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A Comparative Study of non-Markovian Stochastic Processes in Marketing

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An abstract of
A thesis submitted to the Faculty of the
James T. Laney School of Graduate Studies of Emory University
in partial fulfillment of the requirements for the degree of
Master of Science
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Abstract

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Non-Markovian(NM) stochastic processes exist widely in nature; however, they have been largely ignored in traditional marketing research. In this thesis, we investigate the consequences of such NM behaviors both theoretically and experimentally. A stochastic model analogous to the Ising model in statistical physics was used to explain gaps between the Markovian and non-Markovian data discovered in real surveys. Analytically a fixed point relation between parameters is derived using the central limit theorem, while detailed stochastic simulations are performed using a master equation approach. Results from kinetic Monte Carlo (KMC) simulations and analytical solutions coincide well with each other. The model also has the potential to predict a larger group of marketing outcomes if the model parameters are properly defined.

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Acknowledgements

I would like to acknowledge my advisors Dr. H. George E. Hentschel and Dr. Fereydoon Family for their guidance, knowledge, and motivation for completing this work. I would also like to thank Dr. Zhenyu Zhang for helpful suggestions and conceptual discussions. I appreciate Dr. Yiping Song for designing the survey, collecting data and many useful discussions on potential connections between physics models and marketing phenomena. I'm grateful to Dr. Jianfeng Li for his work on master equations. The cooperation of the University of Tennessee - Knoxville and the School of Management at Fudan University was important to my progress. I wish to thank Dr. Virginia Shadron and Dr. Laura Finzi for providing advice and encouragement, Ms. Grace Song for help to revise this thesis, as well as Dr. Stefan Boettcher for serving on my committee. I am grateful to my family and friends for their support, understanding, and encouragement during this process.

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Glossary

P_i	The purchase probability for the i^{th} respondents in non-Markovian surveys. Under this definition, P_0 will be the final purchase probability for Markovian surveys and P_∞ represents the final purchase probability for non-Markovian surveys.
N_r	The size of information on previous consumers' decisions the last consumer could receive. In our experiment specifically, $N_r = 10$.
r	The actual purchase percentage in the memory. A subscript, e.g. "i", means the real purchase percentage the i^{th} respondent received.

Chapter 1

Introduction

The social sciences have inspired the application of numerous mathematical models to understand their dynamics. In most models, stochastic processes form their fundamentals. For example, Fischer Black and Myron Scholes derived their famous option pricing equation (4) based on the Brownian motion (8; 12; 44) assumption; other stochastic processes, such as the autoregression(AR) process and Markov process, are widely used in modeling the business cycles[1] (20; 21; 28), consumers' rational expectations[2] (31; 42), and regime switching[3] (7; 43; 46) in macroeconomics and finance. Particularly within marketing science, researchers have attempted to introduce stochastic models to examine consumer behavior. For example, D. Luce proposed the choice axiom (32) to model consumer choices among different brands. In his paper, Luce assigned different weights w to different brands based on choices made by survey respondents. He stated that the probability for a consumer to select one brand over another from a collection of many brands only depends on its weight. So far, the most frequently adopted model is the Markov Chain(22; 24; 33; 34; 45). The advantage of this model is that it preserves the stochastic nature of human behavior without introducing many mathematical complexities.

Markov models have, however, failed to address the following crucial aspects in marketing sciences: the time-correlated characteristics of human decision making, the path-dependent characteristics of market development and possible social cue

[1]According to the W.Mitchell (35; 36), the business cycle is defined as expansions occurring at the same time in many economic sectors, followed by similarly general recessions, contractions and revivals.

[2]Rational expectations is a hypothesis in economics which states that agents' predictions of the future value of economically relevant variables, is not systematically wrong. In other words, the error term in agents' expectation follows a random walk with the mean value 0.

[3]Regime-switching model was first proposed by James D. Hamilton(20), in which a Markov chain is used to model switches between periods of high volatility and low volatility of asset returns. In the later time, other regime switching models are used to describe the effects of different US monetary and fiscal policies.

effects given by other market participants. For example in the real market, one company's pricing decision on a particular product does not only depend on the present state of the market, but also relies on previous pricing decisions made by competitors and the company itself. Even when the current market price is low, the company is still unlikely to raise the price if past records reveal a tendency of an unceasing price reduction. Empirical studies also show that social cue effects exist and propagate the diffusion of marketing strategies. Take the example of the marketing strategy Quality circle (QC) which was widely adopted in 1980s. Although the contribution of QC on firms' performance is still ambiguous, approximately 90 percent of the Fortune 500 companies had adopted QC by 1985 (2). This fact implies that the diffusion of QC is largely due to the social cues.

The statement of how social cues effect people's decision making was first proposed by social psychologist L. Festinger (14) in 1954. In the paper, he analyzed the experimental results reported by others and made qualitative hypotheses on how people make decisions when social influences exist:

" To the extend that objective, non-social means are not available, people evaluate their opinions and abilities by comparison respectively with the opinions and abilities of other. " (14)

Quantitative models were later constructed. Boyd *et al.* proposed models explaining the social cue effects from the perspectives of culture and human evolution (6; 23), which "provides an evolution foundation of the psychological mechanism posited by Festinger" (27). A later progress was made by the marketing scientist J. Bendor and physicist B. Huberman (27). They proposed a linear model to explain the diffusion of marketing strategies, which includes social cue effects. In their paper, they assume decision makers have to make decisions from two alternatives A and B, with the probability:

$$P[\textit{agent in period t+1 chooses A}] = \alpha p + (1-\alpha) r_t \tag{1.1}$$

Where p is the probability of choosing A purely under the evidence of its performance, and r_t is the percentage of agents who choose A till the last period. The parameter α stands for the social cue strength and different α's can lead to very different dynamics. They conclude in their model, that if $\alpha = 0$, the results given by the model will return to the Polya's urn process (40), which doesn't have a predetermined equilibrium; while if $\alpha \neq 0$, the final percentage of choosing A will converge to the predetermined equilibrium p. The value of α only affects the convergence speed.

The motivation of our work is to improve on Bendor and Huberman's model, and to understand the diffusion process under social cue effects. In this project, we try to answer the following question: if a consumer happens to know decisions made by other consumers, how should one model his/her decision making process using a simple non-Markovian model.

In exploration of the existing marketing literature, we found no effective model of such non-Markovian process. Instead, three approximation methods have been used to deal with the history-dependent characteristics of human decision making. The first method is to simply ignore the time dependence of the process (9; 19; 38). The second is to translate the dynamic state-dependence component into a static variable or constant (15; 16; 18). For example, in the model of trial sales proposed in (15), the author assumed there existed an upper bound for the fraction of households that purchased a product, which is called the "ceiling" proportion. For a successful grocery product, the fraction of consumers who made a purchase would increase and approach its ceiling value in a decreasing step each period. For each step, the increment was equal to the previous increment multiplied by a change rate $1-r$. A constant was set for the change rate $1-r$ although it was actually time dependent in the real market. The third method is the "High-order Markov Chain" (11; 37; 39), which considers system evolutions with possible correlated state-dependent variables explicitly in the model; thus, this method maintains more information on past periods than the other two. In the existing literature, however, only one or two previous time periods were taken into consideration. Therefore, the high-order Markov Chain approach works well only for the cases when the memory of the dynamics is short and the time correlation coefficients decrease rapidly with previous time periods. Thus, it has the same disadvantage as the Markov approach. The high-order Markov method is quite limited in analyzing long-term historical data.

In contrast, the natural sciences have provided some effective methods to solve non-Markovian problems both theoretically and experimentally: Statistical physics and ensemble theory (29) help us find a system's equilibrium state; Kinetic Monte-Carlo method can be used to simulate non-Markov dynamics (1; 17). These methods help to solve problems such as surface growth (13) and diffusion (3), vacancy diffusion (47) and visco-elasticity of physically cross-linked networks (41). Because of their successes in natural science, these existing methods are used in this study to solve time-correlated problems in marketing and account for the non-Markovian nature of consumer decision making.

In this work, several stochastic models that contain non-Markov effects are proposed, including models similar to the Ising model and the Fermi model in the natural sciences. The non-Markov effect of consumer behavior is computed using the "Master Equation" method. The results obtained are then compared with those from kinetic Monte-Carlo simulations and empirical data collected from a real survey. The approximate solution of the Ising model, and effects of different parameters are also discussed.

In Chapter Two, the real survey and data sets are discussed. Chapter Three proposes two possible models. Next, Chapter Four describes both the analytical results derived from two non-Markov models and the results from simulations. The final chapter offers concluding remarks and directions for future research.

Chapter 2

Experiment

Survey-based case studies are widely used for designing strategies in marketing science(5; 10). For example, firms often use surveys to find out the most profitable price before launching new products. In traditional marketing research, most survey results are Markovian since respondents typically receive time-invariant information such as the function of the product or its price before making their decisions. However, we argue that firms may receive imprecise or even misleading information if the Markovian result is chosen as the starting point for designing marketing strategies for a non-Markovian process. In this chapter, we explore non-Markovian effects in consumer decision making and show large differences between Markovian and non-Markovian results.

We conducted three independent surveys. Each of them contained two sample groups, one for testing Markovian effects and the other for testing non-Markovian effects with other conditions being equal. This survey required participants to complete a questionnaire and indicate whether they would purchase the product or not. The following sections discuss the method in more detail.

2.1 Participants

$2,253$ sophomores and juniors from three different universities in Shanghai(China) were randomly selected as the sample of this study. All respondents in the sample were aged between 19 and 23. The ratio of male and female in the sample was approximately $1:1$. For each survey, we collected 751 valid answers, 375 for the Markovian group and 375 for the non-Markovian group. Both groups shared the result of the first respondent.

2.2 Procedure

In the survey, respondents from both groups were asked of the following questions.

- **Q1.** Which cell phone carrier are you currently using?
- **Q2.** Do you know about the intra-network service offered by China mobile?

The survey continued only when the answer to the first question was "China mobile". If a respondent chose "No" to the second question, the following instruction was shown to them:

China mobile(Shanghai) is offering unlimited intra-network mobile to mobile service for students and faculty members of xxx University[1]. The service fee will be ¥p/month[2]. After activation, you will be assigned an intra-network number(INN); members of this network can make unlimited calls to each other via INN. No other hidden fees.

After reading these instructions, Markovian respondents had to answer the following question:

- **Q3.** Do you want to buy this service?

For respondents in the non-Markovian group, information on decisions made by ten respondents before him/her[3] was also provided to the respondent before he/she answered Q3.

According to the pre-test, we selected ¥10, ¥5and ¥3 as the monthly fee for the three surveys respectively so as to catch the properties of low, medium and high purchase rates.

[1]Here xxx stands for the name of the respondent's university.

[2]The p here stands for the monthly fee for this service, in this experiment, we set the p as a variable. Also note that the symbol ¥ is the same as "Yuan", which is the unit of Chinese currency "RMB"(Short for Renminbi). We'll use both ¥ and "RMB" in later sections without distinction.

[3]The 2nd through 9th respondents received the information on the choices of all previous respondents.

2.3 Real Experiment Data

We plotted the total purchase rates for Markov and non-Markov processes versus the number of respondents at each price level in Fig. (2.1).

