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An abstract of
a thesis submitted to the Faculty of Emory College of Arts and Sciences
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Abstract

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When astrophysical black holes are under perturbation, they emit gravitational waves which are characteristic for the black hole. The frequencies of the waves are called quasinormal modes. There is a large amount of research and physics literature show- ing that it is possible to infer black hole parameters (e.g. mass, angular momentum etc) from the quasinormal modes. The problem is similar to the famous Kac's problem "Can you hear the shape of a drum?" A main difference for the black hole problem, which makes it more challenging and interesting is that in practice only a few quasi- normal modes can be acquired. Thus we are not supposed to use all quasinormal modes to determine the black hole parameter as usually done in the inverse spectral problems.

In this work, we study a non-rotating black hole called the de Sitter-Schwartzchilde black hole, which is characterized only by its mass. We develop mathematical methods to obtain a lower bound of the mass from a single quasinormal mode. In particular, we study Zerilli's equation which describes the black hole perturbation. We estimate the lower bound of resonance width by adapting a method due to Harrell for one dimensional Schr¨odinger equations. The lower bound yields the desired estimate of the mass. For a toy model scattering problem, we show by numerical examples the feasibility of the method.

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Chapter 1

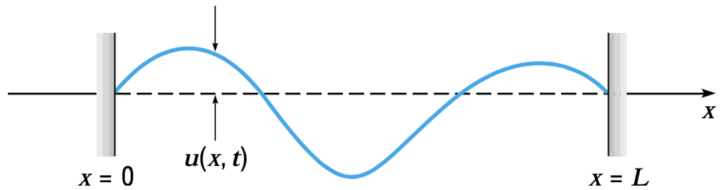
Introduction

1.1 Inverse Spectral Problem

We start with the famous Kac's problem "Can one hear the shape of a drum?" See [1]. To illustrate the idea, we consider the one-dimensional problem of string vibrations for which we can provide an explicit solution. Suppose a string is plucked and produces a sound that we can hear. The string may be thought of as a violin string, a guy wire and so on. See for example [6]. The question is whether we can determine the length of the string from the sound?

We setup the mathematical problem as follows. Suppose that an elastic string of length L is tightly stretched between two supports at the same horizontal level (See Figure 1.1a), so that the x-axis lies along the string. In particular, we consider a simple one string so there are no other additional variables. For $x \in [0, L]$, let $u(x,t)$ be the displacement from the rest position after the string is plucked. Its dynamical significance is most easily seen in the context of the wave equation. $u(x,t)$ satisfies the second order partial differential equation

$$(\partial_t{}^2 - \partial_x{}^2)u(x,t) = 0, \quad x \in [0, L], t \geq 0 \tag{1.1}$$



(a) A vibrating string

(b) Violin String

Figure 1.1: The string vibration problem

By the method of separation variable, it suffices to look for solutions of the form

$$u(x,t) = e^{ikt}w(x)$$

for some $k \in \mathbf{R}$. The motion of the string is determined once we find k and $w(x)$. Note that k is the frequency of the sound produced. By plugging $u(t,x)$ to (1.1), we find that $w(x)$ satisfies the following second order ordinary differential equation:

$$w''(x) + k^2 w(x) = 0, \quad x \in [0, L] \tag{1.2}$$

where k is the frequency of the volume $w(x)$. The equation can be solved once we specify suitable initial and boundary conditions, for example

$$w(0) = w(L) = 0 \tag{1.3}$$

In particular, we consider the situation in which the ends of the string are fixed. Note that equation (1.2) is homogeneous, and the boundary values $w(0), w(1)$ are also zero. It is easy to see that $w = 0$ is a solution for all x regardless of the coefficients k^2 (called trivial solution). However, this is rarely of interest because it just says the string is not plucked. We want to find all the k so there is a non-zero solution. Then k^2 is called an *eigenvalue* and the solution is called an *eigenfunction*. This problem is often called a Sturm-Liouville problem.

The equation can be solved by the method of characteristics. Write the solution of Equation (1.2) as $y = e^{rx}$. Then the characteristic polynomial equation is $r^2 + k^2 = 0$ with roots $r = \pm ki$, so the general solution is

$$w(x) = c_1 e^{ikx} + c_2 e^{-ikx}$$

where c_1, c_2 are constants to be determined. From the first boundary condition $w(0) = 0$ we get

$$c_1 + c_2 = 0$$

From the second boundary condition $w(L) = 0$, we get

$$c_1 e^{ikL} + c_2 e^{-ikL} = 0$$

We obtain that

$$c_1 = -c_2$$

$$c_1(e^{2ikL} - 1) = 0$$

Note that if $c_1 = 0$ then $c_2 = 0$, and we conclude that $w(x) = 0$ is a trivial solution. Since we are interested in nontrivial solutions, we should assume $c_1 \neq 0$. Consequently, we have $e^{2ikL} = 1$. We know that the sine function has the value zero at every integer multiple of π so we can choose n to be any (positive) integer where:

$$k = n\pi/L, n = 0, 1, 2, \cdots .$$

We thus solved the problem. The value $k_n^2 = (n\pi/L)^2$ are the eigenvalues and the corresponding solution

$$u_n(x) = c_1(e^{in\pi x/L} - e^{-in\pi x/L})$$

where c_1 is an arbitrary constant, is the eigenfunction. In fact, we can apply Euler's formula $e^{ix} = \cos x + i \sin x$ to get that

$$w_n(x) = c_1 \sin(n\pi x/L) \tag{1.4}$$

For fixed value of n, the expression $\sin(n\pi x/L)$ is periodic with the period $2L/n$; it therefore represents a vibratory motion of the string having the frequency $n\pi/L$. The factor $\sin(n\pi x/L)$ represents the displacement pattern occurring in the string when it is executing vibrations of the given frequency. Each displacement pattern is called a natural mode of vibration and is periodic in the space variable x; the spatial period $2L/n$ is called the wavelength of the mode of frequency $n\pi x/L$. See [6].

