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a thesis submitted to the Faculty of Emory College of Arts and Sciences
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Abstract

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In the beginning of the nineteenth century, mathematicians became interested in algebraic surfaces in projective spaces. In particular, the remarkable result of Arthur Cayley and George Salmon in 1849 reveals that there are exactly 27 lines on every smooth cubic surface in the projective 3-space over complex numbers. In Chapter 1, we will work out an example of the Fermat surface. In Chapters 2 and 3, we will elaborate on the algebraic geometry machinery, namely fibers, dimensions, Grassmannians, and special classes of morphisms. In Chapter 4, we will prove the theorem of Cayley and Salmon in the context of modern algebraic geometry, inheriting Salmon's construction of the ‘’incidence variety’’ in his original proof. Meanwhile, we show that there exists a line on every cubic surface in the projective 3-space over complex numbers.

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Contents

Chapter 1

Introduction

In the nineteenth century, mathematicians became interested in studying algebraic surfaces in projective spaces, in particular $\mathbb{P}^3_k$. In the classical sense, a cubic surface is the set of zeroes of a cubic homogeneous polynomial in $\mathbb{P}^3$:

$$S = \{(x : y : z : w) \in \mathbb{P}^3_k | f(x : y : z : w) = 0\}.$$

The passion in the study of cubic surfaces, however, grew enormously after English mathematician Arthur Cayley and George Salmon made the following discovery:

Theorem 1.0.1. *Every smooth cubic surface in $\mathbb{P}^3_{\mathbb{C}}$ has exactly 27 lines.*

Cayley started a correspondence with Salmon after he discovered that the lines on cubic surfaces are finite. Salmon quickly replied with proof that the number should be 27 [5]. Some praise this result for marking the beginning of modern algebraic geometry [1]. Indeed, this phenomenon is distinct to cubic surfaces - hyperplanes and quadrics have infinitely many lines on them, whereas a general surface of degree at least 4 has none [section 2 in [10]]. In addition, the wonder of this statement lies not only in its precision and generality, but also in its honesty - all 27 lines are distinct without concerns to count "correctly" regarding multiplicities. In terms of consequences, this statement connects with many interesting mathematics topics, including classical constructions such as the Pascal's Mystical Hexagon Theorem, as well as themes in modern algebraic geometry like deformation theory and intersection theory, and even in string theory [3]. In this thesis, we expose Professor Ravi Vakil's proof in the language of modern algebraic geometry, as in Chapter 27 of his Magnus Opus *Foundation of Algebraic Geometry: The Rising Sea* [9] (we refer to this book as "Vakil" in the rest of this thesis). The first half of the proof will inherit the incidence variety construction 4 in Salmon's original proof in his letter to Cayley [5], resulting in a non-trivial statement for *all* cubic surfaces:

Theorem 1.0.2. *Every cubic surface has at least one line.*

The remainder of the proof consists of a sequence of beautiful, condensed arguments relying heavily on profound algebraic geometry theorems (or on their consequences). Indeed, the author had the pleasure of learning a great deal of algebraic geometry in the process of studying the proof. We will sometimes refer to other classical texts including Professor Robin Hartshorne's *Algebraic Geometry* [2] ("Hartshorne"), Professor Hideyuki Matsumura's *Commutative Ring Theory* [4] ("Matsumura"), Professor Igor Shafarevich's *Basic Algebraic Geometry* [7] ("Shafarevich"), Professor George Salmon's *Lessons Introductory to the Modern Higher Algebra* [6], and articles in Stacks Project [8].

We will be solely working over the algebraically closed field of complex numbers of characteristic 0, $k = \mathbb{C}$. Every ring will be commutative and unital. Unless otherwise stated, we may assume every scheme is integral of finite type over k whenever necessary.

1.1 The Fermat Surface

Before jumping into the sophistication of scheme theory, we might as well check the validity of the statement using the *Fermat cubic surface* in $\mathbb{P}^3_{\mathbb{C}}$ defined by $x^3 + y^3 +$

$z^3 + w^3 = 0$, an irreducible homogeneous polynomial of degree 3 in 4 variables. We define a *hyperplane* in $\mathbb{P}^3_{\mathbb{C}}$ to be the locus of a linear equation, and a *line* in $\mathbb{P}^3_{\mathbb{C}}$ to be the intersection of two distinct hyperplanes. Equivalently, it is then the common zero set of two linear equations that are not scalar multiples of each other. Take any such line which is the intersection of hyperplanes $H_1 : a_1x + a_2y + a_3z + a_4w = 0$ and $H_2 : b_1x + b_2y + b_3z + b_4w = 0$. Thus, we obtain that the matrix

$$\begin{bmatrix} a_1 & a_2 & a_3 & a_4 \\ b_1 & b_2 & b_3 & b_4 \end{bmatrix}$$

of rank 2. Without loss of generality, we may assume that the minor

$$\begin{bmatrix} a_1 & a_2 \\ b_1 & b_2 \end{bmatrix}$$

has rank 2. Thus a few elementary row operations give the row-echelon form which we view as the coefficient matrix of two new hyperplanes whose intersection coincides with the old ones:

$$\begin{bmatrix} 1 & 0 & a'_3 & a'_4 \\ 0 & 1 & b'_3 & b'_4 \end{bmatrix}$$

Substituting $x = -a'_3z - a'_4w = Az + Bw$ and $y = -b'_3z - b'_4w = Cz + Dw$ into the Fermat surface we obtain

$$(Az + Bw)^3 + (Cz + Dw)^3 + z^3 + w^3 = 0$$

which vanish identically for all choices of z, w in $\mathbb{C}$. Further equating coefficients gives us

$$A^3 + C^3 + 1 = B^3 + D^3 + 1 = A^2B + C^2D = AB^2 + CD^2 = 0,$$

which is a necessary and sufficient condition for the line to lie on the Fermat surface.

Claim 1.1.1. *The complex numbers A, B, C, D satisfying the relations*

$$\begin{cases} A^3 + C^3 + 1 = 0 & (1) \\ B^3 + D^3 + 1 = 0 & (2) \\ A^2B + C^2D = 0 & (3) \\ AB^2 + CD^2 = 0 & (4) \end{cases} \tag{1.1}$$

cannot be all nonzero.

Proof. Proceed by contradiction and assume that A, B, C, D are all nonzero. From (3) and (4) we get $A^2B = -C^2D$ and $AB^2 = -CD^2$. Multiplying these two equations on both sides we obtain $(AB)^3 = (CD)^3$. Taking cubic roots, we see that $AB = \omega CD$ for some cubic root of unity ω. Substituting this identity into (3), we get $AB(A+\omega^{-1}C) = 0$. But by our hypothesis, $AB \neq 0$, so $A = -\omega^{-1}C$. Yet substituting $A = -\omega^{-1}C$ back to (1) gets $1 = 0$, a contradiction. Therefore, A, B, C, D cannot be all nonzero. □

This allows us to solve the above system of equations by substituting variables with zeroes. Without loss of generality, let $A = 0$. Thus $C = \omega$ must be a cubic root of -1. But then by (3), $D = 0$. This forces $B = \omega'$ to be also a cubic root of -1. Therefore, $(A, B, C, D) = (0, \omega, \omega', 0)$ renders 9 solutions as ω and ω' have 3 choices respectively. A similar argument starting with the assumption $C = 0$ gives the rest 9 solutions of the form $(A, B, C, D) = (\omega, 0, 0, \omega')$ where ω and ω' are cubic roots of -1.

Having solved all possibilities for the quadruples A, B, C, D, we look back to count the number of lines. Indeed, the 4 by 2 matrix representation for a line will have a row echelon reduced form consisting of a 2 by 2 minor being the identity matrix, and the other two columns constituting a matrix with two (not necessarily distinct) cubic roots of unity and two zeros. In other words, each row will have exactly a 1, a cubic root of unity, and two zeroes, whereas no two nonzero values exist in the same column. This demonstrates that all lines fall into one of the following 3 cases:

$$\begin{cases} x = \omega \cdot y \\ z = \omega' \cdot w \end{cases} \tag{1.2}$$

$$\begin{cases} x = \omega \cdot z \\ y = \omega' \cdot w \end{cases} \tag{1.3}$$

$$\begin{cases} x = \omega \cdot w \\ y = \omega' \cdot z \end{cases} \tag{1.4}$$

where ω and ω' denote a cubic root of unity. To see this, we consider

$$\begin{bmatrix} 1 & 0 & \omega & 0 \\ 0 & 1 & 0 & \omega' \end{bmatrix}$$

which belongs to the case (1.3). Indeed, the positions of the two nonzero elements in each row indicate which two variables of x, y, z, w should be related by a factor of a cubic root of unity! In conclusion, as each case has $3^2 = 9$ possible choices for ω and ω', the total number of lines is exactly 27. Interestingly, the line count on the Fermat surface not only serves as an example, but is also an indispensable ingredient of the proof of the Main Theorem 1.0.1, so we record it for reference.