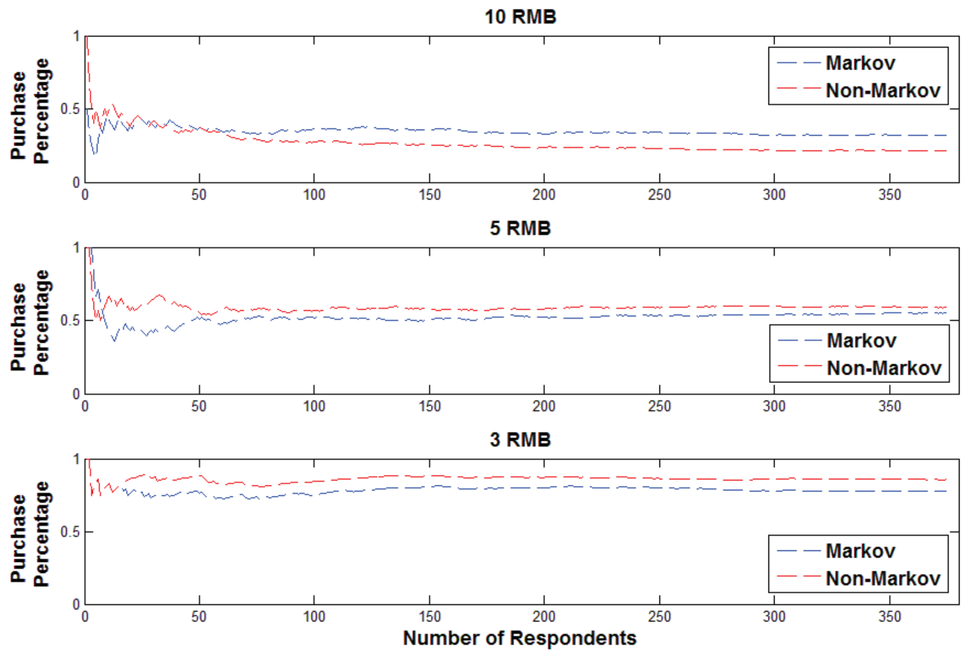


Figure 2.1: Markovian and non-Markovian purchase rates from real survey

From this figure, it's obvious that purchase rates for both Markov(P_0) and non-Markov processes(P_∞) converge to some value lying in $[0, 1]$. As expected, final purchase rates decreased as the price increased for both Markov and non-Markov processes. However, differences between P_0 and P_∞ are found at the same price level in all three sets of data. In Fig. (2.2), we plotted Δ Purchase rate[1] versus number of respondents for all three price levels.

[1] Δ Purchase rate = non-Markovian Purchase Rate - Markovian Purchase Rate

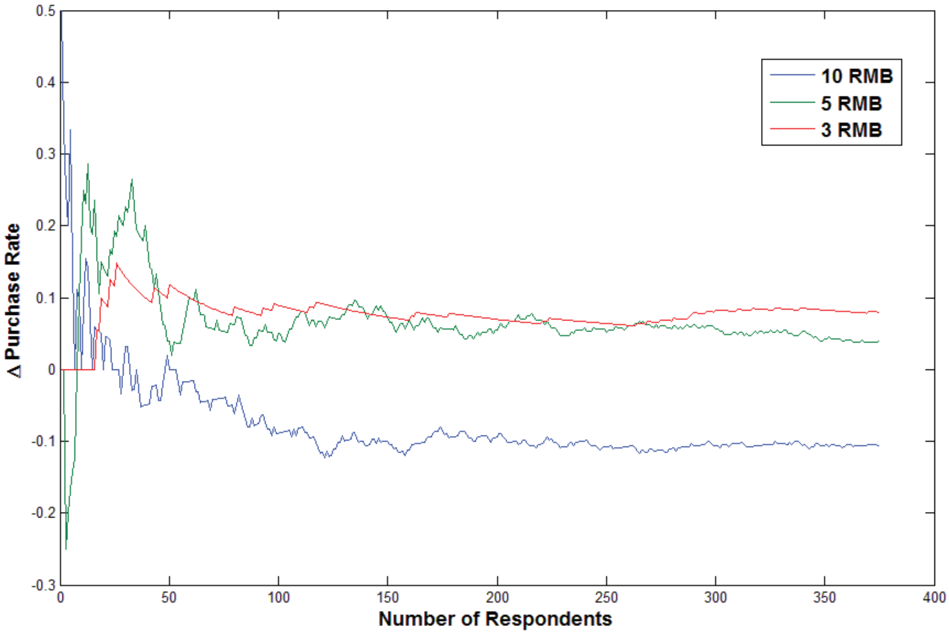


Figure 2.2: Difference between Markovian and non-Markovian purchase rates.

Significant differences are found between the Markovian purchase rate P_0 and the non-Markovian purchase rate P_∞. For example, the purchase rate of the Markovian process of ¥10 converges approximately to the value 32.2%, while the non-Markovian purchase percentage at this price level has its equilibrium at 21.6%. The difference between the two is around 10%, which is nearly 50% of the non-Markovian equilibrium value. When the price is set as ¥5, the gap between Markovian and non-Markovian purchase rate is 4%, which is the smallest among the three; meanwhile,¥3 produces a gap around 8%.

Chapter 3

Models

Although non-Markovian surveys capture more information about respondent behaviors, they are usually difficult and sometimes even infeasible to conduct. Thus, it's appealing if the non-Markovian result can be predicted from Markovian data, which is usually easier to collect. In this chapter, we will build models to achieve this goal; before the actual construction process, we first introduce the model settings, including the definition of the basic parameter and terminology.

3.1 Model Settings, Basic Parameters and Terminology

To model the non-Markovian stochastic process of respondents' decision making, we first use the binary random variable B_i to denote the final decision of the ith respondent in the survey with the subscript "M" for "*Markovian*" and "NM" for "*non-Markovian*". Let B_i in both cases satisfy the following:

$$B_i = \begin{cases} 1 & \text{if the respondent decides to buy} \\ 0 & \text{if the respondent decides not to buy} \end{cases}$$

Then, the result of the survey will form a chain filled with "1" and "0". Assume all respondents' purchase probabilities in the Markov process at a given price follow an independent identical binary distribution[1]. Then

$$B_{i,M} = \begin{cases} 1 & \text{with probability } P_0 \\ 0 & \text{with probability } 1 - P_0 \end{cases}$$

[1]This is also referred to as respondents' innate purchase probability.

By maximum likelihood estimation(see e.g. (30)), we have:

$$P_0 = \langle \frac{1}{N} \sum_{i=1}^{N} B_{i,M} \rangle$$

If we examine the definition of P_0 more carefully, we'll find it shares the same meaning as the sample's final purchase rate. As shown in the last chapter, the final purchase rate in each sample converges, with each value read from data as follow:

$$\begin{cases} P_0(B_{i,M} = 1|price = ¥10) = 0.322 \\ P_0(B_{i,M} = 1|price = ¥5) = 0.548 \\ P_0(B_{i,M} = 1|price = ¥3) = 0.781 \end{cases}$$

The non-Markovian purchase probability P_i for the i^{th} respondent should also depend on decisions made by previous N_r respondents which form his/her memory. Assume the decision of each respondent in the memory has the same weight $1/N_r$; then the overall effect on the last respondent should relates to the purchase percentage r_i in the memory; by using previous notations, it can be defined as the following:

Definition 3.1.1. Let r_i be the purchase percentage in the memory which is received by the he i^{th} respondent.

$$r_i = \frac{1}{N_r} \sum_{j=1}^{N_r} B_{i-j,NM} \tag{3.1}$$

To keep consistent with the survey, we set $B_{1,NM} = B_{1,M}$ and the superscript on $\sum$ changes to $i-1$ for the case $i \leq N_r$.

In the rest of this chapter, we'll propose several models which can provide us with P_i if given P_0 and r_i.

3.2 Ising Model

3.2.1 Original Ising Model

The original Ising Model was first proposed by E. Ising(26) to describe behaviors of a ferromagnetic system. In his model, a discrete random variable called spin was introduced which can take the value of either +1 or -1. Spins, denoted as S, are arranged in a chain lattice and interact only with its nearest neighbors.

Although spin is an abstract concept in Quantum Mechanics, one can simply consider it as a little magnet which can be arranged either aligned or anti-aligned. The energy of system can also be defined as:

$$E = -\sum_{\langle i,j \rangle} J_{ij} S_i S_j \tag{3.2}$$

where $\langle i,j \rangle$ means sum over all spins that are connected and the positive coupling coefficient J_{ij} describes the strength of the interaction between sites i and j.

If we go even further to consider an Ising system in an external magnetic field h_0, the energy now becomes:

$$E = -\sum_{\langle i,j \rangle} J_{ij} S_i S_j - h_0 \sum_j S_j \tag{3.3}$$

Proposition 3.2.1. *The configuration of site j depends only on its local effective field h_j.*

Proof. If we divide the system into two parts, site j and the rest, the total energy of the system can be rewritten as:

$$E = -\left(\sum_{\langle i',j' \rangle} J_{i'j'} S_{i'} S_{j'} + h_0 \sum_{j'} S_{j'} \right) - \left(\sum_i J_{ij} S_i + h_0 \right) \cdot S_j \tag{3.4}$$

$\langle i',j' \rangle$ means sum over all sites i' and j' that are connected with each other but different from j. Index i in the second summation is all sites connected with site j.

Definition 3.2.2. Define energy of the whole system excluding site j as:

$$E_{r,j} = -\left(\sum_{\langle i',j' \rangle} J_{i'j'} S_{i'} S_{j'} + h_0 \sum_{j'} S_{j'} \right)$$

Definition 3.2.3. Define the local effective field[1] at site j as

$$h_j = \sum_i J_{ij} S_i + h_0 \tag{3.5}$$

Then the expression of energy in Eq.(3.3) will become:

$$E = E_{r,j} - h_j \cdot S_j \tag{3.6}$$

[1] Def.(3.2.3) actually tells us the effective magnetic field is the superposition of two parts. The first term stands for the internal magnetic field created by interactions between different sites; while the second term is the external field at that point.

Based on the ensemble theory(see e.g.(25; 29)) in statistical physics, the probability for the spin of cite j taking the value 1 will be[1]:

$$P(S_j = 1) = \frac{e^{-\beta E_{r,j} + \beta h_j}}{e^{-\beta E_{r,j} + \beta h_j} + e^{-\beta E_{r,j} - \beta h_j}} \tag{3.7}$$

Cancel the factor $e^{-\beta E_{r,j}}$ in both denominator and numerator, and we'll have:

$$P(S_j = 1) = \frac{e^{\beta h_j}}{e^{\beta h_j} + e^{-\beta h_j}} \tag{3.8}$$

Similarly, we could also prove the probability of $S_j = -1$ also depends only on its local effective field, which reads,

$$P(S_j = -1) = \frac{e^{-\beta h_j}}{e^{\beta h_j} + e^{-\beta h_j}} \tag{3.9}$$

□

3.2.2 From spin-spin coupling to the generalized Non-Markovian Model

If we assume that customers have no bias to purchase, it can be read from Fig.(2.1) that if $P_0 > \frac{1}{2}$, social cues tended to enhance customer's probability of purchase, while for $P_0 < \frac{1}{2}$, social cues diminished the likelihood of purchase. For example, in the ¥10 case, social cues increased the $P_0 = 0.77$ by 8% to $P_\infty = 0.85$; while for $P_0 = 0.55$ hardly any change was observed with $P_\infty = 0.57$ and finally for $P_0 = 0.32$, the social cues reduced the fraction of people responding positively to $P_\infty = 0.21$.

An Ising system is actually a system with spin-spin coupling interactions. Like the social cue effects above, spins with interactions in an Ising model are more likely to take the same configuration and form uniform clusters in their dynamic processes. Thus, the definition of spin here bears an analogy to consumer's decision variable in the potential non-Markovian model.

The following simple transformation between Ising spin S_i and purchase random variable B_i gives us a better understanding of their similarity:

$$S_i = 2B_i - 1$$

By this transformation, spin $S_i = 1$ and -1 will correlate with purchase and not purchase decisions respectively. If most previous respondents chose to purchase, the i^{th} consumer will receive a r_i close to 1. From Eq.(3.8), he/she will be inclined to purchase and vice versa.

Besides the spin definition, the Ising model can be further modified to incorporate the non-Markovian effect.

[1]Here the parameter $\beta = \frac{1}{k_B T}$, where k_B is the Boltzmann constant and T is the temperature of the system.

Definition 3.2.4. Define the local effective field on the j^{th} consumer in the same manner as the effective magnetic field on spin j in Eq.(3.5):

$$h_j = \frac{J}{N_r} \sum_{i=1}^{N_r} S_{j-i} + h_0 \tag{3.10}$$

Using a similar decomposition as in Def.(3.2.3), h_j is made up of an internal field $\frac{J}{N_r} \sum_{i=1}^{N_r} S_{j-i}$ and an external field h_0.

