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Quasi-Isometric Properties of Graph Braid Groups

By

Praphat Xavier Fernandes
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Aaron Abrams, Ph.D.
Advisor

Raman Parimala, Ph.D.
Committee Member

Emily Hamilton, Ph.D.
Committee Member

Accepted:

Lisa A. Tedesco, Ph.D.
Dean of the James T. Laney School of Graduate Studies

Date

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Abstract

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In my thesis I initiate the study of the quasi-isometric properties of the 2 dimensional graph braid groups. I do this by studying the behaviour of flats in the geometric model spaces of the graph braid groups, which happen to be CAT(0) cube complexes. I define a quasi-isometric invariant of these graph braid groups called the intersection complex. In certain cases it is possible to calculate the dimension of this intersection complex from the underlying graph of the graph braid group. And I use the dimension of the intersection complex to prove that the family of graph braid groups $B_2(K_n)$ are quasi-isometrically distinct for all n. I also show that the dimension of the intersection complex for a graph braid group takes on every possible non-negative integer value.

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For my parents Keith and Sriwan, and my sister Vilaiwan

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Chapter 1

Introduction

Our goal in this paper is to initiate a study of the quasi-isometric properties of graph braid groups. In a series of papers, Farley and Sabalka [10, 11, 9] study the cohomology of graph braid groups and derive presentations for the groups. Farber and others [2, 8] have studied the homology of these groups. Embeddings of graph braid groups into RAAG's and classical braid groups are studied by Crisp and Wiest in [7], and Scrimshaw shows in [18] that for more than 4 strands the classical braid groups cannot embed into graph braid groups. Thus far, little is known about the quasi-isometric properties of the graph braid groups.

Similar to the case of a right angled Artin group, a graph braid group $B_n(\Gamma)$ is quasi-isometric to a finite dimensional CAT(0) cube complex $\overline{D_n(\Gamma)}$. For the case $n = 2$ this suggests that the tools developed by Bestvina, Kleiner and Sageev in [3] and [4] for studying the quasi-isometric properties of certain 2-dimensional right angled Artin groups via their cubical CAT(0) model spaces may be adapted to the study of graph braid groups. Inspired by ideas from [3], we associate to each graph Γ a finite dimensional simplicial complex called the intersection complex $I(\Gamma)$, that encodes the intersection pattern of maximal product subcomplexes in $\overline{D_2(\Gamma)}$. The dimension of $I(\Gamma)$ is finite and a quasi-isometry invariant of $B_2(\Gamma)$.

Letting K_n denote the complete graph with n vertices, we show that the dimension of $I(K_n)$ is $2^{n-6}-1$ for $n \geq 6$. This implies that the spaces $\overline{D_2(K_n)}$

are quasi-isometrically distinct for all n. We also show that the dimension of $I(\Gamma)$ can take on any non-negative integer value.

Chapter 2

Quasi-Isometries

Given two metric spaces (X, d_X) and (Y, d_Y), a quasi-isometric embedding $\phi : X \to Y$ is a map which is not necessarily continuous, but for which a pair of constants $K, L > 0$ exist, which satisfy the following equation for every pair of points x_1, x_2 in X:

$$\frac{1}{K} d_X(x_1, x_2) - L \leq d_Y(\phi(x_1), \phi(x_2)) \leq K d_X(x_1, x_2) + L$$

If in addition there exists a constant $C > 0$ such that for each $y \in Y$ there exists a point $x_y \in X$ such that $y \in N_C(\phi(x_y))$, or in other words that $Y \subset N_C(\phi(X))$, then we say that X and Y are quasi-isometric and that the map $\phi : X \to Y$ is a quasi-isometry. Given such a quasi-isometry $\phi : X \to Y$, there exists a quasi-isometry $\psi : Y \to X$ and a constant $D > 0$ such that for every $x \in X$ and for every $y \in Y$ the following two equations hold:

$$d_X(x, \psi \circ \phi(x)) < D$$

$$d_Y(y, \phi \circ \psi(y)) < D$$

We call such a map $\psi : Y \to X$ a quasi-isometric inverse to $\phi : X \to Y$, because while the compositions $\psi \circ \phi : X \to X$ and $\phi \circ \psi : Y \to Y$ are not equal to the identity maps on X and Y respectively, they are a bounded distance from those identity maps.

Example 2.1. *Consider the metric space $\mathbb{R}$ with the standard Euclidean distance. And consider the metric space $\mathbb{Z}$ made up of the integers, with the standard Euclidean distance inherited from $\mathbb{R}$. The inclusion map $\iota : \mathbb{Z} \hookrightarrow \mathbb{R}$ is a quasi-isometry. And the map given by the greatest integer function, $\lfloor \cdot \rfloor : \mathbb{R} \to \mathbb{Z}$ is its quasi-isometric inverse. Thus the metric spaces $\mathbb{R}$ and $\mathbb{Z}$ are quasi-isometric.*

Given a finitely generated group G with generating set $A = \langle a_1, ..., a_m \rangle$, then one may define a metric d_A on the group G with respect to the generating set A as follows:

$$d_A(g,h) := \{ \text{ Shortest word length in } A \text{ which represent the element } gh^{-1} \}$$

However the metric d_A depends upon the choice of generating set A. Choosing a different generating set $B = \langle b_1, ..., b_n \rangle$ gives us a different metric d_B. However it is easy to see that the two metric spaces that one gets by endowing G with the two different metrics, (G, d_A) and (G, d_B) are quasi-isometric as metric spaces [13].

Example 2.2. *Consider the group $\mathbb{Z}$ with respect to two generating sets $A =< 1 >$ and $B =< 2, 3 >$. And consider the two elements $0, 1 \in \mathbb{Z}$. We have that $d_A(0, 1) = 1$ while $d_B(0, 1) = 2$. However for all $m, n \in \mathbb{Z}$ we have that $\frac{1}{3} d_A(m, n) \leq d_B(m, n) \leq 3 d_A(m, n)$.*

Thus we may study finitely generated groups as geometric objects by studying them as metric spaces up to quasi-isometry. The study of groups as metric spaces up to quasi-isometry was initiated by Gromov [13] and is a major thrust of geometric group theory.

We often find ourselves in the case where we have a nice action of the group G we are trying to study, on a metric space (X, d). If we restrict the kinds of metric spaces that our group G is allowed to act upon then this action (if it satisfies certain conditions) gives us a quasi-isometry between our group and the metric space.

We want to consider metric spaces in which all closed balls are compact, these are called proper metric spaces.

Definition 2.3. (Proper Metric Space) *A metric space (X, d) is proper if closed balls in the metric space are compact.*

A path $\gamma : [a, b] \to X$ in a metric space (X, d) is called rectifiable if the quantity $l(\gamma)$ is finite:

$$l(\gamma) := \sup \{\sum d(\gamma(x_i), \gamma(x_{i+1}))\}$$

Where the supremum ranges over all possible partitions $\mathcal{P} = \{x_0, ..., x_k\}$ of the interval $[a, b]$.

A geodesic between two points x, y in the metric space X is a path γ such that $\gamma(a) = x, \gamma(b) = y$ and $l(\gamma) = d(x, y)$. We want to restrict our attention further, to metric spaces in which there is a geodesic path between every pair of points, such spaces are called geodesic spaces.

Definition 2.4. (Geodesic Space) *A metric space X is said to be a geodesic space if there exists a geodesic γ between every pair of points in X.*

And we restrict the group actions that we consider to co-compact and properly discontinuous actions.

Definition 2.5. (Co-compact Action) *The action of a group G on a metric space (X, d) is said to be co-compact, if the quotient space X/G is compact in the quotient topology.*

Definition 2.6. (Properly Discontinuous Action) *A group G is said to act properly discontinuously on a metric space (X, d) if given any compact set $K \subset X$, for all but finitely many $g \in G$ we have that $g \cdot K \cap K = \phi$.*

Definition 2.7. (Geometric Action) *A group G acting properly discontinuously and co-compactly by isometries on a metric space (X, d) is said to act geometrically on the metric space X.*

Common examples of geometric ations are the fundamental groups of finite cell complexes acting on their universal covers. We may then apply the Švarc-Milnor Lemma:

Lemma 2.8. *[14]* **(Švarc-Milnor Lemma)** *Let (X, d) be a proper geodesic metric space, and let G be a group which acts geometrically on X. Then the group G is finitely generated and for any fixed $x_0 \in X$, the map $G \to X$ given by $g \mapsto g \cdot x_0$ is a quasi-isometry if we consider G to be a metric space with respect to the word metric induced by some finite generating set.*

Thus the quasi-isometric properties of a group G acting geometrically on a proper geodesic metric space X may be studied via the quasi-isometric properties of the metric space X. A nice class of such metric spaces are CAT(0) spaces, and among these CAT(0) cube complexes are particularly nice.

Chapter 3

CAT(0) Cube Complexes

A geodesic metric space is said to be CAT(0) if every geodesic triangle in X is at least as thin as its comparison triangle with the same side lengths in the Euclidean plane. This fact is illustrated in Figure 3.1, where a geodesic triangle in a CAT(0) space is shown together with its corresponding comparison triangle in Euclidean space. A locally CAT(0) metric space is a metric space where every point has a neighbourhood which is CAT(0). A simply connected locally CAT(0) space is CAT(0) [5]. Thus the universal covers of locally CAT(0) cube complexes are globally CAT(0).

One can construct a metric on a cubical cell complex K by assigning each cube the metric of an Euclidean cube with side of length 1. This leads to a piecewise Euclidean metric on the complex K which we refer to as the cubical metric on K. One of the benefits of dealing with a metric cube complex is that there is a nice combinatorial condition which is equivalent to the metric having non-positive curvature.

In [12], Gromov proves the following theorem:

Theorem 3.1. *[12] The cubical metric on K has non-positive curvature if and only if the link of every vertex is a flag complex.*

A flag complex, i.e. every complete sub-graph in the link spans a simplex.

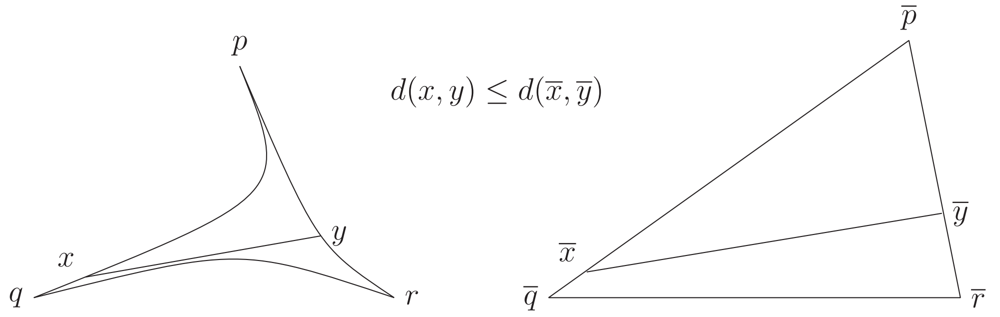

Figure 3.1: Comparison triangles

3.1 Right Angled Artin Groups

Following [5], given a graph Γ, called the defining graph, we define a right angled Artin group (RAAG) A_Γ to be the group with the following presentation, generated by the vertices of Γ, $A_\Gamma = \langle V(\Gamma) | [v, w] \forall$ edge $e \in \Gamma$ with $\partial(e) = \{v, w\}\rangle$.

As outlined in [5], for every RAAG A_Γ with defining graph Γ, one may construct a finite cell complex S_Γ. Start with a wedge of circles attached to a point x_0, one for each vertex v_i of Γ. And label each circle by the corresponding vertex v_i. For each edge $[v_i, v_j]$ in Γ, attach a 2-torus by mapping its standard generators to $v_i v_j {v_i}^{-1} {v_j}^{-1}$. For each triangle in Γ, attach a 3-torus by mapping its standard generators to the circles corresponding to the three vertices of the triangle. Continue in this way, attaching a k-torus for each set of k vertices which span a complete sub-graph of Γ. The resulting complex S_Γ is called the *Salvetti complex* for the RAAG A_Γ and by construction the fundamental group of S_Γ is the RAAG A_Γ. One then has the following theorem from [6]:

Theorem 3.2. *[6] The universal cover of the Salvetti complex, $\overline{S_\Gamma}$, is a CAT(0) cube complex. Hence S_Γ is a $K(\pi, 1)$ space.*

Due to the geometric action of the fundamental group A_Γ on the universal cover $\overline{S_\Gamma}$, Lemma 2.8 gives us a quasi-isometry between A_Γ with the word metric and the universal cover $\overline{S_\Gamma}$ which is a CAT(0) cube complex with the piecewise Euclidean metric.

3.2 Graph Braid Groups

A graph braid group $B_n(\Gamma)$ [1] is defined by analogy with the usual braid group, to be the fundamental group of the configuration space $C_n(\Gamma)$ of n points restricted to a graph Γ. However due to Abrams Stability [1, 17] the configuration space $C_n(\Gamma)$ is homotopy equivalent to a finite cell complex $D_n(\Gamma)$ if the graph Γ is sufficiently subdivided relative to n. In the case of $D_2(\Gamma)$ it is enough for Γ to have no loops or multiple edges between any pair of vertices. Henceforth we shall restrict ourselves to the case of $n = 2$ where we deal with $B_2(\Gamma), D_2(\Gamma)$ and its universal cover $\overline{D_2(\Gamma)}$ and assume that our graph Γ has no loops or multiple edges between pairs of vertices.

In [1] Abrams showed that the universal cover $\overline{D_2(\Gamma)}$ of the discretized configuration space $D_2(\Gamma)$ is a CAT(0) cube complex. Thus implying that $D_2(\Gamma)$ is a finite dimensional $K(\pi, 1)$. Additionally, due to the geometric action of $B_2(\Gamma)$ on $\overline{D_2(\Gamma)}$, we have that $B_2(\Gamma)$ with the word metric is quasi-isometric to the CAT(0) cube complex $\overline{D_2(\Gamma)}$ with the piecewise Euclidean metric.

[1] Usually $B_n(\Gamma)$ refers to the full braid group. However here it refers to the pure braid group which is quasi-isometrically the same as the full braid group. Hence we make no distinction.

3.3 Quasi-Flats in CAT(0) Square Complexes

We now restrict our attention to 2-dimensional CAT(0) cube complexes, or CAT(0) square complexes. Given such a complex K, a quasi-flat Q in K is the image of a quasi-isometric embedding $\phi\colon \mathbb{R}^2 \to K$ of the Euclidean plane into K. In [4], the properties of quasi-flats are studied in a class of well behaved CAT(0) cube complexes under which the spaces $\overline{S_\Gamma}$ and $\overline{D_2(\Gamma)}$ fall. They show that for every quasi-flat Q there exists a locally finite homology class $[F] \in H_2^{lf}(K)$ whose support set $S_{[F]}$ lies a bounded Hausdorff distance from the quasi-flat Q. They go on to analyze the structure of $S_{[F]}$ and find that it consists of a unique cycle of quarter-planes and an additional finite complex that may be ignored.

In [3] a 1-complex is defined that is associated to $\overline{S_\Gamma}$ of a RAAG A_Γ. It is called the quarter-plane complex and has the property that its 1-dimensional cycles correspond to quasi-flats in the CAT(0) cube complex $\overline{S_\Gamma}$. We show that one may define a quarter-plane complex for $\overline{D_2(\Gamma)}$ which also has the property that its 1-dimensional cycles correspond to quasi-flats. Set $K = \overline{D_2(\Gamma)}$ and for a different graph Γ' we set $K' = \overline{D_2(\Gamma')}$. A quasi-isometry $\phi : K \to K'$ induces a bijection between the 1-dimensional cycles of $\mathcal{Q}(K)$ and $\mathcal{Q}(K')$. This parallels the results in [19] where it is shown that for a well behaved CAT(0) 2-complex X, circles in the Tits boundary of X correspond to quasi-flats in X, and a quasi-isometry between two such complexes $\psi : X \to X'$ induces a homeomorphism between the cores of their Tits boundaries (where the core of a Tits boundary $\partial_T X$ is the union of all circles in $\partial_T X$).

The quarter-plane complex $\mathcal{Q}$ is analogous to the core of the Tits boundary as discussed in [19], though it contains additional information in the form of a cellular structure that encodes data about the underlying space $\overline{D_2(\Gamma)}$. This additional cellular information is used to measure the length of a cycle

Σ in $\mathcal{Q}$ by counting the number of 1-cells in the quarter-plane complex which make up the cycle Σ. As it turns out, 1-dimensional cycles in $\mathcal{Q}$ of length 4 correspond to flats in $\overline{D_2(\Gamma)}$. Further, it is shown that the bijection induced on the cycles of the quarter-plane complex by the quasi-isometry ϕ must map 1-dimensional cycles of length 4 to 1-dimensional cycles of length 4. One may view this as an enhancement to the result in [19] in the case of the spaces $\overline{D_2(\Gamma)}$ (for graph braid groups) and $\overline{S_\Gamma}$ (for RAAG's). The interpretation of this statement is that the quasi-isometry ϕ maps flats in K to within bounded Hausdorff distance of flats in K'.

This fact regarding the behaviour of flats under the quasi-isometry ϕ is then used in [3] to define an object called the flat-space associated to a RAAG. For a special class of RAAG's that they call atomic, they show that a quasi-isometry between two atomic RAAG's induces an isometry between their associated flat-spaces. The defining graphs of the atomic RAAG's are then recovered from the local structure of the flat-spaces associated to them and this together with the induced isometry between the flat-spaces is used to show that the RAAG's must be isomorphic.

Unfortunately a similar construction does not obviously suggest itself in the case of the spaces $\overline{D_2(\Gamma)}$. Instead we resort to identifying distinguished subcomplexes (called maximal product subcomplexes) which we show are put into bijection with each other by a quasi-isometry. Intrinsically, a maximal product subcomplex has the structure of a product of trees. Moreover we show that this induced bijection respects large intersecions between these distinguished subcomplexes. Based on this result, we then define a finite dimensional cell complex called the intersection complex $I(\Gamma)$ and show that the quasi-isometry ϕ induces an isometry between the associated intersection complexes.

The intersection complex $I(\Gamma)$ is typically not locally finite, but it is always finite dimensional as long as Γ is finite. Thus the dimension of the intersection complex is a quasi-isometry invariant of the spaces $\overline{D_2(\Gamma)}$. We compute several examples of this invariant. In the case of the complete graph K_n (for $n \geq 6$) we show that the dimension $I(\Gamma)$ is $2^{n-6} - 1$, showing that the spaces $D_2(K_n)$ are quasi-isometrically distinct for each $n \geq 6$. We then construct a family of graphs O_k such that the dimension of $I(O_k)$ is k, so that the invariant is shown to take on every non-negative integer value.

Chapter 4

Graph Braid Groups

We now cover some of the necessary background in graph braid groups. These groups are defined by analogy to the classical braid groups as the fundamental groups of a certain non-compact configuration spaces $C_n(\Gamma)$. We describe a construction of a finite cube complex $D_n(\Gamma)$ that simplifies their study. The complex $D_n(\Gamma)$ is locally CAT(0) and therefore a $K(\pi, 1)$ space, and it is (usually) homotopy equivalent to the original configuration space. After this we develop some of the covering space theory needed to keep track of flats and their intersections in $\overline{D_2(\Gamma)}$.

Following [1] we define the *configuration space* of n points on a graph Γ as follows.

Definition 4.1. (Configuration Space)*[1] The configuration space of n points on a graph Γ, denoted $C_n(\Gamma)$ is defined to be the space $\Gamma \times \cdots \times \Gamma - \Delta$. Here $\Gamma \times \cdots \times \Gamma$ is the cartesian product of n copies of the graph Γ, and $\Delta = \{(x_1, \cdots, x_n) \in \Gamma \times \cdots \times \Gamma | x_i = x_j$ for some $i \neq j\}$.*

Just as the classical braid groups may be defined as the fundamental group of the configuration space of n points in the disc, the graph braid group is defined to be the fundamental group of the configuration space of n points in the graph Γ.