Theorem 1.1.2. *The Fermat surface* $x^3 + y^3 + z^3 + w^3 = 0$ *in* $\mathbb{P}^3$ *has exactly* 27 *lines.*

The above example demonstrates that our inherently geometric statement will be formulated and resolved in algebraic settings.

Chapter 2

Preliminaries

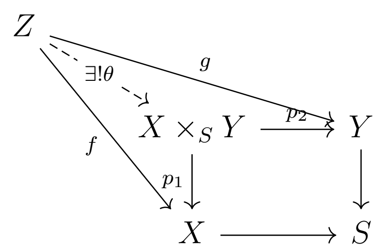
We assume the first definitions and properties of schemes. All rings will be commutative with an identity. The field of complex numbers $k = \mathbb{C}$ is infinite, algebraically closed, and has characteristic 0. A k-variety is an integral, separated scheme of finite type over k. As we solely consider k-varieties, smoothness of a scheme X can be defined as the local property such that the stalk at every point $x \in X$ is a regular local ring - a Noetherian local ring whose minimal generating set of its maximal ideal has the size of its Krull dimension [Vakil 13.2.7 [9]].

We adapt Vakil's terminology for hypersurfaces. A closed subscheme of $\mathbb{P}^n_k$ cut out by a single nonzero homogeneous polynomial equation is called a *hypersurface* in $\mathbb{P}^n_k$. The *degree of a hypersurface* is the degree of the polynomial. In our context, $\mathbb{P}^3_k = \text{Proj}(k[x, y, z, w])$ in which the closed points are quadruples of elements in k modulo scalar multiple relations. A cubic surface in $\mathbb{P}^3_k$ is a closed subscheme determined by a homogeneous cubic polynomial in 4 variables. Indeed, there are 20 monomials in 4 variables of degree 3. Also, two polynomials cut out the same subscheme in $\mathbb{P}^n_k$ if and only if they are scalar multiples of each other. Therefore, we parametrize all smooth cubic surfaces in $\mathbb{P}^3_k$ by $\mathbb{P}^{19}_k$. The remainder of this chapter introduces fibers and Grassmannians as well as their important properties, which would be useful to the proof of the Main Theorem.

2.1 Fibers

We begin with the notion of fiber products.

Definition 2.1.1. *If X, Y are both schemes over a scheme S, then the* fiber product *of X and Y over S, denoted $X \times_S Y$, is a scheme equipped with* projection morphisms *$p_1 : X \times_S Y \to X$ and $p_2 : X \times_S Y \to Y$ which form a commutative diagram with the given morphisms $X \to S$ and $Y \to S$, and such that if Z is any scheme with morphism $f : Z \to X$ and $g : Z \to Y$ which also form a commutative diagram with morphisms $X \to S$ and $Y \to S$, then there exists a unique morphism $\theta : Z \to X \times_S Y$ such that $f = p_1 \circ \theta$ and $g = p_2 \circ \theta$. In circumstances when $S = \mathrm{Spec}(\mathbb{Z})$, S is omitted.*



Observe that the uniqueness of fiber products, if they ever exist, is an immediate formal consequence of their universal property. Yet, if we start with the fiber product $X \times_S Y = \mathrm{Spec}(A \otimes_R B)$ when all $X = \mathrm{Spec}(A), Y = \mathrm{Spec}(B), S = \mathrm{Spec}(R)$ are affine, then we could construct a fiber product for arbitrary schemes by gluing the fiber products of their affine open subsets.

Theorem 2.1.2 (StackProject Lemma 26.17.4 [8])**.** *Let $f : X \to S$ and $g : Y \to S$ be schemes over a scheme S. Let $X \times_S Y$, with projections p and q, be their fiber product. Suppose that $U \subseteq S$, $V \subseteq X$, $W \subseteq Y$ are open subschemes such that $f(V) \subset U$ and $g(W) \subset U$. Then the canonical morphism $V \times_U W \longrightarrow X \times_S Y$ is an open immersion that identifies $V \times_U W$ with $p^{-1}(V) \cap q^{-1}(W) \subseteq X \times_S Y$.*

Proof. We only need to check that $p^{-1}(V) \cap q^{-1}(W)$ satisfies the universal property of $V \times_U W$. $\square$

This fact immediately provides us with an affine covering of $X \times_S Y$.

Corollary 2.1.3 (StackProject Lemma 26.17.4 [8]). *Let $f : X \to S$ and $g : Y \to S$ be morphisms of schemes over a scheme S. Let $S = \bigcup U_i$ be any affine open covering of S. For each $i \in I$, let $f^{-1}(U_i) = \bigcup_{j \in J_i} V_j$ be an affine open covering of $f^{-1}(U_i)$, and let $g^{-1}(U_i) = \bigcup_{k \in K_i} W_k$ be an affine open covering of $g^{-1}(U_i)$. Then $X \times_S Y = \bigcup_{i \in I} \bigcup_{j \in J_i,\, k \in K_i} V_j \times_{U_i} W_k$ is an affine open covering of $X \times_S Y$.*

Using fiber products, we can define the notion of *projective n-space* over a scheme Y, generalizing $\mathbb{P}^n_k$ projective spaces over a field.

Definition 2.1.4. *If Y is any scheme, the projective n-space over Y, denoted $\mathbb{P}^n_Y$, is the fiber product $\mathbb{P}^n_{\mathbb{Z}} \times_{\mathrm{Spec}(\mathbb{Z})} Y$.*

In the case $Y = \mathrm{Spec}(k)$, on the affine covers $D(x_i)$ of $\mathbb{P}^n_{\mathbb{Z}}$, we see that $\mathbb{P}^n_{\mathbb{Z}}|_{D(x_i)} \times Y \cong \mathrm{Spec}(\mathbb{Z}[\frac{x_0}{x_i}, \dots, \frac{x_n}{x_i}]) \times \mathrm{Spec}(k) = \mathrm{Spec}(k[\frac{x_0}{x_i}, \dots, \frac{x_n}{x_i}]) \cong \mathbb{P}^n_k|_{D(x_i)}$, so globally $\mathbb{P}^n_Y \cong \mathbb{P}^n_k$ and this definition agrees with the old Proj construction.

We can also define the *fiber* of a morphism at a point.

Definition 2.1.5 (Hartshorne P89 [2]). *Let $f : X \to Y$ be a morphism of schemes, and let $y \in Y$ be a point. Let $k(y)$ be the residue field of y, and let $\mathrm{Spec}(k(y)) \to Y$ be the natural morphism. Then we define the* fiber *of the morphism f over the point y to be the scheme*

$$X_y = X \times_Y \mathrm{Spec}(k(y)).$$

The fiber X_y is a scheme over $k(y)$, and its underlying topological space is homeomorphic to $f^{-1}(y)$ with induced topology. In our context, k is an algebraically closed field and $k(y)$ is an algebraic extension of k, so conveniently $k = k(y)$.

2.2 Dimension

Definition 2.2.1. *The dimension of a topological space X, $\dim X$, is the supremum of the lengths of chains of irreducible closed subsets in X, starting the index with 0.*

Definition 2.2.2. *The* Krull dimension *of a ring A, is the supremum of the lengths of chains of prime ideals in A, starting the index with 0. The* height *of a prime ideal $\mathfrak{p}$, $\mathrm{ht}(\mathfrak{p})$, is the supremum of the lengths of chains of prime ideals ending at $\mathfrak{p}$, starting the index at 0.*

Definition 2.2.3 (Hartshorne P86 [2])**.** *The dimension of a scheme X, denoted $\dim X$, is its dimension as a topological space. If Z is an irreducible closed subset of X then the* codimension *of Z in X, denoted $\mathrm{codim}(Z, X)$, is the supremum of integers n such that there exists a chain*

$$Z = Z_0 < Z_1 < \cdots < Z_n$$

of distinct closed irreducible subsets of X, beginning with Z. If Y is any closed subset of X, we define

$$\mathrm{codim}(Y, X) = \inf_{Z \subseteq Y} \mathrm{codim}(Z, X)$$

where the infimum is taken over all closed irreducible subsets of Y.