Now it is easy to see that the frequencies $k_n = n\pi/L$ determine the length L. In fact, the lowest frequency $k_1 = \pi/L$ is enough.

1.2 Quasinormal Modes Of A Black Hole

There is a similar problem for astrophysical black holes. When a black hole is perturbed by for example other black holes, it emits gravitational waves with certain frequencies (called quasinormal modes or pure tones), which is similar to the tones produced by a string. In fact, the first gravitational waves detected by LIGO are generated by the collision of two black holes. Fig 1.2 shows the gravitational wave signal observed on September 14, 2015 simultaneously by LIGO Livingston (blue) and LIGO Hanford (red), see [4]. The blue and red waves oscillate together until they get closer enough to "explode", which leads to the highest level of oscillation. After that they decay, and the level of oscillation reduce significantly. This process is a phenomenon called ringdown. See also Fig 1.3 for a clearly visible quasinormal ringing. In particular, one can extract some quasinormal mode associated to the oscillation frequency and decay rate of the wave from the ringdown. An outstanding question is whether one can determine the parameters of the black hole (such as mass, charge, etc) from the quasinormal modes.

The possibility of inferring black hole parameters from quasinormal modes (QNMs) has been explored in physics literatures, see Section 9 of the review paper [5]. For example, for slowly rotating black holes, Detweiler showed by numerical calculation in [12] that the wave parameters for the most damped mode are unique functions of the black hole parameters. Since the success of gravitational wave interferometers, the topic has gained increasing attention, see for instance [13]. One particular motivation for the study is to verify the *black hole no hair theorem* for which two QNMs are needed: one QNM is used to recover the black hole parameter and another QNM is used to test the theorem, see [5, Section 9.7]. Despite some convincing evidences, the theoretical justification is not complete. In general, it is not known which modes are excited and are extractable from the actual black hole ring down signals, see [13, 5].

In this project, we aim to address the problem for a simple black hole model called the Schwarzschild black hole. This kind of black hole is characterized by only one parameter, its mass. So our goal is to estimate the mass from the quasinormal mode. The exterior of the black hole is given by (Y, g) (a Lorentzian manifold) where

$$Y = \mathbb{R}_t \times X, \quad X = (r_+, \infty) \times \mathbb{S}^2$$

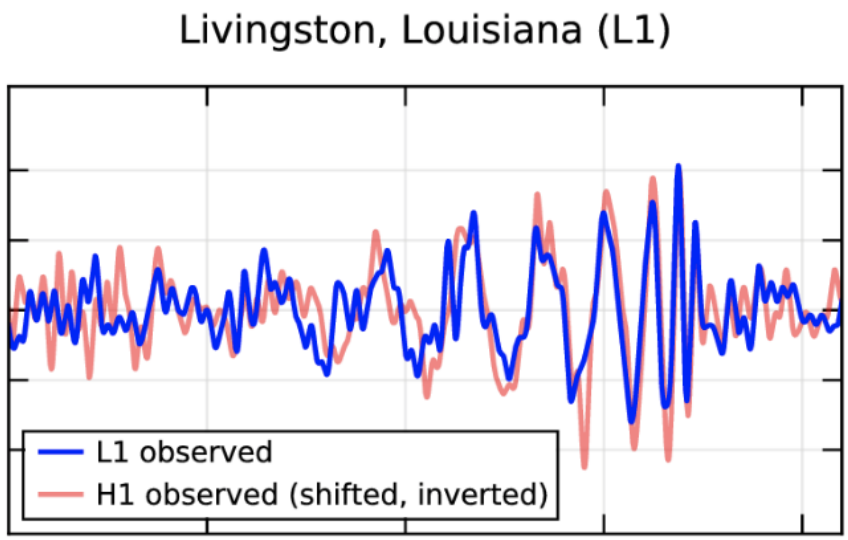


Figure 1.2: Gravitational wave signal observed by LIGO

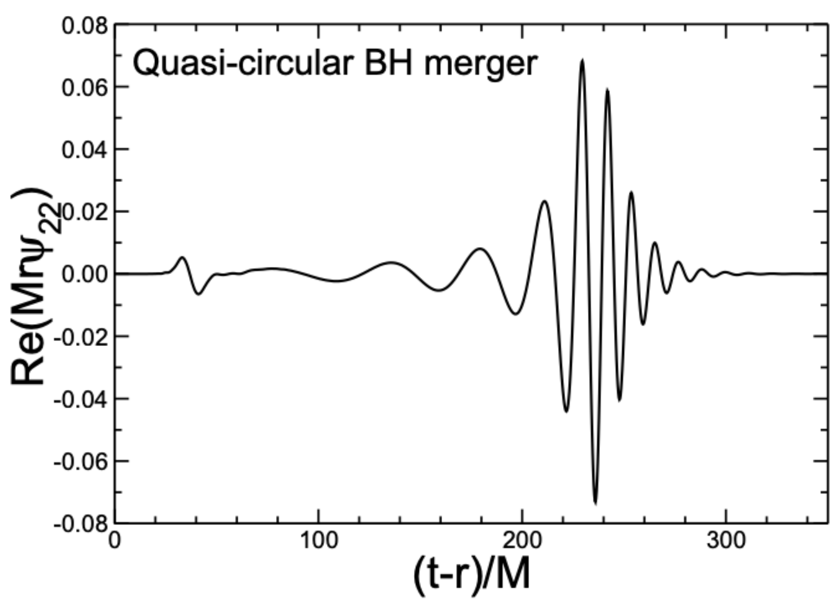


Figure 1.3: Physical processes leading to substantial quasinormal ringing

and

$$g = \alpha^2 dt^2 - \alpha^{-2} dr^2 - r^2 dw^2$$

is the metric where

$$\alpha = (1 - \frac{2m}{r})^{\frac{1}{2}}, r_+ = 2m.$$

Here t, r, w are the coordinates and m is the mass.

