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Chen Chen April 17, 2012

A Study of Option Pricing Models
–
Lognormal or Hyperbolic Levy ?

by

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An abstract of
a thesis submitted to the Faculty of Emory College of Arts and Sciences
of Emory University in partial fulfillment
of the requirements of the degree of
Bachelor of Arts with Honors

Department of Mathematics and Computer Science

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Abstract

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This paper is an investigation into two option pricing models: widely-used Black-Scholes model and one of its augmented extensions – hyperbolic Levy model. Firstly, we have a detailed discussion about the celebrated Black-Scholes model. However, clearly there are many deficiencies in Black-Scholes assumptions. In order to refine Black-Scholes model, Eberlein and Keller(1995) introduce the hyperbolic Levy motion and claim that the new pricing model can provide a better valuation of derivative securities. Following suggestions in that paper, we want to replicate their claims. We perform several statistical tests and show that the hyperbolic distributions can be well fitted to the financial data. This observation suggests us to replace the geometric Brownian motion by the hyperbolic Levy process and build the hyperbolic Levy pricing model. After an introduction into the Levy process theory, we attempt to numerically calculate the value of options according to the hyperbolic Levy model. But it turns out that the price implied by the hyperbolic model cannot be approximated as usual by regular method. This upsetting outcome leads us to look for explanations for the failure of computation, and according to Eberlein, Keller, and Prause (1998) Fast Fourier Transform should be able to efficiently compute the integral. Due to the limited time constraint, we leave it to interested readers. In conclusion, the hyperbolic Levy motion is a better process to fit into empirical data, but it demands an advanced numerical method to compute. Though the observed data is a poorly fit for Black-Scholes, its tractability (elegant solution forms, numerical calculation and other implications) trumps the realism of the hyperbolic Levy model.

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Acknowledgements

I would like to thank Prof. Skip Garibaldi, who served as my advisor for this project and mentored me throughout the year. Without his encouragement, support and guidance, it would be unlikely that I was able to finish the project. I would be also grateful to the other members of my committee, Prof. Emily Hamilton and Prof. Tao Zha, for their comments on the work.

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1. Introduction

Many of the innovations in modern finance have become increasingly dependent on the complex techniques of mathematics, to some extent that problems in financial theories are now driving research in mathematics. Modern option pricing theories are often considered as the subject of "rocket science" and the most mathematically complicated field among all applied areas of finance. In the financial industry most of option valuation models are rooted in a celebrated option pricing formula developed by Fischer Black and Myron Scholes. However, the Black-Scholes formula is far from being perfect. This paper will firstly examine the evolution of Black-Scholes pricing formula, and then test the distributional assumptions upon which the Black-Scholes model is based. After analyzing historical stock prices, it is not hard to find out that the normal distribution assumed by Black and Scholes is a poor fit for the underlying stock returns. Consequently we need to explore alternative distributional assumptions for the returns on stocks. After performing a few statistical tests, it becomes clear that the class of hyperbolic distributions can be well fitted to the empirical data. Based on this observation, we study the Esscher transform of the process with hyperbolic distribution and derive the option pricing formula based on this hyperbolic Levy motion. According to the derived formula, we numerically calculate the value of some stock options. Surprisingly we see that the price of a stock option based on hyperbolic Levy motion cannot be calculated as usual by quadratic methods. This upsetting result leads us to look for the reasons why the quadratic computation method fails, and we suggest that Fast Fourier Transform should be a powerful tool to evaluate the numerical calculation. In summary, while the normal return distribution assumption cannot be accurately justified, the Black-Scholes formula provides an elegant way to compute the option price; the hyperbolic Levy theory is a more precise and accurate model but it demands more sophisticated techniques to handle.

Fundamentally, this paper deals with option pricing. In order to develop an advanced mathematical model to fairly price an option, we need to become familiar with some necessary financial jargons and see how the stock underlying financial derivatives moves.

1.1 Preliminaries

In finance, a derivative is a security whose value is dependent upon or derived from one or more underlying assets. The derivative itself is merely a contract between two parties that specifies conditions (especially the expiration the dates, the values of the underlying assets, and notional amount) under which payoffs are to be made between the parties. Among many financial derivative products, we will mainly focus on a specific kind of derivative – stock option. There are two simple types of stock option: call option and put option. A call option gives the holder the right to buy the underlying stock by a certain date for a pre-specified price. A put option gives the holder the right to sell the underlying stock by a certain date for a pre-specified price. The pre-specified price is known as strike price; the certain date in the contract is known as the expiration date. American options can be exercised at any time up to the expiration date whereas European options can be exercised only on the expiration date. For example, today's date is 16th March 2012 and on 20th April 2012 the holder of the European call option may purchase one XYZ share at strike price K specified in the contract.

In order to gain an intuitive idea for the price of this option, one can imagine on the expiration date 20th April 2012, XYZ share price S might be either above or below the strike price K. Let's say K = 100. If the XYZ share price S is 120 on 20th April 2012, then the holder of the option would be able to purchase the asset for only 100 and get an immediate profit of 20. In this situation, the call would have a value of 20 on the expiration date. On the other hand, if the XYZ share price S is only 90 on 20th April 2012, then no one would want to exercise this call option and the call would be worthless at that time. We can see from the simple example that the value of a call on the expiration date is max{S-K,0}, where K is the strike price and S is the underlying stock price. For notation ease, we will write max{S-K, 0} as $(S-K)^+$. This value is often known as the intrinsic value of a call option. Intrinsic value can be seen as the current exercise value. Apart from its intrinsic value, a call has time value. Time value is the value the option has in addition to its intrinsic value. In other words, it is the premium an investor would pay over its intrinsic value, based on its potential to increase in value before expiring.

Now we have known that a call option's value can be decomposed into two parts: intrinsic value and time value. The next step is to consider what factors would affect a call's price and the boundary for the call's price. Since in this paper we mainly deal with non-dividend paying European calls, we assume there will be no dividend payment for the underlying stock throughout the lifetime of a call option. We claim the following factors will affect the price of a call option: the current stock price S_0, the strike price K, the time to expiration T, the volatility of the stock price σ, and the risk-free interest rate r. We will examine these factor one after another. If a call option is exercised at expiration, the payoff will be the amount by which the stock price is more than the strike price.

Though we cannot know the future share price in advance, it seems reasonable that the higher price today, the more likely the higher price in the future. Thus a call is more valuable if the current stock price increases and the strike price decreases. Now consider the effect of the expiration date and the volatility of stock price. The longer life a call, the more uncertain we are about the underlying stock prices. The same analysis holds for the volatility of the stock price: higher volatility, higher uncertainty. It means the chance that the stock will do very well or very badly increases. The owner of a call benefits from stock prices increase but has limited downside risk in the event of price decreases as the worst case for a call owner is to lose the price of the option. Hence calls price increases as expiration time and volatility increase. The risk-free interest rate affects the price of an option in a less clear-cut way. As interest rate goes up, the expected return required by investors from the stock will increase. In addition, the present value of any future cash flow received by the holder of the option decreases. The combined impact of these two effects is to increase the value of a call. The following table summarizes the above analysis.

Variable	Current Stock Price	Strike Price	Time to expiration	Volatility	Risk-free rate
European Call	+	-	+	+	+

(+ indicates that an increase in the variable causes the option price to increase; - indicates that an increase in the variable causes the option price to decrease)

Knowing that a call's option price is affected by strike price, current stock price, expiration date, volatility, and risk-free rate, i.e. $C = f(S_0, K, T, \sigma, r)$, we now move to derive upper and lower bounds for a call option. This derivation will present us a simple but important concept arbitrage. We claim the stock price is an upper bound to the option price: $C \le S_0$. If this relation were not to hold, a person could easily make a riskless profit by buying the stock and selling the call option. Regardless whether the buyer of the call exercise the option or not, this person will at least have a sure cash inflow of $C - S_0$. A trading strategy like this involves no negative cash flow at any probabilistic or temporal state and a positive cash inflow at least one state. In a complete market, the market prices cannot allow for profitable arbitrage. If an arbitrage exhibits, everyone will want to take advantage of this opportunity and drive the price back to the arbitrage-free state. By using similar analysis, we can establish a lower bound for the price of a European call option on a non-dividend-paying stock is $S_0 - Ke^{-rT}$. In this way, we have established a range for a call's option price, i.e. $S_0 - Ke^{-rT} \le C \le S_0$.

To conclude the preliminary section, we will present a simple one-period binomial asset-pricing model to price a call option. More importantly it provides us a powerful tool to understand arbitrage pricing theory and risk-neutral valuation that underlie in the later part of this paper. The simple binomial situation is described as the following graph:

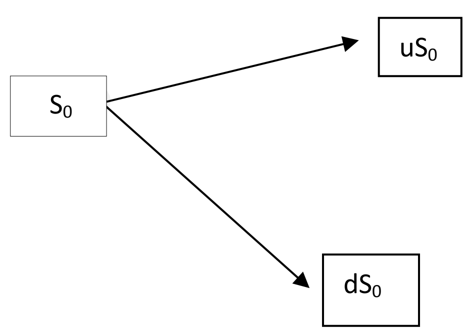


Figure 1: One-period Binomial Model

Suppose that current stock price is S_0 and at expiry the stock can be either moved up to uS_0 or moved down to dS_0. If the stock the price moves up to uS_0, the payoff from the call is f_u; if the stock the price moves down to dS_0, the payoff from the call is f_d. To rule out arbitrage, we must assume $d<1+r<u$. In turns out that a relatively simple argument can be used to price the option in this one-period binomial model. We set up a portfolio consisting of a long position in Δ shares of stock and a short in one option. If the stock goes up, the value of the portfolio at expiry is $uS_0\Delta$ - f_u. If the stock goes down, the value of the portfolio becomes $dS_0\Delta$ - f_d. The two are equal when $uS_0\Delta - f_u = dS_0\Delta - f_d$. In other words, $\Delta = \frac{fu - fd}{uS0 - dS0}$ (1.1.1). In this case, the portfolio is riskless and, to rule out arbitrage, it must earn the risk-free interest rate. That is to say, the present value of the portfolio must be equal to the cost of setting up the portfolio $S_0\Delta$ - C.