We write $\dim X$ and $\dim A$ for topological dimension of a space X and Krull dimension of a ring A, when the context is clear. They agree when X is affine, as the closed irreducible subsets in $\mathrm{Spec}(A)$ have an inclusion-reversing one-to-one correspondence with prime ideals in A.

Example 2.2.4. $\dim \mathrm{Spec}(A) = \dim A$ *for every ring A.*

Example 2.2.5. *If X has discrete topology,* $\dim X = 0$.

For an open subset U of a topological space X, point-set topology tells us that there's a bijection between the irreducible closed subsets of U and the irreducible closed subsets of X intersecting U. Thus we obtain the useful identity:

Proposition 2.2.6. *For any open cover $\{U_\alpha\}$ of X,*

$$\dim X = \sup_\alpha \{\dim U_\alpha\}.$$

Proposition 2.2.7 (Hartshorne II.3.20 [2])**.** *Let X be an integral scheme of finite type over a field k. Then*

1. *For any closed point $x \in X$, $\dim X = \dim \mathcal{O}_{x,X}$.*
2. $\dim X = \text{tr.deg.}(K(X)/k)$.
3. *If Y is a closed subset of X, then $\operatorname{codim}(Y, X) = \inf\{\dim \mathcal{O}_{P,X} | P \in Y\}$.*
4. *If Y is a closed subset of X, then $\dim Y + \operatorname{codim}(Y, X) = \dim X$.*
5. *If U is a non-empty open subset of X, then $\dim U = \dim X$.*
6. *If $k \subset k'$ is a field extension, then every irreducible component of $X' = X \times_k k'$ has dimension $\dim X$.*

The following reputed Noether Normalization Lemma in commutative algebra will show that the dimension of k-varieties are additive over fiber products.

Theorem 2.2.8 (**Noether Normalization Lemma**, Vakil 12.2.4 [9])**.** *Suppose A is an integral domain, finitely generated over a field k. If $\text{tr.deg.}(K(A)/k) = n$, then there are elements $x_1, \ldots, x_n \in A$, algebraically independent over k, such that A is a finite extension of the ring $k[x_1, \ldots, x_n]$.*

Proposition 2.2.9. *Let X, Y be non-empty integral schemes locally of finite type over a field k. Then*

$$\dim X \times_k Y = \dim X + \dim Y.$$

Proof. If $(U_i)_{i\in I}$ and $(V_j)_{j\in J}$ be open affine coverings of X and Y respectively, then 2.2.7 says that $\{U_i \times_k V_j\}_{i,j}$ is an open affine covering of $X \times_k Y$. But for any open sets $U \subseteq X$ and $V \subseteq Y$, $\dim U = \dim X$ and $\dim V = \dim Y$. Thus, by 2.2.6 we may reduce to the case where both X and Y are affine. Let $m := \dim X$ and $n := \dim Y$. Then the Noether Normalization Lemma gives finite injective homomorphism $k[T_1, \ldots T_m] \to \Gamma(X, \mathcal{O}_X)$ and $k[T_{m+1}, \ldots, T_{m+n}] \to \Gamma(Y, \mathcal{O}_Y)$. Taking the tensor product of two maps gives again a finite and injective homomorphism from $k[T_1, \ldots, T_{m+n}]$ to $\Gamma(X \times_k Y, \mathcal{O}_{X\times_k Y})$. But a finite, injective ring map between integral domains is integral. Therefore, by the Going-Up Theorem and the Going-Down Theorem,

$$\dim X \times_k Y = \dim k[T_1, \ldots, T_{m+n}] = m + n = \dim X + \dim Y.$$

□

Corollary 2.2.10. *Let K be a field extension of k. Let X be an integral scheme locally of finite type over a field k. Then* $\dim X = \dim X \times_k K$.

In the previous section, we have defined the fiber of a morphism at a point. A crucial inequality in our proof of 1.0.1 regarding the dimension of fibers of a morphism is listed as Exercise 3.22 in Hartshorne [2]. We offer a solution here. Firstly, we need a commutative algebra result from Matsumura's *Commutative Ring Theory* [4]:

Theorem 2.2.11 (Matsumura Theorem 15.1 [4])**.** *Let $\varphi : A \to B$ be a homomorphism of Noetherian rings, $\mathfrak{q}$ a prime ideal in B, and $\mathfrak{p} = \varphi^{-1}(\mathfrak{q})$. Then*

$$\mathrm{ht}(\mathfrak{q}) \leq \mathrm{ht}(\mathfrak{p}) + \dim B_{\mathfrak{q}}/\mathfrak{p}B_{\mathfrak{q}}.$$

Theorem 2.2.12. *Let $f : X \to Y$ be a morphism of integral scheme of finite type over a field k. Then the following holds.*

1. *Let Y' be a closed irreducible subset of Y, whose generic point η' is contained*

in $f(X)$*. If* Z *is any irreducible component of* $f^{-1}(Y')$*, then* $\eta' \in f(Z)$ *and* $\operatorname{codim}(Z, X) \leq \operatorname{codim}(Y', Y)$*.*

2. *Let* $e = \dim X - \dim Y$ *be the* relative dimension *of* X *over* Y*. For any point* $y \in f(X)$*, every irreducible component of the fiber* X_y *has dimension at least* e*.*

Proof. (1). Let ξ be the unique generic point of Z. Observe that $f(\xi) = \eta'$. Indeed, by continuity of f we have $f(Z) = f(\overline{\xi}) \subseteq \overline{f(\xi)}$. Hence, $Y' = \overline{f(Z)} \subseteq \overline{f(\xi)}$ and therefore $f(\xi)$ must be η'. Now, by 2.2.7 we know that $\operatorname{codim}(Z, X) = \dim \mathcal{O}_{\xi,X}$ and, similarly, $\operatorname{codim}(Y', Y) = \dim \mathcal{O}_{\eta',Y}$. Therefore, the statement is local in nature and thus reduces to the case where X, Y are both affine. Hence, assume $X = \operatorname{Spec}(B), Y = \operatorname{Spec}(A)$, $Y' = V(\mathfrak{p})$, and f is a morphism of affine schemes induced the ring map $\varphi : A \to B$. Then, $f^{-1}(Y') = V(\mathfrak{p}B)$ and $Z = V(\mathfrak{q})$ for some prime ideal $\mathfrak{q} \subseteq B$. Next, by 2.2.11 we see that it remains to prove that $\dim B_{\mathfrak{q}}/\mathfrak{p}B_{\mathfrak{q}}$ is 0. Indeed, by the assumption Z is the maximal irreducible subset in $f^{-1}(Y') = V(\mathfrak{p}B)$, so $\mathfrak{q}$ is the minimal prime in $V(\mathfrak{p}B)$. It follows that no other primes are strictly between $\mathfrak{p}B_{\mathfrak{q}}$ and $\mathfrak{q}B_{\mathfrak{q}}$, completing the proof.

(2). Let $Y' = \overline{\{y\}}$ and $W \subseteq X_y$ be an irreducible component of X_y.

Claim 2.2.13. *The closure of* W *in* X *is an irreducible component of* $f^{-1}(Y')$*.*

Proof. First, observe that Y' is a closed set containing W, so $\overline{W}$ is an irreducible closed subset of Y'. Now, assume $\overline{W} \subset T \subset f^{-1}(Y')$ for some irreducible component T of $f^{-1}(Y')$, then by the same argument as in (a), the unique generic point ξ of T maps to the unique generic point y of Y' under f. Thus $\xi \in X_y$. But then the closure of ξ in X_y, an irreducible set, would contain W. By the fact that W is an irreducible component, $\xi \in W$ and $T = \overline{\{\xi\}} \subseteq \overline{W} \subseteq T$, so $\overline{W} = T$. □

Thus by (1), we obtain that $\operatorname{codim}(\overline{W}, X) \leq \operatorname{codim}(Y', Y)$. Furthermore, by 2.2.7

(2) and additivity of transcendence degree over towers $K(\overline{W})/k(y)/k$, we notice that

$$\dim(\overline{W}) = \text{tr.deg.}(K(\overline{W})/k) = \text{tr.deg.}(K(\overline{W})/k(y)) + \text{tr.deg.}(k(y)/k) = \dim W + \dim Y'.$$

These combined with 2.2.7 (4) gives

$$\dim(W) = \dim X - \dim Y + \text{codim}(Y', Y) - \text{codim}(\overline{W}, X) \geq \dim X - \dim Y = e.$$

□

Although the original statement in Hartshorne assumes f to be a dominant morphism, it's actually not necessary for part (1) and (2) to suit our purposes. When X, Y are both varieties over k, part (2) in the above theorem immediately yields the fiber inequality of the utmost importance in the proof of the main theorem.

