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Connections between mock modular forms and vertex operator algebras

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Doctor of Philosophy

Mathematics

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B.S., Indiana University, 2015

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An abstract of

A dissertation submitted to the Faculty of the

James T. Laney School of Graduate Studies of Emory University

in partial fulfillment of the requirements for the degree of

Doctor of Philosophy

in Mathematics

2020

Abstract

Connections between mock modular forms and vertex operator algebras

By Lea Beneish

The results in this dissertation come in two flavors, first we aim to strengthen the analogy between monstrous and umbral moonshine using vertex operator algebras, and second we derive structural results on vertex operator algebras using mock modular forms.

Towards strengthening the analogy between umbral and monstrous moonshine, we reframe Mathieu moonshine by repackaging the Mathieu moonshine mock modular forms in a few different ways, verifying the existence of corresponding modules, and giving various applications including connections with arithmetic. We produce vertex operator algebra constructions of some of these modules.

Using results from orbifold theory and from the theory of mock modular forms, we derive new structural results on vertex operator algebras. In joint work with Victor Manuel Aricheta, we study the asymptotic structure of sequences of G-modules where G are finite automorphism groups of certain vertex operator algebras (in particular this holds for umbral moonshine modules). And in joint work with Michael Mertens, we use Weierstrass mock modular forms to relate a dimension formula for certain vertex operator algebras to the arithmetic of modular curves.

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Acknowledgments

I would like to thank my family, friends, mentors, and fellow students. Without their support I would not have been able to complete this dissertation.

To my partner, Jackson Morrow, I am grateful to you for always believing in me and supporting me. Thank you for cooking so many delicious meals and snacks for me. Thank you for always encouraging me and putting up with my hyperbolic complaints. You mean the world to me.

To my parents, Julie Azoulay Beneish and Messod Daniel Beneish, thank you for always supporting my career goals, from when I wanted to be a truck driver until now that I want to be a mathematician. Thank you for constantly believing in me, for everything you have taught me, and for your help with so many of my (damn) emails.

I am thankful to the rest of my family, including Abigail Azoulay, Adam Azoulay, Daniel Azoulay, Elie Azoulay, Michael Barrie, Regine Beja, Adrienne Beneish, Esther Beneish, Lea Beneish, Raquel (Kelly) Beneish, Etty (Tika) Benveniste, Jennifer Chandler-Azoulay, William Chin, Celine Gehue, Liliane Hasson, Hadas Mate, Martin Mate, Casey Morrow, and Tom Nicholson.

To my advisor, John Duncan, I am extremely grateful for all of your support and patience these past five years. You have taught me so much over the past few years both in helping me understand technical details and in helping me to always see the bigger picture. Thank you for taking the time to explain things to me, often even the same thing more than once. I appreciate all the time you spent answering the, literal thousands of emails I sent you and carefully reading and editing the even larger number of drafts of papers and statements that I wrote. Thank you also for always looking out for opportunities for me and always having my best interests in mind.

I am also thankful to David Borthwick, John Duncan and David Zureick-Brown for agreeing to be members of my dissertation committee. Thank you for all of your help and support in the job application process, before, during, and after I had the job offers.

To David Zureick-Brown, thank you for everything you have taught me especially over the past year, for co-organizing a conference with me, for always encouraging me, and for all of the mathematical and professional advice throughout graduate school.

To David Borthwick, thank you for a great class in differential geometry, for helping me navigate the PhD program, and for supporting the summer program that I organize.

To Ken Ono, I am grateful for the opportunities you provided me in the first few years of graduate school and for everything I learned in the weekly student number theory seminar.

To David Borthwick, Bree Ettinger, James Nagy, Ken Ono, Larry Rolen, Lars Ruthotto, Robert Schneider, Juan Villeta-Garcia, thank you for very kindly agreeing to speak at Emory Week of Mathematical Exploration.

To Raman Parimala thank you for giving a fantastic public lecture at MAAIM and for all of your support and kindness over the years.

To David Borthwick, John Duncan, Shanshuang Yang, Suresh Venapally, and David Zureick-Brown thank you for the excellent courses you taught while I was at Emory.

To James Nagy and Vaidy Sunderam, thank you for creating a great atmosphere for me as a graduate student and for supporting the conference and summer programs that I organized.

To Kye Ellington, Terry Ingram, Kristen Schroeder, and Clare Sterling, thank you for all of your help throughout my time at Emory, and especially for the support in organizing MAAIM.

To Victor Manuel Aricheta and Michael Mertens, thank you for working with me, I could not have asked for better collaborators.

I am grateful to Victor Manuel Aricheta, Scott Carnahan, Robert Griess, Soon-Yi Kang, Jeffrey Lagarias, Hannah Larson, Robert Lemke Oliver, Kimball Martin, Michael Mertens, Sven Möller, Jackson Morrow, Ken Ono, Nils Scheithauer, Hanson Smith, Frank Thorne, and David Zureick-Brown for mathematical discussions that helped my research.

To my friends Victor Manuel Aricheta, Hannah Larson, Jackson Morrow, and Samy Wu, thank you for always being there for me and eating sushi with me. Thank you also to my other friends and classmates who made my time at Emory happier, including Jack Barlow, Alessandro Barone, Olivia Beckwith, Mike Cerchia, Alexander Clifton, Maddie Dawsey, Brad Elliott, Kalib Elliott, Xiangxi Gao, Michael Griffin, Jayanth Guhan, Sofia Guzzetti, Bastian Haase, Yunyi Hu, Seokho Jin, Kelvin Kan, Noam Kantor, Ariel Keller, Chris Keyes, Maryam Khaqan, Massi Lupo Pasini, Wenjun Ma, Michael Mertens, Sumit Mishra, Zitong Pei, Campbell Reiter, Tomer Reiter, Marcelo Sales, Robert Schneider, Warren Shull, Juvaria Tariq, Jesse Thorner, Sarah Trebat-Leder, Ian Wagner, Jialin Wang, and Shawn Wang.

Contents

Chapter 1

Introduction

In this dissertation, we describe results that aim to strengthen the analogy between monstrous and umbral moonshine using vertex operator algebras, and we derive structural results on vertex operator algebras using mock modular forms. We explain what we mean by this in what follows.

1.1 On the analogy between monstrous and umbral moonshine

Moonshine refers to unexpected connections between finite simple groups and modular forms. The first such instance was observed by McKay and Thompson in the 1970s and involves the monster group $\mathbb{M}$ and the modular j-invariant. This observation led to the monstrous moonshine conjecture of Thompson [Tho79] and Conway–Norton [CN79]. This conjecture postulates the existence of an infinite-dimensional graded module

$$V^\natural = \bigoplus_n V_n^\natural$$

such that for each g in the monster group, the graded trace function

$$T_g(\tau) := \sum_{n=-1}^{\infty} \operatorname{tr}(g \mid V_n^{\natural})q^n$$

is the unique modular function that generates the genus zero function field arising from a specific subgroup Γ_g of $SL_2(\mathbb{R})$, normalized such that $T_g(\tau) = q^{-1} + O(q)$ (where $\tau \in \mathbb{H}$ and $q = e^{2\pi i\tau}$) [CN79].

In the next few years, Frenkel, Lepowsky, and Meurman [FLM85,FLM84,FLM88] constructed $V^{\natural}$ as a vertex operator algebra (VOA). Then in 1992, Borcherds used the theory of vertex operator algebras and generalized Kac–Moody algebras (also known as Borcherds–Kac–Moody algebras [Bor88]) to show that $V^{\natural}$ has the properties conjectured by Conway and Norton and thus proved the monstrous moonshine conjecture [Bor92].

Since the proof of the monstrous moonshine conjecture, several other examples of moonshine phenomena have been discovered. Most significant for this work is the 2010 observation by Eguchi, Ooguri, and Tachikawa [EOT11] of a connection between the largest Mathieu group M_{24} and the elliptic genus of $K3$ surfaces. More precisely, they noticed that the low order multiplicities of superconformal algebra characters in the $K3$ elliptic genus are simple linear combinations of irreducible representations of M_{24}. This led them to conjecture that there exists an infinite-dimensional graded M_{24}-module

$$K^{\natural} = \bigoplus_n K_n^{\natural}$$

whose trace functions, denoted $H_g(\tau)$, are certain mock modular forms. We refer to [BFOR17, Fol17] for background on mock modular forms. In analogy with the work of Conway–Norton, work of Cheng, Eguchi–Hikami, and Gaberdiel–Hohenegger–Volpato [Che10,GHV10b,GHV10a,EH11] determined the mock modular forms $H_g(\tau)$ and then in 2012 Gannon [Gan16] proved the existence of the associated module $K^{\natural}$.

The analogy between the monster and M_{24} extends further when one considers the relationship between these groups and even unimodular positive-definite lattices of rank 24. The Leech lattice Λ [Lee64, Lee67] was proven by Conway [Con69] to be the unique such lattice with no root vectors. It is closely related to the monster. In fact, the Leech lattice was involved in both the construction of the monster by Griess [Gri82], and in the construction of the monster module [FLM88,FLM85,FLM84]. The group M_{24} is closely related to another such lattice, the (unique up to isomorphism) even unimodular lattice with rank 24 and root system A_1^{24} [Nie73]; M_{24} can be realized as the automorphism group of that lattice modulo the normal subgroup generated by reflections in roots. Cheng, Duncan, and Harvey [CDH14a] conjectured that this relationship generalizes. More precisely, they formulated the umbral moonshine conjecture, stating that M_{24} moonshine belongs to a class of 23 moonshines, each corresponding to one of the 23 Niemeier lattices with root systems of full rank [Nie73]. The existence of these umbral moonshine modules was proven in 2015 by Duncan, Griffin, and Ono [DGO15b]. There has been recent progress in constructing umbral moonshine modules by Anagiannis–Cheng–Harrison, Cheng–Duncan, Duncan–Harvey, and Duncan–O'Desky (see [DH17, DO18, CD19, ACH19]), however the umbral moonshine theory does not yet include explicit module constructions in all of its cases.

With the intention to further develop the analogy between the monster and M_{24}, we start with Mathieu moonshine and reframe M_{24} moonshine in terms of trace functions that are weight two quasimodular forms in place of mock modular forms. Restricting to M_{23} we find expressions for these forms that contain arithmetic information. We generalize this type of expression to $\mathbb{Z}/N\mathbb{Z}$ for arbitrary N prime, from which we can observe more connections of a similar kind. At the expense of arithmetic connections, we give a second set of quasimodular trace functions for another $\mathbb{Z}/N\mathbb{Z}$-module which are given only in terms of Eisenstein series. We construct these modules explicitly as tensor products of Heisenberg and Clifford module vertex operator algebras.

We offer another modification of the quasimodular functions that retains their arithmetic content and we prove the existence of the corresponding module with those trace functions. We give a vertex operator algebra construction for these modules.

The quasimodular forms that we give as trace functions of an M_{24}-module come from multiplying the functions $\hat{H}_g(\tau)$, the completions of the mock modular forms $H_g(\tau)$, by $\eta^3(\tau)$ to bring the weight to 2, and then taking the holomorphic projection. In this way we define weight two quasimodular forms $Q_g(\tau)$ for every $g \in M_{24}$. The first example of a weight two quasimodular form is

$$E_2(\tau) = 1 - 24\sum_{n=1}^{\infty} \sigma_1(n)q^n,$$

the usual weight 2 Eisenstein series. In our setting, each coefficient of $-2E_2(\tau)$ has a natural interpretation as the dimension of a virtual M_{24}-module. This is because our $Q_e(\tau)$ is equal to $-2E_2(\tau)$ and we prove the following:

Theorem A. *There exists a virtual graded M_{24}-module $V = \bigoplus_n V_n$ such that*

$$Q_g(\tau) = \sum_{n=0}^{\infty} \mathrm{tr}(g \mid V_n)q^n.$$

Interestingly, if one restricts to the trace functions of M_{23}, a subgroup of M_{24} whose group elements have fixed points in their permutation representations, the $Q_g(\tau)$ have convenient expressions in terms of Eisenstein series and cusp forms. These expressions are of the form

$$Q_g(\tau) = -2E_{2,N}(\tau) + \frac{N}{n_N} G_N(\tau) \tag{1.1.1}$$

where we use N to denote the order of g, we let $n_N := \mathrm{num}\left(\frac{N-1}{12}\right)$, $E_{2,N}(\tau)$ is an expression in terms of Eisenstein series (cf. (4.2.1)), and G_N is a specific cusp form of weight 2 for $\Gamma_0(N)$ with integer coefficients.

The trace functions of this module involve weight two cusp forms and so they contain arithmetic information. As an example, for certain N, the Jacobian of $X_0(N)$, denoted $J_0(N)$, is an elliptic curve, and so the integrality of the trace functions is equivalent to certain divisibility conditions on the number of $\mathbb{F}_p$ points on these curves (this holds even when the dimension of $J_0(N)$ is greater than 1, but we restrict to elliptic curves here for simplicity). These results depend on the cooperation of Eisenstein series and cusp forms to sum to integral coefficients. For example we have the following (known, see Appendix A of [Kat81]) divisibility conditions arising from M_{23}:

Corollary B.

1. *For* $p \neq 11$, *we have* $5 \mid \#J_0(11)(\mathbb{F}_p)$.
2. *For* $p \neq 2, 7$ *we have* $3 \mid \#J_0(14)(\mathbb{F}_p)$.
3. *For* $p \neq 3, 5$, *we have* $4 \mid \#J_0(15)(\mathbb{F}_p)$.

Moreover, the pattern we observe in the denominators of the cusp forms reflects a result of Mazur on a congruence between Eisenstein series and cusp forms in the cases where N is prime [Maz77]. The type of expression in (1.1.1) can be generalized, and in fact, the formula for $Q_g(\tau)$ does not depend on M_{23} and can be defined for arbitrary N. We restrict our focus to N prime and prove the existence of a $\mathbb{Z}/N\mathbb{Z}$-module with trace functions:

$$f_g^{(N)}(\tau) := \begin{cases} -\dfrac{\ell_N}{n_N} E_2(\tau) & \text{if } g = e, \\ -\dfrac{\ell_N}{n_N} E_{2,N}(\tau) - \dfrac{N}{n_N} G_N(\tau) & \text{if } g \neq e, \end{cases} \tag{1.1.2}$$

where $\ell_N := \operatorname{num}\left(\dfrac{N^2-1}{24}\right)$. The functions $f_g^{(N)}(\tau)$ are quasimodular forms of weight 2 with integral coefficients defined in terms of Eisenstein series and cusp forms. The $E_{2,N}(\tau)$ (cf. (4.2.4)) are again defined in terms of Eisenstein series and the $G_N(\tau)$ are certain cusp forms of level N and weight 2 with integer coefficients (cf. Section 4.2, Proposition 4.2.1). With these definitions we state the following result:

Theorem C. *Let N be a prime and $f_g^{(N)}(\tau)$ be as in (1.1.2). Then there exists a virtual graded $\mathbb{Z}/N\mathbb{Z}$-module $V^{(N)} = \bigoplus_n V_n^{(N)}$ such that*

$$f_g^{(N)}(\tau) = \sum_{n=0}^{\infty} \mathrm{tr}(g \mid V_n^{(N)})q^n.$$

As a consequence, for N prime, one can observe many more examples analogous to those in Corollary B arising from the trace functions $f_g^{(N)}(\tau)$.

In the trace function for $\mathbb{Z}/N\mathbb{Z}$ in Theorem 7.2.1, the multiple in front of $E_{2,N}(\tau)$ is $\dfrac{\ell_N}{n_N}$ (recall that $\ell_N = \mathrm{num}\left(\frac{N^2-1}{24}\right)$ and $n_N = \mathrm{num}\left(\frac{N-1}{12}\right)$). We have that ℓ_N is the minimal number which clears the denominators of $E_{2,N}(\tau)$, and we can further divide by n_N because we find a cusp form that satisfies a congruence modulo n_N. If we restrict our functions to be only in terms of Eisenstein series and do not use cusp forms, we can no longer divide by n_N and instead have trace functions as follows: for N prime, let

$$F_g^{(N)}(\tau) := \begin{cases} -\ell_N E_2(\tau) & \text{if } g = e, \\ -\ell_N E_{2,N}(\tau) & \text{if } g \neq e. \end{cases} \tag{1.1.3}$$

These are weight two purely Eisenstein quasimodular trace functions for a $\mathbb{Z}/N\mathbb{Z}$-module. Although this comes at the cost of arithmetic connections arising from cusp forms, we give a vertex operator algebra construction for this module. For each N, this module is $W_{\mathrm{tw}}^{(N)}$, a twisted module for the vertex operator algebra which we denote $W^{(N)}$. The vertex operator algebras $W^{(N)}$ are defined as tensor products of Heisenberg and Clifford module vertex algebras. The precise construction of $W^{(N)}$ is given in Section 4.4.

Theorem D. *For N prime, the canonically twisted module $W_{\mathrm{tw}}^{(N)} = \bigoplus_n W_{\mathrm{tw},n}^{(N)}$ of the vertex operator algebra $W^{(N)}$ is an infinite dimensional virtual graded module for*

$\mathbb{Z}/N\mathbb{Z}$ *such that*

$$F_g^{(N)}(\tau) = \sum_{n=0}^{\infty} \mathrm{tr}(g \mid W_{\mathrm{tw},n}^{(N)}) q^n.$$

We now describe our modification of the functions $Q_g(\tau)$ so that we can give a module construction that retains the arithmetic information from the cusp forms. Namely, we add $\chi(g)\,(\eta^3(\tau)\mu(\tau,z) + 2F_2(\tau))$ to $Q_g(\tau)$, where $\eta(\tau)$ is the Dedekind eta function, $\mu(\tau,z)$ is an Appell–Lerch sum, $\chi(g)$ is the number of fixed points of g in the 24-dimensional permutation representation of M_{24}, and $F_2(\tau)$ is defined to be

$$F_2(\tau) := \sum_{\substack{r>s>0 \\ r-s \text{ odd}}} s q^{rs/2}.$$

This allows us to define the following meromorphic Jacobi forms

$$M_g(\tau,z) := Q_g(\tau) + \chi(g)\left(\eta^3(\tau)\mu(\tau,z) + 2F_2(\tau)\right),$$

associated to each element $g \in M_{24}$. In Proposition 5.1.3, we prove the existence of a module for which suitable expansions of the $M_g(\tau,z)$ are trace functions. Because of their relation to $Q_g(\tau)$, these trace functions contain arithmetic information.

Although on the surface, the relationship between $H_g(\tau)$ and $M_g(\tau,z)$ may seem distant, we claim it is natural. We show that one can equivalently define $M_g(\tau,z)$ by writing $M_g(\tau,z) = H_g(\tau)\eta^3(\tau) + \chi(g)\eta^3(\tau)\mu(\tau,z)$. This type of expression is an example of a canonical decomposition of a meromorphic Jacboi form into a "finite" part and "polar" part established by Zwegers in [Zwe02] and Dabholkar, Murthy, and Zagier in [DMZ12]. The relationship between meromorphic Jacobi forms and umbral moonshine was first discussed by Cheng, Duncan, and Harvey in [CDH14b]. A special case of this is that the mock modular forms $H_g(\tau)$ occur as the "finite parts" of meromorphic Jacobi forms.

Finding a construction of a module whose trace functions are meromorphic Jacobi forms associated to vector valued mock modular forms in umbral moonshine is considered a natural alternative to finding a construction of a module whose trace functions are the vector valued mock modular forms. In fact, to describe this, Duncan and O'Desky coined the term "meromorphic module problem" in [DO18] when they solved this problem for the cases of umbral moonshine corresponding to the Niemeier lattices with root systems $A_6^{\oplus 4}$ and $A_{12}^{\oplus 2}$ (and partially for the cases corresponding to $A_3^{\oplus 8}$ and $A_4^{\oplus 6}$).

For $g \in M_{24}$ such that $[g] \neq$ 3B, 4C, 6B, 12B, 21A, 21B, 23A, or 23B, (where we use the ATLAS notation in [Con85] for conjugacy classes of M_{24}), we give concrete constructions of modules whose trace functions are $\widetilde{M}_g(\tau, z)$, where the $\widetilde{M}_g(\tau, z)$ are defined to be the Fourier expansions of $M_g(\tau, z)$ in the domain $0 < -\mathrm{Im}(z) < \mathrm{Im}(\tau)$.

Theorem E. *For subgroups of M_{24} consisting only of elements $g \in M_{24}$ such that $[g] \neq$ 3B, 4C, 6B, 12B, 21A, 21B, 23A, or 23B, and such that each element fixes a 4-dimensional space in the* 24*-dimensional permutation representation of M_{24}, we have the following module construction:*

$$\widetilde{A}(\mathfrak{p})_{\mathrm{tw}} \otimes \mathcal{W}(\mathfrak{b})_{\mathrm{tw}} \otimes T$$

is an infinite-dimensional, bigraded, virtual module with trace functions as follows:

$$\lim_{\gamma \to -1} \mathrm{tr}\left(\widehat{g}\mathfrak{z}\widetilde{p}(0)\gamma^{J_{12}(0)} y^{J(0)} q^{L(0)-c/24} \mid \widetilde{A}(\mathfrak{p})_{\mathrm{tw}} \otimes \mathcal{W}(\mathfrak{b})_{\mathrm{tw}} \otimes T \right) = \widetilde{M}_g(\tau, z).$$

We describe this construction in two steps. First we construct a related module, $\widetilde{A}(\mathfrak{p})_{\mathrm{tw}} \otimes \mathcal{W}(\mathfrak{b})_{\mathrm{tw}} \otimes V_{\mathrm{tw}}^{s\natural}$, which is the tensor product of a Clifford module, a Weyl module, and a Conway module of [DMC15]. For their definitions and the definitions of the operators that we take the trace of, see Section 5.2. This gives module constructions for subgroups of M_{24} that do not contain elements in the conjugacy classes 3B, 4C, 6B, 12B, 21A, 21B, 23A, or 23B and such that the 24-dimensional permutation representation of M_{24} has a fixed 4-dimensional space when restricted to that subgroup. For example, this gives a module construction for group $\mathbf{L_3(4)} \simeq \mathbf{M_{21}}$, one of the simple subgroups of M_{24}. This does not, however, give a module construction for $\mathbf{M_{11}}$ because the 24-dimensional permutation representation of M_{24} restricted to $\mathbf{M_{11}}$ only fixes a 3-dimensional space.

To remedy cases such as $\mathbf{M_{11}}$ and arrive at the module constructions given in the Main Theorem, we apply a method of Anagiannis, Cheng, and Harrison [ACH19]. For these subgroups, we still require that each element of the subgroup fixes a 4-space but not that the whole subgroup fixes the same 4-space (we still require that the subgroup does not contain elements in the aforementioned conjugacy classes). Here we use $\widetilde{A}(\mathfrak{p})_{\mathrm{tw}}$ and $\mathcal{W}(\mathfrak{b})_{\mathrm{tw}}$ as before (defined in Section 5.2) and T is a modification of $V^{s\natural}_{\mathrm{tw}}$ which we define in Section 5.3. This, for example, gives module constructions for $\mathbf{M_{22}}:\mathbf{2}$ a maximal subgroup of M_{24}, for the smallest sporadic simple group $\mathbf{M_{11}}$, and for groups $\mathbf{2^4}:\mathbf{A_7}$ and $\mathbf{A_8}$ which are maximal subgroups of M_{23}.

The module construction for $\mathbf{M_{11}}$ gives an explicit realization of the trace functions $\widetilde{M}_g(\tau, z)$ whose integrality is equivalent to divisibility conditions on the number of $\mathbb{F}_p$ points on the Jacobian of the modular curve $X_0(11)$, denoted $J_0(11)$. The same is true with $\mathbf{M_{22}}:\mathbf{2}$ and $\mathbf{2^4}:\mathbf{A_7}$ for $J_0(14)$ and with $\mathbf{A_8}$ for $J_0(15)$.

Note that our module gives an explicit construction of the restriction of the Mathieu moonshine module to the subgroup $\mathbf{2^4}:\mathbf{A_7}$, which has also played a prominent role in the symmetry surfing program initiated by Taormina and Wendland in [TW10, TW13, TW15a, TW15b, TW19]. It would be interesting to compare our method to theirs.

1.2 Structural results on vertex operator algebras

Much of the motivation to study vertex operator algebras (VOAs) originates from the construction of the moonshine module. Since the construction of $V^\natural$, VOAs have played a central role in various other instances of moonshine, such as umbral moonshine (see [DH17, DO18, CD19, ACH19]) and Conway moonshine [DMC15]. Vertex operator algebras have also since appeared in the study of infinite-dimensional Lie algebras.

The weight 1 space of a VOA of CFT-type is known to have the structure of a Lie algebra. Schellekens [Sch93] gave a list of 71 Lie algebras, known as Schellekens' list. This list is related to the classification of holomorphic, strongly rational VOAs of central charge 24 (see Chapter 3 for definitions of these terms). This classification can be thought of in analogy to Niemeier's classification of even unimodular lattices of rank 24, with these Lie algebras as the analogues of the root systems of the even unimodular lattices of rank 24.

In the next two sections, we discuss structural results related to these types of VOAs. The results in 1.2.1 are joint work with Michael Mertens and the results in 1.2.2 are joint work with Victor Manuel Aricheta.

1.2.1 On a dimension formula

In the context of classifying holomorphic, strongly rational VOAs of central charge 24 and proving the "completeness" of Schellekens' list, which is now known to contain every possible V_1-space of such a VOA by work of various people (see for instance the introduction of [EMSarb] for references), van Ekeren, Möller, and Scheithauer [EMSarb] find a dimension formula for orbifold VOAs of central charge 24. A special case of their formula had previously been established by Möller in his thesis [M1̈6] and for the sake of simplicity, we only give this special case here. The reader is referred to Chapter 3 and the references given there for the relevant definitions.

Theorem (van Ekeren, Möller, and Scheithauer)**.** *Let V be a holomorphic, strongly rational VOA additionally satisfying the positivity assumption and let $G = \langle g \rangle$ be a cyclic group of automorphisms of V of order N with g of type $N\{0\}$. Denote by V^G the fixed point VOA of V under the action of G and let $V^{\mathrm{orb}(g)}$ be the orbifold vertex operator algebra. Furthermore, assume that V has central charge $c = 24$. Then for $N = 2, 3, 5, 7, 13$, we have the dimension formula*

$$\dim V_1 + \dim V_1^{\mathrm{orb}(g)} = 24 + (N+1)\dim V_1^G - \frac{24}{N-1}\sum_{k=1}^{N-1}\sigma(N-k)\sum_{i\in\mathbb{Z}/N\mathbb{Z}\setminus\{0\}}\dim V(g^i)_{k/N},$$

where $\sigma(m) = \sum_{d|m} d$ denotes the usual divisor sum function.

The proof of this result and of the extension of the result in [EMSarb] for all N such that the modular curve $X_0(N)$ has genus 0, i.e. $N \in \{2, ..., 10, 12, 13, 16, 18, 25\}$, is essentially obtained by writing the character ch_{V^G} explicitly in terms of the Hauptmodul for the group $\Gamma_0(N)$.

We give an extension of the dimension formula in [EMSarb] to levels N where there is no Hauptmodul, but rather where the modular curve $X_0(N)$ has genus 1. In those cases, the modular curve is an elliptic curve E of conductor N defined over $\mathbb{Q}$, which (over $\mathbb{C}$) is isomorphic to the torus $\mathbb{C}/\Lambda_E$ for a full lattice $\Lambda_E \subset \mathbb{C}$ called the period lattice of E. Denote by $\widehat{\zeta}(\Lambda_E; z)$ the associated completed Weierstrass zeta function (see Section 2.4 for the precise definition). For simplicity, we state the theorem just for the prime levels in question. The analogous statement for square-free composite levels is given in Theorem 6.2.5.

Theorem F. *Let V be a holomorphic, strongly rational vertex operator algebra of central charge 24. Let $G = \langle g \rangle$ be a cyclic group of automorphisms of V of order $p \in \{11, 17, 19\}$ such that g is of type $p\{0\}$. Further let $E = X_0(p)$ be the $\Gamma_0(p)$-optimal elliptic curve of conductor p. Then with the assumptions and notations in*

Chapter 3, we have the following dimension formula:

$$\dim V_1 + \dim V_1^{\mathrm{orb}(g)} = (p+1)\dim V_1^G - (p-1)C_E + C_E \sum_{i=1}^{p-1}\sum_{j=1}^{p-1} \sigma(p-j)\dim V(g^i)_{j/p},$$

where we set

$$C_E := -\frac{3-\#E(\mathbb{F}_2)}{2} - \widehat{\zeta}\left(\Lambda_E; L(E,1)\right).$$

In particular, this dimension formula relates invariants of the underlying modular curve to the theory of VOAs. We can now exploit knowledge about arithmetic properties of these invariants to derive a simpler dimension formula in the following way.

To the best of the author's knowledge the question of the rationality of the value $\widehat{\zeta}\left(\Lambda_E; L(E,1)\right)$ has not been investigated so far. It is known due to a classical result of Schneider [Sch57, Chapter II, §4, Satz 15] that the value of the uncompleted zeta function $\zeta\left(\Lambda_E; L(E,1)\right)$ is in fact transcendental. Since all the other quantities in the above dimension formula are clearly rational, we obtain the following immediate corollary.

Corollary G. *Assume the notations as in Theorem F. If we have*

$$\sum_{i=1}^{p-1}\sum_{j=1}^{p-1} \sigma(p-j)\dim V(g^i)_{j/p} \neq p-1$$

for some VOA V as in Theorem F, then the value $\widehat{\zeta}\left(\Lambda_E; L(E,1)\right)$ is rational.

Computing the zeta values in Corollary G numerically, we find that within computational precision $\widehat{\zeta}\left(\Lambda_E; L(E,1)\right) = 17/5,\ 2,\ 4/3$ for $p = 11,\ 17,\ 19$, respectively.

In a very recent preprint [MS], Möller, and Scheithauer independently find a completely general dimension formula like the one in Theorem F with no restriction on the order of the cyclic group G using expansions of vector-valued Eisenstein series. In particular, their general result simplifies to the statement of the Theorem from [M16] quoted above for all primes p.

In the proof of Theorem 4.12 of loc. cit., Möller and Scheithauer show that if the order of the automorphism group G is any prime p such that the genus of $X_0(p)$ is greater than zero, one obtains the lower bound

$$\sum_{i=1}^{p-1}\sum_{j=1}^{p-1}\sigma(p-j)\dim V(g^i)_{j/p}\geq p-1.$$

As Möller and Scheithauer informed us, they have produced — using both computer calculations and theoretical considerations based on work by Chenevier and Lannes [CL19] on p-neighbours of Niemeier lattices — explicit examples of suitable VOAs for which the above inequality is strict, wherefore according to Corollary G the values $\widehat{\zeta}\,(\Lambda_E; L(E,1))$ are indeed rational. In fact, comparing to the (extended) version of Möller's result and using the examples Möller and Scheithauer have constructed, one finds the stronger statement that we have indeed the following identity for the constant C_E from Theorem F,

$$C_E = -\frac{24}{p-1}. \tag{1.2.1}$$

From the proof of Theorem F we can infer the following dimension formula as well, which looks similar to that in Corollary G. The proof relies on the so-called Bruinier-Funke pairing (see Proposition 2.1.2).

Theorem H. *Assume the hypotheses and notation from Theorem 6.2.1, except that p may now denote any prime number, and let* $f(\tau)=\sum_{n=1}^{\infty}a(n)e^{2\pi i n\tau}\in S_2(p)$ *be a* *newform with Atkin-Lehner eigenvalue* $\varepsilon \in \{\pm 1\}$*. Then we have*

$$\sum_{i=1}^{p-1}\sum_{j=1}^{p-1} a(p-j)\dim V(g^i)_{j/p} = -\varepsilon p - a(p).$$

Essentially, the formula in Theorem H also appears on [MS, p. 24], but was proven using a different kind of pairing. We note that loosely speaking, one may interpret Theorem F in view of Theorem H as the case where one replaces the newform f by the weight 2 Eisenstein series in $M_2(p)$.

The proof of Theorem F relies on the following result which states that one can express any harmonic Maass form (in the given levels) essentially in terms of Weierstrass mock modular forms and Hecke operators (see Sections 2.2 and 2.4 for details).

Theorem I. *Let* E *denote the strong Weil curve of conductor*

$$N \in \{11, 14, 15, 17, 19, 21\}.$$

Then any harmonic Maass form of weight 0 *for* $\Gamma_0(N)$ *is is a linear combination of images of the completed Weierstrass mock modular form* $\widehat{\mathfrak{Z}}_E$ *associated to the* $\Gamma_0(N)$*-optimal elliptic curve* E *— i.e. in the cases considered* E *is a model for the modular curve* $X_0(N)$ *— under the Hecke operators* T_m *and Atkin-Lehner involutions , or in other words:*

$$H_0(N) \leq \mathrm{span}_{\mathbb{C}}\left\{\widehat{\mathfrak{Z}}_E|W_Q|T_m|B_d \;:\; m \in \mathbb{N}_0,\ Q \mid N,\ d \mid N\right\},$$

where the operators B_d *are defined in Proposition 2.2.2.*

Remark. It is essential in Theorem I that the elliptic curves under consideration are indeed models for the modular curve $X_0(N)$, where N is the respective conductor. In particular, the genus of $X_0(N)$ must be equal to 1. There are six further levels N

with this property, namely $N \in \{20, 24, 27, 32, 36, 49\}$, but our proof does not work in these cases for reasons we explain in Sections 2.2 and 6.1.

Remark. As our proof will show, the statement of Theorem I remains valid for all square-free levels N if one replaces $\widehat{\mathfrak{Z}}_E$ by the Maass-Poincaré series for $\Gamma_0(N)$ which has exactly one simple pole at the cusp ∞. In particular, one may immediately generalize Theorem F to arbitrary primes p and Theorem 6.2.5 to arbitrary square-free numbers N in this fashion. This way, one may obtain rationality results for the constant terms of these series in analogy to Corollary G. It is however not known to the author whether these constant terms are directly related to special values of interesting functions like the Weierstrass zeta function.

1.2.2 On the asymptotic structure of modules

Probably most well known of these holomorphic, strongly rational VOAs of central charge 24 is the monstrous moonshine module $V^\natural$. Duncan, Griffin, and Ono explored another kind of structural result on $V_n^\natural$.

Duncan, Griffin, and Ono studied the asymptotic structure of the homogeneous subspaces $V_n^\natural$ of the monstrous moonshine module [DGO15a]. We show that their results hold more generally, for a sequence V_n of G-modules for some VOA V satisfying certain conditions and G some finite group of automorphisms of V. As another application we analyze the asymptotic structure of umbral moonshine modules.

They found that as $n \to \infty$, the subspaces $V_n^\natural$ tend to a multiple of the regular representation of $\mathbb{M}$. To explain this, let $M_1^{(\mathbb{M})}, \ldots, M_{194}^{(\mathbb{M})}$ be the irreducible representations of $\mathbb{M}$, ordered as in the ATLAS [Con85]. Write

$$V_n^\natural = \mathrm{m}_1(n) M_1^{(\mathbb{M})} \oplus \cdots \oplus \mathrm{m}_{194}(n) M_{194}^{(\mathbb{M})}.$$

For $i = 1, \ldots, 194$, they showed that

$$\mathrm{m}_i(n) \sim \frac{e^{4\pi\sqrt{n}}}{\sqrt{2}|\mathbb{M}|n^{3/4}} \dim M_i^{(\mathbb{M})} \tag{1.2.2}$$

as $n \to \infty$. To show this, they derived an exact series formula for $\mathrm{m}_i(n)$, and the asymptotic above is obtained by isolating the dominant term of the series. In view of this result, Griess posed the following question (cf. Problem 10.9. in [DGO15a]):

> **Griess' Question.** *If we write each homogeneous subspace of each moonshine module, particularly the moonshine module $V^\natural$, as the sum of a free part (free over the group ring of $\mathbb{M}$) and a non-free part, then the non-free part tends to 0 (relative to the free part) as $n \to \infty$. Is there something to be learnt from an analysis of the non-free parts?*

A step in this direction is given by Larson, who found asymptotic formulas for the non-free parts of $V_n^\natural$ [Lar16]. We point out here that by closely reading the asymptotic formulas, one sees that the non-free parts of $V_n^\natural$ tend to a representation of $\mathbb{M}$ whose irreducible components do not include $M_{16}^{(\mathbb{M})}$ and $M_{17}^{(\mathbb{M})}$. We will show that this extends even further (cf. Theorem K). To explain this, we first introduce a definition.

