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Linear Preserver Problems and Cohomological Invariants

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M.A., University of Montana, 2009

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Abstract

Linear Preserver Problems and Cohomological Invariants
By Hernando Bermudez

Let G be a simple linear algebraic group over a field F. In this work we prove several results about G and it's representations. In particular we determine the stabilizer of a polynomial f on an irreducible representation V of G for several interesting pairs (V, f). We also prove that in most cases if f is a polynomial whose stabilizer has identity component G then there is a correspondence between similarity classes of twisted forms of f and twisted forms of G. In a different direction we determine the group of normalized degree 3 cohomological invariants for most G which are neither simply connected nor adjoint.

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Contents

Chapter 1

Introduction

There are two main threads to this dissertation, unified by the underlying theory of linear algebraic groups. The first thread is the application of algebraic group techniques to the solution of linear preserver problems and in turn how understanding the solutions to such problems informs the understanding of the underlying groups. The second thread is the study of the structure of semisimple linear algebraic groups over general fields via their cohomological invariants. In this chapter we aim to establish the notation and state the main facts to be used throughout this dissertation.

1.1 Algebraic Group Schemes

For a more complete treatment of the contents of this section we refer the reader to the books [57], [101] and [96].

Let F be a field, F_{alg} an algebraic closure of F and Γ the absolute Galois group of F. Denote by Alg_F the category of unital, commutative (associative) F-algebras.

Definition 1.1.1. An *(affine) group scheme* over F is a representable functor G from Alg_F to the category of sets which factors through the forgetful functor from the category of groups to the category of sets. Given $R \in \mathrm{Alg}_F$ we write $G(R)$ for the image of R under the functor G.

By Yoneda's lemma the algebra representing G is determined uniquely up to isomorphism and we will denote it by $F[G]$, it has the structure of a *Hopf algebra*.

Definition 1.1.2. Let L/F be a field extension, G a group scheme defined over F. The *scalar extension of* G *to* L is denoted by G_L, it is a group scheme over L represented by $F[G] \otimes L$.

Definition 1.1.3. A group scheme G over F is said to be:

- *Algebraic* if the F-algebra $F[G]$ is finitely generated.
- *Smooth* if the F-algebra $F[G] \otimes F_{\mathrm{alg}}$ is reduced, i.e., it contains no nonzero nilpotent elements.
- *Connected* if the F-algebra $F[G]$ is connected, i.e., it contains no idempotent elements other thatn 0 and 1.

A smooth algebraic group scheme will simply be called an *algebraic group.*

Example 1.1. (1) The *trivial group*, 1, is the functor which maps every F-algebra R to the group $\{e\}$. It is represented by the trivial Hopf Algebra F.

(2) The *multiplicative group*, $\mathbb{G}_m$, is the functor $\mathbb{G}_m(R) = R^{\times}$. It is represented by the algebra $F[t, t^{-1}]$.

(3) Let A be a central simple algebra over F, the functor

$$\mathrm{GL}_1(A) : R \to (A \otimes R)^{\times}$$

is representable by [57, 20.2]. It is called the *general linear group of* A, in particular if $A = \mathrm{End}(V)$ for some vector space V we write $\mathrm{GL}_1(\mathrm{End}(V)) = \mathrm{GL}(V)$ and if $V = F^n$, $\mathrm{GL}_1(\mathrm{End}(F^n)) = \mathrm{GL}_n(F)$.

1.2 Morphisms

A *morphism* of group schemes $f : G \to G'$ is a natural transformation of functors.

Definition 1.2.1. Let $f : G \to G'$ be a morphism of algebraic groups, it induces a morphism $f^* : F[G'] \to F[G]$ of Hopf algebras. We say that f is:

- *Injective* if f^* is surjective.
- *Surjective* if $f^*_{\mathrm{alg}} : F[G'] \otimes F_{\mathrm{alg}} \to F[G] \otimes F_{\mathrm{alg}}$ is injective.

Let G be an algebraic group, J a Hopf ideal of $F[G]$. The quotient $F[G]/J$ is a Hopf algebra and the associated group H is said to be a *subgroup* of G. The subgroup H is said to be *normal* if $H(R)$ is normal in $G(R)$ for every $R \in \mathrm{Alg}_F$.

Definition 1.2.2. Let $f : G \to G'$ be a morphism of algebraic groups.

(1) The *kernel of f* is the subgroup of G corresponding to the Hopf Ideal $f^*(I') \cdot F[G]$ where I' denotes the augmentation ideal of $F[G']$.

(2) The *image of f* is the subgroup of G' corresponding to the Hopf Ideal $\ker(f^*)$ of $F[G']$.

Example 1.2. Let V be a finite dimensional vector space over F, G an algebraic group. A morphism $\rho : G \to \mathrm{GL}(V)$ is called a *representation* of G. If ρ is injective the representation is said to be *faithful.*

Let $f : G \to H$ be a surjective morphism of group schemes with kernel N then by [57, 22.7] the quotient G/N is well defined and is a group scheme if N is normal.

A *(short) exact sequence* of algebraic groups is a sequence:

$$1 \longrightarrow G \xrightarrow{f} G' \xrightarrow{f'} G'' \longrightarrow 1$$

where f' is surjective and f induces an isomorphism of G with the kernel of f'.

1.3 Tori

Definition 1.3.1. A *torus* is an algebraic group T for which there exists an extension L/F such that T_L is isomorphic to a finite product of copies of $\mathbb{G}_m$.

The extension L/F is said to be a *splitting field of* T. If F itself is a splitting field for T, T is said to be *split*. If F_{sep} denotes a separable closure of F, then F_{sep} is a splitting field for every torus defined over F, see [96, Prop. 13.1.1].

Definition 1.3.2. Let G be a group, T a torus which is a subgroup of G. The torus T is said to be *maximal* if it's not properly contained in any other subtorus of G.

A group G is said to be *split* if it contains a split maximal torus. A *character* of a torus T is a homomorphism $\chi : T_{\text{sep}} \to \mathbb{G}_m$. The group of characters of T is denoted by $T^* = \text{Hom}(T_{\text{sep}}, \mathbb{G}_m)$, there is a natural action of Γ on T^* (see [57, 20.16]).

1.4 The Lie Algebra of an Algebraic Group

Let $F[\epsilon]$ denote the F-algebra of *dual numbers* that is the F-vector space with basis $\{1, \epsilon\}$ where $\epsilon^2 = 1$. There is a unique algebra homomorphism $\kappa : F[\epsilon] \to F$ with $\kappa(\epsilon) = 1$. The kernel of the induced map $G(\kappa) : G(F[\epsilon]) \to G(F)$ has the structure of a *Lie algebra*, see [57, 21.A], it is called the *Lie algebra of* G denoted by $\text{Lie}(G)$.

For every $R \in \text{Alg}_F$ the group $G(R)$ acts on $\text{Lie}(G) \otimes R$ by conjugation, and thus one obtains a representation:

$$\text{Ad} : G \longrightarrow \text{GL}(\text{Lie}(G))$$

called the *adjoint representation* of G.

1.5 Split Semisimple Algebraic Groups

An algebraic group G is said to be *solvable* if the abstract group $G(F_{\text{alg}})$ is solvable. A group $G \neq 1$ is *semisimple* if it is connected and the only connected normal subgroups of G_{alg} are G_{alg} and 1.

Let G be a split semisimple group, and let T be a split maximal torus in G. There is a decomposition:

$$\text{Lie}(G) = \oplus_{\chi \in T^*} V_\chi$$

Where V_χ is the set of $v \in \text{Lie}(G)$ for which $\text{Ad}(G)(t)(v) = \chi(t)v$ for all $t \in T$.

Definition 1.5.1. The nonzero $\chi \in T^*$ for which V_χ is nonzero are called the *roots* of G.

By [57, 25.1] the set of all roots of G has the structure of a *root system*[1] which will be denoted by $\Phi(G)$. The root system $\Phi(G)$ is, up to isomorphism, independent of the choice of T, we therefore refer to $\Phi(G)$ as *the root system of* G.

Let Λ_r denote the root lattice of $\Phi(G)$ and Λ the abstract weight lattice, then we have that $\Lambda_r \subseteq T^* \subseteq \Lambda$ by [57, 25.2]. Further given two split semisimple groups G and G' with corresponding maximal tori T and T' by [57, 25.3] G and G' are isomorphic iff $(\Phi(G), T^*) \cong (\Phi(G'), T'^*)$.

Definition 1.5.2. A semisimple group G is said to be *simply connected* if $T^* = \Lambda$. It is said to be *adjoint* if $T^* = \Lambda_r$.

[1]We refer the reader to [57, §24] for the relevant facts about root systems

Definition 1.5.3. Let $G \neq 1$ be a semisimple group. G is said to be *simple* if G_{alg} has no connected normal subgroups other than G and 1.

Because G_{alg} may still have some discrete normal subgroups, simple groups are sometimes also called *absolutely almost simple groups*. By [57, 25.8] a semisimple group G is simple iff its root system is irreducible. Corresponding to the classification of irreducible systems we say a group has type A_n, B_n, C_n, D_n, E_6, E_7, E_8, G_2 or F_4 if the corresponding root system has that type.

1.6 Representations

Let G be an algebraic group and $\rho : G \to \mathrm{GL}(V)$ a representation, ρ imbues V with the structure of an $F[G]$-comodule. If the only $F[G]$-subcomodules of V are 1 and V, ρ is said to be an *irreducible representation* of G.

Let G be split semisimple, $\rho : G \to \mathrm{GL}(V)$ be an irreducible representation and consider the decomposition

$$V = \oplus_{\chi \in T^*} V_\chi$$

the χ for which V_χ is nontrivial are the *weights of* ρ. They are abstract weights, that is elements of the abstract weight lattice Λ. With respect to the ordering in the lattice there is a largest weight of ρ, called the *highest weight of* ρ, it is a dominant weight. By [57, 27.1] there is a one-to-one correspondence between isomorphism classes of irreducible representations of G and dominant weights which are in T^*.

Example 1.6.1. Let G be a semisimple group and $\rho : G \to \mathrm{GL}(V)$ a representation. The *normalizer of G in* $\mathrm{GL}(V)$ denoted by $N_{\mathrm{GL}(V)}(G)$ is the sub group-scheme of $\mathrm{GL}(V)$ whose A-points are the sets of elements of $\mathrm{GL}(V)(A)$ which normalize the group $G(A)$ in the abstract group-theoretic sense. It is a closed subgroup-scheme of $\mathrm{GL}(V)$ containing G by [20, §A.1].

1.7 Tits Algebras

Let G be a semisimple group, not necessarily split. An *algebra representation* of G, is a homomorphism of groups $\rho : G \to \mathrm{GL}_1(A)$ for some central simple algebra A. The algebras A for which such an algebra representation exists are called the *Tits algebras of* G. By [57, §27], there is an action of Γ on Λ called the $*$-*action*, so that there is a one-to-one correspondence between isomorphism classes of algebra representations of G and dominant weights in T^* which are fixed by the $*$-action.

1.8 Galois Cohomology

Let Γ be a profinite group, and A a discrete topological space on which Γ acts. The action is called *continuous* if the stabilizer of each point is an open subgroup of Γ. A discrete topological space with a continuous action of Γ is called a Γ-*set*, if A is also a group and Γ acts morphically on A we say A is a Γ-*group*, if A is abelian we further say A is a Γ-*module*.

Let $\Gamma-$groups and $\Gamma-$mod denote respectively the categories of Γ-groups and Γ-modules where Γ is the absolute Galois group of F. By [91, Ch. 5] There are covariant functors

$$\begin{aligned} &\mathrm{H}^n(\Gamma, -) : \Gamma - \mathrm{mod} \longrightarrow \mathrm{Groups} && \text{for } n \geq 0 \\ &\mathrm{H}^n(\Gamma, -) : \Gamma - \mathrm{groups} \longrightarrow \mathrm{Pointed\ Sets} && \text{for } n = 0, 1 \end{aligned}$$

called the *nth Galois cohomology functors of* Γ. The two types of functors agree when they're both defined.

Let A and B be Γ-groups with A normal in B and put $C = B/A$, by [91, Prop. 38] there exists a map δ^0 so that the following long exact sequence of pointed sets is exact

$$0 \longrightarrow \mathrm{H}^0(\Gamma, A) \longrightarrow \mathrm{H}^0(\Gamma, B) \longrightarrow \mathrm{H}^0(\Gamma, A) \xrightarrow{\delta^0}$$

$$\mathrm{H}^1(\Gamma, A) \longrightarrow \mathrm{H}^1(\Gamma, B) \longrightarrow \mathrm{H}^1(\Gamma, C)$$

Further if the group A lies in the center of B, then there is a further map (δ^1 so that the following sequence is exact (see [91, Pr. 43])

$$0 \longrightarrow \mathrm{H}^0(\Gamma, A) \longrightarrow \mathrm{H}^0(\Gamma, B) \longrightarrow \mathrm{H}^0(\Gamma, A) \xrightarrow{\delta}$$

$$\mathrm{H}^1(\Gamma, A) \longrightarrow \mathrm{H}^1(\Gamma, B) \longrightarrow \mathrm{H}^1(\Gamma, C) \xrightarrow{\delta^1} \mathrm{H}^2(\Gamma, A)$$

1.9 Twisted Forms

Let Γ be the absolute Galois group of F, and let A be a Γ-group. A Γ-group A' is said to be a *twisted form of* A if there exists a field extension E/F so that $A'_E \cong A_E$ as Γ_E-groups.

By [57, §28.C] a Γ-group A' is a twisted form of A if and only if there exists an element $\sigma \in \mathrm{H}^1(\Gamma, \mathrm{Aut}(A))$ so that the action of Γ on A' can be obtained by twisting the action of Γ on A through σ.

1.10 Galois Cohomology of Algebraic Groups

Let G be an algebraic group defined over F, the abstract group $G(F_{\mathrm{sep}})$ is naturally a Γ-group. We write $\mathrm{H}^n(F, G)$ for the Galois cohomology set $\mathrm{H}^n(\Gamma, G(F_{\mathrm{sep}}))$.

Let G be a split semisimple group. A twisted form G' of G is said to be an *inner form* of G if G' is obtained by twisting G with an element which is in the image of the map

$$\alpha_G : \mathrm{H}^1(F, \overline{G}) \to \mathrm{H}^1(F, \mathrm{Aut}(G_{\mathrm{sep}}))$$

where $\overline{G}$ is the adjoint group of G, and α_G is induced by the action of $\overline{G}$ on G by inner automorphisms.

Chapter 2

Linear Preservers and Representations with a 1-dimensional Ring of Invariants

(The results in this chapter are joint with Skip Garibaldi and Victor Larsen and were first published in Transactions of the AMS in Volume 366, Number 9, September 2014 published by the American Mathematical Society©[1].)

In an 1897 paper [30], Frobenius proved that every linear transformation of the n-by-n real matrices that preserves the determinant is of the form

$$X \mapsto AXB \quad \text{or} \quad X \mapsto AX^t B$$

for some $A, B \in \mathrm{GL}_n(\mathbb{R})$ such that $\det(AB) = 1$; that is, the obvious ones are the only ones. This is the basic example of a solution to a *linear preserver problem* (LPP): one is given a finite-dimensional vector space V over a field K and a polynomial function $f : V \to K$ and one wants to determine the linear transformations of V that preserve f. Since Frobenius, many such problems have been solved, see for example the surveys [59], [76], [60], and [63]. We develop here a general method that solves several new problems, see Examples 2.3.15 and 2.3.16 and Corollaries 2.8.11, 2.9.3, 2.9.5, 2.9.7, 2.9.8, and 2.9.11.

Our method is to introduce an auxiliary group $G \subset \mathrm{GL}(V)$ that is semisimple and such that V is an irreducible representation or Weyl module of G. In section 2.2, we determine the normalizer $N_{\mathrm{GL}(V)}(G)$ of G in $\mathrm{GL}(V)$. We prove in Theorem 2.3.2 that this subgroup equals the stabilizer of a closed G-orbit $\mathscr{O}$ in the projective space $\mathbb{P}(V)$ — i.e., $\mathrm{Stab}_{\mathrm{GL}(V)}(\mathscr{O}) = N_{\mathrm{GL}(V)}(G)$ — under a mild technical assumption on the isotropy subgroup. Using this result, in sections 2.8 and 2.9, we solve two families of LPPs by reducing the problems in each family to determining this stabilizer. These two families consist of representations V with a 1-dimensional ring of G-invariant functions generated by f and are examples of prehomogeneous vector spaces of parabolic type; the two families correspond to the cases where the unipotent radical U of the parabolic subgroup is abelian (i.e., $[U, U] = 0$) or $[U, U]$ is 1-dimensional respectively, and we use the general results on representations in these families from [82], [83], and [45].

Besides obtaining new results, we also recover many known solutions to linear preserver problems. The generality of our method is in contrast to many of the proofs in the literature, which typically are highly dependent on the particular choice of V and f. (The arguments in [77] and [43] are notable exceptions.) Further, we require only very weak assumptions about the field K (at most we require that K is infinite or has characteristic $\neq 2, 3$) and determine the preserver precisely (and not just its identity component or Lie algebra).

Applications of solutions to LPPs

Linear preserver problems arise naturally in algebra, sometimes in non-obvious ways. For example, every associative division algebra D that is finite-dimensional over its center K has a "generic characteristic polynomial" generalizing the notion of characteristic polynomial on n-by-n matrices. Its coefficients are polynomial functions $E_r : D \to K$ for $1 \leq r \leq \sqrt{\dim_K D}$ where E_r has degree r. By determining the preserver of E_r, Waterhouse proved that D is determined up to isomorphism or anti-isomorphism by E_r for any $r \geq 3$, see [102] or [106, Cor. 4]. This in turn gives a result on the essential dimension of central simple algebras, see [29].

2.1 Irreducible representations and the closed orbit

We now describe the basic setup that will be used throughout the paper, providing details and examples for the convenience of the reader who is a non-specialist in semisimple groups.

Let $\widetilde{G}$ be a split semisimple linear algebraic group over a field K and fix a representation $\rho\colon \widetilde{G} \to \mathrm{GL}(V)$—our definition of semisimple includes that $\widetilde{G}$ is connected. Table A below lists some examples of pairs $(\widetilde{G}, V)$ that we will consider. For notational simplicity, we focus on the image G of $\widetilde{G}$ in $\mathrm{GL}(V)$. This group is also split semisimple. We will assume that V is an irreducible representation or is a Weyl module in the sense of [54, p. 183]. (If $\operatorname{char} K = 0$, the two notions coincide.) In either case, $\mathrm{End}_G(V) = K$, see loc. cit. when V is a Weyl module.

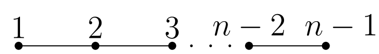
Fix a pinning of G in the sense of [22, §XXIII.1] (called a "framing" in [12]); this includes choosing a split maximal K-torus T, a set of simple roots Δ of G with respect to T, and a corresponding Borel subgroup B. Recall that T^* is naturally included in the weight lattice and there are bijections between dominant weights in T^*, equivalence classes of irreducible representations of G, and equivalence classes of Weyl modules of G [54, II.2.4]. Put $\lambda \in T^*$ for the highest weight of V and v^+ for a highest weight vector in V.

The stabilizer of Kv^+ in G contains the Borel subgroup B, so it is a parabolic subgroup P. The orbit $\mathscr{O}$ of Kv^+ is identified with the projective variety G/P, so $\mathscr{O}$ is *closed* in $\mathbb{P}(V)$.

Definition 2.1.1. We call an element $x \in V$ *minimal* if Kx belongs to $\mathscr{O}$.

Example 2.1.2 (exterior powers)**.** Take $\widetilde{G} = \mathrm{SL}_n$ and $V = \wedge^d(K^n)$ for some d between 1 and n. The group acts via $\rho(g)(v_1 \wedge v_2 \wedge \cdots \wedge v_d) = gv_1 \wedge gv_2 \wedge \cdots \wedge gv_d$ for $v_1, \ldots, v_d \in K^n$. The image G of $\widetilde{G}$ is equal to SL_n / μ_e where μ_e is the group scheme of e-th roots of unity for $e := \gcd(d, n)$. (Recall that, as a scheme, μ_e is $\operatorname{Spec} K[x]/(x^e - 1)$, so it is smooth if and only if $\operatorname{char} K$ does not divide e.) For T and B, we take the image in G of the diagonal and upper-triangular matrices, respectively. The only line stabilized by B is the span of $v^+ = e_1 \wedge \cdots \wedge e_d$, where e_i denotes the element of K^n with a 1 in the i-th position and zeros elsewhere.

The group $\widetilde{G}$ is of type A_{n-1} and its Dynkin diagram Δ is



where we have labeled each vertex with the number i of the corresponding fundamental weight ω_i according to the numbering from [11]. With respect to this numbering, v^+ has weight $\lambda = \omega_d$. The representation V is irreducible because ω_d is minuscule [54, II.2.15].

An element of V is *decomposable* if it can be written as $v_1 \wedge \cdots \wedge v_d$ for some $v_i \in K^n$. As SL_n acts transitively on the d-dimensional subspaces of K^n, we conclude that the minimal elements in V are the nonzero decomposable vectors.

In the special case $d = 2$, we may identify V with the vector space Skew_n of n-by-n alternating matrices — i.e., skew-symmetric matrices with zeros on the diagonal (the extra condition is necessary if $\operatorname{char} K = 2$) — where SL_n acts via $\rho(g)v = gvg^t$. Then v^+ corresponds to $E_{12} - E_{21}$, where E_{ij} denotes a matrix with a 1 in the (i, j)-entry and zeros elsewhere. From this, we see that the minimal elements are the alternating matrices of rank 2.

Example 2.1.3 ("symmetric powers"). Take $\widetilde{G} = \mathrm{SL}_n$ (with fundamental weights numbered as in the previous example) and take V to be the Weyl module with highest weight $\lambda := d\omega_1$ for some $d \geq 1$. The group G is SL_n / μ_e as in the previous example. If $\operatorname{char} K$ is zero or $> d$, this representation is irreducible by, for example, [42, p. 50], and in that case V can be identified with the d-th symmetric power $S^d(K^n)$ of the tautological representation, the highest weight line is spanned by $v^+ = e_1^d$, and minimal elements are the d-th powers of nonzero elements of K^n.

When $d = 2$ and $\operatorname{char} K \neq 2$, we may identify V with the vector space Symm_n of n-by-n symmetric matrices, where SL_n acts by $\rho(g)v = gvg^t$, v^+ corresponds to E_{11}, and the minimal elements are symmetric matrices of rank 1.

Returning to the case of general G and V, we have:

Example 2.1.4. The collection of minimal elements is nonempty and $G(K)$-invariant, so it spans a $G(K)$-invariant subspace of V that contains v^+, hence it must be all of V. Therefore, *there is a basis of V consisting of minimal elements.*

Said differently, the set $\{x \in V \mid Kx \in \mathscr{O}(K)\}$ spans V. This property characterizes $\mathscr{O}$, regardless of K:

Lemma 2.1.5. *If $\mathscr{O}'$ is a closed G-orbit in $\mathbb{P}(V)$ and the set $\{x \in V \mid Kx \in \mathscr{O}'(K)\}$ spans V, then $\mathscr{O}' = \mathscr{O}$.*

Proof. For K_{alg} an algebraic closure of K, the set $\{x \in V \otimes K_{\mathrm{alg}} \mid K_{\mathrm{alg}} x \in \mathscr{O}'(K_{\mathrm{alg}})\}$ spans $V \otimes K_{\mathrm{alg}}$. As it suffices to verify $\mathscr{O} = \mathscr{O}'$ over K_{alg}, we may assume that K is algebraically closed.

