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3-connected, Claw-free, Generalized Net-free graphs are Hamiltonian

By

Susan Rae Janiszewski
Doctor of Philosophy

Mathematics

Ronald Gould, Ph.D.
Advisor

Dwight Duffus, Ph.D.
Committee Member

Vojtěch Rödl, Ph.D.
Committee Member

Accepted:

Lisa A. Tedesco, Ph.D.
Dean of the Graduate School

Date

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Susan Rae Janiszewski
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A dissertation submitted to the Faculty of the Graduate School
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Abstract

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Given a family $\mathcal{F} = \{H_1, H_2, \ldots, H_k\}$ of graphs, we say that a graph is $\mathcal{F}$-free if G contains no subgraph isomorphic to any H_i, $i = 1, 2, \ldots, k$. The graphs in the set $\mathcal{F}$ are known as *forbidden subgraphs*. In this work, we continue to classify pairs of forbidden subgraphs that imply a 3-connected graph is hamiltonian. First, we reduce the number of possible forbidden pairs by presenting families of graphs that are 3-connected and not hamiltonian. Of particular interest is the graph $K_{1,3}$, also known as the *claw*, as we show that it must be included in any forbidden pair. Secondly, we let $N_{i,j,k}$ denote the generalized net, which is the graph obtained by rooting vertex-disjoint paths of length i, j, and k at the vertices of a triangle. We show that 3-connected, $\{K_{1,3}, N_{i,j,0}\}$-free graphs are hamiltonian for $i, j \neq 0, i + j \leq 9$ and $\{K_{1,3}, N_{3,3,3}\}$-free graphs are hamiltonian. These results are best possible in the sense that no path of length i can be replaced by $i+1$ in the above net graphs. When combined with previously known results, this completes the classification of generalized nets such that a graph being $\{K_{1,3}, N_{i,j,k}\}$-free implies hamiltonicity.

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Chapter 1

Introduction

We assume from the beginning that the reader has a basic knowledge of graph theory. Throughout, we use C_k to denote a cycle with k vertices and P_k to denote a path on k vertices. The *length* of a path refers to the number of edges in the path. We define a generalized net, denoted $N_{i,j,k}$, to be a triangle with vertex disjoint paths of length i, j, and k rooted at the vertices of the triangle, and we define the graph Ł$_k$ to be the graph formed by joining two vertex disjoint triangles with a path containing k edges.

Throughout, the circumference of a graph G, denoted $c(G)$, refers to the longest cycle in G. We will also use directed cycle notation in which $\overrightarrow{C}$ indicates that we are traveling around the cycle so that the indices on the vertices of the cycle are increasing and $\overleftarrow{C}$ indicates that we are traveling around the cycle in the opposite direction. For all other terms and notation not defined in this work, see [2].

A graph G with $n \geq 3$ vertices is *hamiltonian* if G contains a cycle of length n. A *hamiltonian path* in G is a path on n vertices. A graph is *hamiltonian connected* if each pair of vertices are the endpoints of a hamiltonian path. A graph G is said to be *connected* if there exists a path between any two vertices and is k*-connected* if the removal of any set of size at most $k-1$ results in a connected graph. Likewise, a graph G is said to be k*-edge-connected* if the removal of at most $k-1$ edges results in a connected graph. Throughout we will use κ and κ' to denote connectivity and edge-connectivity, respectively.

There are many problems in graph theory that focus on determining what graph properties imply a graph with given connectivity is hamiltonian or hamiltonian connected. One of the richest areas of these problems concerns families of forbidden subgraphs, which is the focus of this dissertation.

Given a family $\mathcal{F} = \{H_1, H_2, \ldots, H_k\}$ of graphs we say that a graph is $\mathcal{F}$-free if G contains no subgraph isomorphic to any H_i, $i = 1, 2, \ldots, k$. In particular, if $\mathcal{F} = \{H\}$, we say that G is H-free. The graphs in the set $\mathcal{F}$ are known as *forbidden subgraphs*.

One graph that is commonly included in families of forbidden subgraphs that imply hamiltonian properties is $K_{1,3}$, also known as the *claw*. In [14], Matthews and Sumner presented their famous conjecture on claw-free, 4-connected graphs, which can be found below. This conjecture is still open more than 25 years after it was first published.

Conjecture 1.1. *[14] If G is a 4-connected, $K_{1,3}$-free graph, then G is hamiltonian.*

The most recent progress towards proving the Matthews-Sumner conjecture is due to Kaiser and Vrana [11]. In their paper, they lower the previous best known connectivity requirement from 6 to 5, provided a minumum degree condition is met.

Theorem 1.2. *[11] Every 5-connected claw-free graph of minimum degree at least 6 is hamiltonian.*

They further show that these conditions are sufficient to guarantee not only hamiltonicity, but hamiltonian-connectedness as well.

Theorem 1.3. *[11] Every 5-connected claw-free graph of minimum degree at least 6 is hamiltonian-connected.*

While it is not known whether 4-connected, claw-free is enough to guarantee hamiltonicity, it can easily be shown that 3-connected, claw-free is not sufficient. Several families of 3-connected, claw-free graphs that are not hamiltonian are presented in Chapter 3.

The first results regarding forbidden pairs involved graphs that are 2-connected. Bedrossian determined a complete classification of forbidden pairs which imply a 2-connected graph is hamiltonian in [1]. Faudree and Gould further generalized these results in [6] for sufficiently large graphs. The following theorems summarize their results.

Theorem 1.4. *[6] Suppose A is a connected graph and G is a 2-connected graph. Then G is A-free implies G is hamiltonian if, and only if, $A = P_3$.*

Theorem 1.5. *[6] Let R and S be connected graphs ($R, S \neq P_3$), and G a 2-connected graph of order $n \geq 10$. Then G is $\{R, S\}$-free implies G is hamiltonian if, and only if, $R = K_{1,3}$ and S is one of the graphs $C_3, P_4, P_5, P_6, Z_1, Z_2, Z_3, B, N,$ or W (see Figure 1.1.)*

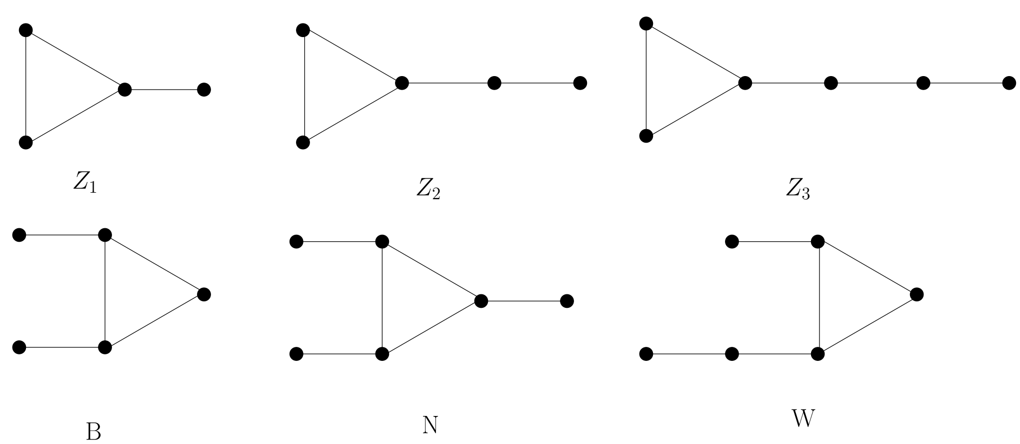

Figure 1.1: Common Forbidden Graphs for 2-connected Graphs

After considering which forbidden pairs imply a 2-connected graph is hamiltonian, it is natural to consider which forbidden pairs of graphs imply a 3-connected graph is hamiltonian. A complete classification of these pairs is not yet known, but several individual results have been determined. The claw, once again, shows up as a graph of importance. In Chapter 3, we will show that it is necessary to include the claw in any forbidden pair.

One of the first results determining a forbidden pair that implies a 3-connected graph is hamiltonian is due to Pfender and Łuczak [13]. The result is the best possible in the sense that P_{11} cannot be replaced by P_{12}.

Theorem 1.6. *[13] Every 3-connected $\{K_{1,3}, P_{11}\}$-free graph is hamiltonian.*

Another result, due to Lai, Xiong, Yan, and Yan [12], involves the graph Z_k, where Z_k is the generalized net $N_{k,0,0}$. Once again, this result is best possible since it can be shown that Z_8 cannot be replaced with Z_9.

Theorem 1.7. *[12] Every 3-connected $\{K_{1,3}, Z_8\}$-free graph is hamiltonian.*

The most recent set of known results are due to Hu and Lin in [10] and [9]. In these papers they begin to explore forbidden pairs which include the claw and a generalized net $N_{i,j,k}$.

Theorem 1.8. *[10] Every 3-connected $\{N_{5,2,2}, K_{1,3}\}$-free or $\{N_{4,3,2}, K_{1,3}\}$-free graph is hamiltonian.*

Theorem 1.9. *[9] Every 3-connected $\{N_{7,1,1}, K_{1,3}\}$-free, $\{N_{6,2,1}, K_{1,3}\}$-free, $\{N_{5,3,1}, K_{1,3}\}$-free, or $\{N_{4,4,1}, K_{1,3}\}$-free graph is hamiltonian*

It can be shown that these results are best possible in the sense that in each pair the graph $N_{i,j,k}$ cannot be replaced with $N_{i+1,j,k}$, $N_{i,j+1,k}$, or $N_{i,j,k+1}$.

A graph G is considered to be *pancyclic* if G contains a cycle of length k for all k, $3 \leq k \leq n$. In [7], Gould, Łuczak, and Pfender classified all forbidden pairs that imply a 3-connected graph is pancyclic. They found six pairs $\{X, Y\}$, where X must be $K_{1,3}$ and Y is one of P_7, $N_{i,j,k}$ where $i + j + k \leq 4$, or Ł$_1$. Of particular note is the graph Ł$_1$, since it is not a subgraph of any of the previously mentioned results.

Theorem 1.10. *[7] Every 3-connected $\{K_{1,3}, L_1\}$-free graph is pancyclic, and therefore hamiltonian.*

In this dissertation, we further classify the pairs of graphs such that G being 3-connected and $\{X, Y\}$-free implies G is hamiltonian. The results are presented in two parts. First, we present infinite families of graphs that are 3-connected and not hamiltonian in order to reduce the list of possible forbidden pairs in Chapter 3. Secondly, we prove Theorem 1.11 in Chapter 4 and Theorem 1.12 in Chapter 5. These theorems, combined with Theorems 1.7, 1.8, 1.9 and the results in Chapter 3, give a complete classification of which generalized net graphs $N_{i,j,k}$ can form a forbidden pair $\{X, Y\}$ such that a 3-connected, $\{X, Y\}$-free graph is hamiltonian.

Theorem 1.11. *Let i, j be non-zero with $i + j = 9$. Then every 3-connected, claw-free, $N_{i,j,0}$-free graph G is hamiltonian.*

Theorem 1.12. *Every 3-connected, claw-free $N_{3,3,3}$-free graph G is hamiltonian.*

Chapter 2

Background

In this chapter, we discuss the necessary tools and theorems that we will utilize in the proofs of Theorem 1.11 and Theorem 1.12.

One of the tools we will utilize is a closure concept introduced by Ryjáček [15]. Let G be a graph, let v be a vertex of G, and let $N(v)$ denote the neighborhood of v. A vertex $v \in V(G)$ is said to be *locally connected* if the neighborhood of v induces a connected subgraph in G. The *local completion of* G *at* v, denoted G'_v, is obtained by adding in the edges $\{xy \mid x, y \in N(v)$ and $xy \notin E(G)\}$. The Ryjáček closure is obtained by recursively performing the local completion operation at vertices that are locally connected until there are no such vertices remaining. Many useful properties of the Ryjáček closure in claw-free graphs are summarized in Theorem 2.1.

Theorem 2.1. *[15] Let G be a claw-free graph. Then the following are true:*

- *$cl(G)$ is uniquely determined.*
- *$cl(G)$ is the line graph of a triangle-free graph.*
- *$c(cl(G)) = c(G)$.*
- *G is hamiltonian if and only if $cl(G)$ is hamiltonian.*

Further research by Brousek, Ryjáček, and Favaron in [3] showed that certain classes of graphs were stable under the closure operation. Of particular interest to us is the result stated in Theorem 2.2.

Theorem 2.2. *[3] Let G be claw-free. If G is $N_{i,j,k}$-free then $cl(G)$ is also $N_{i,j,k}$-free.*

Combining Theorems 2.1 and 2.2, we see that when determining whether a 3-connected graph G which is $\{K_{1,3}, N_{i,j,k}\}$-free is hamiltonian, it is sufficient to show that $cl(G)$ is hamiltonian. Thus, we need only consider those graphs which are closed under the Ryjáček closure operation.

Another well-known theorem that will be of use to us is due to Harary and Nash [8]. Here we define the *line graph* of a graph G, denoted $L(G)$, to be the graph with $E(G)$ as its vertex set and two vertices are connected if and only if the corresponding edges in G are incident with one another.

We say a graph is *eulerian* if it is connected and every vertex has even degree. An eulerian subgraph H of a graph G is said to be a *dominating eulerian subgraph* if every edge of G has at least one endpoint in $V(H)$. Lastly, we define a graph to be *supereulerian* if it contains a spanning eulerian subgraph.

Theorem 2.3. *[8] Let G be a connected graph with at least 3 edges. The line graph $L(G)$ is hamiltonian if and only if G has a dominating eulerian subgraph.*

Pairing this result with the above observation that we need only consider closed graphs, we see that finding a hamiltonian cycle in a 3-connected, $\{K_{1,3}, N_{i,j,k}\}$-free graph G is equivalent to finding a dominating eulerian subgraph in the inverse line graph of $cl(G)$. We denote the inverse line graph of a graph H by $L^{-1}(H)$.

Let $X \subseteq E(G)$ be a subset of edges of G. The *contraction*, denoted G/X, is the graph obtained by identifying the two ends of each edge in X and deleting the resulting loops. For a given subgraph K, we will use G/K to denote $G/E(K)$. The *preimage* of a vertex $v \in V(G/K)$ is the set of all edges that were contracted onto v. A vertex v in a contraction of G is said to be a *nontrivial vertex* if its preimage contains at least one edge.

Let G be a graph such that $\kappa(L(G)) \geq 3$ and $L(G)$ is not complete. The core of the graph G, denoted G_0, can be found by contracting all pendant edges and contracting one of xy or yz for each path xyz where $d_G(y) = 2$. After all contractions have been made, any edge that remains after contracting either xy or yz as described above is called a *nontrivial edge*. Items (a)-(c) in the following theorem were presented by Shao in [16], with item (d) added by Lai, Xiong, Yan, and Yan in [12].

Theorem 2.4. *[12][16] Let G be a graph with $\kappa(L(G)) \geq 3$ and $L(G)$ is not complete. Let G_0 be the core of G, then each of the following holds:*

(a) G_0 is nontrivial and $\delta(G_0) \geq \kappa'(G_0) \geq 3$.

(b) G_0 is well-defined.

(c) If G_0 has a spanning eulerian subgraph, then G has a dominating eulerian subgraph.

(d) If G_0 has a dominating eulerian subgraph containing all nontrivial vertices and both ends of each nontrivial edge, then G has a dominating eulerian subgraph.

We will also use the notion of a *collapsible graph*, which was first introduced by Catlin in [4]. Let $O(G)$ denote the set of odd vertices in G. Given a subset $R \subseteq V(G)$ with $|R|$ even, a subgraph H of G is said to be an *R-subgraph* if both $O(H) = R$ and $G - E(H)$ is connected. A graph G is said to be *collapsible* if for any even subset R of G, G has an R-subgraph. Catlin further showed in [4] that every vertex in a graph G is in a unique maximal collapsible subgraph of G. We form the *reduction* of G by contracting all of the maximal collapsible subgraphs. We use G' to denote the reduced graph.

Theorem 2.5. *[4] Let G be a connected graph and let H be a collapsible subgraph of G. Then each of the following holds:*

(a) G is collapsible if and only if G/H is collapsible. In particular, G is collapsible if and only if the reduction G' is K_1;

(b) G is reduced if and only if G has no nontrivial collapsible subgraphs;

(c) $g(G') \geq 4$ and $\delta(G') \leq 3$;

(d) G is supereulerian if and only if G' is supereulerian.

Theorem 2.6. *[4] Let G be a connected graph and let H be a collapsible subgraph of G. Let the reduction of G, denoted G', have $g(G') \geq 4$, then G is supereulerian if and only if G' is supereulerian.*

Lastly, the method of proof we will use will involve examining graphs of each possible circumference. To aid in this process, we will use the following two results:

Theorem 2.7. *[5] If G is a 3-edge-connected graph with at most 13 vertices, then either G is collapsible or G is contractible to K_2, $K_{1,2}$, or the Petersen graph.*

Theorem 2.8. *[12] Let G be 3-edge-connected. If $c(G) \leq 8$, then G is supereulerian.*

Chapter 3

Further Classification of Pairs of Forbidden Graphs

In this section, we present two new theorems that reduce the number of possible pairs such that that a graph being $\{X, Y\}$-free implies a 3-connected graph is hamiltonian. Theorem 3.1 gives that in any pair $\{X, Y\}$, one of the graphs must be $K_{1,3}$. Theorem 3.2 focuses on reducing the number of graphs that can be paired with $K_{1,3}$ to form a forbidden pair.

We begin by defining five graphs which are 3-connected, claw-free, and not hamiltonian.

- $G_1 = L(H)$ where H is the graph obtained from the Petersen graph by adding one pendant edge to every vertex of P.

- $G_2 =$ the graph obtained by replacing each vertex of the Petersen graph with a K_3.

- $G_2' =$ the graph obtained by replacing all but one vertex of the Petersen graph with a K_3 and the replacing the remaining vertex with a K_t for $t \geq 4$.

- $G_3 =$ the graph obtained by blowing up each vertex of the Petersen graph to a K_t with $t \geq 6$ and replacing each edge between two K_t subgraphs with Ł$_1$.

- G_4 = the graph obtained by blowing up each vertex of the Petersen graph to a K_t with $t \geq 6$ and replacing each edge between two K_t subgraphs with an hourglass, i.e. two triangles joined at a common vertex.

The graphs G_1, G_2, G_2', G_3, and G_4 are pictures in Figure 3.1. Note that the labeled vertices in G_1 are identifications.

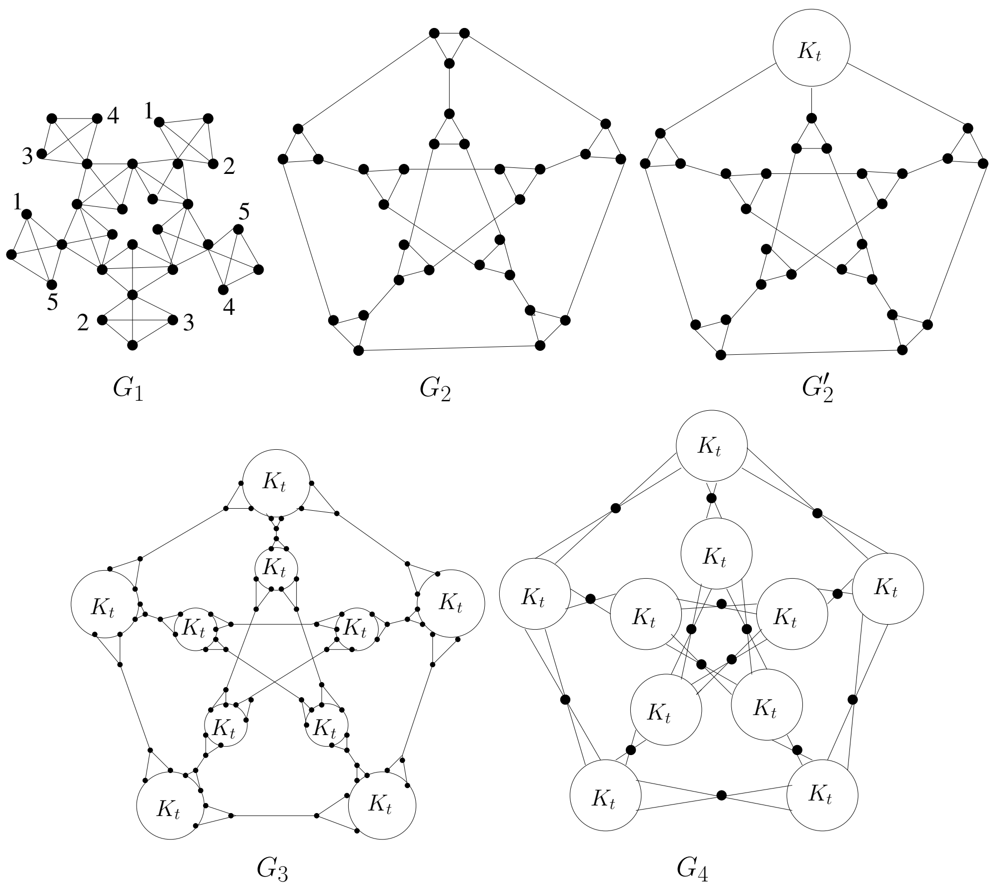

Figure 3.1: 3-connected, claw-free graphs that are not hamiltonian

We also note that the graphs in Figure 3.2, while not claw-free, are also 3-connected and not hamiltonian. In addition, the Petersen graph and the complete bipartite graph $K_{n,m}$ with $n, m \geq 3$ and $n \neq m$ are also 3-connected, not claw-free, and not hamiltonian.