Let (K_n) be a sequence of finite-dimensional representations of a finite group G, and suppose $c_g(n) := \mathrm{tr}(g|K_n) \in \mathbb{R}$ for all $g \in G$ and all n. We say that the sequence (K_n) has *dominant identity trace* if for every $g \in G$ that is not equal to the identity element e, we have $c_g(n) = o(c_e(n))$ as $n \to \infty$. Examples of such sequences are the sequences $(V_n^\natural)$ and $(K_n^\natural)$. Certain vertex operator algebras, for which the monstrous moonshine module is a special case, also have sequences of homogeneous subspaces that have dominant identity trace (cf. Theorem N).

The following theorem shows that if a sequence (K_n) has dominant identity trace, then the subspaces K_n tend to a multiple of the regular representation of G. Thus, Griess' question makes sense for these sequences.

Theorem J. *Let e be the identity element in G, and let $M_1, \ldots, M_s$ be the irreducible representations of G. Write*

$$K_n = \mathrm{m}_1(n)M_1 \oplus \mathrm{m}_2(n)M_2 \oplus \cdots \oplus \mathrm{m}_s(n)M_s.$$

If (K_n) has dominant identity trace, then

$$\mathrm{m}_i(n) \sim \frac{1}{|G|} \dim K_n \dim M_i$$

as $n \to \infty$. Therefore,

$$\lim_{n\to\infty} \frac{\mathrm{m}_i(n)}{\sum_{j=1}^{s} \mathrm{m}_j(n)} = \frac{\dim M_i}{\sum_{j=1}^{s} \dim M_j}.$$

Remark. In some cases, $\dim K_n$ has known asymptotics in terms of simple functions. For example, in the case of monstrous moonshine the asymptotics for the coefficients of the j-function yield (1.2.2). As another example, if $(K_n^\natural)$ is the sequence of homogeneous subspaces of the Mathieu moonshine module, then $\dim K_n^\natural$ may be written as a Rademacher series (cf. Section 2.3). This yields

$$\dim K_n^\natural \sim \frac{4}{\sqrt{8n-1}} \exp\left(\frac{\pi\sqrt{8n-1}}{2}\right)$$

as $n \to \infty$. Therefore if $M_i^{(M_{24})}$ denotes an irreducible representation of M_{24} and $\mathrm{m}_i(n)$ denotes the multiplicity of $M_i^{(M_{24})}$ in $K_n^\natural$, then

$$\mathrm{m}_i(n) \sim \frac{4e^{\pi\sqrt{8n-1}/2}}{|M_{24}|\sqrt{8n-1}} \dim M_i^{(M_{24})}$$

as $n \to \infty$.

The statement of Theorem J, that K_n tends to a multiple of the regular representation of G, is obtained by an analysis which involves only the identity element of G. By performing an analysis which includes all the other elements of G, we find other representations of G—sensitive to some initial condition—which we can view as natural analogues of the regular representation. More precisely, we have the following result.

Theorem K. *Let (K_n) be a sequence of finite-dimensional representations of a finite group G and suppose $c_g(n) := tr(g|K_n) \in \mathbb{R}$ for all $g \in G$ and all n. Let (n_i) be a sequence of integers such that given $g \in G$, the signs $sgn(c_g(n_i))$ are independent of i. If (K_n) has dominant identity trace, then there exist G-modules $L_1, L_2, \ldots, L_t$ (depending on the signs $sgn(c_g(n_i))$) where*

- *L_1 is the regular representation of G, and*
- *the irreducible components of L_{i+1} form a subset of the irreducible components of L_i (for $1 \leq i < t$),*

such that

1. *for some nonnegative integer-valued functions $r_1(n_i), \ldots, r_t(n_i)$ and G-module $L_\epsilon(n_i)$ with bounded multiplicity functions, we have the decomposition*

$$K_{n_i} = r_1(n_i)L_1 \oplus r_2(n_i)L_2 \oplus \cdots \oplus r_t(n_i)L_t \oplus L_\epsilon(n_i),$$

2. *and the module $K_{n_i} \oplus (-r_1(n_i))L_1 \oplus \cdots \oplus (-r_l(n_i))L_l$ tends to a multiple of the representation L_{l+1} (for $1 \leq l \leq t-1$) as $i \to \infty$.*

This result shows that the representations L_j's have a curious property that they are expressed in terms of fewer and fewer irreducible representations of G. In other words, by looking at the sequence $L_1, L_2, \ldots$, we find that the irreducible representations disappear in some order. Thus we find that moonshine for a group naturally equips its irreducible representations with partial orders. This is the part that speaks to Griess' question.

Example. Let $H_g^{(M_{24})}(\tau)$ be the Mathieu moonshine graded trace functions. Consider the action of $A_5 \subset M_{24}$ on the Mathieu moonshine module $K^\natural = \bigoplus K_n^\natural$ with the following graded trace functions:

$$\begin{aligned}
H_{1A}^{(A_5)} &= H_{1A}^{(M_{24})} \\
H_{2A}^{(A_5)} &= H_{2A}^{(M_{24})} \\
H_{3A}^{(A_5)} &= H_{3A}^{(M_{24})} \\
H_{5A}^{(A_5)} &= H_{5A}^{(M_{24})} \\
H_{5B}^{(A_5)} &= H_{5A}^{(M_{24})}.
\end{aligned}$$

Let $M_1^{(A_5)}, \ldots, M_5^{(A_5)}$ be the irreducible representations of A_5, labelled as in GAP's SmallGroup library [GAP18]. If $n_i = 10 + 30i$, then as $i \to \infty$ the discussion in Section 2 shows that $K_{n_i}^\natural$ naturally decomposes as an A_5-module into

$$K_{n_i}^\natural = r_1(n_i)L_1 \oplus r_2(n_i)L_2 \oplus r_3(n_i)L_3 \oplus L_\epsilon(n_i)$$

where: L_1 is the regular representation of A_5; L_2 is a representation of A_5 whose irreducible decomposition is in terms of $M_1^{(A_5)}$, $M_4^{(A_5)}$, and $M_5^{(A_5)}$; L_3 is a representation of A_5 whose irreducible decomposition is in terms of $M_1^{(A_5)}$ and $M_5^{(A_5)}$; and L_ϵ have bounded multiplicity functions. Hence, moonshine on A_5 gives us the partial ordering of (blocks of) irreducible modules: $\{M_2^{(A_5)}, M_3^{(A_5)}\}$, followed by $\{M_4^{(A_5)}\}$, and then by $\{M_1^{(A_5)}, M_5^{(A_5)}\}$. As another example, for $n_i = 21+30i$, moonshine on A_5 gives us the partial ordering of (blocks of) irreducible modules: $\{M_1^{(A_5)}\}$, followed by $\{M_5^{(A_5)}\}$, and then by $\{M_2^{(A_5)}, M_3^{(A_5)}, M_4^{(A_5)}\}$.

An example where the ordering of the irreducible modules does not depend on the congruence class of n is the following: Consider the action of $S_3 \subset \mathbb{M}$ on the monstrous moonshine module $V^\natural$ with graded trace functions

$$H_{1A}^{(S_3)} = T_{1A}$$
$$H_{2A}^{(S_3)} = T_{2A}$$
$$H_{3A}^{(S_3)} = T_{3A}$$

where T_g is the hauptmodul corresponding to $g \in \mathbb{M}$ via monstrous moonshine. This gives the ordering $\{M_2^{(S_3)}\}$, $\{M_3^{(S_3)}\}$, $\{M_1^{(S_3)}\}$, where $M_1^{(S_3)}$, $M_2^{(S_3)}$, $M_3^{(S_3)}$ are the irreducible representations of S_3 ordered by increasing dimension (starting with the trivial representation).

Griess' question is open ended, and the quantitative answer we offer is one of many partial answers to this problem. In fact, when Griess posed the problem, the original intention was to know whether a complete understanding of the asymptotics could provide clues to a richer algebraic structure surrounding the group and the graded module for it.

In the course of proving Theorem K, we have also obtained the asymptotics for the multiplicities of the irreducible components of the non-free parts of K_n. We record this in the following theorem. Here, the set $\mathcal{C}_2$ is the collection of conjugacy classes of G whose corresponding $c_g(n)$'s have the second fastest growth. (See the Proof of Theorem K and Theorem L, in particular (7.2.3), for a precise definition of $\mathcal{C}_2$.)

Theorem L. *Let $M_1, \dots, M_s$ be the irreducible representations of a finite group G and let $\chi_1, \dots, \chi_s$ be their respective characters. Let (K_n) be a sequence of G-modules that has dominant identity trace. Suppose $c_g(n) := tr(g|K_n) \in \mathbb{R}$ for all $g \in G$ and all n. Denote by K'_n the non-free part of K_n, and write*

$$K'_n = \mathrm{m}'_1(n)M_1 \oplus \mathrm{m}'_2(n)M_2 \oplus \cdots \oplus \mathrm{m}'_s(n)M_s.$$

Suppose that (n_j) is a sequence of integers such that given $g \in \mathcal{C}_2$, the signs $sgn(c_g(n_j))$ are independent of j. Then as $j \to \infty$

$$\mathrm{m}_i'(n_j) \sim \frac{1}{|G|} \sum_{[g] \in \mathcal{C}_2} |[g]| f_i'(g) c_g(n_j).$$

Here

$$f_i'(g) := \overline{\chi_i(g)} - \frac{\dim M_i}{\dim M_{j'}} \overline{\chi_{j'}(g)}$$

where j' is a j that minimizes

$$\frac{\sum_{g \in \mathcal{C}_2} |[g]| \overline{\chi_j(g)} \mathrm{sgn}(c_g(n))}{\dim M_j}$$

as $n \to \infty$.

We discuss in Section 7.3 some cases where our results apply. First, we show that our results apply to umbral moonshine. In umbral moonshine the trace functions are mock modular forms which have coefficients that can be expressed in terms of Kloosterman sums weighted by Bessel functions. Thus the coefficients of the trace functions have known asymptotics for which our theory applies. As an explicit example, we use Theorem L to obtain the asymptotics of the multiplicities of the non-free part of the Mathieu moonshine module, which we record here as a corollary.

Corollary M. *Let $K^\natural = \bigoplus K_n^\natural$ be the Mathieu moonshine module, and let K_n' be the non-free part of $K_n^\natural$. Let $M_1^{(M_{24})}, \ldots, M_{26}^{(M_{24})}$ be the irreducible representations of M_{24}, and let $\chi_1, \ldots, \chi_{26}$ be their respective characters. Write*

$$K_n' = m_1'(n) M_1^{(M_{24})} \oplus m_2'(n) M_2^{(M_{24})} \oplus \cdots \oplus m_{26}'(n) M_{26}^{(M_{24})}.$$

Then

$$m_i'(n) \sim (-1)^{n+1} \frac{\sqrt{2}\, e^{\frac{\pi}{4}\sqrt{8n-1}}}{\sqrt{8n-1}} \left(\frac{|2A|}{|M_{24}|} \left(\chi_i(2A) - \frac{\chi_j(2A)}{\dim M_j^{(M_{24})}} \dim M_i^{(M_{24})} \right) \right.$$
$$\left. - \frac{|2B|}{|M_{24}|} \left(\chi_i(2B) - \frac{\chi_j(2B)}{\dim M_j^{(M_{24})}} \dim M_i^{(M_{24})} \right) \right)$$

as $n \to \infty$, where $j = 1$ if n is even and $j = 2$ if n is odd.

Finally, we show that our results apply to a sequence V_n of G-modules for some vertex operator algebra $V = \bigoplus V_n$ and G a finite group of automorphisms of V. More specifically, we have the following result.

Theorem N. *Let $V = \bigoplus V_n$ be a holomorphic, C_2-cofinite, and self-dual vertex operator algebra. Let G be a finite group of automorphisms of V. Let $g \in G$ and denote by $V(g)$ the unique (up to equivalence) g-twisted sector of V. If the conformal weights satisfy $\rho(V(g)) > \rho(V)$ for all $g \neq e$, then the sequence (V_n) of G-modules has dominant identity trace. Consequently, V_n tends to a multiple of the regular representation as $n \to \infty$.*

We note here that the assumption on the conformal weights is conjectured by [Möl18] to always hold for V a holomorphic, C_2-cofinite vertex operator algebra, and G a finite group of automorphism of V. Section 7.4 provides cases where this assumption are known to hold.

Chapter 2

Background–mock modular forms

2.1 Harmonic Maass forms and mock modular forms

Harmonic Maass forms are real-analytic functions f on the upper half plane that transform like weight k modular forms and are annihilated by the weight k hyperbolic Laplacian operator. In this way, it is possible to think of Harmonic Maass forms as non-holomorphic generalizations of classical modular forms.

In this section, we briefly recall some basic definitions and facts about mock modular forms and harmonic Maass forms. For more detailed information as well as references to original works, the reader may consult for example the book [BFOR17].

A *harmonic Maass form* of weight $k \in \mathbb{Z}$ for the group $\Gamma_0(N)$ is a smooth function $f : \mathfrak{H} \to \mathbb{C}$ satisfying the following three conditions:

1. f is invariant under the weight k slash operator,

$$f|_k \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) := (c\tau + d)^{-k} f\left(\frac{a\tau + b}{c\tau + d}\right) = f(\tau) \quad \text{for all } \tau \in \mathfrak{H} \text{ and } \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_0(N).$$

2. f is annihilated by the weight k hyperbolic Laplacian ($\tau = x + iy$),

$$\Delta_k f := \left[-y^2\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) + iky\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)\right] f \equiv 0.$$

3. f has at most linear exponential growth at the cusps, i.e. there exists a polynomial $H \in \mathbb{C}[X]$ such that $f - H(q^{-1})$ has exponential decay towards infinity and analogous conditions hold at all other cusps.

The space of these forms is denoted by $H_k(N)$. The subspaces $S_k(N) \subseteq M_k(N) \subseteq M_k^!(N) \subseteq H_k(N)$ denote the spaces of cusp forms, modular forms, and weakly holomorphic modular forms. It is sometimes convenient to relax the conditions to allow poles in the upper half-plane, in which case we speak of *polar* harmonic Maass forms.

These functions naturally split into a holomorphic and a non-holomorphic part [BFOR17, Lemma 4.3], $f = f^+ + f^-$. The holomorphic part of a harmonic Maass form is called a *mock modular form.* If f is a polar harmonic Maass form we call f^+ a polar mock modular form. Vice versa, given a mock modular form f, we refer to the harmonic Maass form $\widehat{f}$ having it as its holomorphic part as the *(modular) completion* of f.

The non-holomorphic part of a harmonic Maass form is related to a cusp form called the *shadow* of the corresponding mock modular form [BFOR17, Theorem 5.10].

Proposition 2.1.1. *The operator $\xi_k = 2iy^k\overline{\frac{\partial}{\partial\overline{\tau}}}$ defines a surjective $\mathbb{C}$-antilinear map*

$$H_k(N) \twoheadrightarrow S_{2-k}(N)$$

with kernel $M_k^!(N)$.

An important tool obtained from the ξ-operator is the so-called *Bruinier-Funke* *pairing*, defined by

$$\{\cdot,\cdot\} : M_k(N) \times H_{2-k}(N) \to \mathbb{C},\ \{g,f\} := \langle g, \xi_{2-k} f\rangle,$$

where for $g_1, g_2 \in M_k(N)$ such that $g_1 g_2$ is a cusp form we define

$$\langle g_1, g_2\rangle := \frac{1}{[\mathrm{SL}_2(\mathbb{Z}) : \Gamma_0(N)]} \int_{\Gamma_0(N)\backslash\mathfrak{H}} g_1(\tau)\overline{g_2(\tau)} y^k \frac{dxdy}{y^2}$$

as the classical Petersson scalar product. With this we have the following important result (see [BFOR17, Proposition 5.10]), which follows essentially from an application of Stokes's Theorem.

Proposition 2.1.2. *Let $g \in M_k(N)$ and $f \in H_{2-k}(N)$. For a cusp $\mathfrak{a}$ of $\Gamma_0(N)$ of width h, fix $\gamma \in \mathrm{SL}_2(\mathbb{Z})$ with $\gamma.(i\infty) = \mathfrak{a}$ and consider the Fourier expansions*

$$(g|\gamma)(\tau) = \sum_{n=0}^{\infty} a_{\mathfrak{a}}(n) q^{n/h} \quad \textit{and} \quad (f|\gamma)^{+}(\tau) = \sum_{m \gg -\infty} b_{\mathfrak{a}}(n) q^{n/h}.$$

Then we have

$$\{g,f\} = \sum_{\mathfrak{a}} \sum_{n\leq 0} a_{\mathfrak{a}}(-n) b_{\mathfrak{a}}(n).$$

An easy and well-known consequence of this is the following corollary.

Corollary 2.1.3. *A harmonic Maass form in $H_{2-k}(N)$ with no pole at any cusp is a holomorphic modular form.*

2.2 Operators on (mock) modular forms

We first review the definitions and some basic properties of Hecke operators. For this, consider for any $N, m \in \mathbb{N}$ the set

$$\mathcal{M}_m(N) = \left\{ M = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathbb{Z}^{2\times 2} \ :\ \det M = m,\ N \mid c,\ \gcd(a,N) = 1 \right\}.$$

The group $\Gamma_0(N)$ acts on $\mathcal{M}_m(N)$ by left-multiplication and we let $\beta_1, ..., \beta_s$ denote a set of coset representatives of this action. For any function $f : \mathfrak{H} \to \mathbb{C}$ transforming like a modular form of weight $k \in \mathbb{Z}$ under $\Gamma_0(N)$ we then define the m-th Hecke operator acting on f by

$$f|T_m^{(N)} = f|T_m = m^{k/2-1} \sum_{\beta \in \Gamma_0(N) \backslash \mathcal{M}_m(N)} f|_k \beta, \tag{2.2.1}$$

where we extend the action of the weight k slash operator to matrices with positive discriminant in the usual way by

$$(f|_k\gamma)(\tau) = (\det\gamma)^{k/2}(c\tau+d)^{-k} f\left(\frac{a\tau+b}{c\tau+d}\right).$$

We usually omit the indication of the level of the Hecke operator if it is clear from context or not relevant for the action.

These operators form a commutative algebra and they are multiplicative, i.e. one has $T_mT_n = T_{mn}$ for any coprime m, n. Their action on Fourier expansions is particularly easy to describe when $m = p$ is prime. Then we have

$$f|T_p = \begin{cases} f|U_p + p^{k-1} f|B_p & \text{if } p \nmid N \\ f|U_p & \text{if } p \mid N \end{cases}$$

where for $f(\tau) = \sum_{n\in\mathbb{Z}} a_f(n, y)q^n$ we set

$$(f|B_m)(\tau) = f(m\tau) = \sum_{n\in\mathbb{Z}} a_f(n, y)q^{mn} \qquad \text{and} \qquad (f|U_m)(\tau) = \sum_{n\in\mathbb{Z}} a_f(mn, y/m)q^n.$$

We record the following easy corollary ([CS17, Corollary 13.3.10]) of the Multiplicity 1-Theorem (see for instance [CS17, Theorem 13.3.9]).

Lemma 2.2.1. *Let H be an operator on the space $S_k(N)$ of weight k cusp forms for*

$\Gamma_0(N)$ *which commutes with all Hecke operators* T_p *for* $p \nmid N$. *Then any newform is an eigenfunction of* H.

Another important set of operators is given by the *Atkin-Lehner involutions*. For any exact divisor Q of N, i.e. $Q \mid N$ and $\gcd(Q, N/Q) = 1$, we define the Atkin-Lehner operator via the matrix

$$W_Q = \frac{1}{\sqrt{Q}} \begin{pmatrix} Qx & y \\ Nz & Qt \end{pmatrix}, \tag{2.2.2}$$

where $x, y, z, t \in \mathbb{Z}$ are chosen so that $\det W_Q = 1$. In the following proposition, we collect several well-known properties of these operators which will become important in the proof of Theorem 6.1.1. These can be found for instance in [CS17, Lemma 6.6.4, Proposition 13.2.6].

Proposition 2.2.2. *Let* $m \in \mathbb{N}$ *and* Q, Q' *exact divisors of* N *and let* $f : \mathfrak{H} \to \mathbb{C}$ *be a function transforming like a modular form of weight* $k \in 2\mathbb{Z}$ *for* $\Gamma_0(N)$. *Then the following are true.*

(i) As matrices, we have $W_Q = B_Q^{-1}\gamma = \gamma' B_Q$ *for* $\gamma, \gamma' \in \Gamma_0(N/Q)$ *and where we set* $B_m := \frac{1}{\sqrt{m}} \left(\begin{smallmatrix} m & 0 \\ 0 & 1 \end{smallmatrix} \right)$.

(ii) W_Q *normalizes* $\Gamma_0(N)$.

(iii) We have $W_Q^2 \in \Gamma_0(N)$ *and* $f|W_Q|W_{Q'} = f|W_{Q'}|W_Q = f|W_{QQ'}$

(iv) For $\gcd(m, Q) = 1$ *we have* $f|W_Q|T_m = f|T_m|W_Q$.

(v) For $\gcd(m, Q) = 1$ *we have* $f|W_Q|B_m = f|B_m|W_Q$.

(vi) If $Q = p$ *is prime, then* $f|U_p + p^{k/2-1} f|W_p$ *transforms like a modular form for* $\Gamma_0(N/p)$.

2.3 Mock modular forms as Rademacher sums

Convergent Rademacher sums are a large source of examples of mock modular forms. We briefly describe them here following the notation of Cheng and Duncan in [CD14] and refer the reader to loc. cit. for a more detailed discussion.

Rademacher sums are natural generalizations of Poincaré series. The idea behind Poincaré series is that one can construct a symmetric function starting with a non-symmetric function by summing its images under the desired group of symmetries. More precisely, the idea is to sum over images of a function f that is already invariant under a large enough subgroup of the full group of symmetries, so that it makes sense to restrict the summation to coset representatives.

A first example illustrating this takes the function $f(\tau) := e(m\tau)$ (here $e(x) := e^{2\pi i x}$), where m is an integer and $\tau \in \mathfrak{H}$. The function $f(\tau)$ is invariant under the subgroup Γ_∞ of $SL_2(\mathbb{Z})$ consisting of the upper triangular matrices in $SL_2(\mathbb{Z})$. In this case we take the sum

$$\sum_{\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_\infty \backslash SL_2(\mathbb{Z})} e\left(m\frac{a\tau+b}{c\tau+d}\right)\left(\frac{1}{c\tau+d}\right)^k,$$

where k is an integer. This sum is $SL_2(\mathbb{Z})$ invariant and when $k > 2$ the sum is absolutely convergent for any $\tau \in \mathfrak{H}$. Therefore we obtain a weakly holomorphic modular form on $SL_2(\mathbb{Z})$ of weight k.

When $k \leq 2$ such a sum is not absolutely convergent and some modification is needed. Rademacher first established a conditionally convergent expression for the normalized j-invariant in [Rad39]. This approach has since been generalized to apply to modular and mock modular forms of various weights and various subgroups of $SL_2(\mathbb{R})$ (see [Nie74, DF11, CD14]). For simplicity, here we discuss the case where $\Gamma = SL_2(\mathbb{Z})$.

In the case when $k = 2$, the summation can be reordered so that it is conditionally convergent and gives a weakly holomorphic function on the upper half plane. To do this we first define for $K > 0$ the set Γ_{K,K^2} as follows

$$\Gamma_{K,K^2} := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma \mid \ |c| < K, |d| < K^2 \right\}.$$

We can then define the weight k, index m Rademacher sum as follows

$$R^{[m]}_{k,\Gamma}(\tau) = \lim_{K\to\infty} \sum_{\gamma\in\Gamma_\infty\backslash\Gamma_{K,K^2}} e\left(m\frac{a\tau+b}{c\tau+d}\right)\left(\frac{1}{c\tau+d}\right)^k.$$

In the cases where $k < 2$ the convergence problem is more serious. To obtain a convergent expression, in addition to reordering the summation, we need to modify the terms by adding a regularization factor. Thus we have

$$R^{[m]}_{k,\Gamma}(\tau) = c^{[m]}_{k,\Gamma}(0) + \lim_{K\to\infty} \sum_{\gamma\in\Gamma_\infty\backslash\Gamma_{K,K^2}} e\left(m\frac{a\tau+b}{c\tau+d}\right) r^{[m]}_k(\gamma,\tau)\left(\frac{1}{c\tau+d}\right)^k,$$

where $c^{[m]}_{k,\Gamma}(0)$ and $r^{[m]}_k(\gamma,\tau)$ are given explicitly in equations (2.29) and (2.26) of [CD14], respectively.

These Rademacher sums have a pole of order m at infinity and a Fourier expansion of the form

$$R^{[m]}_{k,\Gamma}(\tau) = q^m + \sum_{n\geq 0} c^{[m]}_{k,\Gamma}(n)q^n$$

The Fourier coefficients of these Rademacher sums can be recovered using Rademacher series. To express the Rademacher series we first we define the set

$$\Gamma^\times_K := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma \mid \ 0 < |c| < K \right\}.$$

The Rademacher series $c_{k,\Gamma}^{[m]}(n)$ are given by

$$c_{k,\Gamma}^{[m]}(n) = \lim_{k\to\infty} \sum_{\gamma\in\Gamma_\infty\backslash\Gamma^\times/\Gamma_\infty} K_\gamma(\mu,\nu)B_\gamma(n),$$

where

$$K_\gamma(m,n) = e\left(n\frac{a}{c}\right)e\left(m\frac{d}{c}\right),$$

and

$$B_\gamma(m,n) = \begin{cases} e\left(\frac{-k}{4}\right)\sum_{j\geq 0}\frac{2\pi}{c^{2j+k}}\frac{(-m)^j}{j!}\frac{n^{j+k-1}}{\Gamma(j+k)}, & \text{if } k\geq 1 \\ e\left(\frac{-k}{4}\right)\sum_{j\geq 0}\frac{2\pi}{c^{2j+2-k}}\frac{(-m)^{j+1-k}n^j}{j!\Gamma(j+2-k)} & \text{if } k\leq 1, \end{cases}$$

where Γ is the usual Gamma function k is any weight $k \in \mathbb{R}$.

We remark that expressions we describe can be generalized so that the Rademacher sums are defined to have poles at other cusps (see [DF11]). These and the weight 1/2 Rademacher sum construction of the Mathieu moonshine mock modular forms by [CD14] will be especially useful in Chapter 7.

2.4 Weierstrass mock modular forms

In this section, we briefly recall the construction of Weierstrass mock modular forms. The idea for this construction is due to Guerzhoy [Gue15, Gue14] and was developed further by Alfes, Griffin, Ono, and Rolen [AGOR15].

Let E be an elliptic curve defined over $\mathbb{Q}$ of conductor N defined by the Weierstrass equation

$$E:\ y^2 = 4x^3 - g_2x - g_3.$$

As mentioned in the introduction, this curve (considered over $\mathbb{C}$) is isomorphic to a flat torus C/Λ_E, where $\Lambda_E \subset \mathbb{C}$ is a 2-dimensional $\mathbb{Z}$-lattice. This isomorphism is given by

$$\mathbb{C}/\Lambda_E \to E,\ z + \Lambda_E \mapsto \begin{cases} (\wp(\Lambda_E; z), \wp'(\Lambda_E; z)) & \text{if } z \notin \Lambda_E \\ \mathcal{O} & \text{otherwise,} \end{cases}$$

where

$$\wp(\Lambda_E; z) = \frac{1}{z^2} + \sum_{\omega \in \Lambda_E \setminus \{0\}} \left(\frac{1}{(z-\omega)^2} - \frac{1}{\omega^2} \right)$$

denotes the Weierstrass $\wp$-function and $\mathcal{O} \in E$ denotes the point at infinity. Recall that $\wp(\Lambda_E; z + \omega) = \wp(\Lambda_E; z)$ for all $\omega \in \Lambda_E$ and in fact the field of all *elliptic functions*, i.e. meromorphic functions with this exact periodicity property, is given by $\mathbb{C}(\wp)[\wp']$, where $\wp$ satisfies the differential equation

$$(\wp')^2 = 4\wp^3 - g_2 - g_3.$$

The $\wp$-function has poles of order 2 with residue 0 at all lattice points by construction. Its Laurent expansion around 0 is given by

$$\wp(\Lambda_E; z) = \frac{1}{z^2} + \sum_{n=2}^{\infty} (2n-1) G_{2n}(\Lambda_E) z^{2n-2},$$

where for integers $k > 2$, $G_{2n}(\Lambda_E) = \sum_{\omega \in \Lambda_E \setminus \{0\}} \omega^{-k}$ denotes the weight k *Eisenstein series* of Λ_E, which is of course 0 if k is odd. The negative antiderivative of the Weierstrass $\wp$-function, called the Weierstrass ζ-function, therefore has simple poles at all lattice points and nowhere else and is given by

$$\zeta(\Lambda_E; z) = \frac{1}{z} + \sum_{\omega \in \Lambda_E \setminus \{0\}} \left(\frac{1}{z-\omega} + \frac{1}{\omega} + \frac{z}{\omega^2} \right) = \frac{1}{z} - \sum_{n=2}^{\infty} G_{2n}(\Lambda_E) z^{2n-1}.$$

However, by Liouville's famous theorems on elliptic functions, there cannot be an elliptic function with simple poles only at lattice points and nowhere else, so $\zeta(\Lambda_E; z)$ is not quite an elliptic function. It was first observed by Eisenstein (in a special case) that there is a canonical way to complete the Weierstrass ζ-function to a function which has the periodicity behaviour of an elliptic function at the expense of no longer being holomorphic. In order to define Eisenstein's completed Weierstrass ζ-function let

$$G_2^*(\Lambda_E) = \lim_{s\to 0} \sum_{\omega\in\Lambda_E\setminus\{0\}} \omega^{-2}|\omega|^{-2s}$$

denote the completed[1] Eisenstein series of weight 2. By the famous modularity theorem, there is a newform $f_E \in S_2(N)$ with integer Fourier coefficients associated to E such that the L-functions of E and f_E agree, which by Eichler-Shimura theory yields a polynomial map

$$\phi_E : X_0(N) \to \mathbb{C}/\Lambda_E,$$

the *modular parametrization* of E. Then the non-holomorphic function

$$\widehat{\zeta}(\Lambda_E; z) = \zeta(\Lambda_E; z) - G_2^*(\Lambda_E)z - \frac{\deg \phi_E}{4\pi\|f_E\|^2}\overline{z},$$

where $\|\cdot\|$ denotes the Petersson norm, satisfies $\widehat{\zeta}(\Lambda_E; z+\omega) = \widehat{\zeta}(\Lambda_E; z)$ for all $z \in \mathbb{C}\setminus\Lambda_E$ and $\omega \in \Lambda_E$.

The newform f_E has a Fourier expansion $f_E(\tau) = \sum_{n=1}^{\infty} a_E(n)q^n$ with $q = e^{2\pi i\tau}$. Denoting by

$$\mathcal{E}_E(\tau) = -2\pi i \int_\tau^\infty f_E(t)dt = \sum_{n=1}^{\infty} \frac{a_E(n)}{n} q^n$$

the *Eichler integral* of f_E, one finds the following result [AGOR15, Theorem 1.1].

Theorem 2.4.1. *The function*

$$\mathfrak{Z}_E(\tau) = \zeta(\Lambda_E; \mathcal{E}_E(\tau)) - G_2^*(\Lambda_E)\mathcal{E}_E(\tau),$$

called the Weierstrass mock modular form *is a polar mock modular form of weight* 0

[1] The sum defining the Eisenstein series is no longer absolutely convergent for $k = 2$. The modification here is sometimes called Hecke's trick [Hec27].

for the group $\Gamma_0(N)$*. To be more precise, there exists a meromorphic modular function* M_E *for* $\Gamma_0(N)$ *such that the function*

$$\widehat{\mathfrak{Z}}_E(\tau) = \widehat{\zeta}(\Lambda_E; \mathcal{E}_E(\tau)) - M_E(\tau)$$

is a harmonic Maaß form of weight 0 *for* $\Gamma_0(N)$*.*

It is immediately clear from the definition that the function $\mathfrak{Z}_E$ has poles precisely where the value of the Eichler integral $\mathcal{E}_E(\tau)$ lies in the period lattice Λ_E. It is an open problem to classify those points τ in the complex upper half-plane $\mathfrak{H}$ where this occurs, but the following lemma, whose proof can be found for example in [AM18], allows us to rule out poles in the situation where E and the modular curve $X_0(N)$ are actually isomorphic, so where the degree of ϕ_E is 1.

Lemma 2.4.2. *Let* E *be the strong Weil curve of conductor* N *such that* $X_0(N)$ *has genus* 1*, i.e.* $N \in \{11, 14, 15, 17, 19, 20, 21, 24, 27, 32, 36, 49\}$*. Then the Weierstrass mock modular form* $\mathfrak{Z}_E$ *has no poles in* $\mathfrak{H}$*.*

For the purpose of this paper, it is important to consider the behaviour of the (completed) Weierstrass mock modular form at other cusps than infinity. For this, we need the following slight generalization of [AGOR15, Theorem 1.2].

Proposition 2.4.3. *Let* $\nu \in \mathsf{N}(\Gamma_0(N))$*, the normalizer of* $\Gamma_0(N)$ *in* $\mathrm{SL}_2(\mathbb{R})$*, which commutes with all Hecke operators* T_p *with prime* $p \nmid N$*. Then we have*

$$\left(\widehat{\mathfrak{Z}}_E|_0\nu\right)(\tau) = \widehat{\zeta}(\Lambda_E; \lambda_\nu(\mathcal{E}_E(\tau) - \Omega_{\nu^{-1}}(f_E)))$$

where $\Omega_\nu(f_E) = -2\pi i \int_{\nu^{-1}\infty}^{\infty} f_E(t)dt$ *and* λ_ν *is the eigenvalue of* f_E *under* ν *(see Lemma 2.2.1). In particular for* $-\lambda_\nu\Omega_\nu(f_E) \notin \Lambda_E$ *we find the asymptotic*

$$(\mathfrak{Z}_E|_0\nu)(iy) \sim \widehat{\zeta}(-\lambda_\nu\Omega_\nu(f_E)) + \exp(-\alpha y) \qquad \text{as } y \to \infty,$$

for some $\alpha > 0$.

Proof. In [AGOR15, Theorem 1.2], the result is stated for ν an Atkin-Lehner involution. The exact same proof goes through, only applying Lemma 2.2.1 in the substitutions. In fact, the computation goes through even for any elliptic curve $E/\mathbb{Q}$ and matrix $\sigma \in \mathrm{SL}_2(\mathbb{R})$ such that $\sigma\Gamma_0(N)\sigma^{-1} \cap \Gamma_0(N)$ has finite index in $\Gamma_0(N)$: One finds

$$\begin{aligned}(\widehat{\mathfrak{Z}}_E|_0\sigma)(\tau) &= \widehat{\zeta}\left(\Lambda_E; -2\pi i \int_{\sigma.\tau}^{i\infty} f_E(z)dz\right) \\ &= \widehat{\zeta}\left(\Lambda_E; -2\pi i \int_{\tau}^{i\infty} (f_E|_2\sigma)(z)dz + 2\pi i \int_{\sigma^{-1}.(i\infty)}^{i\infty} (f_E|_2\sigma)(z)dz\right) \qquad (2.4.1) \\ &= \widehat{\zeta}\left(\Lambda_E; -2\pi i \int_{\tau}^{i\infty} (f_E|_2\sigma)(z)dz + \Omega_{\sigma^{-1}}(f_E)\right).\end{aligned}$$

The claim then follows immediately using Lemma 2.2.1. □

Remark. In the case of interest to us, the space $S_2(N)$ is one-dimensional. Since $\mathsf{N}(\Gamma_0(N))$ acts on the space of cusp forms, a cusp form in those levels must be an eigenfunction under any element of the normalizer, so the proof goes through here without the appeal to the Multiplicity-one theorem and Lemma 2.2.1.