Definition 4.2. (Graph Braid Group) *[1] The n-strand graph braid group $B_n(\Gamma)$ of a graph Γ is defined to be the fundamental group of the configuration space $C_n(\Gamma)$ of n points on the graph Γ.*

As we mentioned earlier, $\pi_1(C_n(\Gamma), *)$ is usually called the pure graph braid group. However it is a finite index subgroup of the full graph braid group and is thus quasi-isometric to it. Hence we make no distinction and refer to it as the graph braid group.

One of the drawbacks of the space $C_n(\Gamma)$ is that it is non-compact and does not admit an easy combinatorial description. So following [1] again, we define the *discretized configuration space* of n points on a graph Γ.

Definition 4.3. (Discretized Configuration Space) *The discretized configuration space of n points on a graph Γ is the subcomplex of $\Gamma \times \cdots \times \Gamma$ (n times) defined by $D_n(\Gamma) := \{\sigma \in \Gamma \times \cdots \times \Gamma | \sigma = \sigma_1 \times \cdots \times \sigma_n$ where σ_i are cells in Γ and $\forall i \neq j, \overline{\sigma_i} \cap \overline{\sigma_j} = \emptyset\}$.*

The following theorem from [17] which is a refinement of a result in [1], tells us when the discretized configuration space $D_n(\Gamma)$ is a good approximation of the configuration space $C_n(\Gamma)$.

Theorem 4.4. *[17]* **(Abrams Stability)** *For any $n > 1$ and any graph Γ with at least n vertices, $C_n(\Gamma)$ deformation retracts to $D_n(\Gamma)$ if and only if:*

1. *Each path between two vertices of valence not equal to 2 passes through at least $n-1$ edges.*

2. *Each homotopically non-trivial path from a vertex to itself passes through at least $n+1$ edges.*

In this paper we discuss only the case $n = 2$, for which the above theorem tells us that as long as the graph has no loops or multiple edges between pairs of vertices, the spaces $D_2(\Gamma)$ and $C_2(\Gamma)$ are homotopy equivalent and thus

$B_2(\Gamma)$ is the fundamental group of $D_2(\Gamma)$. We will return to the following four examples of discretized configuration spaces throughout the rest of this paper.

First we define an induced sub-graph of a graph Γ as this will help us to simplify notation.

Definition 4.5. (Induced Subgraph) *An induced subgraph of a graph* Γ *is a subgraph* $\Gamma' \subset \Gamma$ *such that given vertices* v, w *in* Γ', (v, w) *is an edge in* Γ' *if and only if* (v, w) *is an edge in* Γ. *The induced subgraph* Γ' *is said to be induced by its vertex set* $S = V(\Gamma') \subset V(\Gamma)$. *We also write* $\Gamma[S]$ *for the subgraph of* Γ *induced by* S.

Definition 4.6. (Induced Cycle) *A cyclic subgraph of* Γ *which is induced by its vertices is called an induced cycle.*

Example 4.7. (K_5) *For each edge* E *in the graph* K_5, *there is a unique largest subgraph* C_E *which is disjoint from* E. *The graph* C_E *is a* 3*-cycle in* K_5. *Thus we have that* $E \times C_E$ *is a subcomplex of* $D_2(K_5)$, *for each edge* E. *These complexes look like triangular tubes as shown in Figure 4.1.*

For each vertex v *in* K_5, *there exists a unique largest subgraph* K_v *which is disjoint from* v. *The graph* K_v *is isomrphic to* K_4, *and* $v \times K_v$ *is a subcomplex of* $D_2(K_5)$ *for each vertex* v.

In order to see how these subcomplexes fit together, we use the graph K_5 *as a blueprint. Each vertex of* K_5 *corresponds to a* 1*-dimensional subcomplex* K_v *of* $D_2(K_5)$ *shaped like a tetrahedron. The tetrahedral* 1*-complex associated to each vertex is shown in Figure 4.2. Each of the* 4 *edges* E_i *which terminate at* v *corresponds to a subcomplex* $E_i \times C_{E_i}$ *which is a triangular shaped tube that intersects the subcomplex* $v \times K_v$ *in* $v \times C_{E_i}$. *Each tetrahedral* 1*-complex is the meeting point of* 4 *different triangular tubes, as shown in Figure 4.3. The resulting cell complex* $D_2(K_5)$ *is thus a manifold. In fact it is a genus* 6 *orientable surface.*

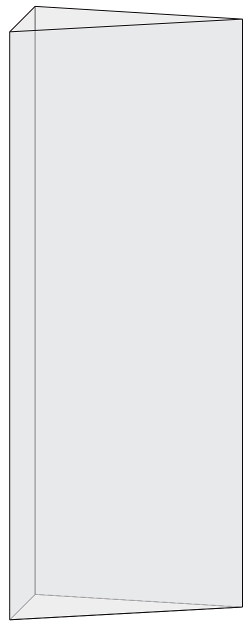

Figure 4.1: Triangular pipe in $D_2(K_5)$

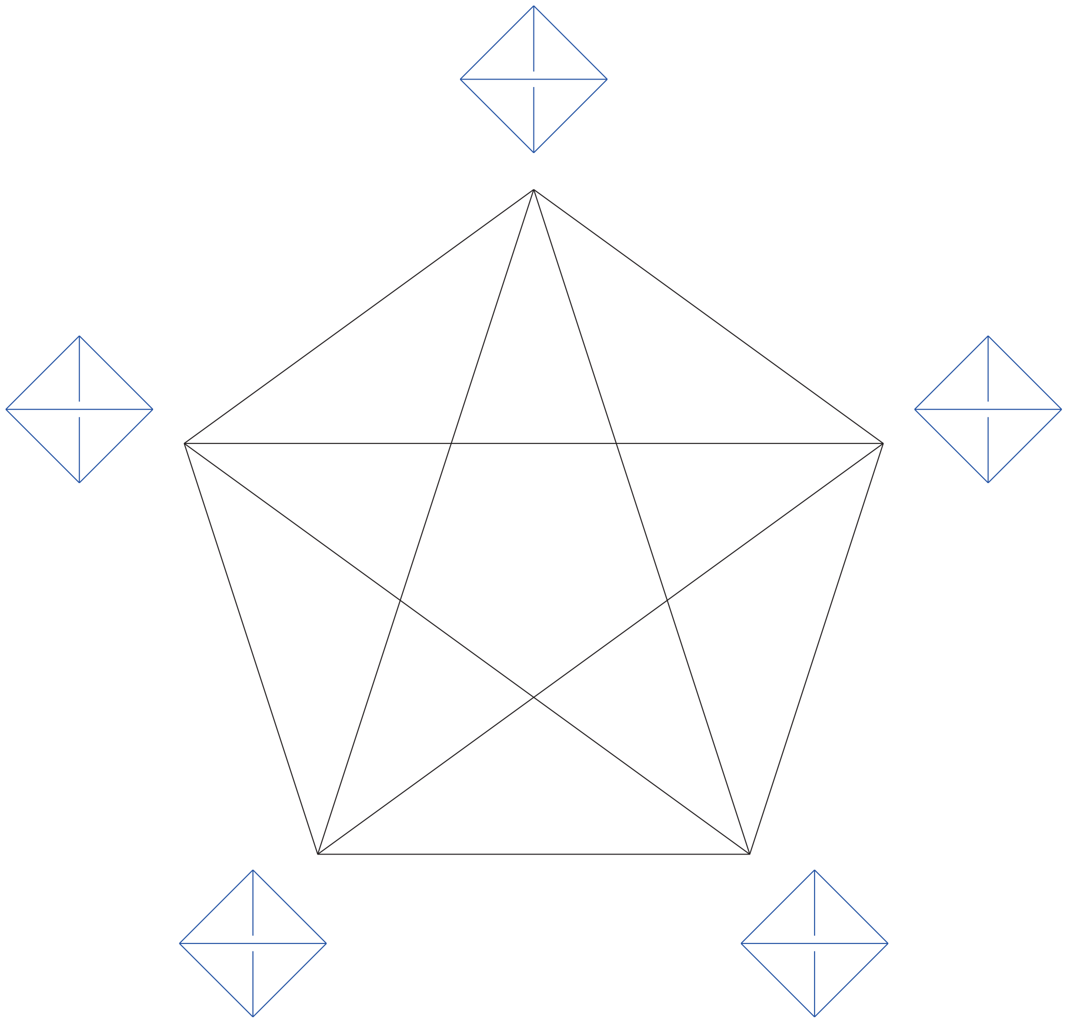

Figure 4.2: Construction of $D_2(K_5)$

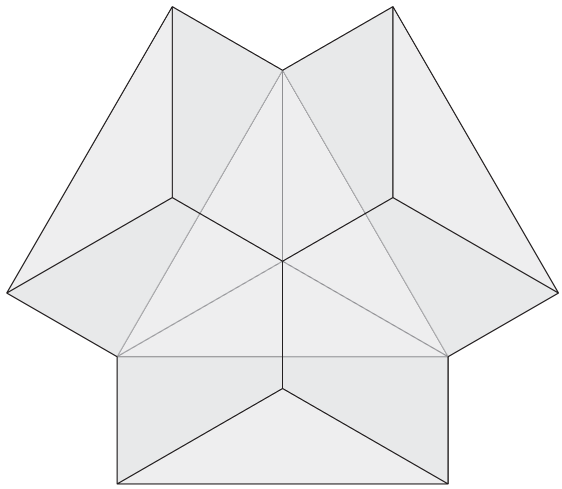

Figure 4.3: Four triangular pipes attached to each face of a tetrahedron.

Example 4.8. (K_6) *There are* 20 3*-cycles in* K_6*, and for each of these* C_i*'s there is a unique* 3*-cycle* C_i' *such that* C_i *and* C_i' *are disjoint. Thus the subcomplexes* $C_i \times C_i'$ *are subcomplexes of* $D_2(K_6)$*. Also note that* $C_i \times C_i'$ *are the largest subcomplexes of the form* $\Gamma_1 \times \Gamma_2$ *where* Γ_1, Γ_2 *are disjoint subgraphs of* K_6 *with all vertices of valence* ≥ 2*.*

Example 4.9. (K_7) *There are* 35 3*-cycles in* K_6*. Given a pair of disjoint 3-cycles* C, C' *we have that the subcomplex* $C \times C'$ *is a subcomplex of* $D_2(K_7)$*. The subcomplexes* $C \times C'$ *are each contained in two subcomplexes* $K_C \times C'$ *and* $C \times K_{C'}$*, where* K_C *is the unique maximal subgraph having all vertices of valence* ≥ 2 *and disjoint from* C'*, and* $K_{C'}$ *is the unique maximal subgraph having all vertices of valence* ≥ 2 *and disjoint from* C*. Also note that both the subcomplexes* $K_C \times C'$ *and* $C \times K_{C'}$ *are maximal (with respect to inclusion) subcomplexes of the form* $\Gamma_1 \times \Gamma_2$ *where* Γ_1, Γ_2 *are disjoint subgraphs with all vertices of valence* ≥ 2*.*

Example 4.10. (K_8) *There are* 56 3*-cycles in* K_8*. Each cycle is a subgraph* *of K_8 induced by the three vertices in the cycle, thus each 3-cycle is of the form $\Gamma[v_1, v_2, v_3]$. Given a pair of disjoint 3-cycles in K_8, of the form $\Gamma[v_1, v_2, v_3]$ and $\Gamma[w_1, w_2, w_3]$. The sub-complex $T = \Gamma[v_1, v_2, v_3] \times \Gamma[w_1, w_2, w_3]$ is a sub-complex of $D_2(K_8)$. Let u_1, u_2 be the remaining vertices of K_8 which are not part of either of this pair of $3 - cycles$. Then there are four sub-complexes of the form:*

$$K_1 = \Gamma[v_1, v_2, v_3, u_1] \times \Gamma[w_1, w_2, w_3, u_2]$$

$$K_2 = \Gamma[v_1, v_2, v_3, u_2] \times \Gamma[w_1, w_2, w_3, u_1]$$

$$K_3 = \Gamma[v_1, v_2, v_3, u_1, u_2] \times \Gamma[w_1, w_2, w_3]$$

$$K_4 = \Gamma[v_1, v_2, v_3] \times \Gamma[w_1, w_2, w_3, u_1, u_2]$$

Lifts of these subcomplexes to the universal cover are later shown to be maximal.

The sub-complex T is contained in each of K_1, K_2, K_3 and K_4. And thus $T \subset K_1 \cap K_2 \cap K_3 \cap K_4$.

Later we shall see that lifts of these are the only maximal product subcomplexes which contain lifts of T.

Re-visiting the examples above, let us examine what their corresponding universal covers look like.

Example 4.11. (K_5) *The universal cover of a smooth constant curvature surface of genus 6 is isometric to the hyperbolic plane $\mathbb{H}^2$. However $D_2(K_5)$ is not smooth, but has a piecewise Euclidean metric, with each 2-cell isometric to I^2.*

In the universal cover $\overline{D_2(K_5)}$ each 0-cell is the meeting point of 6 square cells as shown in Figure 4.4. These 6 square cells cover the subcomplex highlighted in blue in Figure 4.5. The space $\overline{D_2(K_5)}$ is quasi-isometric to $\mathbb{H}^2$.

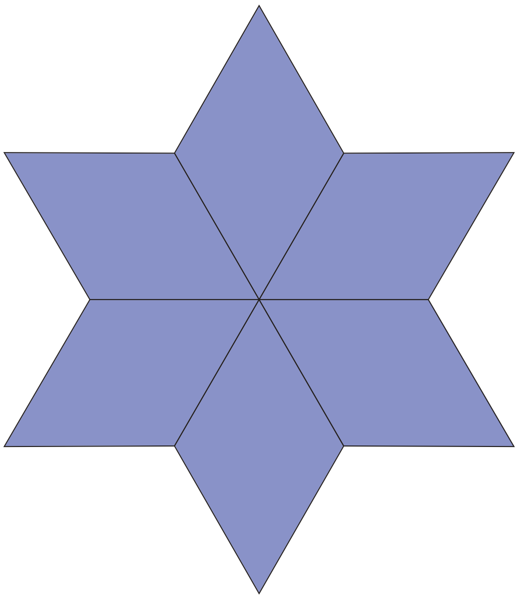

Figure 4.4: Neighbourhood of a 0-cell in $\overline{D_2(K_5)}$

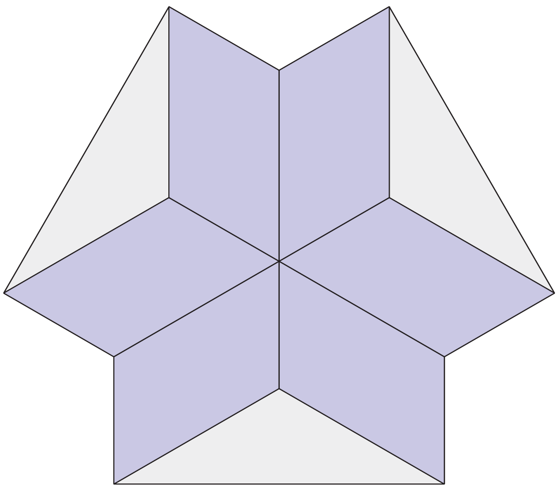

Figure 4.5: The neighbourhood of a 0-cell in $D_2(K_5)$

Example 4.12. (K_6) *Each subcomplex of $D_2(K_6)$ which is a product $C \times C'$ of 3-cycles in $D_2(K_6)$ is a torus. In the universal cover, each of these subcomplexes lifts to infinitely many disjoint subcomplexes isometric to $\mathbb{R}^2$, i.e. each of the subcomplexes $C \times C'$ lifts to infinitely many disjoint flats in $\overline{D_2(K_6)}$.*

Example 4.13. (K_7) *Each subcomplex of $D_2(K_7)$ which is a product $C \times C'$ of 3-cycles is a torus. Each of these lifts to infinitely many flat subcomplexes in $\overline{D_2(K_7)}$.*

For each such subcomplex $C \times C'$, the subcomplexes $K_C \times C'$ and $C \times K_{C'}$ both contain $C \times C'$. Each of these lifts to infinitely many copies of a complex of the form $T_1 \times \mathbb{R}$ or $\mathbb{R} \times T_2$. Here T_1 covers K_C while $\mathbb{R}$ covers C', and T_2 covers $K_{C'}$ while $\mathbb{R}$ covers C. For each flat subcomplex F which is a lift of the subcomplex $C \times C'$, there are exactly two subcomplexes of the form $T_1 \times \mathbb{R}$ and $\mathbb{R} \times T_2$ which are lifts of $K_C \times C'$ and $C \times K_{C'}$ and which contain F.

Example 4.14. (K_8) *Each subcomplex T of $D_2(K_8)$ which is a product of disjoint 3-cycles is a torus. Each of these tori lifts to infinitely many flat subcomplexes in $\overline{D_2(K_8)}$. And each of the complexes K_1, K_2, K_3 and K_4 lift to sub-complexes of $\overline{D_2(K_8)}$ which are products of trees. And each of the lifts of T is contained in exactly four sub-complexes which are lifts of K_1, K_2, K_3 and K_4.*

We now need to talk about $D_2(\overline{\Gamma})$, an intermediate covering space of $D_2(\Gamma)$ that sits somewhere below the universal cover $\overline{D_2(\Gamma)}$. This covering space is useful because it allows us to put Euclidean co-ordinates onto flats in $\overline{D_2(\Gamma)}$. However in order to do this we need to take a brief digression into configurations of coloured graphs, which we call coloured configuration spaces. The definition of the discretized configuration space $D_2(\Gamma)$, can be generalized to take into account the graph colouring, so that for a coloured graph Γ_χ we can define a coloured discretized configuration space $D_2(\Gamma_\chi)$.

A coloured graph is one in which each vertex has been assigned a colour. Adjacent vertices are allowed to have the same colour. When no colouring is specified on a graph, we may assume that it has the standard colouring in which each vertex is assigned a distinct colour.

Definition 4.15. (Graph Colouring) *A k-colouring of a graph Γ is a surjective map $\chi : V(\Gamma) \to \{1, 2, \ldots, k\}$ from the set of vertices to a set of k distinct colours. The colour of a vertex v is $\chi(v)$. A graph Γ with a colouring χ is denoted as Γ_χ. Two k-colourings χ and κ are equivalent if there exists a k-permutation θ such that $\chi = \theta \circ \kappa$. When we refer to a k-colouring of a graph Γ, we shall henceforth be referring to an equivalence class of colourings. A graph with no specified colouring is said to have the standard colouring, in which each of its vertices is assigned a distinct colour.*

The discretized coloured configuration space is defined to be a subset of the product $\Gamma \times ... \times \Gamma$, where a cell $\sigma \in D_n(\Gamma_\chi)$ which is of the form $\sigma = \sigma_1 \times ... \times \sigma_n$, is such that no two distinct cells σ_i, σ_j have vertices which are the same colour.

Definition 4.16. (Discretized Coloured Configuration Space) *The discretized coloured configuration space of n points on the coloured graph Γ_χ is the subcomplex of $\Gamma \times \cdots \times \Gamma$ (n times) defined by $D_n(\Gamma_\chi) := \{\sigma \in \Gamma \times ... \times \Gamma | \sigma = \sigma_1 \times \cdots \times \sigma_n$ where σ_i are cells in Γ, and $\forall i \neq j, \chi(V(\sigma_i)) \cap \chi(V(\sigma_j)) = \phi\}$. Note that if κ is the standard colouring on Γ, then $D_n(\Gamma_\kappa) = D_n(\Gamma)$.*

If we consider the universal covering tree $\overline{\Gamma}$ of the graph Γ, we may lift the standard graph colouring to a colouring of $\overline{\Gamma}$ in which every vertex of $\overline{\Gamma}$ in the fibre of $v \in \Gamma$ is assigned the same colour as v. We call $\overline{\Gamma}$ with this colouring the universal coloured covering tree.