By the Borel Fixed Point Theorem, there is a line $Kx \in \mathscr{O}'(K)$ that is stabilized by B, hence x is a weight vector for some $\mu \in T^*$, which is dominant because $\mathrm{Hom}_G(V(\mu), V)$ is nonzero [54, p. 183, Lemma 2.13(a)]. All the weights of the G-submodule of V generated by x are $\leq \mu$ by loc. cit., but this submodule is all of V, so μ must equal λ and $Kv^+ = Kx$. $\square$

When V is irreducible, the spanning condition is not necessary, because the set $\{x \in V \mid Kx \in \mathscr{O}'(K)\}$ spans V for every orbit $\mathscr{O}'$ with $\mathscr{O}'(K)$ nonempty, hence the well-known result: *$\mathscr{O}$ is the unique closed G-orbit in $\mathbb{P}(V)$.*

When V is a Weyl module, the spanning condition is necessary. Indeed, if there is an exact sequence of representations $1 \to A \to V \to B \to 1$ where A and B are irreducible, then the closed orbit in $\mathbb{P}(A)$ gives a closed orbit in $\mathbb{P}(V)$ distinct from $\mathscr{O}$. This occurs, for example, when $G = \mathrm{SL}_2$ over a field K of characteristic 3 and V is the Weyl module with highest weight 3, in which case we additionally have that the two closed orbits are isomorphic as varieties (to $\mathbb{P}^1$).

It is harmless to identify $\mathbb{G}_m$ — the algebraic group with S-points $S^\times$ — with the subvariety of scalar matrices in $\mathrm{GL}(V)$.

Corollary 2.1.6. *The sub-group-scheme of $\mathrm{GL}(V)$ fixing $\mathscr{O}$ elementwise is the group $\mathbb{G}_m$ of scalar matrices. In particular, it is smooth.*

Proof. Put H for the sub-group-scheme fixing $\mathscr{O}$. For each $h \in H(K)$, there is a morphism $\mathscr{O} \to \mathbb{G}_m$ defined by $[v] \mapsto hv/v$. As $\mathscr{O}$ is projective connected and $\mathbb{G}_m$ is affine, the image must be a point. That is, there is a $c \in K^\times$ such that $hv = cv$ for every minimal $v \in V$. Example 2.1.4 gives that h is a scalar matrix.

A similar argument shows that H is smooth. Put $K[\varepsilon]$ for the dual numbers (with $\varepsilon^2 = 0$) and suppose $h = 1+x\varepsilon$ is in $H(K[\varepsilon])$; x can naturally be viewed as a (possibly non-invertible) linear transformation of V. The equation $(1+x\varepsilon)v = v+\varepsilon\lambda_v v$ defines a morphism $\mathscr{O} \to \mathbb{A}^1$ via $[v] \mapsto \lambda_v$. This map must be constant, therefore x is the scalar λ_v and $\dim \mathrm{Lie}(H) = 1$, i.e., H is smooth. From this and equality of K_{alg}-points $\mathbb{G}_m(K_{\mathrm{alg}}) = H(K_{\mathrm{alg}})$, we deduce that $\mathbb{G}_m = H$. $\square$

Corollary 2.1.7. *If K is infinite, then the subgroup of $\mathrm{GL}(V)(K)$ fixing $\mathscr{O}(K)$ elementwise consists of the scalar matrices.*

Proof. Let $g \in \mathrm{GL}(V)(K)$ fix $\mathscr{O}(K)$. Then, as $\mathscr{O}$ is a rational variety [8, 21.20(ii)] and K is infinite, $\mathscr{O}(K)$ is dense in $\mathscr{O}$ and the automorphism of $\mathbb{P}(V)$ induced by g fixes $\mathscr{O}$ as a variety. Corollary 2.1.6 gives that g is a scalar matrix. $\square$

For analogous results proved in various special cases, see for example the end of [23], Prop. 8 in [52], or Cor. 6.3 in [28].

Section 2.3 below is concerned with calculating the *stabilizer* $\mathrm{Stab}_{\mathrm{GL}(V)}(\mathscr{O})$ of $\mathscr{O}$ in $\mathrm{GL}(V)$, which is a group scheme whose K-points are the elements of $\mathrm{GL}(V)(K)$ that normalize $\mathscr{O}(R)$ for every commutative K-algebra R. We make some general remarks about it here.

Lemma 2.1.8. *If K is infinite and $T \in \mathrm{GL}(V)(K)$ stabilizes $\mathscr{O}(K)$, then T is a K-point of $\mathrm{Stab}_{\mathrm{GL}(V)}(\mathscr{O})$.*

Proof. Identical to the proof of Corollary 2.1.7. $\square$

In the literature on linear preserver problems, authors are sometimes concerned with calculating all matrices that stabilize $\mathscr{O}$ and not just the invertible ones. Note that a linear transformation T may preserve $\mathscr{O}(K)$ but fail to be invertible because $(\ker T) \cap \mathscr{O}$ may have no K-points; this happens even in the very nice case where $k = \mathbb{R}$, see e.g. [111, Example 1]. However, it is sufficient to check that T stabilizes the collection of minimal elements in $V \otimes K_{\mathrm{alg}}$:

Proposition 2.1.9. *If a linear transformation T of V stabilizes $\mathscr{O}(K_{\mathrm{alg}})$, then T is invertible and is a K-point of $\mathrm{Stab}_{\mathrm{GL}(V)}(\mathscr{O})$.*

Proof. We adapt the argument from [16, p. 322]. The set X of minimal elements in V has closure $\bar{X} = X \cup \{0\}$, an irreducible subvariety of V.

Note that $T(\bar{X})$ is a closed subvariety of $\bar{X}$, as can be seen by considering the morphism induced by T on the image $\mathscr{O}$ of X in $\mathbb{P}(V)$. The fiber of $T\colon \bar{X} \to T(\bar{X})$ over 0 is just $\{0\}$, and we deduce that $\dim T(\bar{X}) = \dim \bar{X}$ [49, Th. 4.1], hence $T(\bar{X}) = \bar{X}$. As $\bar{X}(K)$ contains a spanning set for V (Example 2.1.4), T is invertible. Lemma 2.1.8 shows that T is contained in $\mathrm{GL}(V)(K) \cap \mathrm{Stab}_{\mathrm{GL}(V)}(\mathscr{O})(K_{\mathrm{alg}})$, i.e., $\mathrm{Stab}_{\mathrm{GL}(V)}(\mathscr{O})(K)$. $\square$

In older times, this was proved by hand for each choice of G and V, see for example [110].

2.2 The normalizer of G in $\mathrm{GL}(V)$

The purpose of this section is to precisely describe the structure of the normalizer of G in $\mathrm{GL}(V)$, Proposition 2.2.2 below. We maintain the notation and hypotheses of section 2.1.

Write $\mathrm{Aut}(\Delta)$ for the automorphism group of the Dynkin diagram of G. (This is an abuse of notation in that we have already defined Δ to be the set of simple roots, i.e., the vertex set of the Dynkin diagram.) We write $\mathrm{Aut}(\Delta, \lambda)$ for the subgroup of $\mathrm{Aut}(\Delta)$ fixing λ.

Example 2.2.1. Returning to Examples 2.1.2 and 2.1.3, the group $\mathrm{Aut}(\Delta)$ acts on the weights by permuting the fundamental weights according to its action on the diagram; we find that $\mathrm{Aut}(\Delta, \lambda) = 1$ for V with highest weight $d\omega_1$, but for $V = \wedge^d(K^n)$ we have $\mathrm{Aut}(\Delta, \lambda) = 1$ for $n \neq 2d$, and $\mathbb{Z}/2\mathbb{Z}$ for $n = 2d$ (in particular for SL_4 acting on the 4-by-4 alternating matrices).

Write $\mathrm{Aut}(G, \lambda)$ for the inverse image of $\mathrm{Aut}(\Delta, \lambda)$ under the map $\mathrm{Aut}(G) \to \mathrm{Aut}(\Delta)$. To spell this out, recall that given split maximal K-tori T_1, T_2 and Borel K-subgroups B_1, B_2 in G such that $T_i \subset B_i$, there is a $g \in G(K)$ so that $gT_1g^{-1} = T_2$ and $gB_1g^{-1} = B_2$ [8, 19.2, 20.9(i)]. Therefore, given an automorphism ϕ of G, we may compose it with conjugation by an element of $G(K)$ to produce an element ϕ' such that $\phi'(T) = T$ and $\phi'(B) = B$. The automorphism ϕ' is determined up to conjugation by an element of T, so the action of ϕ' on T^* is uniquely determined by ϕ. Then $\mathrm{Aut}(G, \lambda)(K)$ is the collection of $\phi \in \mathrm{Aut}(G)(K)$ such that $\phi'(\lambda) = \lambda$.

The pinning induces a homomorphism i embedding $\mathrm{Aut}(\Delta)$ in the automorphism group of the simply connected cover of G [22, XXIII.4, Th. 4.1]. Further, writing Z for the center of G, we have:

Proposition 2.2.2. *The map i induces a homomorphism* $\mathrm{Aut}(\Delta, \lambda) \to \mathrm{Aut}(G, \lambda)$ *and an injection γ such that the diagram*

$$\begin{tikzcd} & & \mathrm{Aut}(\Delta,\lambda) \arrow[d, "\gamma"'] \arrow[dr, "i"] & & \\ 1 \arrow[r] & \mathbb{G}_m \arrow[r] & N_{\mathrm{GL}(V)}(G) \arrow[r, "\mathrm{Int}"] & \mathrm{Aut}(G,\lambda) \arrow[r] & 1 \end{tikzcd}$$

commutes, the horizontal sequence is exact, and Int *is surjective on K-points. Furthermore, $N_{\mathrm{GL}(V)}(G)$ is smooth and γ identifies $N_{\mathrm{GL}(V)}(G)$ with $((\mathbb{G}_m \times G)/Z) \rtimes \mathrm{Aut}(\Delta, \lambda)$.*

In the statement, we wrote Int for the map such that $\mathrm{Int}(n)(g) = ngn^{-1}$ for $n \in N_{\mathrm{GL}(V)}(G)(R)$ and $g \in G(R)$, for every K-algebra R. For a definition of short exact sequences of affine group schemes and their basic properties, see for example [57, p. 341].

Proof. For the purpose of this proof, write $\widetilde{G}$ for the simply connected cover of G and $\widetilde{Z}$ for its center. For $\pi \in \mathrm{Aut}(\Delta, \lambda)$, $i(\pi) \in \mathrm{Aut}(\widetilde{G})$ normalizes $\ker \lambda|_{\widetilde{Z}}$, which is the kernel of $\widetilde{G} \to G$. Hence $i(\pi)$ induces an automorphism of G. As i is a section of the natural homomorphism $\mathrm{Aut}(\widetilde{G}) \to \mathrm{Aut}(\Delta)$, it is also a section of the natural homomorphism $\mathrm{Aut}(G, \lambda) \to \mathrm{Aut}(\Delta, \lambda)$. It follows from this discussion that i identifies $\mathrm{Aut}(G, \lambda)$ with $(G/Z) \rtimes \mathrm{Aut}(\Delta, \lambda)$.

We claim that $(G/Z)(K)$ is in the image of $N_{GL(V)}(G)(K)$. Indeed, the normalizer contains G and the scalar matrices $\mathbb{G}_m$, and these two groups have intersection Z. This gives an exact sequence

$$1 \longrightarrow \mathbb{G}_m \longrightarrow (\mathbb{G}_m \times G)/Z \xrightarrow{\mathrm{Int}} G/Z \longrightarrow 1. \tag{2.2.3}$$

Applying Galois cohomology gives an exact sequence

$$((\mathbb{G}_m \times G)/Z)(K) \xrightarrow{\mathrm{Int}} (G/Z)(K) \longrightarrow H^1(K, \mathbb{G}_m),$$

where the last term is zero by Hilbert's Theorem 90 [57, 29.3]. As the first term is contained in $N_{\mathrm{GL}(V)}(G)(K)$, we have verified the claim.

For $n \in N_{\mathrm{GL}(V)}(G)(K)$, $\mathrm{Int}(n)$ is an automorphism of G and modifying it by conjugation by an element of $G(K)$, we may assume that $\mathrm{Int}(n)$ normalizes B and T. As $\mathrm{Int}(n)$ defines an equivalence of the irreducible representations or Weyl modules with highest weights λ and ${}^n\lambda$, we deduce that $\mathrm{Int}(n)$ belongs to $\mathrm{Aut}(G, \lambda)(K)$. Running this argument backwards shows that Int is surjective on K-points. This completes also the proof that the sequence is exact and that $N_{\mathrm{GL}(V)}(G)$ is smooth (because $\mathbb{G}_m$ and $\mathrm{Aut}(G, \lambda)$ are [57, 22.12]).

To construct γ, we take $\pi \in \mathrm{Aut}(\Delta, \lambda)(K)$. The element n such that $\mathrm{Int}(n) = i(\pi)$ is determined up to a factor in $K^\times$; we pick n so that $nv^+ = v^+$ and put $\gamma(\pi) := n$. To verify that it is a homomorphism, note that $\mathrm{Int}(\gamma(\pi_1\pi_2)) = \mathrm{Int}(\gamma(\pi_1)\gamma(\pi_2))$, so $\gamma(\pi_1\pi_2)$ and $\gamma(\pi_1)\gamma(\pi_2)$ differ by at most a factor in $K^\times$. But both elements of $\mathrm{GL}(V)$ fix v^+, so they are equal.

For the final claim, note that if $\pi \in \mathrm{Aut}(\Delta, \lambda)$ is such that $\gamma(\pi)$ is in the identity component of $N_{\mathrm{GL}(V)}(G)$, then $\mathrm{Int}\,\gamma(\pi) = i(\pi)$ belongs to the identity component of $\mathrm{Aut}(G, \lambda)$, i.e., to G/Z, and we conclude that the semidirect product N' of $N_{\mathrm{GL}(V)}(G)^\circ$ and $\gamma(\mathrm{Aut}(\Delta, \lambda))$ is identified with a subgroup of the normalizer. Furthermore, writing π_0 to mean the component group, we have

$$\gamma(\mathrm{Aut}(\Delta, \lambda)) = \pi_0(N') \subseteq \pi_0(N_{\mathrm{GL}(V)}(G)) = \pi_0(\mathrm{Aut}(G, \lambda)) = i(\mathrm{Aut}(\Delta, \lambda)),$$

so N' equals $N_{\mathrm{GL}(V)}(G)$. Exactness of (2.2.3) completes the proof. $\square$

Corollary 2.2.4. *If K is algebraically closed, then $N_{\mathrm{GL}(V)}(G)(K)$ is generated by $G(K)$, $K^\times$, and $\gamma(\mathrm{Aut}(\Delta, \lambda))$.* □

In many of the examples considered below, the following holds:

$$\text{There is a connected reductive group } \widetilde{L} \text{ such that } \widetilde{G} \text{ is the derived group of } \widetilde{L} \text{ and } \rho \text{ extends to a homomorphism } \widetilde{\rho} : \widetilde{L} \to \mathrm{GL}(V) \text{ such that } \operatorname{im}\widetilde{\rho} \text{ contains the scalar matrices and } \ker\widetilde{\rho} \text{ is a split torus.} \tag{2.2.5}$$

This allows us to make the more attractive statement, which holds with no hypotheses on K:

Corollary 2.2.6. *Assuming* (2.2.5)*, $N_{\mathrm{GL}(V)}(G)(K)$ is the subgroup of* $\mathrm{GL}(V)$ *generated by $\widetilde{\rho}(\widetilde{L}(K))$ and $\gamma(\mathrm{Aut}(\Delta, \lambda))$.*

Proof. The sequence $1 \to \ker\widetilde{\rho} \to \widetilde{L} \xrightarrow{\widetilde{\rho}} N_{\mathrm{GL}(V)}(G)^\circ \to 1$ is exact by the preceding corollary, so $\widetilde{L}(K) \to N_{\mathrm{GL}(V)}(G)^\circ(K) \to H^1(K, \ker\widetilde{\rho})$ is exact. But the last term is 1 by Hilbert 90 because $\ker\widetilde{\rho}$ is a split torus. □

2.3 Linear transformations preserving minimal elements

We maintain the notation and hypotheses of section 2.1. We will determine the stabilizer $\mathrm{Stab}_{\mathrm{GL}(V)}(\mathscr{O})$ of $\mathscr{O}$ in $\mathrm{GL}(V)$ as an affine group scheme.

Example 2.3.1. If $n \in \mathrm{GL}(V)$ normalizes G, then for every minimal x, the G-orbit of nx in $\mathbb{P}(V)$ is closed and spans V, hence nx is also minimal by Lemma 2.1.5. That is, $N_{\mathrm{GL}(V)}(G)(K)$ is contained in $\mathrm{Stab}_{\mathrm{GL}(V)}(\mathscr{O})(K)$.

Under a technical hypothesis spelled out in Definition 2.3.3, we can say that this containment is an equality. Recall that P is the parabolic subgroup of G stabilizing the highest weight line Kv^+.

Theorem 2.3.2. $\mathrm{Stab}_{\mathrm{GL}(V)}(\mathscr{O}) = N_{\mathrm{GL}(V)}(G)$ *if* P *is not exceptional.*

We delay the proof temporarily.

Definition 2.3.3. Following [21], we define:

(1) If G is simple, then P is *exceptional* in the following cases:

  (a) G has type C_ℓ with $\ell \geq 2$ and P has Levi subgroup of type $C_{\ell-1}$.

  (b) G has type B_ℓ with $\ell \geq 2$ and P has Levi subgroup of type $A_{\ell-1}$.

  (c) G has type G_2 and P is the stabilizer of the highest weight vector in the 7-dimensional fundamental Weyl module.

(2) If G is not simple, we write its adjoint group as $\overline{G}_1 \times \cdots \times \overline{G}_r$ where each $\overline{G}_i$ is simple. We say that P is exceptional if at least one of its images in $\overline{G}_1, \ldots, \overline{G}_r$ is exceptional.

For each of the representations listed in Table A, P is not exceptional, so in these cases Theorem 2.3.2 applies and $N_{\mathrm{GL}(V)}(G) = \mathrm{Stab}_{\mathrm{GL}(V)}(\mathscr{O})$.

All three cases in 2.3.3(1) give genuine exceptions to Theorem 2.3.2. Item (1a) includes the case where $G = \mathrm{Sp}_{2\ell}$ for $\ell \geq 2$ and V is the natural representation. In that case, every nonzero vector is a minimal element, so $\mathrm{Stab}_{\mathrm{GL}_{2\ell}}(\mathscr{O})$ is all of $\mathrm{GL}_{2\ell}$, but $N_{\mathrm{GL}_{2\ell}}(\mathrm{Sp}_{2\ell})$ is $\mathbb{G}_m.\mathrm{Sp}_{2\ell}$, which has dimension only $2\ell^2 + \ell + 1$. Item (1b) is addressed in §2.10. Item (1c) includes the case where G is the automorphism group of the split octonions and V is the space of trace zero octonions. In that case, the closed G-orbit is the quadric in $\mathbb{P}(V)$ defined by the quadratic norm form q [14, 9.2], hence $\mathrm{Stab}_{\mathrm{GL}(V)}(\mathscr{O})$ is the group of similarities of q; this has dimension 22 as opposed to $\dim \mathbb{G}_m.G = 15$.

Remark. Demazure includes a fourth item in his version of 2.3.3, namely that $G \neq 1$ and $P = G$, which would appear as (1d) in 2.3.3 above. But this case cannot occur here due to our assumption that the representation V is faithful.

Proof of Theorem 2.3.2. We abbreviate $N := N_{\mathrm{GL}(V)}(G)$ and $S := \mathrm{Stab}_{\mathrm{GL}(V)}(\mathscr{O})$. Consider the diagram

$$\begin{tikzcd}
1 \arrow[r] & \mathbb{G}_m \arrow[r] \arrow[d, equal] & N^\circ \arrow[r, "\mathrm{Int}"] \arrow[d, dashed] & \mathrm{Aut}(G)^\circ \arrow[r] \arrow[d, "c"] & 1 \\
1 \arrow[r] & \mathbb{G}_m \arrow[r] & S^\circ \arrow[r, "\alpha"] & \mathrm{Aut}(G/P)^\circ \arrow[r] & 1
\end{tikzcd} \tag{2.3.4}$$

where c is given by

$$c(\mathrm{Int}(x))gP = xgP \quad \text{for } g \in G(K) \text{ and } \mathrm{Int}(x) \in \mathrm{Aut}(G)^\circ(K).$$

The top sequence is exact by Proposition 2.2.2. As P is not exceptional, [21, Th. 1] gives that c is an isomorphism, hence α is surjective. From this and Corollary 2.1.6 we see that the bottom sequence is exact and in particular, S° is smooth (because $\mathrm{Aut}(G/P)^\circ$ and $\mathbb{G}_m$ are). Because N° is also smooth, the inclusion $N^\circ(K) \subseteq S^\circ(K)$ given by Example 2.3.1 provides an inclusion of algebraic groups, represented by the dashed arrow in (2.3.4). The inequalities

$$1 + \dim \mathrm{Aut}(G)^\circ = \dim N^\circ \leq \dim S^\circ = 1 + \dim \mathrm{Aut}(G)^\circ,$$

give that $S^\circ = N^\circ$.

Recall from Proposition 2.2.2 that N° is reductive with semisimple part G, hence G is a characteristic subgroup of S°. As S normalizes S°, we deduce that S normalizes G. □

We now describe the matrices in $\mathrm{GL}(V)$ that stabilize $\mathscr{O}(K)$ for various interesting choices of V and $\mathscr{O}$. We rely on Lemma 2.1.8 for the fact that this stabilizer equals the (a priori smaller) group $\mathrm{Stab}_{\mathrm{GL}(V)}(\mathscr{O})(K)$ described by Theorem 2.3.2 and Proposition 2.2.2, so we assume K is infinite. However, this hypothesis is not necessary and can be avoided using arguments from finite group theory, see [39] for a unified argument or the references preceding each statement for a special argument in each case.

For the following result compare [61, Cor. 2], [104, Th. 6.5], [105, Th. 11], or [43, Cor. 6.2].

Corollary 2.3.5 (symmetric matrices). *Suppose K is infinite and has characteristic $\neq 2$. Every invertible linear transformation of $\mathrm{Symm}_n(K)$ that sends rank 1 matrices to rank 1 matrices is of the form*

$$X \mapsto rPXP^t \quad \textit{for some } r \in K^\times \textit{ and } P \in \mathrm{GL}_n(K). \tag{2.3.6}$$

Proof. Take $\widetilde{G} = \mathrm{GL}_n$ as in Example 2.1.3. In the notation of (2.2.5), take $\widetilde{L} = \mathbb{G}_m \times \mathrm{GL}_n$ and for $(r, P) \in K^\times \times \mathrm{GL}_n(K)$, define $\widetilde{\rho}(r, P)$ as in (2.3.6). For every commutative K-algebra R, the set of R-points of $\ker \widetilde{\rho}$ is $\{(t^2, t^{-1}) \mid t \in R^\times\}$. As $\mathrm{Aut}(\Delta, \lambda) = 1$, combining Corollary 2.2.6 and Theorem 2.3.2 gives the claim. □

For the next result compare [67, Th. 3], [104, Th. 5.5], or [43, Cor. 7.3].