It is complicated to describe how the black hole can be perturbed which we do not discuss here. See for example [2]. However, the evolution of the perturbation can be fully described by the Zerilli's equation

$$\frac{d^2}{dx^2} u(x) + (\sigma^2 - V(x))u(x) = 0, \quad x \in \mathbb{R} \tag{1.5}$$

where

$$V(x) = \frac{2n^2(n+1)r^3 + 6n^2mr^2 + 18nm^2r + 18m^3}{r^3(nr+3m)^3}(1 - \frac{2m}{r}) \tag{1.6}$$

and

$$x = r + 2m \log(\frac{r}{2m} - 1), \quad r \in [2m, \infty)$$
$$n = \frac{1}{2}(l-1)(l+2), \quad l = 1, 2, \cdots .$$

The equation (1.5) describes the stationary wave similar to the string vibration problem. As x approaches to the infinity, equation (1.5) allows two independent solutions with the asymptotic behaviors $Z_\pm \to e^{\pm i\sigma x}$. As we assume a time dependence of the form $e^{\pm i\sigma t}$, Z_- represents an outgoing wave and Z_+ represents an ingoing wave. See Chapter 2. According to the definition in Chandrasekhar and Detweiler [8],

Definition 1.2.1. *A quasinormal mode is one which belongs to a complex σ with* $\mathrm{Re}(\sigma) \geq 0$ *such that it represents a purely outgoing wave at $+\infty$ and a purely ingoing wave at $-\infty$.*

Suppose we are given a single or finitely many quasinormal modes. For example, one can extract quasinormal modes from the ringdown, see [5]. The question is whether one can determine the mass of the black hole.

1.3 The Outline

In spirit, the problem is similar to the string vibration problem. However, there are important differences which makes the problem very challenging and interesting.

First, the string vibration problem is posed on a finite interval, while the Zerilli's equation is considered on the real line. So there is no boundary condition for solving the equation but instead, we need to understand the decay of the solution at $x \to \pm\infty$. This naturally leads us to the "scattering theory". We will consider the one-dimensional scattering theory in Chapter 2. As a consequence of this difference, recall that the frequency in the string vibration problem is always real, however, the quasinormal mode is complex. We write the quasinormal mode σ as the form of a complex number: $\alpha + i\beta, \alpha, \beta \in \mathbf{R}$. The wave is described by

$$u(x,t) = e^{i\sigma t} w(x) = e^{i(\alpha+i\beta)t} w(x)$$

$$= e^{it\alpha} e^{-t\beta} w(x)$$

where $e^{it\alpha}$ represents the oscillation of the wave and $e^{-t\beta}$ represents the decay rate.

Second, because of the $V(x)$ term in Zerilli's equation, one cannot obtain explicit solutions as in the string vibration problem. Thus it is very difficult to find the direct relation of the mass and the quasinormal modes. In Chapter 2, we will consider simpler examples for which the relation can be found, and we use it to demonstrate the possibility of recovering scattering parameters from a single quasinormal mode. For Zerilli's equation, we will establish some estimate of the mass instead of "determine" the mass. This will be done in Chapter 3.

Finally, we remark that we are interested in recovering the mass from only one or a few quasinormal modes. This is very different from the usual inverse spectral/resonance problem for which the whole set is used to determine the parameters. In fact,

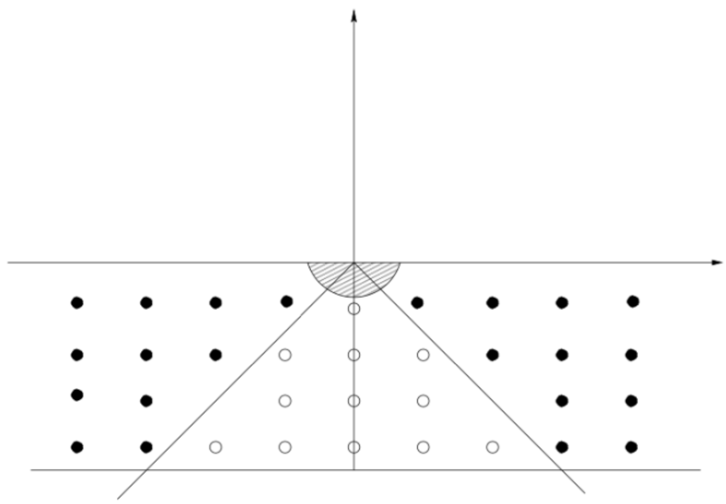


Figure 1.4: The lattice of pseudo-poles approximating resonances (dark dots) in a conic neighbourhood of the continuous spectrum

there is a large literature on distribution of resonances for large angular momentum. For example, Theorem in [11] states that there exists $K > 0, \theta > 0$ such that for any $C > 0$ there is an injective map $\widetilde{b}$ from the set

$$(\pm l \pm \frac{1}{2} - \frac{i}{2}(k + \frac{1}{2}))\frac{1}{3^{3/2}m}$$

in to the set of quasinormal modes such that all the poles in

$$\Omega_C = \{\lambda : \operatorname{Im}\lambda > -C, |\lambda| > K, \operatorname{Im}\lambda > -\theta|\operatorname{Re}\lambda|\}$$

are in the image of $\widetilde{b}$ and for $\widetilde{b}(\mu) \in \Omega_C$, we have $\widetilde{b}(\mu) - \mu \to 0$ as $|\mu| \to \infty$. See Figure. Thus it is not difficult to see that the mass m is determined from a sequence of quasinormal modes.

Chapter 2

Scattering Theory

In this chapter, we will analyze a one dimensional scattering problem and illustrate that it is possible to determine scattering parameters from resonances.