Definition 4.17. (The Universal Coloured Covering Tree) *The universal coloured covering tree $\overline{\Gamma_{\overline{\chi}}}$ of a coloured graph Γ_χ, is the universal covering* *tree $\overline{\Gamma}$ with covering map $p : \overline{\Gamma} \to \Gamma$, and induced colouring $\chi \circ p : V(\overline{\Gamma}) \to \{1, 2, ..., k\}$. If Γ has the standard colouring, then the universal coloured covering tree with its induced colouring is simply denoted as $\overline{\Gamma}$.*

There is a natural map $p_c : D_2(\overline{\Gamma}) \to D_2(\Gamma)$ which is a covering map. However it is important to note that $D_2(\overline{\Gamma})$ need not be the universal cover of $D_2(\Gamma)$ as it is not simply connected in most cases.

Proposition 4.18. *[1] Given a graph Γ with the standard colouring, and its corresponding universal coloured covering tree $\overline{\Gamma}$, the following diagram of covering spaces commutes:*

$$\begin{tikzcd}
D_2(\overline{\Gamma}) \arrow[r, hook, "i"] \arrow[d, "p_c"'] & \overline{\Gamma} \times \overline{\Gamma} \arrow[d, "p \times p"] \\
D_2(\Gamma) \arrow[r, hook, "i"'] & \Gamma \times \Gamma
\end{tikzcd}$$

Where the map p_c: $D_2(\overline{\Gamma}) \to D_2(\Gamma)$ is a restriction of $p \times p$ to the subspace $D_2(\overline{\Gamma})$.

The universal cover $\overline{D_2(\Gamma)}$ also covers the space $D_2(\overline{\Gamma})$ and thus we also have the following commutative diagram of covering spaces:

$$\begin{tikzcd}
\overline{D_2(\Gamma)} \arrow[dd, "s"'] \arrow[dr, "r"] & \\
& D_2(\overline{\Gamma}) \arrow[dl, "p_c"] \\
D_2(\Gamma) &
\end{tikzcd}$$

Definition 4.19. **(Flat)** *A flat F in a metric space X is a subspace which is isometric to $\mathbb{R}^2$.*

Definition 4.20. **(Isolated Flat)** *[16] A flat F in a metric space X is said to be isolated if for every other flat F' in X and for every $D > 0$ we have that $N_D(F) \cap N_D(F')$ is bounded.*

Lemma 4.21. **(Flat Subcomplex Lemma)** *Let $F \subset \overline{D_2(\Gamma)}$ be a flat. Then F is a subcomplex of $\overline{D_2(\Gamma)}$.*

Proof. There is an isometry $f : \mathbb{R}^2 \to \overline{D_2(\Gamma)}$ such that the flat $F = \mathrm{Im}(f)$.

Given a square cell S we would like to show that if $\mathrm{Int}(S) \cap F \neq \emptyset$ then we must have $S \subset F$. So let S be one such square cell and pick $p \in \mathrm{Int}(S) \cap F$. Let $q \in \mathbb{R}^2$ be the pre-image of p under f. There exists $N_\epsilon(q)$ an ϵ-neighbourhood of q in $\mathbb{R}^2$ which is mapped isometrically by f to $N_\epsilon(p)$ an ϵ-neighbourhood of p in $\overline{D_2(\Gamma)}$. Since $\overline{D_2(\Gamma)}$ is a cell-cpmplex and since $p \in \mathrm{Int}(S)$, $\epsilon > 0$ can be chosen so that $N_\epsilon(p) \subset \mathrm{Int}(S)$. And since $N_\epsilon(p)$ is the image under the isometry f of the neighbourhood $N_\epsilon(q)$ we have that $N_\epsilon(p) \subset S \cap F$.

Given an arbitrary point $x' \in S$ there exists a point $x \in N_\epsilon(p)$ such that the geodesic segment $[p, x']$ is an extension of the geodesic segment $[p, x]$. Note that $[p, x] \subset N_\epsilon(p) \subset S$ and that $[p, x'] \subset S$ and is the unique extension of $[p, x]$ in $D_2(\Gamma)$, as this is contained in a single square cell S of Γ.

We also have that $[p, x] \subset N_\epsilon(p) \subset F$ so there exists a point $y \in N_\epsilon(q) \subset \mathbb{R}^2$ such that the geodesic segment $[q, y]$ is mapped isometrically by f to the geodesic segment $[p, x]$ in $N_\epsilon(p)$. The geodesic segment $[q, y]$ extends uniquely to a geodesic segment $\alpha : [0, \infty) \to \mathbb{R}^2$. And since $f : \mathbb{R}^2 \to \overline{D_2(\Gamma)}$ is an isometry, the map $f \circ \alpha : [0, \infty) \to \overline{D_2(\Gamma)}$ is also a geodesic ray in $\overline{D_2(\Gamma)}$ which is an extension of $[p, x]$.

Since the geodesic segment $[p, x']$ is a unique extension of $[p, x]$ in $\overline{D_2(\Gamma)}$(as it lies within a single square cell S), we must have that $[p, x']$ coincides with a sub-segment of $f \circ \alpha$. In other words, there must exist $t, t' \in [0, \infty)$ such that $f \circ \alpha([0,t]) = [p, x]$ and $f \circ \alpha([0,t']) = [p, x']$. Thus we must have that $[p, x'] \subset \mathrm{Im}(f)$, i.e. that $[p, x] \subset F$. And thus $x' \in F$.

Since x' was an arbitrary point in the square cell S we may conclude that $S \subset F$. And thus F must be a sub-complex of $\overline{D_2(\Gamma)}$.

$\square$

Definition 4.22. (Standard Cell Structure) *Consider* $\mathbb{R}$*(resp.* $\mathbb{R}^+ = \{x \in \mathbb{R} \mid x \geq 0\}$*) with a cell structure with a* 0*-cell* σ_i *and a* 1*-cell* τ_i *(isometric to* I*) for every integer* i*, such that the boundary* $\partial\tau_i = \{\sigma_i, \sigma_{i+1}\}$*. We shall call this the* standard cell structure *on* $\mathbb{R}$ *and* $\mathbb{R}^+$*.*

Definition 4.23. (Product Cell Structure) *[15] Given two cell complexes* K_1 *and* K_2 *with countably many open cells, the topological space* $K = K_1 \times K_2$ *may be given a unique cell structure in which each cell* σ *of* K *is a product of cells* σ_1 *and* σ_2 *from* K_1 *and* K_2*, with characteristic map* $\psi_1 \times \psi_2 \colon \sigma \to K$*, where* $\psi_1 \colon \sigma_1 \to K_1$ *and* $\psi_2 \colon \sigma_2 \to K_2$ *are the characteristic maps of* σ_1 *and* σ_2 *respectively. We call this unique cell structure on* K *the* product cell structure.

Definition 4.24. (Quarter-Plane) *[3] Consider* $\mathbb{R}^+$ *with the standard cell structure, and give the product* $\mathbb{R}^+ \times \mathbb{R}^+$ *the product cell structure. A subcomplex of* $\overline{D_2(\Gamma)}$ *or* $D_2(\overline{\Gamma})$ *which is isometric to the complex* $\mathbb{R}^+ \times \mathbb{R}^+$ *is called a* quarter-plane.

In order to prove that quarter-planes in $\overline{D_2(\Gamma)}$ are lifts of subcomplexes of $D_2(\overline{\Gamma})$ which are products of chromatically disjoint geodesic rays in $\overline{\Gamma}$, we need to analyze how subcomplexes isometric to Euclidean cubical strips and half strips behave under the covering map $r : \overline{D_2(\Gamma)} \to D_2(\overline{\Gamma})$

Definition 4.25. (Infinite Square Strip) *An Infinite Strip(or half-strip) is a cell complex of the form* $\mathbb{R} \times I$ *(or* $\mathbb{R}^+ \times I$*) where* $\mathbb{R}$ *and* $\mathbb{R}^+$ *are given the usual cell structure.*

Lemma 4.26. *If $\Delta \subset \overline{D_2(\Gamma)}$ is a subcomplex isometric to an infinite squre strip or half-strip, then $r(\Delta)$ is isometric to an infinite strip or half-strip and r restricted to Δ is an isometry.*

Proof. Let Δ be an infinite square strip in $\overline{D_2(\Gamma)}$. Consider any two adjacent square cells S and S' in Δ. And look at their images under the covering map $s\colon \overline{D_2(\Gamma)} \to D_2(\Gamma)$. Both $s(S)$ and $s(S')$ must be adjacent square cells in $D_2(\Gamma)$ with $s(S) = e \times f$ and $s(S') = e \times g$ where e, f and g are all edges in Γ with e being disjoint from f, g and f, g being adjacent. Thus $\pi_2(s(\Delta)) = \gamma$ where γ is an edge-path in Γ which is disjoint from the edge e. If we now consider the image of S and S' under the covering map $r\colon \overline{D_2(\Gamma)} \to D_2(\overline{\Gamma})$ then $r(S)$ and $r(S')$ must be adjacent squares in $D_2(\overline{\Gamma})$ which cover the squares $s(S)$ and $s(S')$, with $r(S) = \overline{e} \times \overline{f}$ and $r(S') = \overline{e} \times \overline{g}$ where $\overline{e}, \overline{f}, \overline{g}$ cover e, f, g and thus $\overline{e}$ must be chromatically disjoint from $\overline{f}, \overline{g}$. Also $\overline{f}$ and $\overline{g}$ must be adjacent. Because $\overline{\gamma}$ is a tree, we have that $\pi_2(r(\Delta)) = \overline{\gamma}$, where $\overline{\gamma}$ is a geodesic in $\overline{\Gamma}$ which covers γ and which is chromatically disjoint from $\overline{e}$. And since all the squares in $r(\Delta)$ are products of the edge $\overline{e}$ with an edge from the path $\overline{\gamma}$ we may conclude that $r(\Delta) = \overline{e} \times \overline{\gamma}$. Thus Δ covers a strip in $D_2(\overline{\Gamma})$ and since strips are simply connected r must restrict to a homeomorphism and hence an isometry on Δ. The proof for when Δ is a half-strip is identical. □

Proposition 4.27. *If E is a quarter-plane in $\overline{D_2(\Gamma)}$, then there exist chromatically disjoint geodesic rays $\alpha, \beta \in \overline{\Gamma}$ such that $p(E) = \alpha \times \beta$.*

Proof. Since E is a subcomplex its boundary is the union of two geodesics in the 1-skeleton of $\overline{D_2(\Gamma)}$. The quarter-plane E is the union of infinitely many square half-strips $\{\Delta_0, \Delta_1, ..., \Delta_i, \Delta_{i+1}, ...\}$ with $\Delta_i = \overline{e_i} \times \mathbb{R}^+$ such that Δ_i, Δ_{i+1} are adjacent to each other along $\overline{e_i(1)} \times \mathbb{R}^+$ and $\overline{e_{i+1}(0)} \times \mathbb{R}^+$ respectively. By Lemma 4.26 we have that $r(\Delta_i) = e_i \times \beta_i$ in $D_2(\overline{\Gamma})$. Now Δ_i, Δ_{i+1} being adjacent along $\overline{e_i(1)} \times \mathbb{R}^+$ and $\overline{e_{i+1}(0)} \times \mathbb{R}^+$ implies that the half-strips $r(\Delta_i), r(\Delta_{i+1})$ are adjacent along $e_i(1) \times \beta_i$ and $e_{i+1}(0) \times \beta_{i+1}$, so we must have that $\beta_i = \beta_{i+1} = \beta$ for all i. Thus $r(\Delta_i) = e_i \times \beta$ where e_i and β are chromatically disjoint, and $r(\Delta_i), r(\Delta_{i+1})$ adjacent imply that e_i, e_{i+1} are adjacent, so that the cell complex formed by the union of all the e_i's is a geodesic ray α which is chromatically disjoint from β. The half-strips $\{r(\Delta_i)\}$ all fit together to form a quarter-plane, so that $r(E) = \alpha \times \beta$. □

Lemma 4.28. *Let E be a quarter-plane in $\overline{D_2(\Gamma)}$. Then there exists a quarter-plane $E' \subset E$ and a flat F in $\overline{D_2(\Gamma)}$ such that $E' \subset F$, and $\pi_1(s(F)) = \pi_1(s(E'))$ and $\pi_2(s(F)) = \pi_2(s(E'))$.*

Proof. E is the isometric image of $f : [0, \infty) \times [0, \infty) \to \overline{D_2(\Gamma)}$.

Consider the geodesic rays $\alpha = f|_{[0,\infty)\times\{0\}}$ and $\beta = f|_{\{0\}\times[0,\infty)}$. Let $\Gamma_1 = \mathrm{Im}(\pi_1 \circ s \circ \alpha)$ and $\Gamma_2 = \mathrm{Im}(\pi_2 \circ s \circ \beta)$. Then Γ_1, Γ_2 are two disjoint sub-graphs of Γ and $s(E) = \Gamma_1 \times \Gamma_2$. Pick vertices $v \in \Gamma_1$ and $w \in \Gamma_2$ such that v, w have valence ≥ 2 in Γ_1 and Γ_2 respectively. And (v, w) is a 0-cell in $s(E) = \Gamma_1 \times \Gamma_2$.

Let u be a 0-cell in E which is a lift of the 0-cell (v, w). Then there exist integers (s, t) in $[0, \infty) \times [0, \infty)$ so that $f(s, t) = u$, i.e. $s(f(s, t)) = (v, w) \in \Gamma_1 \times \Gamma_2 \subset D_2(\Gamma)$. Consider the map $g = f|_{[s,\infty)\times[t,\infty)}$, this is an isometry $g : [s, \infty) \times [t, \infty) \to \overline{D_2(\Gamma)}$ and $\mathrm{Im}(g) \subset \mathrm{Im}(f)$. Moreover $\mathrm{Im}(g)$ is a sub-complex. Thus $E' = \mathrm{Im}(g)$ is a quarter-plane complex in $\overline{D_2(\Gamma)}$, and by construction $\Gamma'_1 = \pi_1(s(E'))$ and $\Gamma'_2 = \pi_2(s(E'))$ are disjoint sub-graphs of Γ, both of whose vertices all have valence ≥ 2.

Now $s \circ g : [s, \infty) \times [t, \infty) \to \Gamma'_1 \times \Gamma'_2$ is a local isometry and can be extended to a cellular map $h : \mathbb{R} \times \mathbb{R} \to \Gamma'_1 \times \Gamma'_2$ which is also a local isometry. And there is a unique lift of h, call it $\overline{h} : \mathbb{R} \times \mathbb{R} \to \overline{D_2(\Gamma)}$ which contains the image of g, i.e. E'. Thus there is a flat $F = \mathrm{Im}(\overline{h})$ such that $E' \subset F$. And by construction $\pi_1(s(F)) = \pi_1(s(E')) = \Gamma'_1$ and $\pi_2(s(F)) = \pi_2(s(E')) = \Gamma'_2$.

□

Chapter 5

Quarter-Plane Complex

Two given quarter-plane subcomplexes of $\overline{D_2(\Gamma)}$ fit together in one of three ways. They are either equivalent, incident to each other along a pure singular geodesic, or divergent from each other.

Lemma 5.1. *[3] Let $E = \alpha \times \beta$ and $E' = \alpha' \times \beta'$ be quarter-planes in $\overline{D_2(\Gamma)}$. Then one of the following holds.*

1. *(Equivalent) There is a quarter-plane $E'' \subset E \cap E'$ so that we have $Hd(E'', E) < \infty$ and $Hd(E'', E') < \infty$.*

2. *(Incident) There are constants $A, B \in (0, \infty)$ such that after re-labelling the factors of E and E' if necessary, we have α is asymptotic to α' and $\forall p \in E$ and $p' \in E'$, $d(p, E') \geq A(d(p, \alpha) - B)$ and $d(p', E) \geq A(d(p', \alpha') - B)$.*

3. *(Divergent) The distance function d_E grows linearly on E' and vice-versa, i.e. $\exists A, B \in (0, \infty)$ and $p \in E, p' \in E'$ such that $\forall x \in E, x' \in E'$ we have $d(x, E') \geq A(d(x, p) - B)$ and $d(x', E) \geq A(d(x', p') - B)$.*

Given a quarter-plane E, we let $[E]$ denote its equivalence class. The following lemma establishes the fact that the notions of incidence and divergence are well defined when referring to equivalence classes of quarter-planes, so that it makes sense to talk about a pair of quarter-plane equivalence classes $[E], [F]$ being incident or divergent.

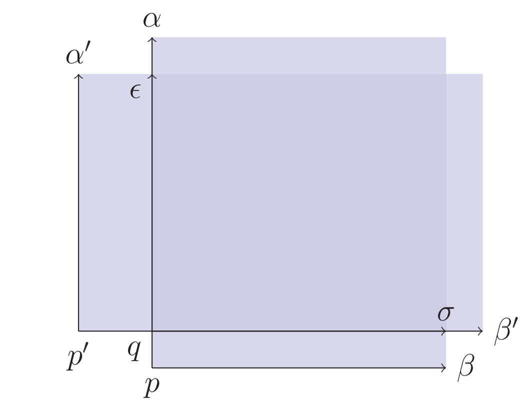

Figure 5.1: Equivalent Quarter-Planes

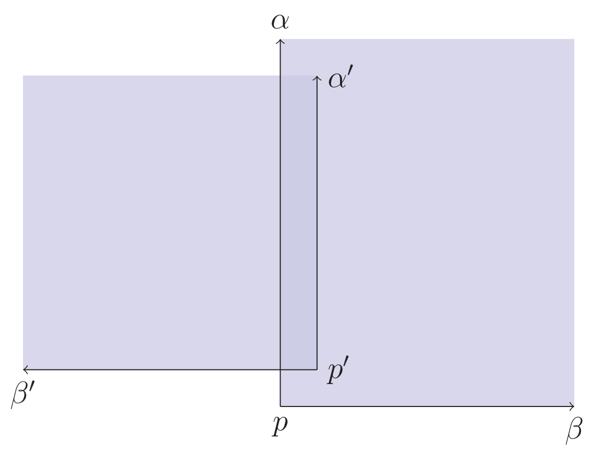

Figure 5.2: Incident Quarter-Planes

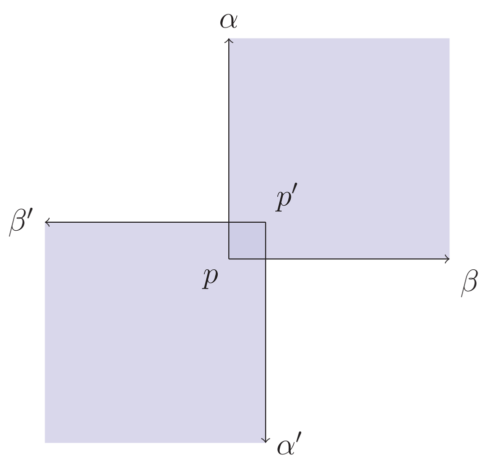

Figure 5.3: Divergent Quarter-Planes

Lemma 5.2. *Consider quarter-planes E and F. Let $E = \alpha \times \beta$ and $F = \tau \times \delta$. Let E' and F' be quarter-planes which are equivalent to E and F, with $E' = \alpha' \times \beta'$ and $F' = \tau' \times \delta'$. Then the following must be true:*

1. *If E is equivalent to F then $\alpha \sim \tau$ and $\beta \sim \delta$.*

2. *If E and F are incident with $\alpha \sim \tau$ then E' and F' are incident with $\alpha' \sim \tau'$.*

3. *If E and F are divergent then E' and F' are divergent.*

Proof. We break the proof up according to the three cases.