It follows that $S_0\Delta - C = (uS_0\Delta - f_u)e^{-rT}$. Substituting from (1.1.1),

we obtain $C = S_0\left(\frac{fu - fd}{uS0 - dS0}\right)(1 - ue^{-rT}) + f_u e^{-rT}$

or C = $e^{-rT}[pf_u + (1-p)f_d]$ (1.1.2) where $p = \frac{e^{rT} - d}{u - d}$ (1.1.3)

The option pricing formula in (1.1.2) does not involve the probabilities of the stock price moving up or down. For instance, the option will be worth the same when the probability of an upward movement is 0.5 as the value when the stock moving up probability is 0.7. The reason is that we are valuing an option in the terms of the price of the underlying stock. The probabilities of up or down in the future are already taken account into the stock price: there is no need to consider them again.

Now we will introduce a very crucial principle in pricing derivatives known as risk-neutral valuation. Returning to equation (1.1.2), the parameter p can be interpreted as the probability of an up movement and 1-p is the probability of down movement. Under this interpretation, C = $e^{-rT}[pf_u + (1-p)f_d]$ shows that the call price is the expected present value of its payoff under up and down scenario probabilities p and 1-p. It means we can price the option as if investors are indifferent to its risk. That is to say investors do not require compensation for assuming risk nor are they willing to pay extra for risk. Under the risk-neutral world the expected return on any investment is the risk-free rate. Again, note p and 1-p are not actual probabilities of up and down movement. We only interpret them as if p and 1-p are up and down movement probabilities. This important principle will be discussed in other models again in the later part of this paper.

1.2 Classical Modeling of Stock Movements

As mentioned above, we can simply model stock prices as if the price movement at each step is governed by a one-step binomial tree, and the one-step model can be generalized to a multistep binomial tree. In this way, we can treat each binomial step separately and work back from the end of the life of the option to the beginning to derive today's option price. However, the binomial model is not very accurate to capture the movements of the stock price in real life. It is entirely possible at each step there are more than two possible outcomes for the stock price. It is necessary for us to look for a better way to mathematically describe the evolution the stock price.

Stochastic process is introduced here to develop an accurate way to model the stock movements. Any variable whose value changes over time in an uncertain way is said to follow a stochastic process. Similar to random variables, stochastic processes can also be classified as continuous variable or discrete variable. In a continuous-variable process, the variable can take any value within a certain range, whereas in a discrete-variable process, the values of the variable are restricted to some possible values. In the rest of this section, we will present a step-by-step approach to understand the classical continuous-time stochastic process that describes the evolution of stock price.

1.2.1 The Markov Property and Martingale

A Markov process is a process where only the current value of a variable is relevant for predicting the future. The mathematical definition is as follows:

Def The process X is a ***Markov chain*** if it satisfies the Markov condition:

$$\mathbb{p}(X_n = s \mid X_0 = x_0, X_1 = x_1, \dots, X_{n-1} = x_{n-1}) = \mathbb{p}(X_n = s \mid X_{n-1} = x_{n-1})$$

$$for\ all\ n \geq 1\ and\ all\ s, x_1, \dots, x_{n-1} \in S \qquad (1.2.1)$$

If a variable is following a Markov process, then the past history of the variable and the way how if emerged from the past to the present are irrelevant. Stock prices are usually assumed to follow a Markov process. For instance, supposing that GOOGLE stock price is $300 now and the stock price follows a Markov process, our predictions for the future should not be affected by the path followed by the price in the past. This Markov property of stock prices is consistent with the weak form of market efficiency hypothesis. It states that all the information about historical prices of is contained in present price of the stock. If the weak for of market efficiency were not true, technical analysis (i.e. interpreting charts of the past history of stock prices) could be used to predict and beat a market. In fact there is very little evidence that they are able to make above-average returns.

Another important concept that we will use is martingale in the later part of the paper. ***Martingale*** is basically defined as follows:

Def A sequence $\{S_n : n \geq 1\}$ is a ***martingale*** with respect to the sequence $\{X_n : n \geq 1\}$ if, for all $n \geq 1$:

(a). $E|S_n| < \infty$

(b) $E(S_{n+1} \mid X_1, \ldots, X_n) = S_n$ (1.2.2)

Suppose S_n represents one's capital at time n. Martingale means conditional on past information upon time n, one will expect no change in his present capital. In other words, a martingale has no tendency to rise or fall as its value at the current time is the expected value in next period. In the case of stock prices, in the real markets stock prices should have a tendency to go up and have a higher rate of return than the risk-free rate in order to compensate investor's risk (i.e. $E(S_{n+1}) > S_n$). Thus stock prices in the real world are ***semi martingales***. However, in the later part of the paper, we will see under a new world – risk neutral world, how we will make the discounted stock price a martingale and use it to price options.

1.2.2 Wiener Processes

A particular type of Markov stochastic process with a mean change of zero and a variance rate of 1.0 per year is called ***Wiener process***. The Wiener process is characterized by the following two properties:

Supposing a variable z follows a Wiener process, then

PROPERTY 1 The change Δz during a small period of time Δt is

$$\Delta z = \epsilon\sqrt{\Delta t} \quad (1.2.2)$$

where ϵ has standard normal distribution $\mathcal{Z}$ (0,1).

PROPERTY 2 The values of Δz for any tow different short intervals of time, Δt, are independent.

From these two properties, it is easy to see that Δz itself has a normal distribution $\mathcal{N}$ (0 ,Δt) and that z follows a Markov process.

Now consider the change in the value of z during a relatively long period of time T, i.e. $z(T) - z(0)$. It can be seen as the sum of Δz in N small time intervals, where $N = \frac{T}{\Delta t}$. Thus $z(T) - z(0) = \sum_{i=1}^{N} \epsilon_i \sqrt{\Delta t}$ (1.2.3), where the ϵ_i (i=1, 2, 3…, N) are distributed $\mathcal{Z}$ (0,1).

By the additive property of normal distribution, it immediately follows that $z(T) - z(0) \sim \mathcal{N}(0, T)$.

The mean change per unit time for a stochastic process is known as the drift rate and the variance per unit time known as the variance rate. The basic Wiener process, , that has been defined as in (1.2.2) with a drift rate of zero and a variance rate of 1.0. Now we will define a generalized Wiener process for a variable in terms of dz as $dx = adt + bdz$ (1.2.4), where a and b are constants. Implicit in (1.2.4) is that x has an expected drift rate of a per unit of time and a variance rate of b^2.

To illustrate these two processes, we make a simulation with a = 0.3 and b =1.5.

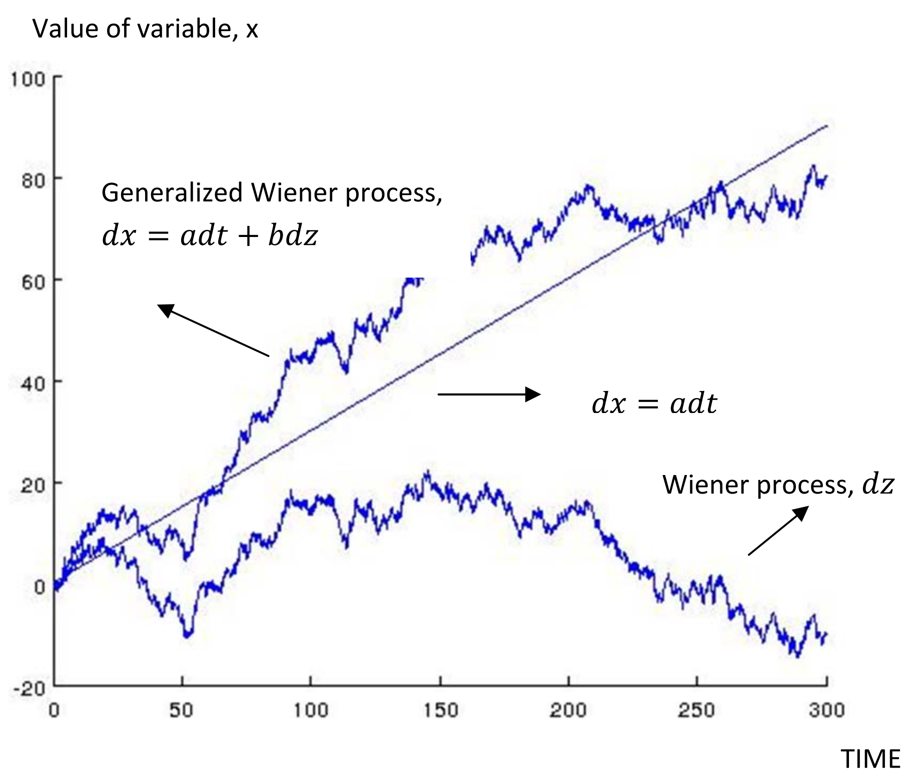


Figure 2: Simulation of Wiener Process

1.2.3 The Classical Model for the Stock Price

In this section we will discuss the stochastic process usually assumed for the price of a non-dividend-paying stock. Seemingly a stock price follows a generalized Wiener process. However, this is a misrepresentation of the evolution of stock prices. A key aspect of stock prices is that the expected percentage return required by an investor from a stock is independent of the stock's price. For instance, if an investor requires a 10% annual expected return when stock A's price is $20, then fixing other conditions constant, he will also require a 10% annual expected return for stock A even when stock A's price is $50. Thus a reasonable assumption is that the expected drift divided by the stock price, μ, is constant. Mathematically, if the coefficient of dz is zero, then this assumption implies that $\frac{dS}{S} = \mu dt$(1.2.5). More specifically, if S_0 and S_T are the stock price at time 0 and T, then by integrating both sides of (1.2.5) we get $S_T = S_0 e^{\mu T}$ (1.2.6). It means that when there is no uncertainty, the stock price grows at a continuously compounded rate of μ per unit of time. Of course, there is uncertainty in reality. Similarly it is reasonable to assume the variability of the percentage return during a short length of time, Δt, is the same regardless of the stock price. Adding this assumption to our previous model, it leads to the model

$$\frac{dS}{S} = \mu dt + \sigma dz \ (1.2.7)$$

Equation (1.2.7) is the most widely used and classical model of stock price behavior. The variable μ is the stock's expected rate of return and the variable σ is the volatility of the stock price. (1.2.7) represents the stock price process in the real world and in a risk-neutral world, μ equals the risk-free rate r. Observing that stock prices are restricted to discrete values (e.g. it can only take multiples of a cent), we need to adapt the continuous model (1.2.7) to a discrete case. So the discrete time version of the model is

$$\frac{\Delta S}{S} = \mu\Delta t + \sigma\epsilon\sqrt{\Delta t} \qquad (1.2.7)$$

and $\frac{\Delta S}{S} \sim \mathcal{N}(\mu\Delta t, \sigma^2\Delta t)$ (1.2.8)

1.2.4 Ito's Lemma

As mentioned in the preliminaries, the price of a stock option is a function of the underlying stock's price and time. In other words, the price of an option is a function of the stochastic variables: underlying stock and time. There is an important result about the behavior of functions of stochastic variables. It is known as Ito's lemma and paves the way for our derivation of Black-Scholes formula in the next section.