Corollary 2.4.4. *For all levels* $N \in \{11, 14, 15, 17, 19, 20, 21, 24, 27, 32, 36\}$*, the completed Weierstrass mock modular form* $\widehat{\mathfrak{Z}}_E$ *for the strong Weil curve* E *of conductor* N *has a simple pole at* ∞ *and is constant at all other cusps.*

Proof. For the given N, the normalizer $\mathsf{N}(\Gamma_0(N))$ acts transitively on the cusps of $\Gamma_0(N)$. Note that for the square-free levels, it is well-known that the Atkin-Lehner operators already act transitively on the cusps. By computing the relevant periods $\Omega_\nu(f_E)$ explicitly[2], we see that none of them are in Λ_E and the claim follows. □

Remark. For the remaining level 49, the normalizer does not act transitively on cusps.

[2]The authors used the `mfsymboleval` command in PARI/GP [Gro18] for this.

However, one can use the fact that

$$\frac{1}{2\pi i}\frac{\partial}{\partial\tau}\widehat{\mathfrak{Z}}_E(\tau) = \frac{1}{g_{49}(\tau)}\left(\frac{1}{2400}E_4(7\tau) - \frac{2401}{2400}E_4(49\tau) + G_{49}(\tau)\right),$$

where $g_{49}(\tau) = q + q^2 - q^4 - 3q^8 - 3q^9 + O(q^{11})$ denotes the unique newform in $S_2(49)$ and $G_{49}(\tau) = -q + q^3 - q^4 - q^5 - q^6 + 49/10q^7 + 5q^8 + q^9 - 6q^{10} + 7q^{11} + O(q^{12}) \in S_4(49)$, is a weakly holomorphic modular form of weight 2. The fact that the derivative of $\widehat{\mathfrak{Z}}_E$ is a weakly holomorphic modular form is a general consequence of Bol's identity and for the identification one notices that by (2.4.1), the function $\widehat{\mathfrak{Z}}_E$ and therefore its derivative can have at most a simple pole at any cusp, so that $\frac{1}{2\pi i}\frac{\partial}{\partial\tau}\widehat{\mathfrak{Z}}_E \cdot g_{49}$ is a holomorphic modular form of level 49. Due to this identity, it can be checked that its only pole is at infinity (in fact, it vanishes at all other cusps), wherefore, since differentiation commutes with the action of $\mathrm{SL}_2(\mathbb{R})$ in weight 0 and doesn't introduce or add any poles, Corollary 2.4.4 is also true for $N = 49$. The same argument would of course also work in the cases covered by Corollary 2.4.4.

Chapter 3

Background–vertex operator algebras

The construction of the moonshine module $V^\natural$ [FLM84, FLM85, FLM88] has greatly motivated the study of vertex operator algebras (VOAs). The problem of orbifolding a conformal field theory with respect to an automorphism rose to prominence contemporaneously in physics [DHVW85, DHVW86]. The construction of $V^\natural$ was subsequently interpreted as the first example of an orbifold model that is not equivalent to a lattice vertex operator algebra [FLM88]. For G a group of automorphisms of V, the study of the fixed point sub-VOA V^G and its representation theory is referred to as *orbifold theory*. We refer the reader to [DRX17, EMSara, MÏ6] for details on cyclic orbifold theory for holomorphic VOAs and give a short summary below.

We first recall some basic definitions and properties of VOAs and their twisted modules. We refer the reader to [FBZ04] [FLM88] and [LL04] for more details.

3.1 Vertex operator algebras: basics and definitions

A *vertex operator algebra* (VOA) V is a complex vector space with a $\mathbb{Z}$-grading, bounded from below, equipped with two distinguished vectors $\mathbf{1}$ and ω called the vacuum element and the conformal vector, respectively. Further, for each vector $v \in V$ there is a map $Y(\cdot, z) : V \to \mathrm{End}(V)[\![z, z^{-1}]\!]$ assigning a formal power series $Y(v, z) :=$

$\sum_{n\in\mathbb{Z}} v(n)z^{-n-1}$ (which we call a vertex operator) to v. The tuple $(V, \mathbf{1}, \omega, Y)$ must satisfy the following axioms for $u, v, w \in V$:

- $u(n)v = 0$ for n sufficiently large,
- $Y(\mathbf{1}, \mathbf{z}) = \mathbf{1}$,
- $Y(v, z)\mathbf{1} \in \mathbf{V}[\![\mathbf{z}]\!]$ and $\lim_{z\to 0} Y(v, z)\mathbf{1} = \mathbf{v}$,
- The Jacobi identity

$$z_0^{-1}\delta\left(\frac{z_1 - z_2}{z_0}\right) Y(u, z_1)Y(v, z_2)w - z_0^{-1}\delta\left(\frac{z_2 - z_1}{-z_0}\right) Y(v, z_2)Y(u, z_1)w = z_2^{-1}\delta\left(\frac{z_1 - z_0}{z_2}\right) Y(Y(u, z_0)v, z_2)w$$

  where $\delta(\cdot)$ is the formal delta series.

- The coefficients of the vertex operator attached to the conformal vector generate a copy of the Virasoro algebra of central charge c. In other words, if $Y(\omega, z) := \sum_{n\in\mathbb{Z}} L(n)z^{-n-2}$ then $[L(m), L(n)] = (m-n)L(m+n) + \frac{1}{12}(m^3 - m)\delta_{m+n,0}c$, and we refer to c as the *central charge* of V.
- This grading on V coincides with the eigenspaces of the $L(0)$ operator, by which we mean that $V_n := \{v \in V \mid L(0)v = nv\}$.
- The smallest $n \in \mathbb{C}$ for which $V_n \neq 0$ is called the *conformal weight* of V and is denoted $\rho(V)$;
- $\dfrac{d}{dz}Y(v, z) = Y(L(-1)v, z)$.

We say V is of *CFT-type* if $\rho(V) = 0$ and $V_0 = \mathbb{C}\mathbf{1}$.

There is a notion of a module over a VOA. A *V-module* is a vector space M equipped with an operation $Y_M : V \to \mathrm{End}(M)[\![z^{\pm 1}]\!]$ which assigns to each $v \in V$ a formal power series $Y_M(v, z) := \sum_{n\in\mathbb{Z}} v^M(n)z^{-n-1}$ subject to several axioms (see section 5.1 of [FBZ04]). A module M whose only submodules are 0 and itself is called *simple* or *irreducible*. A VOA V for which every admissible V-module decomposes into a direct sum of (ordinary) irreducibles is called *rational* and we say that V is *holomorphic* if it is rational and has a unique irreducible module (which must necessarily be V itself). Given a V-module W with a grading, it is possible to define a V-module W', that is (as a vector space) the graded dual space of W (for a definition of the dual module we refer to Section 5.2 of [FHL93]). We say a vertex algebra V is *self-dual* if the module V is isomorphic to its dual V' (as a V-module). In [Zhu96], Zhu introduced a finiteness condition on a VOA V, we say V is C_2*-cofinite* if $C_2(V) := \mathrm{span}\{v(2)w \mid v, w \in V\}$ has finite codimension in V. A VOA is called *strongly rational* if it is rational, C_2-cofinite, self-dual, and of CFT-type.

3.2 Orbifold theory

For G a finite group of automorphisms of V and $g \in G$, one can define a g-twisted module $V(g)$ of V (see Section 3 of [DLM00]). By [DLM00], for V a C_2-cofinite holomorphic VOA and $G = \langle g \rangle$ a cyclic group of automorphisms of V, V posesses a unique simple g^i-twisted V-module, which we call $V(g^i)$, for each $i \in \mathbb{Z}/N\mathbb{Z}$ for N the order of g. By Proposition 4.2.3 of [MÏ6] (see also [DLM00]) there is a representation

$$\phi_i \colon G \to \mathrm{Aut}_{\mathbb{C}}(V(g^i))$$

of G on the vector space $V(g^i)$ such that $\phi_i(g) Y_{V(g^i)}(v,z)\phi_i^{-1}(g) = Y_{V(g^i)}$ for all $i \in \mathbb{Z}/N\mathbb{Z}$ and $v \in V$. This representation is unique up to an N-th root of unity. The eigenspace of $\phi_i(g)$ in $V(g^i)$ corresponding to the eigenvalue $e^{(2\pi i)j/N}$ is denoted by $W^{(i,j)}$ and as $\mathbb{C}[G]$-modules, we have that $V(g^i) = \bigoplus_j W^{(i,j)}$.

The fixed point sub-VOA $V^G = W^{(0,0)}$ of V is defined to be the vectors in V which are fixed pointwise under the action of G.

The main theorem of orbifold theory [CM16, Miy15, DM97] is that if V is strongly rational and G is a finite, solvable group of automorphisms of V, then the fixed-point VOA V^G is strongly rational as well.

For all $i, j \in \mathbb{Z}/N\mathbb{Z}$, the $W^{(i,j)}$ are irreducible V^G-modules [MT04] and further, by the classification of irreducible modules in [MÏ6], there are exactly n^2 irreducible V^G-modules (namely, the $W^{(i,j)}$). We make the additional assumption that g has type $N\{0\}$ (a certain condition on the conformal weights of the g-twisted modules, see Definition 4.7.4 of [MÏ6]), which gives us that the conformal weights obey $\rho(V(g)) \in (1/N)\mathbb{Z}$. This enables us to choose representations ϕ_i such that the conformal weights of $W^{(i,j)}$ obey $\rho(W^{(i,j)}) \in (ij/N)\mathbb{Z}$.

We also assume that V^G satisfies the positivity assumption, which states that for a simple VOA V, the conformal weights of any irreducible V-module $W \neq V$ are positive and the conformal weight of V is zero.

If V^G satisfies the positivity assumption, the orbifold VOA of V with respect to g is defined to be

$$V^{\mathrm{orb}(g)} := \bigoplus_{i \in \mathbb{Z}/N\mathbb{Z}} W^{(i,0)}.$$

Note that if V is strongly rational, then $V^{\mathrm{orb}(g)}$ has the structure of a holomorphic, strongly rational VOA of the same central charge as V.

3.3 Modular invariance of characters

For V a rational, C_2-cofinite vertex operator algebra, Zhu [Zhu96] showed that the characters of the irreducible modules of V form a vector-valued modular function for $SL_2(\mathbb{Z})$ with a multiplier system. Dong–Lin–Ng [DLN15] showed that if V is also self-dual the characters of the irreducible modules of V are modular functions for congruence subgroups of $SL_2(\mathbb{Z})$. Dong–Li–Mason [DLM00] establish the modular-invariance of the characters of twisted irreducible modules. We refer to their works

[Zhu96, DLM00, DLN15] for details on the modular invariance of irreducible (twisted) modules for self dual C_2-cofinite VOAs.

Let V be a self dual, holomorphic VOA of CFT type with central charge c and G a finite group of automorphisms of V then for $g \in G$ we denote by $V(g)$ its unique irreducible g-twisted module. If $h \in G$ commutes with g, then h induces an action on $V(g)$ (which is well-defined up to a scalar factor). A special case of the trace functions defined by Dong–Li–Mason in [DLM00] can then be defined as follows:

$$Z(g,h;\tau) := \sum_{n=0}^{\infty} \mathrm{tr}(h \mid V(g)_{\frac{n}{\mathrm{ord}(g)}+\rho(V(g))}) q^{\frac{n}{\mathrm{ord}(g)}+\rho(V(g))-\frac{c}{24}}.$$

The $Z(g,h;\tau)$ are holomorphic on the upper half plane and modular for some congruence subgroup of $SL_2(\mathbb{Z})$. We will consider these trace functions in Chapter 7.

Next we recall the several results from [EMSara, EMSarb] that will be useful to us in Chapter 6.

Let V be a strongly rational, holomorphic vertex operator algebra with central charge divisible by 24, the characters of the irreducible V^G-modules $\mathrm{ch}_{W^{(i,j)}}(\tau) = \mathrm{tr}_{W^{(i,j)}} q^{L(0)-c/24}$ are holomorphic on the upper half-plane and modular of weight 0 for $\Gamma_0(N)$ ([EMSara, Theorem 5.1]).

Proposition 3.3.1. *The characters* $\mathrm{ch}_{W^{(i,j)}}(\tau)$ *form a vector-valued modular form of weight* 0 *for the Weil representation associated to the finite quadratic module* $(\mathbb{Z}/N\mathbb{Z}) \times (\mathbb{Z}/N\mathbb{Z})$ *endowed with the quadratic form* $q((i,j)) = ij/N + \mathbb{Z}$. *In particular, their transformation properties under the standard generators* $S = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$ *and* $T = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$ *of* $\mathrm{SL}_2(\mathbb{Z})$ *are given by*

$$\mathrm{ch}_{W^{(i,j)}}(S.\tau) = \frac{1}{n} \sum_{k,\ell \in \mathbb{Z}/N\mathbb{Z}} e^{(2\pi i)(i\ell+jk)/N} \mathrm{ch}_{W^{(k,\ell)}}(\tau),$$

and

$$\mathrm{ch}_{W^{(i,j)}}(T.\tau) = e^{(2\pi i)ij/N}\mathrm{ch}_{W^{(i,j)}}(\tau).$$

From equation (7) of [EMSarb], we have the following transformation property.

Proposition 3.3.2. *The character* $\mathrm{ch}_{V^G}(\tau)$ *is a modular function for* $\Gamma_0(N)$ *and moreover, for a matrix* $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$*, sending* ∞ *to the cusp* $\mathfrak{a} = a/c$ *with* $c \mid N$ *and* $\gcd(a,c) = 1$*, we have*[1]

$$\mathrm{ch}_{W^{(0,0)}}(\gamma.\tau) = \frac{c}{N}\sum_{i,j\in\mathbb{Z}/(N/c)\mathbb{Z}} e^{(2\pi i)dcij/N}\mathrm{ch}_{W^{(ci,cj)}}(\tau).$$

For a cusp $\mathfrak{a}$ of $\Gamma_0(N)$, van Ekeren, Möller, and Scheithauer [EMSarb] define the function

$$F_{\mathfrak{a}}(\tau) := \sum_{\substack{\gamma\in\Gamma_0(N)\backslash\mathrm{SL}_2(\mathbb{Z}) \\ \gamma.\infty=\mathfrak{a}}} \mathrm{ch}_{W^{(0,0)}}(\gamma.\tau). \tag{3.3.1}$$

In [EMSarb, Proposition 3.6], they give the following general identity for this function $F_{\mathfrak{a}}$ which is essential in establishing the dimension formulas.

Proposition 3.3.3. *The function* $F_{\mathfrak{a}}$ *defined in* (3.3.1) *satisfies the identity*

$$\sum_{\mathfrak{a}} F_{\mathfrak{a}}(\tau) = \sum_{d\mid N} \frac{\varphi(\gcd(d,N/d))}{\gcd(d,N/d)}\,\mathrm{ch}_{V^{\mathrm{orb}(g^d)}}(\tau),$$

where the sum over $\mathfrak{a}$ *runs over a set of representatives of cusps of* $\Gamma_0(N)$ *and* $\varphi(n) := \#(\mathbb{Z}/n\mathbb{Z})^*$ *denotes Euler's totient function.*

[1] Note that in [EMSarb, Equation (7)] there is an erroneous minus sign in the exponential.

Chapter 4

Quasimodular moonshine and arithmetic connections

In this chapter, we prove the existence of a module for M_{24}, whose trace functions are weight two quasimodular forms. Restricting to the subgroup fixing a point, we see that the integrality of these functions is equivalent to certain divisibility conditions on the number of $\mathbb{F}_p$ points on Jacobians of modular curves. Extending these expressions to arbitrary primes, we find trace functions for modules of cyclic groups of prime order with similar connections. Moreover, for cyclic groups, we give an explicit vertex operator algebra construction whose trace functions are given only in terms of weight two Eisenstein series. These results come from [Ben19b].

4.1 Quasimodular M_{24} forms

The setup of the M_{24} functions starts with building forms of weight 2 from the original Mathieu moonshine functions. More specifically, the forms come from multiplying the completions $\hat{H}_g(\tau)$ of the mock modular forms $H_g(\tau)$ from Mathieu moonshine by $\eta^3(\tau)$. Note that $\hat{H}_g(\tau)\eta^3(\tau)$ is a non holomorphic modular form of weight 2, and since it does not have singularities at cusps, we can apply holomorphic projection to extract something holomorphic. In weight 2, holomorphic projection results in a quasimodular form. For more information about mock modular forms, their completions, and holomorphic projection, we refer the reader to [BFOR17].

To give the expression for the holomorphic projection explicitly, we first define a function $F_2(\tau)$ as follows:

$$F_2(\tau) := \sum_{\substack{r>s>0 \\ r-s \text{ odd}}} s q^{rs/2}.$$

The holomorphic projections of the $\hat{H}_g(\tau)\eta^3(\tau)$ are given in terms of $H_g(\tau)\eta^3(\tau)$ and some multiple of $F_2(\tau)$ for each $g \in M_{24}$. We will see that these expressions are quasimodular forms. For the first case, Dabholkar, Murthy, and Zagier give a formula in [DMZ12] that (when rearranged) says

$$H_e(\tau)\eta^3(\tau) - 48F_2(\tau) = -2E_2(\tau). \tag{4.1.1}$$

Remark. Dabholkar, Murthy, and Zagier also define a higher weight analogue of $F_2^{(k)}$ for $k \geq 2$. For our purposes $k = 2$.

A theorem of Mertens, when specialized to these functions, gives explicitly that the holomorphic projection of $\hat{H}_e(\tau)\eta^3(\tau)$ is equal to the left hand side of (4.1.1) and is a quasimodular form [Mer16]. Mertens' theorem can be applied to the other functions $H_g(\tau)$, and in fact, we have such a formula more generally, for any $g \in M_{24}$. Let $\chi(g)$ be the number of fixed points of g in the 24-dimensional permutation representation of M_{24}. We define

$$Q_g(\tau) := H_g(\tau)\eta^3(\tau) - 2\chi(g)F_2(\tau).$$

This is consistent with the $g = e$ case given above because the number of fixed points in that case is $\chi(e) = 24$. We will show that these functions are quasimodular.

To be precise in the description of the $Q_g(\tau)$, we first define ρ_g, a function from $\Gamma_0(|g|)$ to $\mathbb{C}$ given by $\rho_g(\gamma) := \exp\left(2\pi i\left(-\frac{cd}{|g|h}\right)\right)$, where h is the minimal length among cycles in the cycle shape of g and c, d are the entries of the lower row of a matrix γ in $\Gamma_0(|g|)$.

Proposition 4.1.1. *The $Q_g(\tau)$ are quasi-modular of weight 2 on $\Gamma_0(|g|)$, with multiplier system ρ_g.*

Proof. The following explicit formula

$$H_g(\tau) = \frac{\chi(g)}{24} H_e(\tau) - \frac{T(g)}{\eta^3(\tau)}. \tag{4.1.2}$$

was obtained in [Che10, GHV10b, GHV10a, EH11] (see Section 3 of [CD12]). When rearranged, the above formula relates $H_e(\tau)\eta^3(\tau)$ to each of the $H_g(\tau)\eta^3(\tau)$. The $T(g)$ are weight 2 forms on $\Gamma_0(|g|)$ with multiplier ρ_g and their explicit expressions are given in Appendix B.3.1 of [DGO15b]. Combining these with the equation (4.1.1) for $H_e(\tau)\eta^3(\tau)$ gives that the functions $H_g(\tau)\eta^3(\tau)$ are quasimodular of weight 2. □

Now that we have described our new functions $Q_g(\tau)$ we show that there exists an M_{24}-module for which these are the graded trace functions.

Theorem 4.1.2. *There exists a virtual graded M_{24}-module $V = \bigoplus_n V_n$ such that*

$$Q_g(\tau) = \sum_{n=0}^{\infty} \mathrm{tr}(g \mid V_n) q^n.$$

Proof. First we show that the $Q_g(\tau)$ have integral coefficients. Gannon [Gan16] shows that the functions $H_g(\tau)$ have integral coefficients. It is known that $\eta^3(\tau)$ has integral coefficients, and $F_2(\tau)$ also has integral coefficients. So we know that

$$Q_g(\tau) := H_g(\tau)\eta^3(\tau) - 2\chi(g)F_2(\tau)$$

must have integral coefficients.

Next we show that the multiplicities $\mathrm{m}_i^Q(n)$ of the M_{24} irreducible representations in the class functions defined by the coefficients of $Q_g(\tau)$ are integral.

Gannon shows that the multiplicity generating function

$$\sum_{n>0} \mathrm{m}_i^H(n)q^n = \frac{1}{|M_{24}|}\sum_{g\in M_{24}} H_g(\tau)\overline{\chi_i(g)} \tag{4.1.3}$$

(with χ_i an irreducible character of M_{24}) has integral coefficients. What we need to show is that the coefficients $\mathrm{m}_i^Q(n)$ are integral, where

$$\sum_{n>0} \mathrm{m}_i^Q(n)q^n = \frac{1}{|M_{24}|}\sum_{g\in M_{24}} \left[H_g(\tau)\eta^3(\tau) - 2\chi(g)F_2(\tau)\right]\overline{\chi_i(g)}. \tag{4.1.4}$$

To do this, we can split the right hand side of equation (5.1.35) into two parts. First consider $\frac{1}{|M_{24}|}\sum_{g\in M_{24}} H_g(\tau)\eta^3(\tau)\overline{\chi_i(g)}$. This differs from (5.1.34) only from multiplying by $\eta^3(\tau)$, which does not change the integrality. So it suffices to show that $\frac{1}{|M_{24}|}\sum_{g\in M_{24}} \chi(g)F_2(\tau)\overline{\chi_i(g)}$ has integral coefficients. This is the same as showing that

$$F_2(\tau)\frac{1}{|M_{24}|}\sum_{g\in M_{24}} \chi(g)\overline{\chi_i(g)} = F_2(\tau)\langle\chi,\chi_i\rangle$$

has integral coefficients. We already know that $F_2(\tau)$ has integral coefficients. The integrality of $\langle\chi,\chi_i\rangle$ can be seen from the fact that $\chi(g)$ is a character of a module, and so $\langle\chi,\chi_i\rangle$ is the multiplicity of χ_i in χ, which is necessarily integral. Thus the $\mathrm{m}_i^Q(n)$ from (5.1.35) are integral.

□

4.2 More general framework

In this section we show that for certain conjugacy classes $[g]$, the $Q_g(\tau)$ have convenient expressions containing arithmetic information. Further, we show that this type of expression can be generalized to be in terms of an arbitrary prime N.

If we restrict to M_{23}, a subgroup of M_{24} for which all g have $\chi(g) \neq 0$, we can give an alternate expression for the corresponding $Q_g(\tau)$. First, let

$$E_{2,N}(\tau) := \frac{1}{i(N)\varphi(N)} \sum_{M|N} \mu\left(\frac{N}{M}\right) M^2 E_2(M\tau) \tag{4.2.1}$$

where N is defined to be the order of g, $i(N)$ is the index of $\Gamma_0(N)$ in $SL_2(\mathbb{Z})$, and φ is the Euler totient function. Note that these functions $E_{2,N}(\tau)$ are quasimodular of weight 2 on $\Gamma_0(N)$.

For each $N = |g|$ with $g \in M_{23}$, let $G_N(\tau)$ denote the specific cusp form of level N given explicitly in the appendix of [Ben19b]. We also let $n_N := \text{num}\left(\frac{N-1}{12}\right)$. Then we have the following formula:

$$H_g(\tau)\eta^3(\tau) - 2\chi(g)F_2(\tau) = -2E_{2,N}(\tau) + \frac{N}{n_N}G_N(\tau).$$

This can be easily checked by comparing case-by-case to formula $(B.24)$ in [DGO15b]. From this it follows that for $g \in M_{23}$

$$Q_g(\tau) = -2E_{2,N}(\tau) + \frac{N}{n_N}G_N(\tau). \tag{4.2.2}$$

Note that the formula for $Q_g(\tau)$ above is defined in terms of $N = |g|$ but the expression does not depend on M_{23} at all. We can define such functions $Q_N(\tau)$ for arbitrary prime N with suitable cusp forms $G_N(\tau)$ that come from a result of Mazur. In what follows we show precisely how to define the $Q_N(\tau)$ including how to use the result of Mazur to determine the cusp form.

The multiples of the cusp forms $G_N(\tau)$ in the expressions (4.2.2) have denominators equal to n_N (recall, $n_N = \text{num}\left(\frac{N-1}{12}\right)$). In the trace functions where N is prime, these denominators reflect a result of a congruence between Eisenstein series and cusp forms that is due to Mazur. Due to this result, we are able to define quasimodular forms as in (4.2.2) and show the existence of a $\mathbb{Z}/N\mathbb{Z}$-module such that these forms are its trace functions.

First, we give a more elementary argument for Mazur's result (Proposition 5.12 of [Maz77]) that there exists a cusp form congruent to the Eisenstein series of level N for N prime. Denote the mth coefficient of the normalized Eisenstein series $\frac{1}{24}(NE_2(N\tau) - E_2(\tau))$ by $\sigma_N(m)$. To prove the existence of a cusp form whose coefficients are congruent to $\sigma_N(m)$, we will use theta series, defined in [Gro87]. Let $i, j \in \{1 \dots n\}$ where n is the number of left ideal classes in the quaternion algebra over $\mathbb{Q}$ ramified at the two places N and ∞. Then the theta series are defined as

$$f_{ij} := \frac{1}{2w_j} + \sum_{m \geq 1} B_{ij}(m) q^m, \tag{4.2.3}$$

where $B_{ij}(m)$ are entries of the Brandt matrix of degree m, and w_j are integers that correspond to the cardinalities of certain groups. We will use the fact that these f_{ij} are functions with integral coefficients (except the constant term) on the upper half plane. In fact, the f_{ij} span $M_2(\Gamma_0(N))$ and a certain explicit linear combination of f_{ij} recovers the normalized Eisenstein series. See [Qua11] and [Mar17] for related results using these theta series. We will use them to prove the following proposition.

Proposition 4.2.1. *Let N be prime. Then there exists a cusp form $g(\tau) = \sum_{m>0} c_g(m) q^m$ of weight 2 on $\Gamma_0(N)$ with integer coefficients such that*

$$c_g(m) \equiv \sigma_N(m) \pmod{n_N}$$

for all $m > 0$.

Proof. First we treat the cases $N = 2, 3$. For these cases, $n_N = 1$ and the space of weight two cusp forms on $\Gamma_0(N)$ is empty. This means $c_g(m) = 0$ for all m, and since

$\sigma_N(m)$ are necessarily integers, the statement $0 \equiv \sigma_N(m) \pmod 1$ is true for all m. For the rest of the cases, we use that any modular form of weight 2 can be written as a linear combination of the f_{ij}, defined in (4.2.3). We would like to find a linear combination of f_{ij} with constant term zero, that is, a cusp form, which is congruent to the Eisenstein series modulo n_N. Note that finding such a linear combination with constant term zero ensures we have a cusp form. This is because in weight 2, the sum of the constant terms of a modular form at all cusps must be zero. Since we have that N is prime, $\Gamma_0(N)$ has only two cusps, so it is sufficient to check the vanishing at one cusp. Recall (equation (5.7) in [Gro87]) that the Eisenstein series $\frac{1}{24}(NE_2(N\tau) - E_2(\tau))$ is given by

$$\sum_{j=1}^{n} f_{ij} = \frac{N-1}{24} + \sum_{m>0} \sigma_N(m) q^m = \frac{N-1}{24} + \sum_{m>0} \sum_{j=1}^{n} B_{ij}(m) q^m,$$

for any $i \in \{1 \ldots n\}$.

We will use the fact that all primes $N > 3$ are $1, 5, 7, 11 \pmod{12}$.

The first case is $N \equiv 1 \pmod{12}$ which implies that $N-1 \equiv 0 \pmod{12}$ so $12 \mid N-1$.

Here we take

$$g(\tau) = \left(\sum_{j=1}^{n} f_{1j} \right) - w_1 f_{11} n_N.$$

The constant term of $2w_1 f_{11}$ is 1 and so the constant term of $-w_1 f_{11} n_N$ is $-n_N \frac{1}{2} = -\frac{N-1}{24}$ since $12 \mid N-1$, $(N-1, 12) = 12$. Thus we have:

$$g(\tau) = \frac{N-1}{24} + \sum_{m>0} \sum_{j=1}^{n} B_{1j}(m) q^m - \frac{N-1}{24} - w_1 n_N \sum_{m>0} B_{1j}(m) q^m,$$

and we can cancel the constant terms to get

$$g(\tau) = \sum_{m>0} \sum_{j=1}^{n} B_{1j}(m) q^m - w_1 n_N \sum_{m>0} B_{1j}(m) q^m \equiv \sum_{m>0} \sum_{j=1}^{n} B_{1j}(m) q^m \pmod{n_N},$$

where the left hand side has integer coefficients and the right hand side is equal to the normalized Eisenstein series minus its constant term.

The next case is $N \equiv 5 \pmod{12}$ so $(N-1, 12) = 4$. In particular, $N-1$ is coprime to 3.

Here we take

$$g(\tau) = \left(3\sum_{j=1}^{n} f_{1j}\right) - w_1 f_{11} n_N.$$

The constant term of $2w_1 f_{11}$ is 1 and so the constant term of $-w_1 f_{11} n_N$ is $-n_N \frac{1}{2} = -\frac{3(N-1)}{24}$ since $(N-1, 12) = 4$. Thus we have:

$$g(\tau) = \frac{3(N-1)}{24} + 3\sum_{m>0}\sum_{j=1}^{n} B_{1j}(m)q^m - \frac{3(N-1)}{24} - w_1 n_N \sum_{m>0} B_{1j}(m)q^m,$$

and we can cancel the constant terms to get

$$g(\tau) = 3\sum_{m>0}\sum_{j=1}^{n} B_{1j}(m)q^m - w_1 n_N \sum_{m>0} B_{1j}(m)q^m \equiv 3\sum_{m>0}\sum_{j=1}^{n} B_{1j}(m)q^m \pmod{n_N}.$$

Since we mentioned that $3 \nmid N-1$, then $3 \nmid n_N$ and so $(n_N, 3) = 1$. Multiplying both sides by the inverse of $3 \pmod{n_N}$ leaves us with a cusp form with integer coefficients congruent to the normalized Eisenstein series (minus its constant term) modulo n_N.

The remaining cases, $N = 7, 11 \pmod{12}$ follow similarly.

Note that, for $N = 7 \pmod{12}$, we have $(N-1, 12) = 6$ and we use

$$g(\tau) = \left(2\sum_{j=1}^{n} f_{1j}\right) - w_1 f_{11} n_N.$$

Again we will need that 2 is invertible modulo n_N and since in this case $n_N = \frac{N-1}{6} =$

$2k+1$, we see that $(n_N, 2) = 1$ so 2 is invertible modulo n_N.

And finally, for $N = 11 \pmod{12}$, we have $(N-1, 12) = 2$ and we use

$$g(\tau) = \left(6\sum_{j=1}^{n} f_{1j}\right) - w_1 f_{11} n_N.$$

Lastly we will need that 6 is invertible modulo n_N and since in this case $n_N = \frac{N-1}{2} = 6k+5$, we see that $(n_N, 6) = 1$ so 6 is invertible modulo n_N. □

Now, we have that there exists a cusp form of level N with integral coefficients which is congruent to $\frac{1}{24}(NE_2(N\tau) - E_2(\tau))$ modulo n_N except for the constant term. We will use this result in the next proposition.

First we require some notation, let $\ell_N := \text{num}\left(\frac{N^2-1}{24}\right)$. This is the minimum positive integer that clears denominators of $E_{2,N}(\tau)$ because multiplying $E_{2,N}(\tau)$ by N^2-1 clears denominators and 24 is the largest number we can divide by that does not hurt integrality (all coefficients of $E_{2,N}$ except the constant term are divisible by 24). Note that for $N > 3$, $\ell_N = \left(\frac{N^2-1}{24}\right)$ and for $N = 2, 3$, $\ell_N = 1$.

Remark. Hanson Smith noted that ℓ_N is an upper bound for the genus of $X_1(N)$ ($N > 3$ prime).

With the notation defined above we prove the following proposition.

Proposition 4.2.2. *Let ℓ_N and n_N be as above. Then $\ell_N E_{2,N}$ has integral coefficients and there exists a cusp form $G_N(\tau)$ of level N and weight 2 with integral coefficients such that*

$$\ell_N E_{2,N}(\tau) \equiv -NG_N(\tau) \pmod{n_N}.$$

Proof. We first treat the cases $N = 2, 3$, since for these we have $\ell_N = 1$. We would like to show that $E_{2,2}(\tau)$ and $E_{2,3}(\tau)$ are congruent to $0 \pmod{n_N}$ because the space of weight 2 cusp forms of levels 2 and 3 are both empty. Since $n_2 = n_3 = 1$, this follows because the coefficients of $E_{2,2}(\tau)$ and $E_{2,3}(\tau)$ are integers.

Now we continue with the remaining cases (and can assume $N > 3$). When N is prime, we have the following simplified expression for $E_{2,N}(\tau)$ (cf. (4.2.1)):

$$E_{2,N}(\tau) = \frac{1}{(N+1)(N-1)}\left(N^2 E_2(N\tau) - E_2(\tau)\right). \tag{4.2.4}$$

We can rearrange this to get a more convenient expression for $E_{2,N}$ as follows:

$$E_{2,N}(\tau) = \frac{N}{(N+1)(N-1)}\left(NE_2(N\tau) - E_2(\tau)\right) + \frac{1}{N+1}E_2(\tau).$$

Then we can multiply $E_{2,N}$ by ℓ_N to get

$$\ell_N E_{2,N}(\tau) = \frac{N}{24}\left(NE_2(N\tau) - E_2(\tau)\right) + \frac{N-1}{24}E_2(\tau).$$

By the congruence in the previous proposition we have that there exists a cusp form $g(\tau)$ with integral coefficients of weight 2 and level N such that

$$\left(\frac{1}{24}\left(NE_2(N\tau) - E_2(\tau)\right)\right) - \frac{N-1}{24} = g(\tau) + K(\tau)n_N \tag{4.2.5}$$

where $K(\tau)$ is some q-series with integer coefficients. We subtract $\dfrac{N-1}{24}$ which is equal to the constant term of $\left(\dfrac{1}{24}\left(NE_2(N\tau) - E_2(\tau)\right)\right)$ so that the left hand side of (4.2.5) has integer coefficients.