Corollary 2.2.14. *Let X, Y be varieties over a field k, and let $f : X \to Y$ be a dominant morphism. Then for any point $x \in X$, $y = f(x)$, we have*

$$\dim X - \dim Y \leq \dim f^{-1}(y).$$

2.3 Grassmannians

For an n-dimensional vector space V over a field k with a basis $f_1 \dots f_n$, the *projectivization* of V, $\mathbb{P}(V)$, is the set of one-dimensional subspaces of V. If we introduce coordinates $\xi_1, \dots, \xi_n$ for V, then an element in $\mathbb{P}(V)$ could be identified with *homogenous coordinates* $(\xi_1 : \cdots : \xi_n)$ as a point in $\mathbb{P}^{n-1}$. Analogous to $\mathbb{P}(V)$, which parametrizes the 1-dimensional subspaces of V, we would want to parametrize all its r-dimensional subspaces. So we define the *Grassmannian*, denoted $\text{Gr}(r, V)$, to be the collection of r-dimensional subspaces of V. For an r-dimensional subspace W, choose a basis $\{w_1 \dots w_r\}$. An element $x \in \bigwedge^r V$ is *decomposable* if $x = v_1 \wedge \cdots \wedge v_r$ for some

$v_i \in V$, i.e. it can be written as a wedge product of r vectors of V. The set of wedge products $\{f_{i_1} \wedge \cdots \wedge f_{i_r}\}_{1 \leq i_1 < \cdots < i_r \leq n}$ form a basis of cardinality $\binom{n}{r}$ for the k-vector space $\bigwedge^r V$. But then $\bigwedge^r W$ is a 1-dimensional subspace by the inclusion $W \hookrightarrow V$ for whom the singleton $\{w_1 \wedge \cdots \wedge w_r\}$ is a basis. Therefore, $\bigwedge^r W$ corresponds to a point in the projectivization $\mathbb{P}(\bigwedge^r V)$. Conversely, let $[x]$ be a point in $\mathbb{P}(\bigwedge^r V)$ such that $x = v_1 \wedge \cdots \wedge v_r$ is a decomposable, nonzero element of $\bigwedge^r V$. Then, $\{v_1 \dots v_r\}$ is a set of r-linearly independent vectors and whose span is a r-dimensional subspace of V. This establishes a one-to-one correspondence between $\mathrm{Gr}(r, V)$ and points in $\mathbb{P}(\bigwedge^r V)$ with decomposable representations in $\bigwedge^r V$. In fact, if we identify $\mathbb{P}(\bigwedge^r V)$ with $\mathbb{P}^{\binom{n}{r}-1}$ using the homogeneous coordinates given by the basis $\{f_{i_1} \wedge \cdots \wedge f_{i_r}\}_{1 \leq i_1 < \cdots < i_r \leq n}$, we obtain the *Plücker embedding* of $\mathrm{Gr}(r, V)$ into $\mathbb{P}^{\binom{n}{r}-1}$. The fact that the image of Plücker embedding is closed certifies the Grassmannian $\mathrm{Gr}(r, V)$ to be a projective variety [7]. In the context when the vector space is k^n, we would write $\mathrm{Gr}(r, n)$.

For our purposes, we would focus on the Grassmannian $\mathrm{Gr}(2, 4)$. A useful interpretation of Grassmannian $\mathrm{Gr}(2, 4)$ is through its natural open affine cover of $\mathbb{A}^4_k$'s. Just as in the example of the Fermat surface, with a choice of basis a 2-dimensional subspace in k^4 could be expressed as a 2×4 matrix such that at least one submatrix consisting of columns i, j has rank 2. Let $U_{i,j}$ be the set of all such subspaces. After row operations the 2×2 minor consisting of columns i, j becomes the identity matrix, and the other 4 entries could be any element of k. This gives an isomorphism from $U_{i,j}$ to $\mathbb{A}^4_k$. It also shows that $\mathrm{Gr}(2, 4)$ is smooth as $\mathbb{A}^4_k$ is smooth, and smoothness is a local property. Lastly, we notice that $\mathrm{Gr}(2, 4)$ actually parametrizes projective lines in $\mathbb{P}^3$, as a projective line in $\mathbb{P}^3$ and $\mathbb{P}^3$ itself are respectively 2-dimensional subspaces in k^4 and k^4 modulo the equivalence relations among nonzero scalar multiples. In that regard, we write $\mathbb{G}(1, 3) := \mathrm{Gr}(2, 4)$ and view it as the collection of projective lines in $\mathbb{P}^3$. We record the above facts of $\mathrm{Gr}(2, 4)$ for future reference.

Proposition 2.3.1. $\mathbb{G}(1, 3)$ *is a smooth, irreducible projective variety with a finite* *open affine cover* $\{U_{i,j}\}_{1\leq i<j\leq 4}$*, each isomorphic to* $\mathbb{A}^4$*.*

Chapter 3

Morphisms

Our main goal in this chapter is to define the notion of projective, finite, flat, and smooth morphisms, which respectively characterize the morphisms involved in the proof of the Main Theorem 1.0.1, as well as to state some theorems about the connections between these properties. They are built on profound algebro-geometric facts like the Grothendieck Coherence Theorem and the Fundamental Theorem of Elimination Theory, as well as numerous non-trivial theorems commutative algebra. These results will lead to a highly condensed and crucial part of the proof of the Main Theorem. It reflects a central philosophy of modern algebraic geometry: absolute concepts of spaces are replaced by relative notions via morphisms between them, revealing deep structural insights.

3.1 Projective and finite fibers implies finite

There are many definitions for projective morphism, we adapt the choice in Hartshorne for our purposes. We may assume our schemes X, Y are both integral and of finite type over k.

Definition 3.1.1 (Hartshorne P150 [2]). *A morphism of schemes $f : X \to Y$ is* *projective if it factors into a closed immersion $i : X \to \mathbb{P}^n_Y$ for some n, followed by the projection $\mathbb{P}^n_Y \to Y$.*

Definition 3.1.2. *A morphism of schemes $f : X \to Y$ is affine if for every affine open set U of Y, $f^{-1}(U)$ is an affine open set in X.*

If a ring homomorphism $\varphi : B \to A$ makes A into a finite B-module, then A is called a *finite B-algebra*. This is stronger than A being a *finitely generated B-algebra* because each element of A can be written as a linear combination of a finite generating set.

Definition 3.1.3. *A morphism of schemes $f : X \to Y$ is finite if for every affine open set $U = \mathrm{Spec}(B)$ of Y, $f^{-1}(U) = \mathrm{Spec}(A)$ for a finite B-algebra A.*

Example 3.1.4. *If L/K is a field extension, then the map $f : \mathrm{Spec}(L) \to \mathrm{Spec}(K)$ is always affine, and it's finite if the extension L/K is finite.*

Affineness and finiteness are properties of a morphism satisfied for every affine open set in Y. It suffices to check these properties on any affine open cover of Y [Vakil Proposition 8.3.4 [9]]. We call a map of topological spaces sending closed sets to closed sets to be *closed*.

Theorem 3.1.5 (**The Fundamental Theorem of Elimination Theory**, Theorem 8.4.10 in Vakil [9])**.** *The morphism $\mathbb{P}^n_A \to \mathrm{Spec}(A)$ is closed.*

Lemma 3.1.6. *Given a finite collection of points in $\mathbb{P}^n_k$, there is a hypersurface H that avoids all of them.*

Proof. Proceed by induction on the number of points. Given one point, we can choose any hyperplane that avoids it. Assume the proposition is true for any collection of points of size m, and we would like to avoid $\{p_1, \ldots, p_{m+1}\}$. For each i, $1 \leq i \leq m+1$, choose a hypersurface f_i that avoids all the m points except p_i. If any f_i also avoids p_i, we are done; otherwise, $f = \sum_{1 \le i \le m+1} f_1 \dots \hat{f}_i \dots f_{m+1}$ is a hypersurface that avoids all points. □

We will also need the following fact, which is a special case of the Grothendieck Coherence Theorem.

Theorem 3.1.7 (**The Grothendieck Coherence Theorem, special case**, Vakil Theorem 18.9.1 [9]). *If $f : X \to Y$ is a projective morphism of locally Noetherian schemes, then the pushforward of a coherent sheaf on X is a coherent sheaf on Y.*

Immediately, it gives that an affine projective morphism is finite.