Corollary 2.3.7 (alternating matrices). *Suppose K infinite. For $n \geq 2$ and $n \neq 4$, every invertible linear transformation of $\mathrm{Skew}_n(K)$ that sends rank 2 matrices to rank 2 matrices is of the form (2.3.6). If $n = 4$, then every invertible linear transformation of $\mathrm{Skew}_n(K)$ that sends rank 2 matrices to rank 2 matrices is as in (2.3.6) or is*

$$X \mapsto rPX^*P^t \quad \textit{for some } r \in K^\times,\ P \in \mathrm{GL}_n(K), \tag{2.3.8}$$

and

$$\begin{pmatrix} 0 & x_1 & x_2 & x_3 \\ -x_1 & 0 & x_4 & x_5 \\ -x_2 & -x_4 & 0 & x_6 \\ -x_3 & -x_5 & -x_6 & 0 \end{pmatrix}^* = \begin{pmatrix} 0 & x_1 & -x_2 & -x_4 \\ -x_1 & 0 & -x_3 & -x_5 \\ x_2 & x_3 & 0 & x_6 \\ x_4 & x_5 & -x_6 & 0 \end{pmatrix}. \tag{2.3.9}$$

The map $*$ is a Hodge star operator, which is not uniquely determined. Said differently, one can replace $*$ with its composition by any map as in (2.3.6). Therefore, one finds slightly different formulas in other sources, such as [67, p. 921] and [76, p. 15].

Proof of Corollary 2.3.7. We use the same $\widetilde{L}$ and $\widetilde{\rho}$ from the proof of the preceding corollary, substituting $\mathrm{Skew}_n(K)$ for V. If $n \neq 4$, $\mathrm{Aut}(\Delta, \lambda) = 1$ and the proof is complete. Otherwise $n = 4$, $\mathrm{Aut}(\Delta, \lambda) = \mathbb{Z}/2\mathbb{Z}$ and we are tasked with finding the image of the nonidentity element π of $\mathrm{Aut}(\Delta, \lambda)$ under the map γ from Proposition 2.2.2. The element $*$ of $\mathrm{GL}(V)$ fixes $v^+ = E_{12} - E_{21}$. Furthermore, one checks that $\mathrm{Int}(*)$ normalizes the maximal torus T and permutes the root subgroups (described in [12, §VIII.13.3]) as indicated by the action of π on Δ; it follows that $*$ normalizes G, hence $\gamma(\pi) = *$. $\square$

For the next result compare [48], [65, Th. 1], [73, Th. 1], or [104, Th. 3.5].

Corollary 2.3.10 (rectangular matrices)**.** *Suppose K infinite. For $m, n \geq 2$ and $m \neq n$, every invertible linear transformation of the m-by-n matrices with entries in K that sends rank 1 matrices to rank 1 matrices is of the form*

$$X \mapsto AXB \quad \textit{for some } A \in \mathrm{GL}_m(K) \textit{ and } B \in \mathrm{GL}_n(K). \tag{2.3.11}$$

For $n = m \geq 2$, every invertible linear transformation of $M_n(K)$ is of the form (2.3.11) or is

$$X \mapsto AX^tB \quad \textit{for some } A, B \in \mathrm{GL}_n(K). \tag{2.3.12}$$

Sketch of proof. Here one takes $\widetilde{G} := \mathrm{SL}_m \times \mathrm{SL}_n$ and $\widetilde{L} := \mathrm{GL}_m \times \mathrm{GL}_n$ acting on the space V of m-by-n matrices via $\widetilde{\rho}(A, B)X = AXB^t$ and imitates otherwise the proofs of Corollaries 2.3.5 and 2.3.7. $\square$

Example 2.3.13 (homogeneous polynomials of degree d)**.** Assume $\mathrm{char}\, K = 0$ or $> d$ and continue the notation of Example 2.1.3. Viewing K^n as the dual of a vector space, the representation V becomes the vector space of homogeneous polynomials of degree d in n variables. As $\mathrm{Aut}(\Delta, \lambda) = \{\mathrm{Id}_\Delta\}$, Theorem 2.3.2 gives when K is infinite: *the collection of linear transformations of V that preserve the set of d-th powers of nonzero linear forms is the compositum of G and the scalar matrices in* $\mathrm{GL}(V)$. Compare [93, Th. 10.5.5].

For K of arbitrary characteristic, we can instead identify V with the dual of $S^d(K^n)$ [54, II.2.13–16]; we leave the explicit description in this case to the reader.

Example 2.3.14 (exterior powers)**.** Take $\widetilde{G} = \mathrm{SL}_n$ and $V = \wedge^d K^n$ for some $1 \leq d < n$ as in Example 2.1.2, with K infinite. Theorem 2.3.2 gives for $d \neq n/2$: *the collection of linear transformations of V that preserve the set of nonzero decomposable vectors is the compositum of SL_n and the scalar transformations.* (In case $d = n/2$, every linear transformation that preserves the decomposable vectors is as in the previous sentence, or is the composition of such a transformation with a Hodge star operator.) Compare [75, 3.1, 3.2], [108], or [43, Cor. 7.3].

The hypothesis on V in Theorem 2.3.2—that P is not exceptional—is weak enough that many other examples can also be treated readily. For example, one can recover the stabilizer of the decomposable tensors in $\otimes_{i=1}^r K^{n_i}$ as in [109, Th. 3.8]. We also have the following:

Example 2.3.15 (pure spinors)**.** Take $\widetilde{G} = \mathrm{Spin}_{2n}$ for some $n \geq 3$ (so $\widetilde{G}$ is of type D_n) and take V to be a half-spin representation as defined in [18] or [12, §VIII.13.4(IV)]. This representation is injective (i.e., $G = \widetilde{G}$) if n is odd, and has kernel μ_2 if n is even; in this latter case, the image G is called a half-spin or semi-spin group. The minimal elements in V are the *pure spinors* as defined in [18, §3.1].

The representation V is irreducible (regardless of the characteristic of K) because it is minuscule. As $\mathrm{Aut}(\Delta, \lambda) = \{\mathrm{Id}_\Delta\}$, Theorem 2.3.2 gives: if K is infinite, *the collection of linear transformations of V that preserve the set of pure spinors is the compositum of G and the scalar matrices.*

Example 2.3.16 (minimal nilpotents)**.** Let $\widetilde{G}$ be a split, simple, and simply connected group and take $V = \mathrm{Lie}(\widetilde{G})$. This is a Weyl module for $\widetilde{G}$, and it is irreducible if $\operatorname{char} K$ is very good for $\widetilde{G}$. The minimal elements in $\operatorname{Lie}(\widetilde{G})$ are called *minimal nilpotents*.

As λ is the highest root, $\operatorname{Aut}(\Delta, \lambda) = \operatorname{Aut}(\Delta)$. Theorem 2.3.2 gives that, for K infinite, the collection of linear transformations of $\operatorname{Lie}(\widetilde{G})$ that preserve the minimal nilpotents is the compositum of the adjoint group G, the scalar transformations, and a copy of $\operatorname{Aut}(\Delta)$.

To summarize what we observed in this section: the minimal elements are stabilized by the normalizer of G in $\mathrm{GL}(V)$ (Example 2.3.1). We proved that in many cases the normalizer of G is exactly the stabilizer of the minimal elements (Theorem 2.3.2), using Demazure's description of the automorphism group of projective homogeneous varieties. From this and §2.2, one can read off the group of linear transformations that preserve the minimal elements in many cases.

2.4 The stabilizer in $\mathrm{PGL}(V)$

Theorem 2.3.2 has a clean reformulation in terms of subgroups of $\mathrm{PGL}(V)$. We maintain the hypotheses on G and V from §2.1. Put $\overline{G}$ for the image G/Z of G in $\mathrm{PGL}(V)$.

Theorem 2.4.1. *There are natural inclusions of smooth algebraic groups*

$$N_{\mathrm{PGL}(V)}(\overline{G}) \hookrightarrow \mathrm{Stab}_{\mathrm{PGL}(V)}(\mathscr{O}) \hookrightarrow \operatorname{Aut}(\mathscr{O}).$$

If P is not exceptional, then both maps are isomorphisms. If $\overline{G}$ is simple and V is irreducible, then the second map is an isomorphism.

Proof. We first claim that the group $N_{\mathrm{PGL}(V)}(\overline{G})$ is the semidirect product of $\overline{G}$ and the image of $\gamma(\operatorname{Aut}(\Delta, \lambda))$ in $\mathrm{PGL}(V)$. For this, the exact sequence $1 \to \mathbb{G}_m \to \mathrm{GL}(V) \to \mathrm{PGL}(V) \to 1$ identifies $N_{\mathrm{PGL}(V)}(\overline{G})$ with the image of $N_{\mathrm{GL}(V)}(\mathbb{G}_m \cdot G)$ in $\mathrm{PGL}(V)$. As $\mathbb{G}_m$ and G are the center and derived subgroup of $\mathbb{G}_m \cdot G$ respectively, $N_{\mathrm{GL}(V)}(\mathbb{G}_m \cdot G) = N_{\mathrm{GL}(V)}(\mathbb{G}_m) \cap N_{\mathrm{GL}(V)}(G) = N_{\mathrm{GL}(V)}(G)$. The claim follows from Proposition 2.2.2.

The previous paragraph combines with the proof of Theorem 2.3.2 to give the existence of the arrows, as well as the statement that both are isomorphisms if P is not exceptional. If $\overline{G}$ is simple and V is irreducible (and P is exceptional), then there is a larger simple group $\overline{G}'$ contained in $\mathrm{PGL}(V)$ so that the stabilizer P' in $\overline{G}'$ of a point in $\mathscr{O}$ is a parabolic subgroup that is not exceptional [90, (8.1), (8.14)], and we are done by the previous case. □

2.5 Interlude: non-split groups

So far, we have assumed that the group G is split. We now explain how to remove this hypothesis. Suppose for the duration of this section that G is a semisimple group over K with a faithful representation $\rho\colon G \to \mathrm{GL}(V)$, and that, after base change to a separable closure K_{sep} of K, ρ is irreducible or a Weyl module.

We can fix a pinning of $G \times K_{\mathrm{sep}}$, a highest weight vector $v^+ \in V \otimes K_{\mathrm{sep}}$, a parabolic $P := \mathrm{Stab}_{G \times K_{\mathrm{sep}}}(K_{\mathrm{sep}} v^+)$, and a closed orbit $\mathscr{O} \in \mathbb{P}(V) \times K_{\mathrm{sep}}$ as in §2.1.

Proposition 2.5.1. *The closed G-orbit $\mathscr{O}$ is defined over K and* $\mathrm{Stab}_{\mathrm{GL}(V)}(\mathscr{O}) = N_{\mathrm{GL}(V)}(G)$.

Proof. For $\sigma \in \mathrm{Gal}(K_{\mathrm{sep}}/k)$, the action of G on V commutes with σ, so $\sigma(\mathscr{O})$ is a closed G-orbit in $\mathbb{P}(V)$ whose elements span $V \otimes K_{\mathrm{sep}}$, ergo $\sigma(\mathscr{O}(K_{\mathrm{sep}})) = \mathscr{O}(K_{\mathrm{sep}})$. By the Galois criterion for rationality [8, AG.14.4], $\mathscr{O}$ is defined over K. The group schemes $N_{\mathrm{GL}(V)}(G)$ and $\mathrm{Stab}_{\mathrm{GL}(V)}(\mathscr{O})$ are both defined over K, and the claimed equality is by Theorem 2.3.2. □

Suppose now that the representation $V \otimes K_{\mathrm{sep}}$ is as in Table A. We will prove in Propositions 2.8.1, 2.9.1, and 2.11.1 that

$$\mathrm{Stab}_{\mathrm{GL}(V)}(f) \subset \mathrm{Stab}_{\mathrm{GL}(V)}(\{f = 0\}) = N_{\mathrm{GL}(V)}(G)$$

as group schemes over K_{sep}, and it follows from Proposition 2.5.1 that these relationships also hold over K.

2.6 Representations with a one-dimensional ring of invariants

We have completed our study of linear transformations that preserve minimal elements, and we now move on to considering linear preserver problems (LPPs) as described in the introduction. We maintain the notation of section 2.1, so $\widetilde{G}$ is a split semisimple algebraic group over the field K and $\rho\colon \widetilde{G} \to \mathrm{GL}(V)$ is an irreducible representation or a Weyl module. The $d = 2$ cases from Examples 2.1.2 and 2.1.3 are special in that the ring $K[V]^G$ of G-invariant polynomial functions on V equals $K[f]$ for a nonconstant homogeneous polynomial f. (We say that f is G-invariant if every $g \in G(K_{\mathrm{alg}})$ preserves f, where we use the typical algebraist's definition that an element $g \in G(K)$ *preserves* f if $f \circ g = f$ as polynomials.)

The basic facts about this situation are given by the following proposition, which is well known for $K = \mathbb{C}$, see e.g. [79, Prop. 12]. The quotient V/G is defined to be the variety $\operatorname{Spec} K[V]^G$.

Proposition 2.6.1. *The following are equivalent:*

(1) $\dim V/G = 1$.

(2) There is a dense open $G(K_{\mathrm{alg}})$-orbit in $\mathbb{P}(V)(K_{\mathrm{alg}})$ but not in $V \otimes K_{\mathrm{alg}}$.

(3) $K[V]^G = K[f]$ *for some homogeneous* $f \in K[V] \setminus K$.

(4) V/G is isomorphic to the affine line $\mathbb{A}^1$.

Proof. Assuming (1), Theorem 4 in [88] gives that the fraction field of $K[V]^G$ is $K(f)$ for some homogeneous f, and as in the proof of that theorem we deduce (3). (3) implies (4) because the polynomial f (as an element of $K[V]$) is transcendental over K, and (4) trivially implies (1).

Now suppose (2). Recall that there is a G-invariant dense subset U of $V \otimes K_{\mathrm{alg}}$ such that two elements of U have the same image in V/G iff they are in the same $G(K_{\mathrm{alg}})$-orbit. So, if L is a line in V that is in the open orbit in $\mathbb{P}(V)(K_{\mathrm{alg}})$, then $G(K_{\mathrm{alg}}) \cdot L$ contains a nonempty open subset of V, hence contains a nonempty open subset of U; it follows that the map $L \to V/G$ is dominant, hence that $\dim V/G$ is 0 or 1. But for an orbit X of maximal dimension in V, we have $\dim V/G = \dim V - \dim X$, so if $\dim V/G$ is 0, there is a dense orbit in $V \otimes K_{\mathrm{alg}}$. (1) is proved.

Finally suppose (3) holds; we prove (2). The map $f : V \otimes K_{\mathrm{alg}} \to K_{\mathrm{alg}}$ is G-invariant and nonconstant, so there is no dense orbit in $V \otimes K_{\mathrm{alg}}$. As f is homogeneous and separates the $G(K_{\mathrm{alg}})$-orbits in U, it follows that the dense image of $f^{-1}(K_{\mathrm{alg}}^\times) \cap U$ in $\mathbb{P}(V)$ is a single (open) $G(K_{\mathrm{alg}})$-orbit. $\square$

Representations where the conditions in the proposition hold are closely related to the prehomogeneous vector spaces studied in [87], the θ-groups studied by Vinberg as in [80], and the internal Chevalley modules from [3].

All pairs (G, V) with one-dimensional ring of invariants, G simple, and $K = \mathbb{C}$ are listed on pages 260–262 of [80]. In sections 2.8 and 2.9, we will solve the LPP for the representations listed in Table A. That table does not include all possibilities from [80], and we make some remarks about the remaining entries in section 2.10.

Internal Chevalley modules over $\mathbb{Z}$

Let H be a split simple linear algebraic group over $\mathbb{Z}$ and fix a fundamental dominant weight ω of H with respect to some maximal torus T. The choice of ω defines a parabolic subgroup P of H and a Levi subgroup of P (generated by T and the root subgroups for roots orthogonal to ω). We define $\widetilde{G}$ to be the derived subgroup of L and V to be the submodule of $\mathrm{Lie}(H)$ generated by the root subalgebras for roots α with $\langle \alpha, \omega \rangle = 1$; it is a free $\mathbb{Z}$-module of finite rank. Following the notation of §2.1, we define G to be the (scheme-theoretic) image of $\widetilde{G}$ in $\mathrm{GL}(V)$.

By [92, Th. 2], the ring $\mathbb{Z}[V]^G$ of G-invariant polynomial functions in $\mathbb{Z}[V]$ is finitely generated. We write simply $K[V]^G$ for the ring of $(G \times K)$-invariant polynomial functions on $V \otimes K$; there is a natural inclusion $\mathbb{Z}[V]^G \otimes K \to K[V]^G$, but it need not be surjective when K has prime characteristic, see Example 2.6.3.

Proposition 2.6.2. *Let G and V be as in the previous two paragraphs and suppose that* $\dim \mathbb{C}[V]^G = 1$. *Then:*

(1) For every field K, $\dim K[V]^G = 1$.

(2) $\mathbb{Z}[V]^G = \mathbb{Z}[f]$ *for a nonzero, indivisible polynomial f that is determined up to sign.*

Proof. The map $\mathbb{Z}[V]^G \otimes \mathbb{Q} \to \mathbb{Q}[V]^G$ is an isomorphism because $\mathbb{Q}$ is flat over $\mathbb{Z}$ [92, Lemma 2], and similarly for $\mathbb{Q}[V]^G \otimes_{\mathbb{Q}} \mathbb{C} \xrightarrow{\sim} \mathbb{C}[V]^G$. Hence the fiber of $\mathrm{Spec}(\mathbb{Z}[V]^G) \to \mathrm{Spec}(\mathbb{Z})$ over the generic point has dimension 1 and upper semi-continuity gives $\dim K[V]^G \geq 1$ for all K. On the other hand, Th. 2f and the remark on p.560 of [3] give that $\dim K[V]^G \leq 1$, proving (1).

Proposition 2.6.1 gives that $\mathrm{Spec}(K[V]^G)$ is isomorphic to $\mathbb{A}^1$ for every K, hence $\mathbb{Z}[V]^G$ is "locally polynomial" by [56], i.e., for every prime p, $\mathbb{Z}_{(p)}[V]^G$

is isomorphic to polynomials in one variable over $\mathbb{Z}_{(p)}$. Claim (2) now follows by [25, 3.12]. $\square$

In order to determine a formula for (or the degree of) the generator f of $\mathbb{Z}[V]^G$, it suffices to do so for its image in $\mathbb{C}[V]^G$. The degree can be looked up in [80] or [55, Table II] or can be calculated using [87, p. 65, Prop. 15].

The representations in Table A

We now verify that the representations in Table A are irreducible. If $K = \mathbb{C}$ then this is well known, so we assume that $\operatorname{char} K$ is a prime and we apply results from [3]. Fix a particular G, V, and K from Table A for consideration; we can obtain G as in the previous subsection by taking H to be simply connected of type C_n; D_n; E_7; A_{2n-1}; B or D; G_2; E_6; F_4; E_7; E_8; and B or D respectively. By [3] (using our assumption on $\operatorname{char} K$ in case H has roots of different lengths), V is an irreducible representation of G, as claimed.

Furthermore, in each of these cases, $\dim \mathbb{C}[V]^G = 1$ by [87] or [55, §2], so by Proposition 2.6.2(1) $\dim K[V]^G = 1$. We claim that *the image of the polynomial f from Prop. 2.6.2(2) generates $K[V]^G$ as a K-algebra.* To see this, note that $K[V]^G = K[h]$ for some nonzero homogeneous h, so $f = ch^r$ for some $c \in K^\times$ and $r \geq 1$. Now it suffices to verify that f is irreducible in $K[V]$ (as is well known for the determinant from line 4), or to find an element $v \in V \otimes K_{\mathrm{alg}}[t]$ such that $f(v) \in K_{\mathrm{alg}}^\times t$ (as can be done from (2.8.3) for lines 1–5 or from the formula for f from [13, p. 87] or [28, p. 314] for lines 6–11); the claim is proved.

Example 2.6.3 (binary cubics)**.** Suppose $\operatorname{char} K \neq 2, 3$ and consider the vector space V of binary cubic forms; it is the irreducible representation $V = S^3((K^2)^*)$ of $G = \mathrm{SL}_2$ from line 5 of Table A. In the notation of the three preceding paragraphs, one takes H split of type G_2. The maps that send (x, y) to x^3, x^2y, xy^2, y^3 are a basis for the R-module V, and a formula for f is given in [107, §46, (10)] or [44, (14.33)]:

$$f(a_0x^3+a_1x^2y+a_2xy^2+a_3y^3) = a_1^2a_2^2+18a_0a_1a_2a_3-4a_0a_2^3-4a_1^3a_3-27a_0^2a_3^2. \tag{2.6.4}$$

This is the *discriminant* of the cubic form.

We remark that binary cubic forms (and f) can be identified with cubic algebras (and their discriminant algebras) as described in [32, §4] or [46]. Furthermore, this representation is irreducible also in case $\operatorname{char} K = 2$ by Steinberg's tensor product theorem [54, II.3.17]. We ignore these variations below.

The formula for f in (2.6.4) illustrates how the map $\mathbb{Z}[V]^G \otimes K \to K[V]^G$ need not be surjective: when K has characteristic 2, the image of f is $(a_1a_2+a_0a_3)^2$. A similar phenomenon happens for all the representations on lines 6–11 of Table A, as can be seen from the general formula for f in [13] or [28].

Example 2.6.5. The group $\widetilde{G} = \mathrm{SL}_2 \times \mathrm{SO}_n$ for $n \geq 4$ acts naturally on $V = K^2 \otimes K^n$. Suppose $\operatorname{char} K \neq 2$. Then V is irreducible and in the notation of this section, one takes $H = \mathrm{Spin}_{n+4}$. We may identify V with the 2-by-n matrices so that $\widetilde{G}$ acts via $\rho(g_1, g_2)X = g_1Xg_2^t$. There is a symmetric $S \in \mathrm{GL}_n(K)$ so that the K-points of SO_n are the $g_2 \in \mathrm{SL}_n(K)$ such that $g_2^tSg_2 = S$. It follows that the polynomial map $f : V \to K$ defined by $f(X) := \det(XSX^t)$ is invariant under $\widetilde{G}$. It generates $\mathbb{C}[V]^G$ as argued in [87, pp. 109, 110], so $K[V]^G = K[f]$.

In the smallest case $n = 4$, one can equivalently take $\widetilde{G} = \mathrm{SL}_2 \times \mathrm{SL}_2 \times \mathrm{SL}_2$ and $V = K^2 \otimes K^2 \otimes K^2$. In that case, f is Cayley's hyperdeterminant defined in [15]. It appears, for example, in quantum information theory to measure the entanglement of a 3-qubit system [74].

2.7 Transformations that preserve minimal elements and f

We maintain the assumptions of the previous sections, and from here on we assume furthermore that V is irreducible and the ring $K[V]^G$ of G-invariant polynomial functions on V is generated by a non-constant homogeneous element that we denote by f. Since V is an irreducible representation of G and f is not constant, the subspace consisting of $r \in V$ such that $f(r+v) = f(v)$ for all $v \in V$ must be zero. It follows easily from this that every linear transformation preserving f is *invertible*, as noted in [103, §1], hence the collection of linear transformations ϕ of V that preserve f is the group of K-points of the closed sub-group-scheme $\mathrm{Stab}_{\mathrm{GL}(V)}(f)$ of $\mathrm{GL}(V)$. We call it the *preserver* of f; the classical linear preserver problem is to determine the K-points of this group.