Definition 3.2.5. Define the generalized energy of the j^{th} consumer's choice in the same way as the energy of a magnet in a magnetic field.

$$E_j = -h_j \cdot S_j \tag{3.11}$$

In Ising model, flip spins to a higher energy configuration, e.g., flipping S_j from -1 to +1, have to overcome an energy barrier which could only happen under some certain probability as in Eq.(3.8). Similarly, the energy in Def.(3.2.5) can be understood as a psychological barrier E_j for the j^{th} consumer to overcome to make positive decision. Def.(3.2.4) and Def.(3.2.5) are the generalized local effective field and energy that include non-Markovian effects. Under this definition, Markovian and non-Markovian effects are separate and have counterparts in physics respectively. The internal field contains the non-Markovian part while the external field reflects the Markovian effect. In the case when social cues are absent, only the external component h_0 resides and determines the consumer's choice; when respondents received both types of information, the term h_j together gives the purchase probability for the j^{th} customer.

As in Eq.(3.8), the Markovian purchase probability for the j^{th} customer is given by

$$P_{0,j}(S_j = 1) = \frac{e^{\beta h_0}}{e^{\beta h_0} + e^{-\beta h_0}}; \tag{3.12}$$

while the non-Markovian probability is equal to

$$\begin{aligned} P(S_j = 1) \equiv P_j &= \frac{e^{\beta h_j}}{e^{\beta h_j} + e^{-\beta h_j}}, \\ P(S_j = -1) \equiv 1 - P_j &= \frac{e^{-\beta h_j}}{e^{\beta h_j} + e^{-\beta h_j}} \end{aligned} \tag{3.13}$$

As shown in Eq.(3.13), a consumer's purchase probability depends on the factor βh_j. The combination of the parameter J and the temperature parameter T gives an effective social cue strength, which indicates how seriously a consumer will take into account others' decisions. A consumer is more likely to follow the majority of others' decisions when this product is larger and vice versa.

3.3 Fermi Model

Another potential model has a similar functional form as the Fermi distribution in statistical physics. Recall the Ising model in previous sections, after receiving information r_i, consumer i will compare it with the benchmark point 50% to make his/her decision; however, in the Fermi model, this 50% is replaced by consumer's innate purchase probability P_0. We assume in the Fermi model that a real purchase percentage which exceeds consumer's innate probability P_0 will increase their likelihood of purchasing, while a r below P_0 will lower the respondent's final purchase probability and a real purchase percentage equal to P_0 will have no effects on a respondent's final decision.

Let the non-Markovian purchase probability of i^{th} consumer satisfy the Fermi distribution:

$$P_i = \frac{\left(\frac{P_0}{1-P_0}\right) e^{\beta(r_i - P_0)}}{1 + \left(\frac{P_0}{1-P_0}\right) e^{\beta(r_i - P_0)}} \tag{3.14}$$

Proposition 3.3.1. *If the purchase probability for the ith customer satisfies the Fermi distribution, the purchase probability P_i should have the following properties:*

$$\begin{cases} P_i > P_0 & \text{if } r_i > P_0 \\ P_i = P_0 & \text{if } r_i = P_0 \\ P_i < P_0 & \text{if } r_i < P_0 \end{cases} \tag{3.15}$$

Proof. The proof can be completed by showing the monotonicity property of the Fermi function. □

For the nontrivial case that $\beta > 0$, plotting P_i versus r_i given P_0 fixed gives an "S-shape" curve. The "temperature-like" parameter β, which stands for the strength of social cues, determines the skewness of the curve. The following extreme case will help us understand this point.

The first is $\beta = 0$, which means consumer's decision isn't affected by social cues. At this moment the exponential terms vanish and the Fermi probability reduces to the Markovian case:

$$P_i = P_0.$$

In this case the curve of function P_i reduces to a horizontal line without skewness.

If we consider another extreme, $\beta \to \infty$, then the consumer will simply follow how most others decide.

$$P_i = \begin{cases} 1 & \text{if } r_i > P_0 \\ 0 & \text{if } r_i < P_0, \end{cases} \tag{3.16}$$

At this time, P_i becomes a step-like function with an infinite slope at the point $r_i = P_0$. By P_i's continuity, for a finite real β, P_i will have a "S-shape" curve and in principle the model parameter β could be determined by fitting to real experimental data.

Chapter 4

Main Results from Modeling

In this chapter, we present the main results of the models proposed in Chapter 3, including the numerical results of the Ising and Fermi model, analytical result of the Ising model, which also includes its approximation solution.

4.1 Ising Model

4.1.1 Analytical Solution, Central Limit Approximating Approach

Taking the Ising Model, we'll try to draw some conclusions about the relations between variables P_∞, P_0, J and N_r.

From Eq.(3.13), the actual purchase probability for the n^{th} consumer should be

$$P_n = \langle \frac{\exp h_n(S_{n-1}, \ldots, S_{n-N_r})}{[\exp h_n(S_{n-1}, \ldots, S_{n-N_r}) + \exp -h_n(S_{n-1}, \ldots, S_{n-N_r})]} \rangle, \tag{4.1}$$

The $\langle\rangle$ represents an average of all possible 2^{N_r} outcomes of the sample configurations. For a small sample memory, an average over a binomial distribution is required. As N_r increases, tails of the probability distribution will decrease rapidly and all possible configurations will centralize around its expectation value. In this case, the central limit theorem could be applied and a Gaussian distribution will be a good approximation.

For simplicity, define the random variable $\xi_i = (1/N_r)\sum_{j=1}^{N_r} S_{i-j}$, then rewrite the social cue field in terms of ξ_i as $J\xi_i$. The average of this variable $\langle\xi_i\rangle$ and its variance $\sigma_i^2 = \langle \xi_i^2 \rangle - \langle \xi_i \rangle^2$ can be calculated as

$$\begin{aligned} \langle \xi_i \rangle &= \left[(2/N_r) \sum_{j=1}^{N_r} P_{i-j} \right] - 1 \\ \sigma_{\xi_i}^2 &= (4/N_r^2) \sum_{j=1}^{N_r} P_{i-j}(1 - P_{i-j}). \end{aligned} \tag{4.2}$$

Using the Gaussian approximation for ξ_n, with parameters given in Eq.(4.2), gives for the probability distribution of ξ_0:

$$P_{Gauss,i}(\xi_i) = \frac{1}{\sqrt{2\pi\sigma_{\xi_i}^2}} \exp\left[-\frac{(\xi_i - \langle \xi_i \rangle)^2}{(2\sigma_{\xi_i}^2)} \right] \tag{4.3}$$

If we now replace the R.H.S of Eq.(4.1) by using the definition of expectation, and use the probability distribution given by Eq.(4.3), we have:

$$P_i = \int_{-\infty}^{\infty} d\xi_i P_{Gauss,i}(\xi_i) \frac{\exp(h_0 + J\xi_i)}{[\exp(h_0 + J\xi_i) + \exp -(h_0 + J\xi_i)]}. \tag{4.4}$$

The dynamical process given by Eq.(4.4) can only be solved numerically. However, in the case of large i, which stands for the asymptotic purchase probability at equilibrium, we can simplify further.

In the limit as $i \to \infty$, Eq.(4.2) becomes:

$$\begin{aligned} \langle \xi_\infty \rangle &= 2P_\infty - 1 \\ \sigma_\infty^2 &= (4/N_r)P_\infty(1 - P_\infty). \end{aligned} \tag{4.5}$$

and Eq.(4.4) becomes a fixed point equation

$$2P_\infty - 1 = (\frac{1}{\sqrt{2\pi}}) \int_{-\infty}^{\infty} dt \exp\left(-t^2/2\right) \tanh\left[h_0 + J(2P_\infty - 1) + (2J\sqrt{P_\infty(1 - P_\infty)/N_r}t\right]. \tag{4.6}$$

In principle we can now solve these nonlinear self-consistent equations numerically, but it proves more insightful to expand Eq.(4.6) to the second order in J and derive a quadratic equation which can be solved explicitly. Introducing the notation $x_0 = 2P_0 - 1$ and $x_\infty = 2P_\infty - 1$, a quadratic equation for x_∞ of the form can be found

$$[J^2 x_0(1 - x_0^2)(1 - 1/N_r)]x_\infty^2 + [1 - (1 - x_0^2)J]x_\infty - x_0[1 - (1 - x_0^2)J^2/N_r] = 0 \tag{4.7}$$

Containing all information on how P_∞ depends on J, N_r, and P_0. Details of the result will be shown in Sec.(4.1.3)

4.1.2 Analytical Solution, Master Equation Approach

In this section, we present the "master equations" for the Ising model and discuss a possible algorithm to solve these equations. The binary random variable B_i in this section has the same meaning as the B_i introduced in Sec.(3.1). In addition, we will also have the following variables:

Definition 4.1.1. Define n as the number of customers that responded and k as the number of customers who decided to BUY.

Divide the n customers into two parts, the first $n - N_r$ customers and the last N_r customers, as shown in Fig.(4.1). Then we have the definition of "q-function".

Definition 4.1.2. q-function ($q_n(k, B_1, B_2, B_3, \ldots, B_{N_r})$) is the probability of finding a n-customer sequence with exactly k 1-value customers in the first $n - N_r$ customer queue, and with last N_r customers with decisions of exactly $(B_1, B_2, B_3 \ldots B_{N_r})$.

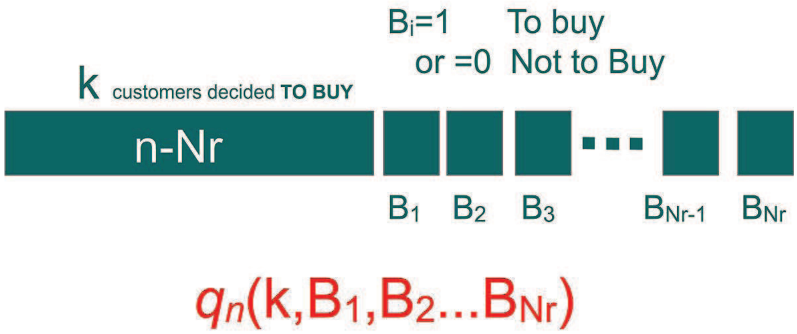


Figure 4.1: Definition of q function.

Both buy and not buy decision in the binary probability distribution for the jth customer, we have:

$$P_j(h_0, r_j, B_j) = \left[\frac{e^{\beta h_j}}{e^{\beta h_j} + e^{-\beta h_j}}\right]^{B_j} \left[\frac{e^{-\beta h_j}}{e^{\beta h_j} + e^{-\beta h_j}}\right]^{1-B_j} \tag{4.8}$$

where h_j has the same meaning as in Def.(3.2.5).

Then the "master equation" can be expressed as:

$$\begin{aligned} q_n\left(k, B_1, B_2, \ldots B_{N_r}\right) = {} & q_{n-1}\left(k, 0, B_1, B_2, \ldots B_{N_r-1}\right) P_{N_r}\left(h_0, \frac{1}{N_r}\sum_{j=1}^{N_r-1} B_j, B_{N_r}\right) \\ & + q_{n-1}\left(k-1, 1, B_1, B_2, \ldots B_{N_r-1}\right) P_{N_r}\left(h_0, \frac{1}{N_r}\left(1 + \sum_{j=1}^{N_r-1} B_j\right), B_{N_r}\right) \end{aligned} \tag{4.9}$$

For the $n \leq N_r$ case, we have the corresponding equation:

$$q_n(B_1, B_2, \ldots B_n) = q_{n-1}(B_1, B_2, \ldots B_{n-1}) P_n \left(h_0, \frac{1}{n-1} \sum_{j=1}^{n-1} B_j, B_n \right) \tag{4.10}$$

with the initial condition:

$$q_1(B_1) = P_0^{B_1} (1 - P_0)^{1-B_1} \tag{4.11}$$

Eq.(4.8), Eq.(4.9), Eq.(4.10) and Eq.(4.11) form the fundamental law of this non-Markovian dynamics, based on which we can design an algorithm to calculate the q function.

