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Extremal Problems for Graphs and Hypergraphs

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Abstract

Extremal Problems for Graphs and Hypergraphs

By William Walter Kay

In this thesis we explore extremal results for graphs and hypergraphs.

Graphs: We say that (finite) sets X, $Y \subseteq \mathbb{Q}$ have *order type* $\tau \in [3]^\ell$ if $|X \cup Y| = \ell$ and $\tau_i = 1$, 2, or 3 whenever the ith element of $X \cup Y$ is in $X \setminus Y$, $Y \setminus X$, or $X \cap Y$ respectively. In this case, we write $\tau(X, Y) = \tau$. The type graph $G(n, \tau)$ is the graph with vertex set $\binom{[n]}{k}$ where X is adjacent to Y if and only if $\tau(X, Y) = \tau$ or $\tau(Y, X) = \tau$. The chromatic number of type graphs have been studied extensively by Erdős, Hajnal, Rado, and others. More recently, Avart, Łuczak, and Rödl asked if there was a general formula for the chromatic number based τ. We compute $\chi(G(n, \tau))$ asymptotically. This is joint work with Avart, Reiher, and Rödl.

Hypergraphs: An oriented k-uniform hypergraph (or k-graph) $\mathcal{H} = (V, \mathcal{E})$ is a vertex set V coupled with a collection of k-tuples $\mathcal{E} \subseteq V^k$. Given $<$ a total order on V, we say that $\overline{X} = (x_1, x_2, \ldots, x_k) \in \mathcal{E}$ is *consistent* with $<$ if $x_1 < x_2 < \ldots < x_k$. We say that $\mathcal{H}$ has *Property O* if for every total order on V there exists some edge $\overline{X} \in \mathcal{E}$ that is consistent with $<$. We examine bounds on the minimum number of edges in an oriented k-graph that has Property O.

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Before I came to Emory, I was in the Master's program at the University of South Carolina. There, I met a number of people that pushed me to be the best person I could be professionally. At USC, Aaron Dutle was two years ahead of me, and was constantly the mathematician that I wanted to become. While I wish we had been in the same program for longer, I can earnestly say that the best part about not being in Aaron's class is never having to compete with him for a job. He is a groomsman in my wedding, and one of my best friends. Andrew Vincent was the happy hour captain, one heck of a number theorist, and eventually my upstairs neighbor. We could not get away from interesting problems once we heard about them, and I honestly believe that (along with Aaron), if we had a policy in our office that we could only work on open problems our C.V.s would be a few lines longer. John Schulte (another groomsman) was one of the first people I met in my class at USC. We were fast friends, trivia partners, and eventually neighbors as well. When grad school gets hard, camaraderie is key and John provided boatloads of it. Thanks bud. Ed Boehnlein (another groomsman- noticing a theme?) was on his way in when I was leaving, but we bonded strongly during that year and have had even more great times after that. Lakehouse, anyone? Speaking of the lakehouse crew (who provided much sanity) I have to thank Blake and Ann. These two never miss a trip and put me up whenever I need a place to crash in Columbia.

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Contents

Chapter 1

Introduction

My research interests lie in a field called *extremal combinatorics*. Loosely speaking, extremal problems look to maximize or minimize some combinatorial property subject to some constraint. For example, "maximize the number of edges in a graph subject to the constraint that it has no triangle as a subgraph." In this chapter, we introduce all of the definitions that we need to describe our results. At the end, we include a brief glossary of technical terms.

1.1 Graphs and Hypergraphs

Here we describe the principal combinatorial objects of interest.

1.1.1 Graphs

A graph $G = (V, E)$ is an ordered pair where V is a (typically finite) set called the *vertex set* of G and $E \subseteq \binom{V}{2}$ is a collection of pairs of vertices called the *edge set* of G. Graphs are relational structures that model any real world phenomenon where the relationship between pairs of objects is of interest. For example, the internet graph is one where the vertex set represents the websites on the internet and the edges represent websites with a link from one to another. The study of graphs is called *graph theory*, and constitutes a major research area in the field of discrete mathematics. Unless otherwise stated, notation and terminology will be as presented in Diestel [Die10].

1.1.2 Hypergraphs

Given an integer $k \geq 2$, a k-uniform hypergraph (or k-graph) $\mathcal{H} = (V, \mathcal{E})$ is an ordered pair where V is a (typically finite) set called the *vertex set* of G and $\mathcal{E} \subseteq \binom{V}{k}$ is a collection of k-sets of vertices called the *edge set* of $\mathcal{H}$. Graphs are k-graphs in the case that $k = 2$.

1.2 Extremal Combinatorics and an Overview of Results

Extremal combinatorics is a branch of combinatorics which studies the maximization or minimization of some combinatorial property subject to some other constraint. For example, "maximize the number of edges in a graph subject to the constraint that it has no triangle as a subgraph." This particular question was answered by Mantel [Man07] and generalized for any complete graph by Turán [Tur41]. In this section, we will detail several relevant types of extremal problems in combinatorics. We will conclude the section with an overview of results contained in this document.

1.2.1 Chromatic Number

Given a graph $G = (V, E)$, we define the *chromatic number* of G (denoted $\chi(G)$) to be the least integer so that there exists a function $f : V \to [\chi(G)]$ so that if $\{u, v\} \in E$, $f(u) \neq f(v)$. Informally, $\chi(G)$ is the fewest number of colors one requires to color the vertex set of V so that adjacent vertices receive different colors. The chromatic number has been studied in a variety of contexts. For example, the celebrated *Four Color Theorem* [AH89] shows that the chromatic number of a planar graph can be no more than 4. Erdős used probabilistic methods to show that for any g and ℓ, there exist graphs whose shortest cycle (or *girth*) is at least g and whose chromatic number is at least ℓ [Erd59]. The chromatic number can be generalized to other combinatorial objects such as hypergraphs (as we will see in Section 1.2.2).

1.2.2 Edge Cardinality

Given a collection of k-graphs $\mathcal{C}$ which satisfy some property, one can ask which members of $\mathcal{C}$ have the most or fewest edges? Alternatively, among the k-graphs which are not members of $\mathcal{C}$, which ones have the most or fewest edges? These types of extremal questions ignore the size of the vertex set, and focus only on the size of the edge sets. These extremal problems have a rich history. Perhaps the most well studied question is the following due to Bernstein [Ber07] and popularized by Erdős [Erd63] [Erd64]. We say that a k-graph has *Property B* if its vertex set can be colored with two colors so that no edge receives only one color (i.e., no edge is *monochromatic*). Let $m(k)$ denote the minimum number of edges in a k-graph which *fails* to have Property B. In other words, if $\mathcal{H}$ is a k-graph with fewer than $m(k)$ edges, then the vertices of $\mathcal{H}$ can be two colored so that no edge is monochromatic. On the other hand, there exists a k-graph with $m(k)$ edges which fails to have Property B. In 1963, Erdős [Erd63] [Erd64] established that $2^k \leq m(k) \leq k^2 2^k$, and the upper bound on $m(k)$ remains the best known. In 1978, Beck [Bec78] used a probabilistic coloring and recoloring scheme to show that $m(k) = \Omega(k^{1/3}2^k)$, improving the lower bound on $m(k)$ (which was reproved by Spencer [Spe81]). In 2000, Radhakrishnan and Srinivasan [RS00] showed $m(k) = \Omega(2^k\sqrt{k/\ln k})$, establishing the best known lower bound on $m(k)$ (which was reproved by Cherkashin and Kozik [CK15]).

1.2.3 Overview of Results

This thesis focuses on two problems in extremal combinatorics–one of each of the aforementioned types. Here we will provide the background and statements of results related to each distinct problem. In the subsequent two chapters we present the corresponding two projects in further detail.

Type Graphs

Chapter 2 focuses on the chromatic number of finite type graphs. Given (finite) sets X, $Y \subseteq \mathbb{Q}$ with $|X \cup Y| = \ell$, we say that X and Y have order type $\tau = (\tau_1, \tau_2, \ldots, \tau_\ell) \in [3]^\ell$ if $X \cup Y = \{z_1 < z_2 < \ldots < z_\ell\}$ and for every $i \in [\ell]$:

$$\tau_i = \begin{cases} 1 & \text{if } z_i \in X \setminus Y, \\ 2 & \text{if } z_i \in Y \setminus X, \\ 3 & \text{if } z_i \in X \cap Y. \end{cases}$$

In this case, we write $\tau(X, Y) = \tau$. Note $|X| = |Y| = k$ if and only if

$$|\{i : \tau_i \in \{1, 3\}| = |\{i : \tau_i \in \{2, 3\}\}| = k.$$

We call any such τ an *order type* of *width* k. For a fixed $\tau \in [3]^\ell$ of width k, the type graph $G(n, \tau)$ is the graph whose vertex set is $\binom{[n]}{k}$ and X is adjacent to Y if and only if $\tau(X, Y) = \tau$ or $\tau(Y, X) = \tau$.

The chromatic number of the finite type graphs has been studied for specific types. For example, in 1968, Erdős and Hajnal [EH68] showed that for the so-called *generalized shift graphs* we have:

$$\chi(G(n, 1\underbrace{3\ldots3}_{k-1}2)) = (1 + o(1)) \log_{(k-1)} n$$

is

where $\log_{(t)} n$ denotes the t-fold iterated binary logarithm. The shift graphs can be seen to have no short odd cycles, and so these shift graphs provide an explicit example of graphs with arbitrarily large odd girth and chromatic number. In 2014, Avart, Łuczak, and Rödl [ALR14] asked about the asymptotic nature of the chromatic number of finite type graphs in general. In the next chapter, we provide an algorithm that, on input τ (a finte type of width k), produces a natural number $\beta = \beta(\tau)$ so that:

$$\chi(G(n,\tau)) = \Theta(\log_{(\beta)} n)$$

This is joint work with Avart, Reiher, and Rödl [AKRR17].

1.2.4 Oriented k-Graphs

Chapter 3.1 focuses on an extremal problem of oriented k-uniform hypergraphs (or oriented k-graphs). An oriented k-graph is a pair $\mathcal{H} = (V, \mathcal{E})$ where V is called the *vertex set* and $\mathcal{E} \subseteq V^k$ is called the edge set. We further require that all the members of each edge are distinct, and no two edges have all the same members. Given $<$ a total order on V, we say that $\overline{X} = (x_1, x_2, \ldots, x_k) \in \mathcal{E}$ is *consistent* with $<$ if $x_1 < x_2 < \ldots < x_k$. We say that $\mathcal{H}$ has *Property O* if for every total order on V there exists some edge $\overline{X} \in \mathcal{E}$ which is consistent with $<$. Let $f(k)$ denote the minimum number of edges in an oriented k-graph which has Property O. We have the bounds:

$$k! \leq f(k) \leq (k^2 \ln k)k!$$

We further show that for a specific value of n, the complete oriented k-graphs on n vertices have $k^{1/2}k!(1+o(1))$ edges and almost all such graphs fail to have Property O. We conclude the chapter with an explicit construction of oriented k-graphs which have

Property O, and comment on small values of $f(k)$. This is joint work with Duffus and Rödl [DKR17].

1.3 Glossary of Technical Terms

In this document, it is convenient to describe functions asymptotically–that is in terms of their rate of growth. Here we include a table of "Big O" notation. Let $f(x)$ and $g(x)$ be given non-negative functions.