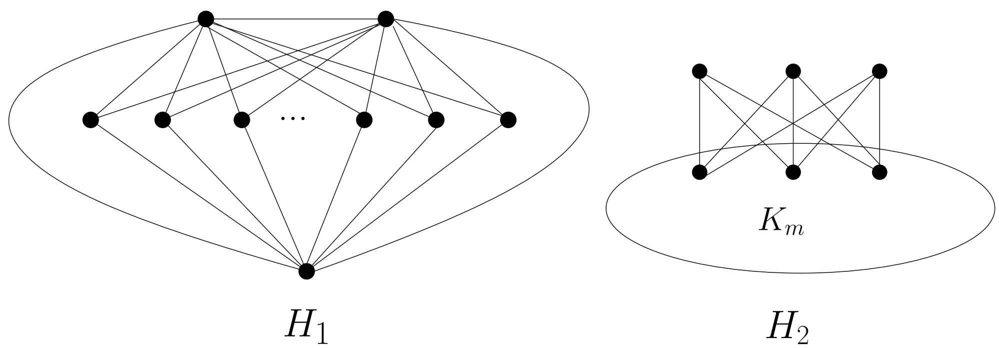

Figure 3.2: 3-connected graphs that are not claw-free and are not hamiltonian

Theorem 3.1. *Let X and Y be connected graphs with neither X nor Y equal to P_3, and let G be a 3-connected graph. If G being $\{X, Y\}$-free implies G is hamiltonian, then either X or Y must be $K_{1,3}$.*

Proof. We will proceed by considering two cases.

Case 1: Assume that X contains an induced P_4. This implies that Y must be a subgraph of every 3-connected, nonhamiltonian graph that is P_4-free. Since the complete bipartite graph $K_{n,m}$ with $n, m \geq 3$ and $n \neq m$ is P_4-free, we automatically get that Y must be a subgraph of $K_{n,m}$. The graph H_1 is P_4-free, which when paired with Y being a subgraph of $K_{n,m}$ forces Y to be a star. The largest star in H_2 is a $K_{1,3}$, thus we conclude that Y must be $K_{1,3}$.

Case 2: Assume that X does not contain an induced P_4. We can further split this case into two subcases: either X contains a cycle or it does not.

Clearly, if X contains no cycles, it must be a tree. The condition that X does not contain an induced P_4 forces X to be a star. The Petersen graph has $\Delta(G) = 3$, which forces X to have $\Delta(G) \leq 3$. Thus X must be $K_{1,3}$.

If X contains a cycle, it must be a triangle or a C_4 since any larger cycle contains an induced P_4. If X contains a triangle, then Y must be a subgraph of both the Petersen graph and $K_{3,m}$ since they are both triangle-free. From here we can easily see that Y must be $K_{1,3}$. If X contains a C_4, then Y must be a subgraph of both the Petersen graph and H_2 since they are both C_4-free. Once again, it is clear by inspection that Y must be $K_{1,3}$.

□

Now that we have established that $K_{1,3}$ must be included in any forbidden pair that implies a 3-connected graph is hamiltonian, we proceed by assuming that $X = K_{1,3}$ and reduce the number of graphs Y that can complete a forbidden pair.

Theorem 3.2. *Let Y be a connected graph with $Y \neq P_3$, and let G be a 3-connected graph. If G being $\{K_{1,3}, Y\}$-free implies G is hamiltonian, then G satisfies each of the following conditions:*

(a) $\Delta(Y) \leq 3$*;*

(b) The longest induced path in Y has at most 11 vertices;

(c) Y contains no cycles of length at least 4;

(d) The distance between two triangles in Y is odd;

(e) Y contains at most two triangles;

(f) If Y contains exactly two triangles, then it must be one of $Ł_1$, $Ł_3$, or $Ł_5$;

(g) Y is claw-free.

Proof. Since all of the graphs in Figure 3.1 are claw-free and non-hamiltonian, it must be the case that Y is an induced subgraph of each of these graphs.

Property (g) follows naturally from each of these graphs being claw-free. Since $\Delta(G_2) = 3$, Y must satisfy (a). There is no induced P_{12} in G_1, therefore (b) is satisfied. Also, (b) is best possible by Theorem 1.6.

The longest cycle in G_1 is a C_{10}, therefore any cycle in Y must be of length less than or equal to 10. When considering G_2 and G_2', the shortest cycle that is not a C_3 is a C_{10}, thus eliminating the possibility of Y containing a C_k for $4 \leq k \leq 9$. The graph G_3 does not contain a C_{10}, so we deduce that the only cycle Y can have is a C_3 and (c) is satisfied.

Condition (d) arises from the fact that all triangles in G_2 are odd distance apart.

We now look at property (e). When considering G_3, we see that the only triangles that appear distance 1 apart occur between the cliques and the addition of any additional vertex would create a $K_4 - \{e\}$. This clearly cannot happen since G_2 does not have this as a subgraph. Therefore, if there are more than two triangles in Y, each pair of triangles must be distance at least 3 by (d). To prevent the induced path of Y from becoming longer than 11, it must be the case that Y is the graph obtained by taking a triangle, joining vertex disjoint paths of length 3 to two vertices, and identifying the endpoint of each path with a vertex of an additional triangle (see Figure 3.3). However, this particular subgraph does not appear in G_4, thus (e) must be satisfied.

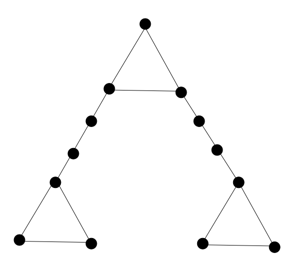

Figure 3.3: Subgraph with 3 triangles.

Lastly, consider property (f). The previous properties guarantee that if Y contains two triangles then it is either an $Ł_k$ or an $Ł_k$ with tree components attached to the vertices of the triangles. We note that G_3 contains no $Ł_1$ or $Ł_5$ subgraphs with additional pendant edges off of the triangles, and G_4 contains no $Ł_3$ or $Ł_7$ subgraphs with additional pendant edges off of the triangles. We need not worry about $Ł_k$ with k even as that would violate (d) or $Ł_k$ with $k \geq 9$ as that would violate (b). To eliminate $Ł_7$, we note that G_1 does not contain $Ł_7$ as an induced subgraph. The graph G_1 contains 10 cliques of size four. If G_1 did contain $Ł_7$ as a subgraph, two cliques would contain triangles. Two additional cliques would not contain any edges since they would contain a vertex of one of the triangles. Each of the remaining seven edges must appear in a unique clique, however there are only six unused cliques remaining in the graph.

This concludes the proof of the theorem.

$\square$

Chapter 4

Claw-Free, $N_{i,j,0}$-free Graphs

In this chapter, we focus on proving Theorem 1.11. Throughout, G will be a graph such that $L(G)$ is 3-connected and claw-free. We will use G_0 to denote the reduced core of G and C to denote a longest cycle in G_0 with vertices labeled by $c_1, c_2, \ldots, c_{c(G_0)}$. If there is more than one cycle of length $c(G_0)$, we will choose C to contain the largest number of nontrivial vertices of G_0.

Let $T_{a,b,c}$ be the tree obtained from taking disjoint paths with a, b, and c vertices and making one endpoint of each adjacent to a new vertex x. It is easy to see that if a graph G has no subgraphs isomorphic to $T_{a,b,c}$ (not necessarily induced), then $L(G)$ is $N_{a-1,b-1,c-1}$-free. By Theorems 2.3 and 2.4, proving Theorem 1.11 is equivalent to showing the following four theorems:

Theorem 4.1. *Let $Y = T_{9,2,1}$ and let G be a connected simple graph without subgraphs isomorphic to Y. Let G_0 be the core of G. If $\kappa(L(G)) \geq 3$, then G_0 has a dominating eulerian subgraph containing all the nontrivial vertices and both end vertices of each nontrivial edge.*

Theorem 4.2. *Let $Y = T_{8,3,1}$ and let G be a connected simple graph without subgraphs isomorphic to Y. Let G_0 be the core of G. If $\kappa(L(G)) \geq 3$, then G_0 has a dominating eulerian subgraph containing all the nontrivial vertices and both end vertices of each nontrivial edge.*

Theorem 4.3. *Let $Y = T_{7,4,1}$ and let G be a connected simple graph without subgraphs isomorphic to Y. Let G_0 be the core of G. If $\kappa(L(G)) \geq 3$, then G_0 has a dominating eulerian subgraph containing all the nontrivial vertices and both end vertices of each nontrivial edge.*

Theorem 4.4. *Let $Y = T_{6,5,1}$ and let G be a connected simple graph without subgraphs isomorphic to Y. Let G_0 be the core of G. If $\kappa(L(G)) \geq 3$, then G_0 has a dominating eulerian subgraph containing all the nontrivial vertices and both end vertices of each nontrivial edge.*

For the proofs of Theorems 4.1, 4.2, 4.3, and 4.4, we will proceed by considering graphs of each possible circumference. As well as considering graphs of each possible circumference, we will treat when C is a dominating cycle and when C is not a dominating cycle separately.

We begin by presenting lemmas that will be used in the proofs of each of the preceding four theorems. The first lemma shows that when $c(G) \geq 12$, the above results are true. When paired with Theorem 2.8, which states that any 3-edge-connected graph with circumference less than or equal to 8 is supereulerian, we see that the only cases left to consider are graphs of circumference 9, 10, and 11.

Lemma 4.5. *Let G be a connected, claw-free graph with no subgraphs isomorphic to $T_{i+1,j+1,1}$ where i, j are non-zero and $i + j = 9$. If $c(G) \geq 12$, then G has a dominating eulerian subgraph containing all vertices.*

Proof. Let C be a cycle of longest length. If there are no vertices off of C then this is the desired dominating eulerian subgraph. Assume not all vertices are on C. Let v be a vertex such that v is not on C but v has a neighbor, x, that is on C. Then x is the center of a $T_{i+1,j+1,1}$, contradicting our assumptions. $\square$

When handling graphs with few vertices, Theorem 2.7 gives that any graph with at most 13 vertices is either contractible to a graph that is supereulerian or is contractible to the Petersen graph. The following lemma applies when the graph is contractible to the Petersen graph.

Lemma 4.6. *If G_0 is the Petersen graph, then either G_0 has a dominating eulerian subgraph containing all nontrivial vertices or G contains a (not necessarily induced) subgraph isomorphic to $T_{i+1,j+1,1}$.*

Proof. Clearly any 9 vertices of the Petersen graph form a cycle that is a dominating eulerian subgraph. If G_0 does not have a dominating eulerian subgraph that contains all nontrivial vertices, it must be the case that all 10 vertices are nontrivial.

If every nontrivial vertex is such because it is the endpoint of a path xyz where $d(y) = 2$, then in G there must be a cycle of length at least 12 and the argument used in the proof of Lemma 4.5 gives a $T_{i+1,j+1,1}$.

Therefore, there must exist a vertex v that is the endpoint of a pendant edge vv' in G. It can be seen that v is the center of a $T_{i,j,1}$ in G_0 since there exists a P_{10} in G_0 where v is vertex $i+1$ along the path. Noting that the first and last vertices along this path were also nontrivial, there must exist vertices in G that are adjacent to those that were contracted to form G_0. This gives the desired $T_{i+1,j+1,1}$ in G.

□

4.1 Lemmas for the case that C is not dominating.

As stated previously, we will treat when C is dominating and when C is not dominating separately in the proofs of the four main theorems in this chapter. In this section, we present lemmas that will be used when C is not a dominating cycle. Each lemma corresponds to a specific graph circumference.

Lemma 4.7. *Let G be a 3-edge-connected graph with $c(G) = 11$ and $g(G) \geq 4$, and let C denote a longest cycle in G which contains the largest number of nontrivial vertices. Assume C is not a dominating cycle and every vertex not on C has at least two neighbors on C. Then G must contain each of the following: $T_{9,2,1}$, $T_{8,3,1}$, $T_{7,4,1}$, and $T_{6,5,1}$.*

Proof. Since C is not dominating there must be an edge xy such that neither x nor y are on the cycle. Since G is 3-edge-connected and by the assumption that each vertex has at most one neighbor off of C, both x and y must have at least two neighbors that lie on C. Any neighbor of y must be at least distance 3 from any neighbor of x, otherwise we create a longer cycle.

Without loss of generality, we can assume that x is adjacent to c_1. Assume that the distance between neighbors of x is 2. Label this second neighbor as c_3. In this case, the neighbors of y can either be $\{c_6, c_8\}$ (symmetrically $\{c_7, c_9\}$) or $\{c_6, c_9\}$. If the distance between neighbors of x is 3, say c_1 and c_4, then the neighbors of y must be $\{c_7, c_9\}$. However, this case is symmetric to the latter case above. This gives two nonsymmetric ways to arrange the neighbors of x and y, which are shown in Figure 4.1.

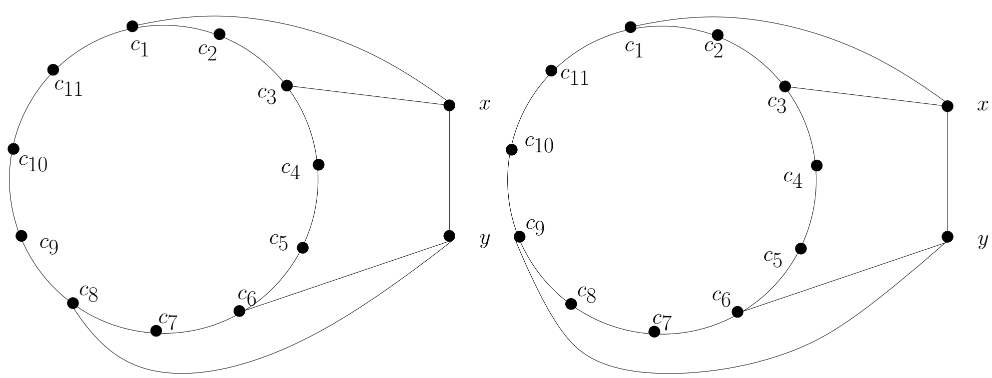

Figure 4.1: Arrangements of neighbors of x and y when $c(G_0) = 11$ and C is not dominating.

In the case where x is adjacent to $\{c_1, c_3\}$ and y is adjacent to $\{c_6, c_8\}$, we get the following trees:

$T_{9,2,1} = T\{c_1 : c_{11}, xy, c_2c_3c_4c_5c_6c_7c_8c_9c_{10}\}$

$T_{8,3,1} = T\{c_1 : x, c_{11}c_{10}c_9, c_2c_3c_4c_5c_6c_7c_8y\}$

$T_{7,4,1} = T\{c_6 : c_7, c_5c_4c_3c_2, yxc_1c_{11}c_{10}c_9c_8\}$

$T_{6,5,1} = T\{c_1 : c_2, c_{11}c_{10}c_9c_8c_7, xyc_6c_5c_4c_3\}$

The only tree above that used an edge that is not also present when x is adjacent to $\{c_1, c_3\}$ and y is adjacent to $\{c_6, c_9\}$ is the $T_{8,3,1}$. In this case, we get $T_{8,3,1} = T\{y : x, c_6c_7c_8, c_9c_{10}c_{11}c_1c_2c_3c_4c_5\}$.

□

Lemma 4.8. *Let G be a 3-edge-connected graph with $c(G) = 10$ and $g(G) \geq 4$, and let C denote a longest cycle in G which contains the largest number of nontrivial vertices. Assume C is not a dominating cycle and every vertex not on C has at least 2 neighbors on C. Then either G has a dominating eulerian subgraph or G must contain each of the following: $T_{9,2,1}$, $T_{8,3,1}$, $T_{7,4,1}$, and $T_{6,5,1}$.*

Proof. Once again, by 3-edge-connectedness, each of x and y must have at least two additional neighbors. Since G is triangle-free and placing a neighbor of x and a neighbor of y less than distance 3 apart creates a longer cycle, it can be seen that there is only one way (up to symmetry) to place these neighbors. Without loss of generality, we can assume that x is adjacent to $\{c_1, c_3\}$ and y is adjacent to $\{c_6, c_8\}$. This configuration is shown in Figure 4.2. With these neighbors of x and y, the vertices of C can be partitioned into three sets with each vertex in a given set being symmetric to the other vertices in the same set: $\{c_1, c_3, c_6, c_8\}$, $\{c_2, c_7\}$, and $\{c_4, c_5, c_9, c_{10}\}$.

If G has at most 13 vertices then it has a dominating eulerian subgraph by Theorem 2.7, thus we can assume that G contains at least 14 vertices. Let v be one of the additional vertices not on C.

Assume v is adjacent to a vertex in the group $\{c_2, c_7\}$, without loss of generality say c_2. Then we get the following trees:

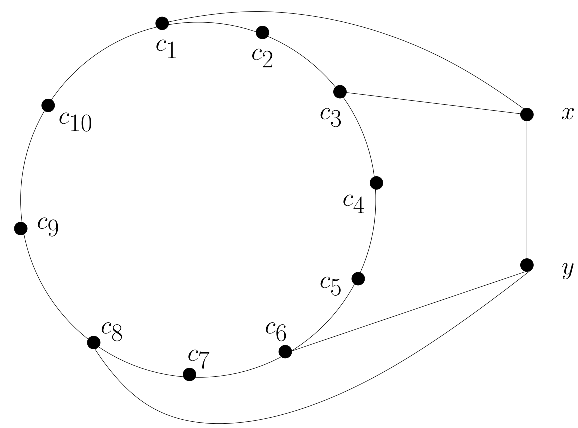

Figure 4.2: Arrangement of neighbors of x and y when $c(G_0) = 10$ and C is not dominating.

$T_{9,2,1} = T\{c_6 : c_7, c_5c_4, yc_8c_9c_{10}c_1xc_3c_2v\}$,
$T_{8,3,1} = T\{x : y, c_1c_2v, c_3c_4c_5c_6c_7c_8c_9c_{10}\}$,
$T_{7,4,1} = T\{c_8 : c_7, yc_6c_5c_4, c_9c_{10}c_1xc_3c_2v\}$,
$T_{6,5,1} = T\{c_2 : v, c_3c_4c_5c_6c_7, c_1c_{10}c_9c_8yx\}$.

If v is adjacent to a vertex in the group $\{c_4, c_5, c_9, c_{10}\}$, without loss of generality say c_4, then we get the following trees:

$T_{9,2,1} = T\{c_4 : v, c_3c_2, c_5c_6c_7c_8c_9c_{10}c_1xy\}$,
$T_{8,3,1} = T\{c_4 : v, c_5c_6c_7, c_3xyc_8c_9c_{10}c_1c_2\}$,
$T_{7,4,1} = T\{c_4 : v, c_5c_6yx, c_3c_2c_1c_{10}c_9c_8c_7\}$,
$T_{6,5,1} = T\{c_8 : c_7, yc_6c_5c_4v, c_9c_{10}c_1c_2c_3x\}$.

