_S$, but are not in general the principal branches of the logarithms of λ and λ_S. We have the following asymptotics as $q \to 0$:

$$\begin{aligned} \mathcal{L}(z) &= \pi i z + 4\log(2) - 8q^{\frac{1}{2}} + O(q) \\ \mathcal{L}_S(z) &= -16q^{\frac{1}{2}} - \frac{64}{3}q^{\frac{3}{2}} + O(q^{\frac{5}{2}}). \end{aligned} \tag{4.9}$$

The functions $\mathcal{L}$ and $\mathcal{L}_S$ satisfy the transformation properties

$$\mathcal{L}|_0 T^{\pm 1} = \mathcal{L} - \mathcal{L}_S \pm \pi i, \quad \mathcal{L}_S|_0 T = -\mathcal{L}_S, \tag{4.10}$$

$$\mathcal{L}|_0 S = \mathcal{L}_S, \qquad \mathcal{L}_S|_0 S = \mathcal{L}, \tag{4.11}$$

where $f|_k T^{-1} = f(z-1)$.

4.2 Proof of Theorem 1.4.5

4.2.1 The +1 eigenfunction construction

In this section we discuss generalizations of Viazovska's +1 eigenfunction construction. Let

$$\phi(z) = \sum c_\phi(n) q^n,$$

be a 1-periodic function on the upper half-plane. The following proposition presents our function of interest in a form where its Fourier transform is easily calculable.

Proposition 4.2.1. *Let $\phi(z)$ be a 1-periodic function that vanishes as $z \to i\infty$ and suppose there is an $r_0 \geq 0$ such that*

$$\phi\left(\frac{i}{t}\right) = O\left(t^{-\frac{d}{2}+2} e^{r_0^2 \pi t}\right) \qquad t \to \infty.$$

Then for $x \in \mathbb{R}^d$

$$\begin{aligned} a(x) := \int_{-1}^{i} \phi\left(-\frac{1}{z+1}\right)(z+1)^{\frac{d}{2}-2} e^{\pi i |x|^2 z} dz + \int_{1}^{i} \phi\left(-\frac{1}{z-1}\right)(z-1)^{\frac{d}{2}-2} e^{\pi i |x|^2 z} dz \\ - 2\int_{0}^{i} \phi\left(-\frac{1}{z}\right) z^{\frac{d}{2}-2} e^{\pi i |x|^2 z} dz + 2\int_{i}^{i\infty} \phi\left(z\right) e^{\pi i |x|^2 z} dz \end{aligned}$$

is a radial Schwartz function and $\widehat{a}(x) = (-i)^{-\frac{d}{2}} a(x)$.

Proof. By hypothesis, $\phi(z)$ decays exponentially as $\mathrm{Im}(z) \to \infty$, all of the above terms will be bounded and a and all of its derivatives will decay exponentially so a is Schwartz.

Because the integrals are absolutely and uniformly convergent we can switch the order of the integrals to see

$$\begin{aligned}\widehat{a}(x) = &\int_{-1}^{i} \phi\left(-\frac{1}{z+1}\right)(z+1)^{\frac{d}{2}-2}(-iz)^{-\frac{d}{2}} e^{\pi i|x|^2\left(-\frac{1}{z}\right)} dz \\ &+ \int_{1}^{i} \phi\left(-\frac{1}{z-1}\right)(z-1)^{\frac{d}{2}-2}(-iz)^{-\frac{d}{2}} e^{\pi i|x|^2\left(-\frac{1}{z}\right)} dz \\ &- 2\int_{0}^{i} \phi\left(-\frac{1}{z}\right) z^{\frac{d}{2}-2}(-iz)^{-\frac{d}{2}} e^{\pi i|x|^2\left(-\frac{1}{z}\right)} dz + 2\int_{i}^{i\infty} \phi(z)(-iz)^{-\frac{d}{2}} e^{\pi i|x|^2\left(-\frac{1}{z}\right)} dz.\end{aligned}$$

Let $w = -\frac{1}{z}$ then

$$\begin{aligned}\widehat{a}(x) &= (-i)^{-\frac{d}{2}} \int_{1}^{i} \phi\left(1 - \frac{1}{w-1}\right)\left(1 - \frac{1}{w}\right)^{\frac{d}{2}-2} w^{\frac{d}{2}-2} e^{\pi i|x|^2 w} dw \\ &+ (-i)^{-\frac{d}{2}} \int_{-1}^{i} \phi\left(1 - \frac{1}{w+1}\right)\left(-1 - \frac{1}{w}\right)^{\frac{d}{2}-2} w^{\frac{d}{2}-2} e^{\pi i|x|^2 w} dw \\ &- 2(-i)^{-\frac{d}{2}} \int_{i\infty}^{i} \phi(w) e^{\pi i|x|^2 w} dw + 2(-i)^{-\frac{d}{2}} \int_{i}^{0} \phi\left(-\frac{1}{w}\right) w^{\frac{d}{2}-2} e^{\pi i|x|^2 w} dw \\ &= (-i)^{-\frac{d}{2}} \int_{1}^{i} \phi\left(-\frac{1}{w-1}\right)(w-1)^{\frac{d}{2}-2} e^{\pi i|x|^2 w} dw \\ &+ (-i)^{-\frac{d}{2}} \int_{-1}^{i} \phi\left(-\frac{1}{w+1}\right)(-w-1)^{\frac{d}{2}-2} e^{\pi i|x|^2 w} dw \\ &+ 2(-i)^{-\frac{d}{2}} \int_{i}^{i\infty} \phi(w) e^{\pi i|x|^2 w} dw - 2(-i)^{-\frac{d}{2}} \int_{0}^{i} \phi\left(-\frac{1}{w}\right) w^{\frac{d}{2}-2} e^{\pi i|x|^2 w} dw \\ &= (-i)^{-\frac{d}{2}} a(x).\end{aligned}$$