Notation	Name	Definition
$f(x) = O(g(x))$	Big O	$\exists\ k, x_0 > 0\ : f(x) \leq k \cdot g(x)\ \forall\ x > x_0$
$f(x) = \Omega(g(x))$	Big Omega	$\exists\ k, x_0 > 0\ : f(x) \geq k \cdot g(x)\ \forall\ x > x_0$
$f(x) = o(g(x))$	Little O	$\lim_{x\to\infty} \frac{f(x)}{g(x)} = 0$
$f(x) = \omega(g(x))$	Little Omega	$\lim_{x\to\infty} \frac{f(x)}{g(x)} = \infty$
$f(x) = \Theta(g(x))$	Theta	$\exists\ k_1, k_2, x_0 > 0\ : k_1 g(x) \leq f(x) \leq k_2 g(x) \forall x > x_0$
$f(x) \sim g(x)$	Asymptotic	$\lim_{x\to\infty} \frac{f(x)}{g(x)} = 1$

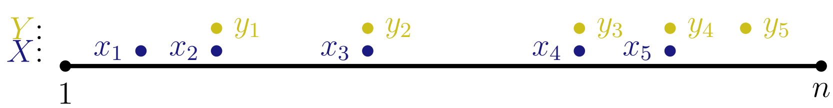
Chapter 2

Type Graphs

For $n > k \geq 2$, the so-called *Shift graph* (denoted $\mathrm{Shift}(n,k)$) is the graph whose vertex set is the k-subsets of $\{1, 2, \ldots, n\}$, and $X = \{x_1, x_2, \ldots x_k\}$ is adjacent to $Y = \{y_1, y_2, \ldots, y_k\}$ if and only if:

$$x_1 < y_1 = x_2 < y_2 = \ldots < y_{k-1} = x_k < y_k$$

Visually, two k-sets are adjacent if their order type looks as follows:

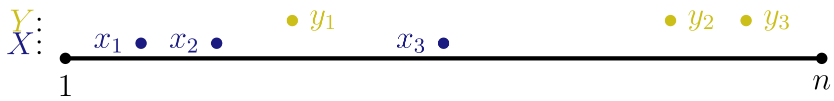


Erdős and Hajnal [EH68] showed that $\chi(\mathrm{Shift}(n,k)) = (1+o(1))\log_{(k-1)}(n)$, where $\log_{(t)}(n)$ is the t-fold iterated binary logarithm. It is easily seen that the shortest odd cycle in $\mathrm{Shift}(n,k)$ has length at least $2k+1$. Consequently, the shift graphs provide an explicit construction of graphs with large *odd girth* and large chromatic number.

Similarly, the *Specker graph* (denoted $\mathrm{Specker}(n,3)$) is the graph whose vertex set is the 3-subsets of $\{1, 2, \ldots, n\}$, and $X = \{x_1, x_2, x_3\}$ is adjacent to $Y = \{y_1, y_2, y_3\}$ if and only if:

$$x_1 < x_2 < y_1 < x_3 < y_2 < y_3$$

Visually, two 3-sets are adjacent if their order type looks as follows:



Erdős and Rado [ER60] showed that for an infinite cardinal κ, $\chi(\text{Specker }(\kappa, 3)) = \kappa$. It is easily seen that the Specker graphs are triangle free. Consequently, the Specker graphs give an example of an (infinite) triangle free graph that has the same chromatic number as the complete graph.

The Shift graphs and Specker graphs hence provide constructions of graphs that serve as interesting extremal examples. In this chapter, we will describe a class of graphs which generalizes both the Specker and Shift graphs, and describe their chromatic numbers asymptotically.

2.1 Preliminaries and Basic Definitions

The goal of this chapter is to establish the chromatic number of the so-called *type-graphs* (see Definition 2.3). For some positive integers n and k with $n \geq k$, the vertex set of these graphs is the k-element subsets of $[n] := \{1, 2, \ldots, n\}$, and two k-sets are adjacent if and only if the mutual position of their elements satisfy some prespecified order pattern. Before defining the type-graphs rigorously, we would like to fix notation concerning order types of pairs of ordered sets. In particular, we will encode such order types as finite sequences of 1's, 2's, and 3's. We will find that it is convenient to define order types for finite subsets of rationals.

Definition 2.1 (order types)**.** *Let X, $Y \subseteq \mathbb{Q}$ be two finite sets with $|X \cup Y| = \ell$ and $X \cup Y = \{z_1 < z_2 < \ldots < z_\ell\}$. We say that the* order type *of the pair (X, Y) is the sequence $\tau = (\tau_1, \tau_2, \ldots, \tau_\ell) \in [3]^\ell$ and write $\tau(X, Y) = \tau$ if for every $i \in [\ell]$ we have:*

$$\tau_i = \begin{cases} 1 & \textit{if } z_i \in X \setminus Y, \\ 2 & \textit{if } z_i \in Y \setminus X, \\ 3 & \textit{if } z_i \in X \cap Y. \end{cases}$$

For example, if $X = \{1, 2, 3, 5\}$ and $Y = \{3, 4, 5\}$ we get $\tau(X, Y) = 11323$. Clearly, for any finite sequence τ of 1's, 2's, and 3's there are sets X, $Y \subseteq \mathbb{Q}$ so that $\tau(X, Y) = \tau$. In fact, one can find such subsets of $\mathbb{N}$, though defining order types for $\mathbb{Q}$ will prove to be technically helpful. The case most relevant to the definition of type-graphs below is $|X| = |Y|$.

Definition 2.2 (fixed width types)**.** *Fix two positive integers k and ℓ. By a type of* width k *and* length ℓ*, we mean the order type of a pair (X, Y) with X, $Y \subseteq \mathbb{Q}$, $|X| = |Y| = k$, and $|X \cup Y| = \ell$.*

So $\tau = 123312$ is a type of width 4 and length 6 that is the order type of e.g., $X = \{1, 3, 4, 7\}$ and $Y = \{2, 3, 4, 9\}$. It is easy to see that any type of width k and length ℓ contains $\ell - k$ 1's, $\ell - k$ 2's, and $2k - \ell$ 3's. As a degenerate case we regard the empty sequence ϵ as an *empty type* of width and length 0. A type is said to be *trivial* if it consists only of 3's, i.e. its width equals its length.

We are now prepared to define the main objects of consideration in this chapter.

Definition 2.3 (type-graphs)**.** *For a nontrivial type τ of width k and an integer $n \geq k$, the* type-graph $G(n, \tau)$ *is the graph with vertex set $\binom{[n]}{k}$ in which two vertices X and Y are adjacent if and only if $\tau(X, Y) = \tau$ or $\tau(Y, X) = \tau$.*

Such graphs and their chromatic numbers have been studied in a variety of contexts. For example, it is known that the chromatic number of the shift graphs $G(n, 132)(= \mathrm{Shift}(n, 2))$ is $\lceil \log n \rceil$, where the base of the logarithm is 2. It is straightforward to check that these graphs are triangle-free, and thus provide explicit examples of triangle-free graphs with arbitrarily large chromatic number. More generally, Erdős and Hajnal [EH68] considered the *generalized* shift graph $G(n, \sigma_k)$, where

$$\sigma_k = 1\underbrace{3\ldots3}_{k-1}2$$

and the infinite analogues which arise when one replaces the finite number n with an arbitrary cardinal number. Concerning the chromatic number of the finite type-graph $G(n, \sigma_k)$ they obtained the following result which we will apply later.

Theorem 2.4 (Erdős, Hajnal)**.** *For any integer $k \geq 2$ we have*

$$\chi(G(n, \sigma_k)) = (1 + o(1)) \log_{(k-1)} n$$

as n tends to infinity.

Strictly speaking, Erdős and Hajnal focused on the case were n is infinite [EH68], but their proof technique applies to finite values of n as well. The thus adapted proof may be found with more details in [DLR95] [MSW15]. In the latter reference the alternative language of ordered Ramsey theory is used. We note that the infinite case of Theorem 2.4 has applications to the computation of infinite Ramsey numbers [EH68] and refer the reader interested in further applications of infinite type-graphs to [Spe57] [ER60] [PR86] [KS05].

Another interesting consequence of Theorem 2.4 is that it provides us with explicit examples of graphs with large *odd girth* and large chromatic number. Any odd cycle in $G(n, \sigma_k)$ has length at least $2k+1$ This line of thought was substantially continued by Nešetřil and Rödl, who used unions of general type-graphs in some of their early work on structural Ramsey theory, see e.g. [NR76].

The problem of determining the chromatic number of general finite type-graphs was recently approached in joint work of Avart, Łuzcak and Rödl [ALR14]. The last section of that article contains a conjecture, restated as Theorem 2.8 below, that predicts this number asymptotically (up to a constant multiplicative factor).

In particular, this conjecture implies that for each nontrivial type τ there exists a nonnegative integer β with $\chi(G(n,\tau)) = \Theta(\log_{(\beta)} n)$ as n tends to inifinity. When intending to calculate β from τ, the first thing one has to do is to express τ as a product of as many other types as possible. The next two definitions help us to talk about this process.

Definition 2.5. *Given two finite sequences $\tau = (\tau_1, \tau_2, \dots, \tau_\ell)$ and $\tau' = (\tau'_1, \tau'_2, \dots, \tau'_{\ell'})$ we write $\tau\tau'$ for their* concatenation $(\tau_1, \tau_2, \dots, \tau_\ell, \tau'_1, \tau'_2, \dots, \tau'_{\ell'})$.

Definition 2.6. *A nonempty type is said to be* irreducible *if it cannot be written as the concatenation of two nonempty types. Otherwise, it is called* reducible.

It should be clear that each nonempty type τ can be written in a unique manner as the concatenation of several irreducible types. In fact, one finds this unique *factorization* of τ by keeping track of the number of 1's and 2's already encountered while reading τ from left to right, and starting a new factor at every moment where these two numbers are equal. For example, the type $\tau = 1332112122$ has the following factorization into irreducible types ρ and ρ':

$$\tau = \underbrace{1332}_{\rho}\underbrace{112122}_{\rho'}$$

As it will turn out most of our work concerning $\chi(G(n,\tau))$ addresses the irreducible case. Once it is solved, the reducible case follows from the irreducible one.

In the next section, we describe an algorithm which partitions any given irreducible type τ into so-called blocks (which will be formally defined in Section 2.2). Notice that if τ is trivial, i.e., a string of 3's, we must have $\tau = 3$ and in this case the number of blocks is 1. On the other hand, any nontrivial irreducible type is going to be partitioned into at least 2 blocks.

The main result of this chapter on irreducible types (which is joint work with Avart, Reiher, and Rödl [AKRR17]) states:

Theorem 2.7. *If τ is a nontrivial irreducible type of width k with b blocks, then*

$$(1+o(1))\log_{(b-2)}\frac{n}{k} \le \chi(G(n,\tau)) \le (2^{(b-2)^2}+o(1))\log_{(b-2)} n$$

and hence

$$\chi(G(n,\tau)) = \Theta(\log_{(b-2)} n).$$

More generally we will obtain the following:

Theorem 2.8. *Let $\tau = \rho_1\rho_2\ldots\rho_t$ be the factorization of an arbitrary nontrivial type τ into irreducible types. Suppose that ρ_i has b_i blocks for $i \in [t]$, and set $b^* = \max(b_1,\ldots,b_t)$. Then we have*

$$\chi(G(n,\tau)) = \Theta(\log_{(b^*-2)} n).$$

The rest of this chapter is structured as follows. In Section 2.2 we describe the block algorithm and thus clarify the meaning of our main result. Then Sections 2.3 and 2.4 are dedicated to the proofs of the lower and upper bounds in Theorem 2.7. Finally, in Section 2.5 we will deduce Theorem 2.8 by means of an argument that considers the categorical product of graphs.

2.2 The Block Algorithm

In this section, we describe an algorithm partitioning the terms of any irreducible type τ into blocks of consecutive terms. We will call this algorithm the *block algorithm* and the partition it produces will be referred to as the *block decomposition* of τ.

As said above, if τ is trivial we have $\tau = 3$ by irreducibility. In this special case we regard τ as consisting of one block only, namely τ itself. If $\tau \neq 3$, then the first digit of τ is either a 1 or a 2 because otherwise we could write $\tau = 3\rho$ for some type

$\rho \neq \epsilon$, contrary to the irreducibility of τ. We call τ *primary* if it starts with a 1 and *secondary* if it starts with a 2.

Given a subsequence B of a type τ that consists of consecutive terms, we write $\mathbf{1}(B)$ to denote the total number of 1's and 3's in B and $\mathbf{2}(B)$ for the total number of 2's and 3's in B.