Therefore, we can assume that v has all its adjacencies in the set $S = \{c_1, c_3, c_6, c_8\}$. Assume that v has at least three adjacencies on C. Since all 3-subsets of S are symmetric, we can assume that v is adjacent to $\{c_1, c_3, c_6\}$. This gives the longer cycle $c_1vc_6c_5c_4c_3xyc_8c_9c_{10}c_1$. Therefore, it must be the case that v has two neighbors on C and one neighbor not on C. Denote the neighbor not on C by v'. Under these conditions we get the following trees:

$$T_{9,2,1} = T\{c_1 : v, xy, c_2c_3c_4c_5c_6c_7c_8c_9c_{10}\},$$
$$T_{8,3,1} = T\{x : y, c_1vv', c_3c_4c_5c_6c_7c_8c_9c_{10}\},$$
$$T_{7,4,1} = T\{c_1 : v, c_{10}c_9c_8c_7, c_2c_3c_4c_5c_6yx\},$$
$$T_{6,5,1} = T\{c_1 : v, c_{10}c_9c_8yx, c_2c_3c_4c_5c_6c_7\}.$$

□

Lemma 4.9. *Let G be a 3-edge-connected graph with $c(G) = 9$ and $g(G) \geq 4$. If C is a longest cycle in G such that every vertex not on C has at least two neighbors on C, then C must be a dominating cycle.*

Proof. Since G is 3-edge-connected, each of x and y must have at least two additional neighbors. It can easily be seen that there is no way to pick the two neighbors of x and the two neighbors of y without either violating G being triangle free or placing a neighbor of x and a neighbor of y less than distance 3 apart, which creates a longer cycle. Both of these contradict our original assumptions.

□

4.2 Lemmas for the case that C is dominating.

Much of the structure of a reduced graph core, G_0, of a specific circumference is determined regardless of whether we assume that G_0 is $T_{9,2,1}$-free, $T_{8,3,1}$-free, $T_{7,4,1}$-free, or $T_{6,5,1}$-free. In this section, we will establish those properties that we will use in each of the four main proofs.

Before presenting any results, we note that in our proofs we are searching for a dominating eulerian circuit that contains all nontrivial vertices. If C contains all nontrivial vertices, it is the dominating eulerian circuit that we are looking for. Therefore, we can assume that in each of these cases there must be at least one vertex w such that w is a nontrivial vertex of G_0 and w

is not on C. This implies there exists a vertex w' such that $w' \notin V(G_0)$ but $ww' \in E(G)$. Since C was chosen to contain the largest number of nontrivial vertices, we get Lemma 4.10.

Lemma 4.10. *Let w be a nontrivial vertex not on C. If w is adjacent to two vertices that are distance 2 apart, say c_{i-1} and c_{i+1}, then the vertex c_i must be nontrivial.*

Proof. The cycle C can be rerouted to include the subpath $c_{i-1}wc_{i+1}$ instead of the subpath $c_{i-1}c_ic_{i+1}$. If c_i were trivial, then the alternate cycle would contain more nontrivial vertices. This contradicts our original choice of C.

□

By 3-edge-connectivity of G_0 and the fact that C is dominating, w must have at least three neighbors on C. When $c(G_0) = 9$, there are 3 nonisomorphic ways to place the three neighbors of w and maintain the property of being triangle-free. These can be classified by the number of vertices between consecutive neighbors of w and are $\{1, 1, 4\}$, $\{1, 2, 3\}$, and $\{2, 2, 2\}$ and are shown in Figure 4.3. Note, that in some of these placements certain vertices on C are forced to be nontrivial by Lemma 4.10, these vertices are represented with triangles. When referring to specific neighbors of w, we will use the labels shown in Figure 4.3.

Lemma 4.11. *Let $c(G_0) = 9$, C be a dominating cycle, and w be a vertex off of C. If the number of vertices between consecutive neighbors of w is given by $\{2, 2, 2\}$, then any other vertex v that is also not on the cycle must have at least 2 neighbors in common with w.*

Proof. Let c_i denote a neighbor of w. If v has neighbors $\{c_{i-1}, c_{i+1}\}$ then $c_{i-1}vc_{i+1}c_iwc_{i+3}\overrightarrow{C}c_{i-1}$ is a C_{10}. Similarly, if v has neighbors $\{c_{i-1}, c_{i+2}\}$ then $c_{i-1}vc_{i+2}c_{i+1}c_iwc_{i+3}\overleftarrow{C}c_{i-1}$ is a C_{11}. Lastly, if v has neighbors $\{c_{i-2}, c_{i+2}\}$ then $c_{i-2}vc_{i+2}c_{i+1}c_iwc_{i+3}\overrightarrow{C}c_{i-2}$ is a C_{10}. Each of these contradict $c(G_0) = 9$.

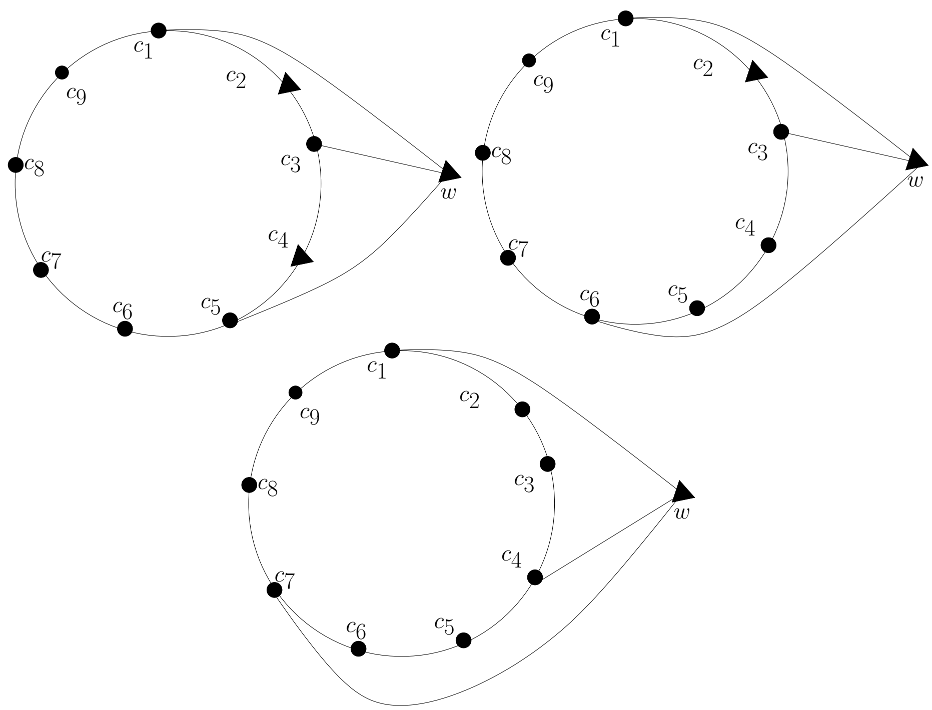

Figure 4.3: Possible neighbor placements on C_9

Therefore, as v has 3 neighbors on C, it must have at least two in common with w. $\square$

When $c(G_0) = 10$, there are four non-isomorphic ways to place the neighbors of w on C which can be categorized by the number of vertices between consecutive neighbors of w. These four ways are $\{1, 1, 5\}$, $\{1, 2, 4\}$, $\{1, 3, 3\}$, and $\{2, 2, 3\}$ and are depicted in Figure 4.4. Any vertex that is forced to be nontrivial by Lemma 4.10 is denoted by a triangle. Once again, when referring to a specific configuration, we will use the labels shown in the figure to identify neighbors of w.

When $c(G_0) = 11$, there are five nonisomorphic ways to place the neighbors of w. The specific configurations are not utilized in any of our proofs, so we

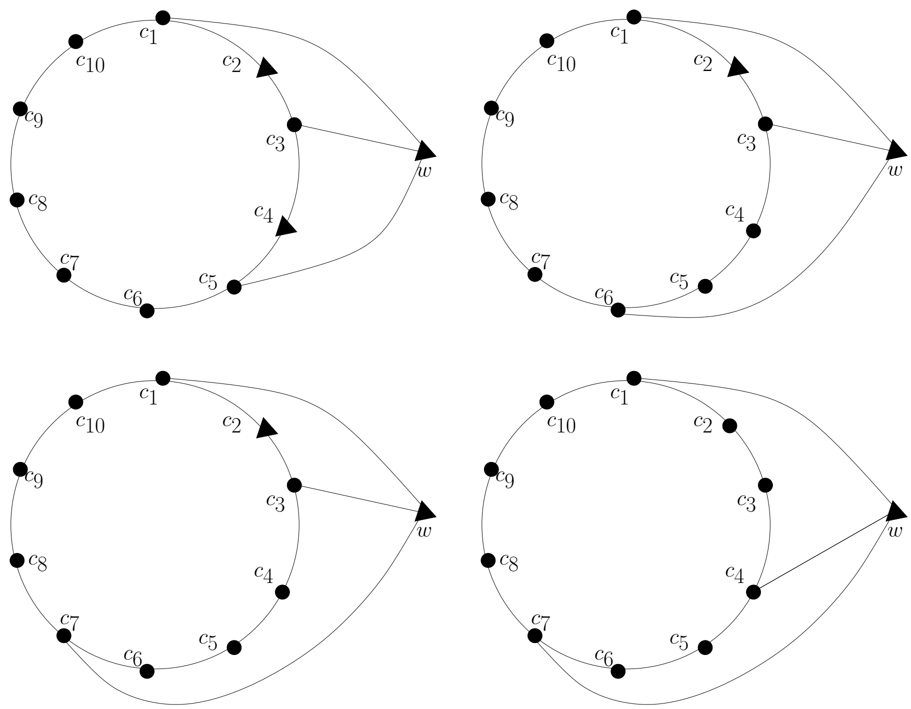

Figure 4.4: Possible neighbor placements on C_{10}

omit these details.

Regardless of the value of $c(G_0)$, if all of the vertices that are not on C happen to have the same set of three adjacencies on C, then the graph G_0 has a dominating eulerian circuit. This is described in Lemma 4.12.

Lemma 4.12. *Given a dominating cycle C in a graph G, if there are at least two vertices not on C and all such vertices have three neighbors on C in common, say $S = \{s_1, s_2, s_3\}$, then G has a spanning eulerian circuit.*

Proof. The exact dominating eulerian circuit depends on the parity of the number of vertices not on C.

First consider when there are an even number of vertices not on C. These can be paired as sets $\{x, y\}$. Begin the circuit by traversing C starting and ending at s_1. Then for each pair $\{x, y\}$, append $s_1xs_2ys_1$ to the end of the circuit. This is clearly a dominating circuit since all vertices of G are included.

Next consider when there are an odd number of vertices not on C. Once again, we begin the trail by traversing C starting and ending at s_1. By assumption, there are at least three vertices not on C. Take three of these vertices, $\{x, y, z\}$, and append $s_1xs_2ys_3zs_1$ to the circuit. If there are vertices remaining, there must be an even number. Therefore, we pair them up and append $s_1xs_2ys_1$ to the circuit for each pair. As with the previous case, the circuit must be a dominating eulerian circuit since all vertices have been included.

□

Corollary 4.13. *Given a dominating cycle C in the reduced core G_0 of a graph G such that there are at least two nontrivial vertices or endpoints of nontrivial edges that are not on C, if all such vertices have three neighbors on C in common, say $S = \{s_1, s_2, s_3\}$, then G has a dominating eulerian circuit that contains all nontrivial vertices and both endpoints of all nontrivial edges.*

Proof. By Theorem 2.4, G has a dominating eulerian circuit if and only if G_0 has a dominating eulerian circuit containing all nontrivial vertices and both endpoints of nontrivial edges. If all nontrivial vertices and endpoints of nontrivial edges have the same adjacencies on C, the dominating eulerian circuit described in the proof of Lemma 4.12 suffices.

□

Before we begin the proofs of the main theorems, we note one more thing about the cases where C is a dominating cycle. In these cases we utilize the existence of a nontrivial vertex w. The edge ww' that is contracted to form G_0 can either be a pendant edge or it could be part of a path of length 3. If an edge that corresponds to a nontrivial vertex is used at the end of a path that contains more than one edge, then it does not matter which type of edge was contracted. We note that in G it might be necessary to end the path with $w'w$ instead of ww' as listed. However, as convention, we will assume that the edge ww' is in the path. In the case that the nontrivial vertex is included in a $T_{i,j,k}$ as the central vertex of degree three, further argument is necessary to show that the desired subgraphs still occur when the edge ww' is part of a path of length 3.

4.3 Proof of Theorem 4.1: $T_{9,2,1}$

The cases where C is a dominating cycle and C is not a dominating cycle will be handled separately. We will split the cases further based on $c(G_0)$. Recall that by Lemma 4.5 and Theorem 2.8, we need only consider $c(G_0) = 9, 10$, and 11.

4.3.1 Case 1: C is not a dominating cycle.

Let xy denote an edge of G_0 such that neither vertex is on C. Without loss of generality we can assume there is a path from x to c_1.

In the case where $c(G_0) = 11$, there is a $T_{9,2,1}$ described by $T\{c_1 : c_{11}, xy, c_2c_3c_4c_5c_6c_7c_8c_9c_{10}\}$. This observation along with Lemmas 4.7, 4.8, and 4.9, give the desired result provided that when $c(G_0) = 10$ every vertex not on C has at least two neighbors on C. This result is presented in Lemma 4.14.

Lemma 4.14. *If G_0 is the reduced core of a graph without subgraphs isomorphic to $T_{9,2,1}$ and $c(G_0) = 10$, then every vertex not on a longest cycle C has at least 2 neighbors on C.*

Proof. First assume there is a vertex w such that none of the neighbors of w can be found on C. Since G_0 is 3-edge connected, w must have at least 3 neighbors. Denote these by x, y, and z. If any of x, y, or z have a neighbor off of C other than w, say z has neighbor z', then there is a $T_{9,2,1}$ centered at w where the paths of length 1 and 2 are y and zz' respectively, and the path of length 9 can be formed by x and the vertices of C. If two neighbors of w share a common neighbor on C, say x and y are both adjacenct to c_1, then $T\{c_1, x, yw, c_2\overrightarrow{C}c_{10}\}$ is a $T_{9,2,1}$. Thus, all neighbors of x, y, and z must be on C and must be distinct. To prevent creating a longer cycle, all neighbors of x, y, and z must be at least distance 4 apart. Clearly there is not enough room to place all the neighbors under these restrictions, so each vertex w must have at least one neighbor on C.

Now assume there is a vertex w that has exactly one neighbor on C. Denote this neighbor by c_1. Since w has at least 3 neighbors, there must be at least 2 not on C, label these as x and y. If x or y have a neighbor off of C besides w, say y has neighbor z, then we have the $T_{9,2,1}$ described by $T\{w : x, yz, c_1c_2c_3c_4c_5c_6c_7c_8c_9\}$. If x and y have a neighbor $c_i \in V(C)$ in common, then c_i is the center of a $T_{9,2,1}$ with the paths of length one and two as x and yw, and the path of length 9 traveling around the cycle. Thus all neighbors of x and y must be distinct. Also note that x and y cannot be adjacent to c_1 since G_0 is triangle-free. They also cannot be adjacent to c_2, c_3, c_9, or c_{10} as that would create a longer cycle. The remaining 5 vertices on the cycle are $\{c_4, c_5, c_6, c_7, c_8\}$. It is clear that there is no way to choose the four neighbors of x and y from these vertices without either creating a longer cycle or a triangle. Therefore, it must be the case that w has at least two neighbors on C.

□

4.3.2 Case 2a: C is a dominating cycle and $c(G_0) = 11$.

Recall that if C contains all nontrivial vertices, then it is the desired dominating eulerian subgraph. So, let w be a nontrivial vertex not on C. Let

w' denote the neighbor of w that is in $V(G)$ but not $V(G_0)$. Without loss of generality, assume that w is adjacent to c_1. Then the subgraph $T\{c_1 : c_{11}, ww', c_2c_3c_4c_5c_6c_7c_8c_9c_{10}\}$ is present.

4.3.3 Case 2b: C is a dominating cycle and $c(G_0) = 10$.

By Theorem 2.7, any graph with at most 13 vertices is supereulerian. Therefore, there must be at least 4 vertices not on C. Let w be a nontrivial vertex not on C with neighbor $w' \in V(G)/V(G_0)$. Let v be one of the other vertices not on C.

If v and w share an adjacency on C, say both are adjacent to c_1, then $T\{c_1 : v, ww', c_2c_3c_4c_5c_6c_7c_8c_9c_{10}\}$ is a $T_{9,2,1}$. Also, if a neighbor of v is distance 2 from a neighbor of w, say w is adjacent to c_1 and v is adjacent to c_3, then there is a $T_{9,2,1}$ described by $T\{c_1 : c_2, ww', c_{10}c_9c_8c_7c_6c_5c_4c_3v\}$.

We will now consider each of the four possible configurations for the neighbors of w as shown in Figure 4.4.

Let the neighbors of w be $\{c_1, c_3, c_5\}$, i.e. they are arranged with $\{1, 1, 5\}$ vertices between consecutive neighbors. By the previous observations that v cannot share neighbors with w or have a neighbor distance 1 or 2 from a neighbor of w, the neighbors of v must come from the set $\{c_2, c_4, c_6, c_8, c_{10}\}$. Since v has at least three neighbors on C, any set of neighbors must include one of $\{c_6, c_8, c_{10}\}$. In each case, the graph contains a $T_{9,2,1}$. If v is adjacent to c_8, the $T_{9,2,1}$ is $T\{c_8 : v, c_7c_6, c_9c_{10}c_1c_2c_3c_4c_5ww'\}$. If v is adjacent to c_{10}, it is $T\{c_{10} : v, c_1c_2, c_9c_8c_7c_6c_5c_4c_3ww'\}$. The case where v is adjacent to c_6 is symmetric to the case where v is adjacent to c_{10}.

Now, let the neighbors of w be $\{c_1, c_3, c_6\}$, i.e. they are arranged with $\{1, 2, 4\}$ vertices between consecutive neighbors. Since v cannot have any neighbors in common with w or distance 2 from a neighbor of w, it must be the case that c_2, c_7, and c_{10} are the three neighbors of v. This results in the longer cycle given by $c_{10}vc_2c_1wc_3c_4c_5c_6c_7c_8c_9c_{10}$.

Let the neighbors of w be $\{c_1, c_3, c_7\}$, i.e. they are arranged with $\{1, 3, 3\}$ vertices between consecutive neighbors. The set of vertices that are not neighbors of w or distance 2 from the neighbors of w consists of $\{c_2, c_4, c_6, c_8, c_{10}\}$. Any set of 3 neighbors of v must include either one of $\{c_4, c_{10}\}$, which are symmetric, or one of $\{c_6, c_8\}$, which are symmetric as well. If v is adjacent to c_4, there is a $T_{9,2,1}$ described by $T\{c_4 : v, c_5c_6, c_3c_2c_1c_{10}c_9c_8c_7ww'\}$. If v is adjacent to c_6, there is a $T_{9,2,1}$ described by $T\{c_6 : v, c_5c_4, c_7c_8c_9c_{10}c_1c_2c_3ww'\}$.

Lastly, consider the case where the neighbors of w are arranged with gaps of size $\{2, 2, 3\}$ and are labeled as c_1, c_4, and c_7. The only vertices that are not neighbors of w or distance 2 from a neighbor of w are c_8 and c_{10}. Therefore, it is not possible to place three neighbors of v without creating a $T_{9,2,1}$.

4.3.4 Case 2c: C is a dominating cycle and $c(G_0) = 9$.

If $|V(G_0)| \leq 13$ then by Lemmas 4.6 and 2.7 either G_0 is supereulerian or contains a $T_{9,2,1}$. So it can be assumed that $|V(G_0)| \geq 14$. This gives at least 5 vertices off of C, at least one of which is nontrivial. As before, we label this vertex w and its contracted neighbor as w'. Consider the three non-symmetric ways to place the neighbors of w as shown in Figure 4.3.