Lemma 2.7.1. *If K is algebraically closed, then*

$$N_{\mathrm{GL}(V)}(G)^\circ(K) \cap \mathrm{Stab}_{\mathrm{GL}(V)}(f)(K) = G(K) \cdot \mu_{\deg f}(K).$$

Proof. By Corollary 2.2.4, every element of $N_{\mathrm{GL}(V)}(G)^\circ(K)$ is a product gz for some $g \in G(K)$ and $z \in K^\times$, and the claim is clear. $\square$

In the notation of (2.2.5), the equation

$$f(\widetilde{\rho}(g)v) = \chi(g) f(v) \quad \text{for all } v \in V \otimes K_{\mathrm{alg}}$$

defines a homomorphism $\chi\colon \widetilde{L} \to \mathbb{G}_m$. Corollary 2.2.6 immediately gives:

Lemma 2.7.2. *Assuming* (2.2.5), *the elements of $N_{\mathrm{GL}(V)}(G)^\circ(K)$ that preserve f are $\widetilde{\rho}(g)$ for $g \in \widetilde{L}(K)$ such that $\chi(g) = 1$.* $\square$

As to the non-identity component of $N_{\mathrm{GL}(V)}(G)$, we have:

#	$\widetilde{G}$	V	$\dim V$	f	$\deg f$	$\operatorname{char} K$
1	SL_n	$S^2(K^n)$	$\binom{n}{2}+n$	det	n	$\neq 2$
2	SL_n (n even, $n \geq 4$)	$\wedge^2(K^n)$	$\binom{n}{2}$	Pf	$n/2$	
3	$\mathrm{E}_6^{\mathrm{sc}}$	minuscule	27	see [53, p. 358]	3	
4	$\mathrm{SL}_n \times \mathrm{SL}_n$	M_n	n^2	det	n	
5	SO_n ($n \geq 3$)	K^n	n		2	$\begin{cases} \neq 2 & \text{if} \\ n \text{ odd} \end{cases}$
6	SL_2	binary cubics	4	see Example 2.6.3	4	$\neq 2, 3$
7	SL_6	$\wedge^3(K^6)$	20	see (2.9.4)	$\vdots$	$\vdots$
8	Sp_6	$\wedge_0^3(K^6)$	14	see (2.9.4)	$\vdots$	$\vdots$
9	Spin_{12}	half-spin	32	see [50, p. 1012]	$\vdots$	$\vdots$
10	$\mathrm{E}_7^{\mathrm{sc}}$	minuscule	56	see [28], [13]	$\vdots$	$\vdots$
11	$\mathrm{SL}_2 \times \mathrm{SO}_n$ ($n \geq 4$)	$K^2 \otimes K^n$	$2n$	see Example 2.6.5	4	$\neq 2, 3$

Table A: Some representations V of groups $\widetilde{G}$ so that f generates the ring of all polynomials on V that are invariant under $\widetilde{G}$. In these cases, we calculate the subgroup $\mathrm{Stab}_{\mathrm{GL}(V)}(f)$ of $\mathrm{GL}(V)$.

Lemma 2.7.3. *There is a homomorphism* $\phi\colon \mathrm{Aut}(\Delta, \lambda) \to \mathbb{G}_m$ *such that* $f(\gamma(\pi)v) = \phi(\pi)f(v)$ *for all* $\pi \in \mathrm{Aut}(\Delta, \lambda)$ *and* $v \in V \otimes E$ *for every extension* E *of* K.

Proof. For a fixed $\pi \in \mathrm{Aut}(\Delta, \lambda)$, define $f_\pi \in K[V]$ via $f_\pi(v) := f(\gamma(\pi)v)$. As

$$f_\pi(gv) = f(\gamma(\pi)gv) = f((i(\pi)(g))\gamma(\pi)v) = f_\pi(v) \quad \text{for all } v \in V \otimes E,$$

and f_π and f are homogeneous of the same degree in $K[V]^G$, we deduce that $f_\pi = \phi(\pi)f$ for some scalar $\phi(\pi) \in K$. As f is nonzero on V, f_π is also nonzero, hence $\phi(\pi) \in K^\times$. $\square$

2.8 Lines 1–5 of Table A

For each of the polynomials f appearing in lines 1–5 of Table A, we will determine the linear transformations of V that preserve f. We prove the following, which is a formal version of an imprecise observation made in [63, p. 840].

Proposition 2.8.1. *For the representations in lines 1–5 of Table A, every linear transformation of* V *that preserves* f *belongs to* $N_{\mathrm{GL}(V)}(G)(K)$.

Because we know so much about these representations, we can check this by hand in each case. This is well known for line 4, is done for line 1 in [26], and a similar argument using the generic minimal polynomial defined in [53, Ch. VI] or [33] works for lines 2 and 3. Alternatively, the proposition follows easily from the following:

Lemma 2.8.2. *Suppose* K *is infinite. For the representations in lines 1–5 of Table A, nonzero* $v \in V$, *and an indeterminate* t, *we have:* v *is minimal if and only if* $\deg f(tv + v') \leq 1$ *for all* $v' \in V$.

Again, this claim can be checked by hand in each case. This is trivial for line 5 and is done for line 4 in [66, Lemma 3.2]—note that the difficult direction in that paper is "if". We give a more uniform proof of the "if" direction based on [82].

Proof of Lemma 2.8.2, "if". Whether or not v is minimal is unchanged upon enlarging K, and the same is true for the other condition (because K is infinite), so we may assume that K is algebraically closed. The representation $G \to \mathrm{GL}(V)$ is not only a representation as in §2.6, it is furthermore of the type considered in [82] or [86, §5] and in particular there is a sequence $u_1, \ldots, u_d$ of weight vectors in V so that every element of V is in the $G(K)$-orbit of some $\sum_{i=1}^{r} c_i u_i$ for $c_i \in K^\times$ [82, Th. 1.2(a))]; an element is minimal if and only if it is in the orbit of $c_1 u_1$ for some $c_1 \in K^\times$; and f vanishes on $\sum_{i=1}^{r} c_i u_i$ if and only if $r < d$ (ibid., Prop. 2.15(b)).

The number d is calculated from root system data (ibid., p. 658), but in each case we see that it equals the degree of the invariant polynomial f computed as described in §2.6. We claim that the restriction of f to the span of the u_i is given by

$$f(\sum_{i=1}^{d} c_i u_i) = c \prod_{i=1}^{d} c_i \quad \text{for some } c \in K^\times. \tag{2.8.3}$$

Indeed, the normalizer of T in G permutes the u_i arbitrarily (ibid., Th. 2.1), so the monomials appearing with a nonzero coefficient in the formula for the restriction of f are stable under the obvious action by the symmetric group on d letters. The condition that $f(\sum_{i=1}^{r} c_i u_i)$ with $c_i \in K^\times$ vanishes if and only if $r < d$ together with the degree of f being d implies the claimed formula.

Finally, if v is non-minimal, then it is in the orbit of $\sum_{i=1}^{r} c_i u_i$ for some $r > 1$, and it is easy to produce a v' so that $\deg f(tv + v') > 1$; this settles the "if" direction. □

Proof of Proposition 2.8.1. Any linear transformation ϕ that preserves f by definition also preserves f over every extension of K. Hence, by Lemma 2.8.2, ϕ preserves minimal elements in $V \otimes K_{\mathrm{alg}}$, i.e., ϕ belongs to $\mathrm{Stab}_{\mathrm{GL}(V)}(\mathscr{O})$, which equals $N_{\mathrm{GL}(V)}(G)$ by Theorem 2.3.2. $\square$

Remark 2.8.4. Lines 1, 2, and 4 of Table A have in common that V can be endowed with a bilinear multiplication that is "strictly power associative" and so V has a generic characteristic polynomial as mentioned above. Write E_r for the coefficient of the characteristic polynomial that is a homogeneous function on V of degree r, so that $E_d = f$. A uniform argument as in Lemma 2.8.2 shows that the preserver in $\mathrm{GL}(V)$ of E_r for each $r \in \{3, ..., d-1\}$ is contained in $\mathrm{Stab}_{\mathrm{GL}(V)}(\mathscr{O})$. We omit the details, but the interested reader can find a precise description of the preserver of E_r in [43, Cor. 6.5] for symmetric matrices (line 1), [67] or [43, Cor. 7.7] for alternating matrices (line 2), and [66], [6], or [106, Cor. 1] for square matrices (line 4). (For line 3, one also has a generic characteristic polynomial, but f is the only coefficient of degree ≥ 3.)

Remark 2.8.5. Lines 1–5 of the table do not exhaust all the representations considered by [82]. The ones we have omitted lack a G-orbit of codimension 1 (ibid., Prop. 3.12) yet there is an open G-orbit in $\mathbb{P}(V)$, hence every G-invariant polynomial on V is constant.

We can now determine the subgroup of $\mathrm{GL}(V)$ of elements that preserve f. Our first result concerns symmetric matrices as in line 1 of the table. Compare [30, §7.III], [26, Th. 1], [61], [104, Th. 6.7], or [43, Cor. 6.3].

Corollary 2.8.6 (symmetric matrices)**.** *For $n \geq 2$ and K of characteristic $\neq 2$, every linear transformation ϕ of* $\mathrm{Symm}_n(K)$ *that preserves the determinant is of the form* (2.3.6) *where* $r^n \det(P)^2 = 1$.

Proof. Combine Proposition 2.8.1, Lemma 2.7.2, and Corollary 2.3.5. $\square$

The next results concern alternating matrices as in line 2 of the table. Compare [67, Th. 3], [104, Th. 5.7], or [43, Cor. 7.4].

Corollary 2.8.7 (alternating n-by-n matrices)**.** *For even $n \geq 6$, every linear transformation ϕ of* $\mathrm{Skew}_n(K)$ *that preserves the Pfaffian is of the form* (2.3.6) *where* $r^{n/2}\det(P) = 1$. □

Corollary 2.8.8 (alternating 4-by-4 matrices)**.** *Every linear transformation of* $\mathrm{Skew}_4(K)$ *that preserves the Pfaffian is of the form* (2.3.6) *or* (2.3.8) *where* $r^{n/2}\det(P) = 1$

Proof. In view of Lemma 2.7.3, it suffices to pick some $X \in \mathrm{Skew}_4(K)$ with $\mathrm{Pf}(X) \neq 0$ and verify that $\mathrm{Pf}(X^*) = \mathrm{Pf}(X)$ for $*$ as in (2.3.9). □

We now determine the linear transformations that preserve the determinant. This is the case famously treated by Frobenius in [30, §7.I] and Dieudonné in [23], and also in [64, Th. 2] and [104, Th. 4.2].

Corollary 2.8.9 (square matrices)**.** *Every linear transformation of $M_n(K)$ that preserves the determinant is of the form* (2.3.11) *or* (2.3.12) *where* $\det(AB) = 1$. □

Every minuscule representation V of a group G of type E_6 has a nonzero G-invariant cubic form f, and G-invariance uniquely determines f up to multiplication by an element of $K^\times$. For the following result, compare [98, 7.3.2] (for $\mathrm{char}\, K \neq 2, 3$) or [1, 5.4]. The analogous (and a priori coarser) result for Lie algebras is [62, 5.5.1].

Corollary 2.8.10 (minuscule representation of E_6)**.** *In the notation of the preceding paragraph, the preserver of f in* $\mathrm{GL}(V)$ *is* $G(K)$.

Proof. Since $\mathrm{Aut}(\Delta, \lambda) = 1$ and μ_3 is in the center of G, Lemma 2.7.1 gives the claim. □

Commuting with the adjoint

For the representations considered in this section, one has a notion of a "classical adjoint" $\operatorname{adj}\colon V \to V$, which is a polynomial map of degree $(\deg f) - 1$. For lines 1, 2, and 4 of the table, the papers [94] and [17] compute the linear transformations on V that commute with this map. We can do the same for line 3, where G is the simply connected split group of type E_6. The group $\operatorname{Aut}(\Delta)$ is $\mathbb{Z}/2\mathbb{Z}$ and we write π for the nonzero element; the subgroup of G of elements fixed by $i(\pi)$ is a split group of type F_4 [14, 7.3] which we denote simply by F_4. The center of G is a copy of μ_3. We find:

Corollary 2.8.11. *If* $\operatorname{char} K \neq 2, 3$*, then the subgroup of* $\operatorname{GL}(V)$ *of elements commuting with the adjoint is* $F_4(K) \cdot \mu_3(K)$*.*

Proof. The minimal elements are precisely the nonzero $v \in V$ so that $\operatorname{adj} v = 0$ [14, 7.10], so any element of $\operatorname{GL}(V)$ that preserves adj necessarily preserves minimal elements, hence belongs to $\mathbb{G}_m.G$. (One could alternatively deduce this using the identity $\operatorname{adj}(\operatorname{adj} v) = f(v)v$.) Further, for any $g \in G$ and $c \in \mathbb{G}_m$, we have $\operatorname{adj}(cgv) = c^2 i(\pi)(g) \operatorname{adj}(v)$ [14, 7.9], hence such a cg commutes with adj if and only if $i(\pi)(g) = c^{-1}g$. In particular, c belongs to G and so is a cube root of unity. That is, the subgroup H of $\operatorname{GL}(V)$ of elements commuting with the adjoint is contained in G, and the image of H in the adjoint group G/μ_3 is contained in the subgroup fixed by $i(\pi)$.

As $F_4 \times \mu_3$ is obviously contained in H, it suffices to show that its image in G/μ_3 is the subgroup fixed by $i(\pi)$. But this subgroup is connected reductive with Lie algebra of type F_4 [14, 7.3], hence is the same as the image of F_4. This proves the claim. $\square$

2.9 Lines 6–11 of Table A

The representations on lines 6–11 of Table A are all of the form considered in [83], [28], and [72]. In particular, the ring of G-invariant polynomials on V is generated by a homogeneous polynomial f of degree 4. These representations appear, for example, when studying electromagnetic black hole charges in various supergravity theories, see [9]. We suppose in this section that the characteristic of K is $\neq 2, 3$; the assumption that the characteristic is $\neq 2$ is so that we may apply the results of [83] and the assumption that the characteristic is $\neq 3$ is a convenience so that we may apply the results of [45]. We will prove:

Proposition 2.9.1. *For the representations in lines 6–11 of Table A, every element of* $\mathrm{GL}(V)$ *that preserves* f *belongs to* $N_{\mathrm{GL}(V)}(G)(K)$*.*

In view of our assumption on the characteristic, we are free to abuse notation and multilinearize f to obtain a symmetric 4-linear form that we also denote by f. Further, for each of these representations, there is a nondegenerate skew-symmetric bilinear form $\langle\,,\rangle$ on V that is invariant under G. This allows us to define a trilinear map $t\colon V \times V \times V \to V$ implicitly by the equation:

$$\langle t(x_1, x_2, x_3), x_4\rangle = f(x_1, x_2, x_3, x_4) \quad \text{for } x_1, x_2, x_3, x_4 \in V.$$

For each $x \in V$, we define a symmetric bilinear form b_x on V via $b_x(v_1, v_2) = f(x, x, v_1, v_2)$. We have:

Lemma 2.9.2. *x is a minimal element if and only if the dimension of the radical of b_x is* $(\dim V) - 1$*.*

Proof. Line 6 is the representation from Example 2.6.3, for which we may check the claim of the lemma by hand. So assume G and V come from one of the lines 7–11; we may apply results from §3–4 of [45].

The radical of b_x has codimension 1 if and only if it is the subspace $y^\perp$ of V of vectors orthogonal (relative to $\langle\,,\rangle$) to some nonzero $y \in V$. That is, if and only if there is a nonzero $y \in V$ such that $f(x, x, y^\perp, z) = 0$ for all $z \in V$. (For "only if", one needs to know that b_x is nonzero for x nonzero, which is Lemma 14 in ibid.) In turn, this is equivalent to: there is a nonzero $y \in V$ such that $t(x, x, z) \in Ky$ for every $z \in V$. But by ibid., Propositions 18 and 20, that is the same as asking for x to be minimal. $\square$

Proof of Proposition 2.9.1. Suppose ϕ preserves f. It defines an isometry between the bilinear forms b_x and $b_{\phi(x)}$ for all $x \in V$. Now apply Lemma 2.9.2 to deduce that ϕ belongs to $\mathrm{Stab}_{\mathrm{GL}(V)}(\mathscr{O})$, hence to $N_{\mathrm{GL}(V)}(G)$ by Theorem 2.3.2. $\square$

We now determine the preserver of the discriminant of binary cubic forms as in line 6 or Example 2.6.3. We omit the details in the proofs of this corollary and the following items because they are entirely similar to earlier proofs.

Corollary 2.9.3 (binary cubics)**.** *Every linear transformation on the vector space of cubic forms $K^2 \to K$ that preserves the discriminant is of the form $q \mapsto cq \circ g$ for some $c \in K^\times$ and $g \in \mathrm{GL}_2(K)$ such that $c^4(\det g)^6 = 1$.* $\square$

Line 7 of the table concerns an SL_6-invariant quartic form f on $\wedge^3 K^6$, for which a (complicated-looking) formula is given in [87, p. 83]; we now give an alternative presentation. Write K^6 as a direct sum $V_2 \oplus V_4$ where V_d has dimension d. There is a natural inclusion $w : V_2 \otimes (\wedge^2 V_4) \to \wedge^3 K^6$ given by $(c, x) \mapsto c \wedge x$. Amongst the line of invariant quartic forms on $\wedge^3 K^6$, there is an element f so that, with respect to a fixed basis a, b of V_2, we have:

$$f(w(a \otimes x + b \otimes y)) = \langle x, y\rangle^2 - 4\,\mathrm{Pf}(x)\,\mathrm{Pf}(y) \qquad (2.9.4)$$

where $\langle x, y\rangle$ denotes the coefficient of t in $\mathrm{Pf}(x + ty)$, i.e., the polarization of the $\mathrm{SL}(V_4)$-invariant quadratic form Pf. (To check the claim (2.9.4), one can either use the formula for f in [87] or one can observe that fw is a nonzero quartic form on $V_2 \otimes (\wedge^2 V_4)$ that is invariant under $\mathrm{SL}(V_2) \times \mathrm{SL}(V_4)$, that there is a unique line of such forms if $K = \mathbb{C}$, and that the right side of (2.9.4) gives such a form.) As every $\mathrm{SL}_6(K)$-orbit in $\wedge^3 K^6$ meets the image of w by [81, Lemma 2.2], equation (2.9.4) is enough to specify f on $\wedge^3 K^6$.

Corollary 2.9.5. *Every linear transformation of $\wedge^3(K^6)$ that preserves the invariant quartic form is of the form*

$$v_1 \wedge v_2 \wedge v_3 \mapsto c(gv_1 \wedge gv_2 \wedge gv_3) \text{ for } c \in K^\times,\ g \in \mathrm{GL}_6 \text{ with } c^4(\det g)^2 = 1 \tag{2.9.6}$$

or the composition of a Hodge star operator with a transformation as in (2.9.6). □

Regarding line 8 of the table, recall that Sp_6 is defined as the subgroup of GL_6 leaving a particular nondegenerate skew-symmetric bilinear form b invariant on its (tautological) representation K^6. We write $\wedge^3_0(K^6)$ for the kernel of the contraction map $\wedge^3(K^6) \to K^6$, cf. [31, §17.1]. The restriction of the SL_6-invariant quartic form on $\wedge^3 K^6$ to $\wedge^3_0(K^6)$ gives an Sp_6-invariant quartic form.

We define GSp_6 to be the subgroup of GL_6 of transformations that scale the bilinear form b by a factor in $K^\times$; it is isomorphic to $(\mathrm{Sp}_6 \times \mathbb{G}_m)/\mu_2$.

Corollary 2.9.7. *Every linear transformation of the space $\wedge^3_0(K^6)$ that preserves the invariant quartic is of the form*

$$v_1 \wedge v_2 \wedge v_3 \mapsto c(gv_1 \wedge gv_2 \wedge gv_3) \quad \text{for some } c \in K^\times \text{ and } g \in \mathrm{GSp}_6(K)$$

with $c^4(\det g)^2 = 1$. □

For the representations on lines 9 and 10, we will prove a result under the assumption that K contains a square root of -1. Alternatively, we could eliminate this hypothesis at the cost of defining a reductive envelope $\widetilde{L}$ of G as we defined GSp_6 for Sp_6 above, i.e., as in (2.2.5).

We write HSpin_{12} for the image G of Spin_{12} under a half-spin representation.

Corollary 2.9.8. *Suppose K contains a square root of -1. Then the subgroup of $\mathrm{GL}_{32}(K)$ of transformations that preserve the HSpin_{12}-invariant quartic form is $\mathrm{HSpin}_{12}(K) \cdot \mu_4(K)$.* □

For the next result, compare [97, Cor. 2.6(i)] or [45, §10]. Those proofs are based on versions of Corollary 2.8.10 for E_6, but our proof does not refer to E_6.

Corollary 2.9.9. *Suppose K contains a square root of -1. Then the subgroup of $\mathrm{GL}_{56}(K)$ of transformations that preserve the $\mathrm{E}_7^{\mathrm{sc}}$-invariant quartic form is $\mathrm{E}_7^{\mathrm{sc}}(K) \cdot \mu_4(K)$.* □

As for line 11, we consider first the case $n = 4$. As the automorphism group of the Dynkin diagram of $\mathrm{SL}_2 \times \mathrm{SL}_2 \times \mathrm{SL}_2$ is the symmetric group $\mathcal{S}_3$, we have the following result (compare [45, §11]):

Corollary 2.9.10. *Every linear transformation of $K^2 \otimes K^2 \otimes K^2$ that preserves the hyperdeterminant is of the form*

$$v_1 \otimes v_2 \otimes v_3 \mapsto g_1 v_1 \otimes g_2 v_2 \otimes g_3 v_3 \quad \textit{for } g_1, g_2, g_3 \in \mathrm{GL}_2(K)$$

such that $\det(g_1 g_2 g_3) = \pm 1$, or is the composition of such a map with a permutation

$$v_1 \otimes v_2 \otimes v_3 \mapsto v_{\sigma(1)} \otimes v_{\sigma(2)} \otimes v_{\sigma(3)} \quad \textit{for } \sigma \in \mathcal{S}_3.$$

Proof. In the notation of (2.2.5), one takes $\widetilde{L}$ to be a product of 3 copies of GL_2 with the obvious $\widetilde{\rho}$; the kernel of $\widetilde{\rho}$ is isomorphic to $\mathbb{G}_m \times \mathbb{G}_m$. In view of Proposition 2.8.1 and §2.7, it suffices to check that the permutations preserve the hyperdeterminant, which is clear from the explicit formula for the hyperdeterminant from, e.g., [74]. □

For the representations on line 11 with $n \geq 5$, we define GO_n to be the algebraic group with R-points the matrices $g \in \mathrm{GL}_n(R)$ such that $g^t S g = \mu(g) S$ for some $\mu(g) \in R^\times$ (for every K-algebra R); it is a reductive envelope of O_n.