Now we are ready to explain how the block algorithm is applied to any primary irreducible type τ. Processing τ from left to right we are to perform the following steps:

(i) The first block B_1 consists of all the initial 1's appearing in τ.

(ii) In general, if the block B_i has just been constructed, the next block B_{i+1} consists of the next consecutive digits of τ such that $\mathbf{2}(B_{i+1}) = \mathbf{1}(B_i)$ and such that subject to this last condition the block B_{i+1} is as long as possible.

(iii) The algorithm stops when all the terms of τ have been placed in a block.

For example, for the type $\tau = 1121112121212222$ we get $B_1 = 11$, $B_2 = 211121$, $B_3 = 212122$, and $B_4 = 22$. One may use appropriate spacing to make the outcome of the block algorithm notationally visible and write, for instance,

$$\tau = 11 \quad 211121 \quad 212122 \quad 22.$$

Similarly the type 131122311222 decomposes into

$$1 \quad 311 \quad 22311 \quad 222$$

and for the type $\sigma_4 = 13332$, the algorithm produces

$$\sigma_4 = 1 \quad 3 \quad 3 \quad 3 \quad 2.$$

Fact 2.9. *When applied to a primary irreducible type* τ *the block algorithm does indeed provide a factorization* $\tau = B_1 B_2 \ldots B_b$ *of* τ *into some nonempty blocks* $B_1, B_2, \ldots, B_b$*, where* $b \geq 2$*. Moreover, we have* $\mathbf{1}(B_b) = 0$*.*

Proof. Since τ starts with a 1, rule (i) gives us a first block $B_1 \neq \epsilon$. Now let i be the largest integer for which the block algorithm produces in its first i steps some nonempty blocks $B_1, B_2, \ldots, B_i$. This happens by an initial application of (i) followed by $i-1$ applications of (ii). Let C denote the finite sequence satisfying

$$\tau = B_1 B_2 \ldots B_i C \tag{2.1}$$

We intend to show that either $C = \epsilon$ so that the algorithm stops, or $0 < \mathbf{1}(B_i) \leq \mathbf{2}(C)$, meaning that the algorithm produces a further nonempty block B_{i+1}. The latter alternative, however, would contradict the maximality of i.

Recall that by construction we have $\mathbf{2}(B_1) = 0$ and $\mathbf{1}(B_j) = \mathbf{2}(B_{j+1})$ for all $j \in [i-1]$. This yields

$$\mathbf{1}(B_1 B_2 \ldots B_{i-1}) = \mathbf{2}(B_1 B_2 \ldots B_i) \tag{2.2}$$

and in combination with (2.1) and $\mathbf{1}(\tau) = \mathbf{2}(\tau)$, it follows that $\mathbf{1}(B_i) \leq \mathbf{1}(B_i C) = \mathbf{2}(C)$. So if $\mathbf{1}(B_i) > 0$ we could use (ii) once more and optain the next nonempty block B_{i+1} to obtain the next nonempty block B_{i+1}, contrary to the maximality of i.

Thus we must have $\mathbf{1}(B_i) = 0$ and (2.2) entails that $B_1 B_2 \ldots B_i$ is a type. By (2.1) and the irreducibility of τ if follows that $C = \epsilon$, meaning that the algorithm stops with a final application of the rule (iii). Now $b = i$, the moreover part was obtained at the beginning of this paragraph, and $b \geq 2$ is clear. □

So far we have only talked about primary types. For dealing with secondary types we use the following symmetry: if τ denotes any finite sequence of 1's, 2's, and 3's, we write τ' for the sequence obtained from τ by replacing all 1's by 2's and vice versa. Evidently if τ is a secondary irreducible type, then τ' is a primary irreducible type and thus we already know how to find its block decomposition $\tau' = B_1 B_2 \dots B_b$. Now we have $\tau = B_1' B_2' \dots B_b'$ and we define this to be the block decomposition of τ. In particular, τ and τ' have the same number of blocks.

Notice that if $\tau(X, Y) = \rho$ holds for some finite sets X, $Y \subseteq \mathbb{Q}$, then $\tau(Y, X) = \rho'$ follows. In particular, for any type τ the two type-graphs $G(n, \tau)$ and $G(n, \tau')$ are the same and thus it suffices to prove Theorem 2.7 for primary τ.

We conclude this section with two statements concerning irreducible types and the block algorithm that will be employed in Sections 2.2, 2.3, and 2.4.

Lemma 2.10. *Suppose that τ is a primary irreducible type of width k and that X, $Y \subseteq \mathbb{Q}$ are two finite sets with $\tau = \tau(X, Y)$. Let $X = \{x_1 < x_2 < \dots < x_k\}$ and $Y = \{y_1 < y_2 < \dots < y_k\}$. Then we have*

(a) $x_i < y_i$ for all $i \in [k]$, and

(b) $x_{i+1} \leq y_i$ for all $i \in [k-1]$.

Proof. Let $\tau = (\tau_1, \tau_2, \dots, \tau_\ell)$, where ℓ denotes the length of τ. We contend that

$$\text{if } i \in [k-1] \text{ and } x_i \leq y_i, \text{ then } x_{i+1} \leq y_i. \tag{2.3}$$

To show this, let y_i be the m-th element in the increasing enumeration of $X \cup Y$. In view of $1 \leq i < k$ we have $1 \leq m < \ell$ and thus $(\tau_1, \tau_2, \dots, \tau_m)$ cannot be a type due to the irreducibility of τ. This in turn yields $|X \cap (-\infty, y_i]| \neq |Y \cap (-\infty, y_i]| = i$. But assuming $x_i \leq y_i$ the number $|X \cap (-\infty, y_i]|$ is at least i, so that altogether it must be at least $i + 1$, which means that $x_{i+1} \leq y_i$. This proves (2.3). Next we show (a) by induction on i. The base case $x_1 < y_1$ follow from τ being primary. For the induction step we suppose that $x_i < y_i$ holds for some $i < k$. Then (2.3) entails $x_{i+1} \le y_i < y_{i+1}$, which concludes the argument.

Finally, (b) is an immediate consequence of (2.3) and (a). □

We now come to the only place in the proof of Theorem 2.7 where the demand from rule (ii) of the block algorithm that the blocks should end with as many 1's as possible is utilized. The purpose of the following lemma is that, roughly speaking, it tells us how the "blocks" of two finite sets X and Y realizing an irreducible type τ overlap with each other. This will be useful in Subsection 2.4.1 for embedding $G(n, \tau)$ into an auxiliary graph whose chromatic number is easier to bound from above.

Lemma 2.11. *Let $\tau = B_1B_2 \dots B_b$ be the block decomposition of some primary irreducible type whose width is k and set $s(i) = \mathbf{2}(B_1, B_2, \dots, B_i)$ for all $i \in [b]$. Then for any two sets X and Y satisfying $\tau = \tau(X, Y)$, say $X = \{x_1 < x_2 < \dots < x_k\}$ and $= \{y_1 < y_2 < \dots < y_k\}$, we have $x_{s(i+1)} < y_{s(i)+1} \le x_{s(i+1)+1}$ for all $i \in [b-2]$.*

Proof. Let $X \cup Y = \{z_1 < z_2 < \dots < z_\ell\}$. Fix any $i \in [b-2]$ and set $\beta = \sum_{j=1}^{i} |B_j|$. By rule (ii) of the block algorithm the block B_{i+1} cannot start with a one and thus we have $z_{\beta+1} \in Y$. In combination with

$$s(i) = \mathbf{2}(B_1B_2 \dots B_i) = |Y \cap (-\infty, z_\beta]|$$

which yields

$$y_{s(i)+1} = z_{\beta+1}. \tag{2.4}$$

Similarly we have

$$s(i+1) = \mathbf{2}(B_1B_2 \dots B_{i+1}) = \mathbf{1}(B_1B_2 \dots B_i) = |X \cap (-\infty, z_\beta]|$$

and thus $x_{s(i+1)} \leq z_\beta$ as well as $z_{\beta+1} \leq x_{s(i+1)+1}$. The desired conclusion follows from these two estimates and (2.4). $\square$

2.3 The Lower Bound – Noncolorability

In this section we will prove the lower bound in Theorem 2.7. So we intend to show that a certain graph $G(n, \tau)$ cannot be colored with a certain "small" number of colors. Recall that for any graph H and any natural number r, the statement $\chi(H) > r$ is equivalent as saying there is no graph homomorphism from H to the complete graph K_r. Thus one strategy to prove such an noncolorabiliy statement is to exhibit a homomorphism from some auxiliary graph G to H, with $\chi(G) > r$ already being known. So in the light of Theorem 2.4 our task reduces to:

Proposition 2.12. *For every nontrivial irreducible type τ of width k with b blocks and every integer $n \geq b$ there is a graph homomorphism*

$$\phi : G(n, \sigma_{b-1}) \to G(kn, \tau).$$

For the construction of such a homomorphism, we will make use of the following.

Fact 2.13. *If B denotes a finite sequence of 1's, 2's, and 3's, and $Y \subseteq \mathbb{Q}$ has size $\mathbf{2}(B)$, then there is a set $X \subseteq \mathbb{Q}$ with $\tau(X, Y) = B$.*

This can easily be shown by induction on the number of 1's appearing in B and we leave the details to the reader.

Proof of Proposition 2.12. As said above we may assume that τ is primary. Let

$$\tau = B_1 B_2 \dots B_b$$

be the block decompostion of τ. We commence by defining recursively an auxiliary sequence R_0, R_1, ..., R_b of finite subsets of $\mathbb{Q}$ with

$$|R_{i-1}| = \mathbf{2}(B_i) \text{ for all } i \in [b]. \tag{2.5}$$

Since B_1 consists exclusively of 1's, such a sequence needs to start with $R_0 = \emptyset$. Once R_{i-1} has been defined for some $i \in [b]$, we use Fact 2.13 to obtain a set $R_i \subseteq \mathbb{Q}$ satisfying $\tau(R_i, R_{i-1}) = B_i$. Notice that for $i < b$ this yields $|R_i| = \mathbf{1}(B_i) = \mathbf{2}(B_{i+1})$, so that the construction may be continued. We also get $|R_b| = \mathbf{1}(B_b) = 0$ and hence $R_b = \emptyset$ from Fact 2.9.

In view of (2.5) we have

$$\sum_{i=0}^{b-1} |R_i| = \sum_{i=1}^{b} \mathbf{2}(B_i) = \mathbf{2}(\tau) = k \tag{2.6}$$

and thus there exist k rational numbers $\alpha_1 < \alpha_2 < \ldots < \alpha_k$ with

$$\cup_{0 \le i < b} R_i \subseteq \{\alpha_1, \alpha_2, \ldots, \alpha_k\}.$$

Pulling this situation back to $[k]$ we define $R_i^* = \{j \in [k] | \alpha_j \in R_i\}$ for all $i \in [b-1]$ as well as $R_0^* = R_b^* = \emptyset$. The main properties of these sets are

$$R_i^* \subseteq [k] \text{ and } \tau(R_i^*, R_{i-1}^*) = B_i \text{ for all } i \in [b]. \tag{2.7}$$

Now we are ready to define the desired map

$$\phi : \binom{[n]}{b-1} \to \binom{[kn]}{k}.$$

Given any integers h_i for $i \in [b-i]$ with $1 \le h_1 < h_2 < \ldots < h_{b-1} \le n$ we set

$$\phi(\{h_1, h_2, \ldots, h_{b-1}\}) = \cup_{i\in[b-1]}\{(h_i - 1)k + j | j \in R_i^*\}.$$

Due to $R_i^* \subseteq [k]$ the right hand side of this formula is indeed a subset of $[kn]$ and by (2.6) its size is k. It remains to check that ϕ maps edges of $G(n, \sigma_{b-1})$ to edges of $G(kn, \tau)$. To this end let any integers h_i for $i \in [b]$ with $1 \le h_1 < h_2 < \ldots < h_b \le n$ be given. Then by (2.7) we have

$$\begin{aligned}\tau(\phi(\{h_1, h_2, \ldots, h_{b-1}\}), \phi(\{h_2, h_3, \ldots, h_b\})) &= \tau(R_1^*, R_0^*)\tau(R_2^*, R_1^*)\ldots\tau(R_b^*, R_{b-1}^*)\\ &= B_1 B_2 \ldots B_b = \tau\end{aligned}$$

as desired.