The q function $q_n(k, B_1, B_2, B_3, \ldots, B_{N_r})$ contains all information of the dynamic process. Thus, the probability distribution of the purchase rate $P(k, n)$ at any total respondents n could be computed as:

$$P(k, n) = \sum_{k' + \sum_i B_i = k} q_n(k', B_1, B_2, \ldots B_{N_r}) \tag{4.12}$$

Which gives the average purchase rate

$$\langle r \rangle_n = \sum_{k=0}^{n} P(k, n) \frac{k}{n}, \tag{4.13}$$

and variance

$$\langle r^2 \rangle - \langle r \rangle^2 = \sum_{k=0}^{n} P(k, n) \frac{k^2}{n^2} - \left(\sum_{k=0}^{n} P(k, n) \frac{k}{n} \right)^2. \tag{4.14}$$

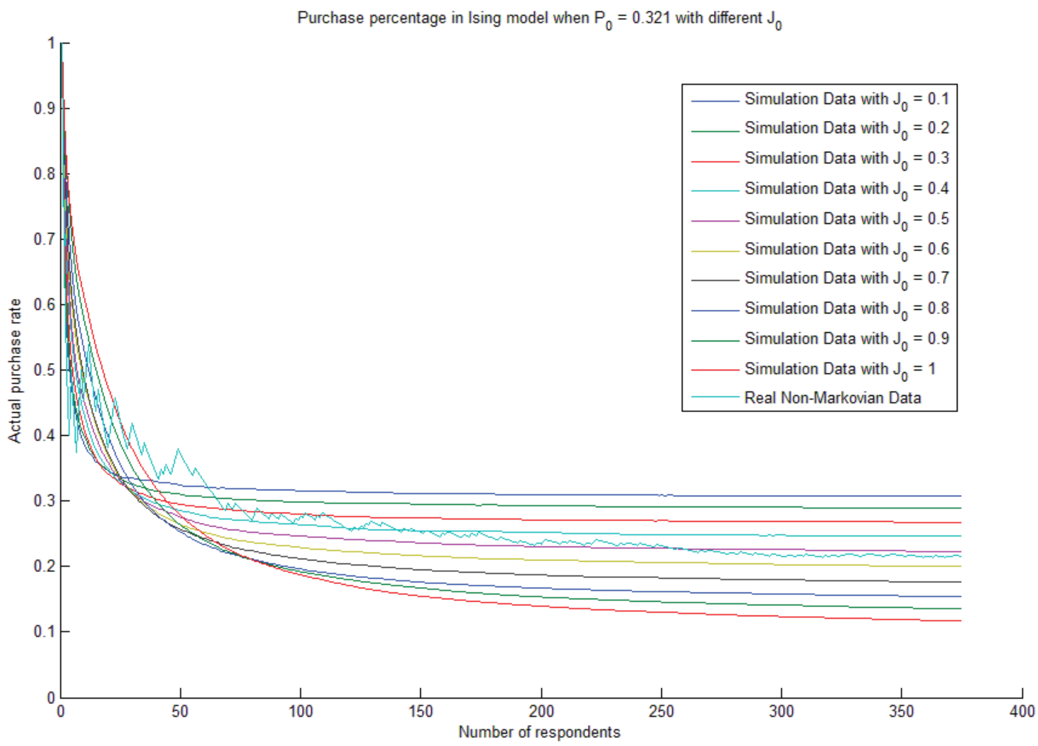
4.1.3 Results and Discussion

In this section, we

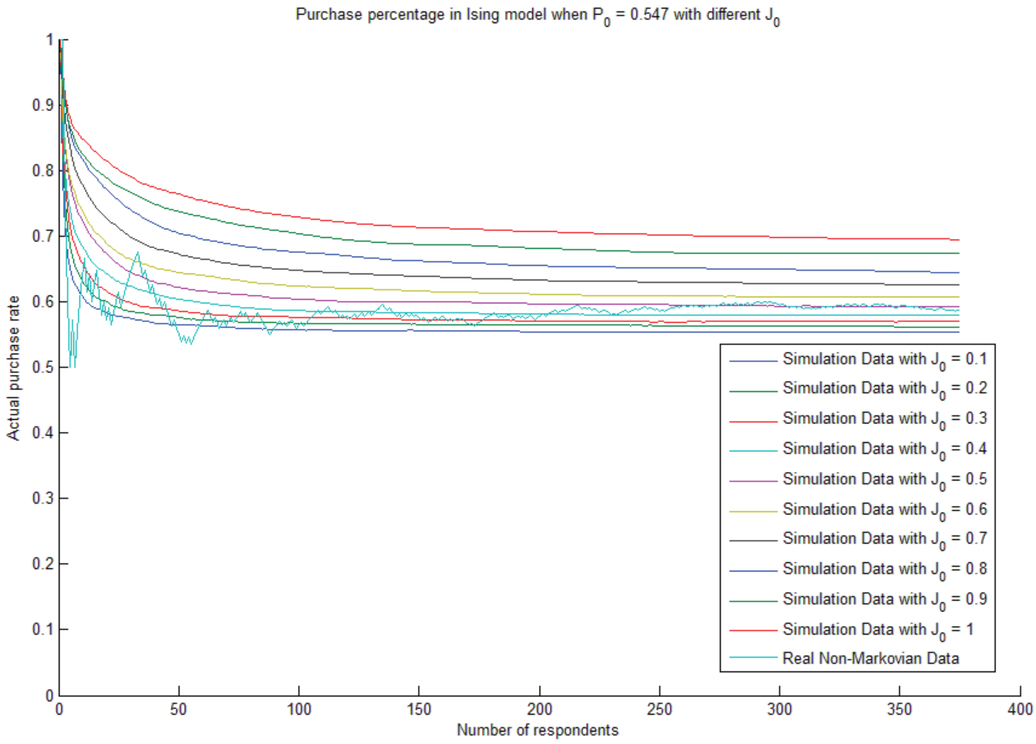
- Verify "Ising" model by comparing simulation results with the real survey data.
- Plot the fixed point relation of "Ising" obtained from the central limit approximation.
- Plot an example distribution derived from the "master equation".

4.1.3.1 Kinetic Monte-Carlo Simulation(KMC)

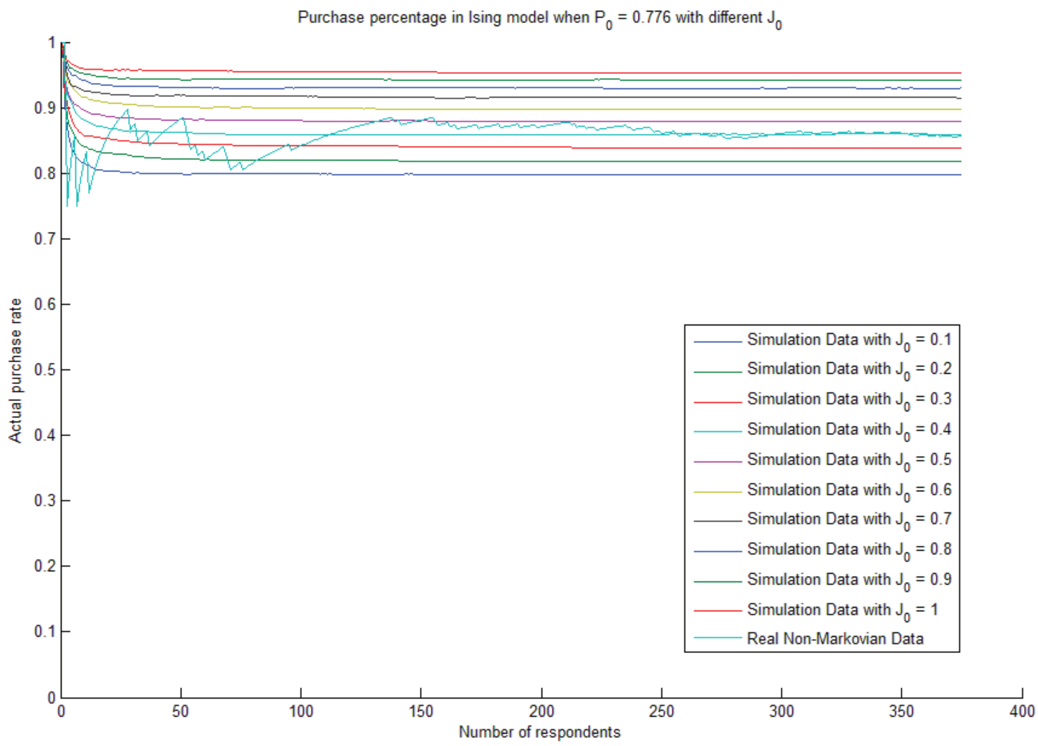
In previous chapters, we constructed the Ising model and introduced several important concepts such as generalized internal field, effective local field and social cue parameter, but in order to conduct a simulation, we have to first determine the value of the model parameter h_0 and J. Although h_0 can be determined by using Eq.(3.12), the other base parameter J can not be derived from the first principles. Leaving this as an open question which will be discussed in Chapter 5, we here present sets of data obtained via KMC simulation by simply choosing different Js. In Fig.(4.2), the values of J are chosen from 0.1 to 1 in increments of 0.1.



(a) $P_0 = 0.321$


Purchase percentage in Ising model when P0 = 0.547 with different J0
Actual purchase rate
Number of respondents
Simulation Data with J0 = 0.1
Simulation Data with J0 = 0.2
Simulation Data with J0 = 0.3
Simulation Data with J0 = 0.4
Simulation Data with J0 = 0.5
Simulation Data with J0 = 0.6
Simulation Data with J0 = 0.7
Simulation Data with J0 = 0.8
Simulation Data with J0 = 0.9
Simulation Data with J0 = 1
Real Non-Markovian Data

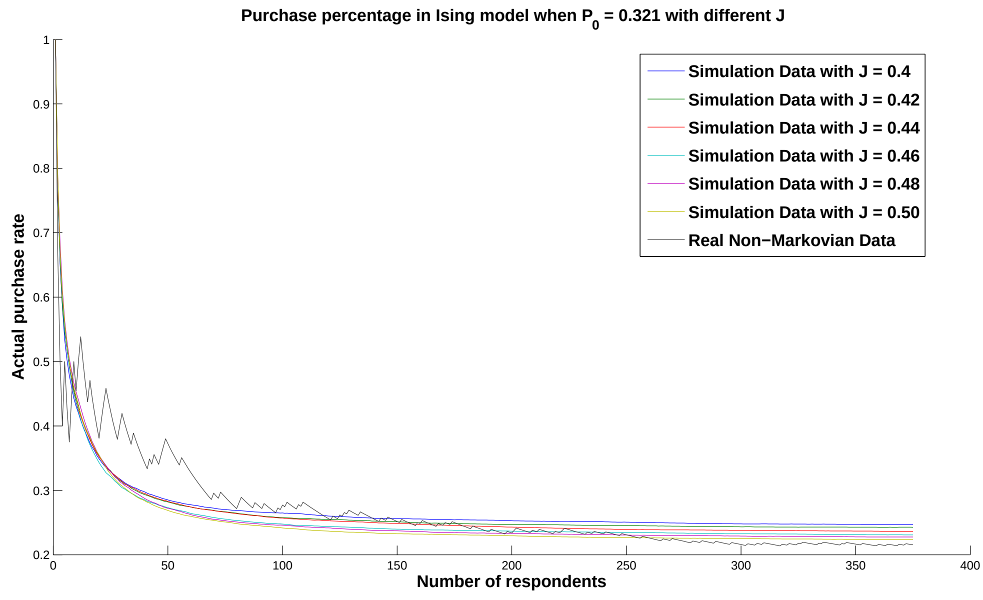
(b) $P_0 = 0.547$



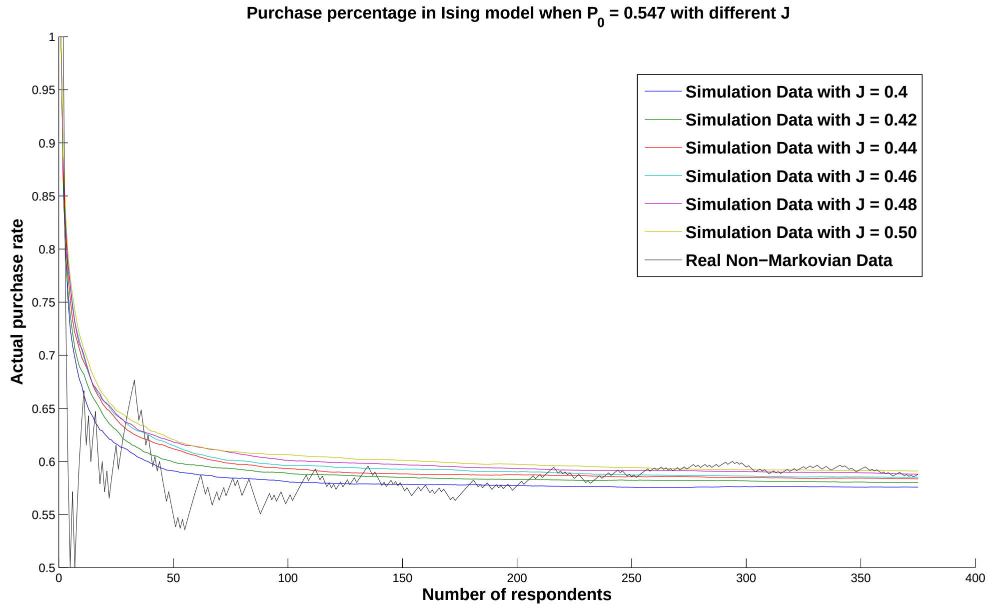
(c) $P_0 = 0.776$

Figure 4.2: Purchase rate in the Ising model at different P_0 with different J.