Corollary 2.9.11. *For $n \geq 5$, every linear transformation of $M_{2n}(K)$ that preserves the degree 4 function from Example 2.6.5 is of the form*

$$X \mapsto g_1 X g_2^t \quad \textit{for } g_1 \in \mathrm{GL}_2(K),\ g_2 \in \mathrm{GO}_n(K) \textit{ with } \det(g_1)\mu(g_2) = \pm 1.$$

Sketch of proof. Note that $\mathrm{Aut}(\Delta, \lambda)$ is naturally identified with the component group of GO_n. □

As a concrete illustration of the remarks in §2.5, we note that Corollary 2.9.11 and Example 2.6.5 go through with no change if we replace the split groups SO_n and O_n with the special orthogonal and orthogonal groups of any nondegenerate symmetric bilinear form, i.e., where the matrix S in Example 2.6.5 is any symmetric invertible matrix. In this way, the corollary gives the stabilizer of f also in the case where $K = \mathbb{R}$ and SO_n is replaced by a real group $\mathrm{SO}(2, n-2)$ or $\mathrm{SO}(6, n-6)$; this situation appears in the study of electromagnetic black hole charges in $\mathcal{N} = 2$ or 4 supergravity, see e.g. [9].

2.10 Some representations omitted from Table A

We have not yet discussed all pairs (G, V) where G is absolutely almost simple, V is an irreducible representation of G, and $K[V]^G$ is generated by a homogeneous polynomial f. For $K = \mathbb{C}$, all such pairs are listed in the the table on pages 260–262 of [80], and we now discuss each of the cases that we have thus far omitted.

Consider one of the groups HSpin_n for $n = 7, 9$ with their natural representations or G_2 with its 7-dimensional representation. In these cases, f has degree 2, i.e., is a quadratic form, so its linear preserver is the orthogonal group $\mathrm{O}(f)$. Note that our Theorem 2.3.2 does not apply to these groups because they correspond to exclusions (1b) and (1c) in Definition 2.3.3.

The natural 32-dimensional representation of the group HSpin_{11} factors through the natural representation of HSpin_{12}. The ring $K[V]^{\mathrm{HSpin}_{12}}$ is also 1-dimensional (as can be seen by the reasoning in §2.6, where H has type E_7) with generator f of degree 4, so clearly the f stabilized by HSpin_{11} is the same as for HSpin_{12} and so the linear preserver of this f is $\mathrm{HSpin}_{12}.\mu_4$ as in Corollary 2.9.8. (For generalizations of this sort of example, see [95].)

The only remaining pairs (G, V) are $(\mathrm{SL}_7, \wedge^3(K^7))$, $(\mathrm{SL}_8, \wedge^3(K^8))$, and HSpin_{14} with its natural 120-dimensional representation. The first representation is noteworthy, because the stabilizer in SL_7 of any element v such that $f(v) \neq 0$ is a group of type G_2, see for example [27, p. 65], [2], or [19]. The orbits in the last representation have been studied over $\mathbb{C}$ in [79], and the fact that $\dim V/G = 1$ has been applied to the theory of quadratic forms in [84] and [85], see also [34]. All three of these representations are irreducible and are not stable under an outer automorphism of G, so applying Theorem 2.3.2, we find without doing any work that $\mathrm{Stab}_{\mathrm{GL}(V)}(\mathscr{O})$ is $\mathbb{G}_m.G$. As to the preserver $\mathrm{Stab}_{\mathrm{GL}(V)}(f)$ in these cases, we omit serious investigation. However, for $K = \mathbb{C}$, one can observe that the identity component G' of $\mathrm{Stab}_{\mathrm{GL}(V)}(f)$ is reductive (because V is an irreducible representation), hence is semisimple (because the center must consist of scalar matrices). It follows from the classification of semisimple groups G' such that $\mathbb{C}[V]^{G'}$ is generated by a single polynomial that $G' = G$. In particular, G is normal in $\mathrm{Stab}_{\mathrm{GL}(V)}(f)$. As $\mathrm{Aut}(\Delta, \lambda) = 1$, it follows that $\mathrm{Stab}_{\mathrm{GL}(V)}(f)$ is contained in $G.\mathbb{G}_m$, i.e., $\mathrm{Stab}_{\mathrm{GL}(V)}(f)$ is $G.\mu_d$, where d is the degree of f (equal to 7, 16, or 8 respectively).

2.11 An alternative formulation of the linear preserver problem

Inspecting the LPP solutions by Frobenius (1897) and Dieudonné (1949) where V is the n-by-n matrices and f is the determinant, one sees that Frobenius determines the preserver of det whereas Dieudonné determines the linear transformations on V that preserve the set of singular matrices. So far, we have been solving Frobenius' version of the problem, but in fact we have also solved Dieudonné's version:

Proposition 2.11.1. *For each of the representations in Table A, the collection of linear transformations preserving the projective variety $f = 0$ is $N_{\mathrm{GL}(V)}(G)$.*

See e.g. [89] for general results on the relationship between the two versions.

Proof. Put S for the sub-group-scheme of $\mathrm{GL}(V)$ preserving the projective variety $f = 0$. Given any $s \in S(K_{\mathrm{alg}})$, ${}^s f$ is in the ideal generated by f and has the same degree as f, hence ${}^s f = cf$ for some $c \in K^\times$ and $c^{-1/\deg f} s$ preserves f. Propositions 2.8.1 and 2.9.1 give that s belongs to $N_{\mathrm{GL}(V)}(G)$.

Conversely, for $n \in \mathrm{GL}(V)$ normalizing G, Corollary 2.2.4 and Lemma 2.7.3 show that ${}^n f$ is a scalar multiple of f. $\square$

Note that the proposition indeed solves Dieudonné's version of the linear preserver problem for the representations in Table A, because we calculated the group $N_{\mathrm{GL}(V)}(G)$ in §2.2.

Chapter 3

Classifying forms of simple groups via their invariant polynomials

(The results in this chapter are joint with Anthony Ruozzi.)

3.1 Introduction

It is a classical result that a nondegenerate quadratic form q over a field of characteristic different from 2 is determined up to scalar multiple by the corresponding orthogonal group $\mathrm{O}(q)$[1]. In this paper we prove an extension of this result for more general algebraic groups. Explicitly, let G be a simple linear algebraic group over a field F and let V be an absolutely irreducible representation of G. A result of Garibaldi-Guralnick [38] states that in "most cases" there exists a homogeneous polynomial f on V which is invariant under the action of G and such that the identity component of the scheme-theoretic stabilizer of f is G. We show, under the same hypotheses, that one can construct a *maximally stable* polynomial which additionally has stabilizer group "as large as possible".

For such an invariant polynomial we show that, in many cases, similarity classes of f classify twisted forms of G for which V is defined over F up to

[1] Although the result is also true for fields of characteristic 2, this doesn't seem to be as well known, see the discussion in [68] and [69]

isomorphism. In particular, if S denotes the stabilizer of f, then the Galois cohomology set $\mathrm{H}^1(F,S)$ classifies twisted forms of f up to isomorphism, and the set $\mathrm{H}^1(F,\mathrm{Aut}(G))$ classifies twisted forms of G up to isomorphism. We note that if f' is a twisted form of f then the identity component of its stabilizer is a twisted form of G, and so the operation of taking the identity component of stabilizer groups gives a map of pointed sets:

$$\rho : \mathrm{H}^1(F,S) \to \mathrm{H}^1(F,\mathrm{Aut}(G)). \tag{3.1.1}$$

Notice that in the case where $S = \mathrm{O}(q)$ the classical result discussed above can be rephrased as stating that the map ρ is injective.

This setup leads to some natural questions:

(1) What are the conditions on a twisted form f' of f to map to G under ρ? That is to say, what are the fibers of ρ?

(2) Which twisted forms of G appear as identity components of stabilizers of forms of f? In other words, what is the image of ρ?

We answer these questions in detail in the sections below culminating in the following theorem,

Theorem 3.1. *Let f be a maximally stable form for a simple group G and put S for the stabilizer of f. The map $\rho_E : \mathrm{H}^1(E,S) \to \mathrm{H}^1(E,\mathrm{Aut}(G))$ is onto with kernel* $\mathrm{Sim}(V_E, f)$ *for all field extensions E of F if and only if the highest weight λ of V is in the root lattice of G and is fixed by every automorphism of the Dynkin diagram of G.*

Proof. The theorem follows immediately from Corollaries 3.5.5 and 3.4.14. □

Given an adjoint group G then with some restrictions on the characteristic of F a representation for which the theorem applies always exists:

Theorem 3.2. *Let G be an adjoint simple linear algebraic group over a field F of very good characteristic for G*[2] *and not 2 or 3 then there exists a polynomial f defined on the adjoint representation $V = \mathrm{Lie}(G)$ such that there is a one-to-one correspondence between twisted forms of G and similarity classes of twisted forms of f.*

Proof. Since the characteristic of F is very good for G, V is irreducible and faithful. Further, since the highest weight of V is the highest root of G which is unique, it is fixed by every automorphism of the Dynkin diagram of G. It follows by Theorem 3.1 that it suffices to show that there exists a maximally stable form f on V. This is the precisely content of Corollary 3.2.8. $\square$

3.2 Constructing invariants with large stabilizer groups

Unless stated otherwise, G will denote a simple algebraic group over a field F and V a faithful absolutely irreducible representation of G with highest weight λ. Throughout f' will denote a G-invariant polynomial defined on V such that the scheme-theoretic stabilizer S of f' has identity component G, so that in particular S is an algebraic group by [57, Pr. 21.10]. Such an f' is known to exist for "most pairs" G, V as above by the results of [38], cf. Corollary 3.2.8 bellow.

In order to get tighter control on the stabilizer we need to describe the structure of the normalizer group $N_{\mathrm{GL}(V)}(G)$. For split G this is done in [7, Prop. 2.2]. We now recall the relevant facts and explain how this can be extended to non-split G.

[2]We say the characteristic of F is very good for G if $\mathrm{Lie}(G)$ is irreducible. By [47, Table 1] this is not the case only if F has characteristic 2 and G has type B_n, C_n, D_n, E_7 or F_4, or F has characteristic 3 and G has type E_6 or G_2 or if G has type A_n and F has characteristic 2 or p with $p \mid n+1$.

Suppose first that G is split. Fix a choice of a maximal split torus T of G and a Borel subgroup B containing T. For any automorphism $\phi \in \mathrm{Aut}(G)$ there exists an element $g \in G(F)$ such that $\phi' = \mathrm{Int}(g) \circ \phi$ maps T and B isomorphically to themselves, where $\mathrm{Int}(g)$ denotes the inner automorphism given by conjugation by g. The action of ϕ' on T^* then induces an automorphism of the set of simple roots of G with respect to T, i.e., a graph automorphism of the Dynkin diagram of G. We denote the group of such automorphisms by $\mathrm{Aut}(\Delta)$. We then have an exact sequence [57, Th. 25.16]

$$1 \longrightarrow \overline{G} \longrightarrow \mathrm{Aut}(G) \xrightarrow{\alpha} \mathrm{Aut}(\Delta). \tag{3.2.1}$$

Suppose now that G is not split, then there is a split group G' defined over F and a cocycle $\eta \in H^1(F, \mathrm{Aut}(G'))$ such that the twisted group G'_η is isomorphic to G. Therefore, twisting the sequence for G' gives a sequence for G and hence a definition of the map

$$\alpha : \mathrm{Aut}(G) \to \mathrm{Aut}(\Delta).$$

We recall from [37, §2] that the group $\mathrm{Aut}(\Delta)$ is finite étale but not necessarily constant, since the absolute Galois group Γ of F acts on $\mathrm{Aut}(\Delta)$ via the $*$-action. We write $\mathrm{Aut}(\Delta, \lambda)$ for the group of automorphisms of Δ which fix λ and $\mathrm{Aut}(G, \lambda)$ for the subgroup of $\mathrm{Aut}(G)$ which α maps to $\mathrm{Aut}(\Delta, \lambda)$. Note that $\mathrm{Aut}(G, \lambda)$ is an smooth because its identity component is the adjoint group $\overline{G}$.

Let Z be the center of G, and note that as subgroups of $\mathrm{GL}(V)$, G and $\mathbb{G}_m$ intersect precisely at Z so the group $(G \times \mathbb{G}_m)/Z$ is well defined. We have the following lemma.

Lemma 3.2.2. *The following diagram is commutative with exact horizontal* *and vertical rows:*

$$\begin{tikzcd}
 & & \mathrm{Aut}(\Delta,\lambda) & & \\
1 \arrow[r] & \mathbb{G}_m \arrow[r] & N_{\mathrm{GL}(V)}(G) \arrow[u, dotted, "\alpha\circ\mathrm{Int}"] \arrow[r, "\mathrm{Int}"] & \mathrm{Aut}(G,\lambda) \arrow[r] \arrow[ul, "\alpha"'] & 1 \\
 & & (G\times\mathbb{G}_m)/Z \arrow[u] & &
\end{tikzcd}$$

Proof. If G is split, this is [7, Prop. 2.2]. The non-split case follows by twisting. □

Definition 3.2.3. Let G be a simple group, V a faithful irreducible representation of G of highest weight λ. An invariant polynomial f defined on V is *maximally stable* if its stabilizer S has identity component G and the following sequence of algebraic groups is exact

$$1 \longrightarrow Z(S) \longrightarrow S \xrightarrow{\mathrm{Int}} \mathrm{Aut}(G,\lambda) \longrightarrow 1.$$

Example 3.2.4. Let G and V be such that $\mathrm{Aut}(\Delta,\lambda)=1$, in particular this is the case if G has type A_1, B_n, C_n, E_7, E_8, F_4 or G_2. Every polynomial $f\in F[V]^G$ is maximally stable. In fact in this case we have $\mathrm{Aut}(G,\lambda)=\mathrm{Inn}(G)\cong\overline{G}$, and the statement follows from the fact that $G\subset S$.

Example 3.2.5. Let (V,q) be an n-dimensional nondegenerate quadratic space over F, we claim that q is maximally stable. By definition the stabilizer group of (V,q) is the orthogonal group $\mathrm{O}(q)$, which has identity component $G=\mathrm{SO}(q)$. The representation V of G has highest weight $\lambda=\lambda_1$ in the numbering from [11].

If q is odd dimensional then $\mathrm{Aut}(\Delta)=1$ and the statement follows from the previous example. If q is even dimensional and $n\neq 4$ then $\mathrm{Aut}(\Delta)=\mathbb{Z}/2\mathbb{Z}$ and every automorphism of Δ fixes λ so therefore we have $\mathrm{Aut}(G,\lambda)=$

$\mathrm{Aut}(G)$. If $n = 4$ then $\mathrm{Aut}(\Delta) = S_3$ but $\mathrm{Aut}(\Delta, \lambda) = \mathbb{Z}/2\mathbb{Z}$ where this group is generated by transposing the two vertices of the Dynkin diagram which don't correspond to λ. To check that the sequence is exact, it suffices to find an element of $\mathrm{O}(q)$ which induces the nontrivial automorphism of the Dynkin diagram generating $\mathrm{Aut}(\Delta, \lambda)$, and for this we can take any isometry of determinant -1 in $\mathrm{O}(q)$.

Example 3.2.6. Let $G = \mathrm{SL}_n$ ($n \geq 2$) and $V = \mathrm{Lie}(G)$ its adjoint representation, then the odd-degree symmetric functions on V are G-invariant but not maximally stable, see [12, Ex. 1, 2].

Proposition 3.2.7. *If there exists an invariant polynomial f' on V whose stabilizer has identity component G, then there exists a maximally stable invariant polynomial f on V.*

Proof. By Lemma 3.2.2 there is an exact sequence,

$$1 \longrightarrow \mathbb{G}_m \longrightarrow N_{\mathrm{GL}(V)}(G) \xrightarrow{\mathrm{Int}} \mathrm{Aut}(G, \lambda) \longrightarrow 1.$$

We note that $N_{\mathrm{GL}(V)}(G)$ contains both G and the scalar matrices $\mathbb{G}_m$ and that the intersection of these two subgroups is precisely the center Z of G. Therefore, the quotient $(G \times \mathbb{G}_m)/Z$ with Z imbedded diagonally is a normal subgroup of $N_{\mathrm{GL}(V)}(G)$. Again by Lemma 3.2.2 there is a map $\psi : N_{\mathrm{GL}(V)}(G) \to \mathrm{Aut}(\Delta, \lambda)$. Choose representatives $n_i \in N_{\mathrm{GL}(V)}(G)$ for the cosets of $(G \times \mathbb{G}_m)/Z$ in $N_{\mathrm{GL}(V)}(G)$. Since the image of ψ is finite, there are only finitely many such n_i. Define

$$f := \prod n_i f'.$$

This definition depends on the choice of the n_i, but since G stabilizes f' and $\mathbb{G}_m$ acts by scalar multiplication, any other choice produces the same form up to scalar multiplication.

We claim that f is the required invariant polynomial. To see this, let S be the stabilizer of f, and consider the action of $n \in N_{\mathrm{GL}(V)}(G)$ on f. By our definition of the n_i there exists a $g \in (G \times \mathbb{G}_m)/Z$ and some i so that $n = n_i g$, note that $g \in (G \times \mathbb{G}_m)/Z$ acts on f by multiplication by $\alpha^d \in F$, where d is the degree of f. We then have

$$n \cdot f = n_i g \cdot f = \alpha^d n_i f = \alpha^d n_i \prod n_j f' = \alpha^d \prod (n_j n_i) f' = \alpha^d \beta f$$

for some $\alpha, \beta \in F^\times$ where the last equality follows since $\{n_i n_j\}$ contains exactly one element of every coset.

To finish note that since $S \subset N_{\mathrm{GL}(V)}(G)$, the identity component of S is G and the kernel of the map $S \to \mathrm{Aut}(G, \lambda)$ is contained in $\mathbb{G}_m$ and thus must be equal to $Z(S)$. It remains to check that $S \to \mathrm{Aut}(G, \lambda)$ is onto as a map of algebraic groups, which can be checked over an algebraic closure F_{alg} of F. To do this, take $\sigma \in \mathrm{Aut}(G, \lambda)(F_{\mathrm{alg}})$. Because the map $N_{\mathrm{GL}(V)}(G) \to \mathrm{Aut}(G, \lambda)$ is onto, there exists a lift $n \in N_{\mathrm{GL}(V)}(F_{\mathrm{alg}})$ of σ and by the discussion above $n \cdot f = \alpha f$ for some scalar $\alpha \in F$. Let $g_\alpha \in \mathbb{G}_m(F_{\mathrm{alg}})$ be the matrix corresponding to $\frac{1}{\alpha^{1/d}}$. Then $n \cdot g_\alpha \in S(F_{\mathrm{alg}})$ maps to σ as required. □

Corollary 3.2.8. *Let G be a simple algebraic group over an infinite field F and V a faithful irreducible representation of G such that (G_{alg}, V_{alg}) is not listed in [38, Tables B,C or D]. If the characteristic of F is large enough then there exists a maximally stable form f defined on V.*

Proof. By [38, Th. 14.1] there exists a form f defined on V so the naive stabilizer of f is G. If the characteristic of F is large enough then the naive stabilizer coincides with the scheme-theoretic stabilizer and the statement follows from the proposition. □

Remark 3.2.9. The proof of the proposition reveals an equivalent way of thinking about a maximally stable form f. Namely, for a maximally stable form f with stabilizer $S \subset \mathrm{GL}(V)$ having connected component G, the stabilizer is so large that elements of $N_{\mathrm{GL}(V)}(G)$ act on f by scalar multiples. That this condition is sufficient follows from the proof of the theorem. Conversely, as G is a normal subgroup of S, $S \subset N_{\mathrm{GL}(V)}(G)$. If f is maximally stable, this gives a diagram:

$$\begin{tikzcd}
1 \arrow[r] & Z(S) \arrow[r] \arrow[d] & S \arrow[r] \arrow[d] & \mathrm{Aut}(G,\lambda) \arrow[r] \arrow[d, equal] & 1 \\
1 \arrow[r] & \mathbb{G}_m \arrow[r] & N_{\mathrm{GL}(V)}(G) \arrow[r] & \mathrm{Aut}(G,\lambda) \arrow[r] & 1
\end{tikzcd}$$

Suppose that $n \in N_{\mathrm{GL}(V)}(F)$ acts by $n \cdot f = f'$. The image g of n in $\mathrm{Aut}(G,\lambda)(F)$ might not be the image of an element of $S(F)$, but it will be over an algebraic closure. Take $s \in S(F_{\mathrm{alg}})$ that maps to g thought of as an element of $\mathrm{Aut}(G,\lambda)(F_{\mathrm{alg}})$. By commutativity, s and n then differ by an element $m_\alpha \in \mathbb{G}_m(F_{\mathrm{alg}})$. That is, over F_{alg}, $n \cdot f = m_\alpha s \cdot f = \alpha^d f = f'$. Since n and f' are defiend over F, $\alpha^d \in F$, and $f' = \alpha^d f$ as homogeneous polynomials over F, as desired.

3.3 Two cohomology sequences

Let G and V be as above with f maximally stable, so in particular, for the stabilizer S of f, we have that the sequence

$$1 \longrightarrow Z(S) \longrightarrow S \xrightarrow{\mathrm{Int}} \mathrm{Aut}(G,\lambda) \longrightarrow 1$$

is exact. Applying Galois cohomology, we obtain the following sequence:

$$\mathrm{H}^1(F, Z(S)) \longrightarrow \mathrm{H}^1(F,S) \xrightarrow{\phi'} \mathrm{H}^1(F, \mathrm{Aut}(G,\lambda)). \tag{3.3.1}$$

Now set H to be the image of the map $\mathrm{Aut}(G) \to \mathrm{Aut}(\Delta)$ modulo the image of the map $\mathrm{Aut}(G,\lambda) \to \mathrm{Aut}(\Delta,\lambda)$. We then have a second exact sequence

$$1 \longrightarrow \mathrm{Aut}(G,\lambda) \longrightarrow \mathrm{Aut}(G) \longrightarrow H \longrightarrow 1.$$

Again passing to cohomology, we get an exact sequence

$$H(F) \longrightarrow \mathrm{H}^1(F, \mathrm{Aut}(G, \lambda)) \xrightarrow{\phi''} \mathrm{H}^1(F, \mathrm{Aut}(G)). \tag{3.3.2}$$

Put

$$\phi = \phi'' \circ \phi' : \mathrm{H}^1(F, S) \to \mathrm{H}^1(F, \mathrm{Aut}(G)).$$

It is simply the map on cohomology induced by the composition

$$\mathrm{Int} : S \to \mathrm{Aut}(G, \lambda) \hookrightarrow \mathrm{Aut}(G).$$

Proposition 3.3.3. *The map ϕ coincides with the map ρ defined in (3.1.1).*

Proof. Let f' be a twisted form of f defined on a vector space V'/F and suppose that $\rho(f') = G'$, i.e., the connected component of the stabilizer of f' is G'. By definition there is an element $g \in \mathrm{Iso}(f', f)_{sep} \subset \mathrm{Iso}(V', V)_{sep}$ such that we have the following equation

$$f'(v_{sep}) = f(g \cdot v_{sep}) \quad \text{for all } v_{sep} \in V'_{sep}.$$

Now let g be such an element. As before we note that the stabilizer S' of f' is a twisted form of S, and therefore there exists an isomorphism

$$\theta : S'_{sep} \simeq S_{sep}.$$

We claim that θ is nothing other than conjugation by g. In fact this just comes down to the observation that if L is an element of $\mathrm{GL}(V')$, then the transformation gLg^{-1} is simply L thought of as a linear transformation of V via the isomorphism g. More explicitly, let $g' \in S'_{sep}$ be a linear transformation which stabilizes f', then we have

$$f(gg'g^{-1} \cdot v_{sep}) = f'(g'g^{-1} \cdot v_{sep}) = f'(g^{-1} \cdot v_{sep}) = f(v_{sep})$$

so $gg'g^{-1} \in S_{sep}$. Its inverse can be constructed similarly. Clearly, θ maps G'_{sep} isomorphically onto G_{sep}, so it induces an element of $\mathrm{Iso}(G', G)_{sep}$.