□

2.4 The Upper Bound – Constructing Colorings

This entire section is dedicated to the proof of the upper bound from Theorem 2.7. The strategy we use is to embed the type-graph $G(n, \tau)$ into some other graph $G_{b-1}(n)$ that depends solely on b and n but not on τ. Thereby the task we are to perform is reduced to the problem of coloring these auxiliary graphs with "few" colors. It seems that this new problem is more susceptable to an inductive treatment than the old one.

2.4.1 Embedding Type-Graphs

We begin by defining the auxiliary graphs $G_b(n)$ mentioned above.

Definition 2.14. *For any positive integers b and n we set*

$$W_b(n) = \{(x_1, x_2, \ldots, x_{2b-1}) | 1 \le x_1 \le x_2 \le \ldots \le x_{2b-1} \le n\}$$

and

$$V_b(n) = \{(x_1, x_2, \ldots, x_{2b-1}) \in W_b(n) | x_1 < x_3 < \ldots < x_{2b-1}\}.$$

By $G_b(n)$ we mean the graph with vertex set $V_b(n)$ in which an unordered pair $e \subseteq V_b(n)$ is declared to be an edge if and only if we can write $e = \{\overline{x}, \overline{y}\}$,
$\overline{x} = (x_1, x_2, \ldots, x_{2b-1})$, and $\overline{y} = (y_1 y_2, \ldots, y_{2b-1})$ such that

(i) $x_1 < y_1 \le x_3 < y_3 \le \ldots \le x_{2b-1} < y_{2b-1}$

(ii) and $x_{j+1} \le y_j$ *for* $j \in [2b-2]$.

It should perhaps be observed that the conditions (i) and (ii) in this definition do not uniquely determine how the elements of the multiset $\{x_1, \ldots, x_{2b-1}\} \cup \{y_1, \ldots, y_{2b-1}\}$ are ordered. This makes it more plausible, of course, that many type-graphs embed homomorphically into $G_b(n)$ and in fact we have the following.

Theorem 2.15. *For any nontrivial irreducible type τ with $b \ge 2$ blocks and every positive integer n there is a graph homomorphism $\phi : G(n, \tau) \to G_{b-1}(n)$.*

Proof. As usual we may assume that τ is a primary type of width k. Let $\tau = B_1 B_2 \ldots B_b$ be its block decomposition and define $s(i) = \mathbf{2}(B_1 B_2 \ldots B_i)$ for any $i \in [b]$. Since

$$0 = s(1) < s(2) < \ldots < s(b) = k\,,$$

there is a map

$$\phi : \binom{[n]}{k} \to V_{b-1}(n)$$

given by

$$\phi(\{x_1, x_2, \ldots, x_k\}) = (x_{s(1)+1}, x_{s(2)}, x_{s(2)+1}, \ldots, x_{s(b-1)}, x_{s(b-1)+1}),$$

whenever $1 \le x_1 < \ldots < x_k \le n$. So roughly speaking ϕ remembers where the "blocks" of such a set $\{x_1, x_2, \ldots, x_k\}$ start and end and forgets everything else.

It remains to verify that ϕ sends edges of $G(n, \tau)$ to edges of $G_{b-1}(n)$. For this purpose let any two vertices X and Y of $G(n, \tau)$ with $\tau(X, Y) = \tau$ be given and write $X = \{x_1 < x_2 < \ldots < x_k\}$ as well as $Y = \{y_1 < y_2 < \ldots < y_k\}$. We need to show that $\{\phi(X), \phi(Y)\}$ is an edge of $G_{b-1}(n)$, i.e., that the clauses (i) and (ii) from Definition 2.14 are satisfied.

Now by Lemma 2.10 (a) we have, in particular, $x_{s(i)+1} < y_{s(i)+1}$ for all $i \in [b-1]$ and Lemma 2.11 tells us that $y_{s(i)+1} \le x_{s(i+1)+1}$ holds for all $i \in [b-2]$. Both statements together yield condition (i) from Definition 2.14.

For the verification of (ii) we consider the cases that the index j is odd or even separately. To deal with the case where j is odd we need to check that $x_{s(i+1)} \le y_{s(i)+1}$ holds for all $i \in [b-2]$, which follows from Lemma 2.11. For even j we need that $x_{s(i+1)+1} \le y_{s(i+1)}$ holds for all $i \in [b-2]$ and this was obtained in Lemma 2.10 (b).

$\square$

Now it is clear that in order to complete the proof of Theorem 2.7 we just need to establish the following result.

Theorem 2.16. *For every positive integer b we have*

$$\chi(G_b(n)) \le (2^{(b-1)^2} + o(1)) \log_{(b-1)} n\,.$$

Throughout the rest of this section we deal with the proof this theorem. We will proceed by induction on b, considering the base cases $b = 1$ and $b = 2$ separately.

They will be established by statement (2.8) and Lemma 2.18 below. The main idea for the induction step is to relate the graphs $G_b(2^n)$ and $G_{b-1}(n)$ to each other. Roughly speaking, we will show that for any $b \geq 3$ the vertex set of the graph $G_b(2^n)$ may be split into about 2^{2b-3} pieces, each of which induces a graph that maps homomorphically into $G_{b-1}(n)$. A precise assertion along these lines is provided by Proposition 2.24 below. For the construction of half of these homomorphisms it will be helpful to bear the following symmetry in mind.

Fact 2.17. *For any positive integers b and n the bijection $\eta : V_b(2^n) \to V_b(2^n)$ given by*

$$(x_1, x_2, \ldots, x_{2b-1}) \mapsto ((2^n+1) - x_{2b-1}, (2^n+1) - x_{2b-2}, \ldots, (2^n+1) - x_1)$$

is an automorphism of $G_b(2^n)$.

We leave the easy proof of this assertion to the reader.

2.4.2 Coloring the Auxiliary Graphs $G_b(n)$

Clearly the graph $G_1(n)$ is nothing else than a complete graph on n vertices. Thus we have

$$\chi(G_1(n)) = n \qquad \text{for every positive integer } n\,. \tag{2.8}$$

The case $b = 2$ of Theorem 2.16 is technically easier than the general case and thus we would like to treat it separately.

Lemma 2.18. *We have $\chi(G_2(n)) \leq 2\lceil \log n \rceil - 1$ for all integers $n \geq 2$.*

Proof. Clearly it suffices to show $\chi(G_2(2^k)) \leq 2k - 1$ for all positive integers k and we shall do so by induction on k. The base case $k = 1$ poses no difficulty because the graph $G_2(2)$ just consists of two isolated vertices. To handle the induction step it is enough to show

$$\chi(G_2(2m)) \le \chi(G_2(m)) + 2 \quad \text{for all } m \ge 2\,. \tag{2.9}$$

Bearing this goal in mind we partition the vertex set of $G_2(2m)$ into the four classes:

$$\begin{aligned} A &= \{(x,y,z) \in V_2(2m) \mid z \le m\}\,, \\ B &= \{(x,y,z) \in V_2(2m) \mid y \le m < z\}\,, \\ C &= \{(x,y,z) \in V_2(2m) \mid x \le m < y\}\,, \\ \text{and} \qquad D &= \{(x,y,z) \in V_2(2m) \mid m < x\}\,. \end{aligned}$$

We also identify subsets of $V_2(2m)$ with the subgraphs of $G_2(2m)$ that they induce. Evidently A is the same as $G_2(m)$, the map $(x,y,z) \mapsto (x+m, y+m, z+m)$ provides an isomorphism between A and D, and there are no edges between A and D. Therefore $A \cup D$ is a disjoint union of two copies of $G_2(m)$ and we have $\chi(A \cup D) = \chi(G_2(m))$. Moreover, using condition (ii) from Definition 2.14 it is easy to check that the sets B and C are independent. This concludes the proof of (2.9) and, thus, the proof of Lemma 2.18. □

Before we proceed to the colouring of $G_b(2^n)$ for $b \ge 3$ we introduce some auxiliary functions.

Lemma 2.19. *Given any integers x and y with $1 \le x < y$ there exist a positive integer f and an odd positive integer q such that*

$$(q-1) \cdot 2^{f-1} < x \le q \cdot 2^{f-1} < y \le (q+1) \cdot 2^{f-1}\,.$$

Moreover, f and q are uniquely determined by x and y so that we may write $f = f(x,y)$ as well as $q = q(x,y)$.

Proof. Let us first prove the existence of f and q. To this end, we pick an integer

n with $y \leq 2^n$. Then we expand $x-1$ and $y-1$ in the binary system using n digits and allowing leading zeros. Say that this yields $x-1 = x_{n-1}\dots x_1x_0$ and $y-1 = y_{n-1}\dots y_1y_0$. Next we compare these expansions from left to right and let $x_{f-1} \neq y_{f-1}$ be the first place where they differ. Notice that $x < y$ ensures $x_{f-1} = 0$ and $y_{f-1} = 1$. Finally we let q be the number with binary representation $q = x_{n-1}\dots x_f 1$.

So formally we have

$$x-1 = \sum_{i=0}^{n-1} x_i \cdot 2^i\,, \quad y-1 = \sum_{i=0}^{n-1} y_i \cdot 2^i\,, \quad q = 1 + \sum_{i=1}^{n-f} x_{f+i-1} 2^i$$

and $x_j = y_j$ for $j \in [f, n-1]$. Clearly, q is odd and

$$(q-1)\cdot 2^{f-1} \leq x-1 < q\cdot 2^{f-1} \leq y-1 < (q+1)\cdot 2^{f-1}\,,$$

wherefore f and q are as desired.

	2^{n-1}	2^{n-2}	$\dots$	2^f	2^{f-1}	2^{f-2}	$\dots$	1
$x-1$	x_{n-1}	x_{n-2}	$\dots$	x_f	0	x_{f-2}	$\dots$	x_0
$y-1$	x_{n-1}	x_{n-2}	$\dots$	x_f	1	y_{f-2}	$\dots$	y_0
$q\cdot 2^{f-1}$	x_{n-1}	x_{n-2}	$\dots$	x_f	1	0	$\dots$	0

The uniqueness of f and q may likewise be shown by studying the binary expansions of $x-1$ and $y-1$. An alternative argument proceeds as follows.

Given x and y, let (f, q) and (f', q') be two pairs with the requested properties. Due to symmetry we may suppose $f \leq f'$. Now we have $(q-1)\cdot 2^{f-1} < x \leq q'\cdot 2^{f'-1}$ and consequently $q \leq q'\cdot 2^{f'-f}$. Similarly $q'\cdot 2^{f'-1} < y \leq (q+1)2^{f-1}$ yields $q'\cdot 2^{f'-f} \leq q$. The combination of both estimates reveals $q = q'\cdot 2^{f'-f}$ but, since q is odd, this if only possible if $f = f'$ and $q = q'$. $\square$

We would like to point out that the uniqueness of f and q will be essential throughout the following arguments. By redoing the above proof of this uniqueness more carefully one can show the following monotonicity property of the function f.

Lemma 2.20. *For any three positive integers x, y, and z such that $x < y \leq z$ the inequality $f(x, y) \leq f(x, z)$ holds.*

Proof. For brevity we set $f = f(x, y)$, $q = q(x, y)$, $f' = f(x, z)$, and $q' = q(x, z)$. Arguing indirectly we assume $f' < f$. Now $(q' - 1) \cdot 2^{f'-1} < x \leq q \cdot 2^{f-1}$ entails $q' \leq q \cdot 2^{f-f'}$ and similarly $q \cdot 2^{f-1} < y \leq z \leq (q' + 1) \cdot 2^{f'-1}$ leads to $q \cdot 2^{f-f'} \leq q'$. Hence we must have $q' = q \cdot 2^{f-f'}$, contrary to the fact that q' is odd. □

The following will be a standard argument later on.