Let the neighbors of w be c_1, c_3, and c_5. We first show that v must have at least one adjacency in common with w. Note that if a vertex v is adjacent to any of the pairs $\{c_{i-1}, c_{i+1}\}$, $\{c_{i-1}, c_{i+2}\}$, $\{c_{i-2}, c_{i+1}\}$ for $i \in \{1, 3, 5\}$ a longer cycle can be found. If we consider v adjacent to c_2, we see that v cannot be adjacent to any vertex from the set $\{c_1, c_3, c_4, c_5, c_8, c_9\}$ since that would create either a triangle or a longer cycle. Therefore v must be adjacent to two of $\{c_6, c_7, c_8\}$. To prevent a triangle, the adjacencies must be c_6 and c_8. This gives rise to the longer cycle $c_1wc_5c_4c_3c_2vc_6c_7c_8c_9c_1$. Therefore v cannot be adjacent to c_2. By symmetry, v also cannot be adjacent to c_4. Since there is no way to place 3 neighbors of v among $\{c_6, c_7, c_8, c_9\}$ without creating a triangle, v must have at least one neighbor from $\{c_1, c_3, c_5\}$.

By Lemma 4.10, both c_2 and c_4 must be nontrivial. If v is adjacent to c_1, we get $T\{c_1 : v, ww', c_9c_8c_7c_6c_5c_4c_3c_2c_2'\}$. The case where v is adjacent to c_5 is symmetric. If v is adjacent to c_3 we get $T\{c_3 : v, ww', c_4c_5c_6c_7c_8c_9c_1c_2c_2'\}$.

Now let the neighbors of w be c_1, c_3, and c_6. By Lemma 4.10, c_2 must be nontrivial. Consider the places where an additional vertex v can be adjacent. If v is adjacent to c_1 there is a $T_{9,2,1}$ described by $T\{c_1 : v, ww', c_9c_8c_7c_6c_5c_4c_3c_2c_2'\}$. The case where v is adjacent to c_3 is similar, with the long path going around the cycle in the opposite direction.

If v was adjacent to c_2 it could not be adjacent to $\{c_1, c_3, c_4, c_5, c_8, c_9\}$ since each of those would create either a triangle or a longer cycle. But this forces v to be adjacent to both c_6 and c_7, which creates a triangle. If v is adjacent to c_4 it cannot be adjacent to $\{c_2, c_3, c_5, c_7, c_8, c_9\}$ since each of these create either a triangle or a longer cycle. The vertex v also cannot be adjacent to c_1, by the argument in the previous paragraph. This leaves only c_6, therefore there is not enough room to place all three adjacencies of v.

The above arguments show that no vertex v can be adjacent to any of $\{c_1, c_2, c_3, c_4\}$. The only way to fit three adjacencies among $\{c_5, c_6, c_7, c_8, c_9\}$ without creating a triangle is to have v adjacent to $\{c_5, c_7, c_9\}$. This configuration leads to the longer cycle $c_1c_2c_3wc_6c_5vc_7c_8c_9c_1$.

The last case to consider is when the neighbors of w are c_1, c_4, and c_7. First assume that there is a vertex v that has all the same adjacencies as w. Let x be a vertex that has an adjacency different than w and v. By symmetry all choices are isomorphic, so let x be adjacent to c_9. This gives a $T_{9,2,1}$ described by $T\{c_1 : v, ww', c_2c_3c_4c_5c_6c_7c_8c_9x\}$. Thus, if one vertex has all the same adjacencies as w, then all of the vertices not on C have the same adjacencies and we have a dominating eulerian circuit by Lemma 4.12.

Lastly, we need to consider the case where no vertex has the same set of adjacencies as w. By Lemma 4.11, each of the vertices not on C must have at least two adjacencies in common with w. Since there are at least four other vertices not on C, there must exist two vertices, say v_1 and v_2, that share the same pair of adjacent vertices with w. Without loss of generality, we can assume these common neighbors to be c_1 and c_4. To keep G_0 triangle-free, the third adjacency of v cannot be one of $\{c_2, c_3, c_5, c_9\}$. The case where v is adjacent to c_7 is handled above, so that leaves c_6 and c_8 as possible adjacencies for v_1 and v_2. Note that being adjacent to c_6 is symmetric to being adjacent to c_8, so there are only two cases to consider.

First suppose that v_1 is adjacent to c_6 and v_2 is adjacent to c_8. In this case there exists a $T_{9,2,1}$ described by $T\{c_4 : c_5, c_3c_2, v_2c_8c_9c_1v_1c_6c_7ww'\}$. Next suppose that v_1 and v_2 are both adjacent to c_6. This gives a $T_{9,2,1}$ described by $T\{c_4 : v_1, c_3c_2, c_5c_6v_2c_1c_9c_8c_7ww'\}$.

This concludes the proof of Theorem 4.1.

□

4.4 Proof of Theorem 4.2: $T_{8,3,1}$

The cases where C is a dominating cycle and C is not a dominating cycle will be handled separately. We will split the cases further based on $c(G_0)$. Recall that by Lemma 4.5 and Theorem 2.8, we need only consider $c(G_0) = 9, 10$, and 11.

4.4.1 Case 1: C is not a dominating cycle.

Lemmas 4.7, 4.8, and 4.9, give the desired result provided that every vertex not on C has at least two neighbors on C when $c(G_0) = 10$ or 11. This result is presented in Lemma 4.15.

Lemma 4.15. *If G_0 is the reduced core of a graph without subgraphs isomorphic to $T_{8,3,1}$ and $c(G_0) = 10$ or 11, then every vertex not on a longest cycle C has at least 2 neighbors on C.*

Proof. First assume that there is a vertex w such that w has no neighbors on C. By 3-edge-connectivity, w must have at least three neighbors. Denote these by x, y, and z. There must be a path from one of these vertices to C. Without loss of generality, we can assume that there is a path from x to c_1. This gives the following $T_{8,3,1}$ (the parentheses denote the path used when $c(G_0) = 11$): $T\{c_1 : (c_{11})c_{10}, xwy, c_2c_3c_4c_5c_6c_7c_8c_9\}$. Therefore every vertex must have at least one neighbor on C.

Assume there is a vertex w that has exactly one neighbor on C. This implies that w must have at least two neighbors not on C, say x and y. Since every vertex has at least one neighbor on C, x must have a neighbor on the cycle, say c_i. This gives a $T_{8,3,1}$ centered at c_i where the paths of length 1 and 8 are obtained by traversing along the cycle and the path of length three is xwy. Thus, each vertex can have at most one neighbor off of C and we have the desired result.

□

4.4.2 Case 2a: C is a dominating cycle and $c(G_0) = 11$.

If all nontrivial vertices of G_0 are on C, then this is the desired dominating eulerian subgraph. Therefore, it must be the case that there is a vertex w not on C that is nontrivial. Let w' denote the vertex that was contracted when forming G_0. Also note that by Theorem 2.7, any graph G_0 with $|V(G_0)| \leq 13$ must be supereulerian, so we can assume that G_0 has at least 14 vertices.

If any two vertices not on C, say v_1 and v_2, have neighbors that are distance 2 apart, say c_1 and c_3 respectively, then c_1 is the center of the $T_{8,3,1}$ described by $T\{c_1 : v_1, c_2c_3v_2, c_{11}c_{10}c_9c_8c_7c_6c_5c_4\}$. If these vertices have neighbors at distance 4, say c_1 and c_5, then $T\{c_1 : v_1, c_2c_3c_4, c_{11}c_{10}c_9c_8c_7c_6c_5v_2\}$ is a $T_{8,3,1}$.

If any vertex v not on C has a neighbor that is distance 1 from a neighbor of w, say w is adjacent to c_1 and v is adjacent to c_2, then there is a $T_{8,3,1}$ described by $T\{c_2 : v, c_1ww', c_3c_4c_5c_6c_7c_8c_9c_{10}c_{11}\}$. If the neighbor of v is distance 5 from a neighbor of w, say w is adjacent to c_1 and v is adjacent to c_6, then there is a $T_{8,3,1}$ described by $T\{c_6 : v, c_5c_4c_3, c_7c_8c_9c_{10}c_{11}c_1ww'\}$.

Combining the results in the previous two paragraphs, if w is adjacent to c_1, any vertex not on C cannot have a neighbor distance 1, 2, 4 or 5 from c_1. This forces all vertices not on C, including w, to have neighbors (without loss of generality) c_1, c_4, and c_9 and G_0 contains a dominating eulerian circuit by Lemma 4.12.

4.4.3 Case 2b: C is a dominating cycle and $c(G_0) = 10$.

Since G_0 cannot be contractible to the Petersen graph, Theorem 2.7 gives that either G_0 is supereulerian or $|V(G_0)| \geq 14$. Therefore, we can assume that G_0 has at least 4 vertices off of C. Let w be a nontrivial vertex. Without loss of generality, we may assume w is adjacent to c_1. Let v denote an additional vertex that is not on C.

If v has a neighbor that is distance 1 from a neighbor of w, without loss of generality say v is adjacent to c_2, then $T\{c_2 : v, c_1ww', c_3c_4c_5c_6c_7c_8c_9c_{10}\}$ is present. If v has a neighbor that is distance 4 from a neighbor of w, without loss of generality say w is adjacent to c_1 and v is adjacent to c_5, then we have $T\{c_5 : v, c_4c_3c_2, c_6c_7c_8c_9c_{10}c_1ww'\}$.

We now consider the possible adjacencies of w as shown in Figure 4.4.

First, assume the adjacencies of w are c_1, c_3, and c_5. There are only two vertices, c_3 and c_8, that are not distance 1 or 4 from a neighbor of w. Thus when placing the three neighbors of v, we must get a $T_{8,3,1}$.

Next, assume the adjacencies of w are c_1, c_3, and c_7. In this case, all vertices of C are either distance 1 or 4 from a neighbor of w. Thus the addition of any vertex v must give a $T_{8,3,1}$.

Finally, assume the adjacencies of w are c_1, c_4, and c_7. In this case the only vertices that are not distance 1 or 4 from a neighbor of w are c_4 and c_9. Once again, there is no way to place three neighbors of v without creating a $T_{8,3,1}$.

Lastly, consider the case where the adjacencies of w are c_1, c_3, and c_6. By Lemma 4.10, c_2 must be nontrivial. In the case where c_2' is not part of the preimage of the cycle C in G, there is a $T_{8,3,1}$ described by $T\{c_2 : c_2', c_1ww', c_3c_4c_5c_6c_7c_8c_9c_{10}\}$. Assuming that c_2' is in the preimage of C and w' is either a pendant vertex or is on the path from w to c_6, there is a $T_{8,3,1}$ described by $T\{w : w', c_1c_2c_2', c_3c_4c_5c_6c_7c_8c_9c_{10}\}$. The case where w' is on the path from w to c_1 is symmetric, with the resulting $T_{8,3,1}$ being described by $T\{w : w', c_6c_5c_4, c_3c_2c_2'c_1c_{10}c_9c_8c_7\}$.

It is now only necessary to consider when w' is located on the path from w to c_3. In this case, we consider the neighbors of an additional vertex v. If v is adjacent to c_2, there is a $T_{8,3,1}$ isomorphic to the one described when c_2' is not on the preimage of C. When v is adjacent to c_4 or c_5 we get $T\{c_4 : v, c_3w'w, c_5c_6c_7c_8c_9c_{10}c_1c_2\}$ or $T\{c_5 : v, c_4c_3w', c_6c_7c_8c_9c_{10}c_1c_2c_2'\}$, respectively. When v is adjacent to c_7 there is a $T_{8,3,1}$ described by $T\{c_7 : v, c_8c_9c_{10}, c_6c_5c_4c_3c_2c_2'c_1w\}$, and when v is adjacent to c_8 there is a $T_{8,3,1}$ described by $T\{c_8 : v, c_7c_6c_5, c_9c_{10}c_1c_2'c_2c_3ww'\}$. The vertices c_{10} and c_9 are symmetric to c_7 and c_8, respectively. This forces all vertices off of C to be adjacent to $S = \{c_1, c_3, c_6\}$ and a dominating eulerian circuit exists by Lemma 4.12.

4.4.4 Case 2c: C is a dominating cycle and $c(G_0) = 9$.

If $|V(G_0)| \leq 13$, then by Lemmas 4.6 and 2.7 either G_0 is supereulerian or contains a $T_{8,3,1}$. So it can be assumed that $|V(G_0)| \geq 14$. This implies at least 5 vertices off of C, at least one of which is nontrivial. As before, we label this vertex w. We consider each of the three ways to place the neighbors of w as shown in Figure 4.3.

First, consider the case where neighbors of w are c_1, c_3, and c_5. By Lemma 4.10, both c_2 and c_4 must be nontrivial. Then the $T_{8,3,1}$ described by $T\{c_4 : c_4', c_3ww', c_5c_6c_7c_8c_9c_1c_2c_2'\}$ is present. Since c_2 and c_4 are symmetric under this configuration, this $T_{8,3,1}$ is sufficient provided at least one of c_2' and c_4' is not located on the preimage of C. If both happen to lie on the preimage of C, then in the case where w' is either a pendant vertex or lies on the path from w to c_1 the $T_{8,3,1}$ described by $T\{w : w', c_5c_4c_4', c_3c_2c_2'c_1c_9c_8c_7c_6\}$ is present. In the case where w' is located on a path between w and c_5, the $T_{8,3,1}$ described by $T\{w : w', c_1c_2c_2', c_3c_4'c_4c_5c_6c_7c_8c_9\}$ is present. This leaves only the case where w' is located on a path between w and c_3.

When both c_2' and c_4' are located on the main cycle and w' lies on a path from w to c_3, we consider an additional vertex v and its possible adjacencies on C. If v is adjacent to c_2 or c_4, there exists a $T_{8,3,1}$ isomorphic to the one described when c_4' is not on the cycle. If v is adjacent to c_6 (or symmetrically c_9), the $T_{8,3,1}$ described by $T\{c_6 : vc_7c_8c_9, c_5c_4c_4'c_3c_2c_2'c_1w\}$ is present, and when v is adjacent to c_7 (or symmetrically c_8), the $T_{8,3,1}$ described by $T\{c_7 : v, c_6c_5w, c_8c_9c_1c_2'c_2c_3c_4'c_4\}$ is present. This implies that all vertices not on C must be adjacent to the set $S = \{c_1, c_3, c_5\}$ and G_0 contains a dominating eulerian circuit by Lemma 4.12.

Next, let the neighbors of w be c_1, c_4, and c_7. By Lemma 4.11, any vertex v not on C must have at least two neighbors in common with w.

If v has exactly 2 neighbors in common with w, by symmetry we can assume c_4 and c_7, then the third neighbor must be either c_2 or c_9. Once again, these cases are symmetric so we may assume the third neighbor is c_2. This gives the following C_9: $c_2c_3c_4wc_1c_9c_8c_7vc_2$. This alternate C_9 implies either c_5 or c_6 is nontrivial. Whether it is c_5 or c_6 that is nontrivial, there is a $T_{8,3,1}$.

When c_5 is nontrivial $T\{c_7 : v, c_6c_5c_5', c_8c_9c_1c_2c_3c_4ww'\}$ is the $T_{8,3,1}$ present. When c_6 is nontrivial, $T\{c_4 : v, c_5c_6c_6', c_3c_2c_1c_9c_8c_7ww'\}$ is present. Therefore, all vertices not on C must have the same three adjacencies and the graph contains a dominating eulerian circuit by Lemma 4.12.

Lastly, consider the case when the neighbors of w are c_1, c_3, and c_6. By Lemma 4.10 the vertex c_2 must be nontrivial.

We first consider possible neighbors of an additional vertex v. If v is adjacent to c_4 or c_9, there is a $T_{8,3,1}$. When v is adjacent to c_4 the $T_{8,3,1}$ is $T\{c_4 : v, c_3ww', c_5c_6c_7c_8c_9c_1c_2c_2'\}$. When v is adjacent to c_9 it is $T\{c_9 : v, c_1c_2c_2', c_8c_7c_6c_5c_4c_3ww'\}$. Note that if c_4 or c_9 were nontrivial vertices, the same $T_{8,3,1}$ subgraphs would be present.

First consider the vertex c_4. Since every vertex in G_0 has degree at least 3 and c_4 is not adjacent to any vertex off of C, it must be the case that c_4 has a chord. The chords c_2c_4 and c_4c_6 create triangles. If the chord is c_4c_7, we get the longer cycle $c_1c_2c_3wc_6c_5c_4c_7c_8c_9c_1$.

If the chord is c_4c_8, then c_7 must be nontrivial since $c_1c_2c_3wc_6c_5c_4c_8c_9c_1$ is an alternate C_9. This gives $T\{w : w', c_3c_2c_2', c_1c_9c_8c_4c_5c_6c_7c_7'\}$ as a $T_{8,3,1}$ when ww' is either a pendant vertex or part of a path from w to c_6. If ww' is part of a path from w to c_1, then $T\{c_4 : c_5, c_3c_2c_2', c_8c_9c_1w'wc_6c_7c_7'\}$ is present. Lastly, if ww' is part of a path from w to c_3, the $T_{8,3,1}$ described by $T\{c_7 : c_7', c_6c_5c_4, c_8c_9c_1ww'c_3c_2c_2'\}$ is present provided c_7' does not lie on the preimage of C and $T\{c_2 : c_2', c_3w'w, c_1c_9c_8c_7c_7'c_6c_5c_4\}$ is present if c_2' does not lie on the preimage of C. If both c_2' and c_7' lie on C, then it is necessary to consider an additional vertex v. We can assume v is not adjacent to either c_2 or c_7 since v would act in the same manner as having c_2' or c_7' not on C. If the vertex v is adjacent to c_3, c_4, c_5, or c_9, there is a $T_{8,3,1}$ present, with the descriptions of them being $T\{c_3 : v, c_2c_2'c_1, w'wc_6c_7'c_7c_8c_4c_5\}$, $T\{c_4 : v, c_3ww', c_5c_6c_7c_7'c_8c_9c_1c_2\}$, $T\{c_3 : c_2, c_4c_5v, w'wc_1c_9c_8c_7c_7'c_6\}$, and $T\{c_9 : v, c_1c_2'c_2, c_8c_7c_7'c_6c_5c_4c_3w'\}$, respectively. This implies that v must be adjacent to the set $S = \{c_1, c_6, c_8\}$ and $T\{c_6 : v, c_7'c_7c_8, c_5c_4c_3c_2c_2'c_1ww'\}$ is present.

If the chord c_4c_9 is present, we get a similar situation with c_5 being nontrivial due to $c_1c_2c_3c_4c_9c_8c_7c_6wc_1$ being an alternate C_9. This gives the $T_{8,3,1}$ described by $T\{w : w', c_1c_2c_2', c_3c_4c_9c_8c_7c_6c_5c_5'\}$ whenever ww' is either a pendant edge or contained in a path from w to c_6. If ww' is on a path from w to c_3, then the $T_{8,3,1}$ described by $T\{c_9 : c_1, c_4c_5c_5', c_8c_7c_6ww'c_3c_2c_2'\}$ is present. Lastly, if ww' is part of a path from w to c_1 and c_2' is not a part of the preimage of C, then the $T_{8,3,1}$ described by $T\{c_2 : c_2', c_1w'w, c_3c_4c_9c_8c_7c_6c_5c_5'\}$ is present. If ww' is part of a path from w to c_1 and c_2' is part of the preimage of C, then it is necessary to consider an additional vertex v. If v is adjacent to any of c_1, c_4, c_6, c_7, c_8, or c_9 there is a $T_{8,3,1}$ with the descriptions given by $T\{c_1 : v, c_9c_8c_7, c_2'c_2c_3c_4c_5c_6ww'\}$, $T\{c_4 : v, c_9c_8c_7, c_5c_6ww'c_1c_2'c_2c_3\}$, $T\{c_6 : v, c_7c_8c_9, c_5c_4c_3c_2c_2'1w'w\}$, $T\{c_7 : v, c_6c_5c_5', c_8c_9c_4c_3c_2c_2'c_1w'\}$, $T\{c_8 : v, c_9c_4c_5, c_7c - 6wc_3c_2c_2'c_1w'\}$, and $T\{c_9 : v, c_4c_5c_5', c_8c_7c_6wc_3c_2c_2'c_1\}$, respectively. This leaves only c_2, c_3, and c_5 as possible neighbors of v, which would violate either the assumption that v has at least three neighbors on C or the assumption that G_0 is triangle-free.