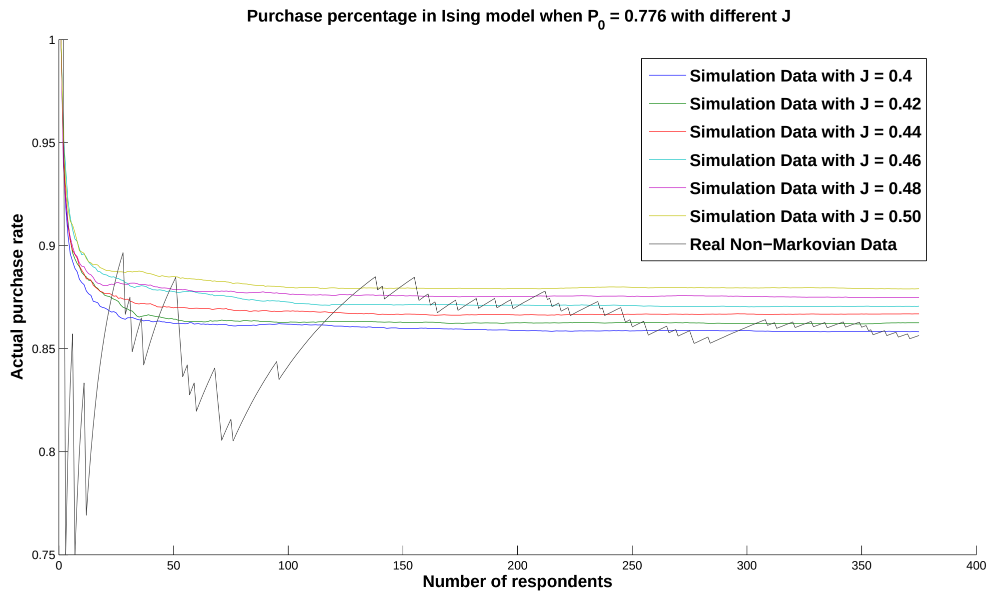
From the Fig.(4.2), it's easy to see there should be a J between [0.4, 0.5] that satisfies: If this J is chosen and fixed in the model, KMC method will give results that coincide with the real survey data. To have a better view, we then simulated sets of data at P_0s with different Js in the interval [0.4,0.5] in increments of 0.02, which is shown in Fig.(4.3).



(a) $P_0 = 0.321$



(b) $P_0 = 0.547$



(c) $P_0 = 0.776$

Figure 4.3: Purchase rate in the Ising model at different P_0 with $J \in [0.4, 0.5]$.

In Fig.(4.3), choose $J \simeq 0.42$ and, given P_0 from the Markovian data, the Ising Model will give a very good predicted P_∞ value coinciding with all three sets of data from the real survey. Therefore, the meaning of the parameter J can now be generalized as the social cue strength of the sample group. Under this definition, J remains a constant for a certain sample group and a particular consumption good, which is a significant finding of this research. In other words, if we can measure J, the non-Markovian P_∞ for a product can be derived or simulated from the Markovian result P_0 without much work or expense.

4.1.3.2 Exact solution

Central limit approximation and the master equation approach attempt to explain the non-Markovian process from two different angles. We combine results from those two approaches and put them together in this section. By the end of this section, we'll show the equivalence of the two solutions.

We solve the fix point relation in Eq.(4.7), and plot the result via MATLAB. After transforming x_0 and x_∞ to the original definition of P_0 and P_∞, a 3-D figure of P_∞ versus P_0 and J are presented in Fig.(4.8) give $N_r = 10$.

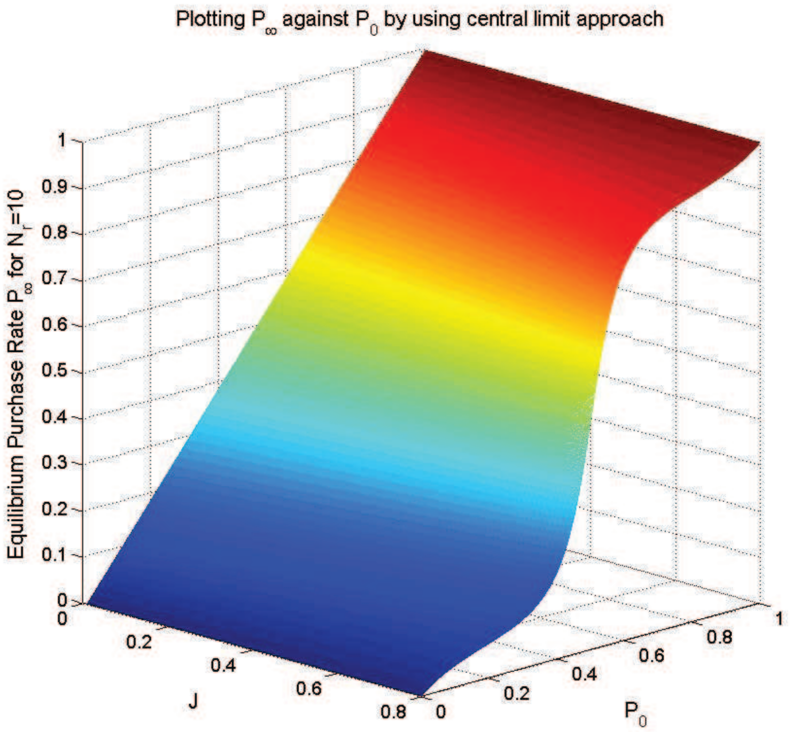


Figure 4.4: Plotting of P_∞ against P_0 and J following the fixed point Eq. (4.7).

We choose the interval of J as $[0, 0.8]$, which includes the optimal interval [0.4,0.5] discussed in the last section. From the figure, it's easy to see the straight line on the left stands for the Markovian case, in which $J = 0$. As J increases, the non-Markovian effect becomes more and more significant and reaches its maximum at the point $J = 0.8$. During this process, the curve of the equilibrium purchase rate P_∞ at a particular J gradually turns "S-shaped", which satisfies all properties of social cues in Sec.(4.1.1).

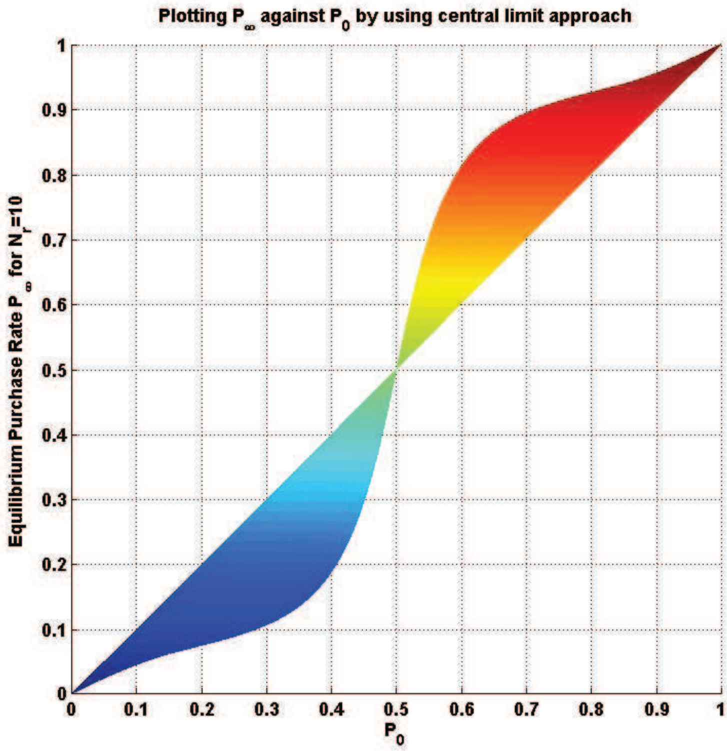


Figure 4.5: Side view of Fig.(4.4) for $N_r = 10$.

To have a better view of the non-Markovian effect, we take the cross section of Fig.(4.4) with a plane parallel to P_0 axis and plot P_∞ versus P_0 in Fig.(4.5). It's clear that the straight line with the slope 1 stands for the Markovian curve with $J = 0$, while the "S-shape" curve represents the curve with $J = 0.8$. The two curves intersect at the point $(0.5, 0.5)$, which means the social cues have no effect when the actual purchase probability is equal to the benchmark point 0.5.

Now let us come back to the solutions of this model by using the master equation approach. In Fig.(4.6)[1], we also plot the non-Markovian equilibrium purchase rate P_∞ versus P_0 and J.

[1]We solved the model and collected data by programming in FORTRAN.

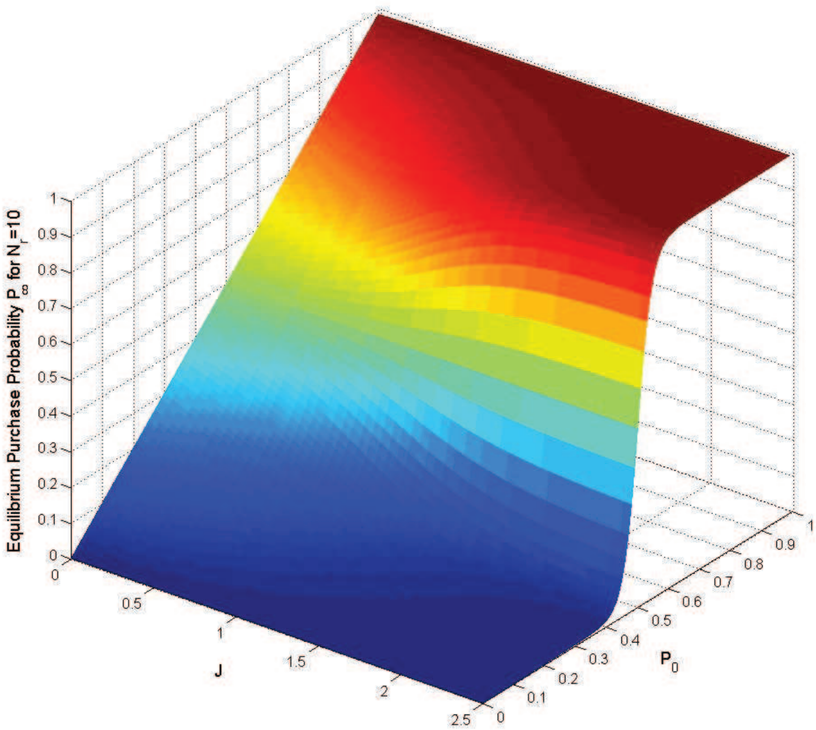


Figure 4.6: Plotting of P_∞ against P_0 and J by using Master Eq.(4.9) and Eq.(4.10)

Similar surface shape as in Fig.(4.4) has been discovered in Fig.(4.6). In Fig.(4.6), we also plotted P_∞ versus P_0 and J for the interval $J \in [0, 2.5]$. As before, in Fig.(4.7) we also plotted its side view as we did in Fig.(4.5). We also find a straight line with the slope 1 in this figure, which corresponds to the left $J = 0$ cross section line in Fig.(4.6). At the cross section at $J = 2.5$, the "S-shape" is clearer and gradually turns into a step function as J approaches ∞. The inclined lines in this figure represents points with same J, and the increment between two such lines is 0.05; while points on the vertical lines have the same P_0.