Thus we have

$$\ell_N E_{2,N}(\tau) = N\left(g(\tau) + \frac{N-1}{24} + K(\tau)n_N\right) + \frac{N-1}{24}E_2(\tau).$$

We define $G_N(\tau) := -g(\tau)$, distribute N, and combine constant terms. Then we can write:

$$\ell_N E_{2,N}(\tau) = -NG_N(\tau) + \frac{N^2-1}{24} + NK(\tau)n_N + \frac{N-1}{24}(E_2(\tau)-1).$$

Since $\frac{N^2-1}{24}$ is an integer multiple of n_N it is 0 (mod n_N). Also, since $N-1$ is a multiple of n_N and each coefficient of $\frac{E_2(\tau)-1}{24}$ is an integer, each coefficient of $\frac{N-1}{24}(E_2(\tau)-1)$ is 0 (mod n_N), and similarly for $NK(\tau)n_N$. So we have shown that $\ell_N E_{2,N}(\tau) \equiv -NG_N(\tau) \pmod{n_N}$. □

Now that we have shown this congruence between $\ell_N E_{2,N}(\tau)$ and $-NG_N(\tau)$ modulo n_N, we have that for N prime, the expression $\frac{\ell_N E_{2,N}(\tau) + NG_N(\tau)}{n_N}$ has integer coefficients. We can call these our

$$Q_N^{(N)}(\tau) := \frac{-\ell_N E_{2,N}(\tau) - NG_N(\tau)}{n_N}, \tag{4.2.6}$$

and we define

$$Q_1^{(N)}(\tau) := \frac{-\ell_N}{n_N} E_{2,1}(\tau). \tag{4.2.7}$$

Note that $Q_1^{(N)}(\tau)$ also has integer coefficients and that $E_{2,1}(\tau) = E_2(\tau)$. These functions (4.2.6) and (4.2.7) will be our trace functions for the $\mathbb{Z}/N\mathbb{Z}$ module. In order to prove the existence of this module we require the following lemma:

Lemma 4.2.3. *Let N be prime. Then*

$$Q_N^{(N)}(\tau) \equiv Q_1^{(N)}(\tau) \pmod{N}.$$

Proof. Again we start with the cases $N = 2, 3$:

For $N = 2$, we have $Q_2^{(2)}(\tau) = -E_{2,2}(\tau) = \frac{-1}{3}(4E_2(2\tau) - E_2(\tau))$ and $Q_1^{(2)}(\tau) = -E_2(\tau)$. To show that $Q_2^{(2)}(\tau) \equiv Q_1^{(2)}(\tau) \pmod 2$, it suffices to show that $3Q_2^{(2)}(\tau) \equiv 3Q_1^{(2)}(\tau)$

(mod 2) because $(2,3)=1$. We see the latter because $3Q_2^{(2)}(\tau)=-4E_2(2\tau)+E_2(\tau)$, $3Q_1^{(2)}(\tau)=-3E_2(\tau)$, and $-3 \pmod 2 = 1 \pmod 2$.

The case $N=3$ is similar, we have $Q_3^{(3)}(\tau)=E_{2,3}(\tau)=\frac{-1}{8}(9E_2(3\tau)-E_2(\tau))$ and $Q_1^{(3)}(\tau)=-E_2(\tau)$. Again, it suffices to show that $8Q_3^{(3)}(\tau)\equiv 8Q_1^{(3)}(\tau) \pmod 3$, which can be seen using the fact that $-8 \pmod 3 = 1 \pmod 3$.

For $N>3$, we will show that

$$Q_N^{(N)}(\tau)n_N \equiv Q_1^{(N)}(\tau)n_N \pmod N.$$

If the above congruence is true then since $(n_N,N)=1$, the congruence in the statement of the lemma will be true. Beginning with the left hand side, we have that

$$Q_N^{(N)}(\tau)n_N=\frac{-1}{24}\big(N^2E_2(N\tau)-E_2(\tau)\big)-NG_N(\tau).$$

We can rewrite this as

$$Q_N^{(N)}(\tau)n_N=\frac{-N^2}{24}E_2(N\tau)+\frac{1}{24}E_2(\tau)-NG_N(\tau),$$

or equivalently

$$Q_N^{(N)}(\tau)n_N=\frac{-N^2}{24}\left(E_2(N\tau)-1\right)-\frac{N^2}{24}+\frac{1}{24}\left(E_2(\tau)-1\right)+\frac{1}{24}-NG_N(\tau).$$

Reducing modulo N gives

$$Q_N^{(N)}(\tau)n_N\equiv\frac{1}{24}(E_2(\tau)-1)-\frac{N^2-1}{24} \pmod N.$$

Next we look at the right hand side. We have

$$Q_1^{(N)}(\tau)n_N = -\ell_N E_{2,1}(\tau) = -\frac{N^2-1}{24}E_2(\tau) = -\frac{N^2}{24}E_2(\tau) + \frac{1}{24}E_2(\tau)$$

which is equal to

$$-\frac{N^2}{24}\left(E_2(\tau)-1\right) + \frac{1}{24}\left(E_2(\tau)-1\right) - \frac{N^2-1}{24}.$$

Reducing modulo N we get

$$Q_1^{(N)}(\tau)n_N \equiv \frac{1}{24}\left(E_2(\tau)-1\right) - \frac{N^2-1}{24} \pmod{N}.$$

And thus

$$Q_N^{(N)}(\tau)n_N \equiv Q_1^{(N)}(\tau)n_N \pmod{N}.$$

□

For N prime, we give quasimodular weight 2 trace functions for $\mathbb{Z}/N\mathbb{Z}$ as follows:

$$f_g^{(N)}(\tau) := \begin{cases} Q_1^{(N)}(\tau) & \text{if } g = e, \\ Q_N^{(N)}(\tau) & \text{if } g \neq e. \end{cases}$$

Theorem 4.2.4. *There exists a virtual graded $\mathbb{Z}/N\mathbb{Z}$-module $V = \bigoplus_n V_n$ such that*

$$f_g^{(N)}(\tau) = \sum_{n=0}^{\infty} \mathrm{tr}(g \mid V_n)q^n$$

where $f_g^{(N)}(\tau)$ is the quasimodular form of weight 2 and level N with integral coefficients as defined as above.

Proof. Showing the existence of this virtual module amounts to showing that the multiplicities of the irreducible representations of $\mathbb{Z}/N\mathbb{Z}$ in the module are integral.

Thus we need to show the integrality of $\frac{1}{N}\sum_{g\in\mathbb{Z}/N\mathbb{Z}} c_g(n)\overline{\chi_i(g)}$, for each i, where $c_g(n)$ are the coefficients of the trace functions for $g \in \mathbb{Z}/N\mathbb{Z}$, and $\chi_i(g)$ are irreducible characters of $\mathbb{Z}/N\mathbb{Z}$. This is equivalent to showing that $\langle \chi_i, f_g^{(N)}(\tau)\rangle$ is integral, for $i \in \{1 \dots N\}$. For χ_1, the character corresponding to the trivial representation, we have

$$\langle \chi_1, f_g^{(N)}(\tau)\rangle = \frac{1}{N}(Q_1^{(N)}(\tau) + (N-1)Q_N^{(N)}(\tau)) = Q_N^{(N)}(\tau) + \frac{1}{N}(Q_1^{(N)}(\tau) - Q_N^{(N)}(\tau)). \tag{4.2.8}$$

In other words, for integrality of (4.2.8) we need that $Q_1^{(N)}(\tau) \equiv Q_N^{(N)}(\tau) \pmod N$. This is true by Lemma 4.2.3.

For all other χ_i, we make use of the fact that, for ζ a primitive Nth root of unity, $\zeta + \cdots + \zeta^{N-1} = -1$ and in particular that $\zeta^k + \cdots + \zeta^{k(N-1)} = -1$ for $1 \leq k \leq N-1$. So all other χ_i's give

$$\langle \chi_i, f\rangle = \frac{1}{N}(Q_1^{(N)}(\tau) - Q_N^{(N)}(\tau)), \tag{4.2.9}$$

which is again integral by the congruence $Q_1^{(N)}(\tau) \equiv Q_N^{(N)}(\tau) \pmod N$. □

4.3 Arithmetic/geometric connections

In this section, we describe some arithmetic connections between the trace functions of the M_{23}-module given in the previous section with expressions in (4.2.2) and the $\mathbb{F}_p$ point counts on (Jacobians of) modular curves. The expressions for $E_{2,N}(\tau)$ are not always integral on their own, and in the levels with cusp forms we saw that adding a multiple (with specific denominator) of a cusp form to $E_{2,N}(\tau)$ gives an expression with integral coefficients. The choices of cusp forms we used in the previous section (given explicitly in the appendix of [Ben19b]) are such that we get integral coefficients when we add $\frac{N}{n_N}G_N$ to $E_{2,N}(\tau)$.

For M_{23} this is summarized below:

(a) $-2E_{2,11}(\tau) + \frac{11}{5}G_{11}(\tau)$has integral coefficients.

(b) $-2E_{2,14}(\tau) + \frac{14}{3}G_{14}(\tau)$ has integral coefficients.

(c) $-2E_{2,15}(\tau) + \frac{15}{4}G_{15}(\tau)$ has integral coefficients.

(d) $-2E_{2,23}(\tau) + \frac{23}{11}G_{23}(\tau)$ has integral coefficients.

Because coefficients of weight 2 cusp forms admit a certain geometric interpretation, these expressions give divisor conditions on the number of $\mathbb{F}_p$ points on Jacobians of modular curves. Let $J_0(N)$ denote the Jacobian of the modular curve $X_0(N)$. For $N = 11, 14, 15$, we have that $J_0(N)$ is an elliptic curve. For $N = 23$ $J_0(N)$ is an abelian surface, namely, the Jacobian of a genus 2 curve.

For simplicity, here, we restrict our attention to N such that $X_0(N)$ are elliptic curves.

The result below relies on the relationship between cusp forms and point counts on elliptic curves. For an introductory reference for elliptic curves, their $\mathbb{F}_p$ points, and the relationship to cusp forms of weight 2, see [Mil06] (in particular, we apply Theorem 7.10 of *loc. cit.*).

Corollary 4.3.1. *We have the following (known) divisibility conditions arising from* M_{23}

1. *For* $p \neq 11$*, we have* $5 \mid \#J_0(11)(\mathbb{F}_p)$*.*

2. *For* $p \neq 2, 7$ *we have* $3 \mid \#J_0(14)(\mathbb{F}_p)$*.*

3. *For* $p \neq 3, 5$*, we have* $4 \mid \#J_0(15)(\mathbb{F}_p)$*.*

Proof. (1) We have that

$$\left(\frac{-121}{60}E_2(11\tau)+\frac{1}{60}E_2(\tau)\right)+\frac{11}{5}G_{11}(\tau)\in\mathbb{Z}[\![q]\!].$$

Using the definition of $E_2(\tau)$, we see that this is equal to

$$-2+\frac{242}{5}\sum_{m=1}^{\infty}\sigma(m)q^{11m}-\frac{2}{5}\sum_{n=1}^{\infty}\sigma(n)q^n+\frac{11}{5}G_{11}(\tau).$$

Note that we defined $G_{11}(\tau)=2\eta^2(\tau)\eta^2(11\tau)$ so all of its coefficients are even, and in fact $G_{11}(\tau)=2\widetilde{G}_{11}(\tau)$ where the $\widetilde{G}_{11}(\tau)=\sum\limits_{n>0}c_{11}(n)$ is the normalized cusp form whose coefficients correspond to the number of $\mathbb{F}_p$ points on the Jacobian of $X_0(11)$, denoted $\#J_0(11)(\mathbb{F}_p)$. The correspondence is given as follows:

$$c_{11}(p)=p+1-\#J_0(11)(\mathbb{F}_p). \tag{4.3.1}$$

We can see that if $11\nmid n$, then the nth coefficient of (a) is $-\frac{2}{5}\sigma(n)+\frac{22}{5}c_{11}(n)\in\mathbb{Z}$. The integrality of the coefficient of $p\neq 11$ then implies that $-2(p+1)+22c_{11}(p)\equiv 0 \pmod 5$. Substituting $c_{11}(p)$ for the right hand side of (4.3.1) gives us that $\#J_0(11)(\mathbb{F}_p)\equiv 0 \pmod 5$.

We note that for $p=11$, we can directly compute $\#J_0(11)(\mathbb{F}_{11})$ with the fact that $c_{11}(11)=1$. Thus we have $\#J_0(11)(\mathbb{F}_{11})=11$.

(2) If $2,7\nmid n$, then the nth coefficient of (b) is $\frac{1}{3}\sigma(n)+\frac{14}{3}c_{14}(n)\in\mathbb{Z}$. The integrality of the coefficient of $p\neq 2,7$ then implies that $p+1+14c_{14}(p)\equiv 0 \pmod 3$. We substitute $c_{14}(p)$ for the right hand side of the following:

$$c_{14}(p)=p+1-\#J_0(14)(\mathbb{F}_p), \tag{4.3.2}$$

and this gives us that $\#J_0(14)(\mathbb{F}_p)\equiv 0 \pmod 3$.

We note that for $p = 2$ and $p = 7$, we can directly compute $\#J_0(14)(\mathbb{F}_2)$ and $\#J_0(14)(\mathbb{F}_7)$ with the facts that $c_{14}(2) = -1$ and $c_{14}(7) = 1$. Thus we have $\#J_0(14)(\mathbb{F}_2) = 4$ and $\#J_0(14)(\mathbb{F}_7) = 7$.

(3) If $3, 5 \nmid n$, then the nth coefficient of (c) is $\frac{1}{4}\sigma(n) + \frac{15}{4}c_{15}(n) \in \mathbb{Z}$. The integrality of the coefficient of $p \neq 3, 5$ then implies that $p+1+15c_{15}(p) \equiv 0 \pmod 4$ We substitute $c_{15}(p)$ for the right hand side of the following:

$$c_{15}(p) = p + 1 - \#J_0(15)(\mathbb{F}_p), \tag{4.3.3}$$

and this gives us that $\#J_0(15)(\mathbb{F}_p) \equiv 0 \pmod 4$.

We note that for $p = 3$ and $p = 5$, we can directly compute $\#J_0(15)(\mathbb{F}_3)$ and $\#J_0(15)(\mathbb{F}_5)$ with the facts that $c_{15}(3) = -1$ and $c_{15}(5) = 1$. Thus we have $\#J_0(15)(\mathbb{F}_3) = 5$ and $\#J_0(15)(\mathbb{F}_5) = 5$. □

We expect infinitely many more such expressions (as in Corollary 4.3.1) arising from the trace functions of the $\mathbb{Z}/N\mathbb{Z}$-modules of Theorem 4.2.4. In the cases above, since the modular curves are elliptic curves and therefore (isomorphic to) their own Jacobians, any divisibility conditions on the number of $\mathbb{F}_p$ points on the Jacobians are equivalent to divisibility conditions on the number of $\mathbb{F}_p$ points on the modular curves themselves. In general, for any prime N, the integrality of trace functions from these $\mathbb{Z}/N\mathbb{Z}$-modules are equivalent to divisibility conditions on the number of $\mathbb{F}_p$ points on the Jacobians of $X_0(N)$ (cf. Theorem 7.10 of [Mil06]).

Remark. The integrality conditions used in the proof or Corollary 4.3.1 implied congruences of the form $-2(p + 1) + 22c_{11}(p) \equiv 0 \pmod 5$. These are equivalent to statements such as $(p + 1) \equiv 0 \pmod 5$ iff $c_{11}(p) \equiv 0 \pmod 5$. Jeffrey Lagarias noted that if one reframes these statements as, for example,

$$p = 4 \pmod 5 \text{ if and only if } 5 \mid c_{11}(p),$$

the expressions can then be written in the following way resembling the Ramanujan congruences:

$$\text{For } p = 5n + 4 \text{ we have } c_{11}(p) = 0 \pmod 5.$$

Remark. In this formulation of the trace functions of the M_{23}-module (4.2.2), we made a choice with the multiple of the cusp form. The denominator is fixed but the choices we made of the numerator are not unique. In fact, we could add any multiple of $NG_N(\tau)$ and still satisfy the congruences necessary for Mathieu moonshine. Therefore, the module here is one in an infinite family of possible modules one can consider. A similar statement holds for the modules of Theorem 4.2.4. It would be interesting to see if stronger results about point counts on modular Jacobians might be obtained by studying these families as a whole.

4.4 An explicit module construction

In the previous sections, the trace functions in Theorem 4.2.4 and those of the M_{23} module (4.2.2) have involved cusp forms. The results of this were some arithmetic/geometric observations in addition to the minimality of the constant $(\frac{\ell_N}{n_N})$ in front of $E_{2,N}(\tau)$ that guarantees integrality of trace functions' coefficients. If we restrict our functions to only involve Eisenstein series, we remove the cusp form contribution (thus we can no longer divide by n_N). We define an alternative set of trace functions for $\mathbb{Z}/N\mathbb{Z}$ in this way. For N prime, we give purely Eisenstein quasimodular weight 2 trace functions for a $\mathbb{Z}/N\mathbb{Z}$-module as follows:

$$F_g^{(N)}(\tau) := \begin{cases} -\ell_N E_2(\tau) & \text{if } g = e, \\ -\ell_N E_{2,N}(\tau) & \text{if } g \neq e. \end{cases}$$

It can be easily seen from the methods in Section 4.2 that there also exists a module for which these are the trace functions. Indeed, we will construct such a module explicitly in this section.

Although the modules with quasimodular trace functions involving cusp forms were arguably more interesting, the advantage of purely Eisenstein quasimodular trace functions is that we can actually give a construction of the module in terms of vertex operator algebras. Moreover, when the space of cusp forms of level N is empty, the purely Eisenstein trace functions are equal to the trace functions in Theorem 4.2.4. Given this, it would be interesting to see if the method presented here may be modified so as to obtain the modules that do include cusp from contributions, discussed in Section 4.2. A partial solution to this appears in Chapter 5.

To construct the purely Eisenstein modules we will find a vertex operator algebra that has the $F_g^{(N)}(\tau)$ as their trace functions. For this we will use two Heisenberg vertex algebras and a Clifford module vertex algebra. First, note that for $N > 3$ we have $\ell_N = \dfrac{N^2-1}{24}$ and

$$\begin{aligned} q\frac{d}{dq}\log\left(\frac{1}{\eta^{(N^2-1)}(\tau)}\right) &= -\frac{N^2-1}{24}E_2(\tau), \text{ and} \\ q\frac{d}{dq}\log\left(\frac{\eta(\tau)}{\eta^N(N\tau)}\right) &= -\frac{N^2-1}{24}E_{2,N}(\tau). \end{aligned}$$

For the remaining cases $N = 2, 3$, we have that $\ell_N = 1$.

When $N = 2$ we have

$$\begin{aligned} q\frac{d}{dq}\log\left(\frac{1}{\eta^{24}(\tau)}\right) &= -E_2(\tau), \text{ and} \\ q\frac{d}{dq}\log\left(\frac{\eta^8(\tau)}{\eta^{16}(2\tau)}\right) &= -E_{2,2}(\tau). \end{aligned}$$

And when $N = 3$ we have

$$q\frac{d}{dq}\log\left(\frac{1}{\eta^{24}(\tau)}\right) = -E_2(\tau), \text{ and}$$
$$q\frac{d}{dq}\log\left(\frac{\eta^{3}(\tau)}{\eta^{9}(3\tau)}\right) = -E_{2,3}(\tau).$$

In what follows, we describe the module construction.

Let D denote the derivative $D(\cdot) := q\frac{d}{dq}(\cdot)$. From the above equations we see that finding a module whose trace functions are equal to $F_g^{(N)}(\tau)$ with $N > 3$ is equivalent to finding a module whose trace functions are equal to:

$$\begin{cases} D\left(\dfrac{1}{\eta^{(N^2-1)}(\tau)}\right)\eta^{N^2}(\tau)\dfrac{1}{\eta(\tau)} & \text{if } g = e, \\ D\left(\dfrac{\eta(\tau)}{\eta^{N}(N\tau)}\right)\eta^{N}(N\tau)\dfrac{1}{\eta(\tau)} & \text{if } g \neq e. \end{cases} \tag{4.4.1}$$

Similarly, we note that finding a module whose trace functions are $F_g^{(2)}(\tau)$ and $F_g^{(3)}(\tau)$ are equivalent to finding a module whose trace functions are equal to:

$$\begin{cases} D\left(\dfrac{1}{\eta^{24}(\tau)}\right)\eta^{32}(\tau)\dfrac{1}{\eta^{8}(\tau)} & \text{if } g = e, \\ D\left(\dfrac{\eta^{8}(\tau)}{\eta^{16}(2\tau)}\right)\eta^{16}(2\tau)\dfrac{1}{\eta^{8}(\tau)} & \text{if } g \neq e, \end{cases} \tag{4.4.2}$$

and

$$\begin{cases} D\left(\dfrac{1}{\eta^{24}(\tau)}\right)\eta^{27}(\tau)\dfrac{1}{\eta^{3}(\tau)} & \text{if } g = e, \\ D\left(\dfrac{\eta^{3}(\tau)}{\eta^{9}(3\tau)}\right)\eta^{9}(3\tau)\dfrac{1}{\eta^{3}(\tau)} & \text{if } g \neq e, \end{cases} \tag{4.4.3}$$

respectively. The next two lemmas indicate how to recover the first of the three factors in each of eqs. (4.4.1) to (4.4.3). To formulate it, let $\mathfrak{h} = \mathbb{C}^{24\ell_N}$ and let γ be an automorphism of order N of $\mathfrak{h}$ such that its characteristic polynomial is $\mathrm{char}_\gamma(x) = \dfrac{(x^N-1)^N}{(x-1)}$ (For $N = 2, 3$, take the characteristic polynomials $\mathrm{char}_\gamma(x) = \dfrac{(x^2-1)^{16}}{(x-1)^8}$ and $\mathrm{char}_\gamma(x) = \dfrac{(x^3-1)^9}{(x-1)^3}$, respectively). Let $V := S(b_i(-n) \mid n > 0;\ i = 1, \ldots, 24\ell_N)$

(where $S(x_1, x_2 \dots) := S(\oplus_{i=1}^{\infty}\mathbb{C}x_i))$ be the Heisenberg vertex algebra for $\mathfrak{h}$ with non-degenerate symmetric bilinear form $\langle\cdot,\cdot\rangle$ fixed by γ. The action of γ on $\mathfrak{h}$ extends naturally to V. See [FBZ04] for more information on the Heisenberg vertex algebra construction. We denote the $L(0)$ operator for this Heisenberg vertex algebra by $L_1(0)$ and let c_1 be its central charge.

Lemma 4.4.1. *For $N > 3$, we have graded trace functions for V as follows:*

$$\operatorname{tr}\left(q^{L_1(0)-\frac{c_1}{24}} \mid V\right) = \frac{1}{\eta^{(N^2-1)}(\tau)}$$

and

$$\operatorname{tr}\left(\gamma q^{L_1(0)-\frac{c_1}{24}} \mid V\right) = \frac{\eta(\tau)}{\eta^N(N\tau)}.$$

Since we instead need the derivatives of those functions, we can take the traces as follows:

Lemma 4.4.2. *For $N > 3$, we have*

$$\operatorname{tr}\left(\left(L_1(0) - \frac{c_1}{24}\right) q^{L_1(0)-\frac{c_1}{24}} \mid V\right) = D\left(\frac{1}{\eta^{(N^2-1)}(\tau)}\right)$$
$$\operatorname{tr}\left(\left(L_1(0) - \frac{c_1}{24}\right) \gamma q^{L_1(0)-\frac{c_1}{24}} \mid V\right) = D\left(\frac{\eta(\tau)}{\eta^N(N\tau)}\right).$$

Note that the two equations above correspond to the first factor of the $g = e$ and $g \neq e$ functions in (4.4.1) and the analogous statements hold for $N = 2, 3$ (eqs. (4.4.2) and (4.4.3)).

Next, for the cases $N > 3$, in order to recover the second factor in each of the functions in eq. (4.4.1), we need a module with trace functions $\eta^{N^2}(\tau)$ and $\eta^N(N\tau)$, respectively. This can be done using a Clifford module vertex algebra. For this construction we follow Duncan and Harvey [DH17]. In this setting, let $\mathfrak{p}$ be a one dimensional complex vector space with a symmetric bilinear form. Let $a(r) := a \otimes t^r$ for $a \in \mathfrak{p}$. Let $\hat{\mathfrak{p}} = \mathfrak{p}[t, t^{-1}]t^{1/2}$ and $\hat{\mathfrak{p}}_{\mathrm{tw}} = \mathfrak{p}[t, t^{-1}]$ with the bilinear form extended so that $\langle a(r), b(s)\rangle = \langle a, b\rangle\delta_{r+s,0}$.

We define Cliff($\mathfrak{p}$) to be the Clifford algebra attached to $\mathfrak{p}$. Let $\hat{\mathfrak{p}}^+ := \mathfrak{p}[t]t^{1/2}$, where $\langle\hat{\mathfrak{p}}^+\rangle$ is a subalgebra of the Clifford algebra and let $\mathbb{C}v$ be a $\langle\hat{\mathfrak{p}}^+\rangle$ module such that that $1v = v$ and $p(r)v = 0$ for $r > 0$. Then we define

$$A(\mathfrak{p}) := \mathrm{Cliff}(\hat{\mathfrak{p}}) \otimes_{\langle\hat{\mathfrak{p}}^+\rangle} \mathbb{C}v,$$

and $A(\mathfrak{p})$ has the structure of a super vertex operator algebra with Virasoro element

$$\omega := \mathfrak{p}(-3/2)\mathfrak{p}(-1/2)v.$$

Let Cliff($\hat{\mathfrak{p}}_{\mathrm{tw}}$) be the Clifford algebra attached to $\hat{\mathfrak{p}}_{\mathrm{tw}}$, let v_{tw} be such that $1v_{\mathrm{tw}} = v_{\mathrm{tw}}$, and let $a(r)v_{\mathrm{tw}} = 0$ for $a \in \mathfrak{p}$ and $r > 0$. Then take $p \in \mathfrak{p}$ such that $\langle p, p\rangle = -2$ and $p(0)^2 = 1$. Define $v_{\mathrm{tw}}^+ := (1 + p(0))v_{\mathrm{tw}}$ so that $p(0)v_{\mathrm{tw}}^+ = v_{\mathrm{tw}}^+$ and let $\hat{\mathfrak{p}}_{\mathrm{tw}}^> := \mathfrak{p}[t]t$. Then we define

$$A(\mathfrak{p})_{\mathrm{tw}}^+ := \mathrm{Cliff}(\hat{\mathfrak{p}}_{\mathrm{tw}}) \otimes_{\langle\hat{\mathfrak{p}}_{\mathrm{tw}}^>\rangle} \mathbb{C}v_{\mathrm{tw}}^+,$$

so that $A(\mathfrak{p})_{\mathrm{tw}}^+$ is isomorphic to $\bigwedge(p(-n) \mid n > 0)v_{\mathrm{tw}}^+$ (where $\bigwedge(x_1, x_2 \ldots) := \bigwedge(\oplus_{i=1}^{\infty}\mathbb{C}x_i)$).

By the reconstruction theorem described in [FBZ04] we can see that $A(\mathfrak{p})_{\mathrm{tw}}$ is a twisted module for $A(\mathfrak{p})$ with fields $Y_{\mathrm{tw}} : A(\mathfrak{p}) \otimes A(\mathfrak{p})_{\mathrm{tw}} \to A(\mathfrak{p})_{\mathrm{tw}}((z^{1/2}))$ where

$$Y_{\mathrm{tw}}(u(-1/2)v, z) = \sum_{n\in\mathbb{Z}} u(n)z^{-n-1/2}$$

with $u \in \mathfrak{p}$. Since $A(\mathfrak{p})_{\mathrm{tw}}^+$ is a submodule of $A(\mathfrak{p})_{\mathrm{tw}}$ (generated by v_{tw}^+), it can be verified that $A(\mathfrak{p})_{\mathrm{tw}}^+$ is a twisted module for $A(\mathfrak{p})$ so that the above map can be restricted to $A(\mathfrak{p})_{\mathrm{tw}}^+$.

Let $L_2(0)$ be the $L(0)$ operator for the Clifford module vertex algebra and c_2 its central charge. Then we can see that $\mathrm{tr}\left(p(0)q^{L_2(0)-\frac{c_2}{24}} \mid A(\mathfrak{p})_{\mathrm{tw}}^+\right) = \eta(\tau)$. We would like a module with graded dimension equal to $\eta(\tau)^{N^2}$ so we will consider a tensor product of these $A(\mathfrak{p})^+_{\mathrm{tw}}$ (we have from [FHL93] that the tensor product of vertex algebras is naturally a vertex algebra).

To do this, we define

$$\widetilde{A}(\mathfrak{p}) := A(\mathfrak{p}_1) \otimes \cdots \otimes A(\mathfrak{p}_{N^2})$$

and

$$\widetilde{A}(\mathfrak{p})_{\mathrm{tw}} := A(\mathfrak{p}_1)^+_{\mathrm{tw}} + \otimes \cdots \otimes A(\mathfrak{p}_{N^2})^+_{\mathrm{tw}}$$

where each $A(\mathfrak{p}_i)^+_{\mathrm{tw}}$ is isomorphic to $\bigwedge(p_i(-n) \mid n > 0)v^+_{\mathrm{tw}}$. Then we can define

$$\widetilde{Y}_{\mathrm{tw}} : \widetilde{A}(\mathfrak{p}) \otimes \widetilde{A}(\mathfrak{p})^+_{\mathrm{tw}} \to \widetilde{A}(\mathfrak{p})^+_{\mathrm{tw}}(\!(z^{1/2})\!)$$

where for $u_i \in A(\mathfrak{p}_i)$

$$\begin{aligned}\widetilde{Y}_{\mathrm{tw}}((u_1 \otimes \cdots \otimes u_{N^2})v, z) &= Y_{\mathrm{tw}}(u_1, z) \otimes \cdots \otimes Y_{\mathrm{tw}}(u_{N^2}, z) \\ &= \sum_{n \in \mathbb{Z}^{N^2}} u_1(n_1) \otimes \cdots \otimes u_{N^2}(n_{N^2}) z^{-n_1 \cdots - n_{N^2} - \frac{N^2}{2}},\end{aligned}$$

with $n = (n_1, \ldots, n_{N^2})$, and finally

$$\widetilde{p}(0) := p_1(0) \otimes \cdots \otimes p_{N^2}(0).$$

Let σ act on $\widetilde{A}(\mathfrak{p})_{\mathrm{tw}}$ by permuting tensor factors with cycle shape N^N. Now we have in the next lemma the second factor of each equation in (4.4.1).

Lemma 4.4.3.

$$\operatorname{tr}\left(\widetilde{p}(0)q^{L_2(0)-\frac{c_2}{24}} \mid \widetilde{A}(\mathfrak{p})_{\mathrm{tw}}\right) = \eta^{N^2}(\tau),$$
$$\operatorname{tr}\left(\sigma\widetilde{p}(0)q^{L_2(0)-\frac{c_2}{24}} \mid \widetilde{A}(\mathfrak{p})_{\mathrm{tw}}\right) = \eta^{N}(N\tau).$$

When $N = 2, 3$, the construction is similar, but we define $\widetilde{A}(\mathfrak{p})$, $\widetilde{A}(\mathfrak{p})_{\mathrm{tw}}$, and $\widetilde{p}(0)$ to have 32 (resp. 27) tensor factors (instead of N^2) and we take instead permutations σ with cycle shape 2^{16} (resp. σ with cycle shape 3^9).

Then for $N = 2$, with $\widetilde{A}(\mathfrak{p})_{\mathrm{tw}} := A(\mathfrak{p}_1)^+_{\mathrm{tw}} \otimes \cdots \otimes A(\mathfrak{p}_{32})^+_{\mathrm{tw}}$, we have:

$$\operatorname{tr}\left(\widetilde{p}(0)q^{L_2(0)-\frac{c_2}{24}} \mid \widetilde{A}(\mathfrak{p})_{\mathrm{tw}}\right) = \eta^{32}(\tau),$$
$$\operatorname{tr}\left(\sigma\widetilde{p}(0)q^{L_2(0)-\frac{c_2}{24}} \mid \widetilde{A}(\mathfrak{p})_{\mathrm{tw}}\right) = \eta^{16}(2\tau).$$

And for $N = 3$ and $\widetilde{A}(\mathfrak{p})_{\mathrm{tw}} := A(\mathfrak{p}_1)^+_{\mathrm{tw}} \otimes \cdots \otimes A(\mathfrak{p}_{27})^+_{\mathrm{tw}}$, we have:

$$\operatorname{tr}\left(\widetilde{p}(0)q^{L_2(0)-\frac{c_2}{24}} \mid \widetilde{A}(\mathfrak{p})_{\mathrm{tw}}\right) = \eta^{27}(\tau),$$
$$\operatorname{tr}\left(\sigma\widetilde{p}(0)q^{L_2(0)-\frac{c_2}{24}} \mid \widetilde{A}(\mathfrak{p})_{\mathrm{tw}}\right) = \eta^{9}(3\tau).$$

Lastly, we recover the third factor in eqs. (4.4.1) to (4.4.3). For this we define $\mathfrak{k}$ to be $\mathbb{C}$ when $N > 3$, $\mathbb{C}^8$ when $N = 2$, and $\mathbb{C}^3$ when $N = 3$. Let $U := S(\mathfrak{k}(-n) \mid n > 0)$ (suitably modified when $N = 2, 3$) be the Heisenberg vertex algebra for $\mathfrak{k}$. We denote the $L(0)$ operator for this Heisenberg vertex algebra by $L_3(0)$ and let c_3 be its central charge. Then we have the following lemma

Lemma 4.4.4. *When $N > 3$, we have*

$$\operatorname{tr}\left(q^{L_3(0)-\frac{c_3}{24}} \mid U\right) = \frac{1}{\eta(\tau)}.$$

And note that for $N = 2, 3$ we get graded dimension equal to $\dfrac{1}{\eta^8(\tau)}$ and $\dfrac{1}{\eta^3(\tau)}$, respectively.

To get a vertex algebra whose trace function is the desired product in eqs. (4.4.1) to (4.4.3), we let

$$W^{(N)} := V \otimes \widetilde{A}(\mathfrak{p}) \otimes U.$$

We take the following canonically twisted module for the vertex algebra $W^{(N)}$:

$$W^{(N)}_{\mathrm{tw}} := V \otimes \widetilde{A}(\mathfrak{p})_{\mathrm{tw}} \otimes U.$$

The actions of γ and σ naturally extend to $W^{(N)}$ and $W^{(N)}_{\mathrm{tw}}$, by letting them act trivially on the factors where they have not already been defined (i.e. γ acts as $\gamma \otimes \mathrm{id} \otimes \mathrm{id}$ and σ acts as $\mathrm{id} \otimes \sigma \otimes \mathrm{id}$). Note that with this definition both γ and σ are automorphisms of $W^{(N)}$, and act equivariantly on the twisted module in the sense that we have $Y_{\mathrm{tw}}(gu, z)gv = gY_{\mathrm{tw}}(u, z)v$ for $u \in W^{(N)}$ and $v \in W^{(N)}_{\mathrm{tw}}$, and g equal to γ or σ.

Lastly, we define the operator $L(0) := L_1(0)+L_2(0)+L_3(0)$ and the central charge $c := c_1 + c_2 + c_3$ where the subscripts indicate the operators and central charges for the Heisenberg and Clifford vertex algebras described above. Then we have that the following forms are equal to the forms in (4.4.1) when $N > 3$, (4.4.2) when $N = 2$, and (4.4.3) when $N = 3$:

$$\begin{cases} \mathrm{tr}\left(\widetilde{p}(0)\left(L_1(0) - \dfrac{c_1}{24}\right) q^{L(0)-\frac{c}{24}} \mid W^{(N)}_{\mathrm{tw}}\right) & \text{if } g = e, \\ \mathrm{tr}\left(\gamma\sigma\widetilde{p}(0)\left(L_1(0) - \dfrac{c_1}{24}\right) q^{L(0)-\frac{c}{24}} \mid W^{(N)}_{\mathrm{tw}}\right) & \text{if } g \neq e. \end{cases}$$

Thus we have constructed a vertex algebra for the $\mathbb{Z}/N\mathbb{Z}$-module with purely Eisenstein quasimodular trace functions.