Case 1 If E and F are equivalent, then there must exist a quarter-plane $E'' \subset E \cap F$, with $E'' = \alpha'' \times \beta''$. Thus $E'' \subset E$ and $E'' \subset F$, so that $\alpha \sim \alpha'', \tau \sim \alpha''$ and $\beta \sim \beta'', \delta \sim \beta''$. And thus by transitivity we must have that $\alpha \sim \tau$ and $\beta \sim \delta$.

Case 2 Suppose E and F are incident. Consider $E' = \alpha' \times \beta'$ and $F' = \tau' \times \delta'$ equivalent to E and F respectively. By Case 1 we have that $\alpha \sim \alpha'$ and $\tau \sim \tau'$. Now E and F are incident along α, τ with $\alpha \sim \tau$. Thus $\alpha \sim \tau$ implies $\alpha' \sim \tau'$ and so E' and F' are incident along α', τ' with $\alpha' \sim \tau'$.

Case 3 Suppose E and F are divergent. Consider $E' = \alpha' \times \beta'$ and $F' = \tau' \times \delta'$ equivalent to E and F respectively. By Case 1 we have that $\alpha \sim \alpha', \beta \sim \beta'$ and $\tau \sim \tau', \delta \sim \delta'$. If E' and F' were not divergent, then they must be equivalent or incident, which would force E and F to be equivalent or incident by Case 1 and 2. Thus we must have that E' and F' are divergent.

□

We now define the Quarter-Plane Complex $\mathcal{Q}$.

Definition 5.3. (Quarter-Plane Complex) *[3] Consider the collection of vertices $\{v_{[\alpha]}\}$, corresponding to the asymptotic classes of singular geodesic rays in $\overline{D_2(\Gamma)}$. If there exists a quarter-plane E in $\overline{D_2(\Gamma)}$ such that α and β are the two singular geodesics making up its boundary, then connect the two vertices corresponding to their asymptotic classes $v_{[\alpha]}, v_{[\beta]}$ by an edge $(v_{[\alpha]}, v_{[\beta]})$. Note that the edges are well-defined, by the previous lemma. The resulting 1-complex is called the quarter-plane complex $\mathcal{Q}(\overline{D_2(\Gamma)})$.*

Definition 5.4. (Hausdorff Equivalence Class) *Given a metric space (X, d) and subsets $A, A' \subset X$. We say that A and A' are Hausdorff Equivalent if there exists $r > 0$ such that $A' \subset N_r(A)$ and $A \subset N_r(A')$. This defines an equivalence relation on the collection of subsets of X.*

In [3] they prove that every quasiflat is bounded Hausdorff distance from a unique cycle of quarter-planes, and that every cycle of quarter-planes is the image of a quasi-isometric embedding. Thus it is enough to deal with cycles of quarter-planes in order to understand quasi-flats.

The following two lemmas from [3] formally establish the bijection between quasi-flats and cycles of quarter-planes.

Lemma 5.5. *[3] Let $Q \subset \overline{D_2(\Gamma)}$ be a quasi-flat. There is a unique cycle of quarter-plane equivalence classes in the quarter-plane complex $[E_1], ..., [E_k] \subset \mathcal{Q}$ such that $\cup_{i=1}^{k} E_i$ is a finite Hausdorff distance from Q. We call this cycle Σ_Q.*

Lemma 5.6. *[3] Every cycle $\Sigma \subset \mathcal{Q}$ arises from a quasi-flat $Q \subset \overline{D_2(\Gamma)}$.*

A brief analysis of the quarter-plane complex $\mathcal{Q}(\overline{D_2(\Gamma)})$ reveals that it only contains even length cycles. This helps us to constrain the behaviour of the cycles of quarter-planes under the quasi-isometry.

Definition 5.7. (i-pure singular geodesic ray) *This is a singular geodesic ray γ in $\overline{D_2(\Gamma)}$ such that $s((\gamma)) = \Gamma_1 \times w$ or $v \times \Gamma_2$ in $D_2(\Gamma)$. If $s((\gamma))$ is of the form $\Gamma_1 \times w$ then it is called 1-pure, if it is of the form $v \times \Gamma_2$ then it is called 2-pure.*

Lemma 5.8. *Given an i-pure singular geodesic ray γ in $\overline{D_2(\Gamma)}$, if τ is a singular geodesic ray which is asymptotic to γ, then there exists t_0 such that $\tau : [t_0, \infty) \to \overline{D_2(\Gamma)}$ is also an i-pure singular geodesic ray.*

Proof. The images of γ and τ are convex sets in $\overline{D_2(\Gamma)}$ and so by Lemma 6.1 they eventually bound a strip, i.e. there exists a t_0 such that for $t \geq t_0$ the rays $\gamma : [t_0, \infty) \to \overline{D_2(\Gamma)}$ and $\tau : [t_0, \infty) \to \overline{D_2(\Gamma)}$ bound an Euclidean strip. Since both these geodesic rays are singular, this implies that the strip is the image of a cellular embedding and that $\tau : [t_0, \infty) \to \overline{D_2(\Gamma)}$ is also i-pure singular. $\square$

Thus given a vertex v in $\mathcal{Q}(\overline{D_2(\Gamma)})$, representing an asymptotic class of singular geodesic rays, either all of the singular geodesic rays in this equivalence class are eventually 1-pure singular or 2-pure singular or none of them are 1 or 2 pure singular. Thus among the asymptotic equivalence classes of singular geodesic rays, those containing 1-pure singular geodesic rays and those containing 2-pure singular geodesic rays form a pair of disjoint distinguished sets of vertices in $\mathcal{Q}(\overline{D_2(\Gamma)})$.

Lemma 5.9. *The quarter-plane complex $\mathcal{Q}(\overline{D_2(\Gamma)})$ has a bi-partite structure. Thus all cycles Σ in $\mathcal{Q}(\overline{D_2(\Gamma)})$ must have even length.*

Proof. Given a quarter-plane equivalence class $[E]$ in $\mathcal{Q}(\overline{D_2(\Gamma)})$, there exists a quarter-plane $E \in [E]$ by Lemma 4.28 which is the image of an isometry $f : [0,\infty) \times [0,\infty) \to \overline{D_2(\Gamma)}$, such that $s(E) = \Gamma_1 \times \Gamma_2 \subset D_2(\Gamma)$ and such that $\gamma = s \circ f : [0,\infty) \times \{0\} \to \overline{D_2(\Gamma)}$ and $\tau = s \circ f : \{0\} \times [0,\infty) \to \overline{D_2(\Gamma)}$ are i and j pure singular respectively with $i \neq j$. Thus $[E]$ is an edge in $\mathcal{Q}(\overline{D_2(\Gamma)})$ between the two distinguished sets of vertices corresponding to the asymptotic classes of eventually 1-pure singular and 2-pure singular geodesic rays. Thus $\mathcal{Q}(\overline{D_2(\Gamma)})$ has a bipartitie structure and thus all cycles Σ must have even length.

$\square$

Consider the graph K_7. If we consider a quarter-plane $E \subset \overline{D_2(K_7)}$ then $r(E)$ in $D_2(\overline{K_7})$ must be a product of two chromatically disjoint geodesic rays α, β in $\overline{\Gamma}$. Since α, β are chromatically disjoint, they must cover disjoint subgraphs of K_7 with vertices of valence ≥ 2. There are two possibilities for such a pair: (K_3, K_4) or (K_3, K_3). Moreover, each disjoint pair of the form (K_3, K_3) is contained in exactly two disjoint pairs of the form (K_4, K_3) and (K_3, K_4) respectively. Consider a lift K of a particular product of the form $K_3 \times K_4$. Each quarter-plane in K is of the form $\alpha \times \beta$ where α covers a path in K_3 and β covers a path in K_4. There are two asymptoticially distinct geodesic rays α, α' which cover paths in K_3, and there are uncountably many asymptotically distinct geodesic rays in K which cover paths in K_4. Thus K contributes a suspended Cantor set to the quarter-plane complex $\mathcal{Q}(\overline{D_2(K_7)})$.

5.1 Coherence

Since cycles of quarter-plane equivalence classes correspond to quasi-flats, any quasi-isometry $\phi : \overline{D_2(\Gamma)} \to \overline{D_2(\Gamma')}$ carries cycles in $\mathcal{Q}(\overline{D_2(\Gamma)})$ to cycles in $\mathcal{Q}(\overline{D_2(\Gamma')})$. Thus in order to better understand the behaviour of such a quasi-isometry we need to understand how the quasi-isometry behaves on intersections of cycles of quarter-plane equivalence classes. Hausdorff equivalence classes behave well with respect to union. Namely, the Hausdorff equivalence class of the union $A \cup B$ of two sets A and B consists of representatives which are each the set union of Hausdorff equivalence class representatives of the sets A and B. The notion of Hausdorff intersection is a more delicate matter. Unless the collection of Hausdorff equivalence classes satisfy a special notion called "coherence" as noted in [3], there may not be a well defined Hausdorff intersection.

Definition 5.10. (Coherent Hausdorff Equivalence Classes) *[3] Given a collection of Hausdorff Equivalence classes $\{[A_\Lambda]\}$, where the representatives $\{A_\Lambda\}$ are subsets of a metric space (X, d). We call the collection of Hausdorff classes coherent if for any finite intersection $A_1 \cap \cdots \cap A_k$, there exists $r_0 > 0$ such that for all $r > r_0$ we have that $[N_r(A_1) \cap \cdots \cap N_r(A_k)] = [N_{r_0}(A_1) \cap \cdots \cap N_{r_0}(A_k)]$. In this case we define $[A_1] \cap \cdots \cap [A_k] := [N_{r_0}(A_1) \cap \cdots \cap N_{r_0}(A_k)]$.*

Next, we prove given a collection of coherent Hausdorff equivalence classes, that the notion of Hausdorff intersection is well defined.

The idea of using the poset of quarter-plane cycle intersections to study the behaviour of a quasi-isometry appears in [3]. And again we choose to include the statement and proofs of facts that were essentially known in [3].

We will show that the collection of quarter-plane equivalence classes is a coherent collection, and so we may consider intersections of cycles of quarter-plane equivalence classes. These Hausdorff equivalence classes may be given a partial ordering that is induced by coarse set inclusion. Our goal in this section is to characterize the kinds of minimal elements which may occur in the poset of cycle intersections of the quarter-plane complex $\mathcal{Q}(\overline{D_2(\Gamma)})$.

Set inclusion induces a partial order on the subsets of a space. One may extend this to Hausdorff equivalence classes by stating that $[A] \subset [B]$ if a representative from $[A]$ is contained in a D-neighbourhood of a representative from $[B]$. This is a well defined notion, independent of the representatives chosen from the Hausdorff equivalence classes.

Definition 5.11. (Hausdorff Inclusion Relation) *Given Hausdorff Equivalence classes $[A]$ and $[B]$ with representatives A, B in a metric space (X, d), we say that $[A] \subset [B]$ if there exists $D > 0$ such that $A \subset N_D(B)$.*

Proposition 5.12. (Well Definedness of Hausdorff Inclusion) *Given subsets A, B in a metric space (X, d) and representatives $A' \in [A]$ and $B' \in [B]$, if there exists $D' > 0$ such that $A' \subset N_{D'}(B')$ then there exists $D > 0$ such that $A \subset N_D(B)$, i.e. $[A'] \subset [B']$.*

Proof. There exists $S, T > 0$ such that $A \subset N_S(A')$ and $B' \subset N_T(B)$. Thus $A \subset N_S(A') \subset N_{D'+S}(B') \subset N_{D'+S+T}(B)$.

Let $D = D' + S + T$. Thus $A \subset N_D(B)$. $\square$

If the Hausdorff equivalence classes in question are from a coherent collection, then the Hausdorff inclusion relation behaves well with respect to the intersection of Hausdorff equivalence classes from the coherent collection. This is essentially the content of the next lemma.

Lemma 5.13. *Given coherent Hausdorff equivalence classes $[A], [B]$ in a metric space (X, d), we have that $[A] \cap [B] \subset [A]$ and $[A] \cap [B] \subset [B]$.*

Proof. Set $S = N_{r_0}(A) \cap N_{r_0}(B) \in [A] \cap [B]$, where $[A] \cap [B] = [N_{r_0}(A) \cap N_{r_0}(B)]$. We have that $S \subset N_{r_0}(A)$ and $S \subset N_{r_0}(B)$. Thus $[S] \subset [A]$ and $[S] \subset [B]$. Thus $[A] \cap [B] \subset [A]$ and $[A] \cap [B] \subset [B]$. □

Lemma 5.14. *Given a quasi-isometry $\phi : X \to X'$ between two metric spaces:*

1. *Given Hausdorff equivalence classes $[A], [B]$ with $[A] \subsetneq [B]$ in X. Then for arbitrary $A \in [A]$ and $B \in [B]$ we have that $[\phi(A)] \subsetneq [\phi(B)]$.*

2. *Given Hausdorff equivalence classes $[A], [B]$ with $[A] \not\subset [B]$. Then for arbitrary $A \in [A]$ and $B \in [B]$ we have that $[\phi(A)] \not\subset [\phi(B)]$.*

3. *Given Hausdorff equivalence classes $[A], [B]$ with $[A] = [B]$. Then for arbitrary $A \in [A]$ and $B \in [B]$ we have that $[\phi(A)] = [\phi(B)]$.*

Proof. The proof for each of the three cases is as follows.

Case 1 Since we have that $[A] \subsetneq [B]$ we have that $[A] \subset [B]$ and $[A] \neq [B]$. Thus there exists $D > 0$ such that $A \subset N_D(B)$ and for all $C > 0$ we have that $B \not\subset N_C(A)$. Thus we have that $\phi(A) \subset \phi(N_D(B))$. Now there exists $D' > 0$ such that $\phi(N_D(B)) \subset N_{D'}(\phi(B))$, so we may conclude that $\phi(A) \subset N_{D'}(\phi(B))$. Now let if possible that there exists $C' > 0$ such that $\phi(B) \subset N_{C'}(\phi(A))$. Using the quasi-isometric inverse to ϕ we can conclude that there exists a $C > 0$ such that $B \subset N_C(A)$ which contradicts the fact that $[A] \neq [B]$. Thus we are forced to conclude that for all $C' > 0$ we have that $\phi(B) \not\subset \phi(A)$. And thus we may conclude that $[\phi(A)] \subsetneq [\phi(B)]$.

Case 2 Since $[A] \not\subset [B]$, for arbitrary representatives $A \in [A]$ and $B \in [B]$ we have that for all $D > 0, A \not\subset N_D(B)$. Now consider $\phi(A)$ and $\phi(B)$.

Let if possible that there exists $D' > 0$ such that $\phi(A) \subset N_{D'}(\phi(B))$. Then using the quasi-isometric inverse of ϕ we can construct a $D > 0$ such that $A \subset N_D(B)$. Contradicting the fact that $[A] \not\subset [B]$. Thus for all $D' > 0$ we have that $\phi(A) \not\subset N_{D'}(\phi(B))$. And thus we may conclude that $[\phi(A)] \not\subset [\phi(B)]$.

Case 3 Since $[A] = [B]$, for arbitrary representatives $A \in [A]$ and $B \in [B]$ there exists $D > 0$ such that $A \subset N_D(B)$ and $B \subset N_D(A)$. Now consider $\phi(A)$ and $\phi(B)$. We have that $\phi(A) \subset \phi(N_D(B))$ and $\phi(B) \subset \phi(N_D(A))$. And due to the fact that ϕ is a quasi-isometry we can construct a $D' > 0$ such that $\phi(N_D(B)) \subset N_{D'}(\phi(B))$ and $\phi(N_D(A)) \subset N_{D'}(\phi(A))$. And thus we have that $\phi(A) \subset N_{D'}(\phi(B))$ and $\phi(B) \subset N_{D'}(\phi(A))$. So that we have $[\phi(A)] = [\phi(B)]$.

□

Proposition 5.15. *Let $[A_1], \cdots, [A_k]$ be a finite collection of Hausdorff equivalence classes drawn from a possibly larger collection of Hausdorff equivalence classes $\{[A_\alpha] | \alpha \in \Lambda\}$.*

1. *Define $[A_1] \cup \cdots \cup [A_k] := [A_1 \cup \cdots \cup A_k]$. This definition is independent of the representatives $A_1, \ldots, A_k$ of the Hausdorff equivalence classes.*

2. *If the collection $\{[A_\alpha] | \alpha \in \Lambda\}$ is coherent then define $[A_1] \cap \cdots \cap [A_k] := [N_r(A_1) \cap \cdots \cap N_r(A_k)]$ where $r > 0$ is the radius of coherence. This definition is independent of the representatives $A_1, \ldots, A_k$ of the Hausdorff equivalence classes.*

3. *Given $[A]$ from the collection $\{[A_\alpha] | \alpha \in \Lambda\}$, if the collection is coherent we have that $[A] \cap ([A_1] \cup \cdots \cup [A_k]) = ([A] \cap [A_1]) \cup \cdots \cup ([A] \cap [A_k])$.*

4. *Given $[A], [B]$ and $[C]$ from the collection $\{[A_\alpha] | \alpha \in \Lambda\}$, with $[A] \subset [B]$, if the collection is coherent then we have that with respect to Hausdorff inclusion $[A] \cap [C] \subset [B] \cap [C]$.*

Proof. Exercise. □

Lemma 5.16. (Hausdorff Intersections of Quarter-Planes) *Given quarter-planes $E_1 = \alpha \times \beta$ and $E_2 = \alpha' \times \beta'$ in $\overline{D_2(\Gamma)}$.*

1. *If E_1 and E_2 are equivalent, then there exists $r_0 > 0$ and a quarter-plane E' such that for all $r > r_0, [N_r(E_1) \cap N_r(E_2)] = [N_{r_0}(E_1) \cap N_{r_0}(E_2)] = [E']$.*

2. *If E_1 and E_2 are incident along α and α' then there exists $r_0 > 0$ such that for all $r > r_0, [N_r(E_1) \cap N_r(E_2)] = [N_{r_0}(E_1) \cap N_{r_0}(E_2)] = [\alpha]$.*

3. *If E_1 and E_2 are divergent, then there exists $r_0 > 0$ and $p \in E_1$ such that for all $r > r_0, [N_r(E_1) \cap N_r(E_2)] = [N_{r_0}(E_1) \cap N_{r_0}(E_2)] = [p]$.*

Proof. Exercise. □

Lemma 5.17. *Given a geodesic ray α and a quarter-plane E, we have that $[\alpha] \cap [E]$ is defined and that $[\alpha] \cap [E] = [p]$ or $[\alpha]$, where $p = \alpha(0)$.*

Proof. Either $\alpha \subset N_{r_0}(E)$ for some $r_0 > 0$, or α is not contained in $N_r(E)$ for any $r > 0$.

If $\alpha \subset N_{r_0}(E)$ for some $r_0 > 0$, then $[N_{r_0}(\alpha) \cap N_{r_0}(E)] = [\alpha]$ and for all $r > r_0$ we have that $[N_r(\alpha) \cap N_r(E)] = [\alpha] = [N_{r_0}(\alpha) \cap N_{r_0}(E)]$. And so in this case $[\alpha] \cap [E]$ is defined and equal to $[\alpha]$.