The only thing we used here is that $\phi(z)$ is 1-periodic. □

In her work, Viazovska used special choices of functions ϕ to show that the resulting $a(x)$ has the additional property that it has double zeros at vectors of length $\sqrt{2k}$, for $k > 1$ and $k > 2$, and a single zero at vectors of length $\sqrt{2}$ and 2 in dimensions 8 and 24 respectively. The significance of this is that the former numbers are the non-minimal length vectors in the E_8 and Leech lattice respectively. Her idea was to relate $a(r)$ satisfying the hypothesis in the proposition above to a function with these specific zeros. The asymptotic behavior of the ϕ combined with the simple characterization zeros of the $\sin^2$ factor in the next proposition offers this description.

Proposition 4.2.2. *Suppose that $\phi(z)$ is a weakly holomorphic quasi-modular form of weight $k = -\frac{d}{2} + 4$ and depth 2 on $SL_2(\mathbb{Z})$ satisfying the conditions of Proposition 4.2.1. Then if $r \geq r_0$ we have that*

$$a(r) = -4\sin^2\left(\frac{\pi r^2}{2}\right)\int_0^{i\infty}\phi\left(-\frac{1}{z}\right)z^{\frac{d}{2}-2}e^{\pi i r^2 z}dz.$$

Proof. By direct calculation we have that

$$\begin{aligned}
&-4\sin^2\left(\frac{\pi r^2}{2}\right)\int_0^{i\infty}\phi\left(-\frac{1}{z}\right)z^{\frac{d}{2}-2}e^{\pi i r^2 z}dz \\
&= \int_0^{i\infty}\phi\left(-\frac{1}{z}\right)z^{\frac{d}{2}-2}e^{\pi i r^2(z+1)}dz - 2\int_0^{i\infty}\phi\left(-\frac{1}{z}\right)z^{\frac{d}{2}-2}e^{\pi i r^2 z}dz \\
&+ \int_0^{i\infty}\phi\left(-\frac{1}{z}\right)z^{\frac{d}{2}-2}e^{\pi i r^2(z-1)}dz \\
&= \int_1^{i\infty+1}\phi\left(-\frac{1}{z-1}\right)(z-1)^{\frac{d}{2}-2}e^{\pi i r^2 z}dz - 2\int_0^{i\infty}\phi\left(-\frac{1}{z}\right)z^{\frac{d}{2}-2}e^{\pi i r^2 z}dz
\end{aligned}$$

$$+\int_{-1}^{i\infty-1}\phi\left(-\frac{1}{z+1}\right)(z+1)^{\frac{d}{2}-2}e^{\pi i r^2 z}dz.$$

We can deform the path of integration because the integrand decays as $\mathrm{Im}(z)\to\infty$ to arrive at

$$\int_1^i \phi\left(-\frac{1}{z-1}\right)(z-1)^{\frac{d}{2}-2}e^{\pi i r^2 z}dz - 2\int_0^i \phi\left(-\frac{1}{z}\right)z^{\frac{d}{2}-2}e^{\pi i r^2 z}dz$$
$$+\int_{-1}^{i}\phi\left(-\frac{1}{z+1}\right)(z+1)^{\frac{d}{2}-2}e^{\pi i r^2 z}dz$$
$$+\int_i^{i\infty}\left[\phi\left(-\frac{1}{z-1}\right)(z-1)^{\frac{d}{2}-2}+\phi\left(-\frac{1}{z+1}\right)(z+1)^{\frac{d}{2}-2}-2\phi\left(-\frac{1}{z}\right)z^{\frac{d}{2}-2}\right]e^{\pi i r^2 z}dz.$$

By using the transformation properties of a depth 2 quasi-modular form we find that this last expression is $a(r)$.

□

4.2.2 The -1 eigenfunction construction

In the previous section we discussed the method Viazovska used to construct Schwartz functions that were eigenfunctions of the Fourier transform with eigenvalue $+1$. Viazovska also used theta functions to construct Schwartz functions with eigenvalue -1 under the Fourier transform. Here we generalize this by studying weakkly holomorphic modular forms on $\Gamma(2)$. For a modular form $\psi(z)\in M_k^!(\Gamma(2))$, let $\psi_\gamma(z) := \psi_I(z)|_k\gamma$.