Itô's lemma:

Suppose that the value of a variable x follows the *Itô's* process

$$dx = a(x,t)dt + b(x,t)dz \qquad (1.2.9)$$

where dz is a Wiener process and a and b are functions of x and t. The variable x has a drift rate of a and a variance rate of b^2 . *Itô's lemma* shows that a function G of x and t follows the process $dG = \left(\frac{\partial G}{\partial x} a + \frac{\partial G}{\partial t} + \frac{1}{2} \frac{\partial^2 G}{\partial x^2} b^2 \right) dt + \frac{\partial G}{\partial x} bdz$ (1.2.10) where dz is the same Wiener process as in (1.2.9). In other words, G also follows an $It\hat{o}'s$ process, with a drift rate of

$$\frac{\partial G}{\partial x} a + \frac{\partial G}{\partial t} + \frac{1}{2} \frac{\partial^2 G}{\partial x^2} b^2$$

and a variance rate of

$$\left(\frac{\partial G}{\partial x} \right)^2 b^2$$

We leave out the proof of this lemma as it is not our main focus in this paper.

As in (1.2.7), we already have $= \mu Sdt + \sigma Sdz$. Applying the lemma to it, we get that the process followed by a function G of S and t is

$$dG = \left(\frac{\partial G}{\partial S} \mu S + \frac{\partial G}{\partial t} + \frac{1}{2} \frac{\partial^2 G}{\partial S^2} \sigma^2 S^2 \right) dt + \frac{\partial G}{\partial S} \sigma Sdz \qquad (1.2.11)$$

Note that both S and G are affected by the same underlying source of uncertaintydz. This will play an important role in the derivation of the Black-Scholes formula.

2. The Black-Scholes Model

The model of stock price behavior used by Black and Scholes is the model we presented in the introductory section. In this section we will cover how we can get the celebrated Black-Scholes formula.

2.1 Lognormal Property of Stock Prices

To develop the Balck-Scholes model, firstly we need to know the lognormal property of stock prices. If we define the function G described in section 1.24 as $G = \ln S$, it is followed from (1.2.11) that $dG = \left(\mu - \frac{\sigma^2}{2}\right)dt + \sigma dz$ (2.1.1)

Since μ and σ are constant, it indicates that $G = \ln S$ follows a generalized Wiener process. The constant drift rate is $\mu - \frac{\sigma^2}{2}$ and constant variance rate is σ^2. Therefore

$$\ln \frac{S_T}{S_0} \sim N[(\mu - \frac{\sigma^2}{2})T, \sigma^2 T] \qquad (2.1.2)$$

In addition, it can be shown shat the expected value E $(S_T) = S_0 e^{\mu T}$ (2.1.2*)

The lognormal property of stock prices provides useful information on the probability distribution of the continuously compounded rate of return earned on a stock between times 0 and T. If we define the continuously compounded rate of return per year between times 0 and T as x, then $S_T = S_0 e^{xT}$. So $x = \frac{1}{T} \ln \frac{S_T}{S_0}$. It means

$$x \sim N[(\mu - \frac{\sigma^2}{2})T, \sigma^2 / T] \qquad (2.1.3)$$

2.2 Black-Scholes Model's Ideas And Assumptions

Before deriving the Black-Scholes formula, it might be better for us to consider the assumptions and ideas lying behind the model. Moreover, in the later part of this paper, we will see some assumptions cannot be justified by empirical data and we need to explore alternative models.

The nature of the arguments in Black-Scholes model is basically the same as the no-arbitrage arguments we used to value stock options in the preliminary section where stock price movements are binomial. We set up a riskless portfolio comprised of the options and the underlying stock and argue that if the portfolio is riskless, then by no-arbitrage the rate of return on the portfolio over a short period of time must be the same as the risk-free interest rate. Note that here we say the portfolio is riskless for only a short period of time. In other words, it remains riskless only for an instantaneously short period of time. To remain riskless, the portfolio must be rebalanced constantly.

The assumptions made in the Black Scholes model are as follows:

1. The stock price follows the process developed (1.2.7)

2. The short selling of securities with full use of proceeds ipermitted.

3. Transactions costs or taxes do not exist. All securities are perfectly divisible.

4. There are no dividend payments for the underlying stock during the life of the option.

5. No arbitrage opportunities exist in the market.

6. The risk-free rate of interest, r, is constant.

As we will see in the later part of this paper, we will re-examine some assumptions and change some assumptions to reach different models.

2.3 Black-Scholes formula's derivation

With the assumptions stated above, now we will see how we can reach the celebrated Black-Scholes formula. If the stock price follows

$$\frac{dS}{S} = \mu dt + \sigma dz \ (1.2.7)$$

Suppose that f is the price of a call option whose underlying stock is S. so f must be a some function of S and t. By the Ito's lemma presented in 1.2.4, we get

$$df = \left(\frac{\partial f}{\partial S} \mu S + \frac{\partial f}{\partial t} + \frac{1}{2} \frac{\partial^2 f}{\partial S^2} \sigma^2 S^2 \right) dt + \frac{\partial f}{\partial S} \sigma S dz \quad (2.3.1)$$

It turns out that we can construct a portfolio consist of the stock and the option so that the Wiener process dz is eliminated. The portfolio is as follows: short one option and long an amount of $\frac{\partial f}{\partial S}$ shares stock. Define $\prod$ as the value of the portfolio:

$\prod = -f + \frac{\partial f}{\partial S} S$ (2.3.2).

Thus the change in the value of the portfolio during the time interval Δt is given by $\Delta \prod = -\Delta f + \frac{\partial f}{\partial S} \Delta S \quad (2.3.3)$

From the discrete version of equation (1.2.7), we can rewrite (2.3.3) as

$$\Delta \prod = (-\frac{\partial f}{\partial t} - \frac{1}{2}\frac{\partial^2 f}{\partial S^2}\sigma^2 S^2)\Delta t \quad (2.3.4)$$

Since equation (2.3.4) does not involve Δz, the noise or uncertainty variable, the portfolio must be riskless during time Δt. By the no-arbitrage assumption listed in the preceding section, the portfolio must instantaneously earn the corresponding short-term risk-free interest rate. It follows that $\Delta \prod = r \prod \Delta t$ (2.3.5), where r is the risk-free interest rate.

Plugging (2.3.4) into (2.3.5), we get

$$(\frac{\partial f}{\partial t} + \frac{1}{2}\frac{\partial^2 f}{\partial S^2}\sigma^2 S^2)\Delta t = r(f - \frac{\partial f}{\partial S}S)\Delta t$$

so that

$$\frac{\partial f}{\partial t} + rS\frac{\partial f}{\partial S} + \frac{1}{2}\frac{\partial^2 f}{\partial S^2}\sigma^2 S^2 = rf \quad (2.3.6)$$

We call equation (2.3.6) the Black-Scholes differential equation. It has many solutions, corresponding to all the different options that can be defined with S as the underlying asset. A particular solution can be obtained when the values of the option at the boundaries of possible values of S and t are specified. In the case of a European call option, the key boundary condition is $f = (S-K)^+$, when t = T.

Another approach to solve (2.3.6) is from the important risk-neutral valuation principle. Looking at equation (2.3.6) again, it does not involve the expected return μ and all the variables involved in (2.3.6) are independent of risk preferences. Since risk preferences do not enter the equation, they cannot affect its solution. Any risk preference can be used to solve the Black-Schole differential equation. So why not assume investors are risk-neutral.

Recalling from our discussion about binomial model in the preliminary section, we conclude that to rule out arbitrage, the call price is the expected present value of its payoff under up and down scenario probabilities p and 1-p, i.e. C = $e^{-rT}[pf_u + (1-p)f_d]$. Moreover it can be shown that if we are under risk-neutral world, i.e. measuring all probabilities in terms of risk-neutral measure, then we can obtain the financial derivative's price by taking the expectation under this new probability distribution and discounting the expected value to the present. This fact is the immediate result from the following theorem and its corollary.

Theorem (***First Fundamental Theorem of Asset Pricing***) If a market model has a risk-neutral probability measure (i.e. a) P and $\tilde{\mathrm{P}}$ are equivalent, and b) under $\tilde{\mathrm{P}}$ the discounted stock price is a martingale), then it does not admit arbitrage.

Corollary: the price at time t of any security that pays V(T) at time T is

$V(t) = \frac{1}{D(t)}\tilde{E}[D(T)V(T)]$ (2.3.7) , where the ***discount process*** is defined as

$D(t) = e^{-\int_0^t R(u)du}$ (2.3.8) and the interest rate process $R(t)$ is adapted.

In the case of an European call option, that is,

$$c = e^{-rT}\tilde{E}[(S-K)^+] \quad (2.3.9)$$

where $\tilde{E}$ denotes the expected value in a risk-neutral world.

To obtain the explicit form for the Black-Scholes formula implied by risk-neutral method in (2.3.9), we need to use the following useful result.

Lemma: If V is log-normally distributed and the standard deviation of $\ln V$ is w, then

$$E[(V-K)^{+}] = E(V)N(d_1) - KN(d_2) \quad (2.3.8)$$

where $d_1 = \frac{\ln\left[\frac{E(V)}{K}\right]+w^2/2}{w}$, $d_2 = \frac{\ln\left[\frac{E(V)}{K}\right]-w^2/2}{w}$ and E denotes the expected value.

This lemma can be shown by directly calculating the expectation. In the calculation we need t0 transform the normal distribution of ln V into the standardized normal distribution and then perform the integration.

Under the stochastic process assumed by Black Scholes, S is log-normal and from (2.1.2*) under risk-neutral measure we have $\tilde{E}(S_T) = S_0 e^{rT}$ and it can also be shown that the standard deviation of $\ln S_T$ is $\sigma\sqrt{T}$ .From the above lemma, equation (2.3.7) implies that

$$c = S_0 N(d_1) - Ke^{-rT} N(d_2) \qquad (2.3.9)$$

where $d_1 = \frac{\ln\left[{S_0}/{K}\right] + (r+\sigma^2/2)T}{\sigma\sqrt{T}}$, and $d_2 = \frac{\ln\left[{S_0}/{K}\right] + (r-\sigma^2/2)T}{\sigma\sqrt{T}}$

This is the celebrated Black-Scholes formula.