To compare the maps ρ and ϕ we begin by recalling that the identification between the set of isomorphism classes of twisted forms of an object (say the homogenous polynomial f) and torsors with coefficients in the automorphism group of that object (in this case denoted by S) is given as follows: suppose that f' is a twisted form of f, then there exists an element $g \in \mathrm{Iso}(f', f)_{sep}$ as above. The identification then maps the element f' to the cocycle $\sigma \mapsto g^{-1} \cdot \sigma(g)$. Next note that, following the remarks before the proposition, the map ϕ is given by taking a cocycle $\alpha_\sigma \in \mathrm{H}^1(F, S)$ and mapping it to the cocycle $\sigma \mapsto \mathrm{Int}(\alpha_\sigma) \in \mathrm{H}^1(F, \mathrm{Aut}(G))$.

Choosing an isomorphism $S_{sep} \simeq \mathrm{Iso}(f', f)_{sep}$ and thus, via θ, an isomorphism $\mathrm{Aut}(G)_{sep} \simeq \mathrm{Iso}(G', G)_{sep}$ we may assume that the cocycle has the form $\alpha_\sigma = [\sigma \mapsto g^{-1} \cdot \sigma(g)]$. We then have

$$\phi(\alpha_\sigma) = [\sigma \mapsto \mathrm{Int}(g^{-1}\sigma(g))] = [\sigma \mapsto \mathrm{Int}(g^{-1}) \circ \mathrm{Int}(\sigma(g))] = [\sigma \mapsto \mathrm{Int}(g^{-1}) \circ \sigma(\mathrm{Int}(g))]$$

which by the previous paragraph is precisely the cocycle corresponding to G'. $\square$

3.4 The Fibers

By Proposition 3.3.3, we now know that the map ρ is induced by a group homomorphism. In this section, we study its fibers using the above long exact sequences in cohomology. Since $\rho = \phi = \phi'' \circ \phi'$, we need to look at the fibers of both of these maps. But first, a definition:

Definition 3.4.1. Two homogeneous forms, f and f', on a vector space V are *similar* with multiplier $\alpha \in F^\times$ if there is an isomorphism $h : V \to V$ such that

$$f'(h(v)) = \alpha f(v) \text{ for all } v \in V.$$

If the multiplier $\alpha = 1$, then the two forms are said to be *isomorphic*.

The same definition extends to twisted forms of a form (V, f). In particular, let (V', f') and (V'', f'') be twisted forms of (V, f). We say that f' is *similar* to f'' with multiplier $\alpha \in F^\times$ if there is an isomorphism $h : V'' \to V'$ such that

$$f'(h(v)) = \alpha f''(v) \text{ for all } v \in V''.$$

Any form similar to f' is then, up to isomorphism, the form $\alpha f'$ for some α, and since the stabilizer group of a form f' is clearly isomorphic to the stabilizer of $\alpha f'$, we reduce to considering similarity up to isomorphism of forms. That is, the set

$$\mathrm{Sim}(V', f') := F^\times / \{\alpha \in F^\times | f' \simeq \alpha f'\}.$$

Clearly, if f' has degree d, then $f' \simeq \alpha^d f'$ for all $\alpha \in F^\times$. In general, however, there can be elements of $F^\times$ trivial in this quotient which are not d-th powers.

To see this, consider $\mathrm{Sim}(V, f)$, and choose an isomorphism $h : V \to V$ such that $f(h(v)) = \alpha f(v)$ for all $v \in V$ and some $\alpha \in F^\times$. It follows that for any $g \in G$,

$$h^{-1}gh \cdot f = h^{-1}g(\alpha f) = h^{-1}(\alpha f) = f.$$

Therefore $h \in N_{\mathrm{GL}(V)}(G)$. For a maximally stable f, every element of the normalizer acts on f by scalar multiplication (see Remark 3.2.9), so the $\alpha \in F^\times$ that occur as $n \cdot f = \alpha f$ for some $n \in N_{\mathrm{GL}(V)}(G)$ are exactly those multipliers for which $f \simeq \alpha f$. After twisting, a similar argument holds for any twisted form of (V, f). We summarize this in a proposition:

Proposition 3.4.2. *Let f be a maximally stable form on V. Then for any twisted form (V', f') over F,*

$$\mathrm{Sim}(V', f') = F^\times / \{\alpha \in F^\times | n \cdot f' = \alpha f' \textit{ for some } n \in N_{\mathrm{GL}(V)}(G)\}.$$

By the definition of ρ, it follows that if two twisted forms f' and f'' of f are similar then $\rho(f') = \rho(f'')$. In particular, any fiber of ρ breaks up into a disjoint union

$$\coprod_i \mathrm{Sim}(V_i, f_i)$$

of similarity classes. The most interesting case is when each fiber consists of a single similarity class, a possibility that we pursue in what follows.

Let us first consider the fibers of ϕ'. To compute them, consider the map

$$F^\times/F^{\times d} \cong \mathrm{H}^1(F, Z(S)) \to \mathrm{H}^1(F, S)$$

from sequence (3.3.1). This map sends $\alpha \in F^\times$ to the isomorphism class of the form αf. That is, the kernel of $\phi' : \mathrm{H}^1(F, S) \to \mathrm{H}^1(F, \mathrm{Aut}(G, \lambda))$ is precisely the similarity classes of f. Choosing any other form (V', f') of f and twisting this exact sequence, we then conclude that $\phi'^{-1}([f']) = \mathrm{Sim}(V', f')$.

If we want the fibers of ρ to coincide with the similarity classes, we need additionally that ϕ'' is injective on

$$\Omega := \ker[H^1(F, \mathrm{Aut}(G, \lambda)) \to H^2(F, Z(S))].$$

Lemma 3.4.3. *If the group H occuring in the sequence*

$$1 \longrightarrow \mathrm{Aut}(G, \lambda) \longrightarrow \mathrm{Aut}(G) \longrightarrow H \longrightarrow 1$$

is trivial then ϕ'' is injective.

Proof. This is immediate from the definitions (see sequence (3.3.2)). □

$H = 1$ happens automatically for G having types B, C, E_7, E_8, F_4, and G_2 since in these cases $\mathrm{Aut}(\Delta) = 1$. So, let us assume that $H \neq 1$. In this case, G has type A, D, or E_6. For the moment, we will exclude the unique case of D_4. With this exclusion, $\mathrm{Aut}(\Delta) = \mathbb{Z}/2\mathbb{Z}$, so the non-triviality of H implies that $\mathrm{Aut}(\Delta, \lambda) = 1$ and $H = \mathrm{Aut}(\Delta)$. Note that for arbitrary group G, the map $\alpha : \mathrm{Aut}(G) \to \mathrm{Aut}(\Delta)$ from sequence (3.2.1) is not onto. However, of the above types, only D_n with n even presents a problem. Of these groups, the only issue is the half-spin group, HSpin, and it is easy to check that H is always trivial in this case. Moreover,

Lemma 3.4.4. *If G is of type A_n ($n \geq 2$), D_n ($n \geq 5$), or E_6 and $H \neq 1$, then since the highest weight λ is defined over F, G must be of inner type.*

Proof. λ being defined over F is the same as the Tits algebra $[A_\lambda]$ being defined over F. So, if G is inner type, this is clearly satisfied. If G is outer type, then usually the Tits algebras are defined over a quadratic field extension with the exception of some particular cases (for example type A_n, n odd: dominant weights mapping to $\frac{n+1}{2} \in \mathbb{Z}/(n+1)\mathbb{Z} = \Lambda/\Lambda^r$ and type D_n: dominant weights mapping to $\overline{e_1} \in \Lambda/\Lambda^r$). In all of these cases, it is straightforward, but tedious, to check that λ fixed by the $*$-action implies that it is fixed by $\mathrm{Aut}(\Delta)$ as well. That is, these cases only occur when $H = 1$. $\square$

Let $\widetilde{G} \to G$ be the simply connected cover of G. The center of $\widetilde{G}$ will be denoted $\widetilde{Z}$. The induced map on the centers $\widetilde{Z} \to Z$ has kernel Z_0. Recall the definition of the Tits class of a group G. $t_G := -\partial(\nu_G)$ where $\nu_G \in H^1(F, \overline{G})$ is the unique element that maps to a quasi-split form of G in $H^1(F, \mathrm{Aut}(G))$ and ∂ is the connecting homomorphism in the cohomology sequence associated to

$$1 \longrightarrow \widetilde{Z} \longrightarrow \widetilde{G} \longrightarrow \overline{G} \longrightarrow 1.$$

For any $\eta \in H^1(F, \overline{G})$, the Tits class of the twisted form G_η is related to the Tits class of G by the following very useful formula

$$t_{G_\eta} = t_G + \partial(\eta). \tag{3.4.5}$$

Proposition 3.4.6. *If G is of type A_n ($n \geq 2$), D_n ($n \geq 5$), or E_6 and $H \neq 1$ then $\phi''|_\Omega$ is injective if and only if t_G and $\partial(\Omega)$ are fixed by the action of* $\mathrm{Aut}(\Delta)$.

Proof. By Lemma 3.4.4, any G satisfying the given properties has inner type. Therefore, every inner form of G also has inner type. From the remarks preceding the lemma, we obtain several immediate simplifications. First, $H \neq 1$ implies that $\mathrm{Aut}(\Delta, \lambda) = 1$, $H = \mathrm{Aut}(\Delta) = \mathbb{Z}/2\mathbb{Z}$, and $\mathrm{Aut}(G) \to \mathrm{Aut}(\Delta)$ is onto for groups of the stated types. Second, $\mathrm{Aut}(G, \lambda) = G/Z := \overline{G}$, and finally, the sequence (3.3.1) becomes

$$1 \longrightarrow Z(S) \longrightarrow S \longrightarrow \overline{G} \longrightarrow 1.$$

Combining this with the sequence associated to the simply connected cover, we get a commutative diagram:

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \widetilde{Z} & \longrightarrow & \widetilde{G} & \longrightarrow & \overline{G} & \longrightarrow & 1 \\ & & \downarrow & & \downarrow & & \| & & \\ 1 & \longrightarrow & Z(S) & \longrightarrow & S & \longrightarrow & \overline{G} & \longrightarrow & 1 \end{array}$$

where the middle map is given by composing the surjection of $\widetilde{G}$ onto G with the inclusion of G into S. Passing to cohomology, we get a commutative square

$$\begin{array}{ccc} H^1(F, \overline{G}) & \xrightarrow{\partial} & H^2(F, \widetilde{Z}) \\ \| & & \downarrow p \\ H^1(F, \overline{G}) & \xrightarrow{d} & H^2(F, Z(S)) \end{array}$$

Since the map $\widetilde{G} \to S$ factors through G, p factors through $H^2(F, Z)$ and $\ker(p) = H^2(F, Z_0)$. Another consequence of the commutativity of this square is that given $\eta \in H^1(F, \overline{G})$, $d(\eta) = 0$ if and only if $\partial(\eta) \in H^2(F, Z_0)$. That is, $\partial(\Omega)$ is always $|Z_0|$-torsion.

Our proof of the injectivity of $\phi''|_\Omega$ relies on the following result of Garibaldi [37, Th. 11, Ex. 17]. There it is shown for a simply connected group that $\phi''|_\Omega$ is injective if and only if $\mathrm{Aut}(G_\eta)(F) \to \mathrm{Aut}(\Delta)(F)$ is onto for every $\eta \in \Omega$. Since $H \neq 1$, this latter condition holds if and only if the Tits class t_{G_η} is fixed by the $\mathrm{Aut}(\Delta)$-action for every $\eta \in \Omega$. If G is not simply connected, then it is a matter of observing that [37, Th. 11] still holds since, from the remarks before the lemma, $\mathrm{Aut}(G) \to \mathrm{Aut}(\Delta)$ is surjective in all our cases. It follows, in general, $\phi''|_\Omega$ is injective if and only if $\mathrm{Aut}(\Delta)$ fixes t_{G_η} for every $\eta \in \Omega$.

Of course, the "trivial form" G is always in Ω, and so, a necessary condition for injectivity is that t_G be fixed by $\mathrm{Aut}(\Delta)$. If G is simply connected, $Z_0 = 1$, so $\ker(d) = \ker(\partial) = \Omega$. From the above formula (3.4.5) for the Tits class of a twisted form, it then follows that t_G fixed by $\mathrm{Aut}(\Delta)$ is also sufficient. In the non-simply connected case, $t_{G_\eta} - t_G = \partial(\eta)$ need not be trivial, so $\phi''|_\Omega$ is injective if and only if $\mathrm{Aut}(\Delta)$ fixes these elements as well. $\square$

Corollary 3.4.7. *Suppose that G is of type A_n $(n \geq 2)$ and the group $H \neq 1$. Then $\phi''|_\Omega$ is injective if and only if t_G and the class of every degree $n+1$ central simple F-algebra A with exponent dividing $|Z_0|$ has exponent ≤ 2 in case A_n.*

Proof. $\mathrm{Aut}(\Delta)$ acts on $\widetilde{Z}$ by -1, so the elements of $H^2(F, \widetilde{Z})$ fixed by this action are precisely the 2-torsion ones. In particular, for $\phi''|_\Omega$ to be injective we need at least that t_G is 2-torsion. This is also sufficient if G is simply connected.

For G not simply connected, $1 < |Z_0| = m \leq n+1$. If $t_G = 0$ or $2|m$ then since every degree $n+1$ central simple F-algebra of exponent dividing m occurs as the Tits class of some twisted for G_η, it follows from formula (3.4.5) and the commutative diagram that $\partial(\Omega)$ consists exactly of the classes of degree $n+1$ central simple F-algebras with exponent dividing m. If t_G has order 2 and 2 is prime to m, then since every degree $n+1$ central simple F-algebra of exponent dividing $2m$ (and exponent at least 2) occurs as the Tits class of some twisted form G_η there is an η such that

$$t_G + \partial(\eta) = t_{G_\eta} = [B] + t_G$$

for B any central simple F-algebra of degree $n+1$ and exponent diciding m. Note that this last equality holds for every choice of B because the exponent m being odd forces the index of B to be at most $\frac{n+1}{2}$. Thus, the class $[B]+t_G$ contains a degree $n+1$ algebra. It then follows as above that $\partial(\Omega)$ consists exactly of the classes of degree $n+1$ central simple F-algebras with exponent dividing m. Finally, the general remarks require that $\partial(\Omega)$ be fixed by $\mathrm{Aut}(\Delta)$. In particular, every central simple algebra of the listed type must have exponent at most 2. $\square$

Corollary 3.4.8. *Suppose that G is of type E_6 and the group $H \neq 1$. Then $\phi''|_\Omega$ is injective if and only if t_G and the class of every degree 27 central simple F-algebra of exponent dividing $|Z_0|$ is trivial.*

Proof. In this case t_{G_η} is 3-torsion for every $\eta \in H^1(F, \overline{G})$, so being fixed by $\mathrm{Aut}(\Delta)$ acting as -1 then implies that it is trivial. If G is simply connected then t_G is fixed by the $\mathrm{Aut}(\Delta)$ action if and only if $t_G = 0$. If G is adjoint, $\Omega = H^1(F, G)$. Since any degree 27 algebra of exponent 3 occurs as the Tits class of some inner twisted form of G and t_{G_η} is fixed by $\mathrm{Aut}(\Delta)$ if and only if it is trivial, it follows that this is only possible if every algebra of this type is split. $\square$

Corollary 3.4.9. *Suppose that G is of type D_n ($n \geq 5$) and the group $H \neq 1$. If $\widetilde{G}$ is isomorphic to* $\mathrm{Spin}(A, \sigma, f)$ *for some central simple F-algebra of degree $2n$ with quadratic pair, then $\phi''|_\Omega$ is injective if and only if A is split and, if G is adjoint, every central simple algebra of degree $2n$ and exponent 2 is also split.*

Proof. The twisted forms of G correspond to degree $2n$ central simple algebras with quadratic pair as in the statement of the proposition. By, [37, Ex. 17], the relations on the Clifford algebras in the Brauer group imply that for any twisted form, G_η, t_{G_η} is fixed by the $\mathrm{Aut}(\Delta)$-action if and only if the corresponding algebra is split.

By our assumption on the irreducibility of V, G cannot be simply connected of type D_n with n even. If it is simply connected with n odd, then from the above remarks, this happens if and only if the algebra splits. If G is adjoint, then as in the previous arguments this is only possible if all algebras of the above type are split. The remaining cases are $G = \mathrm{SO}$ and $G = \mathrm{HSpin}$ (n even). From the remarks before the lemma, our assumptions on H imply that G cannot correspond to a half-spin group. For SO, $Z_0 = \mu_2$ is imbedded diagonally in $\mu_2 \times \mu_2$ (n even) or is the unique order 2 subgroup of μ_4 (n odd). In the first case, $\mathrm{Aut}(\Delta)$ acts by swapping the components and in the second, it acts by -1 on $H^2(F, \widetilde{Z})$. Therefore, if $\eta \in \Omega$, $p(\partial(\eta)) = 0$, so $\partial(\eta) \in H^2(F, Z_0)$, which by the above description is fixed by $\mathrm{Aut}(\Delta)$. Thus, $\phi''|_\Omega$ is injective if and only if t_G is fixed by $\mathrm{Aut}(\Delta)$. This shows that $G = SO(V', q)$ for q a quadratic form on a $2n$-dimensional vector space V'/F. □

Remark 3.4.10. The only remaining case is that of D_4. If it is trialitarian, then G becomes inner only over a field extension of degree 3 or 6 over F. As λ is defined over F, a check as in the lemma shows that this is only possible if $\mathrm{Aut}(\Delta, \lambda) = \mathrm{Aut}(\Delta) = S_3$. That is, $H = 1$ and ϕ'' is injective by the remarks before the lemma. In the remaining case of ${}^2D_4$, $\mathrm{Aut}(\Delta, \lambda) = \mathbb{Z}/2\mathbb{Z}$. However, it is not clear if the desired equivalences in [37, Th. 11] hold, so the argument in the proposition does not apply. We know of no method that works in this case.

Remark 3.4.11. The classical situation of quadratic forms follows immediately from case B_n for odd dimensional forms and the case of SO_{2n} completed in the above proposition. It is worth noting that it only holds, without additional assumptions, for quadratic forms over vector spaces (and not for the more general algebras with quadratic pairs).

In summary, we have proven:

Proposition 3.4.12. *If $H = 1$ or G satisfies the conditions of Proposition*

3.4.6 then

$$\rho : \mathrm{H}^1(F, S) \to \mathrm{H}^1(F, \mathrm{Aut}(G))$$

is an injection.

Corollary 3.4.13. *Let G and H be as in the proposition. Then for (V', f') and (V'', f'') twisted forms of (V, f) with G' and G'' the (respective) connected components of their stabilizers, $G' \simeq G''$ if and only if $f' \simeq \alpha f''$ for some $\alpha \in F^\times$.*

Corollary 3.4.14. *If $H = 1$ or G satisfies the conditions of Proposition 3.4.6 over a field F then for any field extension E/F,*

$$\rho : \mathrm{H}^1(E, S) \to \mathrm{H}^1(E, \mathrm{Aut}(G))$$

is an injection.

Proof. Since the exponent of any element of the Brauer group can only decrease after base extension to E, all of the required properties continue to hold over E, so replacing F by E, the result follows from the proposition. $\square$

3.5 The Image

To complete our picture, we need to determine the image of ρ. Consider the simple group G and any faithful, irreducible representation $\psi : G \to \mathrm{GL}(V)$. The *automorphism group of the representation* ψ, denoted by $\mathrm{Aut}(G, \psi, V)$ is the Zariski closure of the set of pairs $(\xi, \eta) \in \mathrm{Aut}(G) \times \mathrm{Aut}(V)$ which commute with ψ. We define two maps

$$\pi_1 : \mathrm{Aut}(G, \psi, V) \to \mathrm{Aut}(G) \times \mathrm{Aut}(V) \xrightarrow{pr_1} \mathrm{Aut}(G) \tag{3.5.1}$$

$$\pi_2 : \mathrm{Aut}(G, \psi, V) \to \mathrm{Aut}(G) \times \mathrm{Aut}(V) \xrightarrow{pr_2} \mathrm{Aut}(V) \tag{3.5.2}$$

where pr_i denotes the projection into the ith component. We note that the image of π_2 is contained in the normalizer $N_{\mathrm{GL}(V)}(G)$ of G in $\mathrm{GL}(V)$ since this is precisely the condition necessary for an automorphism of V to be compatible with the representation ψ. We are now ready to determine the image of ρ

Proposition 3.5.3. *For a maximally stable polynomial f the image of ρ consists precisely of those twisted forms of G for which the representation V is defined over F.*

Proof. First note that ρ is defined as taking the identity component of the stabilizer of a polynomial defined on V, so every form of G which is in the image of ρ has the representation V defined over F. For the other containment, let G_γ be a form of G for which the representation V is defined over F, that is to say there is a cocycle $\gamma^* \in \mathrm{H}^1(F, \mathrm{Aut}(G, \psi, V))$ so that $\psi_\gamma : G_\gamma \to \mathrm{GL}(V)$ is given by γ^*. We have the following commutative diagram

$$\begin{tikzcd}
\mathrm{Aut}(G,\psi,V) \arrow[rr, "\pi_2"] \arrow[dr, "\pi_1"'] & & N_{\mathrm{GL}(V)}(G) \arrow[dl, "\mathrm{Int}"] \\
& \mathrm{Aut}(G) &
\end{tikzcd}$$

where π_1 and π_2 are the maps defined in (3.5.1). The induced diagram on cohomology is

$$\begin{tikzcd}
\mathrm{H}^1(F,\mathrm{Aut}(G,\psi,V)) \arrow[rr, "p_2"] \arrow[dr, "p_1"'] & & \mathrm{H}^1(F, N_{\mathrm{GL}(V)}(G)) \arrow[dl, "\mathrm{Int}"] \\
& \mathrm{H}^1(F,\mathrm{Aut}(G)) &
\end{tikzcd}$$

and we have that $p_1(\gamma^*) = \gamma$. This implies that γ must come from a cocycle in $\mathrm{H}^1(F, N_{\mathrm{GL}(V)}(G))$, and in fact from a cocycle in $\mathrm{H}^1(F, S)$ as claimed. $\square$

Remark 3.5.4. The argument in Proposition 3.5.3 is basically that found in [100, Prop. 42.4.3]. Note that the proposition in [100] asserts that given a group G, a representation V of G, and a twist G' of G, it is "necessary and sufficient" that G' be a strictly inner twist G in order for V to be defined as a representation of G', but only the "sufficient" is true. Namely, there can be two forms of a group with equivalent faithful F-representations where one is not a strictly inner form of the other. Consider for example the groups SO_{2n} and $\mathrm{SO}(q)$ where q has dimension $2n$ and nontrivial discriminant with their natural representations. The precise problem in the argument in [100] is the statement "the image of the sequence

$$\mathrm{Aut}(H_0, \rho_0, V_0) \to \mathrm{Aut}(H_0) \times \mathrm{Aut}(V_0) \xrightarrow{\mathrm{pr}_2} \mathrm{Aut}(V_0)$$

is contained in the image of ρ_0". In fact, one can only guarantee that the image of this map is contained in the normalizer of the image of ρ_0 as above.