Corollary 3.1.8. *A morphism of schemes $f : X \to Y$ is finite if it is projective and affine.*

Proof. Suppose f is projective and affine. By the special case of the Grothendieck Coherence Theorem 3.1.7, $f_*(\mathcal{O}_X)$ is a coherent $\mathcal{O}_Y$-module. Therefore, the preimage of any affine open set Spec(A) in Y is the spectrum of a finitely generated A-module. □

Finally, we are ready to prove the main result of this section, adapting Vakil's elegant geometric proof of Theorem 18.1.6 in [9].

Theorem 3.1.9 (Vakil Theorem 18.1.6 [9]). *Let $f : X \to Y$ be a projective morphism of integral schemes of finite type over k and all its fibers have finite cardinality. Then f is finite.*

Proof. As finiteness can be checked on any affine open cover, it suffices to show that at each point q in Y, there is an open set V containing q such that the restriction $f^{-1}(V) \to V$ is finite. Let $U = \text{Spec}(A)$ be an affine open set containing q. Hence, we let $Y = U$ and $X = f^{-1}(U)$. By projectivity of f, we can factor f as $X \xrightarrow{i} \mathbb{P}^n_A \xrightarrow{p}$ Spec(A) where i is a closed embedding. By the finite fiber assumption, $f^{-1}(q)$ is a finite set. Now look at the base extension

$$\begin{array}{ccccc} \mathbb{P}^n_{k(q)} & \longrightarrow & \mathbb{P}^n_Y & \longrightarrow & \mathbb{P}^n_{\mathbb{Z}} \\ \downarrow & & \downarrow & & \downarrow \\ \mathrm{Spec}(k(q)) & \longrightarrow & Y & \longrightarrow & \mathrm{Spec}(\mathbb{Z}) \end{array}$$

We notice that $f^{-1}(q) \subseteq \mathbb{P}^n_{k(q)} \subseteq \mathbb{P}^n_Y$ is a finite set of points in $\mathbb{P}^n_{k(q)}$ - the fiber of the map $\mathbb{P}^n_Y \to Y$ over q. Thus, we can apply Lemma 3.1.6 to find a hypersurface H_0 in $\mathbb{P}^n_{k(q)}$ that avoids points in $f^{-1}(q)$. Lift the coefficient of the polynomial defining H_0 from $A_{\mathfrak{q}}/\mathfrak{q}A_{\mathfrak{q}}$ to A, then we obtain a hypersurface H in $\mathbb{P}^n_A$ avoiding $f^{-1}(q)$. Now $H \cap X$ is closed in $\mathbb{P}^n_A$ and by the Fundamental Theorem of Elimination Theory 3.1.5, the image $H' = f(H \cap X)$ is closed in $Y = \mathrm{Spec}(A)$ and doesn't contain q by choice of H. Hence, $V = \mathrm{Spec}(A) \backslash H'$ is an open set containing q. Let V' be an affine open set in V containing q. It suffices to prove that the map $f^{-1}(V') \to \mathbb{P}^n_{V'} - H \to V'$ is finite. By Corollary 3.1.8, it suffices to prove that it's projective and affine. But since the $i(f^{-1}(V')) \cap H = \varnothing$, the map is the same as $f^{-1}(V') \to \mathbb{P}^n_{V'} \to V'$. Meanwhile, $\mathbb{P}^n_{V'} - H$ is a distinguished open subset in $\mathbb{P}^n_{V'}$, which is affine. The first map is a closed embedding, which is affine. Therefore, the composition of the two affine maps is affine. This completes the proof. □

3.2 Flat morphisms

The second half of the chapter delves into the notion of flatness. We recall that if M is an A-module, then the functor $M \otimes_A (\cdot)$ is always right-exact. We say that M is *flat over* A, if $M \otimes_A (\cdot)$ is also left-exact. A ring homomorphism $B \to A$ is flat if A is flat as a B-module. Equivalently by definition, M is a flat A-module if tensoring by M preserves every injective A-module homomorphism.

Example 3.2.1. *Free A-modules are flat. Indeed, if $M \to N$ is an injection, then so is $A^{\oplus I} \otimes_A M = M^{\oplus I} \to N^{\oplus I} = A^{\otimes I} \otimes_A N$.*

Example 3.2.2. *For a multiplicative set $S \subset A$, the ring map $A \to S^{-1}A$ is flat.*

Indeed, as localization is exact, the injectivity of $M \to N$ *implies the injectivity of* $S^{-1}B \otimes_B M = S^{-1}M \to S^{-1}N = S^{-1}B \otimes_B N$.

Remark 3.2.3. *If* M *is a flat* A*-module,* $x \in A$ *a non-zerodivisor, then multiplication by* x *is an injective* A*-module endomoprhism of* M*. Indeed, this could be seen by tensoring* M *with the injective map* $A \xrightarrow{\times x} A$*. This tells us that flat modules are torsion-free.*

In addition, if we're working with finitely generated modules over a principal ideal domain, then the class of flat modules is exactly the class of free modules.

Theorem 3.2.4. *If* A *is a PID,* M *is a finitely generated* A*-module, then the following are equivalent:*

1. M *is flat.*

2. M *is free.*

3. M *is torsion-free.*

Proof. Free modules are clearly flat, using the fact that direct sum of modules distributes over tensor product. Conversely, using the Structure Theorem for Finitely Generated Modules over PID, we obtain that

$$M \cong A^{\oplus r} \oplus \bigoplus_{1 \leq i \leq s} A/(a_i),$$

where $a_i \in A$ are nonzero. But then as M is flat, every one of its summands must be flat. This means that $s = 0$ because $A/(a_i)$ cannot be flat (tensor it with $A \xrightarrow{\times a_i} A$), thus M is free. This also gives the equivalence of flatness and torsion-freeness. □

Flatness has even more good properties - it is transitive and is preserved under base extension and localization.

Theorem 3.2.5 (Theorems in Chapter 7 of Matsumura [4]). *Let M be an A-module, then the following holds.*

1. *(Criterion) M is flat if and only if for every finitely generated ideal $\mathfrak{a} \subseteq A$, the map $\mathfrak{a} \otimes M \to M$ is injective.*

2. *(Base extension) If M is flat over A, $A \to B$ a ring map, then $M \otimes_A B$ is a flat B-module.*

3. *(Transitivity) If B is a flat A-algebra, N a flat B-module, then N is also a flat A-module.*

4. *(Localization) M is flat if and only if $M_\mathfrak{p}$ is flat over $A_\mathfrak{p}$ for every $\mathfrak{p} \in \operatorname{Spec}(A)$.*

Definition 3.2.6. *Let $f : X \to Y$ be a morphism of schemes, and let $\mathscr{F}$ be an $\mathcal{O}_X$-module. We say $\mathscr{F}$ is* flat over Y *at $x \in X$ if $\mathscr{F}_x$ is flat as an $\mathcal{O}_{f(x),Y}$-module via the natural map $f^\# : \mathcal{O}_{f(x),Y} \to \mathcal{O}_{x,X}$. We say $\mathscr{F}$ is flat over Y if $\mathscr{F}$ is flat over Y at every point of X. We say X is flat over Y, or the morphism f is flat, if $\mathcal{O}_X$ is flat over Y.*

Example 3.2.7. *An open immersion is flat.*

The definition of flatness in morphisms of schemes is indeed a natural extension of flatness in modules. It's transitive and is preserved under base change, which is an immediate consequence of the respective facts of flatness in modules.

Theorem 3.2.8 (Theorem III.9.2 in Hartshorne [2]).

1. *Let $A \to B$ be a ring homomorphism, and let M be a B-module. Let $f : X = \operatorname{Spec}(B) \to Y = \operatorname{Spec}(A)$. Let $\mathscr{F} = \tilde{M}$ the quasicoherent sheaf on $\operatorname{Spec}(B)$ associated to M. Then $\mathscr{F}$ is flat over Y if and only if M is flat over A.*

2. *(Base Change) Let $f : X \to Y$ be a morphism, let $\mathscr{F}$ be an $\mathcal{O}_x$-module which his flat over Y, and let $g : Y' \to Y$ be any morphism. Let $X' = X \times_Y Y'$, let*

$f' : X' \to Y'$ be the second projection map, and let $\mathscr{F}' = p_1^*(\mathscr{F})$. Then $\mathscr{F}$ is flat over Y'.