If α is not contained in $N_r(E)$ for any $r > 0$. Then let $p = \alpha(0)$ and $s_0 > d(p, E)$. We have that $[N_{s_0}(\alpha) \cap N_{s_0}(E)] = [p]$, and for all $s > s_0$ we have that $[N_s(\alpha) \cap N_s(E)] = [p] = [N_{s_0}(\alpha) \cap N_{s_0}(E)]$. Thus in this case $[\alpha] \cap [E]$ is defined and equal to $[p]$. □

Lemma 5.18. *Given a point p and a quarter-plane E. We have that $[p] \cap [E]$ is defined and equal to $[p]$.*

Proof. Exercise. □

Proposition 5.19. *The collection of quarter-planes is coherent. That is, given a finite collection of quarter-planes $E_1, \ldots, E_n$ we have that $[E_1] \cap \cdots \cap [E_n] = [E]$ or $[\alpha]$ or $[p]$, where E is some quarter plane, α is a pure singular geodesic and p is a single point.*

Proof. If $n = 2$ the result holds by Lemma 5.16.

Assume that it is true for $n = k$.

If $n = k+1$ then consider $[E_1] \cap \cdots \cap [E_k] \cap [E_{k+1}]$. We have that $[E_1] \cap \cdots \cap [E_k]$ is defined and equal to either $[E]$, $[\alpha]$ or $[p]$, where E is a quarter-plane, α is a pure singular geodesic and p is a point.

Thus $[E_1] \cap \cdots \cap [E_k] \cap [E_{k+1}] = [E] \cap [E_{k+1}]$ or $[\alpha] \cap [E_{k+1}]$ or $[p] \cap [E_{k+1}]$.

If $[E_1] \cap \cdots \cap [E_k] \cap [E_{k+1}] = [E] \cap [E_{k+1}]$ then $[E_1] \cap \cdots \cap [E_k] \cap [E_{k+1}]$ is defined and equal to $[E']$ or $[\alpha]$ or $[p]$, where E' is a quarter-plane, α is a pure singular geodesic, and p is a point, all by Lemma 5.16.

If $[E_1] \cap \cdots \cap [E_k] \cap [E_{k+1}] = [p] \cap [E_{k+1}]$ then $[p] \cap [E_{k+1}]$ is defined and equal to $[p]$. And so $[E_1] \cap \cdots \cap [E_k] \cap [E_{k+1}]$ is defined and equal to $[p]$.

If $[E_1] \cap \cdots \cap [E_k] \cap [E_{k+1}] = [\alpha] \cap [E_{k+1}]$ then by Lemma 5.17 we have that $[\alpha] \cap [E_{k+1}]$ is defined and equal to either $[p]$ or $[\alpha]$. Thus $[E_1] \cap \cdots \cap [E_k] \cap [E_{k+1}]$ is defined and equal to $[p]$ or $[\alpha]$. Where p is a point and α is a pure singular geodesic.

Thus the statement is true for all n. □

The cycles of quarter-plane Hausdorff equivalence classes in the quarter-plane complex form a coherent collection by a previous Lemma. Thus, armed with the above definitions and results we may define the poset of cycle intersections, which is formed by taking intersections of cycles of quarter-plane equivalence classes in the quarter-plane complex, and giving it the poset structure induced on it by the Hausdorff inclusion relation.

5.2 Product subcomplexes and posets

Definition 5.20. (Poset of Cycle Intersections) *Consider the* 1*-dimensional sub-complexes of* $\mathcal{Q}(\overline{D_2(\Gamma)})$ *which can be expressed as the intersection of finitely many cycles* $C_1, ..., C_k$ *in* $\mathcal{Q}(\overline{D_2(\Gamma)})$*. Partially order these* 1*-dimensional complexes by the Hausdorff inclusion relation, to turn them into a poset* $\mathcal{P}$*.*

Our next goal is to classify the types of minimal elements that can occur in the poset $\mathcal{P}$ of cycle intersections of $\mathcal{Q}(\overline{D_2(\Gamma)})$. We approach this by picking an arbitrary quarter-plane equivalence class $[E]$ and looking at a quarter-plane representative, and then examining the types of cycles in $\mathcal{Q}$ which contain $[E]$. In order to do this we examine subcomplexes of $\overline{D_2(\Gamma)}$ which contain quarter-planes, and these are called product subcomplexes.

Definition 5.21. (Product Subcomplex) *A product subcomplex of* $\overline{D_2(\Gamma)}$ *is a subcomplex* K *with a product cell structure induced from* $T_1 \times T_2$ *as in Definition 4.23, where* T_1 *and* T_2 *are infinite trees, each of whose* 1*-cells has length* 1 *and each of whose vertices have valence* ≥ 2*. In particular, product subcomplexes are connected.*

Depending on the type of tree which forms each factor of the product, we may classify product subcomplexes into three types:

1. $T_1 \times T_2$ when both trees contain vertices with valence ≥ 3.
2. $T \times \mathbb{R}$ or $\mathbb{R} \times T$ when only one of the factors is a tree containing vertices of valence ≥ 3.
3. $\mathbb{R} \times \mathbb{R}$ when neither factor contains any vertices of valence ≥ 3.

Definition 5.22. (Standard Product Subcomplex) *A standard product subcomplex is a product subcomplex which is a lift to* $\overline{D_2(\Gamma)}$ *of a subcomplex* K *in* $D_2(\Gamma)$ *of the form* $K = \Gamma_1 \times \Gamma_2$ *where* Γ_1 *and* Γ_2 *are disjoint subgraphs in* Γ*.*

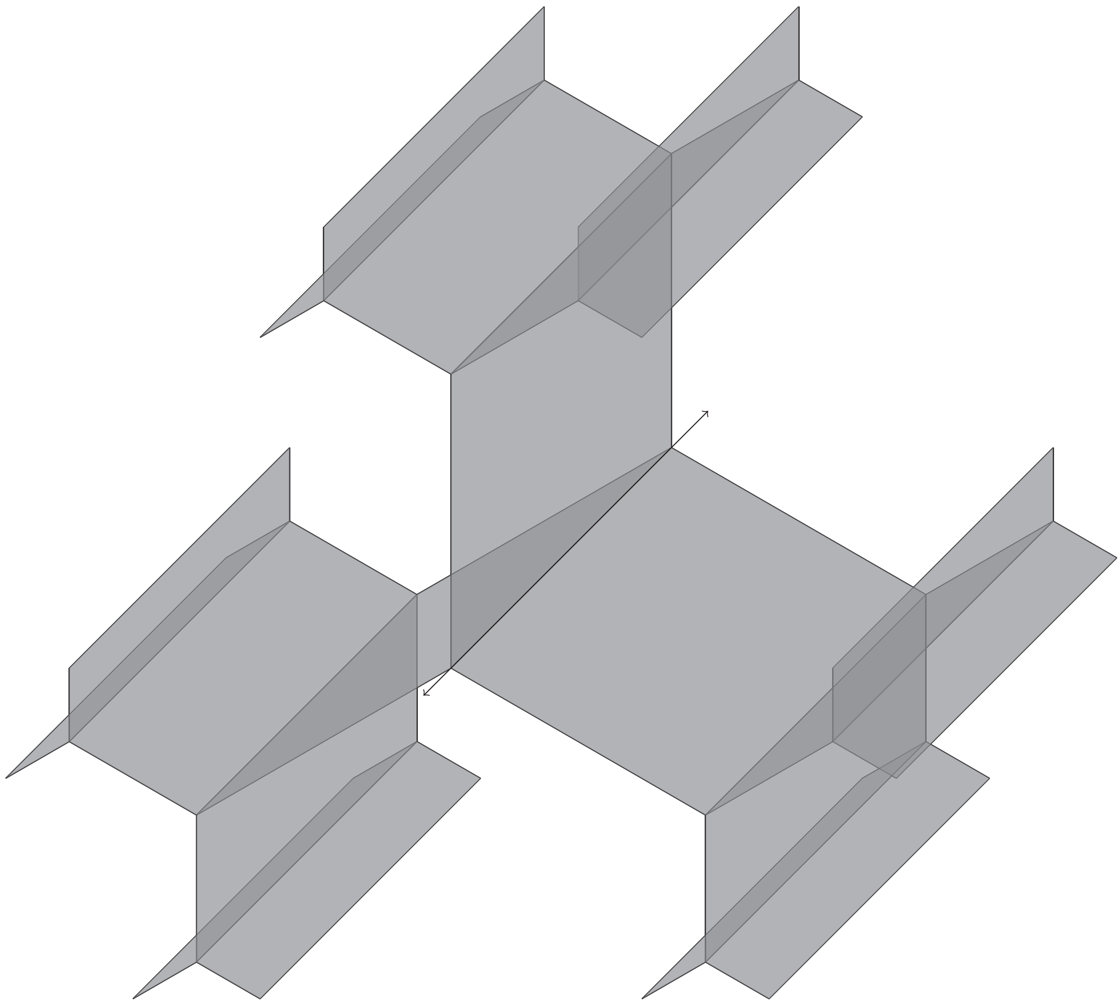

Figure 5.4: Product complex of the form $T \times \mathbb{R}$

Definition 5.23. (Maximal Product Subcomplex) *A maximal product subcomplex K in $\overline{D_2(\Gamma)}$ is a product subcomplex which is maximal in the collection of product subcomplexes with respect to the partial ordering induced on this collection by set inclusion.*

Lemma 5.24. *Let $K = \overline{T_1} \times \overline{T_2}$ be a product subcomplex in $\overline{D_2(\Gamma)}$. Then $r(K)$ in $D_2(\overline{\Gamma})$ is of the form $T_1 \times T_2$ where T_1 and T_2 are chromatically disjoint subtrees of $\overline{\Gamma}$.*

Proof. K may be expressed as a union of square strips $\Delta_{\overline{e},\overline{\gamma}}$ where $\overline{e}$ is an edge in $\overline{T_1}$ and $\overline{\gamma}$ is a geodesic in $\overline{T_2}$. Given two strips $\Delta_{\overline{e},\overline{\gamma_1}}$ and $\Delta_{\overline{e},\overline{\gamma_2}}$, they intersect in a sub-strip or half-strip if and only if $\overline{\gamma_1}$ and $\overline{\gamma_2}$ intersect in a geodesic segment or ray in $\overline{T_2}$. The subcomplex $r(\Delta_{\overline{e},\overline{\gamma}})$ is an embedded subcomplex of $D_2(\overline{\Gamma})$ of the form $e \times \gamma$, where e is an edge in $\overline{\Gamma}$ and γ is a geodesic in $\overline{\Gamma}$ with e, γ chromatically disjoint. Moreover $r(\Delta_{\overline{e},\overline{\gamma_1}} \cap \Delta_{\overline{e},\overline{\gamma_2}}) = r(\Delta_{\overline{e},\overline{\gamma_1}}) \cap r(\Delta_{\overline{e},\overline{\gamma_2}})$. Thus we may conclude that for each $\overline{e}$ in $\overline{T_1}$, the complex $\overline{e} \times \overline{T_2}$ is mapped isometrically to the subcomplex $r(\overline{e} \times \overline{T_2})$ in $D_2(\overline{\Gamma})$. Moreover $r(\overline{e} \times \overline{T_2}) = e \times T_2$ where e is an edge in $\overline{\Gamma}$ and T_2 is a subtree of $\overline{\Gamma}$ which is chromatically disjoint from e. Now consider two subcomplexes of the form $\overline{e} \times \overline{T_2}$ and $\overline{f} \times \overline{T_2}$. These subcomplexes are adjacent along $\overline{e}(0) \times \overline{T_2}$ and $\overline{f}(1) \times \overline{T_2}$ if and only if the edges $\overline{e}$ and $\overline{f}$ are adjacent at $\overline{e}(0)$ and $\overline{f}(1)$. Thus $r(\overline{e} \times \overline{T_2})$ and $r(\overline{f} \times \overline{T_2})$ are adjacent along $e(0) \times T_2$ and $f(1) \times T_2$ if and only of $\overline{e}$ and $\overline{f}$ are adjacent edges in $\overline{T_1}$. Thus, gluing together the subcomplexes $r(\overline{e} \times \overline{T_2})$ as $\overline{e}$ ranges over all the edges of $\overline{T_1}$ accoding to the edge adjacency in $\overline{T_1}$, we get that $r(\overline{T_1} \times \overline{T_2})$ is mapped isometrically to a subcomplex of $D_2(\overline{\Gamma})$ and that $r(\overline{T_1} \times \overline{T_2}) = T_1 \times T_2$ where T_1 and T_2 are chromatically disjoint subtrees of $\overline{\Gamma}$. □

Lemma 5.25. *Given a product subcomplex K in $\overline{D_2(\Gamma)}$, there exists a standard product subcomplex K' in $\overline{D_2(\Gamma)}$ such that $K \subset K'$.*

Proof. Consider the image of K under the covering map s, so that $s(K) = \Gamma_1 \times \Gamma_2$ in $D_2(\Gamma)$. Let K' be the component of $s^{-1}(\Gamma_1 \times \Gamma_2)$ which intersects K. Then $K \subset K'$, and K' is a standard product sub-complex. □

Corollary 5.26. *Maximal product subcomplexes are standard product subcomplexes.*

Proof. Let if possible that a maximal product subcomplex M is not a standard product subcomplex. Then by the previous lemma, there exists a standard product subcomplex M' strictly larger than M and such that $M \subset M'$. This contradicts the fact that M is maximal. Thus we must have that M is a standard product subcomplex. □

Upon examining the three types of product subcomplexes, one may determine what type of minimal element in the poset of cycle intersection classes that a particular quarter-plane equivalence class belongs to by examining the maximal product subcomplex to which a representative quarter-plane belongs. And as it turns out, we may have minimal elements of length $1, 2$ or 4.

Lemma 5.27. *Given a quarter-plane E lying in the intersection of two maximal product subcomplexes M and M', there exists a flat F containing E such that F lies in the intersection of M and M'.*

Proof. Consider $E_1 = \pi_1(s(E))$ and $E_2 = \pi_2(s(E))$. We must have that E_1, E_2 are disjoint subgraphs of Γ. Moreover $s(M) = M_1 \times M_2, s(M') = M_1' \times M_2'$ and $E_1 \subset M_1 \cap M_1', E_2 \subset M_2 \cap M_2'$. Thus all the vertices in E_1 and E_2 have valence ≥ 2. Moreover E_1 and E_2 must both contain non-trivial cycles in Γ, so that $E_1 \times E_2$ lifts to a standard product subcomplex K contained in $M \cap M'$ in $\overline{D_2(\Gamma)}$ with $E \subset K$. We can thus extend the quarter-plane E to a flat $F \subset K$. And thus $E \subset F \subset M \cap M'$. □

Lemma 5.28. *Let E be a quarter-plane contained in a maximal product M of the form $T_1 \times T_2$, where T_1, T_2 are trees with all vertices having valence ≥ 2 and at least one vertex of valence 3. Then there exist Σ_1, Σ_2 two cycles of quarter-plane equivalence classes such that $[E] = \Sigma_1 \cap \Sigma_2$. Thus $[E]$ is a minimal element of length 1 in $\mathcal{P}$.*

Proof. Consider the complex $r(M)$, this must be of the form $S_1 \times S_2$, where S_1, S_2 are chromatically disjoint sub-trees of $\overline{\Gamma}$ both of whose vertices have valence ≥ 2, and both having at least one vertex with valence ≥ 3. The complex $s(E)$ is contained in $s(M)$ and $s(E)$ must be of the form $\alpha \times \beta$, where α, β are chromatically disjoint geodesic rays contained in S_1 and S_2 respectively. Due to the presence in both S_1 and S_2 of vertices with valence ≥ 3, there must exist geodesic rays α_1, α_2 in S_1 and β_1, β_2 in S_2, such that $\alpha \cup \alpha_1, \alpha \cup \alpha_2$ are geodesics in S_1 and $\beta \cup \beta_1, \beta \cup \beta_2$ are geodesics in S_2. Thus we have two flats in $D_2(\overline{\Gamma})$, $F_1 = (\alpha \cup \alpha_1) \times (\beta \cup \beta_1)$ and $F_2 = (\alpha \cup \alpha_2) \times (\beta \cup \beta_2)$ such that $s(E) = \alpha \times \beta = F_1 \cap F_2$. The flats F_1 and F_2 lift to two flats $\overline{F_1}, \overline{F_2}$ in M with $E = \overline{F_1} \cap \overline{F_2}$. Each flat corresponds to a pair of cycles of quarter-plane equivalence classes Σ_1, Σ_2 respectively, with $[E] = \Sigma_1 \cap \Sigma_2$. Thus $[E]$ is a minimal element of length 1 in the post of cycle intersections. □

Lemma 5.29. *Let E be a quarter-plane contained in a maximal product subcomplex $M = \mathbb{R} \times T_2$. If there exists a maximal product subcomplex M' distinct from M, such that E also lies in a maximal product subcomplex M' with $M' = T_1' \times \mathbb{R}$ or $\mathbb{R} \times T_2'$, then M' must have the form $T_1' \times \mathbb{R}$ and E is representative of a length 1 minimal element in $\mathcal{P}$.*

Proof. The complex $s(M)$ covered by M must be of the form $C \times G$, disjoint subgraphs of Γ with C a cycle. Also $s(E)$ is a subcomplex of $s(M)$ covered by E, and $s(E)$ has the form $C \times G'$ where G' is a subgraph of G. Now consider M'. If M' is of the form $\mathbb{R} \times T_2'$ and $E \subset M'$ then the complex $s(M')$ covered by M' must be of the form $C \times H$ with C and H disjoint subgraphs of Γ (where C is the same cycle that appears as a factor of $s(M)$). Since M' is distinct from M we must have that $H \neq G$. However since $E \subset M \cap M'$ we must have that $s(E) \subset s(M) \cap s(M')$ so that G and H have non-empty intersection, in fact $G' \subset G \cap H$. Thus $F = G \cup H$ must be a subgraph of Γ which is disjoint from C, so that the complex $C \times F$ in $D_2(\Gamma)$ lifts to a product subcomplex M'' in $\overline{D_2(\Gamma)}$ which contains both M and M' contradicting the fact that they were both maximal. Thus M' cannot be of the form $\mathbb{R} \times T_2'$. Thus M' must be of the form $T_1' \times \mathbb{R}$, and the complex $s(M')$ covered by M' must be of the form $F \times C'$ where F and C' are disjoint subgraphs of Γ, and C' is a cycle. One can now construct cycles of quarter-planes Σ in M, and Σ' in M' such that $E = \Sigma \cap \Sigma'$. Thus the equivalence class $[E]$ is a minimal element of length 1 in the poset $\mathcal{P}$. $\square$

Lemma 5.30. *Let E be a quarter-plane contained in exactly one maximal product subcomplex M of the form $T_1 \times \mathbb{R}$ or $\mathbb{R} \times T_2$. Then there exists a quarter-plane F incident to E such that any cycle of quarter-planes Σ which contains $[E]$ must also contain $[F]$. Moreover, $m = [E] \cup [F]$ is a minimal element of length 2 in $\mathcal{P}$.*

Proof. Without loss of generality let M be of the form $T_1 \times \mathbb{R}$. And let E be a quarter-plane in M, such that E is contained only in M and no other maximal product sub-complex.

Thus there exists an isometric embedding $m : T_1 \times \mathbb{R} \to \overline{D_2(\Gamma)}$ such that M is the image of m. And the quarter-plane E is the image of the restriction of m to $m : \alpha \times [0, \infty) \to \overline{D_2(\Gamma)}$, where α is a geodesic ray in T_1.

Let F be the image of the restriction of m to $m : \alpha \times (-\infty, 0] \to \overline{D_2(\Gamma)}$. Thus F is a quarter-plane in $\overline{D_2(\Gamma)}$ which is also contained in M and such that F is adjacent to E along $m : \alpha \times \{0\} \to \overline{D_2(\Gamma)}$.