2.1 The Free Scattering

We first consider the free scattering, namely waves are traveling in free space without any perturbation. This is described by the following ODE

$$\frac{d^2}{dx^2}u(x) + \sigma^2 u(x) = 0, \quad x \in \mathbb{R}$$

where $\sigma \in \mathbb{R}$ is constant. Similar to the string vibration problem, the general solution can be found as

$$u(x) = c_1 e^{i\sigma x} + c_2 e^{-i\sigma x}, \quad x \in \mathbb{R}$$

where c_1, c_2 are arbitrary constants. It is helpful to also consider the corresponding wave equation and observe how the waves move. Consider

$$w(t,x) = e^{i\sigma t}u(x) = c_1 e^{i\sigma t + i\sigma x} + c_2 e^{i\sigma t - i\sigma x} \tag{2.1}$$

where $t \in \mathbb{R}$ and $x \in \mathbb{R}$. Here t is the time and x is the position. Then w satisfies the wave equation

$$\partial_t^2 w - \partial_x^2 w = 0$$

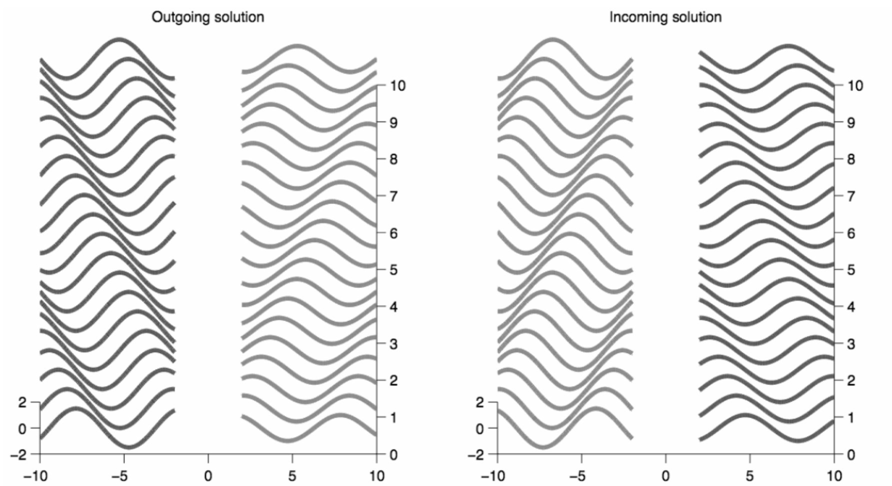


Figure 2.1: Schematic representation of the outgoing (left) and incoming (right) solutions

Among the two functions in $\omega(t,x)$, one represents a wave traveling to the left and one traveling to the right as $t \to +\infty$. To see this, let's look at

$$\omega_1(t,x) = e^{i\sigma t + i\sigma x}$$

If $t + x = c$ a constant, then $w_1(t,x) = e^{i\sigma c}$ is a constant. This wave will travel along $t + x = c$ which is to the left as $t \to +\infty$. See Figure 2.1

Now $u(x)$ is called an outgoing solution if

$$u(x) = c_1 e^{i\sigma x}, x < -R, \quad u(x) = c_2 e^{-i\sigma x}, x > R \tag{2.2}$$

for some $R > 0$ large. So this part goes away to the infinity. The incoming solution is

$$u(x) = b_1 e^{i\sigma x}, x > R, \quad u(x) = b_2 e^{-i\sigma x}, x < -R \tag{2.3}$$

This part comes from the infinity. In scattering theory, we are interested in the behavior of the solution for $x \to \pm\infty$. The behavior at the infinity acts as the boundary condition for the string problem.

2.2 The Potential Scattering

Let's add the potential $V(x)$. To begin with, we take a simple one (the square potential):

$$V(x) = \begin{cases} 1, & x \in [-L, L] \\ 0, & \text{otherwise} \end{cases}$$

Note that the potential is not continuous and is symmetry in $[-L, L]$. Then we consider the equation

$$u''(x) - V(x)u(x) + \sigma^2 u(x) = 0.$$

Physically, one can imagine that the potential function represents a medium or obstacle. When the wave is traveling on $\mathbb{R}$, it can be reflected or transmitted by the potential and goes all the way to the infinity. Outside the support, we have $V = 0$ so the solution should look like

$$u(x) = a_1 e^{i\sigma x} + a_2 e^{-i\sigma x}, \quad |x| > L$$

which is a sum of outgoing and incoming solution. The questions is can we determine the scatterer by observing the outgoing wave.

To understand the process better, let's look at a wave traveling to the right, hit the potential and gets reflected and transmitted. In this case, we look for a solution

$$u_R(x) = \begin{cases} e^{i\sigma x} + re^{-i\sigma x}, x < -L \\ te^{i\sigma x}, x > L \end{cases}$$

Here, r is called the refection amplitude, t is the transmission coefficient. Similarly, we can consider a wave traveling to the left, hit the potential and gets reflected and transmitted. In this case, the solution

$$u_L(x) = \begin{cases} t'e^{-i\sigma x}, x < -L \\ e^{-i\sigma x} + r'e^{i\sigma x}, x > L \end{cases}$$

Here, r' is called the refection amplitude,t' is the transmission coefficient. Note that u_R and u_L can be expressed in terms of the basic outgoing and incoming waves:

$$u_{in}(x) = e^{i\sigma x}, x < -L$$

$$v_{in}(x) = e^{-i\sigma x}, x > L$$

and

$$u_{out}(x) = e^{i\sigma x}, x > L$$

$$v_{out}(x) = e^{-i\sigma x}, x < -L$$

Then the scattering process is encoded in the scattering matrix

$$S = \begin{pmatrix} t & r \\ r' & t' \end{pmatrix}$$

To find the coefficients and the scattering matrix, we look for solution u on $[-L, L]$

of the form

$$u(x) = Ae^{-iqx} + Be^{iqx}, x \in [-L, L]$$

where $q^2 = \sigma^2 - 1$. Then we match the solutions at $x = \pm L$ so the solution is C^1. First we find r', t'. At $x = L$, we have two equations

$$e^{-i\sigma L} + r'e^{i\sigma L} = Ae^{iqL} + Be^{-iqL} \tag{1}$$

$$-i\sigma e^{-i\sigma L} + r'i\sigma e^{i\sigma L} = Aiqe^{iqL} - Biqe^{-iqL} \tag{2}$$