The above observations imply that the chord incident to c_4 must have c_1 as its other endpoint.

Now consider v adjacent to c_5. If v is also adjacent to c_1, we get the longer cycle $c_1vc_5c_4c_3wc_6c_7c_8c_9c_1$. If v is adjacent to c_2, we get the longer cycle $c_1wc_3c_2vc_5c_6c_7c_8c_9c_1$. If v is also adjacent to c_7, we get the longer cycle $c_1c_2c_3wc_6c_5vc_7c_8c_9c_1$. Lastly, if v is also adjacent to c_8, we get the longer cycle $c_1c_2c_3c_4c_5vc_8c_7c_6wc_1$. Taking into account the fact that no vertex is adjacent to c_4 or c_9 when the neighbors of w are arranged in this manner, it must be the case that v is adjacent to $\{c_3, c_5, c_6\}$. Being adjacent to both c_5 and c_6 creates a triangle. We conclude that v cannot be adjacent to c_5. Like c_4, the vertex c_5 must then be incident to some chord within C.

The chord c_5c_9 is the only possible chord that produces a longer cycle, which is given by $c_1c_2c_3wc_6c_7c_8c_9c_5c_4c_1$.

The chord c_2c_5 produces the alternate C_9 given by $c_1wc_3c_2c_5c_6c_7c_8c_9c_1$, which forces c_4 to be nontrivial. This yields a $T_{8,3,1}$ described by $T\{c_4 : c_4', c_3ww', c_5c_6c_7c_8c_9c_1c_2c_2'\}$ if c_4' is not on the preimage of C and the $T_{8,3,1}$ described by $T\{c_2 : c_2', c_1ww', c_3c_4'c_4c_5c_6c_7c_8c_9\}$ in the case that c_4' is located on the preimage of C and c_2' is not. In the case that both c_2' and c_4' are on the cycle and ww' is a pendant edge, the $T_{8,3,1}$ described by $T\{w : w', c_1c_2'c_2, c_3c_4'c_4c_5c_6c_7c_8c_9\}$ is present. If ww' is located on a path from w to c_1 or a path from w to c_6, the subgraph $T\{c_6 : c_5, c_7c_8c_9, ww'c_1c_2'c_2c_3c_4'c_4\}$ is present. Lastly, if the edge ww' is on a path from w to c_3, then the $T_{8,3,1}$ described by $T\{c_3 : w', c_4'c_4c'5, c_2c_2'c_1wc_6c_7c_8c_9\}$ is present. Therefore, the chord c_2c_5 cannot be present.

The chord c_5c_8 also gives an alternate C_9: $c_1c_2c_3c_4c_5c_8c_7c_6wc_1$. This cycle forces c_9 to be nontrivial. In the case where c_9' is not on the cycle, the $T_{8,3,1}$ described by $T\{w : w', c_1c_2c_2', c_3c_4c_5c_6c_7c_8c_9c_9'\}$ is present. This $T_{8,3,1}$ also is present when c_9 is on the cycle and ww' is either a pendant edge in G or lies on a path from w to c_6. In the case where c_9' is on the cycle and ww' is part of a path from w to c_1, the $T_{8,3,1}$ described by $T\{c_4 : c_5, c_3c_2c_2', c_1w'wc_6c_7c_8c_9c_9'\}$ is present. Lastly, when ww' is part of a path from w to c_3, the $T_{8,3,1}$ described by $T\{c_4 : c_5, c_1c_2c_2', c_3w'wc_6c_7c_8c_9c_9'\}$ is present. Therefore, G_0 cannot contain the chord c_5c_8.

It can easily be seen that all chords incident to c_5 not mentioned above create triangles. Since there is no way to place the chords at c_4 and c_5 without creating a longer cycle, a $T_{8,3,1}$, or a triangle, this configuration for the neighbors of w cannot occur.

This concludes the proof of Theorem 4.2.

$\square$

4.5 Proof of Theorem 4.3: $T_{7,4,1}$

The cases where C is a dominating cycle and C is not a dominating cycle will be handled separately. We will split the cases further based on $c(G_0)$. Recall that by Lemma 4.5 and Theorem 2.8, we need only consider $c(G_0) = 9, 10$, and 11.

4.5.1 Case 1: C is not a dominating cycle.

Lemmas 4.7, 4.8, and 4.9, give the desired result provided that when $c(G_0) = 10$ or 11 every vertex not on C has at least two neighbors on C. This result is presented in Lemma 4.16.

Lemma 4.16. *If G_0 is the reduced core of a graph without subgraphs isomorphic to $T_{7,4,1}$ and $c(G_0) = 10$ or 11, then every vertex not on a longest cycle C has at least 2 neighbors on C.*

Proof. We will present the proofs for $c(G_0) = 10$ and $c(G_0) = 11$ together, with parentheses denoting the portions of the paths present for 11 but not 10.

First assume that there is a vertex w such that w has no neighbors on C. This vertex must have at least three neighbors, which we denote x, y, and z. There must be paths from each of the vertices to C, so without loss of generality we can assume there is a path from x to c_1. If any of x, y, or z has a neighbor not on the cycle besides w, say y is adjacent to y', then there is the $T_{7,4,1}$ described by $T\{c_1 : (c_{11})c_{10}, xwyy', c_2c_3c_4c_5c_6c_7c_8\}$. Therefore, we can assume that all neighbors of x, y, and z are on the cycle. To prevent a longer cycle, neighbors of different vertices must either coincide or be at least distance 4 apart. With x adjacent to c_1, there are three nonsymmetric places to choose a neighbor for y: c_1, c_5, or c_6. (Note: these choices are the same whether $c(G_0) = 10$ or 11.) Since y has at least two neighbors on C, it must be adjacent to c_5, c_6, or a vertex symmetric to one of those two. If y is adjacent to c_5, there is a $T_{7,4,1}$ described by $T\{c_1 : x, (c_{11})c_{10}c_9c_8c_7, c_2c_3c_4c_5ywz\}$. If y is adjacent to c_6, there is a $T_{7,4,1}$ described by $T\{c_1 : x, (c_{11})c_{10}c_9c_8c_7, c_2c_3c_4c_5c_6yw\}$. Thus, each vertex must have at least one neighbor on C.

Now assume there is a vertex w with exactly one neighbor on C and neighbors x and y off of C. The vertex x must have a neighbor on C by the above observations. There are three nonsymmetric places to choose this neighbor that do not create a longer cycle: c_4, c_5, and c_6. Each of these gives rise to a $T_{7,4,1}$. If x is adjacent to c_4 we get $T\{w : y, xc_4c_3c_2, c_1(c_{11})c_{10}c_9c_8c_7c_6c_5\}$. If x is adjacent to c_5 we get $T\{w : y, c_1c_2c_3c_4, zc_5c_6c_7c_8c_9c_{10}\}$. Lastly, if x is adjacent to c_6 we get $T\{c_6 : x, c_5c_4c_3c_2, c_7c_8c_9c_{10}(c_{11})c_1wy\}$. Therefore, we can assume that every vertex has at least two neighbors on C.

□

4.5.2 Case 2a: C is a dominating cycle and $c(G_0) = 11$.

Consider the case when C is a dominating cycle. There are at least 3 vertices not on C, at least one of which is nontrivial. Label this nontrivial vertex as w and let one of its neighbors be labeled c_1. We first note that w cannot be adjacent to c_5, or symmetrically c_8, since this creates the $T_{7,4,1}$ described by $T\{w : w', c_1c_2c_3c_4, c_5c_6c_7c_8c_9c_{10}c_{11}\}$.

Let v be an additional vertex not on C. If v is adjacent to c_6 (symmetrically c_7), then $T\{c_1 : w, c_2c_3c_4c_5, c_{11}c_{10}c_9c_8c_7c_6v\}$ is a $T_{7,4,1}$. If v is adjacent to c_3 (symmetrically c_{10}), then $T\{c_3 : v, c_2c_1ww', c_4c_5c_6c_7c_8c_9c_{10}\}$ is a $T_{7,4,1}$. Lastly, if v is adjacent to c_4 (symmetrically c_9), $T\{c_4 : v, c_3c_2c_1w, c_5c_6c_7c_8c_9c_{10}c_{11}\}$ is a $T_{7,4,1}$. Since v cannot be adjacent to consecutive vertices, as that would create a triangle, the only possibilities for neighbors of v are $\{c_1, c_5, c_8\}$, $\{c_2, c_5, c_8\}$, $\{c_2, c_5, c_{11}\}$, $\{c_2, c_8, c_{11}\}$, and $\{c_5, c_8, c_{11}\}$. Note that the cases where v is adjacent to $\{c_2, c_5, c_{11}\}$ and $\{c_5, c_8, c_{11}\}$ are symmetric, as are the cases where v is adjacent to $\{c_2, c_5, c_{11}\}$ and $\{c_2, c_8, c_{11}\}$.

First consider v adjacent to $\{c_2, c_5, c_{11}\}$. If w is adjacent to c_3 or c_4 (symmetrically c_{11} or c_{10}), we get the longer cycles $c_1wc_3c_4c_5c_6c_7c_8c_9c_{10}c_{11}vc_2c_1$ and $c_1wc_4c_5c_6c_7c_8c_9c_{10}c_{11}vc_2c_1$, respectively. This forces w to be adjacent to both c_6 and c_7, which creates a triangle.

Next consider v adjacent to $\{c_2, c_5, c_8\}$. In this case w cannot be adjacent to c_3 or c_4, since that results in the longer cycles $c_1wc_3c_2vc_5c_6c_7c_8c_9c_{10}c_{11}c_1$ and $c_1wc_4c_3c_2vc_5c_6c_7c_8c_9c_{10}c_{11}c_1$, respectively. If w is adjacent to c_6, the longer cycle $c_1wc_6c_5c_4c_3c_2vc_8c_9c_{10}c_{11}c_1$ is present. If w is adjacent to c_7, then the longer cycle $c_1wc_7c_6c_5c_4c_3c_2vc_8c_9c_{10}c_{11}c_1$ is present. This forces w to be adjacent to c_9 and c_{10}, which contradicts G_0 being triangle-free.

Lastly, consider v adjacent to $\{c_1, c_5, c_8\}$. If w is adjacent to c_3 (symmetrically c_{10}), there is a $T_{7,4,1}$ described by $T\{c_5 : v, c_4c_3c_2c_1, c_6c_7c_8c_9c_{10}ww'\}$. If w is adjacent to c_4 (symmetrically c_9), there is a $T_{7,4,1}$ described by $T\{c_9 : w, c_{10}c_{11}c_1v, c_8c_7c_6c_5c_4c_3c_2\}$. Lastly, if w is adjacent to c_6 (symmetrically c_7), there is a $T_{7,4,1}$ described by $T\{c_1 : v, c_2c_3c_4c_5, c_{11}c_{10}c_9c_8c_7c_6w\}$. Paired with the previous restrictions on where neighbors of w can be placed, this shows that there is no way to place the neighbors of w without creating either a longer cycle or a $T_{7,4,1}$.

4.5.3 Case 2b: C is a dominating cycle and $c(G_0) = 10$.

First we observe that an additional vertex v cannot be adjacent to a vertex distance two away from any neighbor of w since this gives rise to a $T_{7,4,1}$. For example, let w be adjacent to c_1 and v be adjacent to c_3. Then $T\{c_3 : v, c_2c_1ww', c_4c_5c_6c_7c_8c_9c_{10}\}$ is the resulting $T_{7,4,1}$. We now proceed by considering the different configurations of the neighbors of w as shown in Figure 4.4.

Let neighbors of w be c_1, c_3, and c_5. By Lemma 4.10, we note that both c_2 and c_4 are nontrivial. When considering where an additional vertex v can be adjacent, we note that there are five vertices that are not distance two from a neighbor of w: c_2, c_4, c_6, c_8, and c_{10}. If v is adjacent to c_6 (symmetrically c_{10}), then $T\{c_6 : v, c_5c_4c_3c_2, c_7c_8c_9c_{10}c_1ww'\}$ is a $T_{7,4,1}$. Therefore v must be adjacent to $\{c_2, c_4, c_8\}$. In this case, $T\{c_2 : v, c_3c_4c_5c_6, vc_8c_9c_{10}c_1ww'\}$ is a $T_{7,4,1}$.

Let the neighbors of w be c_1, c_3, and c_6. When considering which vertices on C can be neighbors of v, the only vertices that are not distance 2 from a neighbor of w are c_2, c_6, c_7, and c_{10}. Since v cannot be adjacent to both c_6 and c_7 as that would create a triangle, two of the three neighbors of v must be c_2 and c_{10}. This creates the longer cycle $c_1wc_3c_4c_5c_6c_7c_8c_9c_{10}c_2c_1$.

Let the neighbors of w be c_1, c_3, and c_7. The vertex c_2 is nontrivial by Lemma 4.10. When c_2' is not on the preimage of C, the $T_{7,4,1}$ described by $T\{c_2 : c_2', c_3c_4c_5c_6, c_1c_{10}c_9c_8c_7ww'\}$ is present. When c_2' is on the cycle and ww' is either a pendant edge or lies on a path from w to c_1, the $T_{7,4,1}$ described by $T\{w : w', c_3c_4c_5c_6, c_7c_8c_9c_{10}c_1c_2c_2'\}$ is present. If ww' is part of a path from w to c_3, then the $T_{7,4,1}$ described by $T\{c_1 : w, c_2'c_2c_3w', c_{10}c_9c_8c_7c_6c_5c_4\}$ is present. Lastly, when ww' is part of a path from w to c_7, it is necessary to consider an additional vertex v. If v is adjacent to c_2, there is the same $T_{8,3,1}$ as if c_2' is not on C. If v is adjacent to any of the vertices c_1, c_3, c_4, c_5, c_6, c_8, or c_9 there is a $T_{7,4,1}$ present, with the descriptions given by $T\{c_1 : v, c_2'c_2c_3w, c_{10}c_9c_8c_7c_6c_5c_4\}$, $T\{c_3 : v, c_2c_2'c_1w, c_4c_5c_6c_7c_8c_9c_{10}\}$, $T\{c_4 : v, c_5c_6c_7w', c_3c_2c_2'c_1c_{10}c_9c_8\}$, $T\{c_5 : v, c_4c_3ww', c_6c_7c_8c_9c_{10}c_1c_2\}$, $T\{c_6 : v, c_5c_4c_3w, c_7c_8c_9c_{10}c_1c_2'c_2\}$, $T\{c_8 : v, c_9c_{10}c_1w, c_7c_6c_5c_4c_3c_2c_2'\}$, and $T\{c_9 : v, c_{10}c_1ww', c_8c_7c_6c_5c_4c_3c_2\}$, respectively. This leaves only c_7 and c_{10} as possible neighbors of v that do not create a $T_{8,3,1}$, which contradicts v having at least 3 neighbors on C.

Lastly, consider when the neighbors of w are c_1, c_4, and c_7. The vertices that are not distance two from a neighbor of w are $\{c_1, c_4, c_7, c_8, c_{10}\}$. To prevent a triangle, v must be adjacent to c_4, one of $\{c_1, c_{10}\}$, and one of $\{c_7, c_8\}$. If v is adjacent to c_{10}, the cycle $c_1c_2c_3c_4vc_{10}c_9c_8c_7wc_1$ is an alternate C_{10}, which forces either c_5 or c_6 to be nontrivial. Since both c_5 and c_6 are distance two from a neighbor of w, if the extra vertex (c_5' or c_6') is a pendant vertex then it acts as v and gives rise to a $T_{7,4,1}$. If the extra vertex is located on C, without loss of generality we can assume this vertex is c_5', then $T\{c_7 : w, c_8c_9c_{10}v, c_6c_5c_5'c_4c_3c_2c_1\}$ is present. The case where v is adjacent to c_8 is symmetric. Therefore we can assume that all vertices off of C have the same set of adjacencies as w and G_0 contains a dominating eulerian circuit by Lemma 4.12.

4.5.4 Case 2c: C is a dominating cycle and $c(G_0) = 9$.

If $|V(G_0)| \leq 13$ then by Lemmas 4.6 and 2.7 either G_0 is supereulerian or contains a $T_{7,4,1}$. So it can be assumed that $|V(G_0)| \geq 14$. This gives at least 5 vertices off of C, at least one of which is nontrivial. As before, we label this vertex w. We proceed by considering the three possible placements of the neighbors of w as shown in Figure 4.3.

Consider w adjacent to $\{c_1, c_3, c_5\}$. By Lemma 4.10, c_2 and c_4 must be nontrivial. Also, an additional vertex v cannot be adjacent to c_1 (symmetrically c_5) since this gives $T\{c_1 : v, c_2c_3ww', c_9c_8c_7c_6c_5c_4c_4'\}$ as a $T_{7,4,1}$.

Let c_i denote one of the neighbors of w. If a vertex v is adjacent to any of the pairs $\{c_{i-1}, c_{i+1}\}$, $\{c_{i-1}, c_{i+2}\}$, or $\{c_{i-2}, c_{i+1}\}$ for $i \in \{1, 3, 5\}$, there is a longer cycle in G_0. Likewise, v cannot be adjacent to both c_2 and c_8 as that creates the longer cycle $c_1wc_3c_4c_5c_6c_7c_8vc_2c_1$. Symmetrically, v cannot be adjacent to both c_4 and c_7. We conclude that v cannot be adjacent to c_2 or c_4 since all possible neighbors either create a triangle or a longer cycle.

Therefore, v must be adjacent to c_3 and either $\{c_6, c_8\}$, $\{c_6, c_9\}$, or $\{c_7, c_9\}$.

Consider v adjacent to c_9. This produces $c_1c_2c_3vc_9c_8c_7c_6c_5wc_1$, which is a longer cycle. The vertices c_9 and c_6 are symmetric, so this handles all of the possibilities.

Now consider w adjacent to $\{c_1, c_3, c_6\}$. By Lemma 4.10, the vertex c_2 must be nontrivial. First note that an additional vertex v cannot be adjacent to c_5 since this gives $T\{c_5 : v, c_4c_3ww', c_6c_7c_8c_9c_1c_2c_2'\}$ as a $T_{7,4,1}$. Likewise, v cannot be adjacent to c_8 since that gives $T\{c_8 : v, c_9c_1c_2c_2', c_7c_6c_5c_4c_3ww'\}$ as a $T_{7,4,1}$. It can easily be seen that if c_i is one of the neighbors of w and v is adjacent to both c_{i-1} and c_{i+1} we get a longer cycle.

If v is adjacent to c_2, the above restrictions eliminate $\{c_1, c_3, c_4, c_5, c_8, c_9\}$ as possible neighbors. This implies that the other two neighbors of v are c_6 and c_7, which contradicts G_0 being triangle-free.

This leaves v adjacent to one of each the following pairs: $\{c_1, c_9\}$, $\{c_3, c_4\}$, and $\{c_6, c_7\}$. Assume v is adjacent to c_7. If v is also adjacent to c_3 we get the longer cycle $c_1wc_6c_5c_4c_3vc_7c_8c_9c_1$. If v is also adjacent to c_4 we get the longer cycle $c_1c_2c_3wc_6c_5c_4c_7c_8c_9c_1$. Thus, v must be adjacent to c_6. If v is adjacent to both c_4 and c_9 we get the longer cycle $c_1c_2c_3c_4vc_9c_8c_7c_6wc_1$. Therefore the adjacencies of v must either be $\{c_3, c_6, c_9\}$ or $\{c_1, c_4, c_6\}$.

Note that if there is a vertex v_1 with adjacencies $\{c_1, c_4, c_6\}$ and a vertex v_2 with adjacencies $\{c_3, c_6, c_9\}$ we get the longer cycle $c_1c_2c_3wc_6c_7c_8c_9v_2c_4v_1c_1$. Therefore we can assume that all vertices that are not on C and are not w must have the same set of adjacencies.