There exist bi-infinite geodesics σ and τ in T_1, which are asymptotically distinct, and such that $\sigma \cap \tau = \alpha$. Let S_1 be the image of the restriction of

m to $m : \sigma \times \mathbb{R} \to \overline{D_2(\Gamma)}$, and let S_2 be the image of the restriction of m to $m : \tau \times \mathbb{R} \to \overline{D_2(\Gamma)}$. So that $S_1 \cap S_2 = E \cup F$. Let Σ_1, Σ_2 be the cycles of quarter-plane equivalence classes in $\mathcal{Q}(\overline{D_2(\Gamma)})$ which correspond to S_1 and S_2 respectively. Then $\Sigma_1 \cap \Sigma_2 = [E] \cup [F]$.

Consider $s(M)$ and $s(E)$. The complex $s(M)$ must be of the form $\Gamma_1 \times C_2$ where C_2 is a sub-graph of Γ which is a induced cycle, and Γ_1 is a sub-graph of Γ. The complex $s(E)$ must be of the form $\Gamma_1' \times C_2$ where Γ_1' is a subgraph of Γ_1.

Now let if possible that there exists a cycle of quarter-plane equivalence classes Σ, such that $[E] \in \Sigma$ but $[F] \notin \Sigma$. Let $[F']$ be the quarter-plane equivalence class in Σ which is incident to $[E]$ along the asymptotic class of the singular geodesic $m : \alpha \times \{0\} \to \overline{D_2(\Gamma)}$. Let F' be a quarter-plane sub-complex belonging to the class $[F']$. Then F' is the image of $f : [0, \infty) \times [0, \infty) \to \overline{D_2(\Gamma)}$, such that $f : [0, \infty) \times \{0\} \to \overline{D_2(\Gamma)}$ is asymptotic to $m : \alpha \times \{0\} \to \overline{D_2(\Gamma)}$ in $\overline{D_2(\Gamma)}$. Thus $s(E) = \Gamma_1' \times C_2$, and $s(F') = \Gamma_1' \times \Gamma_2$ with $\Gamma_2 \neq C_2$. Thus there exists a lift K of the complex $\Gamma_1' \times \Gamma_2$ which contains E and which is not contained in M.

Thus there exists a maximal product sub-complex M' with $E \subset K \subset M'$. This contradicts the fact that M is the only maximal product subcomplex which contains E. Thus such a cycle of quarter-plane equivalence classes Σ, cannot exist. Thus for every cycle of quarter-plane equivalence classes Σ which contains $[E]$ must also contain $[F]$.

Thus $[E] \cup [F]$ is a length 2 minimal element in $\mathcal{P}$.

□

Proposition 5.31. *For any quarter-plane equivalence class $[E]$, there exists a quarter-plane representative $E \in [E]$ such that exactly one of the following holds:*

1. *E is contained in a maximal product subcomplex M of the form $T_1 \times T_2$.*

In this case $[E]$ *is a minimal element of* $\mathcal{P}$ *of length* 1.

2. *E is contained in the intersection of two maximal product subcomplexes* M, M'*,respectively of the form* $\mathbb{R} \times T_2, T_1' \times \mathbb{R}$*. In this case* $[E]$ *is a minimal element of* $\mathcal{P}$ *of length* 1.

3. E *is contained in exactly one maximal product subcomplex* M*, and* M *has the form* $T \times \mathbb{R}$ *or* $\mathbb{R} \times T$*. In this case there exists a quarter-plane equivalence class* $[F]$ *such that* $m = [E] \cup [F]$ *is a minimal element of* $\mathcal{P}$ *of length* 2.

4. E *is contined in exactly one maximal product subcomplex* M*, and* M *has the form* $\mathbb{R} \times \mathbb{R}$*. In this case there exist quarter-plane equivalence classes* $[E_1], [E_2], [E_3]$ *such that* $m = [E] \cup [E_1] \cup [E_2] \cup [E_3]$ *is a minimal element of* $\mathcal{P}$ *of length* 4.

Proof. Given a quarter-plane equivalence class $[E]$ there exists a quarter-plane representative $E \in [E]$ and a flat F such that $E \subset F$, by Lemma 4.28. Now F is a product sub-complex and hence there exists a maximal product sub-complex M such that $F \subset M$. Thus there is a maximal product sub-complex M such that $E \subset M$.

Case 1. If E is contained in a maximal product subcomplex of the form $T_1 \times T_2$ then by Lemma 5.28 $[E]$ is a minimal element of length 1.

Case 2. If E is contained in exactly one maximal product sub-complex M of the form $T_1 \times \mathbb{R}$ or $\mathbb{R} \times T_2$ then by Lemma 5.30, there exists a quarter-plane F which also lies in M such that every cycle of quarter-planes which contains E must also contain F, so that $[E] \cup [F]$ is a length 2 minimal element.

Case 3. If E is contained in two maximal product subcomplexes M and M'. Without loss of generality, if M is of the form $\mathbb{R} \times T_2$ then by Lemma

5.29 then M' must be of the form $T_1 \times \mathbb{R}$ and there exist two cycles of quarter-planes Σ_1, Σ_2 such that $E = \Sigma_1 \cap \Sigma_2$ so that $[E]$ is a length 1 minimal element.

Case 4. If E is contained in exactly one maximal product subcomplex M of the form $\mathbb{R} \times \mathbb{R}$ then E is contained in an isolated flat, which is a cycle of 4 quarter-planes. Let the quarter-plane equivalence classes which are distinct from $[E]$ be $[E_1], [E_2], [E_3]$ so that $m = [E] \cup [E_1] \cup [E_2] \cup [E_3]$ is an isolated cycle in the quarter-plane complex and hence a minimal element of length 4.

□

5.3 Preservation of Flats

Lemma 5.32. *Let $f : X \to X'$ be a quasi-isometry between metric spaces, and let $g : X' \to X$ be the quasi-isometric inverse of f. Let $A, B \subset X$ and $A', B' \subset X'$ and $D > 0$ be such that:*

1. *Hd$(f(A), A') < D$ and Hd$(f(B), B') < D$*
2. *There exist $r_0, r_0' > 0$ such that for all $r > r_0$ and for all $r' > r_0'$ we have $[N_r(A) \cap N_r(B)] = [N_{r_0}(A) \cap N_{r_0}(B)]$ and $[N_{r'}(A') \cap N_{r'}(B')] = [N_{r_0'}(A') \cap N_{r_0'}(B')]$.*

Then there exists $K > 0$ such that Hd$(f(N_{r_0}(A) \cap N_{r_0}(B)), (N_{r_0'}(A') \cap N_{r_0'}(B'))) < K$.

Proof. We have that $\mathrm{Hd}(f(A), A') < D$ and $\mathrm{Hd}(f(B), B') < D$ and thus for $r_0, r_0' > 0$ we have that $\mathrm{Hd}(f(N_{r_0}(A)), N_{r_0'}(A')) < D_0$ and $\mathrm{Hd}(f(N_{r_0}(B)), N_{r_0'}(B')) < D_0$. In particular $f(N_{r_0}(A)) \subset N_{D_0}(N_{r_0'}(A'))$ and $f(N_{r_0}(B)) \subset N_{D_0}(N_{r_0'}(B'))$. Thus we have:

$$f(N_{r_0}(A)\cap N_{r_0}(B)) \subset f(N_{r_0}(A))\cap f(N_{r_0}(B)) \subset N_{D_0}(N_{r'_0}(A'))\cap N_{D_0}(N_{r'_0}(B'))$$

Setting $r' = D_0 + r'_0$ we get:

$$f(N_{r_0}(A) \cap N_{r_0}(B)) \subset N_{r'}(A') \cap N_{r'}(B')$$

And thus since $r' > r'_0$ there exists a $D'_0 > 0$ such that:

$$f(N_{r_0}(A) \cap N_{r_0}(B)) \subset N_{D'_0}(N_{r'_0}(A') \cap N_{r'_0}(B'))$$

Setting $X_0 = N_{r_0}(A) \cap N_{r_0}(B)$ and $X'_0 = N_{r'_0}(A') \cap N_{r'_0}(B')$ we get:

$$f(X_0) \subset N_{D'_0}(X'_0) \tag{5.1}$$

Since $f : X \to X'$ is a quasi-isometry it has a quasi-isometric inverse $g : X' \to X$, and by a similar line of reasoning to that above, there exists a constant $D''_0 > 0$ such that:

$$g(X'_0) \subset N_{D''_0}(X_0)$$

Applying f to both sides we get:

$$f \circ g(X'_0) \subset f(N_{D''_0}(X_0))$$

And since $f \circ g$ is a bounded distance from the identity function on X', there is a constant $C > 0$ such that:

$$X'_0 \subset N_C(f(N_{D''_0}(X_0)))$$

Now $f(N_{D''_0}(X_0))$ is contained in $N_{C''}(f(X_0))$ for some $C'' > 0$ so that we get:

$$X_0' \subset N_{C''}(f(X_0)) \tag{5.2}$$

Putting together equations (5.1) and (5.2) and setting $K = \max\{D_0', C''\}$ we get that

$$\mathrm{Hd}(f(X_0), X_0') < K$$

In other words:

$$\mathrm{Hd}(f(N_{r_0}(A) \cap N_{r_0}(B)), (N_{r_0'}(A') \cap N_{r_0'}(B'))) < K$$

□

Together with the previous Lemma 5.32, we show that a quasi-isometry $\phi : \overline{D_2(\Gamma)} \to \overline{D_2(\Gamma')}$ induces a bijection between the 1-dimensional minimal elements of the posets of cycle intersections of $\overline{D_2(\Gamma)}$ and $\overline{D_2(\Gamma')}$ respectively. The proof is similar to the proof from [3] for RAAGs.

Theorem 5.33. *[3] Given $\phi : \overline{D_2(\Gamma)} \to \overline{D_2(\Gamma')}$ a quasi-isometry. Let $\mathcal{P}$ be the poset of cycle intersections in $\mathcal{Q}(\overline{D_2(\Gamma)})$, and let $\mathcal{P}'$ be the poset of cycle intersections in $\mathcal{Q}(\overline{D_2(\Gamma)})$. Then there exists a $K > 0$ such that for every minimal element $[m] \in \mathcal{P}$, there exists a minimal element $[m_\star] \in \mathcal{P}'$ so that if $m \in [m]$ is a representative of $[m]$, then there is a representative $m_\star \in [m_\star]$ which is a union of quarter-planes, such that $Hd(\phi(m), m_\star) < K$.*

Proof. ϕ maps quasi-flats to quasi-flats. Thus ϕ maps each cycle Σ of quarter-plane equivalence classes in $\mathcal{Q}(\overline{D_2(\Gamma)})$ to a cycle Σ' of quarter-plane equivalence classes in $\mathcal{Q}(\overline{D_2(\Gamma)})$. And by Lemmas 5.5 and 5.6 we have that there exists $D > 0$ such that if $S \in \Sigma, S' \in \Sigma'$ are cycles of quarter-plane representatives then $\mathrm{Hd}(\phi(S), S') < D$.

Let $[m] \in \mathcal{P}$ be minimal. Them $[m]$ is a union of quarter-plane equivalence classes and there exist cycles $\Sigma_1, ..., \Sigma_k$ in $\mathcal{Q}(\overline{D_2(\Gamma)})$ such that $[m] = \Sigma_1 \cap ... \cap$

Σ_k. If $S_i \in \Sigma_i$ are cycles of quarter-plane representatives for each cycle Σ_i then there exists $r_0 > 0$ such that $[m] = [N_{r_0}(S_1) \cap ... \cap N_{r_0}(S_k)]$ and one cna say that $m = N_{r_0}(S_1) \cap ... \cap N_{r_0}(S_k)$ is a representative of $[m]$. This follows from the definition of coherence and from Lemma 5.19.

Let S_i' be the cycles of quarter-planes in $\overline{D_2(\Gamma)}$ such that $\mathrm{Hd}(\phi(S_i), S_i') < D$. For each i let Σ_i' be the cycle of quarter-plane equivalence classes corresponding to S_i'. Then there exists $r_0' > 0$ such that $\Sigma_1' \cap ... \cap \Sigma_k' = [N_{r_0'}(S_1') \cap ... \cap N_{r_0'}(S_k')]$. And by Lemma 5.32 there exists $K > 0$ such that $\mathrm{Hd}(\phi(N_{r_0}(S_1) \cap ... \cap N_{r_0}(S_k)), N_{r_0'}(S_1') \cap ... \cap N_{r_0'}(S_k')) < K$. Let $m_\star = N_{r_0'}(S_1') \cap ... \cap N_{r_0'}(S_k')$. Now $[m_\star] = \Sigma_1' \cap ... \cap \Sigma_k'$ and must have a representative which is the union of at least one quarter-plane, because $[m]$ had a representative which was a union of quarter-planes as $[m] \in \mathcal{P}$. Thus $[m_\star]$ is 1-dimensional and $[m_\star] = \Sigma_1' \cap ... \cap \Sigma_k'$ and so $[m_\star] \in \mathcal{P}'$.

Since $\phi : \overline{D_2(\Gamma)} \to \overline{D_2(\Gamma')}$ is a quasi-isometry, it must preserve the partial ordering induced by Hausdorff inclusion, by Lemma 5.14. Thus $[m] \in \mathcal{P}$ minimal implies that the corresponding $[m_\star] \in \mathcal{P}'$ is minimal.

$\square$

We now focus on proving that this bijection between the 1-dimensional minimal elements of the quarter-plane complexes of $\overline{D_2(\Gamma)}$ and $\overline{D_2(\Gamma')}$ preserves length. As a precursor to proving that, we prove that every length 2 minimal element m is part of a cycle Σ which is made up of one other minimal element, also of length 2, i.e. $\Sigma = m \cup m'$ with m' also having length 2.

Proposition 5.34. *Given a graph Γ, consider the quarter-plane complex $\mathcal{Q}(\overline{D_2(\Gamma)})$. Let m be a minimal element consisting of 2 incident quarter-planes in the poset $\mathcal{P}(\overline{D_2(\Gamma)})$.*

Then there exists a cycle of quarter-planes C in $\mathcal{Q}(\overline{D_2(\Gamma)})$ such that $C = m \cup m'$ where m' is a minimal element in $\mathcal{P}(\overline{D_2(\Gamma)})$ consisting of 2 incident *quarter-planes.*

Proof. A length 2 minimal element consists of two incident quarter-planes $[E] = [\alpha \times \beta]$ and $[F] = [\tau \times \beta']$ where β and β' are asymptotic. Note E, F are contained in a maximal product subcomplex of the form $\mathbb{R} \times T$.

Pick a geodesic ray γ in T such that β and γ are asymptotically distinct. Then let $[E'] = [\alpha \times \gamma]$ and $[F'] = [\tau \times \gamma]$. By Proposition 5.31, $[E'] \cup [F']$ form a minimal element of length 2. Thus $[E] \cup [F] \cup [F'] \cup [E']$ forms a cycle of quarter-planes which is the union of two length 2 minimal elements. $\square$

It now follows quite easily that the bijection induced on $\mathcal{P}$ by a quasi-isometry preserves the lengths of the minimal elements.

Proposition 5.35. *Let graphs Γ, Γ' and a quasi-isometry $\phi : \overline{D_2(\Gamma)} \to \overline{D_2(\Gamma')}$ be given. Suppose m is a 1-dimensional minimal element in $\mathcal{P}(\overline{D_2(\Gamma)})$. There exists a minimal element m' in $\mathcal{P}(\overline{D_2(\Gamma')})$ and a constant $D' > 0$ such that $Hd(\phi(m), m') < D'$, and the length of m is the same as the length of m'.*

Proof. Isolated flats must be mapped to within bounded Hausdorff distance of isolated flats. Thus length 4 minimal elements in $\mathcal{P}$ are mapped bijectively to length 4 minimal elements of $\mathcal{P}'$.

By Proposition 5.34, a length 2 minimal element m_1 is part of a cycle $\Sigma = m_1 \cup m_2$ where m_2 is another length 2 minimal element. Let $\Sigma' = m_1' \cup m_2'$ be the cycle to which Σ is mapped by the quasi-isometry. If m_1' and m_2' are both length 1 minimal elements, this would force them both to be quarter-planes which would be incident to each other along both their boundary geodesics. By Lemma 5.1 this would would force m_1' to be equivalent to m_2' contradicting the fact that Σ' is a cycle. If on the other hand either m_1' or m_2' was a length 1 minimal element while the other was a length 2 minimal element, it would imply that the cycle Σ' was of odd length contradicting Lemma 5.9.

Thus both m'_1 and m'_2 in Σ' must be length 2 minimal elements.

This forces length 1 minimal elements to be mapped to length 1 minimal elements, and we conclude that ϕ preserves the length of minimal elements.

□

The fact that the bijection of minimal elements preserves length allows us to conclude that 4-cycles of quarter-planes in $\overline{D_2(\Gamma)}$ are mapped by the quasi-isometry ϕ to within bounded Hausdorff distance of 4-cycles in $\overline{D_2(\Gamma')}$. Now 4-cycles of quarter-planes have area growth exactly equal to πr^2. This, together with the fact from [4] that support sets with area growth exactly equal to πr^2 must be flats, implies that the image of a flat F in $\overline{D_2(\Gamma)}$ under the quasi-isometry ϕ is bounded Hausdorff distance from a unique flat F' in $\overline{D_2(\Gamma')}$.

Theorem 5.36. *Given* Γ, Γ'*, and a quasi-isometry* $\phi : \overline{D_2(\Gamma)} \to \overline{D_2(\Gamma')}$.

Then there exists a constant $D > 0$ *such that if* F *is a flat in* $\overline{D_2(\Gamma)}$*, then there exists a flat* F' *in* $\overline{D_2(\Gamma')}$ *such that* $Hd(\phi(F), F') < D$.

Proof. Consider the flat F in $\overline{D_2(\Gamma)}$. We have that F is a flat subcomplex of $\overline{D_2(\Gamma)}$. Since F is a union of 4 quarter-planes, it is represented by a 4 cycle in the quarter-plane complex of $\overline{D_2(\Gamma)}$.

There exists a cycle of quarter-planes Σ' such that if F' is a union of representative quarter-planes from this cycle, then $\mathrm{Hd}(\phi(F), F') < D$.

Both F and F' are the union of the same number of minimal elements, which by Proposition 5.35 must have their lengths preserved.

Thus the cycle of quarter-planes corresponding to F' is a 4 cycle.

Consider $p \in F'$. Since F' is a union of 4 quarter-planes we have that $\mathrm{Area}(N_r(p) \cap \Sigma')/(\pi r^2) \to 1$ as $r \to \infty$.

Thus by Theorem 4.1 in [4], F' must be a flat.

□

Chapter 6

Maximal Product Sub-Complexes

In this section we focus on the behaviour of product subcomplexes under quasi-isometries. Our goal is to show that a maximal product subcomplex $K \subset \overline{D_2(\Gamma)}$ is mapped to within bounded Hausdorff distance of a maximal product subcomplex $K' \subset \overline{D_2(\Gamma')}$. To do this, we first show that under a quasi-isometry $\phi : \overline{D_2(\Gamma)} \to \overline{D_2(\Gamma')}$, any product subcomplex $K \subset \overline{D_2(\Gamma)}$ is mapped to within a D-neighbourhood of a product subcomplex $K' \subset \overline{D_2(\Gamma')}$.

Lemma 6.1. *[3] Suppose C, C' are convex subsets of a CAT(0) space X, and let $\Delta = d(C, C')$. Then:*

1. *The sets $Y := \{x \in C | d(x, C') = \Delta\}$ and $Y' := \{x' \in C' | d(x', C) = \Delta\}$ are convex.*

2. *The nearest point map $r : X \to C$ maps Y' isometrically onto Y.*

3. *Y and Y' cobound a convex subset $Z \simeq Y \times [0, \Delta]$.*

4. *If in addition X is a locally finite CAT(0) complex with cocompact isometry group, and C, C' are subcomplexes, then the sets Y and Y' are nonempty, and there is a constant $A > 0$ which depends only on Δ and X, such that if $p \in C$ and $p' \in C'$ with $d(p, Y) \geq 1$ and $d(p', Y') \geq 1$ then $d(p, C') \geq \Delta + Ad(p, Y)$ and $d(p', C) \geq \Delta + Ad(p', Y')$.*

Following [3] we would like to show that if $\phi : \overline{D_2(\Gamma)} \to \overline{D_2(\Gamma')}$ is a quasi-isometry, then there exists a constant $D > 0$ (depending on ϕ) such that any product subcomplex K in $\overline{D_2(\Gamma)}$ is mapped to within a D-neighbourhood of a product subcomplex K' in $\overline{D_2(\Gamma')}$.