Theorem 4.4.5. *Let N be prime. Then $W^{(N)}_{\mathrm{tw}} = \bigoplus_n W^{(N)}_{\mathrm{tw},n}$ is an infinite dimensional* *virtual graded module for* $\mathbb{Z}/N\mathbb{Z}$ *such that*

$$F_g^{(N)}(\tau) = \sum_{n=0}^{\infty} \mathrm{tr}(g \mid W_{\mathrm{tw},n}^{(N)}) q^n.$$

Chapter 5

Module constructions for certain subgroups of the largest Mathieu group

In this chapter, we give vertex operator algebraic module constructions of modules for certain subgroups of M_{24} whose associated trace functions are meromorphic Jacobi forms. These meromorphic Jacobi forms are canonically associated to the mock modular forms of Mathieu moonshine. The construction is related to the Conway moonshine module and employs a technique introduced by Anagiannis–Cheng–Harrison. With this construction we are able to give concrete vertex algebraic realizations of certain cuspidal Hecke eigenforms of weight two. In particular, we give explicit realizations of trace functions whose integralities are equivalent to divisibility conditions on the number of $\mathbb{F}_p$ points on the Jacobians of modular curves. These results come from [Ben19a].

5.1 The functions

In this section we describe the meromorphic Jacobi forms $M_g(\tau, z)$. We will explicitly construct modules for which suitable expansions of the $M_g(\tau, z)$ are the trace functions. The module constructions can be found in Sections 5.2 and 5.3, but first we prove the existence of an overarching virtual M_{24}-module.

In order to define the meromorphic Jacobi forms, we recall a few definitions. Let $\eta(\tau)$ be the Dedekind eta function, defined by

$$\eta(\tau) := q^{1/24} \prod_{n>0} (1 - q^n). \tag{5.1.1}$$

We have the usual Jacobi theta function $\theta_1(\tau, z)$, defined as

$$\theta_1(\tau, z) := -iq^{\frac{1}{8}} y^{\frac{1}{2}} \prod_{n>0} (1 - y^{-1}q^{n-1})(1 - yq^n)(1 - q^n), \tag{5.1.2}$$

where $q = e^{2\pi i\tau}$ and $y = e^{2\pi iz}$. The Appell-Lerch sum $\mu(\tau, z)$ is given by

$$\mu(\tau, z) := \frac{-iy^{1/2}}{\theta_1(\tau, z)} \sum_{n\in\mathbb{Z}} (-1)^n \frac{y^n q^{n(n+1)/2}}{1 - yq^n}. \tag{5.1.3}$$

We recall that $\chi(g)$ is the number of fixed points of g in the 24-dimensional permutation representation of M_{24}, the mock modular forms of weight 1/2 associated to $g \in M_{24}$ from Mathieu moonshine are denoted by $H_g(\tau)$, and $F_2(\tau)$ is defined as follows

$$F_2(\tau) := \sum_{\substack{r>s>0 \\ r-s \text{ odd}}} sq^{rs/2}. \tag{5.1.4}$$

The quasimodular forms $Q_g(\tau)$, for $g \in M_{24}$, that are the holomorphic projection of the completion of the $H_g(\tau)$ multiplied by $\eta^3(\tau)$ (i.e. $\pi_{\text{hol}}(\widehat{H}_g(\tau)\eta^3(\tau))$

from [Ben19b]) can be defined as

$$Q_g(\tau) := H_g(\tau)\eta^3(\tau) - 2\chi(g)F_2(\tau). \tag{5.1.5}$$

We now define

$$\phi_{-2,1}(\tau, z) := -\frac{\theta_1^2(\tau, z)}{\eta^6(\tau)}, \tag{5.1.6}$$

and

$$\phi_{0,1}(\tau, z) := \frac{1}{2} Z_{K3}(\tau, z), \tag{5.1.7}$$

where, from [EOT11], we have

$$Z_{K3}(\tau, z) := 24\mu(\tau, z)\frac{\theta_1^2(\tau, z)}{\eta^3(\tau)} + H_e(\tau)\frac{\theta_1^2(\tau, z)}{\eta^3(\tau)}. \tag{5.1.8}$$

More generally, for $g \in M_{24}$, we define the following weak Jacobi forms:

$$Z_g(\tau, z) := \chi(g)\mu(\tau, z)\frac{\theta_1^2(\tau, z)}{\eta^3(\tau)} + H_g(\tau)\frac{\theta_1^2(\tau, z)}{\eta^3(\tau)}. \tag{5.1.9}$$

In [DMC16], Duncan and Mack-Crane associate weak Jacobi forms $\phi_g(\tau, z)$ of weight zero and index one to symplectic derived equivalences of projective complex K3 surfaces that fix a stability condition in the distinguished space identified by Bridgeland. They identify such automorphisms with elements of $\mathrm{Aut}(\Lambda)$ (the Conway group Co_0) fixing a sublattice of rank greater than or equal to 4. Since M_{24} is a subgroup of Co_0, it is natural to compare the $\phi_g(\tau, z)$ to the weak Jacobi forms $Z_g(\tau, z)$ associated to $g \in M_{24}$, and in fact, these $\phi_g(\tau, z)$ are equal to $Z_g(\tau, z)$ for g in all but 7 of the 26 conjugacy classes of M_{24} (those conjugacy classes are: 3B, 4C, 6B, 12B, 21A, 21B, 23A and 23B). For an explicit expression of $\phi_g(\tau, z)$, see equation (5.3.10).

These $\phi_g(\tau, z)$ are closely related to weight two modular forms $F_g(\tau)$ (not to be confused with $F_2(\tau)$, which is not modular). We take the expression given in Proposition 9.3 of [DMC16] as the definition of $F_g(\tau)$:

$$F_g(\tau) = \frac{\phi_g(\tau, z)}{\phi_{-2,1}(\tau, z)} - \frac{1}{12}\chi(g)\frac{\phi_{0,1}(\tau, z)}{\phi_{-2,1}(\tau, z)}. \tag{5.1.10}$$

We refer to equation (9.19) of [DMC16] for another definition of $F_g(\tau)$.

We begin with the following proposition in which we combine a result of Dabholkar, Murthy, and Zagier [DMZ12], a result of Duncan and Mack-Crane [DMC16], and the functions $Q_g(\tau)$ that were defined in Chapter 4.

Proposition 5.1.1. *For $g \in M_{24}$ such that $[g] \neq$ 3B, 4C, 6B, 12B, 21A, 21B, 23A and 23B we have*

$$\frac{\eta^6(\tau)\phi_g(\tau, z)}{\theta_1^2(\tau, z)} = \chi(g)\left(\eta^3(\tau)\mu(\tau, z) + 2F_2(\tau)\right) + Q_g(\tau)$$

Proof. From equation (8.52) in [DMZ12] we have following:

$$\eta^{-3}(\tau)\frac{\phi_{0,1}(\tau, z)}{\phi_{-2,1}(\tau, z)} = -\frac{12}{\theta_1(\tau, 2z)}Av^{(2)}\left[\frac{1+y}{1-y}\right] - h^{(2)}(\tau). \tag{5.1.11}$$

We note that $2h^{(2)}(\tau) = H_e(\tau)$ and $Av^{(2)}\left[\frac{1+y}{1-y}\right] = \theta_1(\tau, 2z)\mu(\tau, z)$, (see Example 2, Section 8.5 of [DMZ12]) and so we equivalently have

$$\eta^{-3}(\tau)\frac{\phi_{0,1}(\tau, z)}{\phi_{-2,1}(\tau, z)} = -\frac{12}{\theta_1(\tau, 2z)}\theta_1(\tau, 2z)\mu(\tau, z) - \frac{1}{2}H_e(\tau), \tag{5.1.12}$$

which can be rearranged as follows:

$$\frac{\phi_{0,1}(\tau, z)}{\phi_{-2,1}(\tau, z)} = -12\eta^3(\tau)\mu(\tau, z) - \frac{1}{2}\eta^3(\tau)H_e(\tau). \tag{5.1.13}$$

Now, from Proposition 9.3 of [DMC16], we have that

$$\phi_g(\tau,z) = \frac{1}{12}\chi(g)\phi_{0,1}(\tau,z) + F_g(\tau)\phi_{-2,1}(\tau,z), \tag{5.1.14}$$

which is equivalent to

$$\frac{\phi_g(\tau,z)}{\phi_{-2,1}(\tau,z)} = \frac{1}{12}\chi(g)\frac{\phi_{0,1}(\tau,z)}{\phi_{-2,1}(\tau,z)} + F_g(\tau). \tag{5.1.15}$$

Substituting the right hand side of equation (5.1.13) for $\dfrac{\phi_{0,1}(\tau,z)}{\phi_{-2,1}(\tau,z)}$ we obtain

$$\frac{\phi_g(\tau,z)}{\phi_{-2,1}(\tau,z)} = \frac{1}{12}\chi(g)\left(-12\eta^3(\tau)\mu(\tau,z) - \frac{1}{2}\eta^3(\tau)H_e(\tau)\right) + F_g(\tau). \tag{5.1.16}$$

Then we use the identity $\phi_{-2,1}(\tau,z) = -\dfrac{\theta_1^2(\tau,z)}{\eta^6(\tau)}$ to rewrite the above as follows:

$$\frac{-\eta^6(\tau)\phi_g(\tau,z)}{\theta_1^2(\tau,z)} = \frac{1}{12}\chi(g)\left(-12\eta^3(\tau)\mu(\tau,z) - \frac{1}{2}\eta^3(\tau)H_e(\tau)\right) + F_g(\tau), \tag{5.1.17}$$

and simplifying further, we find

$$\frac{-\eta^6(\tau)\phi_g(\tau,z)}{\theta_1^2(\tau,z)} = -\chi(g)\eta^3(\tau)\mu(\tau,z) - \frac{\chi(g)}{24}\eta^3(\tau)H_e(\tau) + F_g(\tau). \tag{5.1.18}$$

We recall the formula (from Appendix B of [DGO15b], also in [Che10, GHV10b, GHV10a, EH11, CD12]) relating $H_g(\tau)$ and $H_e(\tau)$, for $g \in M_{24}$:

$$H_g(\tau)\eta^3(\tau) = \frac{\chi(g)}{24}H_e(\tau)\eta^3(\tau) - F_g(\tau). \tag{5.1.19}$$

This formula gives us that

$$\frac{-\eta^6(\tau)\phi_g(\tau,z)}{\theta_1^2(\tau,z)} = -\chi(g)\eta^3(\tau)\mu(\tau,z) - H_g(\tau)\eta^3(\tau), \tag{5.1.20}$$

or equivalently,

$$\frac{\eta^6(\tau)\phi_g(\tau,z)}{\theta_1^2(\tau,z)} = \chi(g)\eta^3(\tau)\mu(\tau,z) + H_g(\tau)\eta^3(\tau). \tag{5.1.21}$$

Combining this with the quasimodular forms associated to M_{24} in equation (5.1.5), we can write

$$\frac{\eta^6(\tau)\phi_g(\tau,z)}{\theta_1^2(\tau,z)} = \chi(g)(\eta^3(\tau)\mu(\tau,z) + 2F_2(\tau)) + Q_g(\tau). \tag{5.1.22}$$

□

Lemma 5.1.2. *For $g \in M_{24}$ such that $[g] \neq$ 3B, 4C, 6B, 12B, 21A, 21B, 23A and 23B, the $F_g(\tau)$ in equation* (5.1.19) *(and Appendix B of [DGO15b]) are the same as the $F_g(\tau)$ in equation* (9.19) *of [DMC16].*

Proof. For clarity, in the proof of this lemma exclusively, we will write $\widetilde{F_g}(\tau)$ when referring to the $F_g(\tau)$ in [DMC16] and we will write $F_g(\tau)$ for the $F_g(\tau)$ in [DGO15b]. We will show that $\widetilde{F_g}(\tau) = F_g(\tau)$.

From Eguchi, Ooguri, and Tachikawa we have the following expression for the $K3$ elliptic genus

$$Z_{K3}(\tau,z) = 24\mu(\tau,z)\frac{\theta_1^2(\tau,z)}{\eta^3(\tau)} + H_e(\tau)\frac{\theta_1^2(\tau,z)}{\eta^3(\tau)}. \tag{5.1.23}$$

Rearranging the terms of the above equation, we find that

$$\frac{Z_{K3}(\tau,z)\eta^3(\tau)}{\theta_1^2(\tau,z)} = 24\mu(\tau,z) + H_e(\tau), \tag{5.1.24}$$

and then multiplying by $\eta^3(\tau)$ and solving for $H_e(\tau)\eta^3(\tau)$ we have

$$H_e(\tau)\eta^3(\tau) = \frac{Z_{K3}(\tau,z)\eta^6(\tau)}{\theta_1^2(\tau,z)} - 24\mu(\tau,z)\eta^3(\tau). \tag{5.1.25}$$

We can then substitute the right side of the equation above for $H_e(\tau)\eta^3(\tau)$ in equation

(5.1.19) and we obtain:

$$H_g(\tau)\eta^3(\tau) = \frac{\chi(g)}{24}\left(\frac{Z_{K3}(\tau,z)\eta^6(\tau)}{\theta_1^2(\tau,z)} - 24\mu(\tau,z)\eta^3(\tau)\right) - F_g(\tau). \tag{5.1.26}$$

Finally, we solve for $F_g(\tau)$ as follows:

$$F_g(\tau) = \frac{\chi(g)}{24}\frac{Z_{K3}(\tau,z)\eta^6(\tau)}{\theta_1^2(\tau,z)} - \chi(g)\mu(\tau,z)\eta^3(\tau) - H_g(\tau)\eta^3(\tau). \tag{5.1.27}$$

On the other hand, we recall the expression in Proposition 9.3 of [DMC16]:

$$\widetilde{F_g}(\tau) = \frac{\phi_g(\tau,z)}{\phi_{-2,1}(\tau,z)} - \frac{1}{12}\chi(g)\frac{\phi_{0,1}(\tau,z)}{\phi_{-2,1}(\tau,z)}. \tag{5.1.28}$$

We make the substitutions $\phi_{-2,1}(\tau,z) = -\dfrac{\theta_1^2(\tau,z)}{\eta^6(\tau)}$ and $\phi_{0,1}(\tau,z) = \dfrac{1}{2}Z_{K3}(\tau,z)$ and arrive at:

$$\widetilde{F_g}(\tau) = \frac{-\phi_g(\tau,z)\eta^6(\tau)}{\theta_1^2(\tau,z)} + \frac{\chi(g)}{24}\frac{Z_{K3}(\tau,z)\eta^6(\tau)}{\theta_1^2(\tau,z)}. \tag{5.1.29}$$

For $g \in M_{24}$ such that $Z_g(\tau,z) = \phi_g(\tau,z)$, we can substitute $\phi_g(\tau,z)$ for the right hand side of the equation below:

$$Z_g(\tau,z) = \chi(g)\mu(\tau,z)\frac{\theta_1^2(\tau,z)}{\eta^3(\tau)} + H_g(\tau)\frac{\theta_1^2(\tau,z)}{\eta^3(\tau)}. \tag{5.1.30}$$

Thus we have

$$\widetilde{F_g}(\tau) = -\frac{\eta^6(\tau)}{\theta_1^2(\tau,z)}\left(\chi(g)\mu(\tau,z)\frac{\theta_1^2(\tau,z)}{\eta^3(\tau)} + H_g(\tau)\frac{\theta_1^2(\tau,z)}{\eta^3(\tau)}\right) + \frac{\chi(g)}{24}\frac{Z_{K3}(\tau,z)\eta^6(\tau)}{\theta_1^2(\tau,z)}, \tag{5.1.31}$$

which simplifies to

$$\widetilde{F_g}(\tau) = \frac{\chi(g)}{24}\frac{Z_{K3}(\tau,z)\eta^6(\tau)}{\theta_1^2(\tau,z)} - \chi(g)\mu(\tau,z)\eta^3(\tau) - H_g(\tau)\eta^3(\tau). \tag{5.1.32}$$

Therefore, we see that $\widetilde{F_g}(\tau) = F_g(\tau)$. $\square$

Now we have described everything we need to define the functions $M_g(\tau, z)$ as follows:

$$M_g(\tau, z) := H_g(\tau)\eta^3(\tau) + \chi(g)\eta^3(\tau)\mu(\tau, z). \tag{5.1.33}$$

We next show that there exists an M_{24}-module for which suitable expansions of the $M_g(\tau, z)$ are the graded trace functions. We define $\widetilde{M_g}(\tau, z)$ to be the expansion of $M_g(\tau, z)$ in the domain $0 < -\mathrm{Im}(z) < \mathrm{Im}(\tau)$ ($\tau \in \mathbb{H}, z \in \mathbb{C}$).

Proposition 5.1.3. *There exists a virtual bigraded M_{24}-module*

$$V = \bigoplus_{\substack{n,r\in\mathbb{Z} \\ n\geq 0}} V_{n,r}$$

such that

$$\widetilde{M_g}(\tau, z) = \sum_{n,r} \mathrm{tr}(g \mid V_{n,r}) y^r q^n.$$

Proof. For this proof, we restrict to the domain $0 < -\mathrm{Im}(z) < \mathrm{Im}(\tau)$. First we show that the $\widetilde{M_g}(\tau, z)$ have integral coefficients. Gannon [Gan16] shows that the functions $H_g(\tau)$ have integral coefficients (in all of $\mathbb{H}$, and thus in the domain we specify). It follows from the definition of $\eta(\tau)$ that $\eta^3(\tau)$ has integral coefficients. It remains to show that $\chi(g)\eta^3(\tau)\mu(\tau, z)$ has integral coefficients (and from the definition we know the $\chi(g)$ are integers). Because the specified expansion of $\mu(\tau, z)$ is one of the $N = 4$ characters [EOT11], its expansion is known to have integral coefficients. Thus we conclude that the $\widetilde{M_g}(\tau, z)$ have integral coefficients.

Next we show that the multiplicities $\mathrm{m}_i^M(n)$ of the M_{24} irreducible representations in the class functions defined by the coefficients of $\widetilde{M_g}(\tau, z)$ are integral.

Gannon shows that the multiplicity generating function

$$\sum_{n>0} \mathrm{m}_i^H(n)q^n = \frac{1}{|M_{24}|}\sum_{g\in M_{24}} H_g(\tau)\overline{\chi_i(g)} \tag{5.1.34}$$

(with χ_i an irreducible character of M_{24}) has integral coefficients. We need to show that the coefficients $\mathrm{m}_i^M(n)$ are integral, where

$$\sum_{n>0} \mathrm{m}_i^M(n)q^n = \frac{1}{|M_{24}|}\sum_{g\in M_{24}} \left[H_g(\tau)\eta^3(\tau) - \chi(g)\mu(\tau,z)\eta^3(\tau)\right]\overline{\chi_i(g)}. \tag{5.1.35}$$

To do this, we can split the right hand side of equation (5.1.35) into two parts. First consider $\frac{1}{|M_{24}|}\sum_{g\in M_{24}} H_g(\tau)\eta^3(\tau)\overline{\chi_i(g)}$. This differs from (5.1.34) only from multiplying by $\eta^3(\tau)$, which does not change the integrality. So it suffices to show that $\frac{1}{|M_{24}|}\sum_{g\in M_{24}} \chi(g)\mu(\tau,z)\eta^3(\tau)\overline{\chi_i(g)}$ has integral coefficients.

This is the same as showing that

$$\mu(\tau,z)\eta^3(\tau)\frac{1}{|M_{24}|}\sum_{g\in M_{24}} \chi(g)\overline{\chi_i(g)} = \mu(\tau,z)\eta^3(\tau)\langle\chi,\chi_i\rangle$$

has integral coefficients. We already know that $\mu(\tau,z)\eta^3(\tau)$ has integral coefficients (see above). The integrality of $\langle\chi,\chi_i\rangle$ can be seen from the fact that $\chi(g)$ is a character of a module, and so $\langle\chi,\chi_i\rangle$ is the multiplicity of χ_i in χ, which is necessarily integral. Thus the $\mathrm{m}_i^M(n)$ from (5.1.35) are integral. □

Remark. In what follows, we will give module constructions such that the graded trace functions on those modules are equal to $\widetilde{M_g}(\tau,z)$. The condition that $\tau\in\mathbb{H}$, $z\in\mathbb{C}$ be such that $0<-\mathrm{Im}(z)<\mathrm{Im}(\tau)$ is necessary to ensure convergence of the graded dimension functions of the modules. In particular, this is what will allow us to identify the series expansions as their graded dimension functions. We will adopt this restriction of the domain for the rest of the chapter.

5.2 Module construction I

In this section, for $g \in M_{24}$ such that $[g] \neq$ 3B, 4C, 6B, 12B, 21A, 21B, 23A, or 23B, we explicitly construct a module whose trace functions are the $\widetilde{M}_g(\tau, z)$ (see Proposition 5.1.3). This will lead to module constructions for certain subgroups of M_{24} with no elements in any of the above conjugacy classes. We also require that the 24-dimensional permutation representation of M_{24} has a fixed 4-dimensional subspace when restricted to that subgroup.

We use the fact that when $g \in M_{24}$, the following holds: By definition of $Q_g(\tau)$, we see that $M_g(\tau, z) = Q_g(\tau) + \chi(g)\left(\eta^3(\tau)\mu(\tau, z) + 2F_2(\tau)\right)$, and by Proposition 5.1.1 of Section 5.1, for g not in the excluded conjugacy classes as above, we have

$$M_g(\tau, z) = \frac{\phi_g(\tau, z)\eta^6(\tau)}{\theta_1^2(\tau, z)}. \tag{5.2.1}$$

We will split this equation into three factors as follows:

$$M_g(\tau, z) = \left(\phi_g(\tau, z)\right)\left(\eta^4(\tau)\right)\left(\frac{\eta^2(\tau)}{\theta_1^2(\tau, z)}\right). \tag{5.2.2}$$

We postpone the discussion about how to recover the first factor of $M_g(\tau, z)$ for now. The next two lemmas indicate how to recover the second and third of the three factors in Equation 5.2.2.

In order to recover the second factor in (5.2.2), we need a module with graded dimension function $\eta^4(\tau)$. This can be achieved using a Clifford module vertex superalgebra. For this construction we follow Duncan and Harvey [DH17]. We note that the description below can also be found in [Ben19b] and in the previous chapter (Chapter 4). In this setting, let $\mathfrak{p}$ be a one dimensional complex vector space with a non-degenerate symmetric bilinear form. Let $\hat{\mathfrak{p}} := \mathfrak{p}[t, t^{-1}]t^{\frac{1}{2}}$ and $\hat{\mathfrak{p}}_{\mathrm{tw}} := \mathfrak{p}[t, t^{-1}]$, for $a \in \mathfrak{p}$ we write $a(r)$ for at^r with the bilinear form extended so that $\langle a(r), b(s)\rangle = \langle a, b\rangle\delta_{r+s,0}$.

We define $\mathrm{Cliff}(\hat{\mathfrak{p}})$ to be the Clifford algebra attached to $\hat{\mathfrak{p}}$. Let $\hat{\mathfrak{p}}^+ := \mathfrak{p}[t]t^{\frac{1}{2}}$ and let $\langle \hat{\mathfrak{p}}^+ \rangle$ be the subalgebra of the Clifford algebra $\mathrm{Cliff}(\hat{\mathfrak{p}})$ generated by $\hat{\mathfrak{p}}^+$. Take $\mathbb{C}\mathbf{v}$ to be a $\langle \hat{\mathfrak{p}}^+ \rangle$ module such that $1\mathbf{v} = \mathbf{v}$ and $p(r)\mathbf{v} = 0$ for $r > 0$. Then we define

$$A(\mathfrak{p}) := \mathrm{Cliff}(\hat{\mathfrak{p}}) \otimes_{\langle \hat{\mathfrak{p}}^+ \rangle} \mathbb{C}\mathbf{v}, \tag{5.2.3}$$

and $A(\mathfrak{p})$ has the structure of a vertex superalgebra such that $Y(u\left(-\frac{1}{2}\right)\mathbf{v}, z) = \sum_{n\in\mathbb{Z}} u(n+\frac{1}{2})z^{-n-1}$ for $u \in \mathfrak{p}$. $A(\mathfrak{p})$ has the structure of a vertex operator superalgebra with central charge $\frac{1}{2}$ when we equip it with the Virasoro element

$$\omega := p\left(\frac{-3}{2}\right)p\left(\frac{-1}{2}\right)\mathbf{v},$$

for $p \in \mathfrak{p}$ such that $\langle p, p \rangle = -2$.

Let $\mathrm{Cliff}(\hat{\mathfrak{p}}_{\mathrm{tw}})$ be the Clifford algebra attached to $\hat{\mathfrak{p}}_{\mathrm{tw}}$. Define $\hat{\mathfrak{p}}_{\mathrm{tw}}^{>} := \mathfrak{p}[t]t$ and let $\langle \hat{\mathfrak{p}}_{\mathrm{tw}}^{>} \rangle$ be the subalgebra of this Clifford algebra generated by $\hat{\mathfrak{p}}_{\mathrm{tw}}^{>}$. Similarly, define $\hat{\mathfrak{p}}_{\mathrm{tw}}^{-} := \mathfrak{p}[t^{-1}]$ and $\langle \hat{\mathfrak{p}}_{\mathrm{tw}}^{-} \rangle$. Take $\mathbb{C}\mathbf{v}_{\mathrm{tw}}$ to be a $\hat{\mathfrak{p}}_{\mathrm{tw}}^{>}$ module such that $1\mathbf{v}_{\mathrm{tw}} = \mathbf{v}_{\mathrm{tw}}$ and $a(r)\mathbf{v}_{\mathrm{tw}} = 0$ for $a \in \mathfrak{p}$ and $r > 0$. For $p \in \mathfrak{p}$ (as before) such that $\langle p, p \rangle = -2$, we have that $p(0)^2 = 1$ in $\mathrm{Cliff}(\mathfrak{p})$. Define $\mathbf{v}_{\mathrm{tw}}^+ := (1 + p(0))\mathbf{v}_{\mathrm{tw}}$ so that $p(0)\mathbf{v}_{\mathrm{tw}}^+ = \mathbf{v}_{\mathrm{tw}}^+$. Then we define

$$A(\mathfrak{p})_{\mathrm{tw}}^+ := \mathrm{Cliff}(\hat{\mathfrak{p}}_{\mathrm{tw}}) \otimes_{\langle \hat{\mathfrak{p}}_{\mathrm{tw}}^{>} \rangle} \mathbb{C}\mathbf{v}_{\mathrm{tw}}^+, \tag{5.2.4}$$

so that $A(\mathfrak{p})_{\mathrm{tw}}^+$ is isomorphic (as a $\langle \hat{\mathfrak{p}}_{\mathrm{tw}}^{-} \rangle$-module) to $\bigwedge(p(-n) \mid n > 0)\mathbf{v}_{\mathrm{tw}}^+$ (where $\bigwedge(x_1, x_2 \ldots) := \bigwedge(\oplus_{i=1}^{\infty}\mathbb{C}x_i)$).

By the reconstruction theorem described in [FBZ04] we can see that $A(\mathfrak{p})_{\mathrm{tw}}$ is a twisted module for $A(\mathfrak{p})$ with fields $Y_{\mathrm{tw}} \colon A(\mathfrak{p}) \otimes A(\mathfrak{p})_{\mathrm{tw}} \to A(\mathfrak{p})_{\mathrm{tw}}((z^{\frac{1}{2}}))$ such that $Y_{\mathrm{tw}}\left(u\left(\frac{-1}{2}\right)\mathbf{v}, z\right) = \sum_{n\in\mathbb{Z}} u(n)z^{-n-\frac{1}{2}}$ for $u \in \mathfrak{p}$. Since $A(\mathfrak{p})_{\mathrm{tw}}^+$ is a submodule of $A(\mathfrak{p})_{\mathrm{tw}}$ (generated by $\mathbf{v}_{\mathrm{tw}}^+$), it can be verified that $A(\mathfrak{p})_{\mathrm{tw}}^+$ is a twisted module for $A(\mathfrak{p})$ so that the above map can be restricted to $A(\mathfrak{p})_{\mathrm{tw}}^+$. In fact, $A(\mathfrak{p})$ is a canonically twisted

$A(\mathfrak{p})$-module, by which we mean the twisted module for $A(\mathfrak{p})$ with respect to its parity involution (see also [DMC15]).

Let $L_2(0)$ be the $L(0)$ operator for the Clifford module vertex superalgebra and c_2 its central charge. Then we can see that $\operatorname{tr}\left(p(0)q^{L_2(0)-\frac{c_2}{24}} \mid A(\mathfrak{p})^+_{\mathrm{tw}}\right) = \eta(\tau)$. We would like a module with graded dimension equal to $\eta^4(\tau)$ so we will consider a tensor product of these $A(\mathfrak{p})^+_{\mathrm{tw}}$ (we do this as in Chapter 4, defining $\widetilde{A}(\mathfrak{p})_{\mathrm{tw}}$ and $\widetilde{p}(0)$ in the same way as before with 4 in place of N^2).

This completes the proof of the following lemma in which we record the second factor of equation (5.2.2).

Lemma 5.2.1.

$$\operatorname{tr}\left(\widetilde{p}(0)q^{L_2(0)-\frac{c_2}{24}} \mid \widetilde{A}(\mathfrak{p})_{\mathrm{tw}}\right) = \eta^4(\tau).$$

For the third factor, we require a module with graded dimension function given by the expansion of $\dfrac{\eta^2(\tau)}{\theta_1^2(\tau,z)}$ in our usual domain ($\tau \in \mathbb{H}, z \in \mathbb{C}$ such that $0 < -\mathrm{Im}(z) < \mathrm{Im}(\tau)$). To this end, we use a twisted module over a Weyl module vertex operator algebra. We follow Duncan and O'Desky for this construction [DO18]. In this setting, let $\mathfrak{b}$ be a 4-dimensional vector space with a non-degenerate antisymmetric bilinear form. Let $\hat{\mathfrak{b}} := \mathfrak{b}[t,t^{-1}]t^{\frac{1}{2}}$ and $\hat{\mathfrak{b}}_{\mathrm{tw}} := \mathfrak{b}[t,t^{-1}]$, for $b \in \mathfrak{b}$ we write $b(r)$ for bt^r with the bilinear form extended as in the case of the Clifford module vertex operator algebra.

Let $\mathrm{Weyl}(\hat{\mathfrak{b}})$ be the Weyl algebra associated to $\hat{\mathfrak{b}}$ and its antisymmetric bilinear form. Define $\hat{\mathfrak{b}}^+ := \mathfrak{b}[t]t^{\frac{1}{2}}$ and $\hat{\mathfrak{b}}^- := \mathfrak{b}[t^{-1}]t^{\frac{1}{2}}$ so that $\hat{\mathfrak{b}} = \hat{\mathfrak{b}}^+ \bigoplus \hat{\mathfrak{b}}^-$ is a polarization for the antisymmetric bilinear form so that $\hat{\mathfrak{b}}^\pm$ is isotropic. Let $\mathbb{C}\mathbf{v}$ be the unique unital $\langle \hat{\mathfrak{b}}^+ \rangle$-module such that $b\mathbf{v} = 0$ for every $b \in \hat{\mathfrak{b}}^+$.

We define the Weyl module vertex algebra associated to $\mathfrak{b}$ and the antisymmetric bilinear form to be the unique vertex superalgebra structure on

$$\mathcal{W}(\mathfrak{b}) := \mathrm{Weyl}(\hat{\mathfrak{b}}) \otimes_{\langle \hat{\mathfrak{b}}^{+} \rangle} \mathbb{C}\mathbf{v} \tag{5.2.5}$$

such that $Y(b\left(\frac{-1}{2}\right)\mathbf{v}, z) = \sum_{n\in\mathbb{Z}} b(n+\frac{1}{2})z^{-n-1}$ for $b \in \mathfrak{b}$.

Let $\{b_i^{\pm}\}$ be a basis for $\mathfrak{b}^{\pm}$ such that $\langle\langle b_i^{\mp}, b_j^{\pm}\rangle = \pm\delta_{ij}$ where $\langle\langle\cdot,\cdot\rangle$ is the antisymmetric bilinear form on $\mathfrak{b}$. Then define

$$\omega := \frac{1}{2}\sum_i \left(b_i^+\left(\frac{-3}{2}\right)b_i^-\left(\frac{-1}{2}\right) - b_i^+\left(\frac{-1}{2}\right)b_i^-\left(\frac{-3}{2}\right)\right)\mathbf{v}.$$

Then equipped with Virasoro element ω, $\mathcal{W}(\mathfrak{b})$ has the structure of a Weyl module vertex operator algebra.

Similarly, for $\mathfrak{b}^+$ defined to be the span of $\{b_i^+\}$ and $\mathfrak{b}^-$ defined to be the span of $\{b_i^-\}$, we can define $\hat{\mathfrak{b}}^+_{\mathrm{tw}} := \mathfrak{b}^+ \oplus t\mathfrak{b}[t]$ and $\hat{\mathfrak{b}}^-_{\mathrm{tw}} := \mathfrak{b}^- \oplus t^{-1}\mathfrak{b}[t^{-1}]$. Let $\mathbb{C}\mathbf{v}_{\mathrm{tw}}$ be the unique unital $\langle\hat{\mathfrak{b}}^+_{\mathrm{tw}}\rangle$-module such that $b\mathbf{v}_{\mathrm{tw}} = 0$ for every $b \in \hat{\mathfrak{b}}^+_{\mathrm{tw}}$. Then $\mathcal{W}(\mathfrak{b})_{\mathrm{tw}}$ has the structure of a twisted $\mathcal{W}(\mathfrak{b})$-module:

$$\mathcal{W}(\mathfrak{b})_{\mathrm{tw}} := \mathrm{Weyl}(\hat{\mathfrak{b}}_{\mathrm{tw}}) \otimes_{\langle \hat{\mathfrak{b}}^{+}_{\mathrm{tw}} \rangle} \mathbb{C}\mathbf{v}_{\mathrm{tw}} \tag{5.2.6}$$

such that $Y_{\mathrm{tw}}(b\left(\frac{-1}{2}\right), z) = \sum_{n\in\mathbb{Z}} b(n)z^{-n-\frac{1}{2}}$ for $b \in \mathfrak{b}$. $\mathcal{W}(\mathfrak{b})_{\mathrm{tw}}$ is the unique (up to equivalence) irreducible canonically twisted $\mathcal{W}(\mathfrak{b})$-module.

Denote the central charge of $\mathcal{W}(\mathfrak{b})$ by c_3. We also denote by $L_3(n)$ the coefficient of z^{-n-2} in $Y(\omega, z)$ or $Y(\omega, z)_{\mathrm{tw}}$. Letting

$$\jmath := \sum_i b_i^+\left(\frac{-1}{2}\right)b_i^-\left(\frac{-1}{2}\right)\mathbf{v},$$

we denote by $J_3(n)$ the coefficient of z^{-n-1} in $Y(\jmath, z)$ or $Y(\jmath, z)_{\mathrm{tw}}$. The operators $L_3(0)$ and $J_3(0)$ then equip $\mathcal{W}(\mathfrak{b})$ and $\mathcal{W}(\mathfrak{b})_{\mathrm{tw}}$ with a bigrading. We focus on the latter, $\mathcal{W}(\mathfrak{b})_{\mathrm{tw}}$, which has bigrading as follows:

$$(\mathcal{W}(\mathfrak{b})_{\mathrm{tw}})_{n,r} = \{v \in \mathcal{W}(\mathfrak{b})_{\mathrm{tw}} \mid (L_3(0) - \tfrac{c_3}{24})v = nv, J_3(0)v = rv\}. \tag{5.2.7}$$

Then $\mathcal{W}(\mathfrak{b})_{\mathrm{tw}}$ has bigraded dimension as follows:

$$\operatorname{tr}\left(y^{J_3(0)}q^{L_3(0)-\frac{c_3}{24}} \mid \mathcal{W}(\mathfrak{b})_{\mathrm{tw}}\right) = y^{-1}q^{-\frac{1}{6}}\prod_{n>0}(1-y^{-1}q^{n-1})^{-2}(1-yq^n)^{-2}. \tag{5.2.8}$$

Remark. For the above equation to make sense we should expand the right hand side in the domain to which we have restricted, $0 < -\mathrm{Im}(z) < \mathrm{Im}(\tau)$. In other words, each factor of the form $\dfrac{1}{(1-X)}$ should be interpreted as $\sum_{n\geq 0} X^n$.