Proposition 4.2.3. *Let $\psi_I(z)$ be a weight $-\frac{d}{2}+2$ weakly holomorphic modular form on $\Gamma(2)$ that vanishes as $z\to 0$ and suppose that there is an $r_0\geq 0$ such* *that*

$$\psi_I(it) = O(e^{r_0^2 \pi t}) \qquad t \to \infty$$
$$\psi_I(z) = \psi_T(z) + \psi_S(z).$$

Then for $x \in \mathbb{R}^d$,

$$b(x) := \int_{-1}^{i} \psi_T(z) e^{\pi i |x|^2 z} dz + \int_{1}^{i} \psi_T(z) e^{\pi i |x|^2 z} dz$$
$$- 2 \int_0^i \psi_I(z) e^{\pi i |x|^2 z} dz - 2 \int_i^{i\infty} \psi_S(z) e^{\pi i |x|^2 z} dz.$$

is a radial Schwartz function and $\widehat{b}(x) = -(-i)^{-\frac{d}{2}} b(x)$.

Proof. The fact that $b(x)$ is a Schwartz functions follows the same way as before. The Fourier transform of $b(x)$ is given as

$$\widehat{b}(x) = \int_{-1}^{i} \psi_T(z)(-iz)^{-\frac{d}{2}} e^{\pi i |x|^2 \left(-\frac{1}{z}\right)} dz + \int_{1}^{i} \psi_T(z)(-iz)^{-\frac{d}{2}} e^{\pi i |x|^2 \left(-\frac{1}{z}\right)} dz$$
$$- 2 \int_0^i \psi_I(z)(-iz)^{-\frac{d}{2}} e^{\pi i |x|^2 \left(-\frac{1}{z}\right)} dz - 2 \int_i^{i\infty} \psi_S(z)(-iz)^{-\frac{d}{2}} e^{\pi i |x|^2 \left(-\frac{1}{z}\right)} dz.$$

We substitute $w = -\frac{1}{z}$ as before and use the facts

$$\psi_T\left(-\frac{1}{z}\right) = -\psi_T(z) z^{-\frac{d}{2}+2},$$
$$\psi_I\left(-\frac{1}{z}\right) = \psi_S(z) z^{-\frac{d}{2}+2}$$

to show that $\widehat{b}(x) = -(-i)^{\frac{d}{2}} b(x)$. □

Following the same ideas as for $a(x)$, we have

Proposition 4.2.4. *Suppose that $\psi_I(z)$ is a weakly holomorphic modular form of weight $-\frac{d}{2}+2$ on $\Gamma(2)$ satisfying the conditions of Proposition 4.2.3. Then if $r \geq r_0$ we have that*

$$b(r) = -4\sin^2\left(\frac{\pi r^2}{2}\right)\int_0^{i\infty} \psi_I(z)e^{\pi i r^2 z}dz.$$

Proof. The proof follows almost the same as the proof for Proposition 4.2.2. The main points we use to show this are that

$$\psi_I(z-1) = \psi_I(z+1) = \psi_T(z)$$

and

$$\psi_T(z) - \psi_I(z) = -\psi_S(z).$$

□

The following propositions generalize Proposition 4.2.3 and Proposition 4.2.4 to allow us to use $\mathcal{L}(z)$. As we will explain below, this construction was not needed to resolve the sphere packing problem in dimensions 8 and 24, but allows better control over n_- in general in Theorem 1.4.2.

Proposition 4.2.5. *Let $g(z) = f(z)\mathcal{L}(z)$ where $f(z)$ is a weight $-\frac{d}{2}+2$ weakly holomorphic modular form on $SL_2(\mathbb{Z})$. Suppose $g(z)$ vanishes as $z \to 0$ and* *that there is an $r_0 \geq 0$ such that*

$$g(it) = O(te^{r_0^2 \pi t}) \quad t \to \infty.$$

Then for $x \in \mathbb{R}^d$,

$$\begin{aligned} c(x) := &\int_{-1}^{i} g_T(z) e^{\pi i |x|^2 z} dz + \int_{1}^{i} g_{T^{-1}}(z) e^{\pi i |x|^2 z} dz \\ &- 2 \int_{0}^{i} g(z) e^{\pi i |x|^2 z} dz - 2 \int_{i}^{i\infty} g_S(z) e^{\pi i |x|^2 z} dz. \end{aligned}$$

is a radial Schwartz function and $\widehat{c}(x) = -(-i)^{-\frac{d}{2}} c(x)$.