3. *(Transitivity) Let $f : X \to Y$ and $g : Y \to Z$ be morphisms. Let $\mathscr{F}$ be an $\mathcal{O}_x$-module flat over Y, and also Y is flat over Z. Then $\mathscr{F}$ is flat over Z.*

4. *Let X be a noetherian scheme, and $\mathscr{F}$ a coherent $\mathcal{O}_x$-module. Then $\mathscr{F}$ is flat over X if and only if it is locally free.*

3.3 Smooth morphisms

We have defined the smoothness of a scheme X to be the local property such that the stalk at every point $x \in X$ is a regular local ring [Vakil 13.2.7 [9]].

The smoothness of morphisms is another concept. In differential geometry, a differentiable map of manifolds $f : M \to N$ is smooth of relative dimension n if locally the map looks like the projection $U \cong V \times \mathbb{R}^n \to V \subseteq N$. Motivated, the smoothness of morphisms in algebraic geometry is defined in an analogous fashion. We quote Vakil's formulation here.

Definition 3.3.1 (Definition 13.6.2 in Vakil [9]). *A morphism $f : X \to Y$ is* smooth of relative dimension n *if there exist an open cover $\{U_i\}_I$ of X and an affine open cover $\{V_i\}_I$ of Y such that for every i, $f(U_i) \subseteq V_i$ and the following holds. Let $V_i = \operatorname{Spec}(B)$. There exists an $r_i \in \mathbb{N}$, polynomials $f_1, \ldots, f_{r_i} \in B[x_1, \ldots, x_{n+r_i}]$, an open set W in $\operatorname{Spec}(B[x_1, \ldots, x_{n+r_i}]/(f_1, \ldots, f_{r_i}))$ so that the following diagram commutes:*

$$\begin{array}{ccccc} U_i & \xrightarrow{\sim} & W & \xhookrightarrow{open} & \operatorname{Spec} B[x_1, \ldots, x_{n+r_i}]/(f_1, \ldots, f_{r_i}) \\ \downarrow{\scriptstyle \pi|_{U_i}} & & \downarrow{\scriptstyle \rho|_W} & \nearrow{\scriptstyle \rho} & \\ V_i & \xrightarrow{\sim} & \operatorname{Spec} B & & \end{array}$$

where ρ is induced by the natural map $B \to B[x_1, \ldots, x_{n+r_i}]/(f_1, \ldots, f_{r_i})$. In addition, *the Jacobian matrix of the f_i's with respect to the first r variables x_i's*

$$\det\left(\frac{\partial f_j}{\partial x_i}\right)_{i,j\leq r_i}$$

is a non-vanishing function on W. A morphism f is smooth at a point $x \in X$ if there are open sets $U \subseteq X$, $V \subseteq Y$ such that $x \in U$, $f(U) \subseteq V$, and the above condition holds. We say a morphism is smooth *if it's smooth of relative dimension n for some n.*

One may observe that smoothness doesn't depend on global properties of X, Y like separatedness, but is only of local nature on the source. By definition, the locus of any morphism of schemes where it's smooth is an open set.

Example 3.3.2. *By definition, open immersions are étale, and for any ring B, the map $\mathbb{A}^n_B \to \operatorname{Spec}(B)$ is smooth of relative dimension n. For a scheme Y, the projection map $\mathbb{A}^n \times Y \to Y$ is readily seen to be smooth of relative dimension n.*

Just like flatness, smoothness is a good property that defines a reasonable class of morphisms which behaves well under base extension and composition.

Theorem 3.3.3 (Proposition III.10.1 in Hartshorne [2])**.** *Smoothness is preserved under base extension and composition. In particular, if $f : X \to Y$ is smooth of relative dimension m, and $g : Y \to Z$ is smooth of relative dimension n, then $g \circ f$ is smooth of relative dimension $m + n$.*

In fact, smoothness is stronger than flatness.

Theorem 3.3.4 (Theorem 24.8.7 in Vakil [9])**.** *Smooth morphisms are flat.*

The following theorem gives a condition for a morphism of smooth k-varieties to be smooth.

Theorem 3.3.5 (Theorem 23.1 in Matsumura [4]). *If $f : X \to Y$ is a morphism of smooth k-varieties, and all fibers of f are smooth of dimension $e = \dim X - \dim Y$, then f is smooth.*

The non-smooth points of a hypersurface in a projective space are precisely those on which the its Jacobian vanishes, giving a convenient criterion to check smoothness.

Theorem 3.3.6 (Theorem 13.3.B in Vakil [9]). *The non-smooth points of the hypersurface $f = 0$ in $\mathbb{P}^n_k$ correspond to the locus*

$$f = \partial f/\partial x_0 = \partial f/\partial x_1 = \cdots = \partial f/\partial x_n = 0.$$

Chapter 4

Via Dolorosa: The Proof

Now we have developed enough machinery to prove the Main Theorem 1.0.1. We identified the collection of cubic surfaces in $\mathbb{P}^3$ to be $\mathbb{P}^{19}$, and the collection of lines in $\mathbb{P}^3$ to be $\mathbb{G}(1,3)$. We are interested in the information about which lines lie on which cubic surfaces, drawing our attention to the following "incidence variety".

$$\begin{array}{ccccc} & & Z \xrightarrow{\ i\ } \mathbb{P}^{19} \times \mathbb{G}(1,3) & & \\ & \swarrow^{p_2} & & \searrow^{p_1} & \\ \mathbb{P}^{19} & & & & \mathbb{G}(1,3) \end{array}$$

In the above diagram, $Z = \{(X, \ell) : \ell \subseteq X\}$ is a closed subvariety of the fiber product $\mathbb{P}^{19} \times_k \mathbb{G}(1,3)$ encoding the information of incidences of lines and cubic surfaces. The closed points of the scheme $\mathbb{P}^{19} \times_k \mathbb{G}(1,3)$ are in one-to-one correspondence to the Cartesian product of the closed points in $\mathbb{P}^{19}_k$ and $\mathbb{G}(1,3)$ as schemes (which are exactly the parametrized cubic surfaces and lines). Firstly, we would examine the projection map p_1 to $\mathbb{G}(1,3)$ to obtain useful information about Z. Next, we will prove that $p_2 \circ i$ is surjective, leading to the non-trivial fact that every cubic surface contains a line. Finally, we further examine the properties of $p_2 \circ i$ and its fibers, after which the main theorem follows.

By Proposition 2.3.1, $\mathbb{G}(1,3)$ has a cover of 6 affine open sets $U_{i,j}$ for $1 \leq i < j \leq 4$, each isomorphic to $\mathbb{A}^4$. We claim that their preimages under $p_1 \circ i$ in Z are each isomorphic to $\mathbb{P}^{15} \times U_{i,j}$. This fact gives the dimension and smoothness of Z.

Proposition 4.0.1. $\dim Z = 19$.

Proof. Recall in 2.3.1 we showed that $\mathbb{G}(1,3)$ has an open cover of 6 open sets of the form $U_{i,j}$. For $1 \leq i < j \leq 4$, $U_{i,j}$ contains all projective lines whose 2×4 matrix representation has a non-singular 2×2 minor consisting of the i^{th} and j^{th} columns. Without loss of generality, let ℓ be a line in $U_{1,2}$, and could be written as

$$\begin{cases} x = a_3 z + a_4 w \\ y = b_3 z + b_4 w \end{cases} \tag{4.1}$$

Let $X \supseteq \ell$ be a cubic surface defined by a homogeneous cubic polynomial $f(x,y,z,w) = \sum_{i+j+k+\ell=3} \lambda_{i,j,k,\ell} x^i y^j z^k w^\ell = 0$. By substituting Equation 4.1 into f we obtain that $\tilde{f}(z,w) = \sum_{i+j+k+\ell=3} \lambda_{i,j,k,\ell} (a_3 z + a_4 w)^i (b_3 z + b_4 w)^j z^k w^\ell = 0$ for all $z, w \in k$. Therefore, we could expand $\tilde{f}$ in terms of z, w and obtain that the four coefficients - expressions in terms of $\lambda_{i,j,k,\ell}$'s and a_3, a_4, b_3, b_4 - are all zero. Specifically, we get

$$\begin{cases} f_{3,0}(\lambda_{i,j,k,\ell}, a_3, a_4, b_3, b_4)_{i+j+k+\ell=3} = 0 \\ f_{2,1}(\lambda_{i,j,k,\ell}, a_3, a_4, b_3, b_4)_{i+j+k+\ell=3} = 0 \\ f_{1,2}(\lambda_{i,j,k,\ell}, a_3, a_4, b_3, b_4)_{i+j+k+\ell=3} = 0 \\ f_{0,3}(\lambda_{i,j,k,\ell}, a_3, a_4, b_3, b_4)_{i+j+k+\ell=3} = 0 \end{cases} \tag{4.2}$$

where $f_{a,b}(\lambda_{i,j,k,\ell}, a_3, a_4, b_3, b_4)_{i+j+k+\ell=3}$ stands for the coefficient of the term $z^a w^b$. All of the four functions $f_{a,b}$ are linear in each $\lambda_{i,j,k,\ell}$. Therefore, the solution space of Equation 4.2 is isomorphic to $\mathbb{P}^{15}$. But then $Z \cap p_1^{-1}(U_{1,2}) = U_{1,2} \times \mathbb{P}^{15}$, and thus has dimension $15 + 4 = 19$ by the dimension formula 2.2.9. Notice that $\{Z \cap$

$p_1^{-1}(U_{i,j})\}_{1\leq i<j\leq 4}$ is an open cover of Z, and finally by 2.2.6 we conclude that $\dim Z = 19$. ☐

The above proof gives an open cover $\{Z \cap p_1^{-1}(U_{i,j})\}_{1\leq i<j\leq 4}$ which yields important properties of Z.