Lemma 2.21. *For any positive integers $x < y \leq z$ with $f(x, y) = f(x, z)$ we have $q(x, y) = q(x, z)$ and, consequently,*

$$(q - 1) \cdot 2^{f-1} < x \leq q \cdot 2^{f-1} < y \leq z \leq (q + 1) \cdot 2^{f-1},$$

where $f = f(x, y) = f(x, z)$ and $q = q(x, y) = q(x, z)$.

Proof. Define $q = q(x, y)$. Lemma 2.19 gives

$$(q - 1) \cdot 2^{f-1} < x \leq q \cdot 2^{f-1} < y \leq (q + 1) \cdot 2^{f-1}$$

and thus $q \cdot 2^{f-1}$ is the least multiple of 2^{f-1} which is at least x. Due to $f = f(x, z)$ this yields $q(x, z) = q$ and hence $z \leq (q + 1) \cdot 2^{f-1}$. □

Next we record another property of f that shall be used later.

Lemma 2.22. *If any four positive integers t, x, y, and z satisfy $t \leq x < y \leq z$ and $f(x, y) = f(x, z)$, then $f(t, y) = f(t, z)$ holds as well.*

Proof. Setting $f = f(t,z)$ and $q = q(t,z)$ we get

$$(q-1)\cdot 2^{f-1} < t \le q \cdot 2^{f-1} < z \le (q+1)\cdot 2^{f-1}$$

from the definition of these quantities.

Of course the claim would easily follow from $q \cdot 2^{f-1} < y$. So from now on we may assume $y \le q \cdot 2^{f-1}$ towards contradiction. This yields

$$(q-1)\cdot 2^{f-1} < t \le x < y \le q \cdot 2^{f-1} < z \le (q+1)\cdot 2^{f-1}\,,$$

and, in particular, we obtain $f(x,z) = f$ but $f(x,y) \ne f$, thus reaching a contradiction. □

To conclude our discussion of the auxiliary functions f and q we state how they interact with the map η introduced in Fact 2.17.

Fact 2.23. *For any integers x and y with $1 \le x < y \le 2^n$ we have*

$$\begin{aligned} f(x,y) &\in [n]\,, \\ f(2^n+1-y, 2^n+1-x) &= f(x,y)\,, \\ \textit{and} \quad q(2^n+1-y, 2^n+1-x) &= 2^{n+1-f} - q(x,y)\,. \end{aligned}$$

Again we leave the straightforward verification to the reader. We may now return to the problem of coloring the graphs $G_b(2^n)$.

Proposition 2.24. *We have*

$$\chi(G_b(2^n)) \le (2b-6) + 2^{2b-3}\,\chi(G_{b-1}(n))$$

for any integers $n \ge b \ge 3$.

Proof. For any vertex $\overline{x} = (x_1, x_2, \ldots, x_{2b-1})$ of $G_b(2^n)$ we use the abbreviations

$$\begin{aligned} f(\overline{x}) &= f(x_1, x_{2b-1})\,, \\ q(\overline{x}) &= q(x_1, x_{2b-1})\,, \\ T^-(\overline{x}) &= (q(\overline{x}) - 1) \cdot 2^{f(\overline{x})-1}\,, \\ T(\overline{x}) &= q(\overline{x}) \cdot 2^{f(\overline{x})-1}\,, \\ \text{and} \quad T^+(\overline{x}) &= (q(\overline{x}) + 1) \cdot 2^{f(\overline{x})-1}\,. \end{aligned}$$

Recall that by Lemma 2.19 we have

$$T^-(\overline{x}) < x_1 \le T(\overline{x}) < x_{2b-1} \le T^+(\overline{x}) \tag{2.10}$$

for any such vertex $\overline{x}$ and in the first steps of the current proof we will distinguish these vertices according to the position of their other entries x_i with respect to $T(\overline{x})$. To begin with, we partition $V_b(2^n)$ into three sets,

$$V_b(2^n) = A \cup B \cup C\,, \tag{2.11}$$

that are defined by

$$\begin{aligned} A &= \{\overline{x} = (x_1, x_2, \ldots, x_{2b-1}) \in V_b(2^n) \,|\, x_{2b-3} \le T(\overline{x})\}\,, \\ B &= \{\overline{x} = (x_1, x_2, \ldots, x_{2b-1}) \in V_b(2^n) \,|\, x_3 \le T(\overline{x}) < x_{2b-3}\}\,, \\ \text{and} \quad C &= \{\overline{x} = (x_1, x_2, \ldots, x_{2b-1}) \in V_b(2^n) \,|\, T(\overline{x}) < x_3\}\,. \end{aligned}$$

Again we identify subsets of $V_b(2^n)$ with the corresponding induced subgraphs of $G_b(2^n)$. We will use different colors for these three sets and commence by coloring B. This set may be partitioned further into

$$B = B_3 \cup B_4 \cup \ldots \cup B_{2b-4}\,,$$

where

$$B_i = \{\overline{x} = (x_1, x_2, \ldots, x_{2b-1}) \in V_b(2^n)\,|\, x_i \le T(\overline{x}) < x_{i+1}\}$$

for any integer index $i \in [3, 2b-4]$. We claim that each of these $2b-6$ sets is independent. To show this suppose that $\{\overline{x}, \overline{y}\}$ was an edge of $G_b(2^n)$ with $\overline{x}, \overline{y} \in B_i$ for some $i \in [3, 2b-4]$. Let the notation be as in Definition 2.14. By $\overline{x} \in B$, inequality (i) from Definition 2.14, and by (2.10) we have

$$T^-(\overline{x}) < x_1 < y_1 \le x_3 \le T(\overline{x}) < x_{2b-3} < y_{2b-3} \le x_{2b-1} \le T^+(\overline{x})\,,$$

whence $f(y_1, y_{2b-3}) = f(\overline{x})$ and $q(y_1, y_{2b-3}) = q(\overline{x})$. Due to $\overline{y} \in B$ this yields $f(\overline{y}) = f(\overline{x})$ and $q(\overline{y}) = q(\overline{x})$. For this reason $\overline{x}, \overline{y} \in B_i$ imply $y_i \le T(\overline{y}) = T(\overline{x}) < x_{i+1}$, contrary to part (ii) from Definition 2.14. So the sets B_i are indeed independent and we obtain

$$\chi(B) \le 2b - 6\,. \tag{2.12}$$

This accounts for the summand $2b-6$ on the right-hand side of our claim and we may proceed with analyzing A and C. Using Fact 2.23 it is not hard to check that the map η from Fact 2.17 constitutes an isomorphism between A and C, wherefore

$$\chi(A) = \chi(C)\,. \tag{2.13}$$

Now by (2.11), (2.12), and (2.13) we have

$$\chi(G_b(2^n)) \le \chi(A) + \chi(B) + \chi(C) \le (2b-6) + 2\,\chi(A)$$

and thus to finish the current proof we just need to show

$$\chi(A) \leq 2^{2b-4}\, \chi(G_{b-1}(n))\,. \tag{2.14}$$

The main idea for proving this is to split A into at most 2^{2b-4} further sets, each of which is either independent or has the property of being homomorphically mapped into $G_{b-1}(n)$ by a certain function ϕ that is to be introduced next. Observe that by the first statement from Fact 2.23 and by Lemma 2.20 there is a map $\phi : A \to W_{b-1}(n)$ defined by

$$\phi(x_1, x_2, \ldots, x_{2b-1}) = (f(x_1, x_3), f(x_1, x_4), \ldots, f(x_1, x_{2b-1}))$$

for any $(x_1, x_2, \ldots, x_{2b-1}) \in A$.

We call two vertices $\overline{x} = (x_1, x_2, \ldots, x_{2b-1})$ and $\overline{y} = (y_1, y_2, \ldots, y_{2b-1})$ from A equivalent and write $\overline{x} \sim \overline{y}$ if for every integer $i \in [3, 2b-2]$ we have

$$f(x_1, x_i) = f(x_1, x_{i+1}) \;\Leftrightarrow\; f(y_1, y_i) = f(y_1, y_{i+1})\,.$$

It is plain that $\sim$ is an equivalence relation and that the number of its equivalence classes is at most 2^{2b-4}. Thus to conclude the proof of (2.14) we just need to verify the following statement:

$$\text{if } \overline{x}, \overline{y} \in A,\ \overline{x} \sim \overline{y}, \text{ and } \{\overline{x}, \overline{y}\} \in E(G_b(2^n)), \text{ then } \{\phi(\overline{x}), \phi(\overline{y})\} \in E(G_{b-1}(n))\,. \tag{2.15}$$

So let any two equivalent vertices $\overline{x}$ and $\overline{y}$ from A be given and suppose that they are adjacent in $G_b(2^n)$, the notation for this being as in Definition 2.14. For any $i \in [2b-3]$ we set

$$\alpha_i = f(x_1, x_{i+2}) \quad \text{and} \quad \beta_i = f(x_1, y_{i+2})\,. \tag{2.16}$$

Notice that there is no misprint in (2.16) – it is true that $\beta_i = f(y_1, y_{i+2})$ holds as well, and actually this fact is very relevant to our main concern, but it will only be shown at a rather late moment of our argument.

Combining the assumption that $\{\overline{x}, \overline{y}\}$ be an edge of $G_b(2^n)$ with Lemma 2.20 we infer

$$\alpha_1 \le \beta_1 \le \alpha_3 \le \beta_3 \le \ldots \le \alpha_{2b-3} \le \beta_{2b-3} \tag{2.17}$$

as well as

$$\alpha_{j+1} \le \beta_j \text{ for } j \in [2b-4]\,. \tag{2.18}$$

Next we would like to show

$$\alpha_{2b-3} < \beta_{2b-3}\,. \tag{2.19}$$

Assume towards a contradiction that $\alpha_{2b-3} = \beta_{2b-3}$, i.e., $f(\overline{x}) = f(x_1, y_{2b-1})$. Lemma 2.21 yields

$$T^-(\overline{x}) < x_1 \le T(\overline{x}) < x_{2b-1} < y_{2b-1} \le T^+(\overline{x})\,,$$

so in combination with $\{\overline{x}, \overline{y}\}$ being an edge and with $\overline{x} \in A$ we obtain

$$T^-(\overline{x}) < x_1 < y_1 \le x_{2b-3} \le T(\overline{x}) < x_{2b-1} \le y_{2b-2} \le y_{2b-1} \le T^+(\overline{x})\,.$$

It follows that $T(\overline{y}) = T(\overline{x})$ and $f(y_1, y_{2b-2}) = f(y_1, y_{2b-1}) = f(\overline{x})$. Using $\overline{x} \sim \overline{y}$ we may deduce $f(x_1, x_{2b-2}) = f(x_1, x_{2b-1})$. Now Lemma 2.21 shows that $q(x_1, x_{2b-2}) = q(x_1, x_{2b-1})$ holds as well and consequently we have $T(\overline{x}) < x_{2b-2} \le y_{2b-3}$. Thus we get a contradiction to $\overline{y} \in A$, whereby (2.19) is proved.

Extending this result we contend that more generally we have

$$\alpha_i < \beta_i \quad \text{for all } i \in [2b-3]\,. \tag{2.20}$$

Arguing indirectly again, we let i denote the largest counterexample to this claim. Notice that (2.19) tells us $i \le 2b-4$. Set $q = q(x_1, x_{i+2})$, $T^- = (q-1)\cdot 2^{\alpha_i - 1}$, $T = q \cdot 2^{\alpha_i-1}$, and $T^+ = (q+1)\cdot 2^{\alpha_i-1}$. Due to Lemma 2.21 our indirect assumption $\alpha_i = \beta_i$ entails

$$T^- < x_1 \le T < x_{i+2} \le y_{i+2} \le T^+\,,$$

which together with $x_{i+2} \le x_{i+3} \le y_{i+2}$ shows $f(x_1, x_{i+2}) = f(x_1, x_{i+3})$. Now $\overline{x} \sim \overline{y}$ discloses $f(y_1, y_{i+2}) = f(y_1, y_{i+3})$ and by Lemma 2.22 it follows that $f(x_1, y_{i+2}) = f(x_1, y_{i+3})$. Using Lemma 2.21 again we obtain

$$T^- < x_1 \le T < x_{i+3} \le y_{i+3} \le T^+$$

and thus $\alpha_{i+1} = \beta_{i+1}$, contrary to the maximality of i. Thereby (2.20) is proved as well.