Proof. As before we have

$$\begin{aligned} \widehat{c}(x) = &\int_{-1}^{i} g_T(z)(-iz)^{-\frac{d}{2}} e^{\pi i |x|^2 \left(-\frac{1}{z}\right)} dz + \int_{1}^{i} g_{T^{-1}}(z)(-iz)^{-\frac{d}{2}} e^{\pi i |x|^2 \left(-\frac{1}{z}\right)} dz \\ &- 2 \int_{0}^{i} g(z)(-iz)^{-\frac{d}{2}} e^{\pi i |x|^2 \left(-\frac{1}{z}\right)} dz - 2 \int_{i}^{i\infty} g_S(z)(-iz)^{-\frac{d}{2}} e^{\pi i |x|^2 \left(-\frac{1}{z}\right)} dz. \end{aligned}$$

Let $w = -\frac{1}{z}$ to arrive at

$$\begin{aligned} \widehat{c}(x) = &(-i)^{-\frac{d}{2}} \int_{1}^{i} g_T\left(-\frac{1}{w}\right) w^{\frac{d}{2}-2} e^{\pi i |x|^2 w} dw + (-i)^{-\frac{d}{2}} \int_{-1}^{i} g_{T^{-1}}\left(-\frac{1}{w}\right) w^{\frac{d}{2}-2} e^{\pi i |x|^2 w} dw \\ &- 2(-i)^{-\frac{d}{2}} \int_{i\infty}^{i} g\left(-\frac{1}{w}\right) w^{\frac{d}{2}-2} e^{\pi i |x|^2 w} dw - 2(-i)^{-\frac{d}{2}} \int_{i}^{0} g_S\left(-\frac{1}{w}\right) w^{\frac{d}{2}-2} e^{\pi i |x|^2 w} dw. \end{aligned}$$

By using the transformation properties of $\mathcal{L}$ given in equation (4.10) we have that

$$g_T|_{-\frac{d}{2}+2}S = -g_{T^{-1}}$$
$$g_{T^{-1}}|_{-\frac{d}{2}+2}S = -g_T.$$

Using these properties it is clear to see that $\widehat{c}(x) = -(-i)^{-\frac{d}{2}}c(x)$. □

In analogy with the previous propositions we have the following.

Proposition 4.2.6. *Suppose that $g(z)$ is as in Proposition 4.2.5. Then if $r \geq r_0$ we have that*

$$c(r) = -4\sin^2\left(\frac{\pi r^2}{2}\right)\int_0^{i\infty} g(z)e^{\pi i r^2 z}dz.$$

Proof. By direct calculation we have

$$\begin{aligned}
&-4\sin^2\left(\frac{\pi r^2}{2}\right)\int_0^{i\infty} g(z)e^{\pi i r^2 z}dz \\
&= \int_0^{i\infty} g(z)e^{\pi i r^2(z+1)}dz - 2\int_0^{i\infty} g(z)e^{\pi i r^2 z}dz + \int_0^{i\infty} g(z)e^{\pi i r^2(z-1)}dz \\
&= \int_1^{i\infty} g_{T^{-1}}(z)e^{\pi i r^2 z}dz - 2\int_0^{i\infty} g(z)e^{\pi i r^2 z}dz + \int_{-1}^{i\infty} g_T(z)e^{\pi i r^2 z}dz.
\end{aligned}$$

The integrand decays as $z \to i\infty$ so we can deform the path of integration to arrive at

$$\int_1^{i} g_{T^{-1}}(z)e^{\pi i r^2 z}dz - 2\int_0^{i} g(z)e^{\pi i r^2 z}dz + \int_{-1}^{i} g_T(z)e^{\pi i r^2 z}dz$$

$$+\int_i^{i\infty}(g_{T^{-1}}(z)+g_T(z)-2g(z))e^{\pi i r^2 z}dz.$$

By the properties of $\mathcal{L}$ given in equation (4.10) we have

$$g_{T^{\pm1}} = f\mathcal{L}|_{-\frac{d}{2}+2}T^{\pm1} = f(\mathcal{L}-\mathcal{L}_S \pm \pi i).$$

From this it is clear that $g_T + g_{T^{-1}} = 2g - 2g_S$. Using this transformation property completes the proof. □

4.2.3 The zeros of the Schwartz functions

The following proposition gives the conditions needed to control the location of the simple zero and when the double zeros begin for the Schwartz functions.

Proposition 4.2.7. *Assume that the minimal length vector of the lattice of interest has the form $r_0 = \sqrt{2k}$ for some $k \in \mathbb{Z}$. If*

$$g(z) = p(z) + O\left(z^2 e^{2\pi i z}\right)$$

with

$$p(z) = c_0 e^{-r_0^2\pi i z} + c_1 z e^{-(r_0^2-2)\pi i z} + c_2 e^{-(r_0^2-2)\pi i z} + \cdots + c_{2k-1}z + c_{2k}$$

where the c_j are constants and $c_0, c_{2m-1} \neq 0$ for $1 \leq m \leq k$, then if

$$f(r) = -4\sin^2\left(\frac{\pi r^2}{2}\right)\int_0^{i\infty} g(z)e^{\pi i r^2 z}dz$$

$$f\left(\sqrt{2k}\right) = f(r_0) = 0,$$
$$f'\left(\sqrt{2k}\right) = f'(r_0) \neq 0,$$
$$f\left(\sqrt{2m}\right) \neq 0 \qquad 0 \leq m \leq k-1.$$

Proof. If we make the substitution $z = it$ then

$$f(r) = -i^{\frac{d}{2}-1} 4\sin^2\left(\frac{\pi r^2}{2}\right)\left[\int_0^\infty p(it)e^{-\pi r^2 t}dt + \int_0^\infty \left(g(it) - p(it)\right)e^{-\pi r^2 t}dt\right].$$