3. Testing Black-Scholes Assumptions

There is little doubt that Black-Scholes model makes the most important breakthrough in the past few decades in finance and is applied on a large scale in everyday trading operation. The stock price in the Black-Scholes model is assumed to move in the way described by (1.2.7). This implies that $\ln\frac{S_T}{S_0} \sim N[(\mu-\frac{\sigma^2}{2})T, \sigma^2 T]$ (2.1.2) as described in section 2. In other words, the continuously compounded rate of return on a stock is assumed to follow a normal distribution. However, it turns out this assumption is poorly fitted into empirical data.

Here we examine one year's stocks price of GOOGLE(GOOG), IBM and Bank of America Corporation(BAC). As the model is to consider non-dividend stocks, if there is a dividend payment, then we need to adjust the price by adding the dividend back. Thus allowing for the dividend payment, we record observed daily adjusted close prices, denote as S_i, and calculate $u_i = \ln\left(\frac{s_i}{s_{i-1}}\right) for\ i = 1,2,\dots,n.$

a. Quantile-Quantile plots

A qualitative yet powerful tool to test the goodness of fit is quantile-quantile plots. a Q-Q plot is a probability plot, which is a graphical method for comparing two probability distributions by plotting their quantiles against each other. The following figures show normal QQ plots for the returns of GOOG, IBM and BAC.

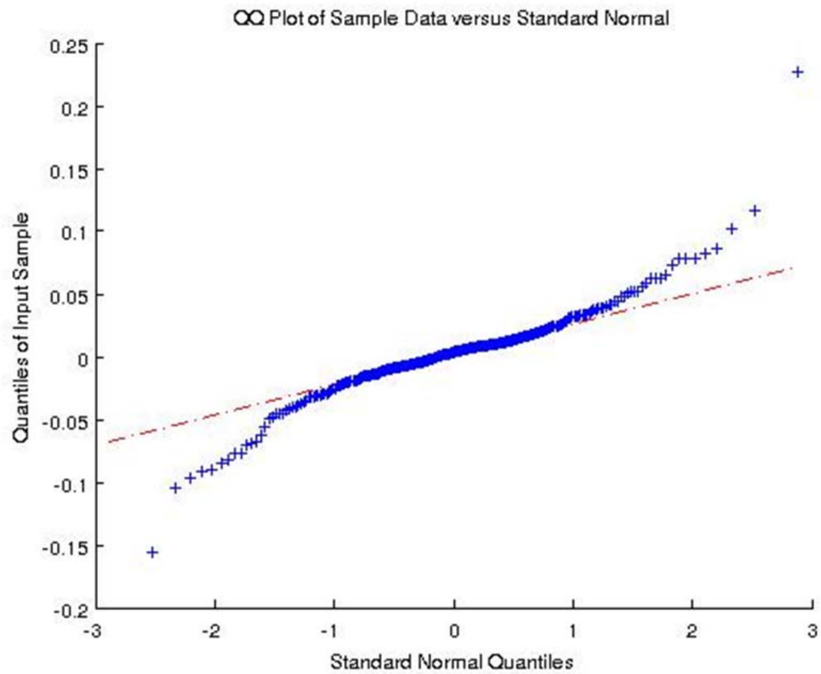


Figure 3: BAC Normal Q-Q Plot

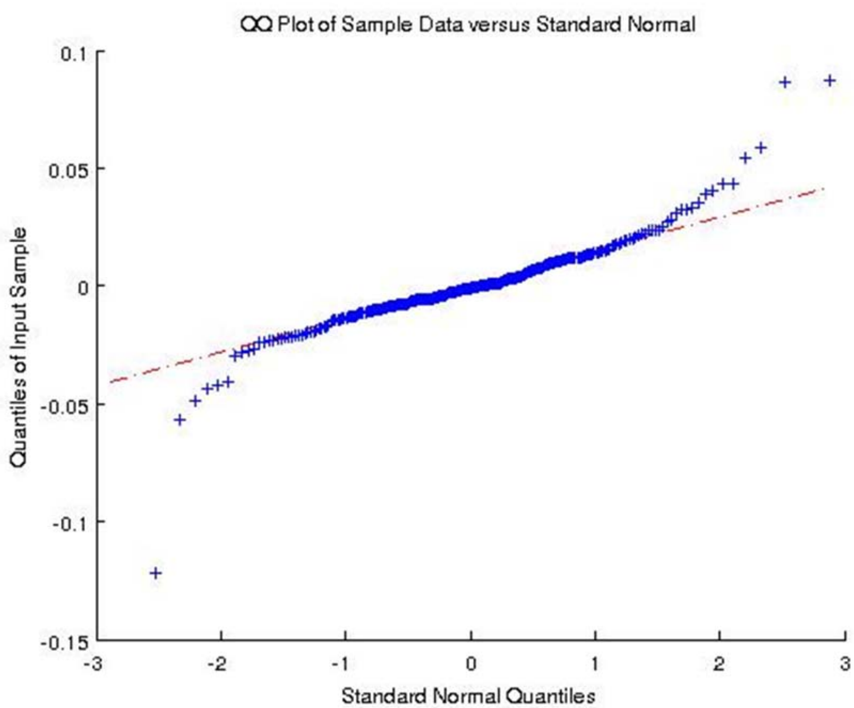


Figure 4: GOOG Normal Q-Q Plot

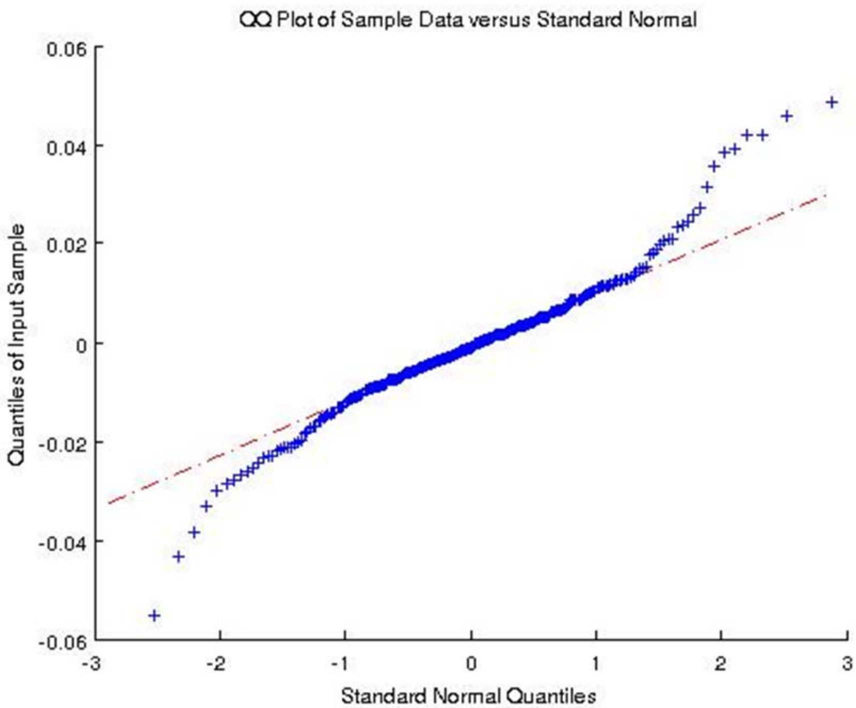


Figure 5: IBM Normal Q-Q Plot

Obviously, the data deviate from the theoretical straight line and there exists large discrepancy between the empirical distribution of the returns of underlying stock and normal distribution.

b. Kurtosis and Skewness

Another standard way of testing for normality is to compute certain moment functions of sample data and to make comparison between the empirical results and the theoretical values for a normal distribution. Here we measure kurtosis and skewness of the sample. If we denote the sample moment of order k by $m_k = \frac{1}{n}\sum_{i=1}^{n}(x_i - \bar{x})^\wedge k$, then the test statistics are given by $\widehat{K} = {}^{m_4}/_{{m_2}^2} - 3$ and $S = {}^{m_3}/_{{m_2}^{3/2}}$. The following table summarizes the test results.

	BAC	GOOG	IBM
Kurtosis (K)	6.3093	8.7923	2.1455
Skewness (S)	0.4491	-0.1840	0.2250

From the above table, the kurtosis and skewness for all these three stocks are far away from zero, which is the theoretical value under the assumption of normality. Again, for all these three stocks, the hypothesis that kurtosis and skewness are both zeros is rejected at the 1% level.

c. Cramér–von Mises criterion

In statistics, the Cramér–von Mises test is a useful tool for testing the goodness of fit of a cumulative distribution function F_1 compared to a given empirical distribution function F_2. The advantage of Cramér–von Mises is that it can compare the empirical distribution not only with normal distribution but also with any other specified distribution. In the hyperbolic package of R (2.13.2 version), the built-in Cramér–von Mises test function will estimate parameters from the financial data and perform the p-value analysis. By testing in R, we obtain the following result:

	BAC	GOOG	IBM
P-value for CvM normality test	4.17E-10	6E-8	4.4E-7

Expectedly, the p-value for all these three stocks is extremely small and we are highly confident to reject the null hypothesis that the continuously compounded rate of return on stocks is normally distributed.

d. Implied Volatility

Besides the lognormal distributional assumption, the other assumptions used in Black-Scholes do not hold either. In the Black-Scholes formula, one cannot directly observe the volatility of the stock prices. One way to obtain the volatility is to estimate it from a history of the stock price. On the other hand: given the current market price of the option, we can calculate the volatility implied by the market price and the corresponding pricing formula and we call this volatility implied volatility. That is to say, in Black-Scholes case, ***implied volatility*** is the volatility that makes the option value calculated by the Black-Scholes formula equal to the market price of that option. If the assumptions except from the lognormal distribution in Black-Scholes were correct, then we should obtain the same volatilities for different options with the same expiry. However the following figure shows this is not true. We plot the implied volatility against strike price for the above three stock call options. Obviously it is not a straight line and look like a U-shaped smile.