Now we are ready to confirm the alternative definition of β_i announced above. That is, for any $i \in [2b-3]$ we claim

$$\beta_i = f(x_1, y_{i+2}) = f(y_1, y_{i+2}) \tag{2.21}$$

where the second equality is to be verified. To see this, set $q = q(x_1, y_{i+2})$, $S^- = (q-1)\cdot 2^{\beta_i-1}$, $S = q\cdot 2^{\beta_i-1}$, and $S^+ = (q+1)\cdot 2^{\beta_i-1}$. Now

$$S^- < x_1 \le S < y_{i+2} \le S^+$$

and $x_3 < y_3 \le y_{i+2}$. Hence $S < x_3$ would entail

$$S^- < x_1 \le S < x_3 \le S^+$$

and, consequently, $\alpha_1 = f(x_1, x_3) = \beta_i \ge \beta_1$, which contradicts the case $i = 1$

of (2.20). This proves $x_1 < y_1 \leq x_3 \leq S$, which in turn establishes (2.21).

Putting everything together, the equations (2.16) and (2.21) yield

$$\phi(\overline{x}) = (\alpha_1, \alpha_2, \ldots, \alpha_{2b-3}) \quad \text{and} \quad \phi(\overline{y}) = (\beta_1, \beta_2, \ldots, \beta_{2b-3})$$

and by (2.20) we may strengthen (2.17) to

$$\alpha_1 < \beta_1 \leq \alpha_3 < \beta_3 \leq \ldots \leq \alpha_{2b-3} < \beta_{2b-3} \,.$$

In particular, this shows that $\phi(\overline{x})$ and $\phi(\overline{y})$ are indeed vertices of $G_{b-1}(n)$ and together with (2.18) it further shows that these two vertices are adjacent. This concludes the proof of (2.15) and, hence, the proof of Proposition 2.24.

□

To summarize, we would like to emphasize again that (2.8), Lemma 2.18 and Proposition 2.24 taken together yield an easy proof of Theorem 2.16 by induction on b. In addition, the combination of Proposition 2.12, Theorem 2.15, and Theorem 2.16 implies Theorem 2.7.

2.5 Reducible Types

Having thus said everything we want to say about the chromatic number of irreducible type-graphs, we devote the present section to the proof of Theorem 2.8. So we consider any nontrivial type τ and let $\tau = \rho_1\rho_2\ldots\rho_t$ be its factorization into irreducible types. For each $i \in [t]$ the number of blocks into which ρ_i decomposes is denoted by b_i and we set $b^* = \max(b_1, \ldots, b_t)$. Finally, let k be the width of τ and let ρ_i have width k_i for $i \in [t]$.

The notation introduced up to this moment will be used throughout this section without being repeated in the numbered statements that will occur.

Recall that our goal is to show

$$\chi(G(n,\tau)) = \Theta(\log_{(b^*-2)} n)\,.$$

Here we have $b^* \geq 2$ because otherwise each factor ρ_i of τ would have to be equal to 3, meaning that τ were trivial. Again we treat the lower bound and the upper bound separately, but this time the latter is easier, so we start with it.

Fact 2.25. *For every $i \in [t]$ and every integer $n \geq k$ there is a graph homomorphism*

$$\phi_i : G(n,\tau) \to G(n,\rho_i)\,.$$

Proof. Set $r = \mathbf{1}(\rho_1\rho_2\ldots\rho_{i-1})$ and $s = \mathbf{1}(\rho_1\rho_2\ldots\rho_i)$. Clearly ρ_i has width $k_i = s - r$, and, since $\rho_1, \rho_2, \ldots, \rho_i$ are types, we also have $r = \mathbf{2}(\rho_1\rho_2\ldots\rho_{i-1})$ and $s = \mathbf{2}(\rho_1\rho_2\ldots\rho_i)$. Now it easy to confirm that the map

$$\phi_i : \binom{[n]}{k} \to \binom{[n]}{k_i}$$

given by

$$\phi_i(\{x_1, x_2, \ldots, x_k\}) = \{x_{r+1}, x_{r+2}, \ldots, x_s\}$$

whenever $1 \leq x_1 < x_2 < \ldots < x_k \leq n$ is a graph homomorphism. $\square$

Applying this, in particular, to some index $i^* \in [t]$ with $b_{i^*} = b^*$ we may deduce the following by means of Theorem 2.7.

Fact 2.26. *As n tends to infinity we have*

$$\chi(G(n,\tau)) \leq (2^{(b^*-2)^2} + o(1)) \log_{(b^*-2)} n\,. \tag{2.22}$$

In the other direction, we will use Proposition 2.12 to homomorphically map the generalized shift graph $G(n, \sigma_{b^*-1})$ into $G(kn, \tau)$.

Fact 2.27. *For every integer $n \geq b^*$ there is a graph homomorphism*

$$\psi : G(n, \sigma_{b^*-1}) \to G(kn, \tau)$$

and, consequently, we have

$$(1 + o(1)) \log_{(b^*-2)} \frac{n}{k} \leq \chi(G(n, \tau)) \,. \tag{2.23}$$

Proof. Let $I = \{i \in [t] \,|\, \rho_i \neq 3\}$ and write $c_i = \sum_{j=1}^{i} k_j$ for every integer $i \in [0, t]$. Recall that we know from Proposition 2.12 that for every index $i \in I$ there exists a homomorphism $\psi_i : G(n, \sigma_{b_i-1}) \to G(k_i n, \rho_i)$. Utilizing these, we define for each $i \in [t]$ a map

$$\widehat{\psi}_i : \binom{[n]}{b^* - 1} \to \binom{[c_{i-1}n + 1, c_i n]}{k_i}$$

by stipulating

$$\widehat{\psi}_i(\{h_1, h_2, \ldots, h_{b^*-1}\}) = \begin{cases} c_{i-1}n + \psi_i(\{h_1, h_2, \ldots, h_{b_i-1}\}\,\mathrm{Br} & \text{if } i \in I, \\ \{c_i n\} & \text{if } i \notin I\,, \end{cases}$$

whenever $1 \leq h_1 < h_2 < \ldots < h_{b-1} \leq n$, where the addition of a number to a set in the upper case is to be performed "elementwise". It is easy to check that the map

$$\psi : \binom{[n]}{b^* - 1} \to \binom{[kn]}{k}$$

given by

$$\psi(X) = \bigcup_{i \in [t]} \widehat{\psi}_i(X)$$

for all $X \in \binom{[n]}{b^*-1}$ is indeed a homomorphism from $G(n, \sigma_{b^*-1})$ to $G(kn, \tau)$.

Formula (2.23) follows from the mere existence of ψ and from Theorem 2.4. $\square$

Owing to (2.22) and (2.23) the proof of Theorem 2.8 is complete.

2.5.1 Concluding Remarks

In this chapter, we have shown that if τ has factorization into irreducible types $\rho_1\rho_2\ldots\rho_t$ where ρ_i has b_i blocks for $i \in [t]$, and set $b^* = \max(b_1, \ldots, b_t)$, then there exist constants c_1 and c_2 so that

$$c_1 \cdot \log_{(b^*-2)}(n) \leq \chi(G(n,\tau)) \leq c_2 \cdot \log_{(b^*-2)}(n).$$

where

$$\frac{1}{k} \leq c_1 \leq c_2 \leq 2^{(b^*-2)^2}.$$

It is natural to ask, what is the chromatic number of $G(n,\tau)$? More precisely, we formulate the following.

Problem 2.28. *Let τ be an irreducibe type with b blocks. Compute precisely c_τ so that*

$$\chi(G(n,\tau)) = (c_\tau + o(1)) \log_{(b-2)} n$$

Since $\tau = 112122$ is an irreducible type with 3 blocks, Theorem 2.7 implies $\chi(\text{Specker}(n,3)) = \Theta(\log n)$. On the other hand, Erdős and Rado [ER60] showed that for κ an infinite cardinal, $\chi(\text{Specker}(\kappa,3)) = \kappa$. Hence, the chromatic number of infinite type-graphs behaves differently than that of finite type-graphs. It is natural to ask the following.

Problem 2.29. *Given an infinite cardinal κ and a type τ, compute precisely $\chi(G(\kappa,\tau))$.*

One can define type hypergraphs in a manner similar to types. Given $X_1, X_2, \ldots, X_t \in \binom{[n]}{k}$, we say that $(\tau_1, \tau_2, \ldots, \tau_\ell) = \tau \in [2^t - 1]^\ell$ is the type of $X_1, X_2, \ldots, X_t$ if $\cup_{j=1}^t X_j = \{z_1 < z_2 < \ldots < z_\ell\}$, and τ_i is the number whose unique binary representation has a 1 in position j if and only if $z_i \in X_j$. Denote the corresponding type-hypergraph $\mathcal{H}(n, \tau)$.

Problem 2.30. *Establish bounds on* $\chi(\mathcal{H}(n, \tau))$.

Chapter 3

Property O

This chapter is motivated by two types of problems concerning hypergraphs. The first is well-known and regards 2-colorable hypergraphs, also said to possess Property B. Several papers have presented bounds on $m(k)$, the minimum number of edges in a k-uniform hypergraph that does not have Property B (see [Bec78] [Spe81] [RS00] [CK15]). The next section contains a brief overview of the background of Property B. The second comes from Ramsey theory, where appropriate properties of graphs containing a given graph with a fixed order can be used to prove negative partition relations for unordered graphs (see [NR75] [NR78] for early papers on this topic). The entirety of this chapter is based on joint work with Duffus and Rödl [DKR17].

3.1 Property B

We give a brief introduction to the work on Property B. The history of this problem is rich, and the probabilistic techniques used in proving the best known upper bound for $m(k)$ serve as a template to produce a probabilistic upper bound on $f(k)$, the minimum number of edges in an oriented k-graph with Property O (to be defined in the next section). In some sense, Property O serves as an ordered analogue to the coloring problem of Property B, and much of the effort in establishing non-trivial lower bounds on $f(k)$ has been motivated by lower bounds on $m(k)$.

Property B is named for Felix Bernstein, who introduced the property in [Ber07]. The first lower bound of 2^{k-1} was established in 1963 by Erdős [Erd63] and served as an early application of the probabilistic method. Suppose that $\mathcal{H} = (V, \mathcal{E})$ is a k-graph with $|\mathcal{E}| < 2^{k-1}$. If we color the vertices red and blue uniformly at random, the probability that any fixed edge is monochromatic is $\frac{1}{2^{k-1}}$. By the union bound, the probability that *some* edge is monochromatic is *at most* $|\mathcal{E}|\frac{1}{2^{k-1}} < 1$ by hypothesis. Hence there is some coloring under which no edge is monochromatic. The first (and current best) upper bound came a year later in 1964 [Erd64], wherein Erdős used probabilistic methods to prove the existence of a k-graph with $O(k^2 2^k)$ edges which fails to have Property B. These probabilistic arguments serve as the archetypes for the bounds on $f(k)$ in Theorem 3.2.

In 1978, Beck [Bec78] improved the lower bound to $\Omega(k^{1/3}2^k)$. The argument was again probabilistic, this time employing the so-called method of *alterations*. Suppose $\mathcal{H} = (V, \mathcal{E})$ is a k-graph with $|\mathcal{E}| = \Omega(k^{1/3}2^k)$. First, color the vertices red and blue uniformly at random. For each vertex v which is in some monochromatic edge, recolor v with probability p, where p is some small prescribed probability. Beck shows that with positive probability, no edge in the recoloring is monochromatic. Beck's argument was essentially probabilistic, but was written as a counting argument. In 1981, Spencer [Spe81] rewrote Beck's argument succinctly in the language of probability theory, and any interested readers are encouraged to read this treatment.