We have that

$$\int_0^\infty p(it)e^{-\pi r^2 t}dt = \int_0^\infty \left(c_0 e^{r_0^2 \pi t} + ic_1 t e^{(r_0^2-2)\pi t} + \cdots + ic_{2k-1}t + c_{2k}\right)e^{-\pi r^2 t}dt$$
$$= \frac{c_0}{\pi(r^2 - r_0^2)} + \frac{ic_1}{\pi^2(r^2 - r_0^2 + 2)^2} + \frac{c_2}{\pi(r^2 - r_0^2 + 2)} + \cdots + \frac{ic_{2k-1}}{\pi^2 r^4} + \frac{c_{2k}}{\pi r^2}.$$

When this term is multiplied by $\sin^2\left(\frac{\pi r^2}{2}\right)$ it is clear that we get a zero at $r = r_0 = \sqrt{2k}$ and that $a\left(\sqrt{2m}\right) \neq 0$ form $0 \leq m \leq k-1$. The first term also ensures that the zero at $r = r_0$ only has order one. It is also clear that $a(r)$ has double zeros at $r = \sqrt{2m}$ for $m > k$.

□

To use this for the $+1$ eigenfunction we replace $g(z)$ by $\phi\left(-\frac{1}{z}\right)z^{\frac{d}{2}-2}$. To use it for the -1 eigenfunction we replace $g(z)$ by $\psi(z)$.

4.3 Proof of Theorem 1.4.2

4.3.1 The $+1$ eigenfunction

In this section we will study when it is possible to construct the $+1$ eigenfunctions. Let $d \equiv 0 \pmod 8$. We can assume that our quasi-modular form $\phi(z)$ is always a holomorphic quasi-modular form divided by some power of $\Delta(z)$. The conditions given above are equivalent to demanding that

$$\widetilde{\phi}(z) = \Delta^n(z)\phi(z)$$

is a weight $-\frac{d}{2} + 4 + 12n$ quasi-modular form of depth 2 on $SL_2(\mathbb{Z})$ such that $\widetilde{\phi}(z) = O(q^{n+1})$ with n minimum. All such forms are of the form

$$\widetilde{\phi}(z) = \sum_{i \geq 1} \alpha_i E_2^{a_i}(z) E_4^{b_i}(z) E_6^{c_i}(z)$$

with atleast one $a_i = 2$, all $a_i \leq 2$, and $2a_i + 4b_i + 6c_i = -\frac{d}{2} + 4 + 12n$ for all i. Equivalently

$$\widetilde{\phi}(z) \in E_2^2 M_{-\frac{d}{2}+12n}(SL_2(\mathbb{Z})) \bigoplus E_2 M_{-\frac{d}{2}+2+12n}(SL_2(\mathbb{Z})) \bigoplus M_{-\frac{d}{2}+4+12n}(SL_2(\mathbb{Z})).$$

The number of such forms is

$$\delta_{d,n} := \dim\left(M_{-\frac{d}{2}+4+12n}(SL_2(\mathbb{Z}))\right) + \dim\left(M_{-\frac{d}{2}+2+12n}(SL_2(\mathbb{Z}))\right) + \dim\left(M_{-\frac{d}{2}+12n}(SL_2(\mathbb{Z}))\right)$$

where

$$\dim(M_k(SL_2(\mathbb{Z}))) = \begin{cases} \lfloor \frac{k}{12} \rfloor + 1 & k \not\equiv 2 \pmod{12} \\ \lfloor \frac{k}{12} \rfloor & k \equiv 2 \pmod{12}. \end{cases}$$

A short calculation shows that $\delta_{d,n} = 3n - \frac{d}{8} + 2$. In order to ensure that $\widetilde{\phi}(z) = O(q^{n+1})$ there needs to be a nontrivial solution to a system of $n+1$ homogeneous equations with $3n - \frac{d}{8} + 2$ variables. Therefore, we must have $2n > \frac{d}{8} - 1$.

Example. *For $d = 8$ we can let $n = 1$ and find that $\widetilde{\phi}(z) = E_2^2E_4^2 - 2E_2E_4E_6 + E_6^2$ which matches the function found in [62].*

For $d = 48$ we can let $n = 3$ and find

$$\begin{aligned}\widetilde{\phi}(z) = \Delta^3(z)\phi(z) &= 1556796748E_2^2(z)E_4^3(z) - 77235475E_2^2(z)E_6^2(z) \\ &- 704733786E_2(z)E_4^2(z)E_6(z) - 1029088507E_4^4(z) + 254261020E_4(z)E_6^2(z) \\ &= -1673465440313507328q^4 + O(q^5).\end{aligned}$$

Dimension $d = 48$ is especially interesting as the bound given by the $+1$ eigenfunction in this case exactly matches the lower bound given by the even unimodular lattice P_{48n}.