Lemma 5.2.2.

$$-\operatorname{tr}\left(y^{J_3(0)}q^{L_3(0)-\frac{c_3}{24}} \mid \mathcal{W}(\mathfrak{b})_{\mathrm{tw}}\right) = \frac{\eta^2(\tau)}{\theta_1^2(\tau,z)}$$

Proof. The equation for $\theta_1(\tau,z)$ (see Section 5.1) implies the following equation for $\dfrac{1}{\theta_1^2(\tau,z)}$:

$$\frac{1}{\theta_1^2(\tau,z)} = -y^{-1}q^{-\frac{1}{4}}\prod_{n>0}(1-y^{-1}q^{n-1})^{-2}(1-yq^n)^{-2}(1-q^n)^{-2}, \tag{5.2.9}$$

Noting that $\eta^2(\tau) = q^{\frac{1}{12}}\prod_{n>0}(1-q^n)^2$ we see that

$$\frac{\eta^2(\tau)}{\theta_1^2(\tau,z)} = -y^{-1}q^{-\frac{1}{6}}\prod_{n>0}(1-y^{-1}q^{n-1})^{-2}(1-yq^n)^{-2}, \tag{5.2.10}$$

and the expansion of this in our specified domain is equal to the graded dimension of $\mathcal{W}(\mathfrak{b})_{\mathrm{tw}}$ (see (5.2.8)). $\square$

To recover the first factor of (5.2.2) we need a module with graded dimension function $\phi_g(\tau,z)$. For this we use the canonically twisted $V^{s\natural}$-module, $V^{s\natural}_{\mathrm{tw}}$, where $V^{s\natural}$ is the unique self-dual, rational, C_2-cofinite vertex operator superalgebra of CFT type with central charge 12 such that $L(0)u = \frac{1}{2}u$ for $u \in V^{s\natural}$ implies $u = 0$ (cf. Theorem 5.15 [Dun07] and Theorem 4.5 [DMC15]).

For full details of the construction of $V^{s\natural}_{\mathrm{tw}}$, we refer the reader to Duncan and Mack-Crane [DMC15]. In what follows we give a brief summary. To define $V^{s\natural}_{\mathrm{tw}}$ here, we start with the construction of Clifford algebra modules (see above), but this time instead of starting with a one-dimensional complex vector space, we take $\mathfrak{a}$ to be a 24-dimensional complex vector space with a non-degenerate symmetric bilinear form. Let $\hat{\mathfrak{a}} := \mathfrak{a}[t, t^{-1}]t^{\frac{1}{2}}$ and $\hat{\mathfrak{a}}_{\mathrm{tw}} := \mathfrak{a}[t, t^{-1}]$, for $a \in \mathfrak{a}$ we write $a(r)$ for at^r with the bilinear form extended as before. We define a polarization $\hat{\mathfrak{a}} = \hat{\mathfrak{a}}^+ \oplus \hat{\mathfrak{a}}^-$ of $\hat{\mathfrak{a}}$ by setting $\hat{\mathfrak{a}}^+ := \mathfrak{a}[t]t^{\frac{1}{2}}$ and $\hat{\mathfrak{a}}^- := \mathfrak{a}[t^{-1}]t^{\frac{-1}{2}}$. Let $\mathbb{C}\mathbf{v}$ be the unique unital $\langle \hat{\mathfrak{a}}^+ \rangle$-module such that $a\mathbf{v} = 0$ for every $a \in \hat{\mathfrak{a}}^+$. Then we can define $A(\mathfrak{a})$ to be the Cliff($\hat{\mathfrak{a}}$)-module:

$$A(\mathfrak{a}) := \mathrm{Cliff}(\hat{\mathfrak{a}}) \otimes_{\langle \hat{\mathfrak{a}}^+ \rangle} \mathbb{C}\mathbf{v}, \tag{5.2.11}$$

where as $\langle \hat{\mathfrak{a}}^- \rangle$-modules, we have the isomorphism $A(\mathfrak{a}) \simeq \bigwedge(\hat{\mathfrak{a}}^-)\mathbf{v}$.

$A(\mathfrak{a})$ has the structure of a vertex superalgebra such that $Y(a\left(-\frac{1}{2}\right)\mathbf{v}, z) = \sum_{n \in \mathbb{Z}} a(n + \frac{1}{2})z^{-n-1}$ for $a \in \mathfrak{a}$. The super space structure $A(\mathfrak{a}) = A(\mathfrak{a})^0 \oplus A(\mathfrak{a})^1$ is given by the parity decomposition on $\bigwedge(\hat{\mathfrak{a}}^-)\mathbf{v}$.

For $\{e_i\}$ an orthonormal basis for $\mathfrak{a}$, the Virasoro element

$$\omega = -\frac{1}{4}\sum_{i=1}^{\dim \mathfrak{a}} e_i\left(-\frac{3}{2}\right) e_i\left(-\frac{1}{2}\right)\mathbf{v},$$

gives $A(\mathfrak{a})$ the structure of a vertex operator superalgebra.

Similarly, for $\mathfrak{a} = \mathfrak{a}^+ \oplus \mathfrak{a}^-$ a polarization of $\mathfrak{a}$ with respect to its non-degenerate symmetric bilinear form, we can define $\hat{\mathfrak{a}}^+_{\mathrm{tw}} := \mathfrak{a}^+ \oplus t\mathfrak{a}[t]$ and $\hat{\mathfrak{a}}^-_{\mathrm{tw}} := \mathfrak{a}^- \oplus t^{-1}\mathfrak{a}[t^{-1}]$. Let $\mathbb{C}\mathbf{v}_{\mathrm{tw}}$ be the unique unital $\langle \hat{\mathfrak{a}}^+ \rangle$-module such that $u\mathbf{v}_{\mathrm{tw}} = 0$ for $u \in \hat{\mathfrak{a}}^+$. Then

$$A(\mathfrak{a})_{\mathrm{tw}} := \mathrm{Cliff}(\hat{\mathfrak{a}}_{\mathrm{tw}}) \otimes_{\langle \hat{\mathfrak{a}}^+_{\mathrm{tw}} \rangle} \mathbb{C}\mathbf{v}_{\mathrm{tw}} \tag{5.2.12}$$

has the structure of a twisted $A(\mathfrak{a})$-module such that $Y_{\mathrm{tw}}(a\left(-\frac{1}{2}\right), z) = \sum_{n\in\mathbb{Z}} a(n)z^{-n-\frac{1}{2}}$ for $a \in \mathfrak{a}$. This is the unique (up to equivalence) irreducible canonically twisted $A(\mathfrak{a})$-module. We again have the isomorphism $A(\mathfrak{a})_{\mathrm{tw}} \simeq \bigwedge(\hat{\mathfrak{a}}_{\mathrm{tw}}^{-})\mathbf{v}_{\mathrm{tw}}$ as $\langle\hat{\mathfrak{a}}_{\mathrm{tw}}^{-}\rangle$-modules.

We will also define a decomposition of $A(\mathfrak{a})_{\mathrm{tw}}$. For this we first define $\mathfrak{z} \in \mathrm{Spin}(\mathfrak{a})$ to be the unique lift of $-\mathrm{Id}_{\mathfrak{a}} \in SO(\mathfrak{a})$ to $\mathrm{Spin}(\mathfrak{a})$ such that $\mathfrak{z}\mathbf{v}_{\mathrm{tw}} = \mathbf{v}_{\mathrm{tw}}$. The element $\mathfrak{z}$ acts on $A(\mathfrak{a})_{\mathrm{tw}}$ with order two and we denote by $A(\mathfrak{a})_{\mathrm{tw}}^{j}$ the eigenspace for this action with eigenvalue $(-1)^j$. We can decompose $A(\mathfrak{a})_{\mathrm{tw}}$ into eigenspaces $A(\mathfrak{a})_{\mathrm{tw}} = A(\mathfrak{a})_{\mathrm{tw}}^{0} \oplus A(\mathfrak{a})_{\mathrm{tw}}^{1}$. For more on the lift to the spin group, we refer the reader to [DMC15]. For the rest of the construction we refer to [DMC16]. Taking $\mathfrak{a} = \Lambda \otimes_{\mathbb{Z}} \mathbb{C}$, we define

$$V^{s\natural} = A(\mathfrak{a})^0 \oplus A(\mathfrak{a})_{\mathrm{tw}}^{1}, \quad V_{\mathrm{tw}}^{s\natural} = A(\mathfrak{a})^1 \oplus A(\mathfrak{a})_{\mathrm{tw}}^{0}. \tag{5.2.13}$$

Denote the central charge of $V^{s\natural}$ by c_1, and denote by $L_1(n)$ the coefficient of z^{-n-2} in $Y(\omega, z)$ or $Y(\omega, z)_{\mathrm{tw}}$.

We will define an additional operator $J_1(n)$ in order to define a bi-grading on $V_{\mathrm{tw}}^{s\natural}$. To define this operator, first let Π be a 4-dimensional subspace of $\Lambda \otimes_{\mathbb{Z}} \mathbb{C}$ and let $\{x, y, z, w\}$ be an orthonormal basis for Π. We then define $a_1^{\pm} = \frac{1}{\sqrt{2}}(x \pm iy)$ and $a_2^{\pm} = \frac{1}{\sqrt{2}}(z \pm iw)$ so that $\langle a_1^{\pm}, a_1^{\mp}\rangle = \langle a_2^{\pm}, a_2^{\mp}\rangle = 1$. Then we let

$$\jmath := \frac{1}{2}a_1^{-}\left(-\frac{1}{2}\right)a_1^{+}\left(-\frac{1}{2}\right)\mathbf{v} + \frac{1}{2}a_2^{-}\left(-\frac{1}{2}\right)a_2^{+}\left(-\frac{1}{2}\right)\mathbf{v},$$

and we denote by $J_1(n)$ the coefficient of z^{-n-1} in $Y(\jmath, z)$ or $Y(\jmath, z)_{\mathrm{tw}}$. The operators $L_1(0)$ and $J_1(0)$ then equip $V_{\mathrm{tw}}^{s\natural}$ with a bigrading as follows:

$$(V_{\mathrm{tw}}^{s\natural})_{n,r} = \{v \in V_{\mathrm{tw}}^{s\natural} \mid (L_1(0) - \tfrac{c}{24})v = nv, J_1(0)v = rv\}. \tag{5.2.14}$$

Taking $\mathfrak{a} = \Lambda \otimes_{\mathbb{Z}} \mathbb{C}$ allows us to identify Co_0 (and therefore M_{24}) with a subgroup of $SO(\mathfrak{a})$. By Proposition 3.1 of [DMC15], for any subgroup G of $SO(\mathfrak{a})$ which is isomorphic to Co_0, there exists a unique lift of G to $\mathrm{Spin}(\mathfrak{a})$ such that the non-trivial central element is $\mathfrak{z}$. We denote this lift by $\widehat{G}$ and for $g \in G$, we denote the lift of g to $\widehat{G}$ by $\widehat{g}$. The spin group acts naturally on $V^{s\natural}$ and $V^{s\natural}_{\mathrm{tw}}$ so we can now state the following lemma:

Lemma 5.2.3. *For $g \in G$ fixing a 4-space in the 24-dimensional permutation representation of M_{24}, we have*

$$\phi_g(\tau, z) = -\mathrm{tr}\left(\mathfrak{z}\widehat{g}y^{J_1(0)}q^{L_1(0)-\frac{c_1}{24}} \mid V^{s\natural}_{\mathrm{tw}}\right). \tag{5.2.15}$$

Define the operators $L(0) := L_1(0) + L_2(0) + L_3(0)$ and $J(0) := J_1(0) + J_3(0)$ and the central charge $c := c_1 + c_2 + c_3$. Combining Lemmas 5.2.1, 5.2.2, and 5.2.3 we can state the following theorem:

Theorem 5.2.4. *$\widetilde{A}(\mathfrak{p})_{\mathrm{tw}} \otimes \mathcal{W}(\mathfrak{b})_{\mathrm{tw}} \otimes V^{s\natural}_{\mathrm{tw}}$ is an infinite dimensional, bigraded, virtual module with trace functions as follows:*

$$\mathrm{tr}\left(\widehat{g}\mathfrak{z}\widetilde{p}(0)y^{J(0)}q^{L(0)-\frac{c}{24}} \mid \widetilde{A}(\mathfrak{p})_{\mathrm{tw}} \otimes \mathcal{W}(\mathfrak{b})_{\mathrm{tw}} \otimes V^{s\natural}_{\mathrm{tw}}\right) = \widetilde{M_g}(\tau, z). \tag{5.2.16}$$

This gives a module construction for any subgroup G of M_{24} for which the 24-dimensional permutation representation of M_{24} restricted to G fixes at least a four dimensional space.

Example. $\widetilde{A}(\mathfrak{p})_{\mathrm{tw}} \otimes \mathcal{W}(\mathfrak{b})_{\mathrm{tw}} \otimes V^{s\natural}_{\mathrm{tw}}$ is a (virtual) module for the group $\mathbf{L_3(4)} \simeq \mathbf{M_{21}}$, one of the simple subgroups of M_{24}. One can see via the following fusion of conjugacy classes

$$[1A, 2A, 3A, 4B, 4B, 4B, 5A, 5A, 7A, 7B]$$

that the 24-dimensional representation of M_{24} restricts to $\mathbf{L_3(4)}$ as $4\psi_1 + 1\psi_2$ (where ψ_i are irreducible representations of $\mathbf{L_3(4)}$ and ψ_1 is the trivial representation). In particular, we see the permutation representation restricted to $\mathbf{L_3(4)}$ fixes a four dimensional space.

5.3 Module construction II

The construction described in the previous section does not apply in cases where the restriction of the 24-dimensional permutation representation to a subgroup of M_{24} does not fix a 4-space. In what follows we give a similar module construction for such subgroups. However, for these subgroups, we still require that each element of the subgroup fixes a 4-space. Note that this is a weaker requirement than asking that the subgroup itself fixes a 4-space, because not every element of the subgroup necessarily fixes the same 4-space. For this construction we apply a method of Anagiannis, Cheng, Harrison [ACH19]. In our context, the idea of the method is to view the theta quotients and the eta quotients in $\phi_g(\tau, z)$ (the graded trace functions of $V^{s\natural}_{\mathrm{tw}}$) as coming from dimensions of different spaces (see (5.3.10)).

We begin by constructing another module T which we show has the same the trace functions as those from $V^{s\natural}_{\mathrm{tw}}$ (recall that $V^{s\natural}_{\mathrm{tw}} = A(\mathfrak{a})^1 \oplus A(\mathfrak{a})^0_{\mathrm{tw}}$).

We define $\mathfrak{f} := \mathbb{C}^4$ and equip it with both a non-degenerate symmetric bilinear form $\langle \cdot, \cdot \rangle$ and a non-degenerate antisymmetric bilinear form $\langle\langle \cdot, \cdot \rangle$. For convenience we make the choice in such a way that a decomposition $\mathfrak{f} = \mathfrak{f}^+ \bigoplus \mathfrak{f}^-$ serves as a polarization for both bilinear forms. Then we may define

$$B = A(\mathfrak{f}) \otimes \mathscr{W}(\mathfrak{f}) \text{ and } B_{\mathrm{tw}} = A(\mathfrak{f})_{\mathrm{tw}} \otimes \mathscr{W}(\mathfrak{f})_{\mathrm{tw}}, \tag{5.3.1}$$

where $A(\mathfrak{f})$ and $\mathscr{W}(\mathfrak{f})$ are defined, as before, to be a Cliff($\hat{\mathfrak{f}}$)-module and a Weyl($\hat{\mathfrak{f}}$)-module associated to $\mathfrak{f}$, each endowed with a vertex superalgebra (resp. vertex algebra) structure.

For $A(\mathfrak{f})$ we let $\{f_i^{\pm}\}$ be a basis for $\mathfrak{f}^{\pm}$ such that $\langle f_i^{\mp}, f_j^{\pm} \rangle = \delta_{ij}$ where $\langle \cdot, \cdot \rangle$ is the non-degenerate symmetric bilinear form on $\mathfrak{f}$. We can then define the elements $\jmath := \sum_i f_i^+(-\frac{1}{2})f_i^-(-\frac{1}{2})\mathbf{v}$ and $\omega := \dfrac{1}{2}\sum_i \left(f_i^+\left(\frac{-3}{2}\right)f_i^-\left(\frac{-1}{2}\right) - f_i^+\left(\frac{-1}{2}\right)f_i^-\left(\frac{-3}{2}\right)\right)\mathbf{v}$ and we denote the corresponding operators by $J_{11}(0)$ and $L_{11}(0)$ and the central charge by c_{11}.

Similarly, for $\mathscr{W}(\mathfrak{f})$ we assume that the antisymmetric bilinear form $\langle\langle\cdot,\cdot\rangle$ on $\mathfrak{f}$ is chosen so that $\langle\langle f_i^{\mp}, f_j^{\pm}\rangle = \pm\delta_{ij}$. We define the elements $\jmath := \sum_i f_i^+(\frac{-1}{2})f_i^-(\frac{-1}{2})\mathbf{v}$ and $\omega := \dfrac{1}{2}\sum_i \left(f_i^+\left(\frac{-3}{2}\right)f_i^-\left(\frac{-1}{2}\right) - f_i^+\left(\frac{-1}{2}\right)f_i^-\left(\frac{-3}{2}\right)\right)\mathbf{v}$ and denote the corresponding operators $J_{12}(0)$ and $L_{12}(0)$ and the central charge c_{12}.

Lastly, $A(\mathfrak{a})$, for $\mathfrak{a} = \Lambda \otimes_{\mathbb{Z}} \mathbb{C}$, along with the conformal vector associated to it has already been defined in the previous section, but here we will rename the $L(0)$ operator and the central charge associated to $A(\mathfrak{a})$ as $L_{13}(0)$ and c_{13}, respectively. We do not define the element $\jmath$ or the operator $J(0)$ for $A(\mathfrak{a})$ because we are no longer assuming that all g in our subgroup fix a single 4-space in $\mathfrak{a}$.

We can now make the definition:

$$T := (B \otimes A(\mathfrak{a}))^1 \oplus (B \otimes A(\mathfrak{a}))^0_{\mathrm{tw}}$$

and we can compute the trace of $g \in M_{24}$ (for g that fix a 4-space, and are in the allowed conjugacy classes) acting on T.

We let $\lambda_i^{\pm 1}$ be the eigenvalues for g acting on $\mathfrak{a}$. Since we are restricting to $g \in M_{24}$ fixing a 4-space of $\mathfrak{a}$, we can assume that for two i we have $\lambda_i = 1$. We also define ν_i to be square roots of the λ_i and $\nu = \prod_{i=1}^{12} \nu_i$. Before we can compute the trace of $\widehat{g}\mathfrak{z}y^{J_1(0)}q^{L_1(0)-\frac{c_1}{24}}$ on T we require a few more definitions.

We recall the product formulas of the Jacobi theta functions

$$\theta_1(\tau, z) := -iq^{\frac{1}{8}}y^{\frac{1}{2}}\prod_{n>0}(1-y^{-1}q^{n-1})(1-yq^n)(1-q^n), \tag{5.3.2}$$

$$\theta_2(\tau,z) := q^{\frac{1}{8}}y^{\frac{1}{2}}\prod_{n>0}(1+y^{-1}q^{n-1})(1+yq^n)(1-q^n), \tag{5.3.3}$$

$$\theta_3(\tau,z) := \prod_{n>0}(1+y^{-1}q^{n-1/2})(1+yq^{n-1/2})(1-q^n), \tag{5.3.4}$$

and

$$\theta_4(\tau,z) := \prod_{n>0}(1-y^{-1}q^{n-1/2})(1-yq^{n-1/2})(1-q^n). \tag{5.3.5}$$

We then recall the definition

$$\eta_g(\tau) := q\prod_{n>0}\prod_{i=1}^{12}(1-\lambda_i^{-1}q^n)(1-\lambda_i q^n), \tag{5.3.6}$$

and note that

$$\frac{\eta_g(\tau/2)}{\eta_g(\tau)} = q^{-\frac{1}{2}}\prod_{n>0}\prod_{i=1}^{12}(1-\lambda_i^{-1}q^{n-\frac{1}{2}})(1-\lambda_i q^{n-\frac{1}{2}}). \tag{5.3.7}$$

We also define

$$C_g = \nu\prod_{i=1}^{12}(1-\lambda_i^{-1}) \tag{5.3.8}$$

and

$$D_g = \nu'\prod_{i=1}^{10}(1-\lambda_i^{-1}), \tag{5.3.9}$$

where ν' is the product $\prod_{i=1}^{10}\nu_i$ (where we choose the labelling so that $\lambda_i^{\pm} = 1$ for $i = 11$ and $i = 12$).

With these definitions, we are able to give the following explicit expression for $\phi_g(\tau,z)$ from Proposition 9.2 of [DMC16]:

$$\phi_g(\tau,z) = -\frac{1}{2}\left(\frac{\theta_4^2(\tau,z)}{\theta_4^2(\tau,0)}\frac{\eta_{-g}(\tau/2)}{\eta_{-g}(\tau)} - \frac{\theta_3^2(\tau,z)}{\theta_3^2(\tau,0)}\frac{\eta_{-g}(\tau/2)}{\eta_{-g}(\tau)}\right) + \frac{1}{2}\left(D_g\eta_g(\tau)\frac{\theta_1^2(\tau,z)}{\eta^6(\tau)} + C_{-g}\eta_{-g}(\tau)\frac{\theta_2^2(\tau,z)}{\theta_2^2(\tau,0)}\right). \tag{5.3.10}$$

In the next few lemmas, we show that the trace of $\widehat{g}\mathfrak{z}y^{J_1(0)}q^{L_1(0)-c_1/24}$ on T is equal to $\phi_g(\tau,z)$.

Lemma 5.3.1. *Let $g \in M_{24}$ such that $\widehat{g}$ fixes a 4-dimensional space of $\mathfrak{a}$. Let $\mathfrak{z}$ denote the parity involution, let c_1 be the central charge, let $L_1(0)$ and $J_{11}(0)$ and $J_{12}(0)$ be operators as before, then*

$$-\lim_{\gamma \to -1} \operatorname{tr}\left(\widehat{g}\mathfrak{z} y^{J_{11}(0)} \gamma^{J_{12}(0)} q^{L_1(0)-\frac{c_1}{24}} \mid (B \otimes A(\mathfrak{a}))^1\right) = -\frac{1}{2}\left(\frac{\theta_4^2(\tau,z)}{\theta_4^2(\tau,0)}\frac{\eta_{-g}(\tau/2)}{\eta_{-g}(\tau)} - \frac{\theta_3^2(\tau,z)}{\theta_3^2(\tau,0)}\frac{\eta_{-g}(\tau/2)}{\eta_{-g}(\tau)}\right). \tag{5.3.11}$$

Proof. We begin by recalling the projection operator $P^1(g) = \frac{1}{2}(g - \mathfrak{z}g)$. This will allow us to compute the graded trace (5.3.11) on $(B \otimes A(\mathfrak{a}))^1$ by using the traces of $\widehat{g}\mathfrak{z} y^{J_1(0)} q^{L_1(0)-\frac{c_1}{24}}$ and $\widehat{g} y^{J_1(0)} q^{L_1(0)-\frac{c_1}{24}}$ on $B \otimes A(\mathfrak{a})$.

We will first compute the traces of $\widehat{g} q^{L_1(0)-\frac{c}{24}}$ and $\widehat{g}\mathfrak{z} q^{L_1(0)-\frac{c}{24}}$ on $A(\mathfrak{a})$. The traces are as follows:

$$\operatorname{tr}\left(\widehat{g} q^{L_{13}(0)-\frac{c_{13}}{24}} \mid A(\mathfrak{a})\right) = q^{-\frac{1}{2}} \prod_{n>0}(1+q^{n-\frac{1}{2}})^4 \prod_{i=1}^{10}(1+\lambda_i q^{n-\frac{1}{2}})(1+\lambda_i^{-1} q^{n-\frac{1}{2}}), \tag{5.3.12}$$

and

$$\operatorname{tr}\left(\widehat{g}\mathfrak{z} q^{L_{13}(0)-\frac{c_{13}}{24}} \mid A(\mathfrak{a})\right) = q^{-\frac{1}{2}} \prod_{n>0}(1-q^{n-\frac{1}{2}})^4 \prod_{i=1}^{10}(1-\lambda_i q^{n-\frac{1}{2}})(1-\lambda_i^{-1} q^{n-\frac{1}{2}}). \tag{5.3.13}$$

Note that $\widehat{g}$ acts trivially on the components of B because $\mathfrak{f}$ is fixed by $\widehat{g}$. So we compute the traces on the components of B as follows:

$$\operatorname{tr}\left(y^{J_{11}(0)} q^{L_{11}(0)-\frac{c_{11}}{24}} \mid A(\mathfrak{f})\right) = q^{-\frac{1}{12}} \prod_{n>0}(1+y^{-1}q^{n-\frac{1}{2}})^2(1+yq^{n-\frac{1}{2}})^2, \tag{5.3.14}$$

$$\operatorname{tr}\left(\mathfrak{z} y^{J_{11}(0)} q^{L_{11}(0)-\frac{c_{11}}{24}} \mid A(\mathfrak{f})\right) = q^{-\frac{1}{12}} \prod_{n>0}(1-y^{-1}q^{n-\frac{1}{2}})^2(1-yq^{n-\frac{1}{2}})^2, \tag{5.3.15}$$

$$\operatorname{tr}\left(\gamma^{J_{12}(0)}q^{L_{12}(0)-\frac{c_{12}}{24}} \mid \mathcal{W}(\mathfrak{f})\right) = q^{\frac{1}{12}}\prod_{n>0}(1-\gamma^{-1}q^{n-\frac{1}{2}})^{-2}(1-\gamma q^{n-\frac{1}{2}})^{-2}, \tag{5.3.16}$$

and

$$\operatorname{tr}\left(\mathfrak{z}\gamma^{J_{12}(0)}q^{L_{12}(0)-\frac{c_{12}}{24}} \mid \mathcal{W}(\mathfrak{f})\right) = q^{\frac{1}{12}}\prod_{n>0}(1+\gamma^{-1}q^{n-\frac{1}{2}})^{-2}(1+\gamma q^{n-\frac{1}{2}})^{-2}. \tag{5.3.17}$$

Define the operators $L_1(0) := L_{11}(0) + L_{12}(0) + L_{13}(0)$ and the central charge $c_1 := c_{11} + c_{12} + c_{13}$.

We can then combine eqs. (5.3.12), (5.3.14) and (5.3.16) and take the limit as $\gamma \to -1$ to compute

$$\begin{aligned}\operatorname{tr}&\Big(\widehat{g}\mathfrak{z}q^{L_1(0)-\frac{c_1}{24}}y^{J_{11}(0)}\gamma^{J_{12}(0)} \mid B\otimes A(\mathfrak{a})\Big)\\ &= q^{-\frac{1}{2}}\prod_{n>0}(1+y^{-1}q^{n-\frac{1}{2}})^2(1+yq^{n-\frac{1}{2}})^2\prod_{i=1}^{10}(1+\lambda_i^{-1}q^{n-\frac{1}{2}})(1+\lambda_i q^{n-\frac{1}{2}})\\ &= \frac{\theta_3^2(\tau,z)}{\theta_3^2(\tau,0)}\frac{\eta_{-g}(\tau/2)}{\eta_{-g}(\tau)}.\end{aligned} \tag{5.3.18}$$

Similarly, we combine eqs. (5.3.13), (5.3.15) and (5.3.17) and take the limit as $\gamma \to -1$, to get:

$$\begin{aligned}\operatorname{tr}&\Big(\widehat{g}\mathfrak{z}q^{L_1(0)-\frac{c_1}{24}}y^{J_{11}(0)}\gamma^{J_{12}(0)} \mid B\otimes A(\mathfrak{a})\Big)\\ &= q^{-\frac{1}{2}}\prod_{n>0}(1-y^{-1}q^{n-\frac{1}{2}})^2(1-yq^{n-\frac{1}{2}})^2\prod_{i=1}^{10}(1-\lambda_i^{-1}q^{n-\frac{1}{2}})(1-\lambda_i q^{n-\frac{1}{2}})\\ &= \frac{\theta_4^2(\tau,z)}{\theta_4^2(\tau,0)}\frac{\eta_{g}(\tau/2)}{\eta_{g}(\tau)}.\end{aligned} \tag{5.3.19}$$

Now that we have computed the traces of $\widehat{g}\mathfrak{z}y^{J_1(0)}q^{L_1(0)-c_1/24}$ and $\widehat{g}y^{J_1(0)}q^{L_1(0)-c_1/24}$

on $B \otimes A(\mathfrak{a})$, we can compute the projection onto $(B \otimes A(\mathfrak{a}))^1$ as follows:

$$\begin{aligned}\operatorname{tr}\Big(\widehat{g}\mathfrak{z}q^{L_1(0)-\frac{c_1}{24}}y^{J_{11}(0)}\gamma^{J_{12}(0)} \mid (B \otimes A(\mathfrak{a}))^1\Big)\\ &= \frac{1}{2}\Big(\operatorname{tr}\left(\widehat{g}\mathfrak{z}q^{L_1(0)-c_1/24}y^{J_{11}(0)}\gamma^{J_{12}(0)} \mid B \otimes A(\mathfrak{a})\right)\\ &- \operatorname{tr}\left(\widehat{g}q^{L_1(0)-c_1/24}y^{J_{11}(0)}\gamma^{J_{12}(0)} \mid B \otimes A(\mathfrak{a})\right)\Big).\end{aligned}$$

Thus, letting $\gamma \to -1$, we have:

$$\operatorname{tr}\Big(\widehat{g}\mathfrak{z}q^{L_1(0)-\frac{c_1}{24}}y^{J_{11}(0)}\gamma^{J_{12}(0)} \mid (B\otimes A(\mathfrak{a}))^1\Big) \to \frac{1}{2}\left(\frac{\theta_4^2(\tau,z)}{\theta_4^2(\tau,0)}\frac{\eta_g(\tau/2)}{\eta_g(\tau)} - \frac{\theta_3^2(\tau,z)}{\theta_3^2(\tau,0)}\frac{\eta_{-g}(\tau/2)}{\eta_{-g}(\tau)}\right). \tag{5.3.20}$$

□

Lemma 5.3.2. *Let $g \in M_{24}$ such that $\widehat{g}$ fixes a 4-dimensional space of $\mathfrak{a}$. Let $\mathfrak{z}$ denote the parity involution, let c_1 be the central charge, let $L_1(0)$ and $J_{11}(0)$ and $J_{12}(0)$ be operators as before, then*

$$-\lim_{\gamma\to-1}\operatorname{tr}\left(\widehat{g}\mathfrak{z}y^{J_{11}(0)}\gamma^{J_{12}(0)}q^{L_1(0)-\frac{c_1}{24}} \mid (B \otimes A(\mathfrak{a}))^0_{\mathrm{tw}}\right) = -\frac{1}{2}\left(D_g\eta_g(\tau)\frac{\theta_1^2(\tau,z)}{\eta^6(\tau)} + C_{-g}\eta_{-g}(\tau)\frac{\theta_2^2(\tau,z)}{\theta_2^2(\tau,0)}\right). \tag{5.3.21}$$

Proof. We begin by recalling the projection operator $P^0(g) = \frac{1}{2}(g + \mathfrak{z}g)$. This will allow us to compute the graded trace (5.3.21) on $(B \otimes A(\mathfrak{a}))^0_{\mathrm{tw}}$ by using the traces of $\widehat{g}\mathfrak{z}y^{J_1(0)}q^{L_1(0)-\frac{c_1}{24}}$ and $\widehat{g}y^{J_1(0)}q^{L_1(0)-\frac{c_1}{24}}$ on $(B \otimes A(\mathfrak{a}))_{\mathrm{tw}}$.

We will first compute the traces of $\widehat{g}q^{L_1(0)-c/24}$ and $\widehat{g}\mathfrak{z}q^{L_1(0)-c/24}$ on $A(\mathfrak{a})_{\mathrm{tw}}$. The traces are as follows:

$$\operatorname{tr}\left(\widehat{g}q^{L_{13}(0)-\frac{c_{13}}{24}} \mid A(\mathfrak{a})_{\mathrm{tw}}\right) = \nu q\prod_{n>0}(1+q^{n-1})^2(1+q^n)^2\prod_{i=1}^{10}(1+\lambda_i q^n)(1+\lambda_i^{-1}q^{n-1}), \tag{5.3.22}$$

$$\operatorname{tr}\left(\widehat{g}\mathfrak{z}q^{L_{13}(0)-\frac{c_{13}}{24}} \mid A(\mathfrak{a})_{\mathrm{tw}}\right) = \nu q \prod_{n>0}(1-q^{n-1})^2(1-q^n)^2 \prod_{i=1}^{10}(1-\lambda_i q^n)(1-\lambda_i^{-1}q^{n-1}). \tag{5.3.23}$$

Note that $\widehat{g}$ acts trivially on the components of B_{tw} because $\mathfrak{f}$ is fixed by $\widehat{g}$. So we compute the traces on the components of B_{tw} as follows:

$$\operatorname{tr}\left(y^{J_{11}(0)}q^{L_{11}(0)-\frac{c_{11}}{24}} \mid A(\mathfrak{f})_{\mathrm{tw}}\right) = yq^{\frac{1}{6}} \prod_{n>0}(1+y^{-1}q^{n-1})^2(1+yq^n)^2, \tag{5.3.24}$$

$$\operatorname{tr}\left(\mathfrak{z}y^{J_{11}(0)}q^{L_{11}(0)-\frac{c_{11}}{24}} \mid A(\mathfrak{f})_{\mathrm{tw}}\right) = yq^{\frac{1}{6}} \prod_{n>0}(1-y^{-1}q^{n-1})^2(1-yq^n)^2, \tag{5.3.25}$$

$$\operatorname{tr}\left(\gamma^{J_{12}(0)}q^{L_{12}(0)-\frac{c_{12}}{24}} \mid \mathcal{W}(\mathfrak{f})_{\mathrm{tw}}\right) = \gamma^{-1}q^{-\frac{1}{6}} \prod_{n>0}(1-\gamma^{-1}q^{n-1})^{-2}(1-\gamma q^n)^{-2}, \tag{5.3.26}$$

and

$$\operatorname{tr}\left(\mathfrak{z}\gamma^{J_{12}(0)}q^{L_{12}(0)-\frac{c_{12}}{24}} \mid \mathcal{W}(\mathfrak{f})_{\mathrm{tw}}\right) = \gamma^{-1}q^{-\frac{1}{6}} \prod_{n>0}(1+\gamma^{-1}q^{n-1})^{-2}(1+\gamma q^n)^{-2}. \tag{5.3.27}$$

As before, we have the operators $L_1(0) := L_{11}(0) + L_{12}(0) + L_{13}(0)$ and the central charge $c_1 := c_{11} + c_{12} + c_{13}$.

We combine eqs. (5.3.22), (5.3.24) and (5.3.26) and let $\gamma \to -1$ to compute

$$\begin{aligned}
&\operatorname{tr}\left(\widehat{g}q^{L_1(0)-\frac{c_1}{24}}y^{J_{11}(0)}\gamma^{J_{12}(0)} \mid (B\otimes A(\mathfrak{a}))_{\mathrm{tw}}\right) \\
&\qquad = -y\nu q \prod_{n>0}\prod_{i=1}^{10}(1+\lambda_i q^n)(1+\lambda_i q^{n-1})(1+y^{-1}q^{n-1})^2(1+yq^n)^2 \\
&\qquad = -C_{-g}\eta_{-g}(\tau)\frac{\theta_2^2(\tau,z)}{\theta_2^2(\tau,0)},
\end{aligned} \tag{5.3.28}$$

and similarly, we combine eqs. (5.3.23), (5.3.25) and (5.3.27) and take the limit as $\gamma \to -1$ to compute

$$\begin{aligned}
&\operatorname{tr}\Big(\widehat{g}\mathfrak{z}q^{L_1(0)-\frac{c_1}{24}}y^{J_{11}(0)}\gamma^{J_{12}(0)} \mid (B\otimes A(\mathfrak{a}))_{\mathrm{tw}}\Big)\\
&\qquad = -y\nu q\prod_{n>0}\prod_{i=1}^{10}(1-\lambda_i q^n)(1-\lambda_i q^{n-1})(1-y^{-1}q^{n-1})^2(1-yq^n)^2\\
&\qquad = D_g\eta_g(\tau)\frac{\theta_1^2(\tau,z)}{\eta^6(\tau)}.
\end{aligned} \tag{5.3.29}$$

We note that letting $\gamma \to -1$ in equation (5.3.29) does not cause any problems with convergence because the double pole that results from taking this limit in the $n = 1$ term of $(1-q^{n-1})^{-2}$ from eq. (5.3.27) is canceled by the double zero coming from the $n = 1$ term $(1-q^{n-1})^2$ in eq. (5.3.23).