Corollary 4.0.2. *Z is a smooth and integral scheme.*

Proof. Firstly, we notice that Z is smooth because it has an open cover consisting of copies of the smooth scheme $\mathbb{P}^{15} \times \mathbb{A}^4$. For the same reason, Z is reduced. As the members of the open cover are irreducible and have non-empty pairwise intersections, Z is also irreducible. Therefore, Z is integral. ☐

Given our knowledge of Z, we are ready to investigate the map $p_2 \circ i$. The fibers of any smooth cubic surface $X \in \mathbb{P}^{19}$ under $p_2 \circ i$ in Z are exactly the lines on X. Therefore, the Main Theorem 1.0.1 is translated to the following statement:

Proposition 4.0.3. *All fibers of smooth cubic curves under $p_2 \circ i$ have exactly 27 elements.*

Firstly, we show that all such fibers of cubic surfaces are nonempty through the next claim.

Claim 4.0.4. *The map $p_2 \circ i$ is surjective.*

Proof. We proceed by proof by contradiction and assume that $p_2 \circ i$ is not surjective. But then, $p_2(Z)$ is a closed, irreducible subscheme of $\mathbb{P}^{19}$. Now, the assumption that $p_2(Z) \subsetneq \mathbb{P}^{19}$ implies $\dim p_2(Z) < \dim \mathbb{P}^{19} = 19$. But we have concluded that $\dim Z = 19$, so if we apply the dimension of fiber inequality 2.2.14 to the map $Z \to p_2(Z)$ and choose $y \in p_2(Z)$ to be the Fermat surface described in the Introduction, we obtain that

$$1 \leq \dim Z - \dim p_2(Z) \leq \dim f^{-1}(y).$$

However, $f^{-1}(y)$ is precisely a finite set of 27 elements, so it must have dimension 0, a contradiction. This completes the proof of $p_2(Z) = \mathbb{P}^{19}$. $\square$

The fact that p_2 is surjective yields the following non-trivial observation.

Corollary 4.0.5. *All cubic surfaces in $\mathbb{P}^3$ contain at least one line.*

The above discussion involves $\mathbb{P}^{19}$, the parameter space of all cubic surfaces in $\mathbb{P}^3$. From now on, we focus on the subset V_{sm} of smooth cubic surfaces whose fibers are of ultimate interest. It's an open subset of $\mathbb{P}^{19}$ because it's the non-vanishing points of the resultant polynomial of all derivatives of a generic cubic homogeneous polynomial [6]. As p_2 is a morphism into the projective space $\mathbb{P}^{19}$, it's by definition a projective morphism. Therefore, its restriction π to an open subset is a projective morphism. We denote $Z_{V_{sm}}$ to be the fiber of V_{sm} in Z. It is a smooth open subscheme of Z since Z is smooth.

For simplicity, we denote the restriction of p_2 to the fiber of V_{sm} to be $\pi : Z_{V_{sm}} \to V_{sm}$ because it will be the focus for the rest of the proof. Our reduction Proposition 4.0.3 requires us to show that all fibers of π have exactly 27 elements. A first step towards this goal is to show that all such fibers are finite. This is eventually done through a point-set topological argument after we prove the crucial lemma that every element in $\pi^{-1}(X_0) = Z_{X_0}$ is isolated and reduced. This lemma will not only be used in proving the finiteness of fibers, but also the smoothness of π. The crux of its proof utilizes an elegant geometric argument. Heuristically, Z_{X_0} is the collection of lines lying on a smooth cubic surface X_0, the statement tells us that every line is "a neighborhood away" from the other lines, so it ensures that the lines are distinct on the surface.

Lemma 4.0.6. *Every point in Z_{X_0} is isolated and reduced.*

Proof. Let $\ell \in Z_{X_0}$ be a line in $\mathbb{P}^3$ lying on X_0. We know that $\mathbb{P}^3$ has an open affine cover with four open sets, each isomorphic to $\mathbb{A}^3$. Without loss of generality, pick $\mathbb{A}^3 \cong \{w \neq 0\} \subseteq \mathbb{P}^3$ and for each copy of $\mathbb{A}^3$, there's a one-to-one correspondence between projective lines intersecting $\{w \neq 0\}$ and the affine lines in $\mathbb{A}^3$. One way to describe this correspondence is through dehomogenizing (resp. homogenizing) the two linear equations defining the line with respect to w. Equipped with this correspondence, we prove the local statement for ℓ in $\mathbb{A}^3$, and by a linear change of coordinates, we could without loss of generality assume ℓ is given by the z-axis in $\mathbb{A}^3$. Moreover, $U_{1,2}$ is a subcollection of projective lines intersecting $\{w \neq 0\}$, and its corresponding lines in $\mathbb{A}^3$ are precisely those with a parametric form $(a, b, 0) + t \cdot (a', b', 1)$ (*). Indeed, for each such line L, its corresponding projective line is given by the matrix

$$\begin{bmatrix} 1 & 0 & -a' & -a \\ 0 & 1 & -b' & -b \end{bmatrix}$$

Therefore, the map $L \mapsto (-a', -a, -b', -b)$ gives a parametrization of lines in $\mathbb{A}^3$ with form (*) using space $\mathbb{A}^4$ that agrees with the isomorphism from $U_{1,2} \subseteq \mathrm{Gr}(2,4)$ to $\mathbb{A}^4$. This agreement allows us to model the neighborhood of ℓ in Z_{X_0} using $\mathbb{A}^4 = \mathrm{Spec} k[a, a', b, b']$. Notice that under this parameterization, the z-axis ℓ corresponds to the origin $(0, 0, 0, 0)$, and its identification with $\mathbb{A}^4$ is consistent with the Grassmannian's open cover of $\mathbb{A}^4$'s. Let X_0 be given by the cubic homogeneous polynomial f, whose image in $\mathbb{A}^3$ is its dehomogenization $\tilde{f} = c_{x^3}x^3 + c_{x^2y}x^2y + \cdots + c_1 = 0$. Then, plugging the parametric form into $\tilde{f}$, we obtain a polynomial in t with coefficients in $k[a, a', b, b']$ that vanishes for all $t \in k$. Extracting coefficients for terms $1, t, t^2, t^3$ and taking into account of the parametrization by $\mathbb{A}^4$, we obtain 4 polynomials f_1, f_2, f_3, f_4 in $k[a, a', b, b']$ such that the line (identified with) (a, a', b, b') lie on X_0 if and only if $f_i(a, a', b, b') = 0$ for all i. Therefore, we reduce to the case $\pi^{-1}(X_0) = \mathrm{Spec}(k[a, a', b, b']/ < f_1, f_2, f_3, f_4 >) \subseteq \mathrm{Spec}(k[a, a', b, b'])$.