We also have the following functorial form of Proposition 3.5.3

Corollary 3.5.5. *Let f be a maximally stable form. The map $\rho_E : \mathrm{H}^1(E, S) \to \mathrm{H}^1(E, \mathrm{Aut}(G))$ is surjective for every field extension E of F if and only if the highest weight λ of V is in the root lattice of G and $\mathrm{Aut}(\Delta, \lambda) = \mathrm{Aut}(\Delta)$.*

Proof. Given a group G over a field F, the results of [99] say that the irreducible representations of G defined over F are in one to one correspondence with the set of dominant weights λ of G which satisfy three conditions: 1) λ is a character of T^*_{sep}, where T is a maximal torus of G, 2) λ is fixed by the $*$-action and 3) the Tits algebra A_λ of G is trivial. Tits further proves that these conditions are satisfied by all dominant weights in the root lattice which are fixed by the $*$-action, and the if direction follows. Now suppose that λ is not in the root lattice, Merkurjev proves in [71] that there exists a field extension E of F and a twisted form G' of G defined over E for which the Tits algebra A_λ has maximal index and hence is not trivial, and this completes the proof. $\square$

3.6 Groups of type E_8, F_4, and G_2

It follows from the results of the previous sections that if the group G is adjoint and its Dynkyn diagram has no nontrivial automorphisms then the map ρ given by a maximally stable form f is onto and has kernel the similarity classes of f. This leads to the following proposition

Proposition 3.6.1. *Let G be a simple group of type E_8, F_4 or G_2 and V any faithful irreducible representation of G except for the representations with highest weight λ_1 for G_2 and λ_4 for F_4. Then if the characteristic of F is large enough there exists a maximally stable form f defined on V, and the map ρ given by f is onto with kernel the similarity classes of f.*

Example 3.6.2. Let G be a simple group of type E_8. As in [38, Th. 4.5], there is a degree 8 homogeneous polynomial f defined on $V = \mathrm{Lie}(G)$ that is invariant under the G-action. Let S be the stabilizer of f in $\mathrm{GL}(V)$; it is generated by G and the 8-th roots of unity. Since $\mathrm{Aut}(\Delta) = 1$, $N_{\mathrm{GL}(V)}(G)$ acts by scalar multiples on f, and thus f is maximally stable by Remark 3.2.9. It follows from the Proposition above that there is a one-to-one correspondence between twisted forms of E_8 and twisted forms of f up to similarity.

When $F = \mathbb{Q}$ it follows by the Hasse principle and the triviality of E_8-torsors over a p-adic field (see for example [78, Ch. 6]) that there are three twisted forms of E_8 over $\mathbb{Q}$, and therefore our result implies that there are up to similarity three twisted forms of the E_8 octic polynomial f.

3.7 Conclusion

The following theorem follows immediately from the results of the previous sections:

Theorem 3.3. *Let G be an absolutely simple linear algebraic group over a field F, and let V be a faithful irreducible representation of G which satisfies the requirements of Proposition 3.4.13. Then, there exists a polynomial f defined on V such that the identity component of the stabilizer of f is G and twisted forms of G with the representation V defined over F are in one to one correspondence with similarity classes of twisted forms of f.*

We also recover as a corollary the following statement:

Corollary 3.7.1. *Let (V, q) be a nondegenerate quadratic space. Then the group $\mathrm{SO}(V, q)$ determines the quadratic form q up to a nonzero scalar multiple.*

Chapter 4

Degree 3 Cohomological Invariants of Split Quasi-Simple Groups that are Neither Simply Connected nor Adjoint

(The results in this chapter are joint with Anthony Ruozzi.)

4.1 Introduction

Let G be a linear algebraic group over a field F. The group of degree n cohomological invariants of G with values in a $\mathrm{Gal}(F_{\mathrm{sep}}/F)$-module A is the set of natural transformations of functors

$$I : H^1(-, G) \to H^n(-, A).$$

An invariant I is called *normalized* if $I(e) = 0$ where e is the trivial G-torsor. The object of interest for us is the group $\mathrm{Inv}^n(G, \mathbb{Q}/\mathbb{Z}(n))_{\mathrm{norm}}$ of normalized invariants of degree n with values in the group $\mathbb{Q}/\mathbb{Z}(n-1)$ which is defined as the direct sum of the colimit over n of the Galois modules $\mu_n^{\otimes 2}$ and a p-component defined via logarithmic de Rham-Witt differentials in the case $p = \mathrm{char}(F) > 0$ (see [51, I.5.7]). For a connected group scheme these invariants are trivial for $n = 1$ and the degree 2 invariants are given by the Picard group [57, §31]. The degree 3 invariants were determined by M. Rost in the case that G is quasi-simple simply connected, and recently by A. Merkurjev [70] in the case where G is adjoint of inner type. In fact Merkurjev does quite a bit more, namely, he provides an exact sequence involving the degree 3 invariants of a semisimple group.

In this paper we use Merkurjev's exact sequence to study the degree 3 invariants for the remaining cases of split quasi-simple groups, namely the groups $G = \mathrm{HSpin}_{4n}$[1] and $G = \mathrm{SL}_n/\mu_m$ (note that the case $G = \mathrm{SO}_{2n}$ has also been computed, cf. [40, Part 1, Ch. VI]). Our study focuses on two main questions: how many independent invariants are there for a given group, and can we give explicit constructions of these new invariants?

The first question is addressed by calculating the groups of decomposable and indecomposable invariants described in Merkurjev's sequence. As a result, we obtain many new invariants for SL_n/μ_m that have never been discussed in the literature. For HSpin_{16}, much more is known. The description of the invariants for HSpin_{4n} allows us to recover these results as well as extend them to arbitrary n. Of particular interest are a formula for an "indecomposable"invariant of HSpin_{4n} in terms of the invariant for PSp_{2n} and the Rost invariant of a twisted Spin group and an extension of the Arason invariant e_3 to algebras with orthogonal involution. One could also ask for an explicit description of the degree 3 invariants and the abelian group generated by them. We answer this, when possible, by restricting the invariants to a suitable subgroup.

[1] Recall that Spin_{4n} has three central subgroups of order 2. The quotient by one of them is SO_{4n}. In our notation, HSpin_{4n} is the quotient by either of the other two.

4.2 Decomposable Invariants

Let G be a semisimple group over a field F. Then there is an exact sequence [70, Thm. 3.9],

$$0 \to \mathrm{CH}^2(BG)_{\mathrm{tors}} \to H^1(F, \hat{C}(1)) \xrightarrow{\sigma} \tag{4.2.1}$$
$$\mathrm{Inv}^3(G, \mathbb{Q}/\mathbb{Z}(2))_{\mathrm{norm}} \to Q(G)/\operatorname{Dec}(G) \xrightarrow{\theta_G^*} H^2(F, \hat{C}(1))$$

where $Q(G)$ is as defined in [57, §31]. For our groups, $Q(G)$ is infinite cyclic with subgroup $\mathrm{Dec}(G)$ (see the next section for the precise definitions).

As Merkurjev observed, this exact sequence describes two types of "invariants". The first are the *decomposable invariants* defined as

$$\mathrm{Inv}^3(G, \mathbb{Q}/\mathbb{Z}(2))_{\mathrm{dec}} := \mathrm{Im}(\sigma)$$

where σ is the map from the sequence (4.2.1). These invariants can be easily understood for our cases of interest by following Merkurjev's arguments for adjoint groups; namely, we define a map α_G as follows. Let $\tilde{G}$ be the universal cover of G with kernel C. For any character $\chi \in \hat{C}(F)$ we consider the pushout

$$\begin{array}{ccccccccc} 1 & \longrightarrow & C & \longrightarrow & \tilde{G} & \longrightarrow & G & \longrightarrow & 1 \\ & & {\scriptstyle\chi}\downarrow & & \downarrow & & \| & & \\ 1 & \longrightarrow & \mathbb{G}_m & \longrightarrow & H_\chi & \longrightarrow & G & \longrightarrow & 1 \end{array}$$

Define a morphism

$$\alpha_G : \mathrm{H}^1(F, G) \to \mathrm{Hom}(\hat{C}(F), \mathrm{H}^2(F, \mathbb{G}_m))$$

by sending a torsor E to the map $\chi \mapsto \partial(E)$ where $\partial : \mathrm{H}^1(F, G) \to \mathrm{H}^2(F, \mathbb{G}_m)$ is the connecting homorphism for the bottom sequence in the diagram. A homomorphism $a \in \mathrm{Hom}(\hat{C}(F), \mathrm{H}^2(F, \mathbb{G}_m))$ will be called *admissible* if $\mathrm{ind}(a(\chi)) \mid \mathrm{ord}(\chi)$ for every $\chi \in \hat{C}(F)$.

Proposition 4.2.2. *For the two families of groups $G = \mathrm{SL}_n / \mu_m$ and $G = \mathrm{HSpin}_{4n}$,*

$$\mathrm{Inv}^3(G, \mathbb{Q}/\mathbb{Z}(2))_{\mathrm{dec}} \simeq \hat{C} \otimes_{\mathbb{Z}} F.$$

Proof. First, we show that every admissible map is in the image of α_G. This can be done exactly as in [70, Prop. 2.4/2.6]. The statement then follows by imitating the proof of [70, Thm. 4.2]. □

Remark. One may wonder whether a similar statement holds for outer groups. Although Merkurjev's sequence is valid for any semisimple group, the proof of [70, Thm. 4.2] does not extend to outer groups and in particular not much is known about $\mathrm{CH}^2(BG)_{\mathrm{tors}}$ in this case which seems to suggest that new techniques will be needed.

4.3 Indecomposable Invariants

The second, and more interesting, class of invariants are the *indecomposable invariants* defined as

$$\mathrm{Inv}^3(G, \mathbb{Q}/\mathbb{Z}(2))_{\mathrm{ind}} := \mathrm{Inv}^3(G, \mathbb{Q}/\mathbb{Z}(2))_{\mathrm{norm}} / \mathrm{Inv}^3(G, \mathbb{Q}/\mathbb{Z}(2))_{\mathrm{dec}}.$$

We note that the elements of this group are not cohomological invariants in the sense described in the introduction. However, it is possible to define an invariant in the sense of [40] from an indecomposable invariant by considering them as maps $H^1(F, G) \to H^3(F, \mathbb{Q}/\mathbb{Z}(2))/P$, where P is the subgroup generated by cup products of Tits algebras of G and elements of the field as in [70, p. 11]. The sequence

$$1 \longrightarrow \mathrm{Inv}^3(G, \mathbb{Q}/\mathbb{Z}(2))_{\mathrm{dec}} \longrightarrow \mathrm{Inv}^3(G, \mathbb{Q}/\mathbb{Z}(2))_{\mathrm{norm}} \longrightarrow \mathrm{Inv}^3(G, \mathbb{Q}/\mathbb{Z}(2))_{\mathrm{ind}} \longrightarrow 1$$

is actually functorial in G as noted in the comments after [70, Rem. 3.10]. That is, for another group G' and a map $G' \to G$, the diagram

$$\begin{tikzcd}
\mathrm{Inv}^3(G,\mathbb{Q}/\mathbb{Z}(2))_{\mathrm{dec}} \arrow[r] \arrow[d] & \mathrm{Inv}^3(G,\mathbb{Q}/\mathbb{Z}(2))_{\mathrm{norm}} \arrow[r] \arrow[d] & \mathrm{Inv}^3(G,\mathbb{Q}/\mathbb{Z}(2))_{\mathrm{ind}} \arrow[d] \\
\mathrm{Inv}^3(G',\mathbb{Q}/\mathbb{Z}(2))_{\mathrm{dec}} \arrow[r] & \mathrm{Inv}^3(G',\mathbb{Q}/\mathbb{Z}(2))_{\mathrm{norm}} \arrow[r] & \mathrm{Inv}^3(G',\mathbb{Q}/\mathbb{Z}(2))_{\mathrm{ind}}
\end{tikzcd} \tag{4.3.1}$$

commutes.

To understand the indecomposable invariants, we need to describe the groups $Q(G)$ and $\mathrm{Dec}(G)$. Let T be a maximal torus in G_{sep}, and let Λ be the $\Gamma = \mathrm{Gal}(F_{\mathrm{sep}}/F)$-lattice corresponding to T, under the usual equivalence of categories between tori and their character modules, equipped with the $*$-action. This action permutes a system of simple roots, cf. [57, §27.A]. It follows that $\Lambda_r \subset \Lambda \subset \Lambda_w$ where Λ_r and Λ_w are, respectively, the root and weight lattices of G. Define the following group

$$Q(G) = (\mathrm{Sym}^2(\Lambda)^W)^\Gamma$$

where W is the Weyl group. It can also be described as the group of Γ-equivariant loops in G [57, §31]. For a simply connected group, $\Lambda = \Lambda_w$, and $Q(G)$ is generated by a single element denoted q [57, Cor. 31.27]. Thus, for any other Λ, since $(\mathrm{Sym}^2(\Lambda)^W) = \mathrm{Sym}^2(\Lambda) \cap (\mathrm{Sym}^2(\Lambda_w)^W)$, there is a unique positive integer ℓ such that $Q(G) = \ell q\mathbb{Z}$. Furthermore, ℓ is the smallest integer such that the quadratic form ℓq takes only integer values on the lattice Λ.

Let n_G be the gcd of all Dynkin indices of all representations of G. The values of this number for absolutely simple simply connected groups can be found in [40, Appendix B]. Since $\ell \mid n_G$,

$$\mathrm{Dec}(G) = n_G q\mathbb{Z}$$

defines a subgroup of $Q(G)$, cf. [70, Ex. 3.5] .

Lemma 4.3.2. *For $G = \mathrm{SL}_n/\mu_m$ and $G = \mathrm{HSpin}_{4n}$,*

$$\mathrm{Inv}^3(G, \mathbb{Q}/\mathbb{Z}(2))_{\mathrm{ind}} \simeq Q(G)/\operatorname{Dec} G.$$

Proof. This follows immediately from Merkurjev's exact sequence (4.2.1) and the remarks following [70, Thm. 3.9], since, in the case of split groups, the map θ_G^* is trivial. □

Therefore, in order to calculate the indecomposable invariants, it suffices to compute this quotient. In the following sections, we compute the groups $Q(G)$ and $\mathrm{Dec}(G)$ for the split groups $G = \mathrm{SL}_n/\mu_{p^r}$ and $G = \mathrm{HSpin}_{4n}$. Throughout, ℓq where $\ell \in \mathbb{Z}^+$ will denote the generator of $Q(G)$.

4.4 SL_n/μ_m

Let p^s be the largest power of p dividing n, r a positive integer with $r \leq s$.

Theorem 4.4.1. *For p odd,*

$$\mathrm{Inv}^3(\mathrm{SL}_n/\mu_{p^r}, \mathbb{Q}/\mathbb{Z}(2))_{\mathrm{ind}} \cong \begin{cases} (\mathbb{Z}/p^r\mathbb{Z})q & \text{if } s \geq 2r \\ (p^{2r-s}\mathbb{Z}/p^r\mathbb{Z})q & \text{if } r < s < 2r \\ 0 & \text{if } s = r \end{cases}$$

and for $p = 2$,

$$\mathrm{Inv}^3(\mathrm{SL}_n/\mu_{2^r}, \mathbb{Q}/\mathbb{Z}(2))_{\mathrm{ind}} \cong \begin{cases} (\mathbb{Z}/2^r\mathbb{Z})q & \text{if } s \geq 2r+1 \\ (2^{2r+1-s}\mathbb{Z}/2^r\mathbb{Z})q & \text{if } r+1 < s < 2r+1 \\ 0 & \text{if } s = r, r+1 \end{cases}$$

Proof. This will follow from formulas (4.4.2) and (4.4.3) below. □

4.4.1 $Q(G)$ for SL_n/μ_m

For SL_n/μ_m, Λ is generated by the coroots along with the element $\tau := \frac{1}{m}(\alpha_1 + 2\alpha_2 + \cdots + (n-1)\alpha_{n-1})$, see [36]. In this case, since all the coroots have the same length, the quadratic form is just given by taking the Gram matrix to be the Cartan matrix and we get

$$q = \sum_{i=1}^{n} w_i^2 - \sum_{i=1}^{n-1} w_i w_{i+1}.$$

We then have

$$\begin{aligned} q(\tau) := &\frac{1}{m^2}\left(\sum_{i=1}^{n-1} i^2 - \sum_{i=1}^{n-2} i(i+1)\right) \\ = &\frac{1}{m^2}\left((n-1)^2 + \frac{(n-2)(n-1)}{2}\right) \\ = &\frac{n(n-1)}{2m^2}. \end{aligned}$$

By definition, $m \mid n$, so the fact that $\gcd(n-1, n) = 1$ implies that we have

$$\ell = \begin{cases} 2m^2/\gcd(2m^2, n) & \text{if } n \text{ is even.} \\ m^2/\gcd(m^2, n) & \text{if } n \text{ is odd.} \end{cases} \tag{4.4.2}$$

Note that if $m = n$ then this agrees with Merkurjev's result for adjoint groups, and also says, of course, that if $m = 1$, i.e. G is simply connected, then $\ell = 1$.

4.4.2 $\mathrm{Dec}(G)$ for SL_n/μ_{p^r}

For a group of type A_{n-1}, we take the set of simple roots $\{\bar{e}_1 - \bar{e}_2, ..., \bar{e}_{n-1} - \bar{e}_n\}$, where the $\bar{e}_i$ are the images of the standard basis vectors e_i for $\mathbb{R}^n$. A dominant character $\chi \in \Lambda$ corresponds to a sum $\sum c_i \bar{e}_i$ with $c_1 \geq c_2 \geq ... \geq$

c_n. Suppose the c_i have distinct values $a_1 > ... > a_k$ with multiplicities $r_1, ..., r_k$. In this case, by [40, pg. 136], n_G can be computed by taking the gcd over all integers

$$N(\chi) = \frac{(n-2)!}{r_1!r_2!...r_k!}[n \sum_i r_i a_i^2 - (\sum_i r_i a_i)^2].$$

We already know some convenient bounds on n_G. First, by [40, Part 2, Lem. 11.4], $m \mid n_G$. Moreover, $n_G \mid \gcd(m^2, 2n)$. To see that it divides the first, choose the dominant character with $c_1 = m$, $c_i = 0$ for $i > 0$. It divides the second by [36, Ex. 1.3] and the observation that the Coxeter number in this case is n.

We consider only the case where $m = p^r$ is a power of a prime and n is arbitrary. In this case, n_G must be a power of p because it divides m^2. Let $n = k \cdot p^s$ where k is coprime to p. If $s = r$, then we can conclude by the above bounds. Namely, if $p \neq 2$ then we have $n_G = p^r$ since we know it is a power of p, and it has to be equal to p^r because it divides $2n$. If $p = 2$, then as $Q(\mathrm{SL}_n / \mu_{2^r})$ has generator $2^{r+1}q$ in this case, we have that $2^{r+1} \mid n_G$. Further, n_G must be 2^{r+1} because it is a power of 2 dividing $2n$.

Thus we reduce to assuming that $s \neq r$. Consider the m-th exterior power of the tautological representation of SL_n. From [10], it has highest weight $\lambda = e_1 + \cdots + e_m$, and in the above notation, $a_1 = 1, a_2 = 0$ and $r_1 = m, r_2 = n - m$, so that

$$\begin{aligned} N(\lambda) &= \frac{(n-2)!}{m!(n-m)!}(n \cdot m - m^2) \\ &= \binom{n-2}{m-1} \\ &= \binom{n}{m}\frac{m(n-m)}{n(n-1)} \end{aligned}$$

as can be found in Dynkin's Tables [24, Table 5]. Appealing to Kummer's

Theorem on the power of a prime dividing the binomial coefficients, we have

$$\operatorname{ord}_p \binom{n}{p^r} = s - r.$$

Using this in the above formula, if $s \neq r$,

$$\begin{aligned} \operatorname{ord}_p \binom{n}{p^r} \frac{p^r(kp^s - p^r)}{kp^s(kp^s - 1)} &= \operatorname{ord}_p \binom{n}{p^r} + 2r - s \\ &= s - r + 2r - s \\ &= r \end{aligned}$$

This computation implies that $n_G = p^r$, since we already knew that $m = p^r \mid n_G$.

In summary,

$$n_{\mathrm{SL}_n/\mu_{p^r}} = \begin{cases} p^r & \text{if } p \neq 2 \text{ or } p = 2 \text{ and } s \neq r. \\ 2^{r+1} & \text{if } p = 2 \text{ and } s = r. \end{cases} \tag{4.4.3}$$

4.4.3 A Fibration

It would be nice to have an explicit description of the indecomposable invariants in this case. However, even the torsors for these groups are difficult to describe. Lacking a reference, we include here a fibration which in some small way explains these objects.

Proposition 4.4.4. *There is a surjection of pointed sets*

$$H^1(F, \mathrm{SL}_n/\mu_m) \twoheadrightarrow \{\text{central simple algebras}/F \text{ of degree } n \text{ with exponent} \mid m\}/\sim$$

where $A \sim B$ if A and B are isomorphic as F-algebras. Moreover, the fiber over the algebra A is isomorphic to $F^\times/\mathrm{Nrd}(A^\times)F^{\times n/m}$.

Proof. Consider the commutative diagram:

$$\begin{tikzcd} 1 \arrow[r] & \mu_m \arrow[r] \arrow[d] & \mathrm{SL}_n \arrow[r] \arrow[d] & \mathrm{SL}_n/\mu_m \arrow[r] \arrow[d] & 1 \\ 1 \arrow[r] & \mathbb{G}_m \arrow[r] & \mathrm{GL}_n \arrow[r] & \mathrm{PGL}_n \arrow[r] & 1 \end{tikzcd}$$

Passing to cohomology, we have that

$$\begin{tikzcd} H^1(F, \mathrm{SL}_n/\mu_m) \arrow[r] \arrow[d, "\phi"'] & H^2(F, \mu_m) \arrow[d] \\ H^1(F, \mathrm{PGL}_n) \arrow[r] & H^2(F, \mathbb{G}_m) \end{tikzcd}$$

commutes. Since

$$\mathrm{Im}(\phi) = \ker\{H^1(F, \mathrm{PGL}_n) \longrightarrow H^2(F, \mu_{n/m})\},$$

it follows that $H^1(F, \mathrm{SL}_n/\mu_m)$ maps onto the isomorphism classes of central simple algebras of degree n with exponent dividing m. The fiber of the top map of the diagram over an algebra A is in the image of $H^1(F, \mathrm{SL}_1(A))$, and this set can be easily computed from the sequence

$$1 \longrightarrow \mathrm{SL}_1(A)(F) \longrightarrow \mathrm{GL}_1(A)(F) \longrightarrow \mathbb{G}_m(F) \longrightarrow H^1(F, \mathrm{SL}_1(A)) \longrightarrow 1.$$

Finally, the map $H^1(F, \mu_m) \to H^1(F, \mathrm{SL}_1(A))$ sends a field element $a \in F^\times$ to $a^{n/m} \in F^\times/\mathrm{Nrd}(A^\times)$. This yields the fibration described in the proposition. $\square$

4.4.4 Examples

If $p = 2$, then any central simple algebra with exponent dividing 2 has an involution of the first kind. This structure has been used to define degree 3 cohomological invariants in some small cases [41], but for odd primes and higher powers, these invariants do not exist in the literature. However, for some values we have a complete description.