Assume all additional vertices v_i are adjacent to $\{c_3, c_6, c_9\}$. The cycle $c_1c_2c_3vc_9c_8c_7c_6wc_1$ is an alternate C_9 that adds w and bypasses c_4 and c_5, implying that one of those two vertices must be nontrivial. First consider when c_5 is the nontrivial vertex. In this case G_0 contains the $T_{7,4,1}$ described by $T\{c_2 : c_2', c_3c_4c_5c_5', c_1wc_6c_7c_8c_9v\}$ when c_2' is not on the preimage of C and $T\{c_9 : v, c_8c_7c_6w, c_1c_2'c_2c_3c_4c_5c_5'\}$ in the case where c_2' is on the preimage of C. Next, consider c_4 as the nontrivial vertex. Since G_0 has minimum degree 3, there must be a chord from c_5. The chord cannot be c_3c_5 or c_5c_7 since G_0 is triangle-free. If the chord is c_2c_5, we get the longer cycle $c_1c_2c_5c_4c_3wc_6c_7c_8c_9c_1$. If the chord is c_5c_8, we get the longer cycle $c_1c_2c_3c_4c_5c_8c_7c_6vc_9c_1$. The chord c_5c_9 also gives $c_1c_2c_3c_4c_5c_9c_8c_7c_6wc_1$ as a longer cycle. This implies that the chord from c_5 must be c_1c_5. The vertex c_4 also needs a chord since G_0 is 3-edge-connected. This chord must be one of c_4c_7, c_4c_8, and c_4c_9. The chord c_4c_7 allows $c_4c_5c_6wc_3c_2c_1c_9c_8c_7c_4$ as a longer cycle. The chord c_4c_8 gives the longer cycle $c_1c_2c_3wc_6vc_9c_8c_4c_5c_1$. Lastly, the chord c_4c_9 gives the longer cycle $c_1c_2c_3vc_6c_7c_8c_9c_4c_5c_1$.

Now assume that all additional vertices v_i are adjacent to $\{c_1, c_4, c_6\}$. We note that the only additional edges are either incident to additional vertices v_i and one of $\{c_1, c_4, c_6\}$ or are chords within C. This graph contains a dominating eulerian circuit, with the description of the circuit depending only on the parity of the number of vertices v_i. If there is an odd number of vertices v_i, begin the dominating eulerian circuit with $c_6wc_3c_2c_1c_9c_8c_7c_6v_1c_4c_5c_6$. Note that one v_i was used in this part of the circuit. Pair up the remaining vertices v_i into pairs $\{x, y\}$ and append $c_6xc_4yc_6$ to the end of the circuit for each pair. If there is an even number of vertices v_i, begin the circuit with $c_6c_7c_8c_9c_1c_2c_3wc_1v_1c_6v_2c_4c_5c_6$. Two vertices v_i were used in this part of the circuit, leaving an even number that we can pair up and append as we did in the odd case. In both the even and odd cases, the circuits must be dominating eulerian circuits since they contain all vertices of G_0.

Lastly, consider the case where w is adjacent to $\{c_1, c_4, c_7\}$. By Lemma 4.11, any additional vertex v must have at least two neighbors in common with w.

Consider v with exactly two adjacencies in common with w. By symmetry, we can assume that these two adjacencies are c_1 and c_4 and the third adjacency is c_6. Assume there is a vertex x not on C that is adjacent to c_9. This vertex x must also be adjacent to c_4 and c_7. This gives a $T_{7,4,1}$ described by

$T\{c_6 : v, c_7c_8c_9x, c_5c_4c_3c_2c_1ww'\}$. Therefore, we can assume there is no vertex adjacent to c_9. The three-edge-connectedness of G_0 implies there must then be a chord at c_9. There are four choices that do not create triangles: c_3c_9, c_4c_9, c_5c_9, and c_6c_9. The chords c_3c_9, c_5c_9, and c_6c_9 create the longer cycles $c_1wc_4c_5c_6c_7c_8c_9c_3c_2c_1$, $c_1c_2c_3c_4vc_6c_5c_9c_8c_7wc_1$, and $c_1c_2c_3c_4c_5c_6c_9c_8c_7wc_1$ respectively.

The chord c_4c_9 gives the alternate longest cycle $c_1wc_7c_8c_9c_4c_5c_6vc_1$ which includes w but omits c_2 and c_3. Therefore, either c_2 or c_3 must be nontrivial. If c_2 is nontrivial and c_2' is not on the preimage of C, we get $T\{c_2 : c_2', c_1vc_6c_5, c_3c_4c_9c_8c_7ww'\}$ as a $T_{7,4,1}$. If c_2' is on the preimage of C, then in the case that ww' is a pendant edge or is part of a path from w to c_1 the $T_{7,4,1}$ described by $T\{w : w', c_4c_5c_6v, c_7c_8c_9c_1c_2'c_2c_3\}$ is present. If ww' is part of a path from w to c_7 or a path from w to c_4, the $T_{7,4,1}$ described by $T\{c_4 : c_9, w'wc_7c_8, c_5c_6vc_1c_2'c_2c_3\}$ is present. If c_3 is nontrivial and c_3' is not on the preimage of C, there is a $T_{7,4,1}$ described by $T\{w : w', c_4c_5c_6v, c_7c_8c_9c_1c_2c_3c_3'\}$. The case where c_3' is on the preimage of C is isomorphic to when c_2' is on the preimage of C, and the above arguments suffice. Therefore, the chord c_4c_9 cannot be present in G_0. Since there is no way to place a chord at c_9 without violating our assumptions, we conclude that v cannot have exactly two adjacencies in common with w when the neighbors of w are $\{c_1, c_4, c_7\}$.

This leaves only the case when all vertices not on the cycle C have the same adjacencies as w. In this case, G_0 is guaranteed to have a spanning eulerian circuit by Lemma 4.12.

This concludes the proof of Theorem 4.3.

□

4.6 Proof of Theorem 4.4: $T_{6,5,1}$

The cases where C is a dominating cycle and C is not a dominating cycle will be handled separately. We will split the cases further based on $c(G_0)$. Recall that by Lemma 4.5 and Theorem 2.8, we need only consider $c(G_0) = 9, 10,$ and 11.

4.6.1 Case 1: C is not a dominating cycle.

Lemmas 4.7, 4.8, and 4.9, give the desired result provided that when $c(G_0) = 10$ or 11 every vertex not on C has at least two neighbors on C. This result is presented in Lemma 4.17.

Lemma 4.17. *If G_0 is the reduced core of a graph without subgraphs isomorphic to $T_{6,5,1}$ and $c(G_0) = 10$ or 11, then every vertex not on a longest cycle C has at least 2 neighbors on C.*

Proof. Once again, we will present the proofs for when $c(G_0) = 10$ and $c(G_0) = 11$ together, with the vertices in parentheses denoting the portions of paths that appear when $c(G_0) = 11$ but not when $c(G_0) = 10$.

We start by assuming that there is a vertex w with no neighbors on C. Let the neighbors of w be x, y, and z, and assume that there is a path from x to c_1. The path from x to c_1 can have at most one other vertex, say x', otherwise this path paired with wy gives a path of length 5, which can be used to create a $T_{6,5,1}$ centered at c_1. Similarly, if there is such a vertex x', then neither y nor z can have any neighbors off of C other than w and possibly x', as this would either create a triangle or give a path of length 5 that could be used to create a $T_{6,5,1}$. Any neighbor of y that is on C must either be c_1 or be at least distance 5 from c_1. This leaves two other choices, c_6 and c_7, which are symmetric. Since y cannot be adjacent to both c_1 and x' as that would create a triangle, y must be adjacent to one of c_6 or c_7.

Without loss of generality, we may assume that y is adjacent to c_6. This gives the $T_{6,5,1}$ described by $T\{w : z, xx'(c_{11})c_{10}c_9c_8, yc_6c_5c_4c_3c_2\}$. Therefore, all neighbors of x, y, and z must be on C.

Now we can assume that x is adjacent to c_1. When considering neighbors of y, there are three nonsymmetric places that do not create a triangle or a longer cycle: c_1, c_5, and c_6. Since y has at least two neighbors on C, it must be adjacent to either c_5, c_6, or a vertex symmetric to one of those choices. If y is adjacent to c_5 we get the $T_{6,5,1}$ described by $T\{w : z, yc_5c_4c_3c_2, xc_1(c_{11})c_{10}c_9c_8c_7\}$. If y is adjacent to c_6 we get the $T_{6,5,1}$ described by $T\{w : z, yc_6c_5c_4c_3, xc_1(c_{11})c_{10}c_9c_8c_7\}$. We conclude that every vertex must have at least one neighbor on C.

Now we wish to show that every vertex has at least two neighbors on C. Assume there is a vertex w with exactly one neighbor on C, say c_1, and neighbors x and y off of C. The vertex x must have a neighbor on C, and there are three nonsymmetric ways to choose it: c_4, c_5, or c_6. If x is adjacent to c_4 there is a $T_{6,5,1}$ described by $T\{c_4 : x, c_3c_2c_1wy, c_5c_6c_7c_8c_9c_{10}\}$. If x is adjacent to c_5, there is a $T_{6,5,1}$ described by $T\{w : y, xc_5c_4c_3c_2, c_1(c_{11})c_{10}c_9c_8c_7c_6\}$. Lastly, if x is adjacent to c_6 then the subgraph $T\{c_1 : w, (c_{11})c_{10}c_9c_8c_7, c_2c_3c_4c_5c_6x\}$ is present.

□

4.6.2 Case 2a: C is a dominating cycle and $c(G_0) = 11$.

Without loss of generality, we may assume that c_1 is one of the neighbors of w, where w is a nontrivial vertex. Note that when considering w, it cannot have adjacencies that are distance 5 apart. For example, if w was adjacent to both c_1 and c_6 this gives $T\{w : w', c_1c_2c_3c_4c_5, c_6c_7c_8c_9c_{10}c_{11}\}$ as a $T_{6,5,1}$ if ww' is a pendant edge or part of a path to the third adjacency of w. If ww' is located on the path from w to c_1 (or symmetrically the path from w to c_6), then the $T_{6,5,1}$ described by $T\{c_1 : w', c_{11}c_{10}c_9c_8c_7, c_2c_3c_4c_5c_6w\}$ is present.

If w is also adjacent to c_3, its third adjacency must be c_5 or c_{10} as these are the only vertices that are not distance 1 or 5 from c_1 or c_3. By symmetry, we may assume c_5. Since we can reroute C to include w and omit either c_2 or c_4, these two vertices must be nontrivial. This gives a $T_{6,5,1}$ described by $T\{w : w', c_1c_2c_3c_4c_4', c_5c_6c_7c_8c_9c_{10}\}$ when ww' is either a pendant edge or part of a path from w to c_3. When ww' is part of a path from w to c_1 (or symmetrically from w to c_5), there is a $T_{6,5,1}$ described by $T\{c_1 : w', c_2c_3c_4c_5w, c_{11}c_{10}c_9c_8c_7c_6\}$. Since c_{10} is symmetric to c_3, we conclude that w cannot be adjacent to either of these vertices.

Therefore, when w is adjacent to c_1 the other two adjacencies must be one of $\{c_4, c_5\}$ and one of $\{c_8, c_9\}$. The three combinations that do not have two adjacencies distance 5 apart are $\{c_1, c_4, c_8\}$, $\{c_1, c_5, c_8\}$, and $\{c_1, c_5, c_9\}$. Note that all three of these cases are symmetric, so we may assume that the adjacencies are $\{c_1, c_4, c_8\}$. When considering an additional vertex v, there are six nonisomorphic places that it can be adjacent to: $c_1, c_2, c_8, c_9, c_{10}$, and c_{11}. In each case, there is a $T_{6,5,1}$.

When v is adjacent to c_1, the $T_{6,5,1}$ is $T\{c_1 : v, c_2c_3c_4c_5c_6, c_{11}c_{10}c_9c_8ww'\}$. When v is adjacent to c_2 it is $T\{c_2 : v, c_3c_4c_5c_6c_7, c_1c_{11}c_{10}c_9c_8w\}$. When v is adjacent to c_8, the $T_{6,5,1}$ is $T\{c_8 : v, wc_1c_{11}c_{10}c_9, c_7c_6c_5c_4c_3c_2\}$. Having v adjacent to c_9 gives $T\{c_9 : v, c_{10}c_{11}c_1ww', c_8c_7c_6c_5c_4c_3\}$. With v adjacent to c_{10}, the $T_{6,5,1}$ is $T\{c_{10} : v, c_9c_8c_7c_6c_5, c_{11}c_1c_2c_3c_4w\}$. Lastly, when v is adjacent to c_{11}, $T\{c_{11} : v, c_{10}c_9c_8ww', c_1c_2c_3c_4c_5c_6\}$ is the $T_{6,5,1}$.

4.6.3 Case 2b: C is a dominating cycle and $c(G_0) = 10$.

As before, we proceed by considering the four ways to place the neighbors of w as shown in Figure 4.4.

Let the neighbors of w be $\{c_1, c_3, c_5\}$. In this case both c_2 and c_4 are nontrivial by Lemma 4.10. If c_2' is not part of the preimage of C (or, by symmetry, if c_4' is not part of the preimage of C), there is a $T_{6,5,1}$ described by $T\{c_2 : c_2', c_3c_4c_5ww', c_1c_{10}c_9c_8c_7c_6\}$. If both c_2' and c_4' are on C, then $T\{c_1 : w, c_2'c_2c_3c_4'c_4, c_{10}c_9c_8c_7c_6c_5\}$ is a $T_{6,5,1}$.

Let the neighbors of w be $\{c_1, c_3, c_6\}$. By Lemma 4.10, c_2 is nontrivial. If c_2' is not part of the preimage of C, then $T\{c_2 : c_2', c_1c_{10}c_9c_8c_7, c_3c_4c_5c_6ww'\}$ is a $T_{6,5,1}$ in G_0. If c_2' is on C, then $T\{w : w', c_6c_7c_8c_9c_{10}, c_1c_2'c_2c_3c_4c_5\}$ is present if ww' is a pendant edge or belongs to a path from w to c_3, $T\{c_6 : w, c_5c_4c_3c_2c_2', c_7c_8c_9c_{10}c_1w'\}$ is present if ww' is part of a path from w to c_1, and $T\{c_6 : w', c_5c_4c_3c_2c_2', c_7c_8c_9c_{10}c_1w\}$ is present if ww' is part of the path from w to c_6.

Let the neighbors of w be $\{c_1, c_3, c_7\}$. Once again, c_2 is nontrivial by Lemma 4.10. Up to symmetry, when considering an additional vertex v there are only 6 different places that v can be adjacent (symmetric vertices are given in parentheses): c_2, $c_3(c_1)$, $c_4(c_{10})$, $c_5(c_9)$, $c_6(c_8)$, or c_7. Each possible choice gives us a $T_{6,5,1}$. If v is adjacent to c_3, the $T_{6,5,1}$ is $T\{c_3 : v, c_2c_1c_{10}c_9c_8, c_4c_5c_6c_7ww'\}$. If v is adjacent to c_4, then the subgraph $T\{c_4 : v, c_5c_6c_7ww', c_3c_2c_1c_{10}c_9c_8\}$ is present. If v is adjacent to c_5, then $T\{c_5 : v, c_6c_7c_8c_9c_{10}, c_4c_3c_2c_1ww'\}$ is present. If v is adjacent to c_6, then $T\{c_6 : v, c_5c_4c_3ww', c_7c_8c_9c_{10}c_1c_2\}$ is present. And lastly, if v is adjacent to c_7 the $T_{6,5,1}$ is described by $T\{c_7 : v, c_6c_5c_4c_3c_2, c_8c_9c_{10}c_1ww'\}$.

The only case left to consider is when the neighbors of w are $\{c_1, c_4, c_7\}$. Up to symmetry there are 6 different ways to place a neighbor of an additional vertex v (symmetric vertices are given in parentheses): $c_1(c_7)$, $c_2(c_6)$, $c_3(c_5)$, c_4, $c_8(c_{10})$, and c_9. When v is adjacent to c_1 the subgraph $T\{c_1 : v, c_2c_3c_4ww', c_{10}c_9c_8c_7c_6c_5\}$ is present. When v is adjacent to c_3 there is a $T_{6,5,1}$ given by $T\{c_3 : v, c_2c_1c_{10}c_9c_8, c_4c_5c_6c_7ww'\}$. When v is adjacent to c_4 then $T\{c_4 : v, c_3c_2c_1ww', c_5c_6c_7c_8c_9c_{10}\}$ is a $T_{6,5,1}$. Lastly, when v is adjacent to c_8 there is a $T_{6,5,1}$ given by $T\{c_8 : v, c_9c_{10}c_1c_2c_3, c_7c_6c_5c_4ww'\}$. This leaves exactly three neighbors for v: c_2, c_6, and c_9. However, when these are the neighbors of v, G_0 contains the longer cycle $c_1c_2vc_6c_5c_4wc_7c_8c_9c_{10}c_1$.

4.6.4 Case 2c: C is a dominating cycle and $c(G_0) = 9$.

If $|V(G_0)| \leq 13$ then by Lemmas 4.6 and 2.7 either G_0 is supereulerian or contains a $T_{6,5,1}$. It can be assumed that $|V(G_0)| \geq 14$. This gives at least 5 vertices off of C, at least one of which is nontrivial. As before, we label this vertex w. We proceed by considering the possible configurations of the neighbors of w as shown in Figure 4.3.

Let the neighbors of w be c_1, c_3, and c_6. By Lemma 4.10, c_2 must be nontrivial. When considering an additional vertex v, first note that it cannot be adjacent to either c_6 or c_7 because both give a $T_{6,5,1}$. If v is adjacent to c_6 the $T_{6,5,1}$ is $T\{c_6 : v, c_5c_4c_3ww', c_7c_8c_9c_1c_2c_2'\}$. If v is adjacent to c_7, then $T\{c_7 : v, c_8c_9c_1c_2c_2', c_6c_5c_4c_3ww'\}$ is present.

Consider v adjacent to c_2. Then v cannot also be adjacent to c_1, c_3, c_4, or c_9 as G_0 is triangle-free. If it were adjacent to c_4, G_0 would have the longer cycle $c_1c_2vc_4c_3wc_6c_7c_8c_9c_1$. Being adjacent to c_5 would create the longer cycle $c_1c_2vc_5c_4c_3wc_6c_7c_8c_9c_1$. This leaves only c_8 and c_9 for the other two adjacencies of v, which contradicts G_0 being triangle-free.

Now consider v adjacent to c_4. This vertex cannot be adjacent to c_3 or c_5 as that would create a triangle, and it has already been established that no vertex can be adjacent to c_2, c_6, or c_7. To prevent a triangle, the other two adjacencies must be c_1 and c_8. This gives the longer cycle $c_1c_2c_3wc_6c_5c_4vc_8c_9c_1$.

Next consider v adjacent to c_5. If v is also adjacent to c_8, then the longer cycle $c_1c_2c_3c_4c_5vc_8c_7c_6wc_1$ is present. If v is adjacent to c_9, then $c_1c_2c_3c_4c_5vc_9c_8c_7c_6wc_1$ is a longer cycle. To keep G_0 triangle-free, the other adjacencies of v must then be c_1 and c_3. This gives $c_1vc_5c_4c_3wc_6c_7c_8c_9c_1$ as a longer cycle.

The only possibility left is to have v adjacent to c_1, c_3, and c_8. In this case v must be nontrivial since C can be rerouted to include v and leave out c_2. This gives the $T_{6,5,1}$ described by $T\{v : v', c_8c_9c_1c_2c_2', c_3c_4c_5c_6ww'\}$ when vv' is either a pendant edge or part of a path from v to c_1. When vv' is part of a path from v to c_3, the $T_{6,5,1}$ described by $T\{c_1 : c_2 : vv'c_3c_4c_5, c_9c_8c_7c_6ww'\}$ is present. Lastly, when vv' is part of a path from v to c_8, either the $T_{6,5,1}$ described by $T\{c_2 : c_2', c_1c_9c_8v'v, c_3c_4c_5c_6ww'\}$ or the $T_{6,5,1}$ described by $T\{c_3 : v, c_4c_5c_6ww', c_2c_2'c_1c_9c_8v'\}$ is present depending on the location of c_2' in G.