At $x = -L$, we have another equations

$$t'e^{i\sigma L} = Ae^{-iqL} + Be^{iqL} \tag{3}$$

$$-it'\sigma e^{i\sigma L} = Aiqe^{-iqL} - Biqe^{iqL} \tag{4}$$

We start the calculation by eliminating coefficient B. By multiplying (1) and then add (2), we have equation (5) as belows:

$$(q - \sigma)e^{-i\sigma L} + r'(q + \sigma)e^{i\sigma L} = 2Aqe^{iqL} \tag{5}$$

Similarly, we can get equation (6) from (3) and (4):

$$t'(q - \sigma)e^{i\sigma L} = 2Aqe^{-iqL} \tag{6}$$

Then we dividing (5) by (6):

$$e^{-i\sigma L} + r'\frac{(q + \sigma)}{(q - \sigma)}e^{i\sigma L} = e^{2iqL}t'e^{i\sigma L} \tag{a}$$

Then we continue the similar process by eliminating A. Thus first multiplying (1) and then minus (2), we have equation (7) as belows:

$$(q+\sigma)e^{-i\sigma L}+r'(q-\sigma)e^{i\sigma L}=2Bqe^{-iqL} \tag{7}$$

Similarly, we can get equation (8) from (3) and (4):

$$t'(q+\sigma)e^{i\sigma L}=2Bqe^{iqL} \tag{8}$$

Then we dividing (7) by (8):

$$e^{-i\sigma L}+r'\frac{(q-\sigma)}{(q+\sigma)}e^{i\sigma L}=e^{-2iqL}t'e^{i\sigma L} \tag{b}$$

Then we divide (a) by (b), we can have the function of r in expression of K:

$$e^{-i\sigma L}+r'\frac{(q+\sigma)}{(q-\sigma)}e^{i\sigma L}=e^{4iqL}(e^{-i\sigma L}+r'\frac{(q-\sigma)}{(q+\sigma)}e^{i\sigma L}$$

$$r'(\frac{(q+\sigma)}{(q-\sigma)}-\frac{(q-\sigma)}{(q+\sigma)}e^{4iqL})=e^{-2i\sigma L}e^{4iqL}-e^{-2i\sigma L}$$

Let $K=\frac{(q+\sigma)}{(q-\sigma)}-\frac{(q-\sigma)}{(q+\sigma)}e^{4iqL}$, r' can be written as:

$$r'=\frac{e^{-2i\sigma L}e^{4iqL}-e^{-2i\sigma L}}{K}$$

and

$$t'=e^{-2i\sigma L-2iqL}+r'\frac{q+\sigma}{q-\sigma}e^{-2iqL}$$

Similarly we can solve the other two coefficients t, r:

$$r=\frac{e^{-2i\sigma L}e^{2iqL}-e^{-2i\sigma L}e^{-2iqL}}{K}$$

and

$$t = e^{-2i\sigma L - 2iqL} + r\frac{q+\sigma}{q-\sigma}e^{-2iqL}$$

and we get the same expression for the denominator K.

Because of the K factor, we see that the scattering matrix S could be singular when $K = 0$. In fact, the scattering matrix can be defined for σ complex. The resonances are defined to be the poles of the scattering matrix which is given by $K = 0$. The resonances are the same as the quasinormal modes in the black hole problem. We remark that one can define resonances as poles of the resolvent.

From the expression of K, we obtain that resonances are solutions of the equation

$$(\frac{q+\sigma}{q-\sigma})^2 = e^{4iqL}$$

where $q^2 = \sigma^2 - 1$. Notice that this equation is difficult to solve. We can use a Matlab code `squarepot.m` to compute the resonances, see [10]. The result is shown in Figure 2.2.

2.3 Recover The Potential From A Resonance

In this section, we show that one can uniquely determine L from a single resonance. We use the equation

$$(\frac{q+\sigma}{q-\sigma})^2 = e^{4iqL}$$

in the previous part. There are some issues for solving L which are related to the choice of branches of complex logarithmic function. So we proceed as follows. First set $q = \alpha + i\beta$ since it is a complex number. We get

$$(\frac{q+\sigma}{q-\sigma})^2 = e^{4i(\alpha+i\beta)L}$$

$$= e^{4i\alpha L - 4\beta L} = e^{4i\alpha L} e^{-4\beta L}$$

By taking the absolute value on the both sides, we can eliminate the part $e^{4i\alpha L}$, since the modulus equals to 1. We get:

$$|\frac{q+\sigma}{q-\sigma}|^2 = e^{-4\beta L}$$

Now we can solve that

$$L = \frac{-1}{2\beta} \ln|\frac{q+\sigma}{q-\sigma}| \tag{c}$$

Suppose we are given a σ with $\operatorname{Im}\sigma \neq 0$. We can find the corresponding q and use the above formula to find L. Below, we show some numerical examples.

We can apply the matlab code `squarepot.m` to check that if we presume the value of L and got a list of solutions by computing resonances in one dimension [10], are we able to get the same value of L from these solutions. In this case we are able to check that the resonances are exactly the poles of the scattering matrix and L can be obtained if we have already known any value of σ.

Example 1: Let assume $L = 1.5$, so the distance between each boundary line and the vertical line of the origin is 1.5. Then we got the corresponding pole locations and a list of solutions(See Fig.2.2).

We can pick 2 solutions from them. First check the 3rd solution of σ which is $2.0114 - 0.8579i$. Then from $q^2 = \sigma^2 - 1$, we have $q = 1.7976 - 0.9600i$. By inserting these values into function (c) we obtain $L = 1.5000$ in matlab, which is the same as the distance. Similarly, when we enter another solution of σ which is $9.3380 - 1.9533i$. The answer is the same.