We start with the Tripod Lemma from [3], which allows us to conclude that a pure singular geodesic γ at the intersection of three flats F_1, F_2 and F_3 gets mapped by a quasi-isometry to within Hausdorff distance D of a pure singular geodesic γ', and that the flats F_1, F_2 and F_3 get mapped to within a D-neighbourhood of the parallel set of γ'.

Lemma 6.2. *[3] Let $\phi : \overline{D_2(\Gamma)} \to \overline{D_2(\Gamma')}$ be a quasi-isometry. Let $Y \subset \overline{D_2(\Gamma)}$ be a subcomplex isometric to $T \times \mathbb{R}$, where T is an infinite tripod with vertex v. Let $\gamma = v \times \mathbb{R}$. There exists a constant D and a singular geodesic $\gamma' \subset \overline{D_2(\Gamma')}$ such that $Hd(\phi(\gamma), \gamma') < D$, and such that $\phi(Y) \subset N_D(\mathbb{P}(\gamma'))$.*

Proof. The product subcomplex Y is the union of three flats $F_1 \cup F_2 \cup F_3$, the intersection of all three being the pure singular geodesic γ. Each of these flats is a cycle of quarter-planes. As we proved in Proposition 5.19, the collection of quarter-plane Hausdorff equivalence classes is coherent. In the case of F_1, F_2, F_3 there exists $r_0 > 0$ such that $N_{r_0}(F_1) \cap N_{r_0}(F_2) \cap N_{r_0}(F_3)$ is quasi-isometric to $\mathbb{R}$ and for all $r \geq r_0, [N_r(F_1) \cap N_r(F_2) \cap N_r(F_3)] = [N_{r_0}(F_1) \cap N_{r_0}(F_2) \cap N_{r_0}(F_3)]$. By Theorem 5.36, there exists a constant $D > 0$ and flats F_1', F_2', F_3' in $\overline{D_2(\Gamma')}$ such that $\mathrm{Hd}(\phi(F_i), F_i') < D$ for $i = 1, 2, 3$. Thus by Lemma 5.32 there exists an $r_0' > 0$ and $K > 0$ such that for all $r' \geq r_0', [N_{r'}(F_1') \cap N_{r'}(F_2') \cap N_{r'}(F_3')] = [N_{r_0'}(F_1') \cap N_{r_0'}(F_2') \cap N_{r_0'}(F_3')]$ and $\mathrm{Hd}(\phi(N_{r_0}(F_1) \cap N_{r_0}(F_2) \cap N_{r_0}(F_3)), N_{r_0'}(F_1') \cap N_{r_0'}(F_2') \cap N_{r_0'}(F_3')) < K$. Since $\phi(N_{r_0}(F_1) \cap N_{r_0}(F_2) \cap N_{r_0}(F_3))$ is quasi-isometric to $\mathbb{R}$ this implies that $S' = N_{r_0'}(F_1') \cap N_{r_0'}(F_2') \cap N_{r_0'}(F_3')$ is quasi-isometric to $\mathbb{R}$ and so must have 2 ends. Moreover S' is a convex subset of a CAT(0) space and hence itself CAT(0). Since S' is 2 ended its Tits boundary is disconnected and hence

S' contains a geodesic γ''. Now $d(\gamma'', F_i') \leq r_0'$ for $i = 1, 2, 3$. Thus there exists a pure singular geodesic γ' such that $\gamma' \subset N_r'(F_1') \cap N_r'(F_2') \cap N_r'(F_3')$ for some $r' > r_0'$. Thus there exists a $D'' > 0$ such that $\mathrm{Hd}(\gamma'', \gamma') < D''$ and so we may conclude that there exists a $D' > 0$ with $\mathrm{Hd}(\phi(\gamma), \gamma') < D'$. Also $F_1', F_2', F_3' \subset \mathbb{P}(\gamma')$ so that $\phi(Y) \subset \mathbb{P}(\gamma')$. □

Proposition 6.3. *Let graphs* Γ, Γ' *and a quasi-isometry* $\phi : \overline{D_2(\Gamma)} \to \overline{D_2(\Gamma')}$ *be given. If* $K \subset \overline{D_2(\Gamma)}$ *is a product sub-complex, then there exists a standard product sub-complex* $K' \subset \overline{D_2(\Gamma')}$ *and a constant* $D \geq 0$ *such that* $\phi(K) \subset N_D(K')$ *in* $\overline{D_2(\Gamma')}$.

Proof. Since K is a product complex in $\overline{D_2(\Gamma)}$, it is a sub-complex of the form $T_1 \times T_2$ in $\overline{D_2(\Gamma)}$, where every vertex in T_1 and T_2 has valence ≥ 2.

We will find a standard product subcomplex $K' \subset \overline{D_2(\Gamma')}$, and a constant $D \geq 0$ such that $\phi(K) \subset N_D(K')$ in $\overline{D_2(\Gamma')}$.

We shall break our proof up into cases, depending on the structure of the trees T_1 and T_2.

Case 1: $T_1 = \mathbb{R}$ and $T_2 = \mathbb{R}$, i.e. all vertices in T_1, T_2 have valence exactly 2.

Then K is a flat, and by Theorem 5.36 there exists a constant $D \geq 0$ and a flat $F' \subset \overline{D_2(\Gamma')}$ such that $Hd(\phi(K), F') < D$. By Lemma 4.21 F' is a product sub-complex and by Lemma 5.25 it is contained in a standard product sub-complex K'. And so we have $\phi(K) \subset N_D(F') \subset N_D(K')$.

Case 2: $T_1 \neq \mathbb{R}$ and $T_2 = \mathbb{R}$, i.e. all vertices in T_2 have valence exactly 2 but there exists at least one vertex in T_1 with valence ≥ 3.

Pick $v \in T_1$ such that degree(v) ≥ 3. Consider the geodesic $v \times \mathbb{R}$. By Lemma 6.2 there must be a singular geodesic τ' in $\overline{D_2(\Gamma)}$ of the form $v' \times \gamma'$ or $\gamma' \times v'$ and a constant $D \geq 0$ such that $Hd(\phi(v \times \mathbb{R}), \tau') < D$ and such that for any geodesic $\alpha \subset T_1$, $\phi(\alpha \times \mathbb{R}) \subset N_D(\mathbb{P}(\tau'))$.

Now $\mathbb{P}(\tau') = T' \times \tau'$ or $\tau' \times T'$, and by Lemma 5.25 there exists a standard product sub-complex K' with $\mathbb{P}(\tau') \subset K'$.

For any point $x \in K$, let α_1 be a bi-infinite geodesic containing x and v. Then let α_2, α_3 be bi-infinite geodesics such that $\alpha_1 \cap \alpha_2 \cap \alpha_3 = v$, so that the flats $F_1 = \alpha_1 \times \mathbb{R}, F_2 = \alpha_2 \times \mathbb{R}, F_3 = \alpha_3 \times \mathbb{R}$ all intersect in the singular geodesic $v \times \mathbb{R}$. Since we have that $\phi(F_1), \phi(F_2), \phi(F_3) \subset N_D(K')$, we must have that $\phi(x) \in N_D(K')$. Since x was arbitrary, we conclude that $\phi(K) \subset N_D(K')$.

Case 3: T_1 and T_2 both have vertices of valence ≥ 3.

Let H be the set of all horizontal geodesics in K, i.e. geodesics of the form $\alpha \times w$, where α is a geodesic in T_1 and w is a vertex of valence ≥ 3 in T_2. Similarly let V be the set of vertical geodesics in K, i.e. geodesics of the form $v \times \beta$ with v a vertex of valence ≥ 3 in T_1 and β a geodesic in T_2. By Lemma 6.2 there exists a constant $D > 0$ and sets of singular geodesics H' and V' in $\overline{D_2(\Gamma')}$ such that for $\gamma \in H$ or V there exists a singular geodesic γ' in H' or V' respectively such that $\mathrm{Hd}(\phi(\gamma), \gamma') < D$.

Let $\mathbb{F}$ be the collection of all the flats in K. An arbitrary flat F in $\mathbb{F}$ is of the form $F = \alpha \times \beta$. By Theorem 5.36 there exists a constant $D > 0$ such that for each flat F in $\mathbb{F}$, there exists a flat F' in $\overline{D_2(\Gamma')}$ such that $\mathrm{Hd}(\phi(F), F') < D$. Let $\mathbb{F}'$ be the collection of all those flats in $\overline{D_2(\Gamma')}$ which are within Hausdorff distance D from $\phi(F)$ for some $F \in \mathbb{F}$. We shall show that there is a standard product subcomplex which contains $\mathbb{F}'$.

For an arbitrary pair of horizontal and vertical geodesics $\gamma_1 = \alpha \times w_1 \in H$ and $\tau_1 = v_1 \times \beta \in V$, there exists a flat $F = \alpha \times \beta$ in $\mathbb{F}$. Pick vertices $v_2 \in \alpha$ and $w_2 \in \beta$ with valence ≥ 3 and consider the paths $[v_1, v_2]$ in T_1 and $[w_1, w_2]$ in T_2. Note that the geodesics $\gamma_2 = \alpha \times w_2$

and $\tau_2 = v_2 \times \beta$ belong to H and V respectively. Also note that γ_1, γ_2 bound the strip $\alpha \times [w_1, w_2]$, while τ_1, τ_2 bound the strip $[v_1, v_2] \times \beta$. So γ_1 is asymptotic to a horizontal geodesic γ_2 in the flat F, while τ_1 is asymptotic to a vertical geodesic τ_2 in the flat F. By Lemma 6.2 there exist pure singular geodesics γ_1', γ_2' in H' and τ_1', τ_2' in V' such that $\mathrm{Hd}(\phi(\gamma_2), \gamma_2') < D$, $\mathrm{Hd}(\phi(\tau_2), \tau_2') < D$, $\mathrm{Hd}(\phi(\gamma_1), \gamma_1') < D$ and $\mathrm{Hd}(\phi(\tau_1), \tau_1') < D$, and there exists a flat F' in $\overline{D_2(\Gamma')}$ with $F' \subset \mathbb{P}(\gamma_2')$ and $F' \subset \mathbb{P}(\tau_2')$ such that $\mathrm{Hd}(\phi(F), F') < D$. Moreover as ϕ is a quasi-isometry, the singular geodesics γ_1', γ_2' must be asymptotic, and likewise τ_1', τ_2' must be asymptotic. Since γ_2 and τ_2 are not asymptotic and both lie in the flat F, the corresponding singular geodesics γ_2' and τ_2' cannot be asymptotic. Thus if $\gamma_1' \in H'$, this forces $\gamma_2' \in H'$ and $\tau_1', \tau_2' \in V'$, whereas if $\gamma_1' \in V'$, this forces $\gamma_2' \in V'$ and $\tau_1', \tau_2' \in H'$. Since the geodesics $\gamma_1 \in H$ and $\tau_1 \in V$ were arbitrary, we may conclude that the sets H' and V' are disjoint and that for any pair of singular geodesics $\gamma' \in H'$, $\tau' \in V'$ there is a flat $F' \in \mathbb{F}'$ containing geodesics asymptotic to γ' and τ'.

Let $\Gamma_1' = \bigcup_{F' \in \mathbb{F}'} \pi_1(F')$ and $\Gamma_2' = \bigcup_{F' \in \mathbb{F}'} \pi_2(F')$. We next claim that Γ_1' and Γ_2' are connected.

To prove the claim, consider two arbitrary horizontal geodesics $\gamma_1 = \alpha_1 \times v_1$ and $\gamma_2 = \alpha_2 \times v_2$. If α_1 and α_2 contain subrays that are asymptotic, then since they are in the tree T_1, these subrays are equal and in particular $\pi_1(s(\gamma_1))$ and $\pi_1(s(\gamma_2))$ have non-empty intersection. So, suppose that α_1 and α_2 do not contain asymptotic subrays.

Let $\beta_i \subset \alpha_i$ be subrays (for $i = 1, 2$). Let β be a bi-infinite geodesic containing subrays asymptotic to each of β_1, β_2. Thus $\beta \times v_1$ and $\beta \times v_2$ must be asymptotic horizontal geodesics. Since they are asymptotic, $\beta \times v_1$ and $\beta \times v_2$ must bound an Euclidean strip and thus $\pi_1(s(\beta \times v_1)) =$

$\pi_1(s(\beta \times v_2))$. Now $\pi_1(s(\beta_i \times v_i))$ and $\pi_1(s(\alpha_i \times v_i))$ (for $i = 1, 2$) have non-empty intersection in Γ_1'. And $\pi_1(s(\beta \times v_1))$ contains both $\pi_1(s(\beta_1 \times v_1))$ and $\pi_1(s(\beta_2 \times v_2))$ in Γ_1'. Thus $\pi(s(\beta \times v_1))$ and $\pi(s(\alpha_i \times v_i))$ (for $i = 1, 2$) have non-empty intersection in Γ_1'. Thus given an arbitrary pair of horizontal geodesic rays, $\alpha_1 \times v_1$ and $\alpha_2 \times v_2$, we were able to find a subgraph $\pi_1(s(\beta \times v_1))$ of Γ_1', such that $\pi_1(s(\alpha_1 \times v_1))$ and $\pi_1(s(\alpha_2 \times v_2))$ both have non-empty intersection with $\pi_1(s(\beta \times v_1))$. So we may conclude that Γ_1' is connected.

Thus we may conclude that Γ_1' is connected, and similar reasoning we shows that Γ_2' is also connected. By construction $\Gamma_1' \cap \Gamma_2' = \emptyset$, so $\Gamma_1' \times \Gamma_2'$ is a sub-complex of $D_2(\Gamma')$ and $s^{-1}(\Gamma_1' \times \Gamma_2')$ is a disjoint union of product sub-complexes $\overline{D_2(\Gamma')}$.

All that remains to be shown is that $\mathbb{F}'$ lies within a single component of $s^{-1}(\Gamma_1' \times \Gamma_2') \subset \overline{D_2(\Gamma')}$.

Given any pair of flats F', G' in $\mathbb{F}'$, let $F, G \in \mathbb{F}$ be such that $\phi(F)$ is in a D-neighborhood of F' and $\phi(G)$ is in a D-neighborhood of G'. There exists a flat $E_{F,G} \in \mathbb{F}$ such that $E_{F,G}$ intersects both F and G in a quarter-plane. Let $E_{F,G}' \in \mathbb{F}'$ be such that $\phi(E_{F,G})$ is in a D-neighborhood of $E_{F,G}'$. By Lemma 5.32 we have that $E_{F,G}'$ must intersect each of F' and G' in a set that is quasi-isometric to a quarter-plane. Thus $F' \cup G' \cup E_{F,G}'$ is connected, and so F', G' and $E_{F,G}'$ must all lie within the same component of $s^{-1}(\Gamma_1' \times \Gamma_2')$. Since this holds for arbitrary flats F', G' in $\mathbb{F}'$, we can conclude that $\mathbb{F}'$ is contained within a single component $K' \subset s^{-1}(\Gamma_1' \times \Gamma_2')$.

Now since K is just a union of all the flats in $\mathbb{F}$ we have $\phi(K) \subset N_D(\mathbb{F}') \subset K'$, and K' is a standard product subcomplex.

$\square$

Theorem 6.4. *Let Γ and Γ' be two graphs and let $\phi : \overline{D_2(\Gamma)} \to \overline{D_2(\Gamma')}$ be a quasi-isometry.*

If $K \subset \overline{D_2(\Gamma)}$ is a maximal product subcomplex, then there exists a maximal product subcomplex $K' \subset \overline{D_2(\Gamma')}$ and a constant $D = D(L, A)$ such that $Hd(\phi(K), K') < D$. Moreover, K and K' are quasi-isometric.

Proof. Let $K \subset \overline{D_2(\Gamma)}$ be a maximal product complex. By Proposition 6.3 there must exist a standard product subcomplex $K' \subset \overline{D_2(\Gamma')}$ such that $\phi(K) \subset N_{D'}(K')$, where D' depends only on ϕ.

Without loss of generality, we may assume that K' is maximal.

Let $\psi : \overline{D_2(\Gamma')} \to \overline{D_2(\Gamma)}$ be a quasi-isometric inverse of ϕ. Again Proposition 6.3 tells us that there must exist a standard product complex $K'' \subset \overline{D_2(\Gamma)}$ and a constant D'' such that $\psi(K') \subset N_{D''}(K'')$.

Now due to the fact that $\phi(K) \subset N_{D'}(K')$ in $\overline{D_2(\Gamma')}$, and $\psi(K') \subset N_{D''}(K'')$ in $\overline{D_2(\Gamma)}$ where ϕ and ψ are (L, A)-quasi-isometric inverses of each other, we must have that $K \subset N_{D'''}(K'')$ in $\overline{D_2(\Gamma)}$, for some $D''' \geq 0$.

Consider the function $d_K : \overline{D_2(\Gamma)} \to \mathbb{R}$, defined as $d_K(x) = d(x, K)$. By Corollary 2.5 in [14] the function d_K is convex. Now d_K restricted to K'' is bounded as $K \subset N_{D'''}(K'')$. Thus since $d_K : K'' \to \mathbb{R}$ is a bounded convex function, it must also be constant, and equal to Δ, for some constant Δ. Thus the complex K'' is a fixed distnce Δ from the complex K.

Both K and K'' are product complexes in $\overline{D_2(\Gamma)}$ and so are convex. Thus the nearest point map $r : K \to K'$ maps K isometrically onto its image $r(K) \subset K''$. Since Δ is the distance between K and K'', then $r(K)$ and K bound a subcomplex of $\overline{D_2(\Gamma)}$ which is isometric to $K \times [0, \Delta]$, by Lemma 6.1. However $\overline{D_2(\Gamma)}$ is a 2-complex, and the subcomplex isometric to $K \times [0, \Delta]$ would contain a 3-cell, as K contains 2-cells. This forces $\Delta = 0$ and so we must have that $K \subset K''$. Since K is a maximal product sub-complex, this implies that $K = K''$.

Thus $\psi(K') \subset N_{D''}(K)$ in $\overline{D_2(\Gamma)}$. Putting this together with the fact that $\phi(K) \subset N_{D'}(K')$ in $\overline{D_2(\Gamma)}$ where both $D' = D'(L, A)$ and $D'' = D''(L, A)$, we can conclude that there exists a constant $D = D(L, A)$ such that $Hd(\phi(K), K') < D$.

As the Hausdorff Distance between K and K' is finite we may conclude that they are quasi-isometric. $\square$

Chapter 7

The Intersection Complex

Using the fact that maximal product sub-complexes are preserved by quasi-isometries and that large intersections of maximal product sub-complexes are preserved, we define the intersection complex of $\overline{D_2(\Gamma)}$ to be a complex which keeps track of the maximal product sub-complexes and their pattern of large intersections in the space $\overline{D_2(\Gamma)}$.

The next lemma talks about how maximal product subcomplexes with large intersections must intersect in a standard product subcomplex. In effect, this gives us a way to keep track of the large intersections of maximal product subcomplexes by examining the underlying graph Γ.