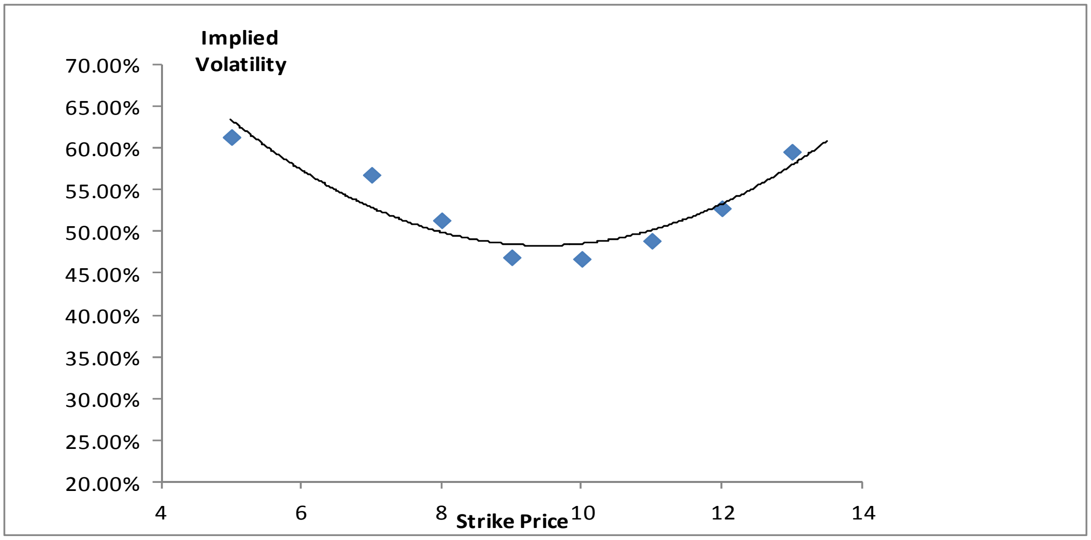


Figure 6: BAC implied volatility

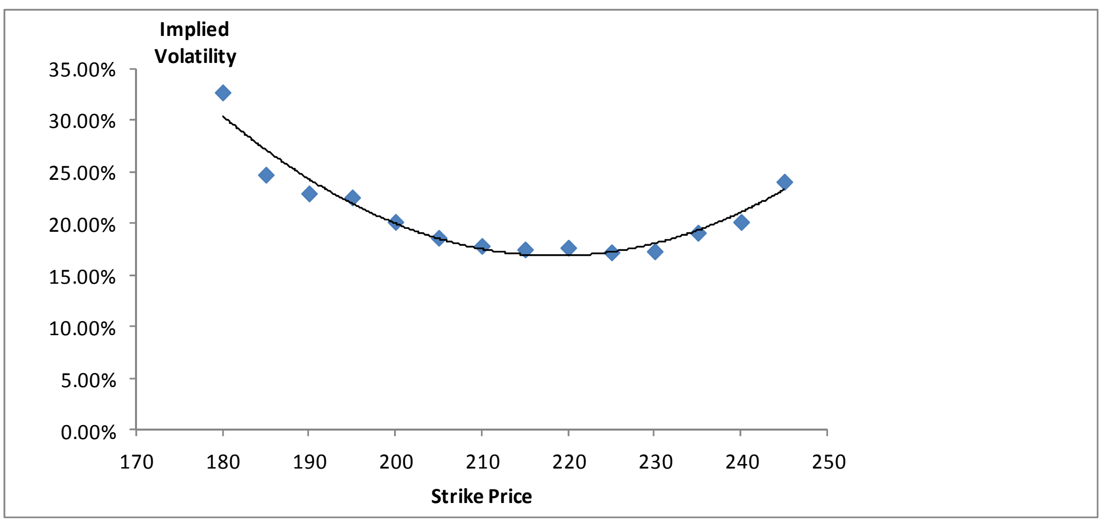


Figure 7: IBM implied volatility

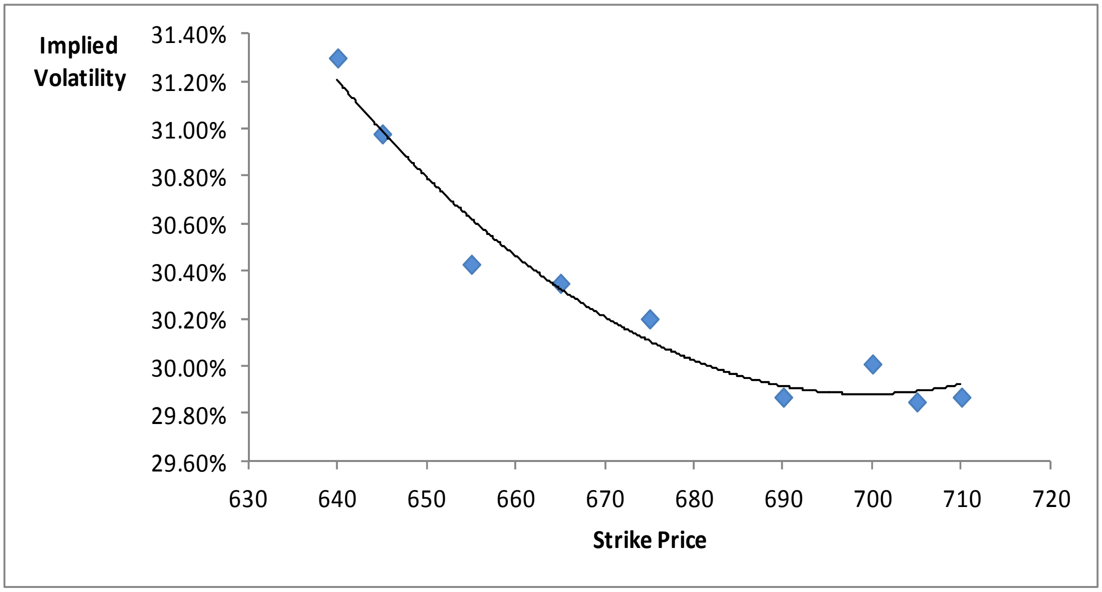


Figure 8: GOOG implied volatility

(All option prices are obtained on March 20th and they will be expired on April 20th)

4. Modeling Stock Price by Discontinuous Lévy Process

One may be curious why the normal distribution assumed by Black-Scholes is poorly fitted into the empirical data. Looking back the stock modeling section (1.2.3) again, it is not hard to find that Wiener process (also called Brownian motion, we will use these two terms interchangeably in the rest of the paper) explained there fails to capture a key characteristic of stock prices. Stock prices are definitely not continuous. They are only allowed to take discrete values. Thus the proposed log normal distribution of total returns by Wiener process is expected to deviate far away from the empirical data. In order to devise a model that is well fitted into empirical result, it is necessary for us to introduce pure jump processes that capture the discrete-value taking characteristic of stock price. But how do we model jumps and what jumps should we model? We believe there are two types we need to consider:

Small jumps term describes the day-to-day jitter that causes minor fluctuations in stock prices;

Big jumps term describes large stock price movements caused by major market upsets arising from, e.g., earthquakes, etc.

It turns out that the hyperbolic Levy process provides a fairly accurate model to capture the above jumps and its assumed distribution of return is well fitted into data.

4.1 The Hyperbolic Density and Empirical Data

In this section, we will introduce the hyperbolic distribution and see how empirical data are fitted. As mentioned above, the hyperbolic Levy motion will be used to model the movement of stock prices and it is a stochastic process associated with the hyperbolic distribution. Hence we need firstly to know the basics about the hyperbolic distribution and see whether it is a good fit to the data. The name hyperbolic derives the fact that their log-density being a hyperbola. The parameterization of the ***hyperbolic density*** is given by

$$f_{(\alpha,\beta,\delta,\mu)}(x) = \frac{\sqrt{\alpha^2-\beta^2}}{2\alpha\delta K_1(\delta\sqrt{\alpha^2-\beta^2})} e^{-\alpha\sqrt{\sigma^2+(x-\mu)^2}+\beta(x-\mu)} \qquad (4.1.1)$$

where K_1 denotes the Modified Bessel function of second kind with index 1. It has four parameters μ, δ >0, and $0 \le |\beta| < \alpha$. Roughly speaking, α and β are parameters that determine the shape; δ and μ are scale and location parameters. The following graph shows the difference between the hyperbolic pdf and the normal pdf with the same mean and standard deviation on the same plot.

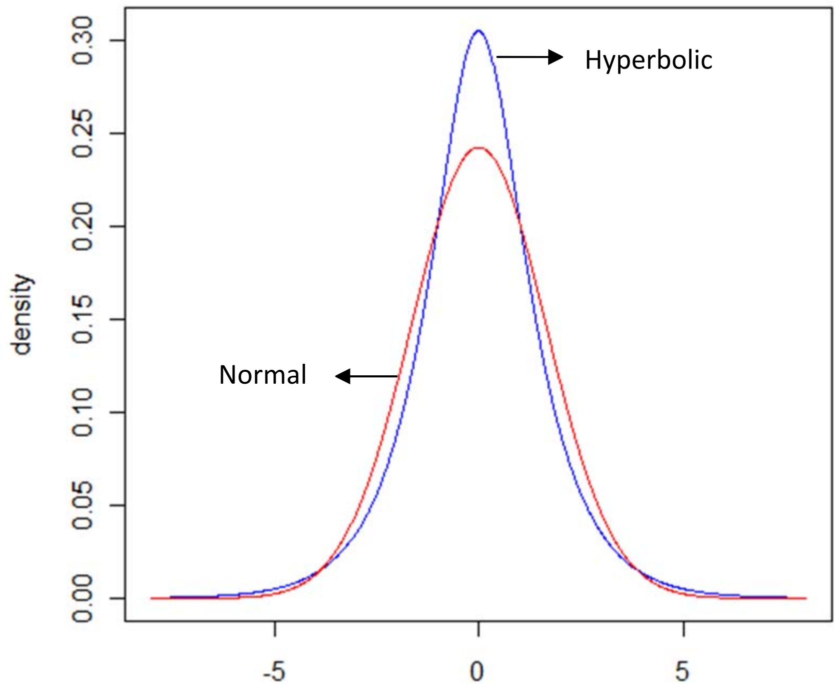


Figure 9: Normal PDF V.S. Hyperbolic PDF

Clearly, the hyperbolic distribution allows for heavier tails and decreases slower than the normal distribution. Thus it is more suitable to model phenomena where large values are more likely to occur than is the case for the normal distribution, and is expected to be better fitted to the financial data than the normal distribution.