Example 2: We can set another value of L and check again to ensure general-

1	1.2060-0.3312i
2	-1.2060-0.3312i
3	2.0114-0.8579i
4	-2.0114-0.8579i
5	3.0283-1.1768i
6	-3.0283-1.1768i
7	4.0748-1.3893i
8	-4.0748-1.3893i
9	5.1271-1.5484i
10	-5.1271-1.5484i
11	6.1803-1.6757i
12	-6.1803-1.6757i
13	7.2333-1.7819i
14	-7.2333-1.7819i
15	8.2859-1.8732i
16	-8.2859-1.8732i
17	9.3380-1.9533i
18	-9.3380-1.9533i
19	10.3896-2.0246i
20	-10.3896-2.0246i

Table 2.1: The Solution List

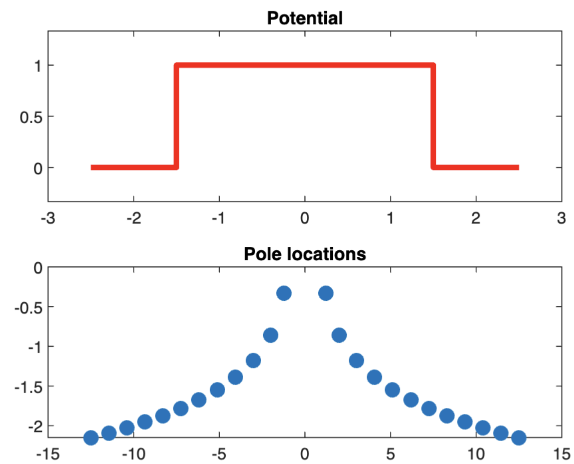


Figure 2.2: 1D potential and its corresponding scattering resonances

ization. Let's say $L = 2.5$. Then the distance between each boundary line and the vertical line of the origin is 2.5. Enter the information into matlab code shows the potential and the solutions in Fig 2.4, and we keep first 20 solutions to check the answer.

The process is the same. Simply pick 2 solutions to check. First check the 5th solution of σ which is $2.0241 - 0.4821i$. We calculate $L = 2.5002$ in matlab, which is close enough to the value of distance. Similarly, when we enter another solution of σ which is $6.2971 - 1.0034i$. The answer $L = 2.4999$ is the almost same.

1	1.1273-0.1053i
2	-1.1273-0.1053i
3	1.5028-0.3099i
4	-1.5028-0.3099i
5	2.0241-0.4821i
6	-2.0241-0.4821i
7	2.6017-0.6101i
8	-2.6017-0.6101i
9	3.2021-0.7084i
10	-3.2021-0.7084i
11	3.8132-0.7876i
12	-3.8132-0.7876i
13	4.4302-0.8536i
14	-4.4302-0.8536i
15	5.0506-0.9101i
16	-5.0506-0.9101i
17	5.6731-0.9595i
18	-5.6731-0.9595i
19	6.2971-1.0034i
20	-6.2971-1.0034i

Table 2.2: The Solution List

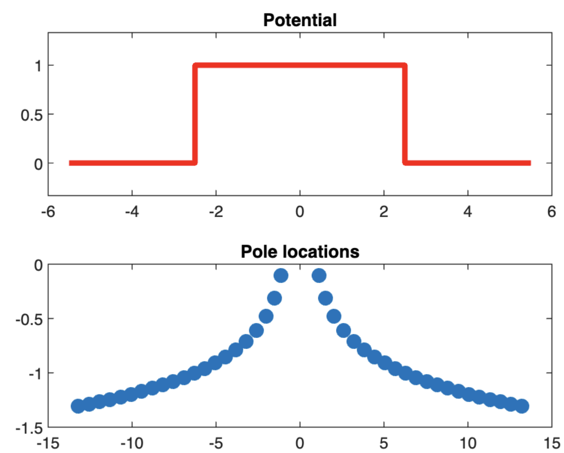


Figure 2.3: 1D potential and its corresponding scattering resonances

Chapter 3

Estimates of Black Hole Mass

3.1 The Main Result

In this chapter, we obtain some theoretical estimate of the resonance in one-dimensional potential scattering. The result can be applied to the Zerilli's equation to yield some estimate of the black hole mass given a single quasinormal mode. We formulate the main result as follows.

Let V be a bounded, real-valued, measurable function of $x \in [-l, l]$ and ψ be a function with absolutely continuous first derivative satisfying

$$-\psi'' + (V - k^2)\psi = 0 \tag{3.1}$$

with the boundary condition that

$$\psi'(-l)/\psi(-l) = -ik, \quad \psi'(l)/\psi(l) = +ik \tag{3.2}$$

where we assume $0 > \arg k > -\pi/4$. In particular, we have $\operatorname{Im} k < 0$ and $\operatorname{Re} k > 0$.

Definition 3.1.1. *When ψ exists, k^2 is called a resonance eigenvalue for $\frac{-d^2}{dx^2} + V$.*

Our goal is to obtain some estimate of resonances in terms of the potential V. For V supported in $[0, L]$, the problem was studied by Harrell [9]. In particular, the paper derives lower bounds in one-dimensional case assuming knowledge of the real part, and such method fit our goals perfectly. We are able to adapt Harrell's method to generate the lower bound for the resonance eigenvalues in one dimension depending only on the support and bounds of V and on the real part of the resonance eigenvalue.

However, there is an issue for applying this to the Zerilli's equation. Zerilli's potential is not compactly supported but decays at the infinity. In particular, solutions corresponding to quasinormal mode have the following asymptotic expansions

$$\begin{aligned} Z &= e^{-ikx}\sum_{j=0}^{\infty}\alpha_j r^{-j}, \quad x \to -\infty \\ Z &= e^{ikx}\sum_{j=0}^{\infty}\beta_j (r-2m)^{j}, \quad x \to \infty \end{aligned} \tag{3.3}$$

where α_j, β_j are constants, see [8]. Therefore, the boundary conditions will not be satisfied exactly but only approximately. So we consider the following condition

$$\psi'(-l)/\psi(-l) = -ik + \delta_-, \quad \psi'(l)/\psi(l) = +ik + \delta_+ \tag{3.4}$$

where $\delta_\pm$ are small complex number, assuming in a form of $\alpha_\pm + i\beta_\pm$ with $\alpha_\pm, \beta_\pm$ real numbers. We assume that $|\beta_\pm| < \operatorname{Re} k$.