Now let the adjacencies of w be c_1, c_4, and c_7. By Lemma 4.11, any vertex v must have at least two adjacencies in common with w.

First consider a vertex v with exactly two adjacencies in common with w. Without loss of generality, we can say these are c_1 and c_4 and the third adjacency is c_8. The graph G_0 has at least 5 vertices off of C, so consider another additional vertex x. It too must have at least two adjacencies in common with w, and when considering the placement of v there are 5 ways (up to symmetry) that we can place the neighbors of x: $\{c_1, c_4, c_8\}$, $\{c_1, c_4, c_6\}$, $\{c_2, c_4, c_7\}$, $\{c_4, c_7, c_9\}$, and $\{c_1, c_4, c_7\}$.

When x is adjacent to $\{c_1, c_4, c_8\}$ the graph G_0 contains the subgraph $T\{c_4 : x, c_5c_6c_7ww', vc_8c_9c_1c_2c_3\}$. When x is adjacent to $\{c_1, c_4, c_6\}$ there is a $T_{6,5,1}$ given by $T\{c_1 : v, c_9c_8c_7ww', c_2c_3c_4c_5c_6x\}$. When x is adjacent to $\{c_2, c_4, c_7\}$ there is a longer cycle $c_1wc_7xc_2c_3c_4vc_8c_9c_1$. When x has adjacencies $\{c_4, c_7, c_9\}$, then $c_1c_2c_3c_4c_5c_6c_7xc_9c_8vc_1$ is a longer cycle. Lastly, when x is adjacent to $\{c_1, c_4, c_7\}$, $T\{c_4 : x, c_5c_6c_7ww', c_3c_2c_1c_9c_8v\}$ is a $T_{6,5,1}$.

Therefore, it must be the case that all vertices off of C must have the same adjacencies as w and G_0 contains a spanning eulerian circuit by Lemma 4.12.

The last case to consider is when the adjacencies of w are c_1, c_3, and c_5. By Lemma 4.10, c_2 and c_4 are both nontrivial.

Now consider where an additional vertex v might be adjacent. If v is adjacent to c_6 (symmetrically c_9), $T\{c_6 : v, c_5c_4c_3ww', c_7c_8c_9c_1c_2c_2'\}$ is a $T_{6,5,1}$.

If v is adajcent to c_7 (symmetrically c_8), there is a $T_{6,5,1}$ given by $T\{c_7 : v, c_8c_9c_1c_2c_2', c_6c_5c_4c_3ww'\}$. Since all vertices off the cycle must have three adjacencies on the cycle, it must be the case that all such vertices are adjacent to c_1, c_3, and c_5 and G_0 contains a spanning eulerian circuit by Lemma 4.12.

This concludes the proof of Theorem 4.4.

□

Chapter 5

Claw-Free, $N_{3,3,3}$-free Graphs

The main focus of this chapter is proving Theorem 1.12. As in the previous chapter, G is a graph such that $L(G)$ is 3-connected and claw-free, G_0 denotes the reduced core of G, and C is a longest cycle in G_0 with vertices labeled by $c_1, c_2, \ldots, c_{c(G_0)}$. If there is more than one cycle of length $c(G_0)$, we choose C to contain the largest number of nontrivial vertices of G_0.

Once again, we use $T_{a,b,c}$ to denote the tree obtained from taking disjoint paths with a, b, and c vertices and making one endpoint of each adjacent to a new vertex x. By the same observations as in the previous chapter, we know that if a graph G has no subgraphs (not necessarily induced) isomorphic to $T_{a,b,c}$, then $L(G)$ is $N_{a-1,b-1,c-1}$-free. Thus, proving Theorem 1.12 is equivalent to proving the following theorem:

Theorem 5.1. *Let $Y = T_{4,4,4}$ and let G be a connected simple graph without subgraphs isomorphic to Y. Let G_0 be the core of G. If $\kappa(L(G)) \geq 3$, then G_0 has a dominating eulerian subgraph containing all the nontrivial vertices and both end vertices of each nontrivial edge.*

The structure of the proof of Theorem 5.1 is similar to the proofs presented in Chapter 4. We divide the proof into cases based on the circumference of the graph. Recall that Theorem 2.8 states that a 3-edge-connected graph with circumference less than or equal to eight is supereulerian. Therefore we need only consider graphs with circumference nine or greater. We divide the cases further by considering when C is a dominating cycle and when C is not a dominating cycle separately.

5.1 C is not a dominating cycle.

We split this section into cases based on circumference. Before considering the cases, we prove Lemmas that handle whenever there exists a path of length 3 such that no vertex on the path is a vertex of C. From there we consider when $c(G_0) \geq 13$, $c(G_0) = 12$, $c(G_0) = 11$, and $c(G_0) = 10$. Note that by Lemma 4.9, if $c(G_0) = 9$ then C must be a dominating cycle. By Theorem 2.8, any 3-edge-connected graph with circumference less than or equal to eight is supereulerian. When combined, these arguments finish the proof for when C is not a dominating cycle. Let $g = c(G_0)$ throughout this section.

First consider when $c(G_0) \geq 10$ and there is a path of 4 vertices off of C. Label this path as $v_1v_2v_3v_4$. If an endpoint of this path is adjacent to C, without loss of generality say v_1 is adjacent to c_1, then there is a $T_{4,4,4}$ centered at c_1. If neither endpoint is connected to C either by an edge or a path that does not include either of the interior vertices, then one of the two cases must be true. The first possibility is that one of v_1v_4, v_1v_3, and v_2v_4 is an edge. In this case, we can find reorder the vertices v_i to find a path of 4 vertices such that an endpoint is adjacent to C. The other possibility is that none of the above edges exist. In this case, the pair of edges v_1v_2 and v_3v_4 serve as a 2-edge-cut, which violates the assumption that G_0 is 3-edge-connected. Therefore, in each of the following cases we can assume that the longest path off of C contains at most three vertices.

The following two lemmas apply when there is a path of length 3 off of C. Lemma 5.2 assumes that the middle vertex does not have an adjacency on C, while Lemma 5.3 handles when the middle vertex does have an adjacency on C.

Lemma 5.2. *If there is a vertex w with neighbors $v_1, v_2, \ldots, v_{d(w)}$ such that none of the vertices v_j are on C, then all of the vertices v_j must have adjacencies c_i and c_{i+4} for some $1 \leq i \leq c(G_0)$.*

Proof. Assume such a vertex w exists. Without loss of generality, we can assume that there is a path from v_1 to c_i since there must be some path from w to C. This path must be comprised of a single edge, otherwise there would be a $T_{4,4,4}$ centered at c_i. Clearly each v_j cannot be adjacent to a vertex v_k as that would violate the assumption that G_0 is triangle-free. Also, each v_j cannot have an additional adjacency off of C as that would imply the existence of a $T_{4,4,4}$ centered at c_i.

Since G_0 is 3-edge-connected, the vertex v_2 requires two additional neighbors, both of which must be on C. If v_2 is adjacent to a vertex that is distance 1, 2, or 3 from c_i, then G_0 contains a longer cycle. Similarly, if v_2 is adjacent to a vertex that is distance greater than 4 from c_i, say x, then G_0 contains the subgraph $T\{c_i : c_{i+1}c_{i+2}c_{i+3}c_{i+4}, c_{i-1}c_{i-2}c_{i-3}c_{i-4}, v_1wv_2x\}$. This leaves c_i, c_{i+4}, and c_{i-4} as possible neighbors of v_2. Due to the symmetry between the vertices v_1 and v_2 and the symmetry between c_{i-4} and c_{i+4}, we can assume that both v_1 and v_2 are adjacent to c_i and c_{i+4}. Since all v_i are symmetric, this implies that all neighbors of w must be adjacent to c_i and c_{i+4}.

□

If $|N_{G_0}(w)| = t$, then the structure described in the preceding proof is a $K_{3,t}$ with partite sets $\{w, c_i, c_{i+5}\}$ and $N_{G_0}(w)$. We will refer to these structures as a $K_{3,t}$ anchored at $\{c_i, c_{i+5}\}$. One observation to make is that none of the vertices v_i can be nontrivial, as that would give the existence of a $T_{4,4,4}$ in G. Also, the edge wv_i cannot be nontrivial as that would imply the existence of a $T_{4,4,4}$ as well. Therefore, if there exists a dominating eulerian circuit of $G_0 \setminus \{w \cup N_{G_0}(w)\}$ that contains all nontrivial vertices, both ends of every nontrivial edge, and both anchor vertices of the $K_{3,t}$, then we can extend this to a dominating eulerian circuit of G_0 with all of the desired properties simply by appending $c_i v_1 w v_2 c_i$ into the middle of the circuit at the appropriate spot.

Lemma 5.3. *If there exists a path of length three, say $v_1v_2v_3$, such that each $v_i \notin V(C)$ has at least one adjacency on C, then it must be the case that $c(G_0) = 10$, v_1 and v_3 are adjacent to c_i and c_{i+4}, and v_2 is adjacent to c_{i-3}.*

Proof. When determining the adjacencies of v_1 and v_2 in the proof of Lemma 5.2, the fact that v_2 did not have any adjacencies on C was not used. Therefore, we can assume, without loss of generality, that both v_1 and v_3 are adjacent to c_1 and c_5 and no other vertices.

Let x denote the neighbor of v_2 that is on C. If x is distance 1 or 2 from either c_1 or c_5 then G_0 contains a longer cycle. If the distance from x to c_1 is greater than 4, then G_0 contains a $T_{4,4,4}$ centered at c_1. The case when the distance from x to c_5 is greater than 4 is symmetric.

If x is distance exactly 4 from c_5, then either x is c_1 and G_0 contains a triangle or x is c_9. Assuming $x = c_9$ and $c(G_0) = g \geq 11$, this gives a $T_{4,4,4}$ described by $T\{v_2 : v_3c_5c_4c_3, c_9c_8c_7c_6, v_1c_1c_gc_{g-1}\}$. Note that if $g < 11$, then c_9 is distance 1 or 2 from c_1 and G_0 contains the $T_{4,4,4}$ described previously. The case where x is distance 4 from c_1 is symmetric.

The above observations imply that x must be distance 3 from both c_1 and c_5. The only way this is possible is if $c(G_0) = 10$ and $x = c_8$, which is the desired result.

$\square$

Now assume that $c(G_0) = 10$, there is a path $v_1v_2v_3$ such that $v_i \notin V(C)$, and the adjacencies of each v_i are as described in Lemma 5.3. Observe that v_2 can have arbitrarily many neighbors v_i off of C and each of these neighbors is symmetric to both v_1 and v_3. None of these neighbors of v_2 can be nontrivial, as that would imply a $T_{4,4,4}$ in G. Likewise, none of the edges v_2v_i can be nontrivial as that would also imply a $T_{4,4,4}$ in G. Therefore, if there exists a dominating eulerian circuit of $G_0 \setminus \{v_2 \cup N_{G_0}(v_2)\}$ that contains all nontrivial vertices, both ends of every nontrivial edge, and both c_i and c_{i+4}, then we can extend this to a dominating eulerian circuit of G_0 with all of the desired properties simply by appending $c_i v_1 v_2 v_3 c_i$ into the middle of the circuit at the appropriate spot.

Since we have shown that any path of length 3 that is not on C can be absorbed into a dominating eulerian circuit provided that the original dominating eulerian circuit contains all vertices of C, it is now only necessary to consider paths of length 2 that are not on C. (Note that if all paths not on C are of length 1, then C is a dominating cycle and this case is handled in the next section.)

By Lemma 5.2, any vertex with no adjacencies on C must be the middle vertex on a path of length 3 not on C. Therefore, we may assume that all vertices have at least one adjacency on C. By assuming that each vertex x not on C has at least one neighbor on C, we can go one step further and assume that x has all but at most one neighbor on C. To see this, assume that x is adjacent to y_1, another vertex not on C. By assumption y_1 has an adjacency on C. If x were to have another neighbor not on C, say y_2, then $y_1 x y_2$ would be a path of length 3 and we can create the desired dominating eulerian circuit.

In each of the following sections, let xy be an edge not dominated by $V(C)$. Without loss of generality, we assume that x is adjacent to c_1.

5.1.1 Case 1: $c(G_0) \geq 13$.

When assuming x is adjacent to c_1, we can easily see that y cannot be adjacent to any vertex that is distance 1 or 2 from c_1 as that would create a longer cycle. Likewise, we can assume that y is not adjacent to any vertex that is distance greater than 4 from c_1 as that would imply a $T_{4,4,4}$ centered at c_1. Therefore, the two neighbors of y must be distance 3 or 4 from c_1. This gives three nonsymmetric possibilities: y is adjacent to c_4 and c_{g-2}, y is adjacent to c_5 and c_{g-2}, and y is adjacent to c_5 and c_{g-3} (recall that $g = c(G_0)$.)

First, assume that y is adjacent to c_4 and c_{g-2}. This graph contains a $T_{4,4,4}$ described by $T\{c_4 : c_3c_2c_1x, c_5c_6c_7c_8, yc_gc_{g-1}c_{g-2}\}$. Note that this $T_{4,4,4}$ is present whenever $c(G_0) \geq 11$.

Next, assume that y is adjacent to c_5 and c_{g-2}. (Note that in this case, y adjacent to c_4 and c_{g-3} is symmetric.) This graph contains a $T_{4,4,4}$ described by $T\{c_5 : c_4c_3c_2c_1, yc_{g-2}c_{g-1}c_g, c_6c_7c_8c_9\}$. This particular $T_{4,4,4}$ is present whenever $c(G_0) \geq 12$.

Lastly, assume that y is adjacent to c_5 and c_{g-3}. This graph contains a $T_{4,4,4}$ described by $T\{c_5 : c_4c_3c_2c_1, c_6c_7c_8c_9, yc_{g-3}c_{g-2}c_{g-1}\}$. This graph is present whenever $g - 3 > 9$, i.e. when $c(G_0) \geq 13$.

5.1.2 Case 2: $c(G_0) = 12$.

By the same arguments as when $c(G_0) = 13$, we can assume that the neighbors of y must be distance 3 or 4 from c_1. The only arrangement of these neighbors that did not produce a $T_{4,4,4}$ in a graph of circumference 12 was when both neighbors were distance 4. Therefore, we may assume that the neighbors of y are c_5 and c_9. Since we assumed no paths of length 3 off of C, the third neighbor of x must be on C. However, since x and y are symmetric, we conclude that any neighbor of x must be distance 4 from both c_5 and c_9. As x is already adjacent to c_1 and there is no other vertex that is distance 4 from both c_5 and c_9, it is easy to see that any choice for a third neighbor of x will either result in a $T_{4,4,4}$, a triangle, or a longer cycle.

5.1.3 Case 3: $c(G_0) = 11$.

Recall that there are two possible configurations of the neighbors of x and y that are shown in Figure 4.1. In both configurations x is adjacent to c_1 and c_3 and y is adjacent to c_6. In this case, G_0 contains the $T_{4,4,4}$ described by $T\{c_6 : c_5c_4c_3c_2, c_7c_8c_9c_{10}, yxc_1c_{11}\}$.

5.1.4 Case 4: $c(G_0) = 10$.

Recall that there is only one possible configuration of the neighbors of x and y that does not violate our assumptions about G_0, and this is shown in Figure 4.2. By Lemma 2.7, the graph G_0 must contain at least 13 vertices. Therefore, there must be an additional vertex, v, that is not on C. If v is adjacent to one of the neighbors of x or y, without loss of generality we can assume c_1, then G_0 contains the $T_{4,4,4}$ described by $T\{c_8 : c_7c_6c_5c_4, c_9c_{10}c_1v, yxc_3c_2\}$. If v is adjacent to a vertex in one of the gaps containing two vertices, say c_4, then G_0 contains $T\{c_1 : c_2c_3c_4v, c_{10}c_9c_8c_7, xyc_6c_5\}$. This implies that v can only be adjacent to vertices in the gaps containing one vertex. Since v must have two adjacencies on C (otherwise there is a path of length 3 off of C), it must be the case that v is adjacent to c_2 and c_7. This implies that G_0 contains the $T_{4,4,4}$ described by $T\{x : c_1c_2vc_7, c_3c_4c_5c_6, yc_8c_9c_{10}\}$.

This concludes the argument for when C is not a dominating cycle.

5.2 C is a dominating cycle.

Since C is a dominating eulerian subgraph, if C contains all nontrivial vertices and both endpoints of every nontrivial edge the theorem is satisfied. Therefore, it must be the case that there is a vertex off of C that is either nontrivial or the endpoint of a nontrivial edge. Throughout this section, let w denote this vertex and w' denote the vertex in G that was contracted to w.

The vertex w' can either be a pendant vertex or belong to a path of length 3. Since we are looking for a $T_{4,4,4}$ subgraph, we do not need to consider separately when a nontrivial vertex is the center of the $T_{4,4,4}$ and when it is not as there are no paths of length 1 involved. Therefore, we will assume that w' (and any other contracted vertex) is a pendant vertex and note that the same subgraphs are present when the vertex is contracted from a path of length 3, with the possible modification of switching the order along the path.

5.2.1 Case 1: $c(G_0) \geq 12$.

Assume $c(G_0)$ is exactly 12 and two neighbors of w are distance 2 apart. Without loss of generality say w is adjacent to c_1 and c_3. Recall that by Lemma 4.10, anytime that two neighbors of w are distance 2 apart the vertex between these neighbors must be nontrivial. Therefore, c_2 must be nontrivial. In this situation, there are four non-symmetric places to put the third neighbor of w: c_5, c_6, c_7, and c_8. These produce the following $T_{4,4,4}$ subgraphs:

Neighbor of w	Resulting Subgraph
c_5	$T\{c_5 : c_4c_3c_2c_2', wc_1c_{12}c_{11}, c_7c_8c_9c_{10}\}$
c_6	$T\{c_6 : c_5c_4c_3c_2, wc_1c_{12}c_{11}, c_7c_8c_9c_{10}\}$
c_7	$T\{c_7 : c_6c_5c_4c_3, wc_1c_2c_2', c_8c_9c_{10}c_{11}\}$
c_8	$T\{c_8 : c_7c_6c_5c_4, wc_1c_2c_3, c_9c_{10}c_{11}c_{12}\}$

Note that if $c(G_0) > 12$ and two neighbors of w are distance 2 apart, we can contract edges along C to get one of the structures considered above. Uncontracting the edges to get back to the original graph will clearly preserve the existence of a $T_{4,4,4}$.

Now consider when $c(G_0) = 12$ and all neighbors of w are distance at least

3 apart. There are three non-symmetric ways to place the neighbors of w: $\{c_1, c_4, c_7\}$, $\{c_1, c_4, c_8\}$, and $\{c_1, c_5, c_9\}$.

In the case where the neighbors are $\{c_1, c_4, c_8\}$ there is a $T_{4,4,4}$ described by $T\{c_8 : wc_1c_2c_3, c_7c_6c_5c_4, c_9c_{10}c_{11}c_{12}\}$. In the case where the neighbors are $\{c_1, c_5, c_9\}$ there is a $T_{4,4,4}$ described by $T\{w : c_1c_2c_3c_4, c_5c_6c_7c_8, c_9c_{10}c_{11}c_{12}\}$. Once again, if we have a core G_0 with $c(G_0) > 12$ and it is possible to contract edges along C to create one of the above two structures, then G_0 must also contain a $T_{4,4,4}$.

In the case where the neighbors of w are $\{c_1, c_4, c_7\}$, it is necessary to consider additional structure of G_0. By Lemma 2.7, either G_0 is supereulerian or contains at least 14 vertices. Therefore, we can assume that G_0 has at least one additional vertex v. If v is adjacent to c_2 (symmetrically c_6), G_0 contains the $T_{4,4,4}$ described by $T\{c_7 : c_6c_5c_4c_3, wc_1c_2v, c_8c_9c_{10}c_{11}\}$. If v is adjacent to c_4 we get $T\{c_1 : c_2c_3c_4v, wc_7c_6c_5, c_{12}c_{11}c_{10}c_9\}$. Lastly, if v is adjacent to c_{10} there is a $T_{4,4,4}$ described by $T\{c_1, c_2c_3c_4c_5, wc_7c_8c_9, c_{12}c_{11}c_{10}v\}$. This leaves $c_1, c_3, c_5, c_7, c_8, c_9, c_{11}$, and c_{12} as possible neighbors of w.