Lemma 7.1. *Let K_1 and K_2 be two standard product sub-complexes with $K_1 \cap K_2 \neq \emptyset$. Let H be a product sub-complex with $H \subset K_1 \cap K_2$. Then there exists a standard product sub-complex H' such that $H \subset H' \subset K_1 \cap K_2$.*

Proof. K_1 and K_2 are standard product sub-complexes. So there exist disjoint sub-graphs Γ_1, Γ_2 and Θ_1, Θ_2 in Γ such that K_1 is a component of $s^{-1}(\Gamma_1 \times \Gamma_2)$ and K_2 is a componenet of $s^{-1}(\Theta_1 \times \Theta_2)$. Since $H \subset K_1 \cap K_2 \neq \emptyset$ we must have that $s(H) \subset s(K_1) \cap s(K_2) \neq \emptyset$. Also since H is a product sub-complex, we have that $s(H) = \Lambda_1 \times \Lambda_2$ for two disjoint sub-graphs Λ_1, Λ_2 in Γ. Also $s(H) \subset s(K_1) = \Gamma_1 \times \Gamma_2$ and $s(H) \subset s(K_2) = \Theta_1 \times \Theta_2$. Each component of $s^{-1}(\Lambda_1 \times \Lambda_2)$ is contained in a component of $s^{-1}(\Gamma_1 \times \Gamma_2)$ and $s^{-1}(\Theta_1 \times \Theta_2)$. And there exists a unique component of $s^{-1}(\Lambda_1 \times \Lambda_2)$ which contains H, say H'. By definition H' is a standard product sub-complex, being a lift of $\Lambda_1 \times \Lambda_2 \subset D_2(\Gamma)$. Moreover we must have that $H \subset H' \subset K_1$ and $H \subset H' \subset K_2$, because $H \subset K_1 \cap K_2$. □

Theorem 7.2. *Let $\phi : \overline{D_2(\Gamma)} \to \overline{D_2(\Gamma')}$ be a quasi-isometry. Let $K_1, \ldots, K_m$ be a collection of maximal product sub-complexes in $\overline{D_2(\Gamma)}$, such that $K_1 \cap \cdots \cap K_m$ contains a standard flat F. Then there exists a constant $D > 0$ and maximal product sub-complexes $K_1', \ldots, K_m'$ in $\overline{D_2(\Gamma')}$ such that $Hd(\phi(K_i), K_i') < D$ for $i = 1, \ldots, m$, and there exists a standard product sub-complex $H' \subset \overline{D_2(\Gamma')}$ such that $H' \subset K_1' \cap \cdots \cap K_m'$.*

Proof. Theorem 6.4 tells us that there exists a constant $D > 0$ and maximal product sub-complexes $K_1', \ldots, K_m'$ in $\overline{D_2(\Gamma')}$ such that $\mathrm{Hd}(\phi(K_i), K_i') < D$ for $i = 1, \ldots, m$. Also by Theorem 5.36 there exists a constant $D' > 0$ and a flat F' in $\overline{D_2(\Gamma')}$ such that $\mathrm{Hd}(\phi(F), F') < D'$. Since $F \subset K_i$ we have that $\phi(F) \subset N_C(K_i')$ and this implies that $F' \subset N_{C'}(K_i')$, for some constants $C, C' > 0$. Since F' and K_i' are both convex sets, we may combine this with the fact that we are in a 2-complex and apply Lemma 6.1 to conclude that $F' \subset K_i'$ for each $i = 1, \ldots, m$. Thus we have that $F' \subset K_1' \cap \cdots \cap K_m'$. Applying Lemma 7.1, we may conclude that there exists a standard product sub-complex H' such that $F' \subset H' \subset K_1' \cap \cdots \cap K_m'$. □

Definition 7.3. (Intersection Complex) *Let Γ be a graph. The intersection complex $I(\Gamma)$ is a simplicial complex defined as follows:*

1. *Begin with a 0-simplex for each maximal product complex of $\overline{D_2(\Gamma)}$.*
2. *Let $k+1$ 0-simplices span a k-simplex if the intersection of the corresponding $k+1$ maximal product subcomplexes contains a flat.*

Lemma 7.4. *For any finite graph Γ, the intersection complex $I(\Gamma)$ is finite dimensional.*

Proof. Consider a flat F in $\overline{D_2(\Gamma)}$ and let K be a product subcomplex of $D_2(\Gamma)$ such that $F \subset K$.

Consider $\pi_1(s(K))$ and $\pi_2(s(K))$ which must be disjoint subgraphs of Γ, both of whose vertices all have valence ≥ 2.

The graph Γ contains finitely many disjoint pairs of subgraphs Γ_1, Γ_2, and for any such pair F can be contained in at most one maximal product subcomplex M with $s(M) = \Gamma_1 \times \Gamma_2$. Thus there is a bound b depending only on Γ on the number of maximal product subcomplexes that can contain F. It follows that the dimension of $I(\Gamma)$ is at most $b - 1$. $\square$

In light of Lemma 7.4 we may define the maximal intersection number to be the dimension of the intersection complex $I(\Gamma)$.

Definition 7.5. (Maximal Intersection Number) *Define the maximal intersection number* $m(\Gamma)$ *to be the dimension of the intersection complex* $I(\Gamma)$.

Corollary 7.6. *Let* Γ *and* Γ' *be graphs. A quasi-isometry* $\phi : \overline{D_2(\Gamma)} \to \overline{D_2(\Gamma')}$ *induces an isometry* $\phi_* : I(\Gamma) \to I(\Gamma')$.

Proof. There exists a constant $D > 0$ such that given a maximal product subcomplex M in $\overline{D_2(\Gamma)}$ there exists a unique maximal product subcomplex M' in $\overline{D_2(\Gamma')}$ such that $\mathrm{Hd}(\phi(M), M') < D$.

If σ, σ' are the 0-simplices corresponding to M and M', then we may define $\phi_*(\sigma) = \sigma'$.

Now consider $\sigma_0, \dots, \sigma_k$ and the corresponding maximal product sub-complexes $M_0, \dots, M_k$. If there exists a flat $F \subset M_1 \cap \cdots \cap M_k$ then $\sigma_0, \dots, \sigma_k$ span a k-simplex σ. Theorem 6.4 tells us that there exists a constant $D > 0$ and maximal product sub-complexes $M'_0, \dots, M'_k$ such that $\mathrm{Hd}(\phi(M_i), M'_i) < D$. Let $\sigma'_0, \dots, \sigma'_k$ be the corresponding 0-simplices in $I(\Gamma')$. Then Theorem 7.2 tells us that there exists a standard product sub-complex $K' \subset M'_0 \cap \cdots \cap M'_k$. This standard product sub-complex H' in $\overline{D_2(\Gamma')}$ such that $H' \subset M'_0 \cap \cdots \cap M'_k$. And thus $\sigma'_0, \ldots, \sigma'_k$ span a k-simplex in $I(\Gamma')$. This allows us to extend ϕ_* inductively to each k-skeleton of $I(\Gamma)$ so that ϕ_* is an isometry between $I(\Gamma)$ and $I(\Gamma')$. $\square$

The fact that $m(\Gamma)$ is a quasi-isometry invariant is now an immediate corollary.

Corollary 7.7. *The maximal intersection number $m(\Gamma)$ is a quasi-isometry invariant of $\overline{D_2(\Gamma)}$*

The next lemma encapsulates the fact that maximal product sub-complexes cover complexes in $D_2(\Gamma)$ which are products of disjoint induced sub-graphs of Γ. The reader may refer back to Definition 4.5 for the definition of an induced subgraph.

Lemma 7.8. *Let M be a maximal product subcomplex of $\overline{D_2(\Gamma)}$. Consider the subgraphs $A = \pi_1(s(M))$ and $B = \pi_2(s(M))$. Then A and B are induced subgraphs of Γ all of whose vertices have valence ≥ 2.*

Proof. Both A and B must be subgraphs with all vertices of valence ≥ 2 as M is a product sub-complex.

If A were not an induced subgraph, then the subgraphs A' induced by the vertex set of A would be strictly larger than A, and there would be a component M' of $s^{-1}(A' \times B)$ that is a product subcomplex properly containing M. This would contradict the fact that M is maximal. Likewise B must be induced. $\square$

The next lemma characterizes the types of subcomplexes of $D_2(\Gamma)$ which lift to maximal product subcomplexes in the universal cover $\overline{D_2(\Gamma)}$. Moreover this characterization is based purely on the underlying graph Γ, and this is what allows us to calculate the invariant $m(\Gamma)$ for concrete examples.

Lemma 7.9. *Let $A, B \subset \Gamma$ be a pair of disjoint connected induced subgraphs of Γ, such that all the vertices of A and B have valence ≥ 2. Let $C = V(\Gamma) \setminus \{V(A) \cup V(B)\}$ and let $A' = \Gamma[V(A) \cup C]$ and $B' = \Gamma[V(B) \cup C]$. Then the following are equivalent:*

1. *Each component of $s^{-1}(A \times B)$ in $\overline{D_2(\Gamma)}$ is a maximal product subcomplex.*

2. *Every cycle Σ contained in A' or B' is contained in A or B.*

Proof. First let us assume that $A \times B \subset D_2(\Gamma)$ lifts to a maximal product sub-complex.

Thus A and B are a pair of disjoint induced sub-graphs by Lemma 7.8.

Let Σ be a cycle in A'. If Σ is not contained in A then Σ must contain a vertex of C as A is an induced sub-graph. Let $A'' = A \cup \Sigma$. Then $A \subset A''$, all vertices of A' have valence ≥ 2, and $A \times B \subset A'' \times B$, which contradicts the hypothesis that components of $s^{-1}(A \times B)$ are maximal. Similarly any cycle in B' must be contained in B.

Conversley let the sub-graphs A and B have the property that any cycles $\Sigma_A \subset A', \Sigma_B \subset B'$ must satisfy $\Sigma_A \subset A$ and $\Sigma_B \subset B$.

Let if possible that a component of a lift of $A \times B$ is not maximal. Then there must exist disjoint subgraphs A'', B'' with all vertices of valence ≥ 2 such that $A'' \subset A'$ and $B'' \subset B'$, with at least one of A'', B'' strictly larger than A or B respectively. Without loss of generality suppose A'' is strictly larger than A. Then there must exist a cycle Σ in A'' which is not contained in A. But A'' is disjoint from B, so this contradicts the hypothesis.

We conclude that no component of $s^{-1}(A \times B)$ is contained in any larger product sub-complex. Thus these are maximal product sub-complexes. $\square$

7.1 Applications

We now apply some of the tools that we have developed for the quasi-isometric classification of the 2-dimensional graph braid groups to some concrete families of examples. First we calculate $m(\Gamma)$ for the family of graphs K_n and show that the invariant takes on distinct values for each n, thus showing that the spaces $\overline{D_2(K_n)}$ and hence the graph braid groups $B_2(K_n)$ are all quasi-isometrically distinct.

Lemma 7.10. *Let $\Gamma = K_n$ for $n \geq 7$ and let M be a maximal product sub-complex of $\overline{D_2(\Gamma)}$. Then there exists a partition of the vertices of K_n into disjoint sets S_1, S_2 such that $s(M) = \Gamma[S_1] \times \Gamma[S_2]$, i.e. $s(M)$ is the product of the disjoint complete subgraphs $\Gamma[S_1], \Gamma[S_2]$ in Γ.*

Proof. Let $\Gamma_1 = \pi_1(s(M))$ and $\Gamma_2 = \pi_2(s(M))$. Since M is a maximal product sub-complex, it is a standard product sub-complex and is a lift of $\Gamma_1 \times \Gamma_2$. Let $S_1 = V(\Gamma_1)$ and $S_2(V(\Gamma_2))$. The sets S_1, S_2 must partition the vertices of Γ, for if there exists a vertex v of $\Gamma - (S_1 \cup S_2)$, then there is also a cycle using v and some vertices of S_1, which would contradict the maximality of M by Lemma 7.9.

Additionally Γ_1 and Γ_2 must be induced by S_1 and S_2 respectively, for otherwise M would not be maximal.

Thus we may conclude that there exists a partition of the vertices of Γ into S_1 and S_2 such that M is a lift of $\Gamma[S_1] \times \Gamma[S_2]$. □

Proposition 7.11. *Let $n \geq 6$. The maximal intersection number of the complete graph K_n is $m(K_n) = 2^{n-6} - 1$.*

Proof. Let $\Gamma = K_n$ for $n \geq 7$, and let $C_1, C_2 \subset \Gamma$ be two disjoint 3-cycles. Let F be a lift of $C_1 \times C_2$ to $\overline{D_2(\Gamma)}$.

For each partition $S_1 \cup S_2$ of the remaining $n-6$ vertices, the product of the two subgraphs induced by $C_1 \cup S_1$ and $C_2 \cup S_2$ is a subcomplex of $D_2(K_n)$.

Components of the lift of this subcomplex to $\overline{D_2(K_n)}$ are maximal product subcomplexes, and exactly one of them contains F. For different partitions $S_1 \cup S_2$, these maximal product complexes are distinct. The number of partitions is 2^{n-6} so $I(K_n)$ contains a simplex of dimension $2^{n-6}-1$. No other maximal product subcomplex contains F, so this simplex is maximal in $I(K_n)$.

Flats in $\overline{D_2(K_n)}$ which do not cover subcomplexes of the form $C_1 \times C_2$ where C_1, C_2 are 3-cycles, are contained in fewer maximal product subcomplexes. This is because for such a flat F' we would have $s(F') = K_1 \times K_2$ with $K_1 \cup K_2$ using more than 6 vertices of K_n, thus limiting the number of maximal product sub-complexes which contained F'.

Thus $m(K_n) = 2^{n-6} - 1$. □

Corollary 7.12. *Let $m, n \geq 2$. If $\overline{D_2(K_m)}$ and $\overline{D_2(K_n)}$ are quasi-isometric, then $m = n$.*

Proof. It is easy to check that $\overline{D_2(K_2)}, \overline{D_2(K_3)}, \overline{D_2(K_4)}$ and $\overline{D_2(K_5)}$ are quasi-isometrically distinct from each other as well as from $\overline{D_2(K_n)}$ when $n \geq 6$.

This, together with Proposition 7.11 tells us that the spaces $\overline{D_2(K_n)}$ are quasi-isometrically distinct for all n. □

Now that the invariant $m(\Gamma)$ has been applied to find an infinite family of quasi-isometrically distinct graph braid groups, it is only natural to wonder what are the possible values that the invariant can take. We answer this question by constructing an infinite family of quasi-isometrically distinct graph braid groups $B_2(O_k)$ such that the invariant takes values $m(O_k) = k$.

Definition 7.13. *A k-orchard is the graph obtained in the following manner:*

1. *Consider a simple path of k vertices with $(k-1)$ edges between them. We call this graph the backbone, and the edges in this graph are called*

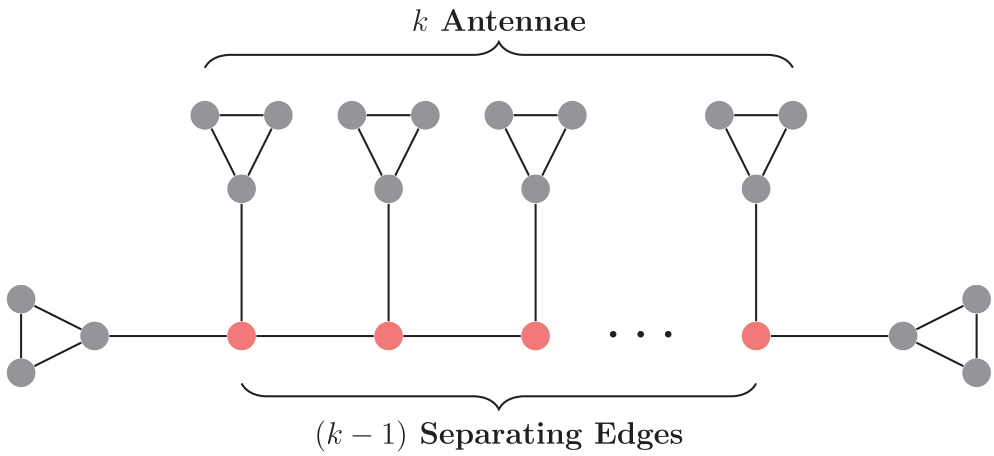

Figure 7.1: A k-orchard.

the separating edges. Note that the backbone has exactly two vertices of valence 1*, called the end vertices.*

2. *Consider k disjoint* 3*-cycles and connect each* 3*-cycle by a single edge (called a connector) to a single vertex from each of the n vertices in the backbone.*

3. *Consider two additional* 3*-cycles and connect each* 3*-cycle by a single edge (also called a connector) to one of the end vertices of the backbone.*

Proposition 7.14. *Consider a k-orchard graph Γ, and let A, B be two disjoint* 3*-cycles in Γ. Let P_{AB} be the shortest edge-path in Γ from A to B. Consider $A \times B \subset D_2(\Gamma)$ and let F be a standard product sub-complex in $\overline{D_2(\Gamma)}$ which is a lift of $A \times B$.*

If s is the total number of separating edges of Γ that lie on P_{AB} then the total number of maximal product subcomplexes in $\overline{D_2(\Gamma)}$ containing F is $s+2$.

Proof. Let e_S be a separating edge in Γ that lies on the path P_{AB}. Note that $\Gamma \setminus e_S$ consists of two disjoint components, Γ_A, Γ_B the components containing

A and B respectively. Also note that the two sets of vertices $V(\Gamma_A)$ and $V(\Gamma_B)$, form a partition of the vertices of Γ, and Γ_A and Γ_B are the subgraphs induced by these vertices. One can check that the subgraphs Γ_A and Γ_B satisfy condition (2) of Lemma 7.9, so each component of $s^{-1}(\Gamma_A \times \Gamma_B)$ is a maximal product subcomplex. Let $M_F(e_S)$ be the component of $s^{-1}(\Gamma_A \times \Gamma_B)$ which contains F. There is one such maximal product subcomplex corresponding to each separating edge e_S, and these are all distinct.

Let e_A and e_B be the two connector edges which connect the cycles A and B to the backbone. Note that $\Gamma \setminus e_A$ has two components A and Γ_B, and $\Gamma \setminus e_B$ has two components Γ_A and B. By a similar line of reasoning as above, the subcomplexes $A \times \Gamma_B$ and $\Gamma_A \times B$ of $D_2(\Gamma)$ lift to maximal product subcomplexes in $\overline{D_2(\Gamma)}$. Let $M_F(e_A), M_F(e_B)$ be the components of these lifts which contain the product subcomplex F. Note that these are distinct and distinct from the $M_F(e_S)$ defined above.

One can check that the only way to partition the vertices of Γ into two sets is by deleting either a separating edge or a connector edge, and considering the vertices of each of the two resulting disjoint components.

If one delets f_S, a separating edge which does not lie in P_{AB} then one obtains disjoint components Γ_1, Γ_2 whose product lifts to a maximal product subcomplex, but which does not contain K, because $\Gamma_1 \times \Gamma_2$ does not contain $A \times B$.

Similarly if one deletes e_C a connector edge which does not touch A or B, then the disjoint components Γ_1, Γ_2, give rise to $\Gamma_1 \times \Gamma_2$ which does not contain $A \times B$. And while $\Gamma_1 \times \Gamma_2$ does lift to a maximal product subcomplex, none of these lifts contain K.

Thus we may conclude that the total number of maximal product subcomplexes that contain the standard product subcmplex K is $s + 2$. □

Corollary 7.15. *Let O_k be the k-orchard. Then $m(O_k) = k$.*

Proof. The longest path in O_k through the separating edges that connects two disjoint cycles A and B has length $k-1$. Thus by Proposition 7.14, the number of maximal product sub-complexes containing a lift of $A \times B$ is $k-1+2=k+1$. Thus the highest dimension of a simplex in $I(O_k)$ is k. $\square$

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