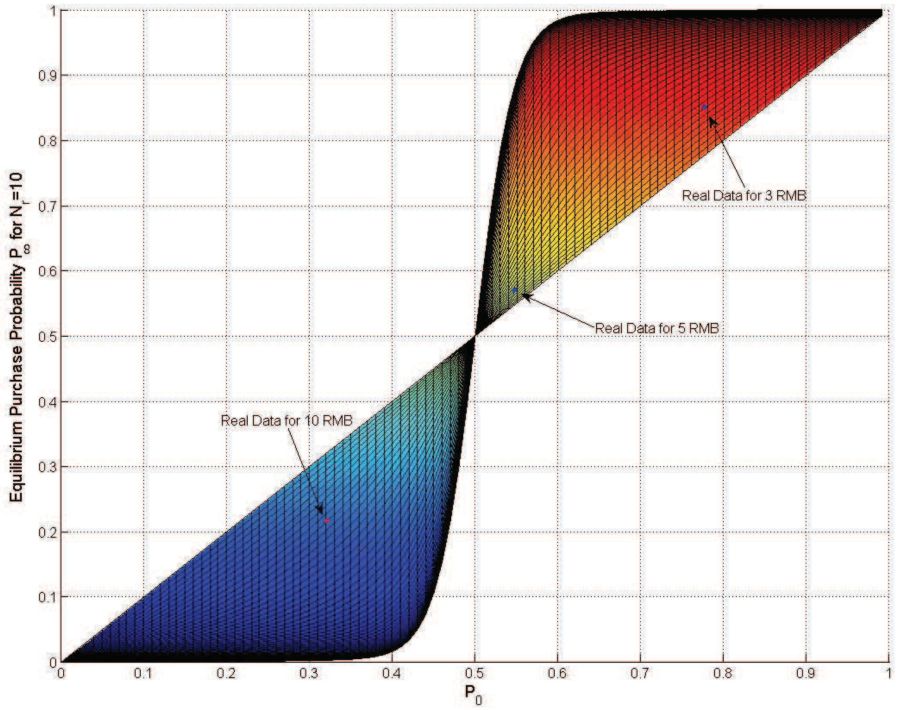


Figure 4.7: Side view of Fig.(4.4) for $N_r = 10$.

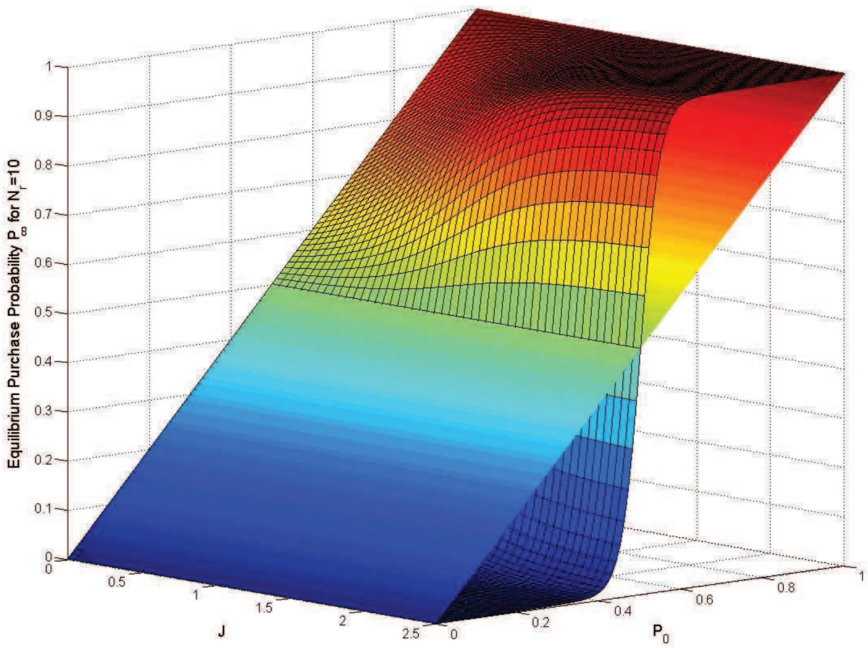


Figure 4.8: P_∞ against P_0 and J from the fixed point Eq. (4.7), here N_r is set to equal to 10.

The previous figures only show two extreme cases at $J = 0$ and the other end of its interval. We'll consider the non-Markovian effect at an arbitrary J in its interval in the next figure. Fig.(4.8) combines the solution to Ising model and the result of traditional Markov model. The surface with crossovers in Fig.(4.8) is the same surface shown in Fig.(4.6), while the inclined plane in the same figure represents the Markovian(MC) results. Fig.(4.9) gives the difference between the KMC and MC results versus P_0. Note that the agreement between the simulation and the experimental data for $J \simeq 0.42$ is very good.

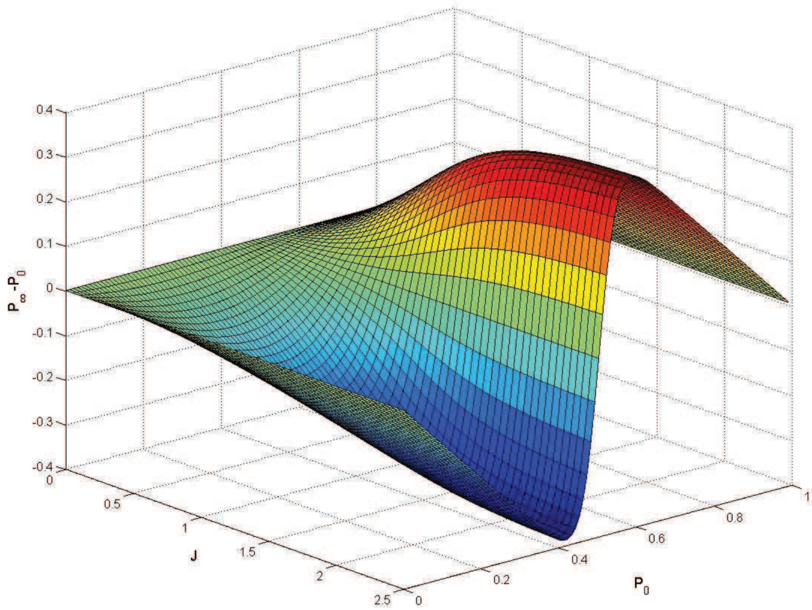


Figure 4.9: Difference between KMC and MC result versus J and P_0.

Besides all the equilibrium properties we discussed, the master equation provides unique information on the dynamic processes that the central limit approach could not. In Fig.(4.10) we show the Probability Distribution Function(PDF) of r at different Markovian P_0 by using the q function $q_n(k, B_1, B_2, B_3, \ldots B_{N_r})$. The parameter J is set to be equal to 0.5 and for each P_0, we conducted simulations for three survey lengths $N = 20, 100$ and 300 for each P_0. One could easily observe that the distribution spreads more widely in a small N case. In all three experiment lengths for which we conducted a simulation, the PDF is not symmetric except for the case $P_0 = 0.5$. The figure also shows for a very large N case, e.g. $N = 300$, PDF at each P_0 diminishes very quickly as r deviates from its expectation value, which also tells why the central limit approach works for our survey.

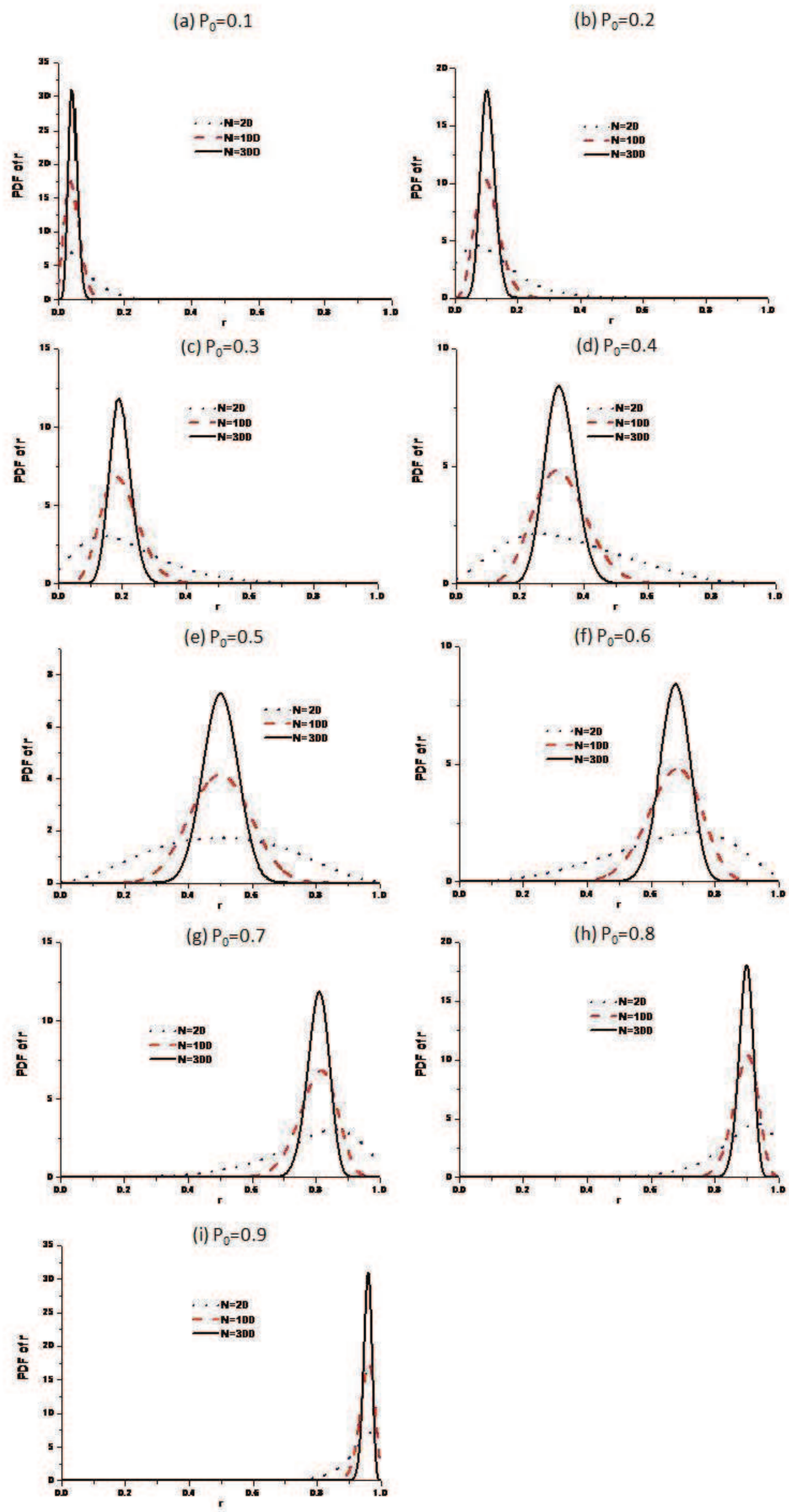
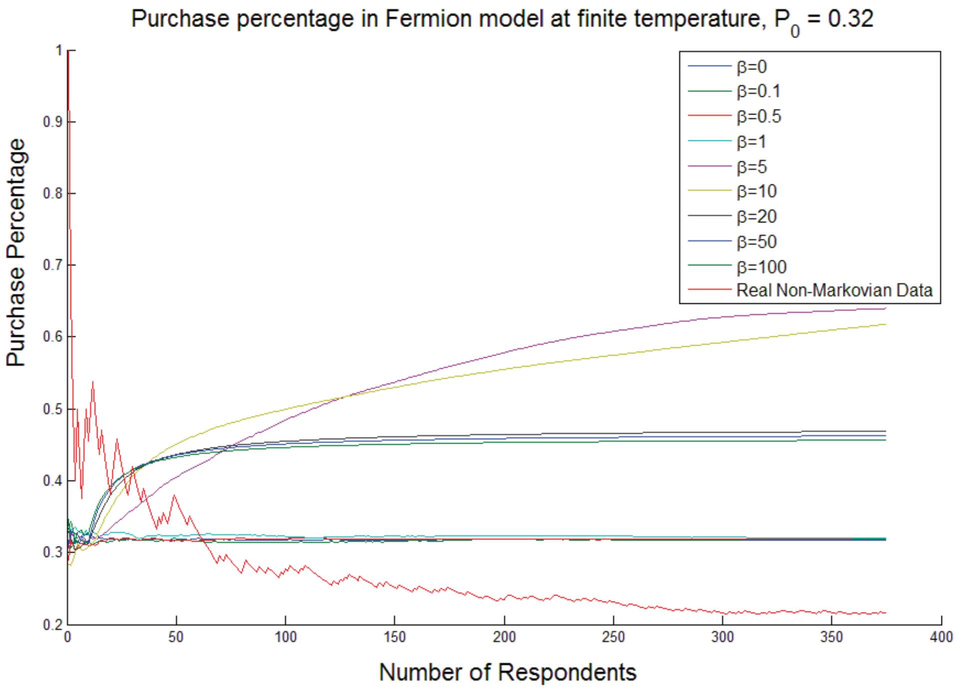