Finally, to show that $(0, 0, 0, 0) \in \mathrm{Spec}(k[a, a', b, b']/ < f_1, f_2, f_3, f_4 >)$ is a reduced and isolated point, it suffices to show that $(0, 0, 0, 0)$ is the only solution to $\overline{f}$, which stands for the image of f in $\frac{k[a,a',b,b']}{(a,a',b,b')^2}[t]$. Equivalently, $\overline{f}$ equals f modulo the terms with degree 2 or above. In other words, if $\overline{f_i}$ stands for f_i modulo the terms with degree 2 or above, then $\overline{f} = \overline{f_1} + \overline{f_2}t + \overline{f_3}t^2 + \overline{f_4}t^3$. As X_0 contains the z-axis ℓ, immediately we obtain that $c_{z^3} = c_{z^2} = c_z = c_1 = 0$. Thus, $\overline{f}(t)$ preserves only the terms with exactly one x or one y involved, hence could be written as

$$\begin{aligned}\overline{f}(t) &= c_x(a+a't) + c_{xz}(a+a't)t + c_{xz^2}(a+a't)t^2 + c_y(b+b't) + c_{yz}(b+b't)t + c_{yz^2}(b+b't)t^2 \\ &= (a+a't)(c_x + c_{xz}t + c_{xz^2}t^2) + (b+b't)(c_y + c_{yz}t + c_{yz^2}t^2) \equiv 0.\end{aligned}$$

For convenience, denote $C_x(t) = c_x + c_{xz}t + c_{xz^2}t^2$ and $C_y(t) = c_y + c_{yz}t + c_{yz^2}t^2$. For the remainder of the proof, we utilize the hypothesis that X_0 is smooth. By the Jacobian Criterion 3.3.6, we see that X_0's regularity at $(0, 0, t_0)$ (or in original projective coordinates, $[0 : 0 : t_0 : 1]$) implies $c_x + c_{xz}t_0 + c_{xz^2}t_0^2 = C_x(t_0)$ and $c_y + c_{yz}t_0 + c_{yz^2}t_0^2 = C_y(t_0)$ are not both 0 for any $t_0 \in k$. In particular, when $t_0 = 0$, we see that c_x and c_y are not both zero. Moreover, c_{xz^2} and c_{yz^2} are not both zero. Indeed, X_0 contains the line $\{x = 0\} \cap \{y = 0\}$ which intersects the plane at infinity $\{w = 0\}$ at the projective point $[0 : 0 : 1 : 0]$. X_0 is smooth at that point, so its Jacobian at $[0 : 0 : 1 : 0]$ cannot be the zero matrix. Finally, as $c_{wz^2} = c_{z^2} = 0$, the two other entries left are c_{xz^2} and c_{yz^2} who cannot be both zeroes.

First, we claim that if c_{xz^2} is nonzero, then $b = b' = 0$. The polynomial $C_x(t)$ has degree 2, so either it has two distinct roots or one root of multiplicity 2. Let's first assume that C_x has two distinct roots t_0, t_1. Then $C_y(t_0)$ and $C_y(t_1)$ are both nonzero, so the unique solution to $b + b't_0 = 0$ and $b + b't_1 = 0$ is $b = b' = 0$. Otherwise, C_x has one root t_0 of multiplicity 2. Not only do we get $b + b't_0 = 0$, we also have $C_x'(t) = 0$. Yet, as $\overline{f}(t)$ vanishes for all t, taking its derivative we that $a'C_x(t) + (a + a't)C_x'(t) + b'C_y(t) + (b + b't)C_y'(t) = 0$. Substitute t_0 gets $b' = 0$. Thus $b = 0$. Likewise, if $c_{yz^2} \neq 0$, the argument above gives $a = a' = 0$. We solved the case when both c_{yz^2} and c_{xz^2} are nonzero. Yet, as we have seen that c_{yz^2} and c_{xz^2}

cannot both be zero, so the remaining case is when exactly one of them is 0. Without loss of generality, assume $c_{yz^2} = 0$ and $c_{xz^2} \neq 0$. Immediately, we get $b = b' = 0$. Therefore, $(a + a't)C_x(t) = (a + a't)(c_x + c_{xz} + c_{xz^2}t^2) \equiv 0$. Because the coefficient of t^3 is $a'c_{xz^2} = 0$, we have $a' = 0$. It follows that the coefficient of t^2 is $ac_{xz^2} = 0$ and finally we have $a = 0$. This completes the proof.

□

Given that all points in Z_{X_0} are isolated, its finite cardinality is a quick topological corollary.

Proposition 4.0.7. *If $X_0 \in V_{sm}$, then Z_{X_0} is a finite set.*

Proof. Firstly, we know that $\mathbb{G}(1,3)$ and $\mathbb{P}^{19}$ each have an open covering consisting of finitely many affine open subsets, so $\mathbb{P}^{19} \times \mathbb{G}(1,3)$ is quasicompact. Therefore, Z as its closed subset is also quasicompact. The preimage Z_{X_0} of a closed point $X_0 \in V_{sm}$ under a continuous map π is closed, so it follows that Z_{X_0} is quasicompact. However, by the above Lemma 4.0.7 we see each point of Z_{X_0} is open, and therefore we obtain an open cover for Z_{X_0} just by taking the collection of all its points as singletons. Finally, its finite subcover must consist of the finite collection of all of Z_{X_0}'s elements, completing the proof. □

By Theorem 3.1.9, a projective morphism with finite fibers is a finite morphism. We claim that π is a smooth morphism, for after which by Theorem 3.3.4 π will be flat.

Theorem 4.0.8. *π is flat.*

Proof. We first prove that π is smooth. The domain and codomain of π, V_{sm} and $Z_{V_{sm}}$, are both smooth. So by 3.3.5 it reduces to proving that all fibers are smooth schemes. Let A be a stalk at any point in any fiber of π. We have just shown in 4.0.6 that A is a reduced local ring of dimension 0. This means the zero ideal of A is its unique maximal ideal. Therefore, A is a field. It follows that all fibers of π are smooth, so π is a smooth morphism. Finally, Theorem 3.3.4 tells us that π is flat. □

Finally, we have enough information to determine that all fibers of smooth cubic hypersurfaces under π have the same size.

Theorem 4.0.9. *All fibers of closed points in V_{sm} under π has the same cardinality.*

Proof. We first solve the affine case where π is induced by the ring homomorphism $\varphi : A \to B$. Let $\mathfrak{m} \in \operatorname{Spec}(A)$ be a maximal ideal of A. Since π is finite, B is a finite A-algebra, thus an integral extension of A.

As both flatness and finiteness are properties preserved under base change, we may further replace A with local ring $A_\mathfrak{m}$ and B with $S^{-1}B$ because the cardinality of fibers of $\mathfrak{m}A_\mathfrak{m}$ under $\tilde{\pi} : \operatorname{Spec}(S^{-1}B) \to \operatorname{Spec}(A_\mathfrak{m})$ is the same as π. B remains an integral extension of A, so the prime ideals of B lying over $\mathfrak{m}$ are maximal. As B is also flat over a Noetherian local domain A, B is a free module over A. Notice that the fiber of $\mathfrak{m}$ is precisely the prime ideals in B containing $\mathfrak{m}B$, so by hypothesis there's only a finite number of them. In addition, we know from 4.0.6 that the fiber $\operatorname{Spec}(B \otimes_A A/\mathfrak{m}) \cong \operatorname{Spec}(B/\mathfrak{m}B)$ is a reduced scheme. Therefore, its zero ideal is a finite intersection of all distinct maximal (prime) ideals in $B/\mathfrak{m}B$. Lift it back to B, we see that $\mathfrak{m}B = \cap_{1\le i\le d}\mathfrak{m}_i = \prod_{1\le i\le d}\mathfrak{m}_i$ for d maximal ideals $\mathfrak{m}_i$ of B which are exactly the distinct maximal (prime) ideals containing $\mathfrak{m}B$. Thus, $d = |\pi^{-1}(\mathfrak{m})|$ is the cardinality of the fiber. But then by the Chinese Remainder Theorem,

$$B/\mathfrak{m}B \cong \prod_{1\le i\le d} B/\mathfrak{m}_i.$$

However, as k is algebraically closed and $B/\mathfrak{m}_\mathfrak{i}$ and $A/\mathfrak{m}$ are both algebraic extensions of k and therefore are both isomorphic to k. It follows that

$$d = \dim_{A/\mathfrak{m}} B/\mathfrak{m}B = \operatorname{rank}_A(B) = [K(X) : K(Y)].$$

where $K(X)$ and $K(Y)$ are respectively the function fields of $X = \text{Spec}(B)$ and $Y = \text{Spec}(A)$. This proves the affine case. Because $Z_{V_{sm}}$ and V_{sm} are integral schemes by Corollary 4.0.2, all of their affine open subsets share the same respective function fields, the fiber cardinality is always $[K(Z_{V_{sm}}) : K(V_{sm})]$ for closed points in $\pi(Z_{V_{sm}})$. □

By Theorem 1.1.2, the fiber of the Fermat surface - a closed point in V_{sm} - has 27 elements, so all fibers of smooth cubic surfaces should have exactly 27 elements, completing the proof of the main theorem.

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