Now consider the stocks we examined in the previous section and see if the hyperbolic distribution provides a better fit.

a. Quantile-Quantile plots

Again, we plot the Q-Q plots for the three stocks analyzed above:

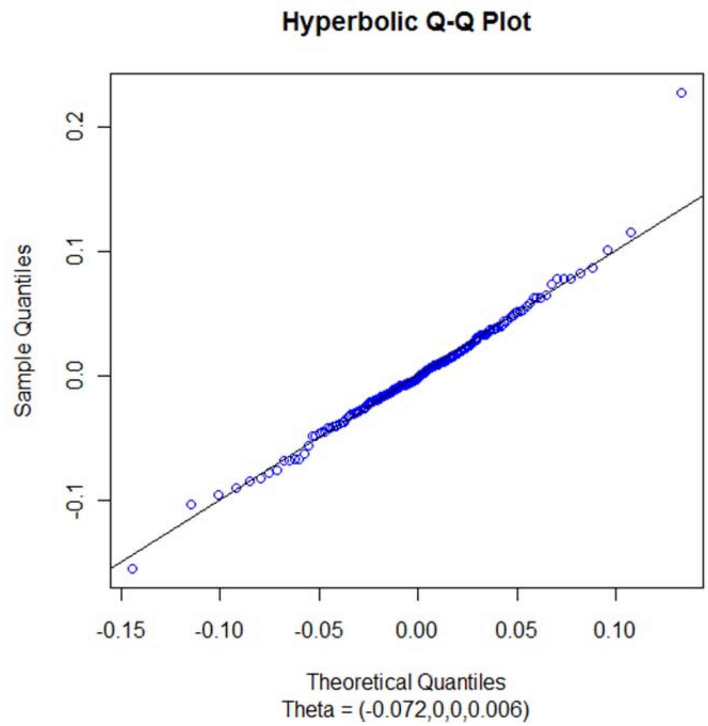


Figure 10: BAC Hyperbolic Q-Q Plot

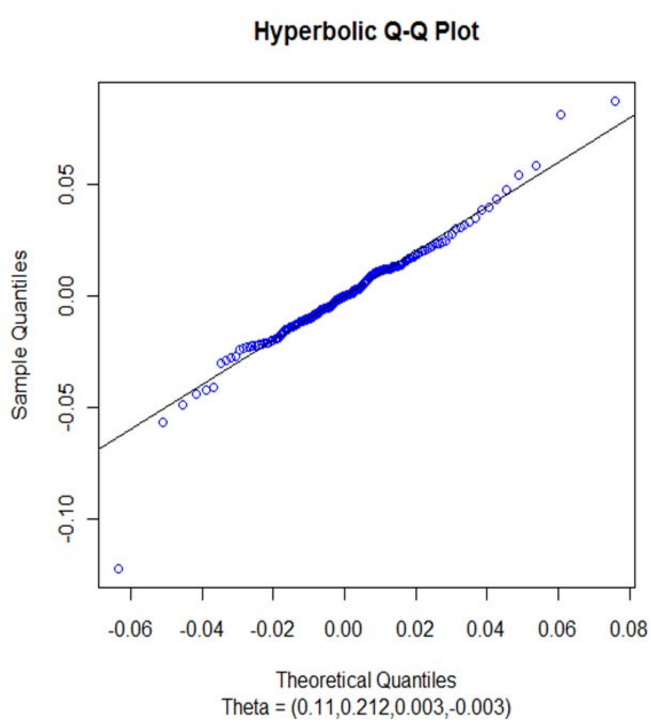


Figure 11: GOOG Hyperbolic Q-Q Plot

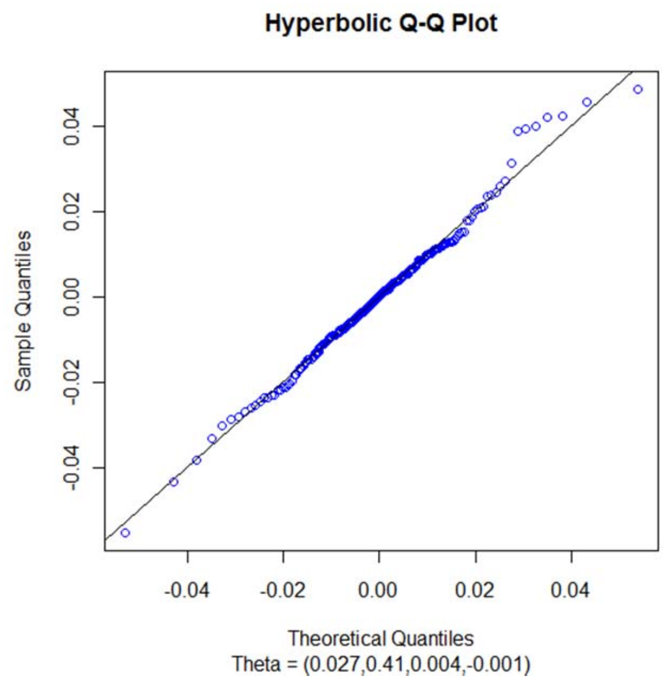


Figure 12: IBM Hyperbolic Q-Q Plot

Obviously, the data are very close to the hyperbolic distribution. The deviation from the theoretical straight line is very small and it is a much better fit than the normal distribution.

b. the Cramér–von Mises test

As with the test for normality, we perform the Cramér–von Mises test again for the hyperbolic distribution. The following table summarizes the result:

	BAC	GOOG	IBM
p-value Cramér–von Mises test for hyperbolic distritution	>0.25	>0.25	>0.25

The p-value is so large that it is extremely unlikely that we obtain the hyperbolic distribution just by chance. With no doubt that the hyperbolic distribution provides a better fit into the data.

Given the empirical results on stock returns where in most cases β = μ = 0, we will mainly concentrate on the symmetric centered cases in the later part of the paper. As a consequence, using $\varsigma = \delta\sqrt{\alpha^2 - \beta^2}$ for notational ease and $\beta = 0$, (4.1.1) can be written as

$$hyp_{\varsigma,\delta}(x) = \frac{1}{2\delta K_1(\varsigma)} \exp(-\varsigma\sqrt{1+(\frac{x}{\delta})^2}) \qquad (4.1.2)$$

4.2 Lévy Processes

The Lévy processes, which include both Poisson process and Brownian motion as special cases, were a class of stochastic processes to be firstly used to investigate trajectories. This modeling process can be borrowed into study of finance and turns out it plays a crucial role in describing stock price movements.

Def A process X is called a Lévy process if it has the following properties:

1. Independent increments: for any u ≤ s < t, $X_t - X_s$ is independent of X_u

2. Stationary increments: for any s,t >0, $X_{s+t} - X_s$ has the same distribution as $X_t - X_0$

3. Continuity in probability: $X_s \to X_t$ in probability as s tends to t.

From the above definition, it is clear that the most common example of a Lévy process is Brownian motion, where $X_t - X_s$ is independently normally distributed with mean of zero and variance of $t-s$. There are other examples like Poisson processes, compound Poisson processes, the *Cauchy process*, gamma processes and the variance gamma process.

One important result that has a direct impact on option pricing is the expression of the characteristic function of X_t, often known as *Lévy-Khintchine formula*:

$$E[iuX_1] = e^{-t\phi(u)} \quad (4.2.1)$$

where t≥0 and ϕ has the following representation:

$\phi(u) = \frac{\sigma^2}{2}u^2 - iau + \int_{|x|\geq 1} (1 - e^{iux})k(dx) + \int_{|x|<1} (1 - e^{iux} + iux)k(dx)$ (4.2.2) where a is called the drift rate of for the Lévy process and k(dx) is a measure on $R - \{0\}$ such that $\int \inf(1, x^2)\ k(dx) < \infty$ and called the Lévy measure of the process X.

Observing that there is a one-to-one correspondence between Lévy processes and characteristic functions described as above. Then we can start build our process with equation (4.2.2) by specifying different characteristic function for X_1. The following building experiment can show that why Lévy processes appear as a wide natural class of candidates for stock prices. Beginning with formula (4.2.2), we build three processes $X^{(1)}$, $X^{(2)}$, and $X^{(3)}$ as follows:

1. Let $X_t^{(1)} = -at + \sigma W_t$ where W_t is a Brownian motion. The characteristic function of $X^{(1)}$ is simple and equal to $\phi_1(u) = iau + \frac{1}{2}\sigma^2 u^2$.

2. $X_t^{(2)} = \sum_{i=1}^{N_t} Y_i$ where N is a Poisson process (The detail about Poisson process can be found at Chapter 11 of Shreve's *Stochastic Calculus for Finance II*) the number whose intensity λ = $\int_{|x|>1} k(dx)$ and Y_i are i.i.d with distribution $\int_{|x|>1} k(dx)$. We call $X_t^{(2)}$ the ***Compound Poisson Process***. It's characteristic function is $e^{\phi_2(u)}$, where

$\phi_2(u) = -\int_{|x|>1} (e^{iux} - 1)k(dx)$

3. $X_t^{(3)}$ is obtained as limit of compounded Poisson processes.

With these built processes, if we look at back again at (4.2.2), it is easy to see that any Lévy process can be written as $X = X^{(1)} + X^{(2)} + X^{(3)}$ (4.2.3). The decomposition exhibited in (4.2.3) illuminates the fact that $X^{(1)}, X^{(2)}$, and $X^{(3)}$ are semi martingales, so is X. As it is known that stock prices are semi martingales under the real probability measure P and it is natural to choose Lévy processes candidates for stock prices. The above discussion shows that in order to get continuity for the process X, the components $X^{(2)}$ and $X^{(3)}$ need to be zero. That is to say the process is reduced to the geometric Brownian motion. Hence, the important property:

The only Lévy process with continuous paths is the geometric Brownian motion.

From previous sections it is clearly seen that if we start with a continuous Lévy process (i.e. geometric Brownian motion) to describe the stock movements, we obtain normality. But the empirical test shows the empirical data deviates far from lognormal return. Expressed differently, it means that it is necessary to introduce discontinuous Lévy processes whenever deviations from normality are clearly exhibited by the data.

4.3 Hyperbolic Lévy Motion model

As mentioned in section 4.1, the hyperbolic distribution is well fitted into the data. Thus it's natural to come up with the idea that we should generate some random process taking the hyperbolic distribution into Levy process. In fact, according to Eberlein and Keller (1995), by the characteristic function defined in (4.2.2), we can generate a Lévy process X_t such that the distribution of X_1 is given by (4.1.1). This process X_t is called hyperbolic Levy motion depending on parameters (α,β,μ, δ) and we will use a standardized X_t with β=μ=0, whose density function is given by (4.1.2), to analyze the underlying stock price movements.