Figure 4.10: PDF of purchase rate r at different P_0, with $J = 0.5$, $N_r = 10$

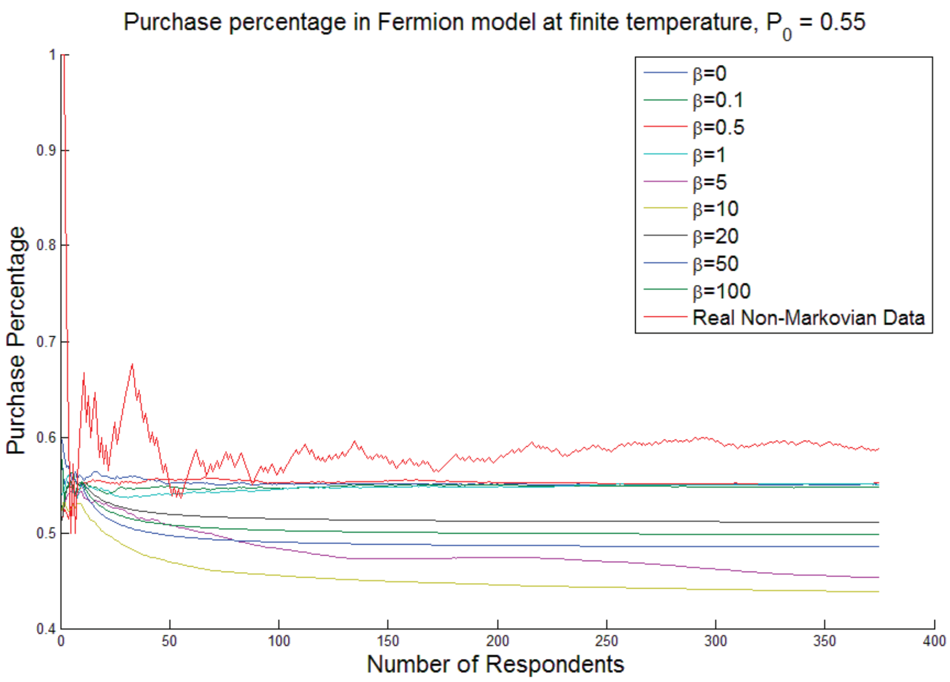
4.2 Fermi Model

4.2.1 Simulation Results and Discussion

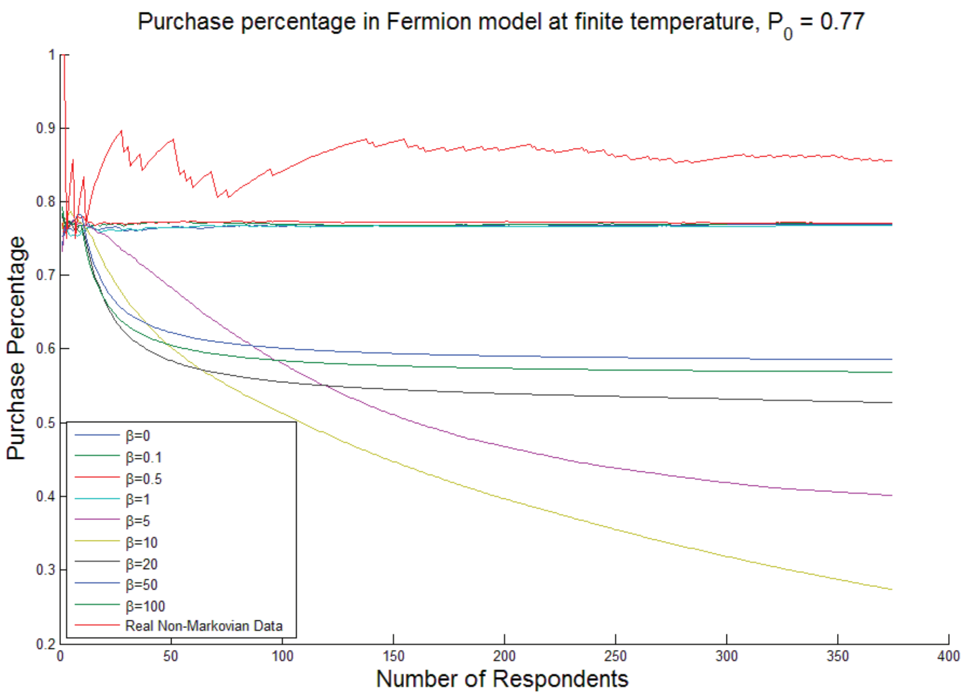
The Fermi model was solved mainly by using KMC simulation. In our simulation, we chose a set of different β at three Markovian probabilities $P_0 = 0.321, 0.547$ and 0.776. In Fig.(4.11), results are presented by showing the purchase percentage versus number of respondents.



(a) $P_0 = 0.321$



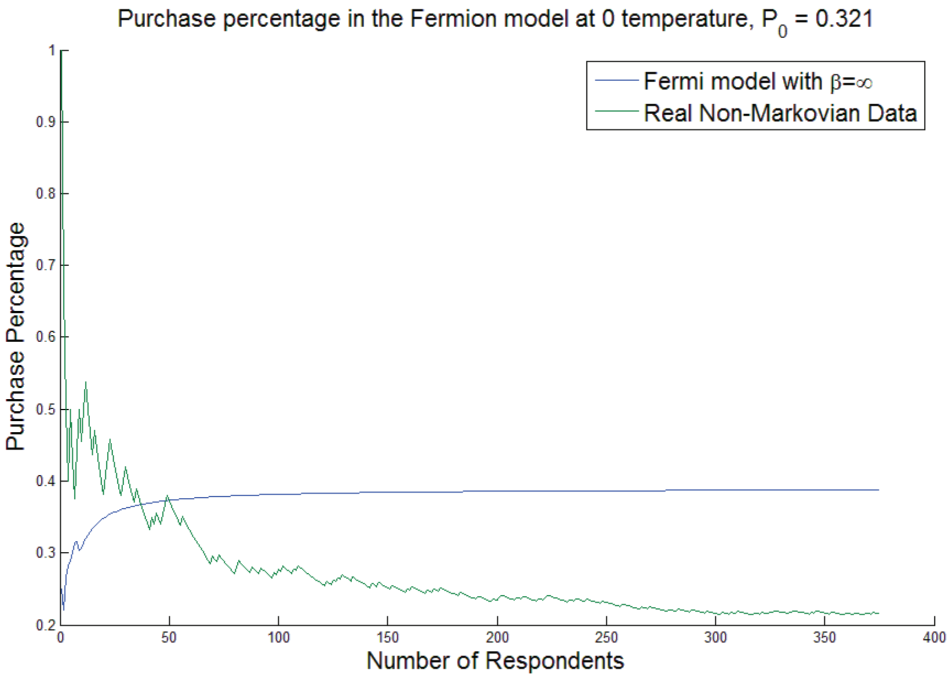
(b) $P_0 = 0.547$



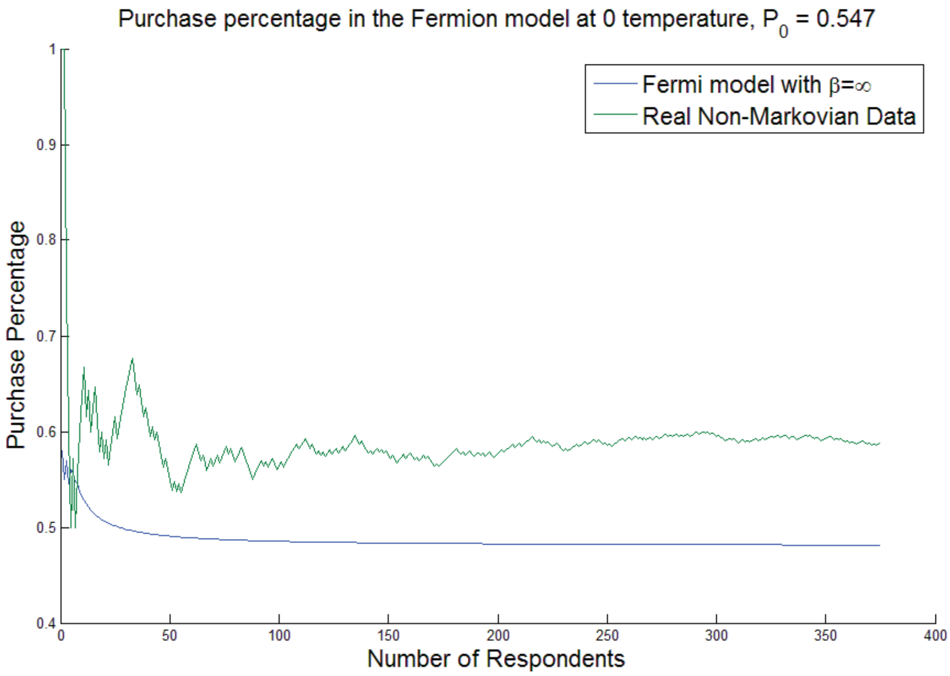
(c) $P_0 = 0.776$

Figure 4.11: Purchase percentage in the Fermi model with different temperatures parameter β.

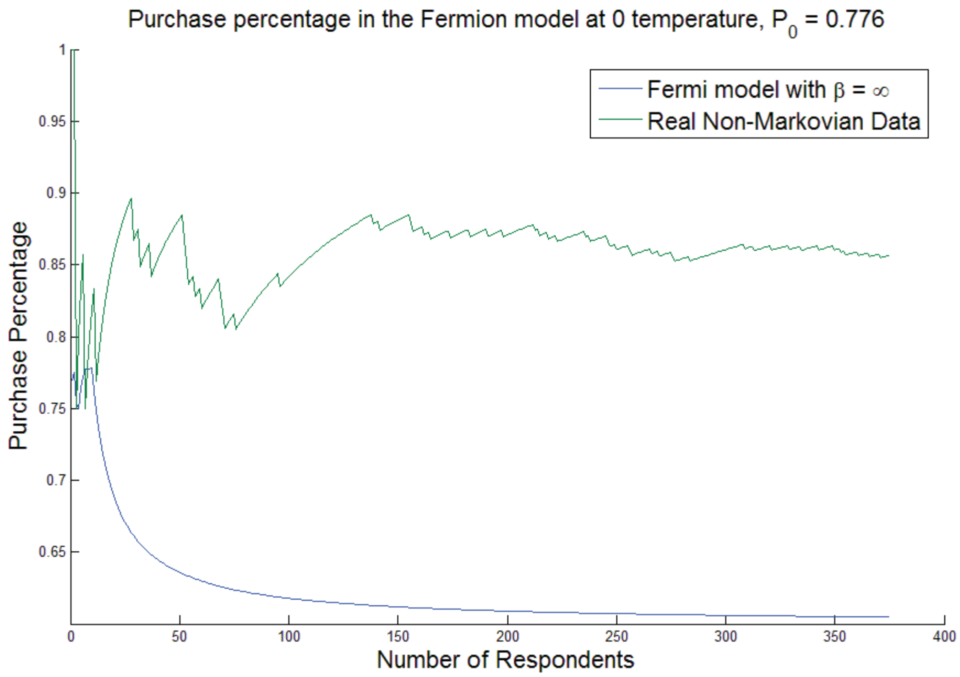
Comparing the simulation result with the real experiment data, one can easily see that $P_0 > 1/2$ will adversely affect P_∞; similarly, $P_0 < 1/2$ will enhance the positive decisions. At this point, we may ask where this pattern comes from and how it happens. To address the first question, we show the dynamic process of the model under the extreme case of Eq.(3.16). In Fig.(4.12), we plot this result for all three P_0.