Now that we have computed the traces of $\widehat{g}\mathfrak{z}y^{J_1(0)}q^{L_1(0)-\frac{c_1}{24}}$ and $\widehat{g}y^{J_1(0)}q^{L_1(0)-\frac{c_1}{24}}$ on $(B\otimes A(\mathfrak{a}))_{\mathrm{tw}}$, we can compute the projection onto $(B\otimes A(\mathfrak{a}))^0_{\mathrm{tw}}$ as follows:

$$\begin{aligned}
&\operatorname{tr}\Big(\widehat{g}\mathfrak{z}q^{L_1(0)-\frac{c_1}{24}}y^{J_{11}(0)}\gamma^{J_{12}(0)} \mid (B\otimes A(\mathfrak{a}))^0_{\mathrm{tw}}\Big)\\
&\qquad = \frac{1}{2}\Big(\operatorname{tr}\Big(\widehat{g}\mathfrak{z}q^{L_1(0)-\frac{c_1}{24}}y^{J_{11}(0)\gamma^{J_{12}(0)}} \mid (B\otimes A(\mathfrak{a}))_{\mathrm{tw}}\Big)\\
&\qquad\quad - \operatorname{tr}\Big(\widehat{g}q^{L_1(0)-\frac{c_1}{24}}y^{J_{11}(0)}\gamma^{J_{12}(0)} \mid (B\otimes A(\mathfrak{a}))_{\mathrm{tw}}\Big)\Big).
\end{aligned} \tag{5.3.30}$$

Thus, letting $\gamma \to -1$, we get

$$\operatorname{tr}\Big(\widehat{g}\mathfrak{z}q^{L_1(0)-\frac{c_1}{24}}y^{J_{11}(0)}\gamma^{J_{12}(0)} \mid (B\otimes A(\mathfrak{a}))^0_{\mathrm{tw}}\Big) \to \frac{1}{2}\left(D_g\eta_g(\tau)\frac{\theta_1^2(\tau,z)}{\eta^6(\tau)} + C_{-g}\eta_{-g}(\tau)\frac{\theta_2^2(\tau,z)}{\theta_2^2(\tau,0)}\right). \tag{5.3.31}$$

$\square$

Lemma 5.3.3. *Let $\widehat{g}$, $\mathfrak{z}$, $J_{11}(0)$, $J_{12}(0)$, $L_1(0)$, let c_1 be as before, then we have*

$$
\begin{aligned}
-\lim_{\gamma\to -1}\operatorname{tr}\Big(\widehat{g}\mathfrak{z}q^{L_1(0)-\frac{c_1}{24}}y^{J_{11}(0)}\gamma^{J_{12}(0)}\mid T\Big)&\\
&= -\frac{1}{2}\left(\frac{\theta_4^2(\tau,z)}{\theta_4^2(\tau,0)}\frac{\eta_g(\tau/2)}{\eta_g(\tau)}-\frac{\theta_3^2(\tau,z)}{\theta_3^2(\tau,0)}\frac{\eta_{-g}(\tau/2)}{\eta_{-g}(\tau)}\right)\\
&\quad -\frac{1}{2}\left(D_g\eta_g(\tau)\frac{\theta_1^2(\tau,z)}{\eta^6(\tau)}+C_{-g}\eta_{-g}(\tau)\frac{\theta_2^2(\tau,z)}{\theta_2^2(\tau,0)}\right)\\
&= \phi_g(\tau,z).
\end{aligned}
\tag{5.3.32}
$$

We omit the proof of this lemma because the statement follows from the two lemmas immediately before it.

Now we have shown that the module T recovers the trace functions $\phi_g(\tau,z)$.

Define the operators $L(0) := L_1(0)+L_2(0)+L_3(0)$ and $J(0) := J_{11}(0)+J_3(0)$ and the central charge $c := c_1+c_2+c_3$. Combining Lemmas 5.2.1, 5.2.2, and 5.3.3 we can state the following theorem.

Theorem 5.3.4. *$\widetilde{A}(\mathfrak{p})_{\mathrm{tw}}\otimes\mathcal{W}(\mathfrak{b})_{\mathrm{tw}}\otimes T$ is an infinite dimensional, bigraded, virtual module with trace functions as follows:*

$$
\lim_{\gamma\to -1}\operatorname{tr}\left(\widehat{g}\mathfrak{z}\widetilde{p}(0)\gamma^{J_{12}(0)}y^{J(0)}q^{L(0)-\frac{c}{24}}\mid \widetilde{A}(\mathfrak{p})_{\mathrm{tw}}\otimes\mathcal{W}(\mathfrak{b})_{\mathrm{tw}}\otimes T\right)=\widetilde{M}_g(\tau,z). \tag{5.3.33}
$$

Example. $\widetilde{A}(\mathfrak{p})_{\mathrm{tw}}\otimes\mathcal{W}(\mathfrak{b})_{\mathrm{tw}}\otimes T$ is a (virtual) module for the group $\mathbf{M_{22}}:\mathbf{2}$, one of the maximal subgroups of M_{24}. One can see via the following fusion of conjugacy classes:

$$[1A,2A,3A,4B,4B,5A,6A,7A,7B,8A,11A,2A,2B,4A,4B,6A,8A,10A,12A,14A,14B]$$

and from looking at the cycle structure of each of these conjugacy classes that each element of $\mathbf{M_{22}}:\mathbf{2}$ fixes a 4-dimensional space. (Although all of $\mathbf{M_{22}}:\mathbf{2}$ only fixes a

2-dimensional space).

Example. $\widetilde{A}(\mathfrak{p})_{\mathrm{tw}} \otimes \mathcal{W}(\mathfrak{b})_{\mathrm{tw}} \otimes V^{s\natural}_{\mathrm{tw}}$ is a (virtual) module for the group $\mathbf{2^4} \colon \mathbf{A_7}$, one of the maximal subgroups of M_{23}. One can see this via the following fusion of conjugacy classes:

$$[1A, 2A, 2A, 4B, 3A, 3A, 6A, 4B, 8A, 5A, 6A, 7A, 14A, 7B, 14B]$$

and from looking at the cycle structure of each of these conjugacy classes that each element of $\mathbf{2^4} \colon \mathbf{A_7}$ fixes a 4-dimensional space. (Although all of $\mathbf{2^4} \colon \mathbf{A_7}$ only fixes a 3-dimensional space).

Example. $\widetilde{A}(\mathfrak{p})_{\mathrm{tw}} \otimes \mathcal{W}(\mathfrak{b})_{\mathrm{tw}} \otimes T$ is a (virtual) module for the group $\mathbf{A_8}$, one of the maximal subgroups of M_{23}. One can see via the following fusion of conjugacy classes:

$$[1A, 2A, 2A, 3A, 3A, 4B, 4B, 5A, 6A, 6A, 7A, 7B, 15A, 15B]$$

and from looking at the cycle structure of each of these conjugacy classes that each element of $\mathbf{A_8}$ fixes a 4-dimensional space. (Although all of $\mathbf{A_8}$ only fixes a 3-dimensional space).

Example. $\widetilde{A}(\mathfrak{p})_{\mathrm{tw}} \otimes \mathcal{W}(\mathfrak{b})_{\mathrm{tw}} \otimes T$ is a (virtual) module for the smallest sporadic group $\mathbf{M_{11}}$, one of the subgroups of M_{24}. One can see via the following fusion of conjugacy classes:

$$[1A, 2A, 3A, 4B, 5A, 6A, 8A, 8A, 11A, 11A]$$

and from looking at the cycle structure of each of these conjugacy classes that each element of $\mathbf{M_{11}}$ fixes a 4-dimensional space. (Although all of $\mathbf{M_{11}}$ only fixes a 3-dimensional space).

Remark. The module for $\mathbf{M_{11}}$ restricts in particular to a module for $\mathbb{Z}/\mathbf{11}\mathbb{Z}$. This gives an explicit realization of the module in [Ben19b] for which the integrality of its trace functions is equivalent to divisibility conditions on the number of $\mathbb{F}_p$ points on $J_0(11)$ because of the cusp forms in the expressions (see equation 3.2 and Appendix A of [Ben19b]).

Similarly the modules for $\mathbf{M_{22}}: \mathbf{2}$ and $\mathbf{2^4}: \mathbf{A_7}$ give explicit realizations of modules for which the integrality of their trace functions are equivalent to divisibility conditions on the number of $\mathbb{F}_p$ points on $J_0(14)$ and the module for $\mathbf{A_8}$ gives an explicit realization of a module for which the integrality of its trace functions is equivalent to divisibility conditions on the number of $\mathbb{F}_p$ points on $J_0(15)$.

Chapter 6

On Weierstrass mock modular forms and a dimension formula for certain vertex operator algebras

In this chapter, we establish several dimension formulas for certain strongly rational, holomorphic vertex operator algebras, complementing previous work by van Ekeren, Möller, and Scheithauer. We use techniques from the theory of mock modular forms and harmonic Maass forms, especially Weierstrass mock modular form. The results in this section are joint work with Michael Mertens and come from [BM19].

6.1 The space of harmonic Maass forms in terms of Weierstrass mock modular forms

In this section we prove the following theorem, which states that any harmonic Maass form (in the given levels) can be expressed in terms of Weierstrass mock modular forms and their images under Hecke operators.

Theorem 6.1.1. *Let E denote the strong Weil curve of conductor*

$$N \in \{11, 14, 15, 17, 19, 21\}.$$

Then any harmonic Maass form of weight 0 for $\Gamma_0(N)$ is is a linear combination of images of the completed Weierstrass mock modular form $\widehat{\mathfrak{Z}}_E$ associated to the $\Gamma_0(N)$-optimal elliptic curve E — i.e. in the cases considered E is a model for the modular curve $X_0(N)$ — under the Hecke operators T_m and Atkin-Lehner involutions , or in other words:

$$H_0(N) \leq \operatorname{span}_{\mathbb{C}} \left\{ \widehat{\mathfrak{Z}}_E | W_Q | T_m | B_d \ : \ m \in \mathbb{N}_0, \ Q \mid N, \ d \mid N \right\},$$

where the operators B_d are defined in Proposition 2.2.2.

Before proceeding to the proof of Theorem 6.1.1, we require a general result on the action of Hecke operators on Poincaré series. Recall that one may formally define a Poincaré series of weight k for the group $\Gamma \leq \mathrm{SL}_2(\mathbb{Z})$ by averaging a suitably periodic seed function $\varphi : \mathfrak{H} \to \mathbb{C}$ over the coset representatives $\Gamma_\infty \backslash \Gamma$ where $\Gamma_\infty = \mathrm{Stab}_\Gamma(\infty)$ denotes the stabilizer of the cusp ∞, i.e.

$$\mathcal{P}(\Gamma, k, \varphi) := \sum_{\gamma \in \Gamma_\infty \backslash \Gamma} \varphi|_k \gamma.$$

If the defining series converges absolutely, this defines a function which transforms like a modular form of weight k under Γ (see for instance [Ono09, Lemma 8.2] and the references therein). Restricting to the case of $\Gamma = \Gamma_0(N)$, one can compute their Fourier expansions fairly explicitly, especially in the most important cases for our purposes, where the seed function is either the exponential function $\varphi(\tau) = \exp(2\pi i m \tau)$, $m \in \mathbb{Z}$, or a modified version of the Whittaker function (see [Ono09, Section 8] for details) yielding harmonic Maass forms. In those cases, the Fourier coefficients of the mth Poincaré take the general form

$$a_m^{(N,k)}(n) = C_k(n/m)^{(k-1)/2} \sum_{c=1}^{\infty} \frac{K(m,n,Nc)}{Nc} \mathcal{J}_k\left(\frac{\sqrt{mn}}{Nc}\right), \tag{6.1.1}$$

where C_k is some constant depending on the weight k, $\mathcal{J}_k$ is a suitable test function[1] so that the sum converges absolutely, and $K(m,n,c)$ denotes the Kloosterman sum

$$K(m,n,c) = \sum_{d\,(c)^*} \exp\left(2\pi i \frac{m\bar{d} + nd}{c}\right) \tag{6.1.2}$$

where the sum runs over all d modulo c with $\gcd(c,d) = 1$ and $d\bar{d} \equiv 1 \pmod{c}$. Kloosterman sums satisfy the so-called *Selberg identity*,

$$K(m,n,c) = \sum_{d|\gcd(m,n,c)} dK(1, mn/d^2, c/d). \tag{6.1.3}$$

This identity was first noted without proof by Selberg [Sel38]. The first published proof was found by Kuznetsov [Kuz80] using his famous summation formula, and an elementary proof was found by Matthes [Mat90]. Using this, we can show the following general result.

Proposition 6.1.2. *Let $k \leq 0$, and $N, \nu \in \mathbb{N}$. Further denote the νth Maass-Poincaré series of weight k and level N normalized so that its principal part at ∞ is given by $q^{-\nu} + O(1)$ by $P_\nu^{(N,k)}(\tau)$. Then we have that*

$$P_\nu^{(N,k)} = \sum_{d|\gcd(N,\nu)} (\nu/d)^{1-k} P_1^{(N/d,k)} | T_{\nu/d}^{(N/d)} | B_d. \tag{6.1.4}$$

Proof. The action of Hecke operators in level N may be solely defined on the space of one-periodic holomorphic functions together with a weight k slash action of $\mathrm{SL}_2(\mathbb{R})$ via their actions on Fourier expansions: For a function $f(\tau) = \sum_{n\in\mathbb{Z}} a(n)q^n$, we have

[1] In the cases considered, it is essentially a Bessel function.

(see for instance [CS17, Proposition 10.2.5]) $(f|T_m^{(N)}) = \sum_{n\in\mathbb{Z}} b(n)q^n$ where

$$b(n) = \sum_{\substack{d|\gcd(m,n)\\ \gcd(d,N)=1}} d^{k-1}a(mn/d^2). \tag{6.1.5}$$

Since a harmonic Maass form is uniquely determined by its holomorphic part, we restrict our attention to the holomorphic part of the right-hand side of (6.1.4). Using (6.1.5), this is given by

$$\sum_{d|\gcd(N,\nu)} (\nu/d)^{1-k} \sum_{n\in\mathbb{Z}} \left[\sum_{\substack{t|\gcd(\nu/d,n)\\ \gcd(N/d,t)=1}} t^{k-1} a_1^{(N/d,k)} \left(\frac{n\nu/d}{t^2}\right) \right] q^{dn}.$$

The principal part of the holomorphic part is easily seen to equal $q^{-\nu}$ as desired, since the only term then surviving in the sum is the one with $d = \gcd(N,\nu)$ and $t = \nu/d$. The nth coefficient for $n > 0$ is given by

$$\nu^{1-k} \sum_{d|\gcd(N,\nu,n)} \left[\sum_{\substack{t|\gcd(\nu/d,n/d)\\ \gcd(N/d,t)=1}} (dt)^{k-1} a_1^{(N/d,k)} \left(\frac{n\nu}{(dt)^2}\right) \right].$$

Plugging in the definition of the coefficient $a_1^{(N/d,k)}\left(\frac{n\nu}{(dt)^2}\right)$ then yields

$$\begin{aligned} &C_k \nu^{1-k} \sum_{d|\gcd(N,\nu,n)} \sum_{\substack{t|\gcd(\nu/d,n/d)\\ \gcd(N/d,t)=1}} (\nu n)^{(k-1)/2} \sum_{c=1}^{\infty} \frac{K\left(1, \frac{\nu n}{(dt)^2}, \frac{N}{d}c\right)}{\frac{N}{d}c} \mathcal{J}_k \left(\frac{\sqrt{\frac{\nu n}{(dt)^2}}}{\frac{N}{d}c} \right) \\ =&C_k (n/\nu)^{(k-1)/2} \sum_{d|\gcd(N,\nu,n)} \sum_{\substack{t|\gcd(\nu/d,n/d)\\ \gcd(N/d,t)=1}} \sum_{c=1}^{\infty} (dt) \frac{K\left(1, \frac{\nu n}{(dt)^2}, \frac{N(ct)}{dt}\right)}{N(ct)} \mathcal{J}_k \left(\frac{\sqrt{\nu n}}{N(ct)} \right). \end{aligned}$$

We may and do assume without loss of generality that the defining series for the coefficient $a_1^{N,k}$ is absolutely convergent, since in the cases of interest to us, this can be achieved essentially by a Hecke-trick-like argument and analytic continuation (see for instance [Bru02, Chapter 1] for details). Noticing further that every common divisor of Nc, ν, n can be uniquely factored into a common divisor d of N, ν, n and a common divisor t of $c, \nu/d, n/d$ which is coprime to N/d, we may rearrange the above sum to obtain the expression

$$C_k(n/\nu)^{(k-1)/2} \sum_{c=1}^{\infty} \left[\sum_{d \mid \gcd(Nc,\nu,n)} dK\left(1, \frac{\nu n}{d^2}, \frac{Nc}{d}\right) \right] \frac{\mathcal{J}_k\left(\frac{\sqrt{\nu n}}{Nc}\right)}{Nc}.$$

By the Selberg identity (6.1.3) we see by comparing to (6.1.1) that this is exactly the coefficient $a_{\nu}^{(N,k)}(n)$. The constant terms may be compared through a similar but easier argument, which we refrain from carrying out here, therefore completing the proof. $\square$

For levels $N \in \{1, ..., 10, 12, 13, 16, 18, 25\}$, where the modular surface $X_0(N)$ has genus 0, and weight $k = 0$, the above result has been shown using ad hoc methods in [BL15, Lemma 2.11], the result for general levels in weight 0 is stated for $\gcd(N, \nu) = 1$ and (incorrectly) for $\nu \mid N$ in [BKL$^+$18, Theorem 1.1] (see also [BKL$^+$]). In [JKK, Theorem 1.1], Jeon, Kang, and Kim give a slightly different proof of Proposition 6.1.2 for the case of weight 0, which is not an important restriction, and use it to prove congruences for the Fourier coefficients of modular functions for genus 1 levels (see loc. cit., Theorem 1.6). The analogous statement for cuspidal Poincaré series may also be obtained from the Petersson coefficient formula together with the fact that Hecke operators are Hermitian with respect to the Petersson inner product on the space of cusp forms, see [CS17, Proposition 10.3.19] for the case $N = 1$, but which is easily generalized to higher levels.

Proof of Theorem 6.1.1. According to Corollary 2.4.4, the Weierstrass mock modular form is (up to a possible additive constant which is of no importance here) the first Maass-Poincaré series in the respective level with its only pole at the cusp ∞. Proposition 6.1.2 shows how to obtain harmonic Maass forms with arbitrarily high pole orders at infinity by the application of Hecke operators, provided that we are able to express the lower-level Poincaré series needed in (6.1.4) in terms of functions of the form $\widehat{\mathfrak{Z}}_E|W_Q|T_m|B_d$, as required in the theorem.

We therefore need to show first that we can generate the Poincaré series $P_1^{(d,0)}$ for all divisors $d \mid N$ for the relevant levels $N \in \{11, 14, 15, 17, 19, 21\}$. By Proposition 2.2.2 (vi), we know that $pf|U_p + f|W_p$ transforms under $\Gamma_0(N/p)$ for any function f transforming like a modular function for $\Gamma_0(N)$. Applying this to the function $P_1^{(N,0)}|W_p$ shows that $P_1^{N,0} + pP_1^{N,0}|W_p|U_p$ is a harmonic Maass form on $\Gamma_0(N/p)$. Viewed as a harmonic Maass form for $\Gamma_0(N)$, this function has a principal part $q^{-1} + O(1)$ at ∞ and by applying the Atkin-Lehner operator $W_p = \gamma \cdot \left(\begin{smallmatrix} \sqrt{p} & 0 \\ 0 & 1/\sqrt{p} \end{smallmatrix}\right)$ (cf. Proposition 2.2.2 (i)) to it once more, we see that it has principal part $(q^{1/p})^p + O(1)$ at the cusp N/p of $\Gamma_0(N)$. The same is true for the Poincaré series $P_1^{N/p,0}$ viewed as a harmonic Maass form for $\Gamma_0(N)$, so the two functions can only differ by a constant by Corollary 2.1.3. Since N is square-free by assumption, we can repeat this process to obtain all Poincaré series $P_1^{d,0}$ in this fashion, keeping in mind the commutation rules for Hecke and Atkin-Lehner operators in Proposition 2.2.2.

Since the Atkin-Lehner operators act transitively on the cusps of $\Gamma_0(N)$, we have shown now that any harmonic Maass form of level N for the given N can be written as a linear combination of functions of the form $P_1^{(N,0)}|W_Q|T_m|B_d|W_{Q'}$ for $Q, Q' \mid N$, $m \in \mathbb{N}_0$, where we set $f|T_0 := 1$ for any function for convenience, and $d \mid N$. In order to complete the proof, we need to show that the application of $W_{Q'}$ may be avoided. The harmonic Maass form in $H_0(N)$ which has a pole of order m at the cusp $W_p\infty$ for a prime $p \mid N$ and nowhere else is given by

$$P_m^{(N,0)}|W_p = \sum_{d|\gcd(N,m)} (m/d) P_1^{(N/d,0)}|T_{m/d}^{(N/d)}|B_d|W_p$$

by Proposition 6.1.2. For a common divisor d of m and N not divisible by p the operators B_d and W_p commute (see e.g. [CS17, Proposition 13.2.6 (d)]). Furthermore, we can write $T^{(N/d)}_{m/d} = U_{p^\ell} T^{(N/d)}_{m'/d}$ with $m = p^\ell m'$ and $p \nmid m'$. The Atkin-Lehner operator W_p commutes with the Hecke operator $T^{(N/d)}_{m'/d}$ and we have for any function f which is invariant under $\Gamma_0(N/d)$ and any integer $r > 0$ that

$$f|U_{p^r}|W_p = f|U_{p^r}|B_p + \frac{1}{p} f|U_{p^{r-1}}|W_p|B_p - \frac{1}{p} f|U_{p^{r-1}}, \tag{6.1.6}$$

which is an immediate consequence of Proposition 2.2.2 (i) and (vi). By induction on r this shows that we may write $P_1^{(N/d,0)}|T^{(N/d)}_{m/d}|W_p$ as a linear combination of functions of the form $P_1^{(N/d,0)}|W_p^\varepsilon|T^{(N/d)}_{\tilde{m}}|B_d$ for $\varepsilon \in \{0,1\}$ and suitable $\tilde{m} \in \mathbb{N}_0$. Since $P_1^{(N/d,0)}$ may be written as a linear combination of functions of the form $P_1^{(N,0)}|W_Q|T_Q$, the same argument as just used shows that also $P_1^{(N/d,0)}|T^{(N/d)}_{m/d}|W_p$ can be written in the form claimed in the theorem.

For a common divisor d of m and N which is divisible by p, we note that we can write $W_p = \left(\frac{1}{\sqrt{p}}\left(\begin{smallmatrix} p & 0 \\ 0 & 1 \end{smallmatrix}\right)\right)^{-1} \gamma$ for a suitable $\gamma \in \Gamma_0(N/p)$. Therefore we have that

$$P_1^{(N/d,0)}|T^{(N/d)}_{m/d}|B_d|W_p = P_1^{(N/d,0)}|T^{(N/d)}_{m/d}|B_{d/p}|\gamma = P_1^{(N/d,0)}|T^{(N/d)}_{m/d}|B_{d/p}$$

since $P_1^{(N/d,0)}|T^{(N/d)}_{m/d}|B_{d/p} \in H_0(\Gamma_0(N/p))$, so that also these summands are of the desired form. In summary, this proves the theorem for all square-free levels N, since the Atkin-Lehner involutions W_p, $p \mid N$ generate the group of all Atkin-Lehner involutions, which act transitively on the cusps.

□

Remark. It is essential in the above proof of Theorem 6.1.1 that the Atkin-Lehner involutions W_Q commute with all Hecke operators T_m with $\gcd(Q, m) = 1$. One could try to extend the the above proof to levels where the full normalizer $\mathsf{N}(\Gamma_0(N))$ acts transitively on the cusps (which is true in addition for levels 20, 24, 27, 32, 36, but not 49, among those where $X_0(N)$ has genus 1). For levels 20 and 24 it is even true that the normalizer commutes with all Hecke operators T_m with $\gcd(m, N) = 1$. Unfortunately, the behaviour with respect to the U-operator (see (6.1.6)) is not such that the above proof immediately generalizes. We point out however that the first part of the proof that one can express all the Poincaré series $P_1^{(d,k)}$ for $d \mid N$ using only elements in the normalizer and Hecke operators together with B-operators, works for levels 20 and 24.

6.2 Dimension formulas

In this section we state and prove our dimension formulas.

Theorem 6.2.1. *Let V be a holomorphic, strongly rational vertex operator algebra of central charge 24. Let $G = \langle g \rangle$ be a cyclic group of automorphisms of V of order $p \in \{11, 17, 19\}$ such that g is of type $p\{0\}$. Further let $E = X_0(p)$ be the $\Gamma_0(p)$-optimal elliptic curve of conductor p. Then with the assumptions and notations in Chapter 3, we have the following dimension formula:*

$$\begin{aligned}\dim V_1 + \dim V_1^{\mathrm{orb}(g)} = (p+1)\dim V_1^G - (p-1)C_E& \\ + C_E \sum_{i=1}^{p-1}\sum_{j=1}^{p-1} \sigma(p-j)\dim V(g^i)_{j/p},&\end{aligned}$$

where we set

$$C_E := -\frac{3 - \#E(\mathbb{F}_2)}{2} - \widehat{\zeta}\left(\Lambda_E; L(E,1)\right).$$

Proof of Theorem 6.2.1. Let $N = p$ be a prime number such that $E := X_0(p)$ has genus 1, i.e. $p \in \{11, 17, 19\}$. From Proposition 3.3.1 we know that ch_{V^G} is a modular function for $\Gamma_0(p)$ without poles in $\mathfrak{H}$ and furthermore the transformation behaviour

$$\mathrm{ch}_{V^G}(S.\tau) = \frac{1}{p} q^{-1} \sum_{n=0} \dim V_n^G q^n + \frac{1}{p} q^{-1} \sum_{i=1}^{p-1} \sum_{n=0}^{\infty} \dim V(\sigma^i)_{n/p} q^{n/p}. \tag{6.2.1}$$

Using this together with Theorem 6.1.1, we can express ch_{V^G} solely in terms of Weierstrass mock modular forms, the Fricke involution W_p, and Hecke operators, more precisely we find, using that $\dim V_0 = 1$,

$$\begin{aligned} \mathrm{ch}_{V^G} = \widehat{\mathfrak{Z}}_E &+ \frac{1}{p} \sum_{i=1}^{p-1} \sum_{j=1}^{p-1} (p-j) \dim V(g^i)_{j/p} \widehat{\mathfrak{Z}}_E | W_p | T_{p-j} \\ &+ \frac{1}{p} \Big((p\widehat{\mathfrak{Z}}_E | U_p + \widehat{\mathfrak{Z}}_E | W_p) | B_p + p\widehat{\mathfrak{Z}}_E | W_p | U_p \Big) + (\dim V_1^G - C), \end{aligned} \tag{6.2.2}$$

with

$$\begin{aligned} C = \dim V_1^G - \Bigg(c_E(0) &+ \frac{c_{E,W_p}(0)}{p} \sum_{i=1}^{p-1} \sum_{j=1}^{p-1} \sigma_1(p-j) \dim V(g^i)_{j/p} \\ &+ \frac{1}{p} \left[p c_E(0) + (p+1) c_{E,W_p}(0) \right] \Bigg), \end{aligned} \tag{6.2.3}$$

where $c_E(0) = -a_E(2)/2$ and $c_{E,W_p}(0) = \widehat{\zeta}(\Lambda_E; L(E,1))$ denote the constant terms of $\widehat{\mathfrak{Z}}_E$ and $\widehat{\mathfrak{Z}}_E | W_p$ respectively, see Section 2.4. Therefore we find, using the relation $S = W_p B_p^{-1}$ and the fact that both $p\widehat{\mathfrak{Z}}_E | U_p + \widehat{\mathfrak{Z}}_E | W_p$ and $\widehat{\mathfrak{Z}}_E + p\widehat{\mathfrak{Z}}_E | W_p | U_p$ transform like modular forms of level 1 by Proposition 2.2.2 (vi) and hence are invariant under S,

$$\begin{aligned} \mathrm{ch}_{V^G} | S = \widehat{\mathfrak{Z}}_E | W_p | B_p^{-1} &+ \frac{1}{p} \sum_{i=1}^{p-1} \sum_{j=1}^{p-1} (p-j) \dim V(g^i)_{j/p} \widehat{\mathfrak{Z}}_E | T_{p-j} | B_p^{-1} \\ &+ \frac{1}{p} \Big(p\widehat{\mathfrak{Z}}_E | U_p | B_p^{-1} + \widehat{\mathfrak{Z}}_E | B_p^{-1} + p\widehat{\mathfrak{Z}}_E | W_p | U_p | B_p^{-1} \Big) + (\dim V_1^G - C). \end{aligned} \tag{6.2.4}$$

Now Proposition 3.3.3 implies that

$$\mathrm{ch}_{V^{\mathrm{orb}(g^N)}} + \mathrm{ch}_{V^{\mathrm{orb}(g)}} = F_0 + F_\infty,$$

for $F_{\mathfrak{a}}$ as in (3.3.1). By definition we have $F_\infty = \mathrm{ch}_{V^G}$, so its constant term equals $\dim V_1^G$. The constant term of F_0 equals p times that of $\mathrm{ch}_{V^G}|S$, since we can write $F_0 = \sum_{j=0}^{p-1} \mathrm{ch}_{V^G}|(ST^j)$ and all summands have the same constant term. Therefore by comparing constant terms we obtain after some simplification

$$\begin{aligned} \dim V_1 + \dim V_1^{\mathrm{orb}(g)} = (p+1)\dim V_1^G - (p-1)\big(c_E(0) - c_{E,W_p}(0)\big) \\ + \big(c_E(0) - c_{E,W_p}(0)\big)\sum_{i=1}^{p-1}\sum_{j=1}^{p-1} \sigma(p-j)\dim V(g^i)_{j/p}. \end{aligned} \tag{6.2.5}$$

Plugging in the definitions of $c_E(0)$ and $c_{E,W_p}(0)$, we arrive at the dimension formula stated in Theorem 6.2.1 □

In particular, this dimension formula relates invariants of the underlying modular curve to the theory of VOAs. Since all the other quantities in the above dimension formula are clearly rational, we obtain the following immediate corollary.

Corollary 6.2.2. *Assume the notations as in Theorem 6.2.1. If we have*

$$\sum_{i=1}^{p-1}\sum_{j=1}^{p-1} \sigma(p-j)\dim V(g^i)_{j/p} \neq p-1$$

for some VOA V as in Theorem 6.2.1, then the value $\widehat{\zeta}\,(\Lambda_E; L(E,1))$ is rational.

From the proof of Theorem 6.2.1 we can infer the following dimension formula as well, which looks similar to that in Corollary 6.2.2. The proof relies on the so-called Bruinier-Funke pairing (see Proposition 2.1.2).

Theorem 6.2.3. *Assume the hypotheses and notation from Theorem 6.2.1, except that p may now denote any prime number, and let $f(\tau) = \sum_{n=1}^{\infty} a(n)e^{2\pi i n\tau} \in S_2(p)$ be a newform with Atkin-Lehner eigenvalue $\varepsilon \in \{\pm 1\}$. Then we have*

$$\sum_{i=1}^{p-1}\sum_{j=1}^{p-1} a(p-j)\dim V(g^i)_{j/p} = -\varepsilon p - a(p).$$

Essentially, the formula in Theorem 6.2.3 also appears on [MS, p. 24], but was proven using a different kind of pairing. We note that loosely speaking, one may interpret Theorem 6.2.1 in view of Theorem 6.2.3 as the case where one replaces the newform f by the weight 2 Eisenstein series in $M_2(p)$.

Proof of Theorem 6.2.3. Since ch_{V^G} is a modular function, its image under the ξ-operator (see Proposition 2.1.1) must be 0. Hence, applying the Bruinier-Funke pairing with the newform f must yield zero as well. Since we have $\mathrm{ch}_{V^G}(\tau) = q^{-1} + O(1)$ and the expansion at the cusp 0 as given in (6.2.1) together with $(f_E|W_p)(\tau) = \varepsilon f(\tau)$ with $\varepsilon \in \{\pm 1\}$, we can apply Proposition 2.1.2 and obtain

$$a(1) + \varepsilon\left(\frac{a(p)}{p} + \frac{1}{p}\sum_{i=1}^{p-1}\sum_{j=1}^{p-1} a(p-j)V(\sigma^i)_{j/p}\right) = 0.$$

Since $a(1) = 1$, the formula claimed in Theorem 6.2.3 follows. $\square$

To conclude, we formulate the dimension formula for the square-free composite levels N such that the modular curve $X_0(N)$ has genus 1, i.e. $N \in \{14, 15, 21\}$. In what follows, we always write $N = p_1p_2$ where p_1 and p_2 are primes. In view of Section 1.2.1, we point out that it is not essential that N has exactly two prime factors and in principle, the method of proof would go through for arbitrary square-free numbers N. However, for the sake of simplicity of the exposition and since all the cases under consideration here are of this form, we restrict to the situation of precisely two distinct prime factors. Before stating the result, we record the following lemma about expansions at other cusps, which is essentially the same as [BL15, Corollary 2.7] and also follows easily from (6.1.6).

Lemma 6.2.4. *Let N be any square-free number and let $f \in H_0(N)$ be a harmonic Maass form satisfying $f(\tau) = q^{-\nu} + c + O(\exp(-\alpha y))$ and $(f|W_Q)(\tau) = c_Q + O(\exp(-\alpha y))$ for all (exact) divisors $Q \mid N$, where $\alpha > 0$ and $c, c_Q \in \mathbb{C}$. Then if $p \mid N$ is a prime with $p \nmid \nu$ and $a \in \mathbb{Z}_{>0}$, then the function $-p^{a+1} f|U_{p^{a+1}}|W_p$ has only a pole at the cusp ∞ and we have the expansion*

$$-p^{a+1}(f|U_{p^{a+1}}|W_p)(\tau) = q^{p^a\nu} - (p-1)\sigma(p^a)c - c_p + O(\exp(-\alpha y)).$$

We now state the dimension formula for composite N.