To describe the dynamic movement of stock price, we want to obtain a similar equation like (1.2.7). Moreover now the model should involve the hyperbolic Levy motion so that the assumption deficiencies in the Black-Scholes can be fixed or at least can be more or less improved. Indeed, if we use the hyperbolic Lévy motion to model stock price, the mathematical equation is as follows:

$$dS_t = \mu S_{t-}dt + \sigma S_{t-}X_t \quad (4.3.1),$$

where X_t is a standardized hyperbolic Lévy motion and S_{t-} is the left side limit.

However, the solution for (4.2.1) may take negative values with positive probability. Thus we need to work out a way to circumvent this problem. According to Eberlin, Keller, and Prause(1998), the reformulated equation is

$$dS_t = \mu S_{t-}dt + \sigma S_{t-}X_t + S_{t-}(e^{\sigma \Delta X_t} - 1 - \sigma \Delta X_t) \quad (4.3.2)$$

Motivated by Ito's formula, we can solve (4.2.2) and obtain the following solution:

$$S_t = S_0 \exp(\mu t + \sigma X_t) \quad (4.3.3)$$

Apparently, this process can no longer take negative values.

4.3.1 Esscher Transform

Having setting up a clear mathematical description of the underlying stock, we now consider how to price the option in the hyperbolic model. The modern technique to perform option pricing is the martingale method as we touched in previous sections. Here we introduce The Esscher transform, which provides a powerful tool to value financial derivatives.

For a probability density function f(x), let h be a real number such that $M(h) = \int_{-\infty}^{\infty} e^{hx} f(x)dx$ (4.3.4) exists and non-zero. As a function in x, $f(x;h) = \frac{e^{hx} f(x)}{M(h)}$ (4.3.5) is a probability function, and it is called the Esscher transform (parameter h) of the original distribution. Note that the original pdf $f(x)$ is related to the new pdf $f(x;h)$ by

$\frac{e^{hx}}{M(h)}$. Since the exponential function is positive, the modified probability measure is equivalent to the original probability; that is to say, both probability measures agree with which events have probability zero. Now consider the Esscher transform (parameter h) of the process $\{X(t)\}$. It is basically the same as the Esscher transform of a single random variable described above. The difference is that now we have a new variable t. The analogy is as follows: $f(x;h) \to f(x,t;h) = \frac{e^{hx} f(x,t)}{M(h,t)}$ and

$M(x;h) \to M(x,t;h) = \frac{M(x+h,t)}{M(h,t)}$.

According to Gerber and Shiu (1994), the Esscher transform is an efficient technique to value option if the log of the underlying stock price follows a certain stochastic processes with stationary and independent increments. That is to say, let S(t) denotes the price of a non-dividend stock at time t. Suppose there is a stochastic process $\{X(t)\}$ such that $S(t) = S(0)e^{X(t)}, \quad t \geq 0$. According to the first fundamental asset pricing theorem, we want the discounted stock price process to be a martingale under some probability measure. In other words, in order to apply the Esscher transform to value derivatives, we seek h^* such that $\{e^{-rt}S(t)\}_{t\geq 0}$ is a martingale with respect to the probability measure corresponding to h^*. In particular,

$$S(0) = E^{h^*}[e^{-rt}S(t)], \text{ where r is the risk free rate.}$$

To solve h^*, we need the following fact.

Fact: $M(x,t;h) = [M(x,1;h)]^t$ (4.3.6)

Thus $1 = e^{-rt}E^{h^*}\left[e^{X(t)}\right] = M(1,t;h^*)$. From (4.3.6), we will have $e^r = M(1,1;h^*)$ or

$$r = \ln[M(1,1;h^*)] \qquad (4.3.7)$$

Therefore, if we can find Esscher parameter h^* in (4.3.7), then the price of the option is simply calculated as the expectation of the discounted payoffs with respect to the equivalent martingale measure. Clearly, both the assumed Wiener process in the Black-Scholes model and the process described in the hyperbolic Levy motion model have analytical expression for their moment generating functions. Hence we can apply the Esscher transform technique to price option price for both Black-Scholes and hyperbolic models. For a European call option, using the Esscher transform arguments described above we obtain,

$$C = E^{h*}[e^{-rT}(S-K)^+] \quad (4.3.6)$$

where h^* is the Esscher parameter such that $(e^{-rt}S_t)_{t\geq 0}$ is a martingale. In the Black-Scholes case S is modeled as (1.2.7) whereas in the hyperbolic model S is modeled as (4.3.2).

<u>4.3.2 Applying Esscher Transform to the Hyperbolic Model</u>

In the case of the hyperbolic model, if we know $M(1,1;\theta^*)$ for the hyperbolic distribution, then it's easy to solve h^*. By the definition of the Esscher transform of stochastic process, we have $M(1,1;\theta^*) = \frac{M(\theta^*+1)}{M(\theta^*)}$.

Lemma: The moment-generating function of the hyperbolic distribution is given by

$$M(x) = e^{\mu x}\frac{\sqrt{\alpha^2-\beta^2}}{K_1(\delta\sqrt{\alpha^2-\beta^2}}\frac{K_1(\delta\sqrt{\alpha^2-(\beta+x)^2}}{\sqrt{\alpha^2-(\beta+x)^2}}, |\beta+x|<\alpha \quad (4.3.7)$$

In the symmetric centered case with $\varsigma = \delta\alpha$,

$$M(x) = \frac{\varsigma}{K_1(\varsigma)} \frac{K_1(\sqrt{\varsigma^2 - (\varsigma x)^2}}{\sqrt{\varsigma^2 - (\varsigma x)^2}}, |x| < \frac{\varsigma}{\delta} \quad (4.3.8)$$

Plug (4.3.8) back into (4.3.7), it is easy to get h^* as the solution of

$$r = \ln \frac{K_1(\sqrt{\varsigma^2 - \delta^2(\theta+1)^2})}{K_1(\sqrt{\varsigma^2 - \delta^2\theta^2})} - \frac{1}{2}\ln \frac{\varsigma^2 - \delta^2(\theta+1)^2}{\varsigma^2 - \delta^2\theta^2}$$

Therefore, we can obtain the option price as in (4.3.6) by taking the expectation of the discounted payoff under the risk neutral measure. The expectation under P^{h^*} can be written explicitly as

$$C_{hyp} = S_0 \int_{\gamma}^{\infty} f_T(x;\theta^* + 1)dx - e^{-rT} K \int_{\gamma}^{\infty} f_T(x;\theta^*)dx \quad (4.3.9)$$

where $\gamma = \ln(K / S_0)$ and $f_t(x;\theta^*)$ is the density of the distribution of X_t under the equivalent martingale measure.

4.4 Numerical Approximation

Having known the option pricing formula for hyperbolic Levy motion explicitly, it is natural to consider applying (4.3.9) to the empirical data and see how these values differ from the market and Black-Scholes prices.

Methodology to compute (4.3.9):

The general goal is to re-write (4.3.9) as an explicit integral with all the parameters that can be obtained from empirical data, and then approximate its value in MATLAB

1. $f_t(x;\theta^*)$ is related to the original function $f_t(x;\theta)=\dfrac{e^{\theta x} f_t(x)}{M(\theta)^t}$ (4.4.1), where M is moment-generating function of the hyperbolic distribution and t is the trading days.

2. According to Elberlin and Keller (1995), the moment generating function of the hyperbolic distribution is given by $M(u)=\dfrac{\varsigma}{K_1(\varsigma)}\dfrac{K_1(\sqrt{\varsigma^2-\delta^2u^2})}{\sqrt{\varsigma^2-\delta^2u^2}}$ (4.4.2) where $\varsigma=\delta\sqrt{\alpha^2-\beta^2}$; α, β are estimated from empirical data. Hence the denominator in (4.4.1) can be numerically computed if we know θ.

3. We obtain the value θ^* by numerically solving the equation

$r=\ln\dfrac{K_1(\sqrt{\varsigma^2-\delta^2(\theta+1)^2})}{K_1(\sqrt{\varsigma^2-\delta^2\theta^2})}-\dfrac{1}{2}\ln\dfrac{\varsigma^2-\delta^2(\theta+1)^2}{\varsigma^2-\delta^2\theta^2}$ (4.4.3) where r is the given daily risk-free interest rate

4. $f_t(x)$ is also given as $f_t(x)=\dfrac{1}{\pi}\int_0^\infty \cos(ux)\phi_t(u)du$ (4.4.4), where ϕ is the characteristic function of the hyperbolic distribution represented as

$\phi(u;\varsigma,\delta)=\dfrac{\varsigma}{K_1(\varsigma)}\dfrac{K_1(\sqrt{\varsigma^2+\delta^2u^2})}{\sqrt{\varsigma^2-\delta^2u^2}}$ (4.4.5) and by the property of characteristic functions, we know how to handle the sums of independent variables and can obtain

$\phi_t(u) = \phi^t(u)$ (4.4.6)

5. From the above four steps, we will be able to re-write the first term in (4.3.9) as

$\frac{S_0 e^{(\theta^*+1)}}{\pi M(\theta^*+1)} \int_{\ln(K/S_0)}^{\infty} \int_0^{\infty} \cos(ux)\phi^t(u)\,du\,dx$ (4.4.7). In the same way we can obtain the expression for the second term in (4.3.9). All the parameters in (4.4.7) can either be obtained from the empirical data or solved from the equations given in the previous four steps.

Implementation: we write a few lines in the MATLAB script to numerically approximate (4.4.7).