Below, we call k^2 a resonance if there is a ψ satisfying (3.1) and (3.4). We let $\epsilon = \operatorname{Im} k^2$ which is called the resonance width. We let $E = \operatorname{Re} k^2$. Also, we use a bar to denote complex congregate. Our main result is

Theorem 3.1.2. *For any $\sigma > 0$,*

$$|\epsilon| \geq \min\left(\sigma, \frac{[2\operatorname{Re} k + (\beta_+ - \beta_-)]e^{-2l^2m^2\sqrt{1+\frac{\sigma^2}{m^4}}}}{2l + (E + \frac{\sigma^2}{2E})l^3/3}\right) \tag{3.5}$$

where $m = \sup_{-l \le x \le l} \sqrt{|V(x) - E|^2 + \epsilon^2}$.

Now we outline the steps to estimate the black hole mass using the above theorem. We recall Zerilli's equation from Chapter 1:

$$\frac{d^2}{dx^2} u(x) + (\sigma^2 - V(x))u(x) = 0, \quad x \in \mathbb{R}$$

where

$$V(x) = \frac{2n^2(n+1)r^3 + 6n^2 M r^2 + 18nM^2 r + 18M^3}{r^3(nr + 3M)^3}(1 - \frac{2M}{r})$$

Here, we changed the notation that M is the mass. Note that r is a function of x:

$$x = r + 2m \log(\frac{r}{2M} - 1), \quad r \in [2M, \infty)$$

Suppose k^2 be a given quasinormal mode for $n = 1$. This is a common assumption, see [5, 13]. We remark that due to the complexity of Zerilli's potential, some of the steps below are better implemented numerically.

Step 1: We choose l large and consider Zerilli's equation on $[-l, l]$. According to (3.3), we can assume the boundary condition (3.4) holds. The dependence of $\delta_\pm$ on l can be made more precise by examining the asymptotic expansion.

Step 2: We apply Theorem 3.1.2 to get an estimate

$$\begin{aligned} e^{2l^2 m^2 \sqrt{1 + \frac{\sigma^2}{m^4}}} &\ge \epsilon^{-1} \frac{[2 \operatorname{Re} k + (\beta_+ - \beta_-)]}{2l + (E + \frac{\sigma^2}{2E}) l^3 / 3} \\ \Longrightarrow m^4 &\ge (2l^2)^{-2} (\ln(\epsilon^{-1} \frac{[2 \operatorname{Re} k + (\beta_+ - \beta_-)]}{2l + (E + \frac{\sigma^2}{2E}) l^3 / 3}))^2 - \sigma^2 \end{aligned} \tag{3.6}$$

We need to choose σ to optimize the lower bound.

Step 3: We use $m = \sup_{-l\le x\le l} \sqrt{|V(x)-E|^2+\epsilon^2}$ and the expression of $V(x)$ to obtain an lower bound of M.

3.2 The Proof

The idea of the proof is to first obtain an expression of the resonance width in terms of the resonant state. Then we estimate the resonant state. These are done in two lemmas below.

Lemma 3.2.1. *Let* $E = \operatorname{Re} k^2, \epsilon = \operatorname{Im} k^2$. *We have*

$$\epsilon = \frac{[(\operatorname{Re} k+\beta)|\psi(-l)|^2 + (\operatorname{Re} k-\beta)|\psi(l)^2|]}{\int_{-l}^{l}|\psi(x)|^2dx} \tag{3.7}$$

Proof. By integration by parts, we have

$$2i\epsilon\int_{-l}^{l}|\psi(x)|^2dx = \int_{-l}^{l}\{\overline{\psi}(x)[\frac{-d^2}{dx^2}+V-E]\psi(x) - \psi(x)[\frac{-d^2}{dx^2}+V-E]\overline{\psi}(x)]\}dx$$

$$= (k^2-\overline{k^2})\int_{-l}^{l}\psi(x)\overline{\psi}xdx$$

$$= \int_{-l}^{l}(-\psi''+V\psi)\overline{\psi}dx - \int_{-l}^{l}\psi(-\overline{\psi''}+V\psi)dx$$

$$= -\overline{\psi}(l)\psi'(l) + \overline{\psi}(-l)\psi'(-l) + \psi(l)\overline{\psi'}(l) - \psi(-l)\overline{\psi'}(-l)$$

$$= 2i\operatorname{Im}(\overline{\psi}(l)\psi'(l)) + 2i\operatorname{Im}(\overline{\psi}(-l)\psi'(-l))$$

$$= 2i\operatorname{Im}((i\operatorname{Re} k + (-1)\operatorname{Im} k + \alpha_+ + i\beta_+)|\psi(l)|^2 - (-ik+\delta_-)|\psi(-l)|^2)$$

$$= 2i((\operatorname{Re} k+\beta_+)|\psi(l)|^2 - (-\operatorname{Re} k+\beta_-)|\psi(-l)|^2)$$

with the boundary condition we have the expression function for ϵ

$$\epsilon = \frac{[(\mathrm{Re}\, k + \beta_+)|\psi(l)|^2 + (\mathrm{Re}\, k - \beta_-)|\psi(-l)^2|]}{\int_{-l}^{l} |\psi(x)|^2 dx} \tag{a}$$

$\square$

Lemma 3.2.2. *For $-l < x \leq l$,*

$$|\psi(x)| \leq \sqrt{1 + ((\mathrm{Im}\, k)^2 + (\mathrm{Re}\, k)^2)x^2}\exp(\int_{-l}^{x} |V(x') - E - i\epsilon|(x - x')dx') \tag{3.8}$$

Proof. By integrating (1) with (2), the equation becomes

$$\psi(x) = 1 - ikx + \int_{-l}^{x} dx_1 \int_{-l}^{x_1} dx_2 (V(x_2) - E - i\epsilon)\psi(x_2))$$

By utilizing the Gronwall inequality, since $\psi, B(x) : [-l, l] \to \mathbb{R}$ are bounded measurable function and $C : [-l, l] \to \mathbb{R}$ is an integrable function with the property above, we have

$$|\psi(x)| \leq B(x)(\int_{-l}^{x} \exp(C(x)')dx')$$

so for $-l \leq x \leq l$

$$|\psi(x)| \leq \sqrt{1 + ((\mathrm{Im}\, k)^2 + (\mathrm{Re}\, k)^2)x^2}\exp(\int_{-l}^{x} |V(x') - E - i\epsilon|(x - x')dx')$$

$\square$

Now we are ready to prove Theorem 3.1.2.