(a) $P_0 = 0.321$



(b) $P_0 = 0.547$



(c) $P_0 = 0.776$

Figure 4.12: Purchase percentage in the Fermi model at 0 temperatures for different P_0.

As shown in the figure, patterns similar to those in Fig.(4.11) can be discovered. Thus we can conclude that the pattern occurs because of the social cue strength parameter β. In other words, the non-Markovian effect leads to this pattern. The answer to the 2nd question correlates with the asymmetry of the consumer's purchase rate r. Similar to Fig.(4.10), when $P_0 > 0.5$, the cumulative probability of getting an $r_i > P_0$ is less than the probability of $r_i < P_0$ and vice versa. Therefore, as we average over our simulation sample with its size tending to ∞, the final purchase rate r and P_0 will go in the opposite direction.

The Fermi model also tells us that the pattern of the final purchase rate is produced by fluctuations; initial conditions are very important in this model. Although a representative customer can have a continuous innate purchase probability, for any particular customer in the survey, the decision is not made by throwing a dice. When customers receive the information r_i and compare buy or not buy options, there's no innate probability P_0 for them to compare with. Since a possible model should fulfill both of the following conditions: the information will increase the likelihood if $r_i > 0.5$ and diminish the possibility if $r_i < 0.5$. The failure of the Fermi model help verify our previous assumption that the consumer's purchase decision represents the unbiased, neutral behavior. Therefore, the Ising model has also been justified by understanding the Fermi model.

Chapter 5

Conclusions and Future Extensions

In this project, different non-Markovian models have been studied and compared with real survey data. Unlike Markovian models in traditional marketing research, non-Markovian models are closer to reality but usually more difficult to solve. Main results were obtained through two models: Ising model and Fermi model. We derived possible purchase probability distributions and analytical solutions via two approaches for the former, while for the latter, numerical method was mainly used. From both analytical and numerical approaches, we concluded that Non-Markovian Ising model was appropriate to describe customers' decision making for a product without purchase bias. However, several questions still need to be answered. A number of possible extensions follows:

5.1 Parameter estimation, the methodology

Although everything looks neat and straightforward in the model, problems arise when we get our hands dirty with the real data. The result of MC method comes from averaging over a large sample. Therefore the external field h_0 in Eq.(3.12) should correspond to the P_0 over a large sample, which is not achievable in real cases. In our simulation, however, statistical errors always exist since we let P_0 be equal to the convergence value of one particular sample. This error grows even larger when we use this imprecise P_0 to determine J in the non-Markovian model. In fact, the current method for parameter determination gives us a J different from its real value. A correct method should determine the parameter J and h_0 simultaneously. Based on this idea, here we present a potential method called "Sum of Square of Residues Minimization"(SSRM).

Given each pair of J and h_0, we take the average over a large sampling simulation. The dynamic process gives us a set of data $(\overline{y}_1, \overline{y}_2, \ldots \overline{y}_n)$, and the square of residues is defined as:

$$SSR = \sum_{i=1}^{n} (y_i - \overline{y}_i)^2, \quad y_i \text{ is the real data at step i.} \tag{5.1}$$

If we choose the appropriate step length, we should be able to minimize SSR and then obtain its corresponding J and P_0. An algorithm such as the steepest descent method could be used to improve the efficiency of the program. Now the optimal values for J and h_0 we obtain are:

$$J = 0.359,$$

How to setup other standard of goodness of fit is left as an open question.

5.2 Unification of Ising and Fermi model

The fundamental difference between the Ising and the Fermi model is the choice of benchmark point in customers' decision making. An absolute point 0.5 is chosen in Ising while a point relative to individual customer P_0 dominates in the Fermi model. Besides this, they are quite similar, for example, both external and internal fields could be defined. Thus we propose the following hypothesis: the Ising and Fermi model could be unified into a larger model which could be used to describe a larger group of consumer behaviors in marketing science. The intuition behind the the models' unification comes from the fact that Eq.(**??**) can also be rewritten via :

$$e^{\alpha} = \frac{P_0}{1 - P_0}$$

If we let the β in Eq.(3.14) be equal to the parameter Φ in Eq.(**??**) and replace η_i by $r_i - P_0$. The Fermi model will have the same expression as Ising model.

Another intuition for supporting this idea comes from a possible connection between physics theory and marketing phenomena. The Fermi model we proposed in this thesis corresponds to Fermi distribution. In quantum mechanics, the Fermi distribution is derived from the Pauli exclusive principle, which prohibits identical particles from occupying the same energy level. A similar principle can be found in luxury marketing, for example, chasing limited edition goods released by tier 1 brands. In this case, purchasing such goods is biased, which makes the assumption in the Fermi model by reasonable. In contrast, Ising relates to the purchase of normal goods which doesn't have bias.

5.3 Correlation in Ising model

As shown in previous chapters, the Ising distribution can be derived from either ensemble theory in statistical physics or via solving differential equation in Sec.(**??**). However, the two approaches help us understand this result from two points of view. The latter focuses on a more microscopic dynamic process while the former introduces the definitions of generalized internal and external fields. From an Ising point of view, Eq.(3.10) shows an explicit correlation between the internal and the external parts. One may argue from the data that the correlation might not be as simple as Eq.(3.10) shows, but Eq.(3.10) can always been considered as the 1st order Taylor expansion of the explicit function. A complete correlation is needed if the current approximation ignores so much reality, but for this specific case, it's not necessary. Other interesting topics include:

- The effect of memory size N_r.
  The basic question we are interested in is how does the final purchase rate change if the memory size N_r changes? Will the memory size parameter N_r affect the final purchase rate or will it only influence its converge rate? Another question is whether there would be an effect if N_r is not a fixed number, but a quantity increasing during the evolution. For example, what if the ith customer knows all information from the first customer? Will the purchase rate converge to some non-zero value in this case?

- The effect of fake information
  In reality, there exists the possibility that merchants choose to provide customers fake information for some reason. For example, TV shopping programs always give audiences the illusion that other customers are more than willing to make a purchase. Will such fake information affect customers' final decision? If so, is this effect positive or negative? And how large is the effect?

Actually in the real world, interesting topic including both points exists. Here we give the online shopping website "Amazon" as an example. As Fig.(5.1) shows, viewers will be given information regarding the percentage of previous viewers who finally purchase the product in most Amazon product pages . Thus, it's a memory size varying problem with N_r increasing as the number of web viewer increases.

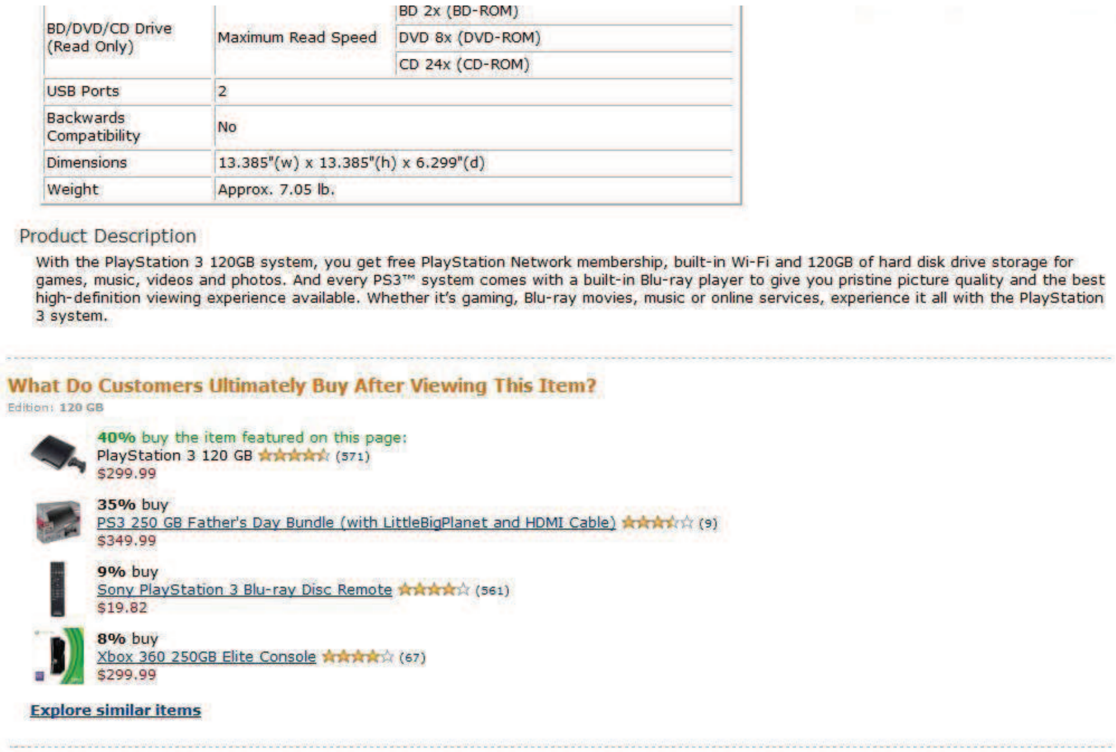


Figure 5.1: An example of Non-Markovian dynamics in real markets.

If the Amazon website chooses not to provide such information, the final purchase rate will be governed by Markovian dynamics. Imagine a fictive product that has an equilibrium rate 26.5% for its Markovian dynamic process. However, the company wants to improve its sales performance by disclosing fake information to the public. For example, they can add 6% artificially to the actual purchase percentage and release the number 32.5% on their website. Then, the unexpected negative non-Markovian effects will arise and the final purchase rate will be pushed even lower, say, to the value 21% as in our 10 RMB case, which is lower than the previous Markovian result. This presumptive experiment tells us that presenting arbitrary fake information may not improve sales. Therefore, extensions on this research would be very useful because of its potential applications in real market strategy designing.

Appendix A

Survey Questionnaire

In this section, we present the questionnaire used for the survey with the monthly fee set to be equal to ¥10.

Q1. Which cell phone carrier are you currently using?

☐ China unicom ☐ China mobile ☐ Both

Q2. Do you know about the intra-network service offered by China mobile?

☐ Yes ☐ No

Unlimited Intra-Network Mobile to Mobile Service

China mobile(Shanghai) is offering unlimited intra-network mobile to mobile service for students and faculty members of your university. The service fee will be ¥10/month. After activation, you will be assigned an intra-network number(INN); members of this network can make unlimited calls to each other via INN. No other hidden fees.

[Tell the respondent decisions made by his/her previous 10 respondents]

Q3. Do you want to buy this service?

☐ Yes ☐ No

Q4. Is the 10/month service fee is expensive for you?

☐Very Cheap ☐Cheap ☐Normal ☐Expensive ☐Very Expensive

(After the respondent answer questions above, tell him/her this survey will be used for research; then finish this survey by completing the following questions)

Q5. Your monthly cell phone cost most likely lies in?

☐ ¥0 ∼ ¥10 ☐ ¥11 ∼ ¥20 ☐ ¥21 ∼ ¥50 ☐ ¥51 ∼ ¥100

☐ ¥101 ∼ ¥200 ☐ ¥201 ∼ ¥500 ☐ More than ¥500

Background information

S1. What's your gender? ☐ Male ☐ Female

S2. What's your age?

S3. Which class are you currently in?

S4. What's your major?

S5. Approximately, how much are your normal expenses every month?

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