Theorem 6.2.5. *With the notation and hypotheses as in Theorem 6.2.1, only that the group $G = \langle g \rangle$ has order $N = p_1p_2$, we have the following dimension formula:*

$$\begin{aligned}
&\dim V_1^{\mathrm{orb}(g)} + \dim V_1^{\mathrm{orb}(g^{p_1})} + \dim V_1^{\mathrm{orb}(g^{p_2})} + \dim V_1 \\
&= [\mathrm{SL}_2(\mathbb{Z}) : \Gamma_0(N)] \dim V_1^G + Nc_{E,N} + p_1 c_{E,p_1} + p_2 c_{E,p_2} \\
&+ \sum_{m=0}^{N-1} \sum_{\substack{i,j \in \mathbb{Z}/N\mathbb{Z} \\ ij \equiv m\ (N)}} \sigma((N-m)') \dim W_{m/N}^{(i,j)} \big[c_{E,N}(0) + p_2^{b_m+1} c_{E,p_1}(0) \\
&\qquad + (p_1^{a_m+1} + p^{a_m}) c_{E,p_2}(0) - (p_1^{a_m+1} + p_1^{a_m} + p_2^{b_m+1} + p_2^{b_m}) c_E(0)\big] \\
&+ \sum_{m=0}^{p_2-1} \sum_{\substack{i,j \in \mathbb{Z}/p_2\mathbb{Z} \\ p_1 ij \equiv m\ (p_2)}} \sigma((N-p_1m)') \dim W_{m/p_2}^{(p_1i,p_1j)} \left[\frac{p_1}{p_2} c_N + p_1 \left(p_2^{b_{p_1m}+1} + p_2^{b_{p_1m}} - \frac{1}{p_2}\right) c_{E,p_1}(0) \right. \\
&\qquad \left. - \left(p_1^{a_{p_1m}+2} + p_1^{a_{p_1m}+1} + \frac{p_1}{p_2}\right) c_{E,p_2}(0) + \left((p_1+1)(p_1^{a_{p_1m}+1} - p_2^{b_{p_1m}+1}) + \frac{p_1}{p_2}\right) c_E(0)\right]
\end{aligned}$$

$$+\sum_{m=0}^{p_1-1}\sum_{\substack{i,j\in\mathbb{Z}/p_1\mathbb{Z}\\ p_2ij\equiv m\,(p_1)}}\sigma((N-p_2m)')\dim W_{m/p_1}^{(p_2i,p_2j)}\left[\frac{p_2}{p_1}c_N - p_2\left(p_2^{b_{p_2m}+1}+p_2^{b_{p_2m}}-\frac{1}{p_1}\right)c_{E,p_1}(0)\right.$$
$$+p_2\left(p_1^{a_{p_2m}+1}+p_1^{a_{p_2m}}-\frac{1}{p_1}\right)c_{E,p_2}(0)$$
$$\left.+\left((p_2+1)p_2^{b_{p_2m}+1}-p_2(p_1^{a_{p_2m}+1}+p_1^{a_{p_2m}})+\frac{p_2}{p_1}\right)c_E(0)\right].$$

where $c_E(0) = -a_E(2)/2$ *denotes the constant term at infinity of* $\widehat{\mathfrak{Z}}_E$ *and* $c_{E,Q}(0)$ *denotes the constant term of* $\widehat{\mathfrak{Z}}_E|W_Q$.

Proof. We fix our Atkin-Lehner operators as $W_{p_\ell} = \left(\begin{smallmatrix} * & * \\ p_\ell & p_{\ell\pm1}\end{smallmatrix}\right)B_{p_\ell}$, where the first matrix, which we denote by γ_{p_ℓ}, is in $\Gamma_0(p_\ell)$, and $W_N = SB_N$. By Proposition 3.3.3 we obtain the following expansions of ch_{V^G} at the cusps of $\Gamma_0(N)$:

$$(\mathrm{ch}_{V^G}|S)(\tau) = \frac{1}{N}\sum_{m=0}^{\infty}\sum_{\substack{i,j\in\mathbb{Z}/N\mathbb{Z}\\ ij\equiv m\,(N)}}\dim W_{m/N}^{(i,j)}q^{m/N-1}$$
$$(\mathrm{ch}_{V^G}|\gamma_{p_\ell})(\tau) = \frac{1}{p_{\ell\pm1}}\sum_{m=0}^{\infty}\sum_{\substack{i,j\in\mathbb{Z}/p_{\ell\pm1}\mathbb{Z}\\ p_\ell^2ij\equiv m\,(N)}}\dim W_{m/N}^{(p_\ell i,p_\ell j)}q^{m/N-1}$$

Note that in the formula for $\mathrm{ch}_{V^G}|\gamma_{p_\ell}$ only those summands survive where m is divisible by p_ℓ.

Using Corollary 2.4.4, Proposition 6.1.2, and Lemma 6.2.4 we can express the unique (up to additive constants) harmonic Maass form in $H_0(N)$ having a pole of order $m = p_1^ap_2^bm'$, $\gcd(N,m')=1$, at ∞ and nowhere else as

$$\begin{aligned}&p_1p_2m(\widehat{\mathfrak{Z}}_E|T_{m'}|U_{p_1^{a+1}}|W_{p_1}|U_{p_2^{b+1}}|W_{p_2})(\tau)\\ &\quad= q^{-m}+\sigma(m')\big((p_1^{a+1}-1)(p_2^{b+1}-1)c_E(0)+(p_2^{b+1}-1)c_{E,p_1}(0)\\ &\qquad+(p_1^{a+1}-1)c_{E,p_2}(0)+c_{E,N}(0)\big)+O(\exp(-\alpha y))\end{aligned} \tag{6.2.6}$$

for some suitable $\alpha > 0$ and $c_E(0)$ and $c_{E,Q}(0)$ denoting the constant terms of $\widehat{\mathfrak{Z}}_E$ and $\widehat{\mathfrak{Z}}_E|W_Q$, respectively. For reference, we also give the following expansions which are needed in order to express ch_{VG} in terms of Weierstrass mock modular forms,

$$p_1p_2m(\widehat{\mathfrak{Z}}_E|T_{m'}|U_{p_1^{a+1}}|U_{p_2^{b+1}})(\tau) = p_1^{a+1}p_2^{b+1}\sigma(m')c_E(0) + O(\exp(-\alpha y)), \tag{6.2.7}$$

$$\begin{aligned} &p_1p_2m(\widehat{\mathfrak{Z}}_E|T_{m'}|U_{p_1^{a+1}}|U_{p_2^{b+1}}|W_{p_2})(\tau) \\ &\qquad = p_1^{a+1}\sigma(m')\left((p_2^{b+1}-1)c_E(0) + c_{E,p_2}(0)\right) + O(\exp(-\alpha y)), \end{aligned} \tag{6.2.8}$$

and

$$\begin{aligned} &p_1p_2m(\widehat{\mathfrak{Z}}_E|T_{m'}|U_{p_1^{a+1}}|W_{p_1}|U_{p_2^{b+1}})(\tau) \\ &\qquad = p_2^{b+1}\sigma(m')\left((p_1^{a+1}-1)c_E(0) + c_{E,p_1}(0)\right) + O(\exp(-\alpha y)). \end{aligned} \tag{6.2.9}$$

Hence we may write

$$\begin{aligned}
\mathrm{ch}_{V^G} = \widehat{\mathfrak{Z}}_E + \frac{1}{N}\sum_{m=0}^{N-1}\sum_{\substack{i,j\in\mathbb{Z}/N\mathbb{Z}\\ ij\equiv m\ (N)}} &\dim W_{m/N}^{(i,j)}(-p_2^{b_m+1}(-p_1^{a_m+1}((N-m)'\widehat{\mathfrak{Z}}_E|T_{(N-m)'})\\
&|U_{p_1^{a_m+1}}|W_{p_1})|U_{p_2^{b_m+1}}|W_{p_2})|W_N\\
+\frac{1}{p_2}\sum_{m=0}^{p_2-1}\sum_{\substack{i,j\in\mathbb{Z}/p_2\mathbb{Z}\\ p_1ij\equiv m\ (p_2)}} &\dim W_{m/p_2}^{(p_1i,p_1j)}(-p_2^{b_{p_1m}+1}(-p_1^{a_{p_1m}+1}((N-p_1m)'\widehat{\mathfrak{Z}}_E|T_{(N-p_1m)'})\\
&|U_{p_1^{a_{p_1m}+1}}|W_{p_1})|U_{p_2^{b_{p_1m}+1}}|W_{p_2})|W_{p_1}\\
+p_2\sum_{m=0}^{p_1-1}\sum_{\substack{i,j\in\mathbb{Z}/p_1\mathbb{Z}\\ p_2ij\equiv m\ (p_1)}} &\dim W_{m/p_1}^{(p_2i,p_2j)}(-p_2^{b_{p_2m}+1}(-p_1^{a_{p_2m}+1}((N-p_2m)'\widehat{\mathfrak{Z}}_E|T_{(N-p_2m)'})\\
&|U_{p_1^{a_{p_2m}+1}}|W_{p_1})|U_{p_2^{b_{p_2m}+1}}|W_{p_2})|W_{p_2}\\
+\dim V_1^G - C&
\end{aligned} \tag{6.2.10}$$

where we write $N-m = p_1^{a_m}p_2^{b_m}(N-m)'$ with $\gcd(N,(N-m)')=1$ and the constant C is defined by

$$\begin{aligned}
C = c_E(0) + \frac{1}{N}\sum_{m=1}^{N-1}\sum_{\substack{i,j\in\mathbb{Z}/N\mathbb{Z}\\ ij\equiv m\ (N)}} &\dim W_{m/N}^{(i,j)}p_1^{a_m+1}p_2^{b_m+1}\sigma((N-m)')c_E(0)\\
+\frac{1}{p_2}\sum_{m=0}^{p_2-1}\sum_{\substack{i,j\in\mathbb{Z}/p_2\mathbb{Z}\\ p_1ij\equiv m\ (p_2)}} &\dim W_{m/p_2}^{(p_1i,p_1j)}p_1^{a_{p_1m}+1}\sigma((N-p_1m)')\left[(p_2^{b_{p_1m}+1}-1)c_E(0)+c_{E,p_2}(0)\right]\\
+\frac{1}{p_1}\sum_{m=0}^{p_1-1}\sum_{\substack{i,j\in\mathbb{Z}/p_1\mathbb{Z}\\ p_2ij\equiv m\ (p_1)}} &\dim W_{m/p_1}^{(p_2i,p_2j)}p_2^{b_{p_2m}+1}\sigma((N-p_2m)')\left[(p_1^{a_{p_2m}+1}-1)c_E(0)+c_{E,p_1}(0)\right].
\end{aligned} \tag{6.2.11}$$

As in the proof of Theorem 6.2.1, we compute also the expansions of ch_{V^G} at all other cusps, which is fairly straightforward from the expression in (6.2.10) using once more the known commutation relations among Hecke and Atkin-Lehner operators in Proposition 2.2.2, so we refrain from giving these expansions explicitly for the sake of brevity.

By Proposition 3.3.3, we have that

$$\sum_{\mathfrak{a}} F_{\mathfrak{a}}(\tau) = \mathrm{ch}_{V^{\mathrm{orb}(g)}}(\tau) + \mathrm{ch}_{V^{\mathrm{orb}(g^{p_1})}}(\tau) + \mathrm{ch}_{V^{\mathrm{orb}(g^{p_2})}}(\tau) + \mathrm{ch}_{V}(\tau),$$

where the sum runs over a complete set of representatives of cusps of $\Gamma_0(N)$, which we may and do fix as $\{1/N(\equiv \infty), 1/p_1, 1/p_2, 1(\equiv 0)\}$. It is easy to see that the constant term of $F_{\mathfrak{a}}$ is precisely the constant term of $\mathrm{ch}_{V^G}|W_{N\mathfrak{a}}$ multiplied by the width of the cusp, which in this case is $N\mathfrak{a}$. Using this observation, we obtain the dimension formula stated in the theorem after some simplification steps. $\square$

Chapter 7

Moonshine modules and a question of Griess

In this chapter, we consider the situation in which a finite group acts on an infinite-dimensional graded module in such a way that the graded trace functions are weakly holomorphic modular forms. Under a mild hypothesis we completely describe the asymptotic module structure of the homogeneous subspaces. As a consequence we find that moonshine for a group gives rise to partial orderings on its irreducible representations. This serves as a first answer to a question posed by Griess. In particular, we show that our hypothesis holds for umbral moonshine and for automorphism groups of certain vertex operator algebras. The results in this section are joint work with Victor Manuel Aricheta and come from [AB19].

7.1 Asymptotic structure of homogeneous subspaces

Let (K_n) be a sequence of finite-dimensional representations of a finite group G, and suppose $c_g(n) := \mathrm{tr}(g|K_n) \in \mathbb{R}$ for all $g \in G$ and all n. We say that the sequence (K_n) has *dominant identity trace* if for every $g \in G$ that is not equal to the identity element e, we have $c_g(n) = o(c_e(n))$ as $n \to \infty$. Examples of such sequences are the sequences $(V_n^\natural)$ and $(K_n^\natural)$ (where $K^\natural$ is the Mathieu moonshine module).

The following theorem shows that if a sequence (K_n) has dominant identity trace, then the subspaces K_n tend to a multiple of the regular representation of G. Thus, Griess' question makes sense for these sequences.

Theorem 7.1.1. *Let e be the identity element in G, and let $M_1, \ldots, M_s$ be the irreducible representations of G. Write*

$$K_n = \mathrm{m}_1(n)M_1 \oplus \mathrm{m}_2(n)M_2 \oplus \cdots \oplus \mathrm{m}_s(n)M_s.$$

If (K_n) has dominant identity trace, then

$$\mathrm{m}_i(n) \sim \frac{1}{|G|} \dim K_n \dim M_i$$

as $n \to \infty$. Therefore,

$$\lim_{n\to\infty} \frac{\mathrm{m}_i(n)}{\sum_{j=1}^{s} \mathrm{m}_j(n)} = \frac{\dim M_i}{\sum_{j=1}^{s} \dim M_j}.$$

Proof. Let $\mathrm{m}_i(n)$ be the multiplicity of M_i in K_n so that

$$K_n = \mathrm{m}_1(n)M_1 \oplus \mathrm{m}_2(n)M_2 \oplus \cdots \oplus \mathrm{m}_s(n)M_s.$$

Let χ_i be the irreducible character of M_i. By the usual orthogonality of characters,

$$\mathrm{m}_i(n) = \frac{1}{|G|} \sum_{g\in G} \overline{\chi_i(g)} c_g(n). \tag{7.1.1}$$

Since (K_n) has dominant identity trace, we see that the formula for $\mathrm{m}_i(n)$ is dominated by the term corresponding to $g = e$, and we recover the asymptotic given in Theorem 7.1.1.

∎

7.2 Partial orders on irreducible representations

We find that moonshine for a group naturally equips its irreducible representations with partial orders.

Theorem 7.2.1. *Let (K_n) be a sequence of finite-dimensional representations of a finite group G and suppose $c_g(n) := tr(g|K_n) \in \mathbb{R}$ for all $g \in G$ and all n. Let (n_i) be a sequence of integers such that given $g \in G$, the signs $sgn(c_g(n_i))$ are independent of i. If (K_n) has dominant identity trace, then there exist G-modules $L_1, L_2, \ldots, L_t$ (depending on the signs $sgn(c_g(n_i))$) where*

- *L_1 is the regular representation of G, and*
- *the irreducible components of L_{i+1} form a subset of the irreducible components of L_i (for $1 \leq i < t$),*

such that

1. *for some nonnegative integer-valued functions $r_1(n_i), \ldots, r_t(n_i)$ and G-module $L_\epsilon(n_i)$ with bounded multiplicity functions, we have the decomposition*

$$K_{n_i} = r_1(n_i)L_1 \oplus r_2(n_i)L_2 \oplus \cdots \oplus r_t(n_i)L_t \oplus L_\epsilon(n_i),$$

2. *and the module $K_{n_i} \oplus (-r_1(n_i))L_1 \oplus \cdots \oplus (-r_l(n_i))L_l$ tends to a multiple of the representation L_{l+1} (for $1 \leq l \leq t-1$) as $i \to \infty$.*

In the course of proving Theorem 7.2.1, we have also obtained the asymptotics for the multiplicities of the irreducible components of the non-free parts of K_n. We record this in the following theorem. Here, the set $\mathcal{C}_2$ is the collection of conjugacy classes of G whose corresponding $c_g(n)$'s have the second fastest growth.

Theorem 7.2.2. *Let $M_1, \ldots, M_s$ be the irreducible representations of a finite group G and let $\chi_1, \ldots, \chi_s$ be their respective characters. Let (K_n) be a sequence of G-modules* *that has dominant identity trace. Suppose* $c_g(n) := tr(g|K_n) \in \mathbb{R}$ *for all* $g \in G$ *and all* n*. Denote by* K'_n *the non-free part of* K_n*, and write*

$$K'_n = \mathrm{m}'_1(n)M_1 \oplus \mathrm{m}'_2(n)M_2 \oplus \cdots \oplus \mathrm{m}'_s(n)M_s.$$

Suppose that (n_j) *is a sequence of integers such that given* $g \in \mathcal{C}_2$*, the signs* $sgn(c_g(n_j))$ *are independent of* j*. Then as* $j \to \infty$

$$\mathrm{m}'_i(n_j) \sim \frac{1}{|G|} \sum_{[g]\in\mathcal{C}_2} |[g]| f'_i(g) c_g(n_j).$$

Here

$$f'_i(g) := \overline{\chi_i(g)} - \frac{\dim M_i}{\dim M_{j'}} \overline{\chi_{j'}(g)}$$

where j' *is a* j *that minimizes*

$$\frac{\sum_{g\in\mathcal{C}_2} |[g]| \overline{\chi_j(g)} \mathrm{sgn}(c_g(n))}{\dim M_j}$$

as $n \to \infty$*.*

Let $\mathcal{C}$ be the set of conjugacy classes of G. We define a partial ordering on the elements of $\mathcal{C}$ as follows: $[h] < [g]$ if and only if $c_h(n) = o(c_g(n))$ as $n \to \infty$, and $[g] = [h]$ if and only if $c_g(n) \sim kc_h(n)$ for some non-zero constant k as $n \to \infty$. In other words, we order the elements of $\mathcal{C}$ by increasing order of growth of the corresponding $c_g(n)$'s.

Let L_1 be the regular representation of G and decompose K_n into the direct sum of representations

$$K_n = r_1(n)L_1 \oplus K_n^{(1)} \tag{7.2.1}$$

where $r_1(n)$ is a nonnegative integer which is as large as possible. (Thus $K_n^{(1)}$ is the non-free part of K_n). Then by definition,

$$\begin{aligned} r_1(n) &:= \min_{1\le j\le s}\left\{\left\lfloor \frac{\mathrm{m}_j(n)}{\dim M_j}\right\rfloor\right\} \\ &= \min_{1\le j\le s}\left\{\left\lfloor \frac{1}{|G|}\left(c_e(n) + \sum_{[g]\in\mathcal{C}, g\neq e} \frac{|[g]|\overline{\chi_j(g)}c_g(n)}{\dim M_j}\right)\right\rfloor\right\} \end{aligned} \tag{7.2.2}$$

Let

$$\mathcal{C}_2 = \{[g]\in\mathcal{C} : [h]\le[g]<[e] \text{ for all } h\in G\setminus\{e\}\}. \tag{7.2.3}$$

That is, $\mathcal{C}_2$ is the collection of conjugacy classes in G whose corresponding $c_g(n)$'s has the second fastest growth rate (with $c_e(n)$ having the fastest growth by assumption). As $n\to\infty$, the dominant terms in the sum in (7.2.2) are the terms corresponding to the conjugacy classes in $\mathcal{C}_2$. Therefore when n is sufficiently large, we may find $r_1(n)$ by finding a j which minimizes

$$\frac{\sum_{[g]\in\mathcal{C}_2} |[g]|\overline{\chi_j(g)}c_g(n)}{\dim M_j},$$

or equivalently, a j which minimizes

$$\frac{\sum_{[g]\in\mathcal{C}_2} |[g]|\overline{\chi_j(g)}\mathrm{sgn}(c_g(n))}{\dim M_j}.$$

Let j_1 be one such j, which exists since the signs are independent of n, so that

$$j_1\in J_1 := \left\{ j\in\{1,\dots,s\} : j \text{ minimizes } \left\lfloor\frac{\mathrm{m}_j(n)}{\dim M_j}\right\rfloor \text{ for } n \text{ sufficiently large}\right\}.$$

Thus, if we write $K_n^{(1)}$ in terms of irreducible representations of G, say

$$K_n^{(1)} = \mathrm{m}_1^{(1)}(n)M_1\oplus\cdots\oplus\mathrm{m}_s^{(1)}(n)M_s,$$

then the multiplicity functions for the non-free part have the following exact formula:

$$\mathrm{m}_i^{(1)}(n) = \mathrm{m}_i(n) - \left\lfloor \frac{\mathrm{m}_{j_1}(n)}{\dim M_{j_1}} \right\rfloor \dim M_i. \tag{7.2.4}$$

Using (7.1.1), we have

$$\mathrm{m}_i^{(1)}(n) \sim \frac{1}{|G|} \sum_{[g]\in\mathcal{C}} |[g]| f_i^{(1)}(g) c_g(n)$$

as $n \to \infty$, where

$$f_i^{(1)}(g) := \left(\overline{\chi_i(g)} - \frac{\dim M_i}{\dim M_{j_1}} \overline{\chi_{j_1}(g)} \right).$$

Note that $f_i^{(1)}(e) = 0$ so that the dominant terms in the asymptotic formula for $\mathrm{m}_i^{(1)}(n)$ are the terms corresponding to the conjugacy classes in $\mathcal{C}_2$. More precisely,

$$\mathrm{m}_i^{(1)}(n) \sim \frac{1}{|G|} \sum_{[g]\in\mathcal{C}_2} |[g]| f_i^{(1)}(g) c_g(n)$$

and this proves Theorem 7.2.2.

Note that when $i \in J_1$, as $n \to \infty$ we have

$$\left\lfloor \frac{\mathrm{m}_{j_1}(n)}{\dim M_{j_1}} \right\rfloor = \left\lfloor \frac{\mathrm{m}_i(n)}{\dim M_i} \right\rfloor$$

so that the multiplicity function (7.2.4) is equal to

$$\mathrm{m}_i^{(1)}(n) = \mathrm{m}_i(n) - \left\lfloor \frac{\mathrm{m}_i(n)}{\dim M_i} \right\rfloor \dim M_i \leq \dim M_i.$$

Hence $K_n^{(1)}$ tends to a multiple of a representation L_2 whose irreducible components do not contain M_i for $i \in J_1$.

Similar to (7.2.1), we can then write

$$K_n^{(1)} = r_2(n)L_2 \oplus K_n^{(2)}$$

where $r_2(n)$ is a nonnegative integer that is as large as possible. Explicitly, if we write

$$L_2 = \bigoplus_{i \notin J_1} \ell_2(i) M_i$$

then we have

$$r_2(n) := \min_{j \notin J_1} \left\{ \left\lfloor \frac{\mathrm{m}_j^{(1)}(n)}{\ell_2(j)} \right\rfloor \right\}.$$

One may take

$$\ell_2(j) = \left\lfloor \sum_{[g] \in \mathcal{C}_2} |[g]| f_j^{(1)}(g) \mathrm{sgn}(c_g(n)) \right\rfloor.$$

We can repeat the arguments as before to show that there exist a set J_3 and a representation L_3 whose irreducible components do not contain M_i for $i \in J_1 \cup J_2$, such that $K_n^{(2)}$ tends to a multiple of L_3.

Proceeding inductively, this gives us the decomposition

$$K_n = r_1(n)L_1 \oplus r_2(n)L_2 \oplus \cdots \oplus r_t(n)L_t \oplus L_\epsilon(n),$$

where the L_j are expressed in terms of fewer and fewer irreducible representations of G, and where $L_\epsilon(n)$ is a representation of G with bounded multiplicity functions. This proves Theorem 7.2.1. $\square$

7.3 Application to umbral moonshine

The umbral moonshine conjecture, proven in [DGO15b], states that for a Niemeier root system X and setting $m = m^X$ where m^X is the Coxeter number of any simple component of X, there is a naturally defined bi-graded infinite-dimensional representation of G^X

$$K^X = \bigoplus_{r \in I^X} \bigoplus_{\substack{D \in \mathbb{Z}, D \leq 0 \\ D = r^2 \pmod{m}}} K^X_{r,-D/4m}$$

such that the graded trace functions

$$H^X_{g,r}(\tau) = -2q^{-1/4m}\delta_{r,1} + \sum_{\substack{D \in \mathbb{Z}, D \leq 0 \\ D = r^2 \pmod{m}}} tr(g|K^X_{r,-D/4m})q^{-D/4m}$$

for $r \in I^X$ are components of *vector-valued* mock modular forms H^X_g. Here I^X depends on the types of components in the root system [DGO15b].

Convenient expressions for mock modular forms can be found using Rademacher sums which are essentially regularized Poincaré series. The theory of Rademacher sums, dating back to the 1930s and originally establishing a conditionally convergent expression for the normalized modular j-invariant, has been generalized to apply to modular and mock modular forms of various weights and various subgroups of $SL_2(\mathbb{R})$ [Rad39, Nie74, DF11, CD14]. Cheng and Duncan considered Niebur's method for constructing Rademacher sums in weight 1/2 and verified that it produces the functions appearing in Mathieu moonshine [CD12].

As an example, we examine the case of Mathieu moonshine. Following Cheng and Duncan [CD14], let n_g be the order of g, let ϵ be the multiplier system for the Dedekind η function, and let ρ_g be a character specified by the minimal cycle length in the cycle shape of g. Then the mock modular form $H_g^{(M_{24})}$ is defined as:

$$\begin{aligned}H_g^{(M_{24})}(\tau) &= -2R^{[-1/8]}_{\Gamma_0(n_g),\rho_g\epsilon^{-3},1/2}(\tau)\\ &= -2q^{-1/8} + 2\sum_{n>0} c_{\Gamma_0(n_g),\rho_g\epsilon^{-3},\frac{1}{2}} q^{n-1/8}.\end{aligned}$$

The coefficients can be expressed in terms of Kloosterman sums weighted by Bessel functions which have known asymptotics for which our theory applies.

As an explicit example, we use Theorem 7.2.2 to obtain the asymptotics of the multiplicities of the non-free part of the Mathieu moonshine module, which we record here as a corollary.

Corollary 7.3.1. *Let $K^\natural = \bigoplus K_n^\natural$ be the Mathieu moonshine module, and let K_n' be the non-free part of $K_n^\natural$. Let $M_1^{(M_{24})}, \ldots, M_{26}^{(M_{24})}$ be the irreducible representations of M_{24}, and let $\chi_1, \ldots, \chi_{26}$ be their respective characters. Write*

$$K_n' = m_1'(n)M_1^{(M_{24})} \oplus m_2'(n)M_2^{(M_{24})} \oplus \cdots \oplus m_{26}'(n)M_{26}^{(M_{24})}.$$

Then

$$\begin{aligned}m_i'(n) \sim (-1)^{n+1}\frac{\sqrt{2}\,e^{\frac{\pi}{4}\sqrt{8n-1}}}{\sqrt{8n-1}}\Bigg(&\frac{|2A|}{|M_{24}|}\left(\chi_i(2A) - \frac{\chi_j(2A)}{\dim M_j^{(M_{24})}}\dim M_i^{(M_{24})}\right)\\ &-\frac{|2B|}{|M_{24}|}\left(\chi_i(2B) - \frac{\chi_j(2B)}{\dim M_j^{(M_{24})}}\dim M_i^{(M_{24})}\right)\Bigg)\end{aligned}$$

as $n \to \infty$, where $j = 1$ if n is even and $j = 2$ if n is odd.

Proof. From the Rademacher sum formulation, we find that the nth coefficient of $H_g^{(M_{24})}$ is

$$c_g(n) = -4\pi \sum_{\substack{c>0\\ 0\le d<c\\ (c,d)=1}} e\left(n\frac{d}{c} - \frac{3s(d,c)}{2}\right) e\left(-\frac{cd}{n_g h_g}\right)\frac{1}{c(8n-1)^{\frac{1}{4}}} I_{\frac{1}{2}}\left(\frac{\pi}{2c}(8n-1)^{\frac{1}{2}}\right).$$

From the asymptotics of the I-Bessel function

$$I_{\frac{1}{2}}(x) \sim \frac{e^x}{\sqrt{2\pi x}},$$

and upon isolating the dominant term of this sum, we get

$$c_g(n) = \mathrm{sgn}(c_g(n)) \frac{4}{n_g^{1/2}\sqrt{8n-1}} \exp\left(\frac{\pi\sqrt{8n-1}}{2n_g}\right) + o\left(\exp\left(\frac{\pi\sqrt{8n-1}}{2n_g}\right)\right).$$

Thus $\mathcal{C}_2 = \{2A, 2B\}$. Also, from the Rademacher sum formulation, we find that $\mathrm{sgn}(c_{2A}(n)) = (-1)^n$ and $\mathrm{sgn}(c_{2B}(n)) = (-1)^{n+1}$. Thus, from the character table of M_{24}, the j that minimizes

$$\frac{\sum_{g\in\mathcal{C}_2} |[g]|\chi_j(g)\mathrm{sgn}(c_g(n))}{\dim M_j^{(M_{24})}}$$

is $j = 1$ when n is even, and $j = 2$ when n is odd. (Note that $\chi_j(2A)$ and $\chi_j(2B)$ are real-valued for any j.) Thus Theorem 7.2.2 in this case becomes:

$$\begin{aligned} \mathrm{m}_i'(n) \sim (-1)^{n+1} \frac{2\sqrt{2}\, e^{\frac{\pi}{4}\sqrt{8n-1}}}{\sqrt{8n-1}} \Bigg(\frac{|2A|}{|M_{24}|} \left(\chi_i(2A) - \frac{\chi_j(2A)}{\dim M_j^{(M_{24})}} \dim M_i^{(M_{24})} \right) \\ - \frac{|2B|}{|M_{24}|} \left(\chi_i(2B) - \frac{\chi_j(2B)}{\dim M_j^{(M_{24})}} \dim M_i^{(M_{24})} \right) \Bigg). \end{aligned}$$

□

Similar to the Mathieu moonshine case, the vector-valued mock modular forms appearing in umbral moonshine, denoted $H_g^{(\ell)}$ for lambency ℓ, can also be conveniently expressed in terms of *vector-valued* Rademacher sums according to

$$H_g^{(\ell)} = -2R_{\Gamma_0(n_g),\psi^{(\ell)}\rho_g^{(\ell)},1/2}$$

which have similar asymptotics to the usual Rademacher sums [CD14, DGO15b].

7.4 Asymptotic regularity for vertex operator algebras

In this section, we consider vertex operator algebras. We refer to Chapter 3 and the references there for basic definitions in the theory of vertex operator algebras. Let $V = \bigoplus V_n$ be a vertex operator algebra and let G be any finite group of automorphisms of V. We show that if V and G satisfy certain natural conditions (to be enumerated shortly), then the sequence (V_n) of G-modules has dominant identity trace. Consequently, Theorem 7.1.1 applies to such a sequence of modules.

The natural conditions for V that we will assume are the following: holomorphic, C_2-cofinite, and of CFT type. By Theorem 2 of [DLM00], if g is an automorphism of V of finite order, then V possesses a unique g-twisted sector (up to equivalence), which is denoted $V(g)$. We shall assume another condition for V and G concerning the conformal weights of the twisted sectors; we assume that the conformal weight $\rho(V)$ of the untwisted sector V is strictly less than the conformal weights $\rho(V(g))$ of the other twisted sectors $V(g)$ for $g \in G \setminus \{e\}$. This minimality condition is conjectured to always hold when V is a holomorphic, C_2-cofinite vertex operator algebra of CFT type and when G is a finite group of automorphisms of V (cf. Conjectures 1.1 and 2.2 in [Möl18]). We have the following proposition.

Theorem 7.4.1. *Let $V = \bigoplus V_n$ be a holomorphic, C_2-cofinite vertex operator algebra of CFT type. Let G be a finite group of automorphisms of V. Let $g \in G$ and denote by $V(g)$ the unique (up to equivalence) g-twisted sector of V. If the conformal weights satisfy $\rho(V(g)) > \rho(V)$ for all $g \neq e$, then the sequence (V_n) of G-modules has dominant identity trace. Consequently, V_n tends to a multiple of the regular* *representation as $n \to \infty$.*

There are examples for which the assumptions in Theorem 1.6 are known to hold. For instance, if W is a holomorphic and C_2-cofinite vertex operator algebra, and if $k \in \mathbb{Z}$, then our assumptions hold for $V = W^{\otimes k}$ and $G \leq S_k \leq \mathrm{Aut}(V)$ (cf. Proposition 4.1 of [Möl18]). Other examples come from lattice vertex operator algebras. Given an even, unimodular, and positive-definite lattice L, our assumptions also hold for $V = V_L$ (the lattice vertex operator algebra associated to L) and $G \leq \mathrm{Aut}(V_L)$ (cf. Proposition 4.2 of [Möl18]). We refer the reader to Section 4.2 of *loc. cit.* for more on the structure of $\mathrm{Aut}(V_L)$. In particular, for these examples of V and G, the sequence (V_n) of G-modules has dominant identity trace, and thus, Theorem 7.1.1 applies.

Proof of Theorem 7.4.1. Let $V := V(e)$, and write $V(g)$ as follows:

$$V(g) = \bigoplus_{n=0}^{\infty} V(g)_{\frac{n}{\mathrm{ord}(g)}+\rho(V(g))}.$$

If $h \in G$ commutes with g, then h induces an action on $V(g)$ (which is well-defined up to a scalar factor). If c is the central charge of V, then the twisted trace functions are the following power series:

$$Z(g,h;\tau) := \sum_{n=0}^{\infty} \mathrm{tr}(h \mid V(g)_{\frac{n}{\mathrm{ord}(g)}+\rho(V(g))})q^{\frac{n}{\mathrm{ord}(g)}+\rho(V(g))-\frac{c}{24}}. \tag{7.4.1}$$

In particular, for all $h \in G$, the coefficients of $Z(e,h;\tau)$ encode the traces of h on the homogenous subspaces of V. We denote by $c_h(n)$ these coefficients, i.e.,

$$Z(e,h;\tau) = \sum_{n=0}^{\infty} c_h(n)q^{n+\rho(V)-\frac{c}{24}}.$$

To prove the proposition, we need to show that $c_h(n) = o(c_e(n))$ for $h \neq e$ as $n \to \infty$. Now, since V is holomorphic and C_2-cofinite, we know from [DLM00] that $Z(g,h;\tau)$

is holomorphic in $\mathbb{H}$, and moreover, if $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in SL_2(\mathbb{Z})$, then

$$Z(g, h; \gamma\tau) = \sigma(g, h, \gamma) Z(g^a h^c, g^b h^d; \tau) \tag{7.4.2}$$

for some constant $\sigma(g, h, \gamma)$. Furthermore, because V is of CFT type and self-dual (since it is holomorphic), the invariance subgroups of these graded trace functions are congruence subgroups [DR18].

Let $h \neq e$. Choose $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in SL_2(\mathbb{Z})$ such that c is not a multiple of the order of h. By (7.4.2), the expansion of $Z(e, h; \tau)$ at the cusp $\gamma\infty$ is $\sigma(e, h, \gamma)Z(h^c, h^d; \tau)$. Since $h^c \neq e$, we have $\rho(V(h^c)) > \rho(V)$ by our assumption on conformal weights. So by (7.4.1), the order of the pole of $Z(e, e; \tau)$ at $\tau = \gamma\infty$ is strictly greater than the order of the pole of $Z(e, h; \tau)$ at $\tau = \gamma\infty$. At any other cusp s, the order of the pole of $Z(e, e; \tau)$ at $\tau = s$ is greater than or equal to the order of the pole of $Z(e, h; \tau)$ at $\tau = s$.

We will use this comparison of the orders of poles at the cusps together with the fact that these twisted trace functions are modular functions on congruence subgroups to compare the asymptotic growths of the coefficients. That is, since the trace functions are modular they can be expressed in terms of Rademacher sums and their coefficients can be expressed as Rademacher series [DF11]. In particular, from the asymptotic expressions in Section 4.2 of [DF11] we can read the growth of $c_h(n)$ at each of the different cusps. This allows us to see how the growth of the coefficients depends on the order of the pole, and we find that the Bessel function (cf. (4.2.1) of [DF11]) is largest when the order of the pole is largest. Thus we have that $c_h(n) = o(c_e(n))$ and so the sequence (V_n) of G-modules has dominant identity trace and Theorem 7.1.1 applies.

□

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