Proof. We estimate $\int_{-l}^{l} |\psi(x)|^2 dx$ in lemma 1. This will give us the lower bound for

ϵ. Use inequality from Lemma 2, we get

$$\int_{-l}^{l}|\psi(x)|^2dx \leq \int_{-l}^{l}(1+((\operatorname{Im}k)^2+(\operatorname{Re}k)^2)x^2)e^{2(\int_{-l}^{x}|V(x')-E-i\epsilon|(x-x')dx')}dx$$

$$\leq \int_{-l}^{l}(1+((\operatorname{Im}k)^2+(\operatorname{Re}k)^2)x^2)e^{2(\int_{-l}^{l}\sqrt{m_l^2}(l-x')dx')}dx$$

$$\leq \int_{-l}^{l}(1+((\operatorname{Im}k)^2+(\operatorname{Re}k)^2)x^2)e^{2l^2m_l^2}dx$$

$$= (2l+((\operatorname{Im}k)^2+(\operatorname{Re}k)^2)l^3/3)e^{2l^2m_l^2} \tag{b}$$

By using the (a) and (b) above, we get

$$|\epsilon| \geq \frac{[(\operatorname{Re}k+\beta_+)|\psi(l)|^2+(\operatorname{Re}k-\beta_-)|\psi(-l)|^2]e^{-2l^2m_l^2}}{2l+((\operatorname{Im}k)^2+(\operatorname{Re}k)^2)l^3/3}$$

Since $\operatorname{Re}k^2=E, \operatorname{Im}k^2=\epsilon,$ we get $|\operatorname{Im}k|=|\frac{\epsilon}{2\operatorname{Re}k}|$. Now we observe the following:

$$\sqrt{E}=\sqrt{(\operatorname{Re}k)^2-(\operatorname{Im}k)^2}<\operatorname{Re}k$$

Also,

$$E+i\epsilon=(\operatorname{Re}k+i\operatorname{Im}k)^2=(\operatorname{Re}k)^2-(\operatorname{Im}k)^2+2i\operatorname{Re}k\operatorname{Im}k \tag{3.9}$$

The real part yields $(\operatorname{Re}k)^2-(\operatorname{Im}k)^2=E$. So we derive that

$$\begin{aligned}(\operatorname{Re}k)^2 &= E+(\operatorname{Im}k)^2\\ &= E+|\frac{\epsilon}{2\operatorname{Re}k}|^2\\ &< E+|\frac{\epsilon}{2\sqrt{E}}|^2\\ &= E+\frac{\epsilon^2}{4E}\end{aligned} \tag{3.10}$$

Finally, we have

$$\sqrt{E} < \operatorname{Re} k < \sqrt{E + \epsilon^2/4E}.$$

In the meantime, since the equation (3.1) is linear it may be supposed that

$$1 = \psi(-l) \le |\psi(l)| \tag{c}$$

By applying (c), we have

$$\begin{aligned} |\epsilon| &\ge \frac{[(\operatorname{Re} k + \beta_+)|\psi(l)|^2 + (\operatorname{Re} k - \beta_-)|\psi(-l)|^2]e^{-2l^2 m_l^2}}{2l + ((\operatorname{Im} k)^2 + (\operatorname{Re} k)^2)l^3/3} \\ &\ge \frac{[2 \operatorname{Re} k + (\beta_+ - \beta_-)]e^{-2l^2 m_l^2}}{2l + ((\frac{\epsilon}{2 \operatorname{Re} k})^2 + (\operatorname{Re} k)^2)l^3/3} \\ &\ge \frac{[2 \operatorname{Re} k + (\beta_+ - \beta_-)]e^{-2l^2 m_l^2}}{2l + (E + \frac{\epsilon^2}{2E})l^3/3} \end{aligned} \tag{d}$$

In addition, since

$$\begin{aligned} m_l^2 &= \sup_{-l \le x \le l} |V(x) - E + i\epsilon| \\ &= \sup_{-l \le x \le l} \sqrt{(V(x) - E)^2 + \epsilon^2} \\ &= m^2 \sup_{-l \le x \le l} \sqrt{1 + \epsilon^2/((V(x) - E)^2)} \\ &= m^2 \sqrt{1 + \frac{\epsilon^2}{m^4}} \end{aligned} \tag{3.11}$$

we have $m_l^2 = m^2\sqrt{1 + \frac{\epsilon^2}{m^4}}$, where

$$m = \sup_{-l \le x \le l} \sqrt{|V(x) - E|^2 + \epsilon^2}$$

Finally, we substitute m_l^2 to get the inequality of Theorem 3: if we have the condition that $\epsilon < \sigma$ then

$$|\epsilon| \geq \frac{[2\,\mathrm{Re}\,k + (\beta_+ - \beta_-)]e^{-2l^2m^2\sqrt{1+\frac{\epsilon^2}{m^4}}}}{2l + (E + \frac{\epsilon^2}{2E})l^3/3}$$

$$\geq \frac{[2\,\mathrm{Re}\,k + (\beta_+ - \beta_-)]e^{-2l^2m^2\sqrt{1+\frac{\sigma^2}{m^4}}}}{2l + (E + \frac{\sigma^2}{2E})l^3/3}$$

This completes the proof of the theorem. □

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