In 2000, Radhakrishnan and Srinivasan [RS00] improved the lower bound to $\Omega(\sqrt{k/\log k}2^k)$ by a modification of Beck's argument, where the vertices that are candidates for recoloring are considered sequentially according to a probabilistic algorithm. The same lower bound was reproven in 2015 by Cherkashin and Kozik [CK15] by a nice application of a random greedy algorithm.

3.2 Definitions and Overview

We would like to determine the minimum number of edges in a oriented uniform hypergraph needed to ensure that for every ordering of the vertex set, some edge is ordered in the same way. In some sense, we seek to establish an ordered analogue to Property B, where the bounds in question are normalized by $k!$ (as opposed to 2^k, as with Property B). Here are the required definitions followed by our results and a conjecture.

Fix a positive integer $k \geq 2$ and a finite set V. An *ordered k-set* $\overline{X}$ is a k-tuple $(x_1, x_2, \ldots, x_k)$ of distinct elements of V; we use X to denote the unordered set $\{x_1, x_2, \ldots, x_k\}$. Given a family of ordered k-sets $\mathcal{E} \subseteq V^k$ with no two k-tuples on the same k-element set, call $\mathcal{H} = (V, \mathcal{E})$ an *oriented k-uniform hypergraph*, or, more briefly, an *oriented k-graph*. In the case that $\mathcal{E}$ contains an ordered k-set for each k-element subset of V, call $\mathcal{H}$ a *k-tournament*. So, a k-tournament is obtained from the complete k-uniform hypergraph $K_n^{(k)}$ by giving each k-set an orientation. For $X \subseteq V$ and a linear order $<$ on V, an ordered k-set $\overline{X} = (x_1, x_2, \ldots x_k)$ is *consistent* with $<$ if $x_1 < x_2 < \ldots < x_k$.

Here is the property that interests us.

Definition 3.1 (Property O)**.** *Given an oriented k-graph $\mathcal{H} = (V, \mathcal{E})$ we say that $\mathcal{H}$ has the ordering property, or* Property O*, if for every (linear) order $<$ of V there exists $\overline{X} \in \mathcal{E}$ that is consistent with $<$.*

For an integer $k \geq 2$, let $f(k)$ be the minimum number of edges in an oriented k-graph with Property O. Here is what we know about bounds for $f(k)$.

Theorem 3.2. *The function $f(k)$ satisfies $k! \leq f(k) \leq (k^2 \ln k)k!$ where the lower bound holds for all k and the upper bound for $k \geq k_0$.*

The upper bound for $f(k)$ is proven in Section 3.3. The lower bound $k! \leq f(k)$ follows from a standard argument. Given any oriented k-graph $\mathcal{H} = ([n], \mathcal{E})$, clearly each $\overline{X} \in \mathcal{E}$ is consistent with

$$(n-k)!\binom{n}{k} = \frac{n!}{k!}$$

orders on $[n]$. Consequently, if $\mathcal{H}$ has Property O then

$$|\mathcal{E}| \cdot \frac{n!}{k!} \ \geq \ n!, \text{ so } |\mathcal{E}| \geq k!.$$

We would like to decide if $f(k)$ is bounded away from $k!$, in analogy with Property B.

Problem 3.3. *Determine whether* $\dfrac{f(k)}{k!} \to \infty$ *as* $k \to \infty$.

We are unable to improve the simple lower bound for $f(k)$ at this point, however we can show that for appropriately chosen k and $n = n(k)$, almost all k-tournaments on n vertices fail to have Property O. This is made precise in Theorem 3.4. Let $\mathcal{T}_{n,k}$ denote the set of all k-tournaments on $[n]$.

Theorem 3.4. *Let* $0 < \alpha < 1$, *let* $c = \frac{2\pi}{3e}e^{e^2/2}$ *and let* $n = (c\alpha)^{1/k}\left(\frac{k}{e}\right)^2 k^{3/2k}$. *Then for* k *sufficiently large at least* $(1-\alpha)|\mathcal{T}_{n,k}|$ *do not have Property O.*

In Section 3.3, we prove the upper bound of $f(k)$ given in Theorem 3.2. In Section 3.4 we prove Theorem 3.4. In Section 3.5 we provide a construction of k-graphs with Property O, investigate the situation for small values of n and k, and pose a few problems.

We close this section with an observation used in both Sections 3.3 and 3.4.

Fact 3.5. *Let*

$$n = \left(\frac{k}{e}\right)^2 (1 + o(1)). \tag{3.1}$$

Then

$$\binom{n}{k} = \left(e^{-e^2/2}\right)\frac{n^k}{k!}(1 + o(1)). \tag{3.2}$$

Proof. Indeed, (3.1) implies that

$$\begin{aligned}
\frac{(n)_k}{n^k} &= \left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\cdots\left(1-\frac{k-1}{n}\right) \\
&= \exp\left(\sum_{j=1}^{k-1}\ln(1-j/n)\right) \\
&= \exp\left((1+o(1))\sum_{j=1}^{k-1}-j/n\right) \\
&= \exp\left((1+o(1))\left(-\binom{k}{2}/n\right)\right) \\
&= \exp\left((1+o(1))\left(-e^2/2\right)\right) \;=\; (1+o(1))e^{-e^2/2},
\end{aligned}$$

so (3.2) holds. □

3.3 Proof of Theorem 3.2

We verify the upper bound in Theorem 3.2 by showing that for k large enough, there exists a k-tournament with $(k^2\ln k)k!$ edges which has Property O. Indeed, we show that for an appropriate choice of n, a randomly selected member of $\mathcal{T}_{n,k}$ has Property O with positive probability.

Let $\mathcal{H} = ([n],\mathcal{E}) \in \mathcal{T}_{n,k}$ be sampled uniformly. Note that this is equivalent to orienting the edges of a k-tournament independently according to the uniform distribution. For a fixed order $<$ on $[n]$ and a fixed $\overline{X} \in \mathcal{E}$, the probability that $\overline{X}$ is not consistent with $<$ is $1-\frac{1}{k!}$. Since the edges of $\mathcal{H}$ are oriented independently, the probability that no edge of $\mathcal{H}$ is consistent with $<$ is $(1-\frac{1}{k!})^{\binom{n}{k}}$. Taking the union bound over all orders on V, we see that the probability that there exists an order $<$ on V so that no edge of $\mathcal{H}$ is consistent with $<$ is at most $n!(1-\frac{1}{k!})^{\binom{n}{k}}$.

The upper bound follows once we verify (3.3) and (3.4), below, for k sufficiently large.

Let $n = \left(\frac{k}{e}\right)^2 \left(\pi \cdot \exp(e^2/2) \cdot k^3 \ln k\right)^{1/k}$. Then

$$\binom{n}{k}\frac{1}{k!} \;\leq\; k^2 \ln k, \text{ and} \tag{3.3}$$

$$n! \left(1 - \frac{1}{k!}\right)^{\binom{n}{k}} \;<\; 1. \tag{3.4}$$

To prove (3.3), we apply Fact 3.5 and the Stirling approximation $k! = (k/e)^k \sqrt{2\pi k}(1 + o(1))$:

$$\begin{aligned}\binom{n}{k}\frac{1}{k!} &= e^{-e^2/2}\left(\frac{(k/e)^{2k}\,\pi \cdot e^{e^2/2} \cdot k^3 \ln k}{(k!)^2}\right)(1 + o(1)) \\ &= \frac{1}{2}k^2 \ln k(1 + o(1)).\end{aligned} \tag{3.5}$$

Hence (3.3) holds for k sufficiently large.

Turning to inequality (3.4), we use the choice of n and (3.5) to infer that

$$\begin{aligned}n \ln n &= 2\left(\frac{k}{e}\right)^2 \ln k(1 + o(1)) \\ &< \binom{n}{k}\frac{1}{k!},\end{aligned} \tag{3.6}$$

for k sufficiently large. We have

$$\begin{aligned}n! \left(1 - \frac{1}{k!}\right)^{\binom{n}{k}} &\leq n^n \left(1 - \frac{1}{k!}\right)^{\binom{n}{k}} \\ &\leq \exp(n \ln n) \cdot \exp\left(-\binom{n}{k}\frac{1}{k!}\right) \\ &= \exp\left(n \ln n - \binom{n}{k}\frac{1}{k!}\right) \\ &< 1.\end{aligned}$$

where the last inequality follows from (3.6). This proves (3.4) and completes the proof of the upper bound.

3.4 Proof of Theorem 3.4

Let α, c, and n be as in the statement of Theorem 3.4. We first obtain an expression for $\binom{n}{k}$, the number of edges in a k-tournament in $\mathcal{T}_{n,k}$, which we use in the proof of Theorem 3.4.

Applying Stirling's formula to $k!$:

$$\begin{aligned} n^k &= c\alpha\left(\frac{k}{e}\right)^{2k} k^{3/2} \\ &= \frac{e^{e^2/2}}{3e}\alpha(k!)^2 k^{1/2}(1+o(1)). \end{aligned} \tag{3.7}$$

On the other hand, by Fact 3.5,

$$n^k = e^{e^2/2}(n)_k(1+o(1)). \tag{3.8}$$

Equate the right hand sides of (3.7) and (3.8) and, for brevity, set $\omega = (\alpha/3e)k^{1/2}(1+o(1))$:

$$\begin{aligned} \binom{n}{k} &= \frac{\alpha}{3e}k^{1/2}k!(1+o(1)) \\ &= \omega k! \end{aligned} \tag{3.9}$$

the estimate we require.

We will show that if T is sampled from $\mathcal{T}_{n,k}$, the set of all k-tournaments on $[n]$, according to the uniform distribution, the probability that T has Property O is at most α. It will follow that at least $(1-\alpha)|\mathcal{T}_{n,k}|$ members of $\mathcal{T}_{n,k}$ fail to have Property

O.

The random sampling of $T = ([n], \mathcal{E})$ from $\mathcal{T}_{n,k}$ is done in two phases. In the first phase we will select k-tuples that are consistent with the natural order $<$ on $[n]$ and in the second phase we will assign to the remaining k-tuples one of the $k! - 1$ remaining orientations.

3.4.1 Phase 1: reveal consistent edges

In phase 1, reveal the set $C(T)$ of the members of $\mathcal{E}$ that are oriented consistently with $<$.

For any $\overline{X} \in \mathcal{E}$, $\mathbb{P}(\overline{X} \in C(T)) = 1/k!$ and thus, by (3.9),

$$E(|C(T)|) = \binom{n}{k}\frac{1}{k!} = \omega.$$

Let A_ω be the event that $|C(T)| \leq \frac{2}{\alpha}\omega$. By Markov's inequality we have

$$\mathbb{P}\left(|C(T)| > \frac{2}{\alpha}\omega\right) < \frac{\alpha}{2} \text{ and so } \mathbb{P}(A_\omega) > 1 - \frac{\alpha}{2}.$$

Assume that A_ω occurs. For each $\overline{X} \in \mathcal{E}$, as before, let $\min X$ be the $<$-least element of X. Define

$$M = \{\min X | \overline{X} \in C(T)\}$$

and note that

$$|M| \leq |C(T)| \leq \frac{2}{\alpha}\omega \ < k - 1.$$

Thus, for each $\overline{X} \in C(T)$, $X \setminus M \neq \emptyset$. Let $W \subseteq [n]$ be obtained by selecting one element from each $X \setminus M$. We now define $<'$ to be the natural order $<$ on each of W and $[n] \setminus W$, and let $u <' v$ for $u \in W$, $v \in [n] \setminus W$.

We claim that no $\overline{X} \in C(T)$ is consistent with $<'$. To see this, let $v \in X \cap W$. On the one hand, $\min X \notin W$ by the way that we selected W. On the other hand,

$v <' \min X$ by the definition of $<'$. However, $\overline{X} \in C(T)$ means precisely that $\overline{X}$ is consistent with $<$, and so $v <' \min X$ is a contradiction.