4.3.2 The -1 eigenfunction

We will follow the same basic argument as in the previous section. Proposition 4.2.3 and Proposition 4.2.5 show that the modular function, $\psi(z)$, we use to construct our Schwartz function must be a sum of a modular form g of weight

$-\frac{d}{2}+2$ on $\Gamma(2)$ such that $g = g_T + g_S$ and a function of the form $f\mathcal{L}$ where f is a modular form of weight $-\frac{d}{2}+2$ on $SL_2(\mathbb{Z})$. In [12] it was shown that this is equivalent to

$$\widetilde{\psi}(z) \in \left(U^2 - V^2\right) M_{-\frac{d}{2}-2+12n}(SL_2(\mathbb{Z})) \bigoplus W M_{-\frac{d}{2}+12n}(SL_2(\mathbb{Z})) \bigoplus \mathcal{L} M_{-\frac{d}{2}+2+12n}(SL_2(\mathbb{Z})),$$

where $\widetilde{\psi}(z) = \Delta^n(z)\psi(z)$. We now want to choose a $\widetilde{\psi}(z)$ in this space such that $\widetilde{\psi}(z)$ has a constant term without a z. This ensures that the Schwartz function will have a simple zero at $\sqrt{2n}$. We also need to ensure $\widetilde{\psi}_S(z) = O(q^{n+\frac{1}{2}})$ so that that $\psi(z)$ vanishes as $z \to 0$. $\widetilde{\psi}_S(z)$ is only supported on half-integral exponents so this gives a system of $n+2$ homogeneous equations. Let

$$\delta'_{d,n} := \dim\left(M_{-\frac{d}{2}-2+12n}(SL_2(\mathbb{Z}))\right) + \dim\left(M_{-\frac{d}{2}+12n}(SL_2(\mathbb{Z}))\right) + \dim\left(M_{-\frac{d}{2}+2+12n}(SL_2(\mathbb{Z}))\right),$$

then a short computation shows $\delta'_{d,n} = 3n - \frac{d}{8} + 1$ and so to guarantee a nontrivial solution we must have $2n \geq \frac{d}{8} + 1$.

Remark. One can ignore the contribution from $\mathcal{L}$ for $d = 8$ and $d = 24$ and get the same minimal value for n. For example, for $d = 8$ using the method described above we find

$$\widetilde{\psi}(z) = \frac{1}{3}\left(U^2 - V^2\right) E_6 + \frac{2}{3} W E_4^2,$$

which is equal to the form used in [62]. For this reason $\mathcal{L}$ did not show up in the constructions in [62] or [11].

Remark. This also shows that for $d = 48$ the minimal possible n is $n = 4$. Therefore, one cannot match the function found on the "plus" side and resolve the sphere packing problem for $d = 48$ using this method.

The following table gives the sphere packing upper bounds obtained using the functions constructed here.

d	Given upper bound	Best upper bound	Best lower bound
8	0.25	0.25	0.25
16	0.235331	0.025	0.0147
24	0.0019	0.0019	0.0019
32	2.8×10^{-3}	1.3×10^{-4}	1.1×10^{-5}
40	1.2×10^{-5}	???	7.9×10^{-8}
48	2.3×10^{-5}	1.1×10^{-6}	2.3×10^{-8}
56	7.3×10^{-8}	???	2.3×10^{-11}
64	1.7×10^{-7}	???	1.3×10^{-12}
72	4.5×10^{-10}	4.0×10^{-10}	3.4×10^{-20}
80	1.1×10^{-9}	???	1.1^{-16}
88	2.8×10^{-12}	???	6.3×10^{-25}
96	7.7×10^{-12}	1.3×10^{-13}	2.7×10^{-27}

Appendix A

First Appendix

The Sage and Mathematica code below implements the procedure described in the proof of Theorem 1.3.1.