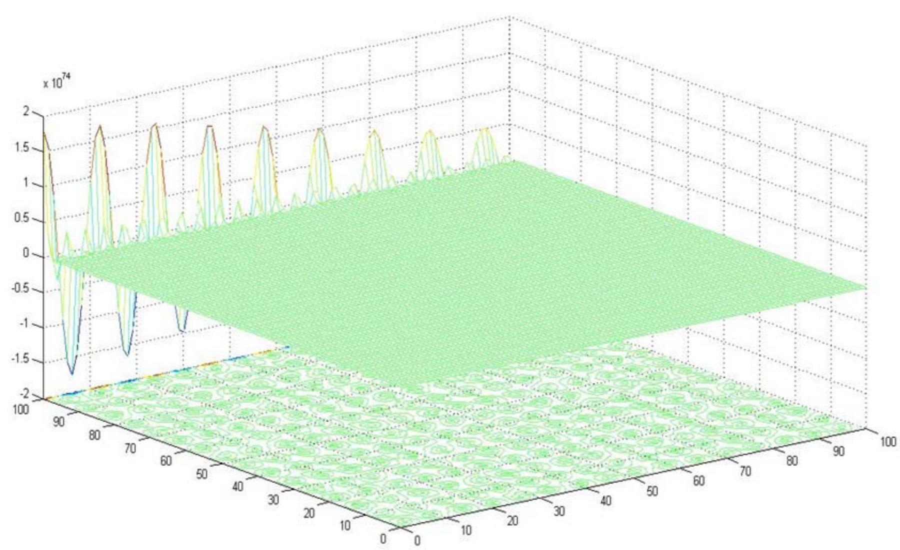
The code is as written in the next page:

```
function result = hyper_update( delta, alpha, beta, r_annual, s0, T, t)

    zeta= delta.*sqrt(alpha.^2-beta.^2);
    r = (1+r_annual).^(1/365)-1;
    fun_theta=@ (u)log((besselk(1,sqrt(zeta.^2-
    delta.^2.*(u+1).^2)))./besselk(1,sqrt(zeta.^2-delta.^2.*u.^2)))-
    1/2*log((zeta.^2-delta.^2.*(u+1)^2)/(zeta.^2-delta.^2.*u.^2))-r;
    theta_star = fzero(fun_theta,0.05);
    phi= @(u)
    zeta/besselk(1,zeta).*besselk(1,sqrt(zeta.^2+delta.^2.*u.^2))./sq
    rt(zeta.^2+delta.^2.*u.^2);
    M = @(u) zeta/besselk(1,zeta).*besselk(1,sqrt(zeta.^2-
    delta.^2.*u.^2))./sqrt(zeta.^2-delta.^2.*u.^2);
    integrnda = @(x,y)
    s0./M(theta_star+1).*exp((theta_star+1).*y).*(1/pi).*cos(x.*y).*p
    hi(x).^t;
    integrndb = @(x,y) exp(-
    r*t)*T./M(theta_star).*exp((theta_star).*y).*(1/pi).*cos(x.*y).*p
    hi(x).^t;
    a = quad2d(integrnda,0,100,log(T/s0),10);
    b = quad2d(integrndb,0,100,log(T/s0),10);
    result = a - b;

end
```

However, the outcome is very frustrating. We cannot obtain any reasonable values for the hyperbolic option values. It is because the behavior of the integrand is bizarre and the regular quadratic approximation method may not be applied to evaluate the integral. The above graph is the plotted integrand.

Looking back at the expression of $f_t(x)$ in (4.5.4), if we plot the density function using the parameters obtained from empirical data (the German stock Deustche Bank from Oct 2nd 1989 to Sep 30th 1992) against time t (in trading days) and parameter θ, we can see that it does not look like a regular pdf it is supposed to be. Are the equations wrong? Of course not. We examine every equation in previous sections carefully. It is because we cannot numerically approximate $f_t(x) = \frac{1}{\pi}\int_0^\infty \cos(ux)\phi_t(u)du$ in a regular way. Thus our plot is an unfaithful representation of the equations because of the unsuccessful numerical integration. The following graphs are the plotted graphs for $f_t(x)$

Figure 13: Integrand of (4.4.7)



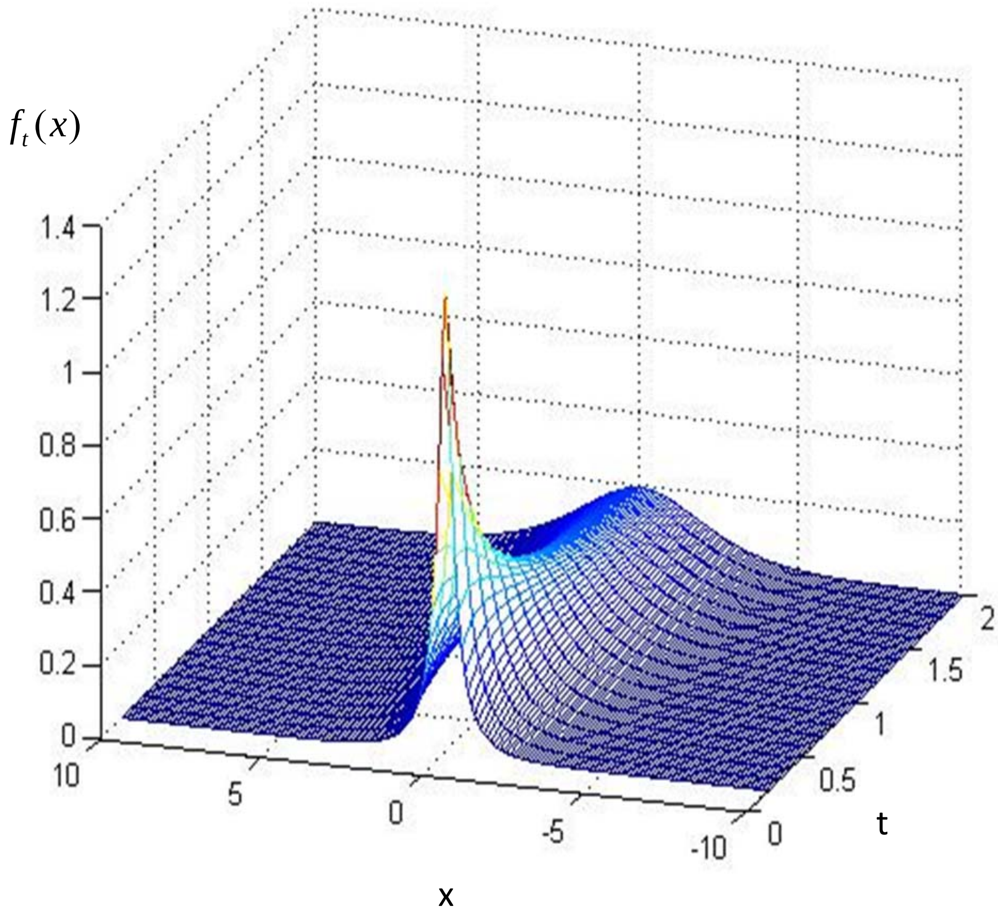


Figure 14: The graph in the special case, where ς = σ = 1, shows what the density function $f_t(x)$ should look like

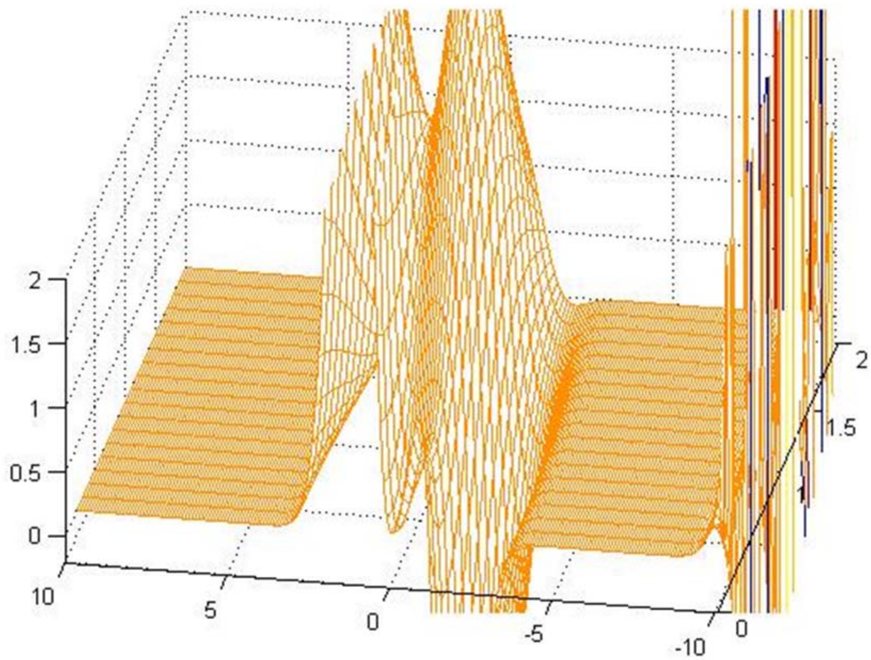


Figure 15: The graph in the case, where ς = 0.3258 and σ = 0.003, shows the density function $f_t(x)$ behaves abnormally

One possible reason for the strange behavior exhibited by the hyperbolic model is that though the hyperbolic Levy motion stochastic variables are identically independently distributed and additive, we do not know the distributional form of the sum of these hyperbolic increments and may lead to some strange result. In contrast, the normal distributional assumption of returns does not present any problem in calculation because we know the sum of normal random variables still follow a normal distribution and we have enough tools to deal this familiar distribution.

For reader's interest, according to Carr(1999), the Fast Fourier Transformation may be a useful technique to perform the numerical approximation when there is an analytical expression for the characteristic function of the risk-neutral density. It may circumvent the issues come across by the quadratic method we used above.

5.Conclusion

Let us summarize what we have discussed above. We investigate the basic models for option pricing - binomial, Black-Scholes, and the hyperbolic model. Each model has its own advantage. Binomial is simple and presents the basic idea for further investigation; Black-Schole can be seen as a continuation of binomial model and shows how risk neutral valuation works in the analysis of derivatives; the hyperbolic has an excellent fit to empirical data and replaces the geometric Brownian motion with pure jump process to better model the evolution of stock prices. It seems that the hyperbolic model is better than the other models as the observed distribution of returns is well fitted and it still has closed form solution. However, after a careful examination of the hyperbolic option pricing formula, we find the hyperbolic model cannot be computed as easily as Black-Scholes. Due to the limited time constraint, we suggest the reader can try the Fast Fourier Transformation and it may provide an accurate approximation to the hyperbolic price. Though these models are completely different from one another, risk-neutral valuation can be used to value all of them. More generally, we discover that the Esscher transform is a powerful tool to value financial derivatives. Any Esscher transform of a stochastic process gives a new probability measure for the process; if we can find the parameter of the Esscher transform such that the discounted value of each underlying security is martingale under the new probability measure, then the price of a derivative is simply calculated as the expectation, with respect to the equivalent martingale measure, of the discounted payoffs. In summary, the underlying tradeoff between distributional assumptions is between tractability (closed-form solutions, and other implications) and more accurate description of stock price movements. Even though we understand that the observed data is not log normal, the tractability of the geometric Brownian motion trumps the realism of other processes. As in our case, the implied prices by the hyperbolic model cannot be easily computed and require advanced computation whereas the Black-Scholes price can be easily to be computed.

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