Theorem 4.4.5. *If* $4 \mid n$*,*

$$\mathrm{Inv}^3(\mathrm{SL}_{2n}/\mu_2, \mathbb{Q}/\mathbb{Z}(2))_{\mathrm{norm}} \simeq F^\times/F^{\times 2} \oplus \mathbb{Z}/2\mathbb{Z}.$$

Moreover, these invariants can be described explicitly by restricting the degree 3 *invariants of* PSp_{2n}*.*

Proof. Using the natural inclusion

$$\mathrm{Sp}_{2n} \subset \mathrm{SL}_{2n}$$

and modding out by the center of Sp_{2n} gives an inclusion

$$\mathrm{PSp}_{2n} \subset \mathrm{SL}_{2n}/\mu_2.$$

The degree 3 invariants of PSp_{2n} were completely described in [41] when $4 \mid n$. We use this description and the above inclusion to do the same for SL_{2n}/μ_2. The group SL_{2n} acts on the $2n$-dimensional vector space $F^{\oplus 2n}$. Let $V = \wedge^2 F^{\oplus 2n}$. SL_{2n} acts canonically on V, and since it is the second exterior power, so does SL_{2n}/μ_2. Over an algebraic closure, there is an open SL_{2n}/μ_2-orbit in $\mathbb{P}(V)$ [80, Summary Table], so by[34, §9.3], there is a surjection

$$H^1(F, N) \longrightarrow H^1(F, \mathrm{SL}_{2n}/\mu_2)$$

where N is the stabilizer of a generic point in the open orbit. By inspection, or again referring to [80, Summary Table], $N = \mathrm{Sp}_{2n}/\mu_2 = \mathrm{PSp}_{2n}$. Thus we have an inclusion

$$\mathrm{Inv}^3(\mathrm{SL}_{2n}/\mu_2, \mathbb{Q}/\mathbb{Z}(2))_{\mathrm{norm}} \hookrightarrow \mathrm{Inv}^3(\mathrm{PSp}_{2n}, \mathbb{Q}/\mathbb{Z}(2))_{\mathrm{norm}}.$$

Now, from the commutative diagram

$$\begin{array}{ccc} H^1(F, \mathrm{PSp}_{2n}) & \longrightarrow & H^2(F, \mu_2) \\ \downarrow & & \| \\ H^1(F, \mathrm{SL}_{2n}/\mu_2) & \longrightarrow & H^2(F, \mu_2), \end{array}$$

and the description of the decomposable invariants in [70, Thm. 4.6], it follows that

$$\operatorname{Inv}^3(\mathrm{SL}_{2n}/\mu_2, \mathbb{Q}/\mathbb{Z}(2))_{\mathrm{dec}} \simeq \operatorname{Inv}^3(\mathrm{PSp}_{2n}, \mathbb{Q}/\mathbb{Z}(2))_{\mathrm{dec}}.$$

By the commutativity of diagram (4.3.1), it follows that

$$\operatorname{Inv}^3(\mathrm{SL}_{2n}/\mu_2, \mathbb{Q}/\mathbb{Z}(2))_{\mathrm{dec}} \hookrightarrow \operatorname{Inv}^3(\mathrm{SL}_{2n}/\mu_2, \mathbb{Q}/\mathbb{Z}(2))_{\mathrm{norm}}$$

also splits, and these groups have the same degree 3 normalized invariants. The statement follows from the description of the invariants in Theorem 4.4.1 and the explicit construction of [41]. □

An Explicit Invariant for SL_8/μ_2

In [41, §5], the authors go further to describe the invariants for PSp_8 via the Rost invariant for a simply connected group of type E_6. A similar calculation can be done for SL_8/μ_2 and a simply connected group of type E_7. First, consider the diagram

$$\begin{array}{ccc} \mathrm{PSp}_8 & \subset & \mathrm{E}_6 \\ \cap & & \cap \\ \mathrm{SL}_8/\mu_2 & \subset & \mathrm{E}_7 \end{array}$$

The inclusion $\mathrm{SL}_8/\mu_2 \hookrightarrow \mathrm{E}_7$ can be obtained via the Borel-de Siebenthal theory of maximal rank subgroups by deleting the vertex labeled 2 which does not disconnect the Dynkin diagram, cf. [58]. We have the corresponding map of simply connected groups

$$\mathrm{SL}_8 \to \mathrm{SL}_8/\mu_2 \hookrightarrow \mathrm{E}_7.$$

By inspection, the Rost multiplier is 1 in this case, so the non-trivial degree 3 indecomposable invariant for SL_8/μ_2 is a restriction of the Rost invariant of E_7. We also remark here that the invariant $\Delta(A, \sigma)$ described in [41] is a generator for the indecomposable invariants of PSp_8 and thus of SL_8/μ_2 by Theorem 4.4.5. The decomposable invariants $x \in F^\times/F^{\times 2}$ are defined on an element $y \in H^1(F, \mathrm{SL}_8/\mu_2)$ as $y \mapsto (x) \cup \partial(y)$ where $\partial : H^1(F, \mathrm{SL}_8/\mu_2) \to H^2(F, \mu_2)$. From the definition of $\Delta(A, \sigma)$ it is difficult to say much more about it. However, in a recent preprint, Demba Barry [4] has shown that the image of this element in $\frac{H^3(F,\mu_2)}{[A]\cdot F^\times}$ is non-zero for an indecomposable algebra of degree 8 and exponent 2. That is, not only is Δ not generated by decomposable invariants, it cannot even be written in a similar manner using cup products.

An Explicit Invariant for SL_9/μ_3

Similarly to the previous example, the indecomposable invariants of SL_9/μ_3 can be described via a simply connected group of type E_8. Again, the Borel-de Siebenthal theory shows that by deleting the vertex labeled 3 which does not disconnect the Dynkin diagram, there is an inclusion of groups

$$\mathrm{SL}_9/\mu_3 \hookrightarrow \mathrm{E}_8.$$

As above, the Rost multiplier for the induced map on simply connected groups is 1. Therefore, a generator for the degree 3 indecomposable invariants of SL_9/μ_3 is the restriction of the Rost invariant of E_8. Moreover, the 3-torsion part of $\mathrm{Inv}_{\mathrm{norm}}(E_8, \mathbb{Q}/\mathbb{Z}(2))$ is isomorphic to $\mathbb{Z}/3\mathbb{Z}$ [40, Part 2, Thm. 16.8], so the restriction gives a splitting:

$$\mathrm{Inv}^3(\mathrm{SL}_9/\mu_3, \mathbb{Q}/\mathbb{Z}(2))_{\mathrm{norm}} \simeq F^\times/F^{\times 3} \oplus \mathbb{Z}/3\mathbb{Z}.$$

4.5 HSpin

Theorem 4.5.1. *We have*

$$\mathrm{Inv}^3(\mathrm{HSpin}_{4n}, \mathbb{Q}/\mathbb{Z}(2))_{\mathrm{ind}} \cong \begin{cases} 0 & \text{if } n > 1 \text{ is odd or } n = 2 \\ 2\mathbb{Z}/4\mathbb{Z} & \text{if } n \equiv 2 \mod 4 \text{ and } n \neq 2 \\ \mathbb{Z}/4\mathbb{Z} & \text{if } n \equiv 0 \mod 4 \end{cases}$$

Proof. This follows from equations (4.5.3) and (4.5.4) below. □

Corollary 4.5.2. *We have*

$$\mathrm{Inv}^3(\mathrm{HSpin}_{16}, \mathbb{Q}/\mathbb{Z}(2))_{\mathrm{norm}} \cong F/F^{\times 2} \oplus \mathbb{Z}/4\mathbb{Z}.$$

Proof. The invariant e_3 for HSpin_{16} constructed in [35] via the inclusion $\mathrm{HSpin}_{16} \to E_8$, gives the required splitting. □

4.5.1 $Q(G)$ for HSpin

As in the previous case, for G of type D_{2n} all the roots have the same lengths, and Λ is generated by the coroots and the additional element $\tau = \frac{1}{2}\sum_{i \text{ odd}} \alpha_i$. The quadratic form q is given by

$$q = \sum_{i=1}^{2n} w_i^2 - \sum_{i=1}^{2n-2} w_i w_{i+1} - w_{2n-2}w_{2n}.$$

Computing as before, $q(\tau) = \frac{n}{4}$, and it follows that

$$\ell = \begin{cases} 1 & \text{if } n \equiv 0 \mod 4. \\ 2 & \text{if } n \equiv 2 \mod 4. \\ 4 & \text{if } n \text{ is odd.} \end{cases} \tag{4.5.3}$$

4.5.2 Dec(G) for HSpin_{4n}

Let $\tilde{G} \to G \to \overline{G}$ be the standard central isogenies where $\tilde{G}$ is the simply connected cover of G and $\overline{G} = G/C(G)$ is adjoint. We have the following relationship for the Dynkin indices:

$$n_{\tilde{G}} \mid n_G \mid n_{\overline{G}}.$$

This follows from the definition: $n_{\tilde{G}}$ is the gcd over all representations given by highest weights in Λ_w, whereas n_G is the gcd taken over all representations whose weights vanish on the kernel of the map $\tilde{G} \to G$. Similarly $n_{\overline{G}}$ is the gcd of representations whose highest weight vanishes on all of the kernel of $\tilde{G} \to \overline{G}$. Note that for $\tilde{G}$, the Dynkin index is equal to the order of the group of degree 3 invariants see [40, Part 2, Thm. 10.7]. Applying this to the case where $G = \mathrm{HSpin}_{4n}$ we get that

$$2 \mid n_{\mathrm{HSpin}_{4n}} \mid 4,$$

since we have $n_{\overline{G}} = n_{\mathrm{PGO}_{4n}} = 4$ from [70] and $n_{\tilde{G}} = n_{\mathrm{Spin}_{4n}} = 2$ or 4 from [40, Appendix B].

Now let χ be a fundamental weight of Spin_{4n}, C the center of Spin_{4n}; then C^*_{sep} consists of four elements, $0, \lambda, \lambda^+$ and λ^- (see [40, p. 146]). Put n_0, n^+ for the $\gcd(N(\chi))$ where the gcd is taken over all characters that restrict to 0, and λ^+ respectively, then one has that $n_{\mathrm{HSpin}_{4n}} = \gcd(n_0, n^+ \cdot \mathrm{ind}(C^+))$. Furthermore from [40, Part 2, Lem. 15.3], we know that $n^+ = 2^{2n-3}$ and n_0 is divisible by 4, and this implies that

$$\begin{aligned} n_{\mathrm{HSpin}_8} &= 2. \\ n_{\mathrm{HSpin}_{4n}} &= 4 \text{ for } n > 2. \end{aligned} \tag{4.5.4}$$

4.6 Restriction of Invariants to Subgroups

4.6.1 Restrictions in terms of $Q(G)/\operatorname{Dec}(G)$

Consider the following diagram of groups

$$\begin{tikzcd}
\mu_2 \arrow[r, equal] \arrow[d] & \mu_2 \arrow[r, equal] \arrow[d] & \mu_2 \arrow[d] \\
\mathrm{Sp}_{2n} \arrow[r, hook] \arrow[d] & \mathrm{SL}_{2n} \arrow[r, hook] \arrow[d] & \mathrm{Spin}_{4n} \arrow[d] \\
\mathrm{PSp}_{2n} \arrow[r, hook] & \mathrm{SL}_{2n}/\mu_2 \arrow[r, hook] & \mathrm{HSpin}_{4n}
\end{tikzcd} \tag{4.6.1}$$

where the left vertical sequence is just the standard isogeny. The inclusions

$$\mathrm{Sp}_{2n} \subset \mathrm{SL}_{2n} \subset \mathrm{Spin}_{4n}$$

can be easily described. The first was treated above. The second comes from deleting an appropriate end vertex of the Dynkin diagram of type D_{2n} to obtain a diagram of type A_{2n-1} in such a way that μ_2 will still sit inside all groups, and we will be able to obtain the inclusion $\mathrm{SL}_{2n}/\mu_2 \subset \mathrm{HSpin}_{4n}$ as claimed. Now notice that by right side of diagram (4.3.1) and the previous results we get a diagram

$$\begin{tikzcd}
\mathrm{Inv}^3(\mathrm{HSpin}_{4n}, \mathbb{Q}/\mathbb{Z}(2))_{\mathrm{ind}} \arrow[r] \arrow[d, "\simeq"] & \mathrm{Inv}^3(\mathrm{SL}_{2n}/\mu_2, \mathbb{Q}/\mathbb{Z}(2))_{\mathrm{ind}} \arrow[r] \arrow[d, "\simeq"] & \mathrm{Inv}^3(\mathrm{PSp}_{2n}, \mathbb{Q}/\mathbb{Z}(2))_{\mathrm{ind}} \arrow[d, "\simeq"] \\
Q(\mathrm{HSpin}_{4n})/\operatorname{Dec}(\mathrm{HSpin}_{4n}) \arrow[r] & Q(\mathrm{SL}_{2n}/\mu_2)/\operatorname{Dec}(\mathrm{SL}_{2n}/\mu_2) \arrow[r] & Q(\mathrm{PSp}_{2n})/\operatorname{Dec}(\mathrm{PSp}_{2n})
\end{tikzcd}$$

The top row of this diagram gives the restriction of the generator of the group of indecomposable invariants of HSpin_{4n} to SL_{2n}/μ_2 and the restriction of the generator of the group of indecomposable invariants of SL_{2n}/μ_2 to PSp_{2n} which was described above. We now compute the other restriction by using the bottom row of the diagram.

Notice that the only interesting case is when $n \equiv 0 \mod 4$ since if n is odd then all of the groups of indecomposable invariants are trivial, and there is nothing to say. Similarly if $n \equiv 2 \mod 4$ then $\mathrm{Inv}^3(\mathrm{HSpin}_{4n}, \mathbb{Q}/\mathbb{Z}(2))_{\mathrm{ind}} \cong \mathbb{Z}/2\mathbb{Z}$ as shown above. However, the other two groups are both trivial, so the generator restricts trivially in these cases. To finish we note that, since the coroots of SL_{2n} are also coroots of Spin_{4n} and they all have the same length, the Rost multiplier of the inclusion $\mathrm{SL}_{2n} \hookrightarrow \mathrm{Spin}_{4n}$ is 1. This means that the map

$$\mathbb{Z}/4\mathbb{Z} \simeq Q(\mathrm{HSpin}_{4n})/\operatorname{Dec}(\mathrm{HSpin}_{4n}) \to Q(\mathrm{SL}_{2n}/\mu_2)/\operatorname{Dec}(\mathrm{SL}_{2n}/\mu_2) \simeq \mathbb{Z}/2\mathbb{Z}$$

maps $1 \in \mathbb{Z}/4\mathbb{Z}$ to $1 \in \mathbb{Z}/2\mathbb{Z}$. Therefore, a generator of the group of indecomposable invariants of HSpin_{4n} restricts to a generator of the indecomposable invariants of SL_{2n}/μ_2.

We can go further and note that since we have

$$\mathrm{Inv}^3(\mathrm{HSpin}_{4n}, \mathbb{Q}/\mathbb{Z}(2))_{\mathrm{dec}} \simeq \mathrm{Inv}^3(\mathrm{SL}_{2n}/\mu_2, \mathbb{Q}/\mathbb{Z}(2))_{\mathrm{dec}} \simeq \mathrm{Inv}^3(\mathrm{PSp}_{2n}, \mathbb{Q}/\mathbb{Z}(2))_{\mathrm{dec}}$$

by applying the five lemma to diagram (4.3.1) we get that the map

$$\mathrm{Inv}^3(\mathrm{HSpin}_{4n}, \mathbb{Q}/\mathbb{Z}(2))_{\mathrm{norm}} \to \mathrm{Inv}^3(\mathrm{SL}_{2n}/\mu_2, \mathbb{Q}/\mathbb{Z}(2))_{\mathrm{norm}}$$

is onto.

4.6.2 Explicit description of restrictions

Consider diagram (4.6.1), by looking at the long cohomology exact sequence we get a diagram:

$$\begin{tikzcd}
H^1(F,\mathrm{Sp}_{2n}) \arrow[r] \arrow[d] & H^1(F,\mathrm{PSp}_{2n}) \arrow[r] \arrow[d] & H^2(F,\mu_2) \arrow[d, equal] \\
H^1(F,\mathrm{SL}_{2n}) \arrow[r] \arrow[d] & H^1(F,\mathrm{SL}_{2n}/\mu_2) \arrow[r] \arrow[d] & H^2(F,\mu_2) \arrow[d, equal] \\
H^1(F,\mathrm{Spin}_{4n}) \arrow[r] & H^1(F,\mathrm{HSpin}_{4n}) \arrow[r] & H^2(F,\mu_2)
\end{tikzcd}$$

Now let $x \in H^1(F, \mathrm{HSpin}_{4n})$ be a class mapping to a class $[A] \in H^2(F, \mu_2)$ with index dividing $2n$; z and element of $H^1(F, \mathrm{PSp}_{2n})$ which also maps to $[A]$, and let x_0 be the image of z in $H^1(F, \mathrm{HSpin}_{4n})$. Twisting the bottom row of this diagram by x_0 and putting Spin_{x_0} for the twist of Spin_{4n} we find a cocycle $y \in H^1(F, \mathrm{Spin}_{x_0})$ which maps to $x \in H^1(F, \mathrm{HSpin}_{4n})$.

Notice that the map on indecomposable invariants induced by the quotient $\mathrm{Spin}_{x_0} \to \mathrm{HSpin}_{x_0}$ is onto because the group $Q(\mathrm{HSpin}_{x_0})$ contains the generator q of $Q(\mathrm{Spin}_{x_0})$. That is, there is a generator e_3' of $\mathrm{Inv}^3(\mathrm{HSpin}_{x_0}, \mathbb{Q}/\mathbb{Z}(2))_{\mathrm{ind}}$ which maps to the Rost invariant e_3^{Spin} of Spin_{x_0}. Now let e_3 be the image of e_3' in $\mathrm{Inv}^3(\mathrm{HSpin}_{4n}, \mathbb{Q}/\mathbb{Z}(2))_{\mathrm{ind}}$ under the isomorphism described in [70, p. 14]. We get an equation

$$e_3(x) = e_3^{\mathrm{Spin}}(y) + e_3(x_0) \in H^3(F, \mathbb{Z}/4\mathbb{Z})/P,$$

where P here is the subgroup defined in section 3 above, namely the subgroup generated by cup products of Tits algebras with elements of the field. By the results of the last section we obtain:

Proposition 4.6.2. *Let* Δ *denote the invariant of* $\mathrm{Inv}^3(\mathrm{PSp}_{2n}, \mathbb{Q}/\mathbb{Z}(2))_{\mathrm{norm}}$ *(this invariant was first constructed in [41]), by the results of the last section we obtain the equation*

$$e_3(x) = \Delta(z) + e_3^{\mathrm{Spin}}(y) \ \in H^3(F, \mathbb{Z}/4\mathbb{Z})/P. \tag{4.6.3}$$

Remark 4.6.4. The case $n = 4$ of the previous proposition was first proven in [35, Cor. 10.2]. For this case, Corollary 4.5.2 allows us to strengthen the statement of the proposition to an equation

$$e_3(x) = \Delta(z) + e_3^{\mathrm{Spin}}(y) \in H^3(F, \mathbb{Z}/4\mathbb{Z}). \tag{4.6.5}$$

4.7 Algebras with orthogonal involution in I^3

In [35], Garibaldi uses a construction of a degree 3 invariant of HSpin_{16} to define an invariant for central simple algebras (A, σ) of degree 16 with an orthogonal involution and to deduce some nice properties. We now wish to show how the results of this paper allow us to recover and extend some of those results to algebras of degree any multiple of 16.

Let (A, σ) be a central simple algebra with orthogonal involution over a field F of characteristic not 2. Over the function field F_A of the Severi-Brauer variety of A, the involution σ is adjoint to a quadratic form q_σ determined up to similarity. We say *σ is in I^n*, if q_σ is in I^n, where I denotes the fundamental ideal in the Witt ring of F_A.

To relate these algebras with the results of this paper we recall that by [35, Lem. 4.1] the pairs (A, σ) which lie in I^3 are exactly those that are in the image of the map $H^1(k, \mathrm{HSpin}_{4n}) \to H^1(k, \mathrm{PGO}_{4n})$.

Now let e_3 be a generator of $\mathrm{Inv}^3(\mathrm{HSpin}_{4n}, \mathbb{Q}/\mathbb{Z}(2))_{\mathrm{ind}}$, and for a given pair (A, σ) fix an element $\eta \in H^1(F, \mathrm{HSpin}_{4n})$ which maps to (A, σ). Define

$$e_3(A, \sigma) := e_3(\eta) \in H^3(F, \mathbb{Q}/\mathbb{Z}(2))/[A] \cdot H^1(F, \mu_2).$$

We note that the quotient on the right hand side implies that the value of $e_3(A, \sigma)$ does not depend on the choice of η since we have the sequence:

$$H^1(F, \mu_2) \to H^1(F, \mathrm{HSpin}_{4n}) \to H^1(F, \mathrm{PGO}_{4n})$$

Put $E(A) := \ker(H^3(F,\mathbb{Z}/4\mathbb{Z}) \to H^3(F_A,\mathbb{Z}/4\mathbb{Z}))$, and let $P = [A]\cdot H^1(F,\mu_2)$ as before. Clearly $P \subset E(A)$ since A splits over F_A. We now obtain the following generalization of [35, Thm. 2.6, Cor. 2.8]

Theorem 4.7.1. *Let (A,σ) be a central simple algebra with orthogonal involution of degree divisible by 16. Then there exists an invariant $e_3(A,\sigma) \in H^3(F,\mathbb{Q}/\mathbb{Z}(2))/E(A)$ such that if K/F splits A, then $\operatorname{res}_{K/F} e_3(A,\sigma)$ is the Arason invariant $e_3(q_{\sigma\otimes K})$. Furthermore (A,σ) is in I^4 if and only if $e_3(A,\sigma)$ is zero.*

Proof. The construction of the invariant was described above. The proofs of the other statements can be taken basically verbatim from [35]. We sketch them here for the reader's convenience. Suppose the algebra A splits over K, then by the results of section 6 above, there exists a class $x \in H^1(K,\mathrm{Spin}_{4n})$ which maps to (A,σ) in $H^1(K,\mathrm{PGO}_{4n})$, and the value of the restriction $\operatorname{res}_{K/F} e_3(A,\sigma)$ is the value of the Rost invariant of x, i.e. the Arason invariant of q_σ. That (A,σ) is in I^4 if and only if $e_3(A,\sigma)$ is zero then follows from the corresponding statement for q_σ and the Arason invariant. □

Remark 4.7.2. It is shown in [5, Thm. 3.9] that a degree 3 invariant restricting to the Arason invariant does not exist for degree 8 algebras with orthogonal involution in I^3. These results are extended in [35, Ex. 2.7] to rule out the existence of an invariant in all degrees not covered by the theorem above.

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