Sage code

```
epsilon_list=[0,0,0.0295,0.021,0.0163,0.0081,0.001] #list of our epsilon choices
error_list=[0,0,[0,12719.9+1.59552*10^8,328255+1.7476*10^8],[0,10559.2+4.30607*10^6,328255+4.
60022*10^6,3.77919*10^6+4.91402*10^6],[0,9026.37+51727.4,328255+54478.9,3.77919*10^6+57374.2,
1.75707*10^7+60420.8],[0,5893.44+1.54878*10^ 6,328255+1.58991*10^ 6,3.77919*10^6+1.63212*10^
6,1.75708*10^7+1.67544*10^ 6,5.37043*10^7+1.71991*10^ 6]] #from Mathematica and Lemma 2.3

#build symbolic expressions for Hankel determinants in terms of power sums s_i
S.<s0,s1,s2,s3,s4,s5,s6,s7,s8>=PolynomialRing(QQ)
ss=[s0,s1,s2,s3,s4,s5,s6,s7,s8]
Matrices=[matrix([ [ss[k] for k in [j..j+i 1]] for j in [0..i 1] ]) for i in [0..5] ]
MM=[M.determinant() for M in Matrices] #Hankel determinant in terms of S_i
AA.<a0,a1,a2,a3,a4,a5>=PolynomialRing(QQ) #the coefficients aj of a polynomial
aa=[a0,a1,a2,a3,a4,a5]

var('w,p,j') #p=pi
c=2*p^2/3
s=10 #point of bounding errors   see Remark following the proof of Theorem 1.1
#define the function R(j,w) that approximates p(n+j)/p(n)
R= exp(j*c*w/(1 + sqrt(1 + j*c*w^2)))*(sqrt(1 + j*c*w^2)  w)/((w  1)*(1 + j*c*w^2)^(3/2))
A=R.series(w,s).truncate() #degree s 1 Taylor polynomial
T.<E1,E2,E3,E4,E5,w,p,j>=PolynomialRing(QQ)
EE=[0,E1,E2,E3,E4,E5]
```

```
A=T(A) #put A in the polynomial ring

def collect_errors(c,err): #minimizes c, given list of bounds on |E_i|
        M=c.monomials()
        C=[a.n() for a in c.coefficients()]
        l=len(C)
        to_sub= dict((EE[i],err[i]) for i in [1..len(err) 1])
        new=[]
        for i in [0..l 1]:
                if M[i].degree(E1)==M[i].degree(E2)==M[i].degree(E3)==M[i].degree(E4)==M[i].
                degree(E5)==0: #a monomial with no error terms will stay the same
                        new.append(C[i]*M[i].subs(p=RR(pi)).n())
                else:
                        new.append( abs(C[i]*M[i].subs(to_sub).subs(p=RR(pi)).n()))
        return min(0,sum(new))

for d in [2,3,4,5]:
        epsilon=epsilon_list[d]
        elem=[( 1)^i*aa[d i]/aa[d] for i in [0..d]] #elem sym functs in roots of sum(a_iX^i)
        for i in [d+1..2*d 2]:
                elem.append(0)
        power_sums=[d] #list of power sums
        for k in [1..2*d 2]: #builds power sums recursively using Newton Girard formulae
                power_sums.append(( 1)^(k 1)*k*elem[k]+sum([( 1)^(k 1+i)*elem[k i]*power_sums
                [i] for i in [1..k 1]]))
        hankel_list=[0,0] #polynomial expression for Hankel det in terms of coefficients aj
        for m in [2..d]:
                to_sub = dict( (ss[i],power_sums[i]) for i in [0..2*m 2] )
                D=MM[m].subs(to_sub)*aa[d]^(2*m 2)
                D=AA(D) #put D back in polynomial ring
                hankel_list.append(D)
        err=error_list[d]
        to_sub = dict((aa[i],binomial(d,i)*(A.subs(j=i)+EE[i]*w^s)) for i in [0..d])
        Delta_is_positive=[]
        for m in [2..d]:
                D=hankel_list[m] #D is D_{d,m}
                Delta=T(D.subs(to_sub)) #with A_s and symbolic errors plugged in
                k=3*m*(m 1)/2
                w=T(w)
                minimized_Delta = sum([Delta.coefficient({w:i}).subs(p=RR(pi)).n()*w^(i k) fo
                r i in [0..k+1]]) + sum([collect_errors(Delta.coefficient({w:i}),err).n()*w^(
                i k) for i in [k+2..(2*m 2)*s] ])
                if minimized_Delta.subs(w=epsilon).n() > 0:
                        Delta_is_positive.append(m)
                else:
                        print d,m,'choose_smaller_epsilon'
        if len(Delta_is_positive)==d 1:
```

```
        print 'For_d_=', d, 'J^{n,d}_is_hyperbolic_for_all_n_>_', floor(1/(c.subs(p=R
        R(pi))*epsilon^2)+1/24)
    else:
        print 'choose_smaller_epsilon'
```

Mathematica code

```
c = 2/3*Pi^2;
R[j_, w_]:= Exp[c*j*w/(1+Sqrt[1+c*j*w^2])](Sqrt[1+c*j* w^2] w)/((w 1)(1+c*j* w^2)^(3/2))
L[w_]:=(1+21*w)/(1 w)*Exp[ 1/(2*w)]+Exp[ 1/w]/(w^2 w^3)
Do[Print[N[Maximize[{R[i, w]*L[w]/(w^10*(1    L[w])),0<=w<=0.0295},w],30]],{i,1,2}]
Do[Print[N[Maximize[{R[i, w]*L[w]/(w^10*(1    L[w])),0<=w<=0.021},w],30]],{i,1,3}]
Do[Print[N[Maximize[{R[i, w]*L[w]/(w^10*(1    L[w])),0<=w<=0.0163},w],30]],{i,1,4}]
Do[Print[N[Maximize[{R[i, w]*L[w]/(w^10*(1    L[w])),0<=w<=0.0081},w],30]],{i,1,5}]
Do[Print[N[Maximize[{Abs[D[R[i,w],{w,10}]]/Factorial[10],0<=w<=0.0295},w],30]],{i,1,2}]
Do[Print[N[Maximize[{Abs[D[R[i,w],{w,10}]]/Factorial[10],0<=w<=0.021},w],30]],{i,1,3}]
Do[Print[N[Maximize[{Abs[D[R[i,w],{w,10}]]/Factorial[10],0<=w<=0.0163},w],30]],{i,1,4}]
Do[Print[N[Maximize[{Abs[D[R[i,w],{w,10}]]/Factorial[10],0<=w<=0.0081},w],30]],{i,1,5}]
Do[If[CountRoots[PartitionsP[i+3]*x^3+3*PartitionsP[i+2]*x^2+3*PartitionsP[i+1]*x+Partitions
 P[i],x]<3,Print[i]],{i,94,344}]
Do[If[CountRoots[PartitionsP[i+4]*x^4+4*PartitionsP[i+3]*x^3+6*PartitionsP[i+2]*x^2+4*Partit
 ionsP[i+1]*x+PartitionsP[i],x]<4,Print[i]],{i,206,572}]
Do[If[CountRoots[PartitionsP[i+5]*x^5+5*PartitionsP[i+4]*x^4+10*PartitionsP[i+3]*x^3+10*Part
 itionsP[i+2]*x^2+5*PartitionsP[i+1]*x+PartitionsP[i],x]<5,Print[i]],{i,381,2105}]
```