3.4.2 Phase 2: reveal inconsistent edges

In phase 2, reveal the orientation of the edges not in $C(T)$. For each $\overline{X} \notin C(T)$ there are $k! - 1$ possible orientations of $\overline{X}$ in T – any one except that given by the natural order $<$. Since T is chosen according to the uniform distribution, each orientation is equally likely and at most one of these is consistent with $<'$. Thus,

$$\mathbb{P}(\overline{X} \text{ is consistent with } <') \leq 1/(k! - 1).$$

Also, if $X \cap W = \emptyset$, then $<$ and $<'$ coincide on K, so $\overline{X}$ cannot be consistent with $<'$. Set $\omega' = \frac{2}{\alpha}\omega \geq |W|$.

Since the only k-tuples which may become consistent with $<'$ are those which have a nonempty intersection with W, in view of (3.9),

$$\mathbb{P}(\exists \overline{X} \notin C(T) \text{ consistent with } <' | A_\omega)$$

is bounded above by

$$\begin{aligned}
\left(\sum_{j=1}^{\omega'} \binom{\omega'}{j}\binom{n-\omega'}{k-j}\right)\frac{1}{k!-1} &= \left(\sum_{j=1}^{\omega'} \binom{\omega'}{j}\binom{n-\omega'}{k-j}\right)\frac{\omega}{\binom{n}{k}}(1+o(1)) \\
&= \omega \sum_{j=1}^{\omega'} \frac{(\omega')_j (n-\omega')_{k-j} k!}{j!(k-j)!(n)_k}(1+o(1)) \\
&\leq \omega \sum_{j=1}^{\omega'} \frac{(k)_j (\omega')_j}{j!(n)_j}(1+o(1)). \qquad (3.10)
\end{aligned}$$

The inequality in (3.10) holds because $j \leq \omega'$ and so $\dfrac{(n-\omega')_{k-j}}{(n)_k} \leq \dfrac{1}{(n)_j}$.

It is straightforward to argue that for all numbers a, b, c satisfying $1 \leq a, b < c$,

$$\frac{(a-1)(b-1)}{c-1} < \frac{ab}{c}.$$

Repeated application of this shows that the expression in (3.10) is bounded above by

$$\begin{aligned}
\omega \sum_{j=1}^{\omega'} \frac{1}{j!} \left(\frac{k\omega'}{n} \right)^j (1+o(1)) &\leq \omega \left(e^{k\omega'/n} - 1 \right) (1+o(1)) \\
&\leq \omega \frac{k\omega'}{n} (1+o(1)) \\
&= \frac{2k\omega^2}{\alpha n} (1+o(1)). \qquad (3.11)
\end{aligned}$$

We apply

$$\omega = (\alpha/3e)k^{1/2}(1+o(1)) \text{ and } n = \left(\frac{k}{e} \right)^2 (1+o(1))$$

to the expression on the right hand side of (3.11) to obtain

$$\begin{aligned}
\frac{2k\omega^2}{\alpha n}(1+o(1)) &= \frac{2\alpha}{9} \frac{(k/e)^2}{n} (1+o(1)) \\
&= \frac{2\alpha}{9}(1+o(1)) \\
&< \frac{\alpha}{2}
\end{aligned}$$

for k sufficiently large. Therefore

$$\mathbb{P}(\exists \overline{X} \notin C(T) \text{ consistent with } <' | A_\omega) < \frac{\alpha}{2}$$

If T has Property O then either A_ω does not occur, or A_ω does occur and some $\overline{X} \notin C(T)$ is consistent with $<'$. Consequently, we have

$$\begin{aligned}\mathbb{P}(T \text{ has Property O}) &\leq \mathbb{P}(A_\omega^c) + \mathbb{P}(A_\omega)\mathbb{P}(\exists \overline{X} \notin C(T) \text{ consistent with } <' | A_\omega) \\ &< \frac{\alpha}{2} + \frac{\alpha}{2} \\ &= \alpha.\end{aligned}$$

Hence, the probability that T fails to have Property O is at least $(1-\alpha)$. Since T is a uniform selection from $\mathcal{T}_{n,k}$, this is equivalent to saying at least $(1-\alpha)|\mathcal{T}_{n,k}|$ members of $\mathcal{T}_{n,k}$ fail to have Property O, as desired. This completes the proof of Theorem 3.4.

3.5 A Construction, Small Values of n, and Problems

We have an upper bound for $f(k)$, the minimum number of edges in oriented k-graphs with Property O, in Theorem 3.2: for $k \geq k_0$, $f(k) \leq (k^2 \ln k)k!$. We have a construction of oriented k-graphs with Property O, for all $k \geq 2$. While these k-graphs have edge sets that are larger than the upper bound obtained by the probabilistic proof in Section 3.3, the hypergraphs are not unreasonably large.

For each $k \geq 2$ we construct an oriented k-graph $\mathcal{G}_k = (V_k, \mathcal{E}_k)$ that has Property O, where

$$|V_k| = 3^{k-1} \text{ and } |\mathcal{E}_k| = 2^{2(k-2)} \cdot 3^{\binom{k-1}{2}+1}. \tag{3.12}$$

To begin, let $\mathcal{G}_2 = (V_2, \mathcal{E}_2)$ be an oriented 3-cycle. It is clear that $\mathcal{G}_2$ has Property O and its vertex and edge sets have the sizes given in (3.12). Notice that the number of edges given in (3.12) provides an upper bound on $f(k)$ which is weaker than the probabilistic upper bound given in Theorem 3.2. This can be seen by application of Stirling's approximation and a crude analysis:

$$\begin{aligned}
(k^2 \log k)k! &\approx (k^2 \log k)\left(\frac{k}{e}\right)^k \sqrt{2\pi k} \\
&< 2^{2(k-2)}\left(\frac{k}{e}\right)^k \\
&< 2^{2(k-2)}\left(3^k\right)^k \\
&= O(2^{2(k-2)}3^{k^2})
\end{aligned}$$

where the last expression is the growth rate of the bound given in (3.12).

Here is the induction hypothesis: $\mathcal{G}_k = (V_k, \mathcal{E}_k)$ is an oriented k-graph with Property O and satisfies the conditions in (3.12). Let X, Y and Z be three disjoint copies of V_k and let $\mathcal{G}_X = (X, \mathcal{E}_X), \mathcal{G}_Y = (Y, \mathcal{E}_Y)$ and $\mathcal{G}_Z = (Z, \mathcal{E}_Z)$ each be isomorphic to $\mathcal{G}_k$. Define $\mathcal{G}_{k+1} = (V_{k+1}, \mathcal{E}_{k+1})$ as follows. (See Figure 3.1)

- Let $V_{k+1} = X \cup Y \cup Z$.
- Let $\mathcal{E}_{k+1}$ be comprised for these four types of $(k+1)-$tuples:

$$\begin{aligned}
T_1 &= \{(x, y_1, y_2, \ldots, y_k) : \ x \in X \text{ and } (y_1, y_2, \ldots, y_k) = \overline{y} \in \mathcal{E}_Y\}; \\
T_2 &= \{(\overline{z}, x) : \ \overline{z} \in \mathcal{E}_Z \text{ and } x \in X\}; \\
T_3 &= \{(\overline{y}, z) : \ \overline{y} \in \mathcal{E}_Y \text{ and } z \in Z\}; \\
T_4 &= \{(\overline{x}, y) : \ \overline{x} \in \mathcal{E}_X \text{ and } y \in Y\}.
\end{aligned}$$

To see that $\mathcal{G}_{k+1}$ has Property O, let $<$ be any linear order on V_{k+1}. We find a member of $\mathcal{E}_{k+1}$ consistent with $<$ as follows.

1. Suppose there is $x \in X$ such that $x < \min Y$. Since $\mathcal{G}_Y$ has Property O there is some $\overline{y} \in \mathcal{E}_Y$ consistent with $<$. Thus, $(x, \overline{y}) \in T_1$ is consistent with $<$.

2. Suppose there is $x \in X$ such that $\max Z < x$. Since $\mathcal{G}_Z$ has Property O there is some $\overline{z} \in \mathcal{E}_Z$ consistent with $<$. Thus, $(\overline{z}, x) \in T_2$ is consistent with $<$.

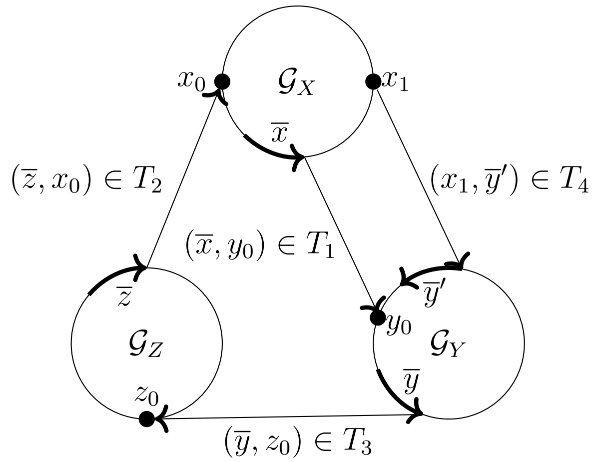


Figure 3.1: Constructing $\mathcal{G}_{k+1}$ from $\mathcal{G}_k$.

By 1 and 2 , we may assume that

$$\text{for all } x \in X \text{ there exist } y_x \in Y \text{ and } z_x \in Z \text{ such that } y_x < x < z_x.$$

Let $x_0 = \max X$. Then $x \leq x_0 < z_{x_0}$ for all $x \in X$.

3. Suppose all $y \in Y$ satisfy $y < z_{x_0}$. Since $\mathcal{G}_Y$ has Property O there is some $\overline{y} \in \mathcal{E}_Y$ consistent with $<$. Then $(\overline{y}, z_{x_0}) \in T_3$ is consistent with $<$.

4. Suppose some $y \in Y$ satisfies $z_{x_0} < y$. Then for all $x \in X$, $x < y$. Since $\mathcal{G}_X$ has Property O there is some $\overline{x} \in \mathcal{E}_X$ consistent with $<$. Then $(\overline{x}, y) \in T_4$ is consistent with $<$.

Therefore, $\mathcal{G}_{k+1}$ has Property O. (Note that 1 - 4 use all types of edges.) Let us see that $\mathcal{G}_{k+1} = (V_{k+1}, \mathcal{E}_{k+1})$ satisfies the conditions in (3.12). First,

$$\begin{aligned} |V_{k+1}| &= 3|V_k| \\ &= 3 \cdot 3^{k-1} \end{aligned}$$

$$= 3^k.$$

Second,

$$\begin{aligned}|\mathcal{E}_{k+1}| &= 4 \cdot |\mathcal{E}_k| \cdot |V_k| \\ &= 4 \cdot 2^{2(k-2)} \cdot 3^{\binom{k-1}{2}+1} \cdot 3^{k-1} \\ &= 2^{2k} \cdot 3^{\binom{k}{2}+1}.\end{aligned}$$

3.5.1 Concluding Remarks

Let $n(k)$ be the minimum number of vertices in a k-tournament with Property O. We have already seen that for any oriented k-graph to have Property O, it must have at least $k!$ edges. Since $\binom{n}{3} \geq 3!$ forces $n \geq 5$, we have $n(3) \geq 5$. An exhaustive computer search shows that there are no 3-tournaments on 5 vertices with Property O. However, the case where $n = 6$ is already much more time consuming. On the other hand, from the construction above, we have an oriented 3-graph on 9 vertices which has Property O. Thus $n(3) \leq 9$. It remains an open question as to whether there exists a 3-tournament with Property O on $n = 6$, 7, or 8 vertices. So, it is natural to pose this:

Problem 3.6. *Find the minimum number of vertices $n(3)$ in a 3-tournament with Property O.*

Returning to the function $f(k)$, we would like to determine $f(3)$, the minimum number of edges in an oriented 3-graph with Property O. It is easily seen that $f(3) > 6$. For k in general, it would be interesting to find a construction that improves the upper bound in Theorem 3.2. Finally, we would like to settle Problem 1, that is, to determine whether $f(k)/k! \to \infty$.

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