Appendix B

Second Appendix

Here we discuss other possible constructions for Schwartz functions which behave well under the Fourier transform. Let $H_n(x)$ be the n-th Hermite polynomial. Let $\phi(z) = \sum_{n\geq n_0} c_\phi(n)q^n$ be a weight $-\frac{d}{2}$ weakly holomorphic modular form. Define

$$h(r) := \int_0^{i\infty} \phi\left(-\frac{1}{z}\right) z^{\frac{d}{2}-1} H_1\left(\sqrt{-2\pi i z}|r|\right) e^{\pi i r^2 z} dz.$$

Proposition B.0.1. *We have that*

$$\widehat{h}(r) = i^{\frac{d}{2}} h(r).$$

Proof. As before we can switch the order of integration. The Fourier transform of $H_1(\sqrt{-2\pi iz}|r|)e^{\pi ir^2z}$ is $\left(\frac{-i}{\sqrt{-iz}}\right)^d e^{\pi ir^2\left(-\frac{1}{z}\right)}$ so we have

$$\widehat{h}(r) = \int_0^{i\infty} \phi\left(-\frac{1}{z}\right) z^{\frac{d}{2}-1} \left(\frac{-i}{\sqrt{-iz}}\right)^d H_1\left(\sqrt{\frac{2\pi}{-iz}}|r|\right) e^{\pi ir^2\left(-\frac{1}{z}\right)}dz.$$

We make the change of variable $z = \frac{1}{w}$ to find

$$\begin{aligned}
\widehat{h}(r) &= -i^{\frac{d}{2}} \int_{i\infty}^{0} \phi(w)w^{-\frac{d}{2}+1+\frac{d}{2}-2} H_1(\sqrt{-2\pi iw}|r|)e^{\pi ir^2w}dw \\
&= i^{\frac{d}{2}} \int_0^{i\infty} \phi(w)w^{-1} H_1(\sqrt{-2\pi iw}|r|)e^{\pi ir^2w}dw \\
&= i^{\frac{d}{2}} \int_0^{i\infty} \phi\left(-\frac{1}{w}\right) w^{\frac{d}{2}-1} H_1(\sqrt{-2\pi iw}|r|)e^{\pi ir^2w}dw.
\end{aligned}$$

□

Proposition B.0.2. *We have that*

$$h(0) = 2\sqrt{2\pi}c_\phi(0).$$

Proof. We use the modular transformation of ϕ and the fact that $H_1(x) = 2x$ to rewrite $h(r)$ as

$$\begin{aligned}
h(r) &= \int_0^{i\infty} \phi(z)z^{-1}\left(2\sqrt{-2\pi iz}|r|\right)e^{\pi ir^2z}dz \\
&= 2\sqrt{2\pi}|r|\int_0^{\infty} \phi(it)t^{-\frac{1}{2}}e^{-\pi r^2t}dt.
\end{aligned}$$

We can evaluate the integral

$$\int_0^\infty t^{-\frac{1}{2}} e^{-\pi r^2 t} dt = \frac{1}{|r|}$$

in order to prove the proposition. □

The following is a generalization of a construction in work by Radchenko and Viazovska [48] on Fourier interpolation on the real line. Much more can actually be said using these kinds functions as building blocks.

Proposition B.0.3. *Let* $g_\epsilon(z) = \sum c_\epsilon(n) q^{\frac{n}{2}}$ *be a weight* $-\frac{d}{2}+2$ *weakly holomorphic modular form such that* $g_\epsilon\left(-\frac{1}{z}\right) = \epsilon(-iz)^{-\frac{d}{2}+2} g_\epsilon(z)$ *for* $\epsilon \in \{+,-\}$. *Define*

$$j_\epsilon(r) := \frac{1}{2}\int_{-1}^{1} g_\epsilon(z) e^{\pi i r^2 z} dz,$$

where the integral is over the semicircle from -1 *to* 1. *Then*

$$\widehat{j}_\epsilon(r) = (-i)^d \epsilon j_\epsilon(r)$$

and

$$j_\epsilon(\sqrt{n}) = c_\epsilon(-n).$$

The proof of this statement is similar to the ones presented in Section 4 so it won't be given here. It should be noted that the behavior under Fourier transformation can be proven without changing the path of the contour integral so g_ϵ could be replaced by other (not necessarily holomorphic) modular objects.

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