Consider v adjacent to c_3. If v is adjacent to any of the vertices c_5, c_8, c_{11}, or c_{12}, we can find a longer cycle in G_0, and if v is adjacent to c_7 or c_9 the graph contains a $T_{4,4,4}$. These are summarized below:

Neighbor of v	Resulting Subgraph
c_5	$c_1c_2c_3vc_5c_4wc_7c_8c_9c_{10}c_{11}c_{12}c_1$
c_7	$T\{c_7 : c_6c_5c_4w, c_8c_9c_{10}c_{11}, vc_3c_2c_1\}$
c_8	$c_1wc_7c_6c_5c_4c_3vc_8c_9c_{10}c_{11}c_{12}c_1$
c_9	$T\{c_3 : c_2c_1ww', c_4c_5c_6c_7, vc_9c_{10}c_{11}\}$
c_{11}	$c_1c_2c_3vc_{11}c_{10}c_9c_8c_7c_6c_5c_4wc_1$
c_{12}	$c_1c_2c_3vc_{12}c_{11}c_{10}c_9c_8c_7c_6c_5c_4wc_1$

Thus, when v is adjacent to c_3, the only other possible neighbor of v that does not create either a longer cycle or a $T_{4,4,4}$ is c_1, thus there is no way to place the remaining neighbors of v. We conclude that v cannot be adjacent to c_3 or (by symmetry) c_5.

Now consider v adjacent to c_1. It has already been determined that v cannot be adjacent to any vertex from the set $\{c_2, c_3, c_4, c_5, c_6, c_{10}\}$. Note that if v is adjacent to c_{12} there is a triangle, and if v is adjacent to c_7 or c_8 there is a $T_{4,4,4}$ as described below. This forces v to be adjacent to $\{c_1, c_9, c_{11}\}$.

Neighbor of v	Resulting Subgraph
c_7	$T\{c_1 : c_2c_3c_4c_5, c_{12}c_{11}c_{10}c_9, vc_7ww'\}$
c_8	$T\{c_1 : c_2c_3c_4c_5, c_{12}c_{11}c_{10}c_9, vc_8c_7c_6\}$

Since G_0 has minimum degree 3 and c_2 does not have an adjacency off of C, there must be a chord incident with c_2. If the chord is also incident to c_4 or c_{12} there is a triangle in G_0. Every other possible chord, when considered with vertex v adjacent to $\{c_1, c_9, c_{11}\}$ also present, gives either a longer cycle or a $T_{4,4,4}$ as summarized below.

Chord	Resulting Subgraph
c_2c_5	$C_{13} = c_2c_3c_4wc_1c_{12}c_{11}c_{10}c_9c_8c_7c_6c_5c_2$
c_2c_6	$T\{c_2 : c_3c_4ww', c_6c_7c_8c_9, c_1c_{12}c_{11}c_{10}\}$
c_2c_7	$T\{c_2 : c_3c_4c_5c_6, c_7c_8c_9c_{10}, c_1c_{12}c_{11}v\}$
c_2c_8	$C_{13} = c_2c_8c_9c_{10}c_{11}c_{12}c_1wc_7c_6c_5c_4c_3c_2$
c_2c_9	$T\{c_2 : c_1c_{12}c_{11}c_{10}, c_9c_8c_7w, c_3c_4c_5c_6\}$
c_2c_{10}	$C_{13} = c_2c_{10}c_{11}c_{12}c_1vc_9c_8c_7c_6c_5c_4c_3c_2$
c_2c_{11}	$T\{c_4 : c_3c_2c_{11}c_{10}, wc_1vc_9, c_5c_6c_7c_8\}$

Therefore, we conclude that v cannot be adjacent to c_1 or (symmetrically) c_7. This leaves only c_8, c_9, c_{11}, and c_{12} as possible adjacencies of v. There is no way to choose three neighbors of v without creating a triangle.

Consider a core G_0 with $c(G_0) = 13$ that could have had one edge of C contracted to create the graph with circumference 12 and w adjacent to c_1, c_4, and c_7 as described above, there are four possibilities up to symmetry: w is adjacent to $\{c_1, c_4, c_7\}$, $\{c_1, c_4, c_8\}$, $\{c_1, c_4, c_9\}$, or $\{c_1, c_5, c_9\}$. In each of the last three cases, it is possible to choose an edge to contract that would give the adjacencies of w as either $\{c_1, c_4, c_8\}$ or $\{c_1, c_5, c_9\}$. This implies that these graphs must contain a $T_{4,4,4}$ by previous argument. In the case where the neighbors of w are $\{c_1, c_4, c_7\}$, the subgraph $T\{c_7 : wc_1c_{13}c_{12}, c_6c_5c_4c_3, c_8c_9c_{10}c_{11}\}$ is present.

Every graph with $c(G_0) > 13$ in which w does not have adjacencies that are distance two apart can be transformed into one of the graphs of circumference 13 described in the previous paragraph by contracting edges along C. Thus, each of these graphs must contain a $T_{4,4,4}$ since uncontracting edges preserves the existence of a $T_{a,b,c}$.

5.2.2 Case 2: $c(G_0) = 11$.

Lemma 2.7 states that any graph with at most thirteen vertices and circumference more than nine is collapsible. Therefore, we may assume that there is both the nontrivial vertex w and at least one other vertex v not on C. We proceed by considering the possible placements of the neighbors of w. Up to symmetry, there are five configurations of neighbors of w. We will label these as $\{c_1, c_3, c_5\}$, $\{c_1, c_3, c_6\}$, $\{c_1, c_3, c_7\}$, $\{c_1, c_4, c_7\}$, and $\{c_1, c_4, c_8\}$.

Case 2a: w is adjacent to $\{c_1, c_3, c_5\}$.

Lemma 4.10 implies that both c_2 and c_4 must be nontrivial vertices. This gives the $T_{4,4,4}$ described by $T\{c_1 : c_2c_3c_4c_4', wc_5c_6c_7, c_{11}c_{10}c_9c_8\}$.

Case 2b: w is adjacent to $\{c_1, c_3, c_6\}$.

Again, Lemma 4.10 implies that c_2 must be nontrivial. We consider where the vertex v can have adjacencies. First note that v cannot be adjacent to c_3, c_7, c_9, or c_{11} as those adjacencies immediately give rise to a $T_{4,4,4}$ (these are summarized in the table below.) Also note that these $T_{4,4,4}$ subgraphs would be present if these vertices were simply nontrivial instead of having an adjacency off of C.

Neighbor of v	Resulting Subgraph
c_3	$T\{c_6 : c_5c_4c_3v, wc_1c_2c_2', c_7c_8c_9c_{10}\}$
c_7	$T\{c_1 : c_2c_3c_4c_5, wc_6c_7v, c_{11}c_{10}c_9c_8\}$
c_9	$T\{c_1 : c_{11}c_{10}c_9v, c_2c_3c_4c_5, wc_6c_7c_8\}$
c_{11}	$T\{c_6 : c_5c_4c_3c_2 : wc_1c_{11}v, c_7c_8c_9c_{10}\}$

Next assume that v is adjacent to c_4. Since G_0 is 3-edge-connected, v must have at least two other neighbors on C. If v is also adjacent to c_5, there is a triangle in G_0. If v is adjacent to c_2 there is the longer cycle $c_2vc_4c_5c_6c_7c_8c_9c_{10}c_1wc_3c_2$, while v being adjacent to c_{10} gives the $T_{4,4,4}$ described by $T\{c_4 : c_3c_2c_1c_{11}, c_5c_6ww', vc_{10}c_9c_8\}$. This leaves only c_1 and c_6 as possible neighbors of v. However, when v is adjacent to $\{c_1, c_4, c_6\}$ there is a $T_{4,4,4}$ described by $T\{c_1 : c_2c_3c_4c_5, c_{11}c_{10}c_9c_8, vc_6ww'\}$. Therefore, v must not be adjacent to c_4.

Now consider v adjacent to c_5. If v is also adjacent to c_1 or c_2 there is the longer cycle $c_2vc_5c_4c_3wc_6c_7c_8c_9c_{10}c_{11}c_1c_2$ or $c_1vc_5c_4c_3wc_6c_7c_8c_9c_{10}c_{11}c_1$, respectively. If v is also adjacent to c_6, there is a triangle. Therefore, v must be adjacent to the set $\{c_5, c_8, c_{10}\}$. This configuration gives the $T_{4,4,4}$ described by $T\{c_8 : c_7c_6ww', vc_5c_4c_3, c_9c_{10}c_{11}c_1\}$. We conclude that v is not adjacent to c_5.

Lastly, consider v adjacent to c_2. Clearly, v cannot also be adjacent to c_1 as that creates a triangle, and if v is adjacent to c_{10} that gives the longer cycle $c_{10}vc_2c_1wc_3c_4c_5c_6c_7c_8c_9c_{10}$. This implies v must be adjacent to $\{c_2, c_6, c_8\}$. This configuration gives the longer cycle $c_2vc_8c_9c_{10}c_{11}c_1wc_6c_5c_4c_3c_2$. Therefore, v is not adjacent to c_2.

The above arguments leave only c_1, c_6, c_8, and c_{10} as possible neighbors of v. First note that if v is adjacent to both c_1 and c_6 there is the $T_{4,4,4}$ described by $T\{c_1 : c_2c_3c_4c_5, vc_6ww', c_{11}c_{10}c_9c_8\}$. So, v must either have $\{c_1, c_8, c_{10}\}$ as its set of adjacencies or be adjacent to $\{c_6, c_8, c_{10}\}$. In either of these cases, if v is nontrivial, there is the $T_{4,4,4}$ described by $T\{c_6, c_7c_8vv', c_5c_4c_3c_2, wc_1c_{11}c_{10}\}$. This implies that the only nontrivial vertex not on C is w and all edges not on C are dominated by the set $\{c_1, c_3, c_6, c_8, c_{10}\}$.

If v is adjacent to $\{c_1, c_8, c_{10}\}$, then $c_1c_2c_3c_4c_5c_6wc_1c_{11}c_{10}c_9c_8cvc_1$ is a the desired dominating eulerian circuit. Note that while the eulerian circuit does not visit the vertex c_7, by previous argument we know that it is neither nontrivial nor the endpoint of a nontrivial edge. Likewise, if v is adjacent to $\{c_6, c_8, c_{10}\}$ the desired dominating eulerian circuit is the trail $c_6c_7c_8c_9c_{10}vc_6wc_1c_2c_3c_4c_5c_6$. This case did not visit c_{11} in the eulerian circuit, but that is not necessary as c_{11} also cannot be nontrivial or the endpoint of a nontrivial edge.

We conclude that when the neighbors of w are arranged in this manner, either G_0 does not meet our assumptions or there exists an appropriate dominating eulerian circuit containing all nontrivial vertices and both endpoints of each nontrivial edge.

Case 2c: w is adjacent to $\{c_1, c_3, c_7\}$.

Once again, Lemma 4.10 implies that c_2 must be a nontrivial vertex. This gives the $T_{4,4,4}$ described by $T\{c_7 : c_6c_5c_4c_3, c_8c_9c_{10}c_{11}, wc_1c_2c_2'\}$. Therefore, the neighbors of w cannot be arranged in this manner without violating the assumptions of the theorem.

Case 2d: w is adjacent to $\{c_1, c_4, c_7\}$.

We proceed by considering the possible adjacencies of v. First we note that if v is adjacent to c_2 (symmetrically c_6), then there is a $T_{4,4,4}$ described by $T\{c_7 : c_6c_5c_4c_3, wc_1c_2v, c_8c_9c_{10}c_{11}\}$. Also, if v is adjacent to c_4 there is a $T_{4,4,4}$ described by $T\{c_7 : c_6c_5c_4v, wc_1c_2c_3, c_8c_9c_{10}c_{11}\}$.

Next consider when v is adjacent to c_3. If v is adjacent to a vertex from the set $S = \{c_5, c_8, c_9, c_{10}, c_{11}\}$, then there is either a longer cycle or a $T_{4,4,4}$ as summarized in the table below. This implies that v must be adjacent to both c_1 and c_7, which gives the $T_{4,4,4}$ described by $T\{c_7 : c_8c_9c_{10}c_{11}, c_6c_5c_4c_3, vc_1ww'\}$. Therefore, v cannot be adjacent to c_3 or (symmetrically) c_5.

Neighbor of v	Resulting Subgraph
c_5	$C_{12} = c_3vc_5c_4wc_7c_8c_9c_{10}c_{11}c_1c_2c_3$
c_8	$C_{12} = c_3vc_8c_9c_{10}c_{11}c_1wc_7c_6c_5c_4c_3$
c_9	$T\{c_9 : c_{10}c_{11}c_1c_2, vc_3c_4w, c_8c_7c_6c_5\}$
c_{10}	$C_{12} = c_1c_2c_3vc_{10}c_9c_8c_7c_6c_5c_4wc_1$
c_{11}	$C_{13} = c_3vc_{11}c_{10}c_9c_8c_7c_6c_5c_4wc_1c_2c_3$

Now consider when v is adjacent to c_1. If v is also adjacent to c_{11} there is a triangle, and if v is adjacent to c_7 there is a $T_{4,4,4}$ described by $T\{c_1 : c_2c_3c_4c_5, vc_7ww', c_{11}c_{10}c_9c_8\}$. Thus the other two neighbors of v must come from the set $\{c_8, c_9, c_{10}\}$. To prevent a triangle, the neighbors must be c_8 and c_{10}. By symmetry, if there is a vertex adjacent to c_7 its set of neighbors must be $\{c_7, c_9, c_{11}\}$.

Likewise, when we consider v adjacent to c_{11} we can deduce that v must be adjacent to $\{c_7, c_9, c_{11}\}$. This can be seen by observing that if v is adjacent to c_1 or c_{10} there is a triangle and the only way to choose two neighbors from the remaining vertices without creating a triangle is to have v adjacent to

$\{c_7, c_9, c_{11}\}$. By symmetry, any vertex v adjacent to c_8 must be adjacent to $\{c_1, c_8, c_{10}\}$.

Lastly, consider v adjacent to c_{10}. It has already been determined that the only other vertices that v could be adjacent to are c_1, c_7, and c_8. We note that if v is adjacent to c_7 we get the $T_{4,4,4}$ described by $T\{c_1 : c_2c_3c_4c_5, vc_7ww', c_{11}c_{10}c_9c_8\}$. Therefore, we conclude that all vertices off of C that are not w are adjacent to either $\{c_1, c_8, c_{10}\}$ or $\{c_7, c_9, c_{11}\}$.

Note that the sets $s_1 = \{c_1, c_8, c_{10}\}$ and $s_2 = \{c_7, c_9, c_{11}\}$ are symmetric and that there cannot be v_1 adjacent to s_1 and v_2 adjacent to s_2 as that permits the longer cycle $c_1c_2c_3c_4c_5c_6c_7v_2c_9c_8v_1c_{10}c_{11}c_1$. So, without loss of generality, we can assume that v is adjacent to s_1. Also note that v must be trivial as v nontrivial gives $T\{c_7 : c_6c_5c_4c_3, wc_1vv', c_8c_9c_{10}c_{11}\}$ as a $T_{4,4,4}$ subgraph. Therefore, w is the only nontrivial vertex not on C and $c_1c_2c_3c_4c_5c_6c_7wc_1vc_8c_9c_{10}c_{11}c_1$ is the desired dominating eulerian circuit.

Case 2e: w is adjacent to $\{c_1, c_4, c_8\}$.

As in the previous case, we proceed by considering the possible neighbors of the vertex v. Immediately, we can eliminate c_2 (symmetrically c_3), c_5 (symmetrically c_{11}), and c_7 (symmetrically c_9), as they give rise to the $T_{4,4,4}$ subgraphs $T\{w : c_1c_{11}c_{10}c_9, c_4c_3c_2v, c_8c_7c_6c_5\}$, $T\{c_8 : c_7c_6c_5v, c_9c_{10}c_{11}c_1, wc_4c_3c_2\}$, and $T\{c_4 : c_5c_6c_7v, wc_8c_9c_{10}, c_3c_2c_1c_{11}\}$, respectively. This leaves only c_1, c_4, c_6, c_8, and c_{10} as possible adjacencies of v.

First consider v adjacent to c_6 (symmetrically c_{10}). If v is also adjacent to c_{10}, then $T\{c_6 : c_7c_8ww', vc_{10}c_{11}c_1, c_5c_4c_3c_2\}$ is present. If v is also adjacent to c_1, then $T\{c_6 : vc_1c_2c_3, c_5c_4ww', c_7c_8c_9c_{10}\}$ is present. This implies that any vertex adjacent to c_6 must also be adjacent to $s_1 = \{c_4, c_6, c_8\}$. Note that v cannot be nontrivial as that would give rise to the $T_{4,4,4}$ described by $T\{c_8 : c_7c_6vv', c_9c_{10}c_{11}c_1, wc_4c_3c_2\}$. By symmetry, any vertex adjacent to c_{10} must be trivial and adjacent to the set $s_2 = \{c_1, c_8, c_{10}\}$.

The only remaining possibility is that the adjacencies of v are the set $s_3 = \{c_1, c_4, c_8\}$. Any vertex adjacent to the set s_3 could possibly be nontrivial.

Putting together the arguments above, we see that the only nontrivial vertices not on C must be adjacent to s_3. If there are at least two nontrivial vertices, there is a dominating eulerian subgraph by Corollary 4.13. If w is the only nontrivial vertex, there must be at least one nontrivial vertex x that is not on C since Lemma 2.7 guarantees at least 14 vertices in G_0. If x is adjacent to s_3, we can pair it with v and use the dominating trail for an even number of nontrivial vertices described in the proof of Corollary 4.13. Without loss of generality, we can then assume x is adjacent to s_1 since the sets s_1 and s_2 are symmetric. In this case, the subgraph $c_8c_7c_6c_5c_4wc_8xc_4c_3c_2c_1c_{11}c_{10}c_9c_8$ is the desired dominating eulerian circuit.

5.2.3 Case 3: $c(G_0) = 10$.

Lemma 2.7 states that any graph with at most thirteen vertices and circumference more than nine is collapsible. Therefore, we may assume that there is both the nontrivial vertex w and at least three other vertices v_i not on C. We proceed by considering the possible placements of the neighbors of w. Up to symmetry, there are four configurations of neighbors of w, as shown in Figure 4.4.

Case 3a: w is adjacent to $\{c_1, c_3, c_5\}$.

We first note that Lemma 4.10 implies that both c_2 and c_4 are nontrivial. Immediately, we see that if v is adjacent to c_6 (symmetrically c_{10}), then the subgraph $T\{c_1 : c_2c_3c_4c_4', wc_5c_6v, c_{10}c_9c_8c_7\}$ is present, and if v is adjacent to c_8 then the $T_{4,4,4}$ described by $T\{c_1 : c_2c_3c_4c_4', wc_5c_6c_7, c_{10}c_9c_8c_7\}$ is present. We note that these subgraphs are also present if the vertices c_6, c_8, and c_{10} are nontrivial.

First consider v adjacent to c_2. Clearly, v adjacent to c_1 or c_3 creates a triangle. If v is adjacent to c_5, then the $T_{4,4,4}$ described by $T\{c_5 : vc_2c_1c_{10}, c_6c_7c_8c_9, c_4c_3ww'\}$ is present. If v is adjacent to c_7, then $c_7vc_2c_3c_4c_5wc_1c_{10}c_9c_8c_7$ is a longer cycle. This leaves only c_4 and c_9 as the possible adjacencies of v, which gives the longer cycle $c_4vc_9c_8c_7c_6c_5wc_1c_2c_3c_4$. This implies that v is not adjacent to c_2 or, by symmetry, c_4.

This leaves the vertices c_1, c_3, c_5, c_7, and c_9 as the only vertices that v can be adjacent to. We first note that none of these vertices v can be nontrivial as v must be adjacent to at least one of c_1, c_3, and c_7, and this would give rise to the $T_{4,4,4}$ subgraphs $T\{c_5 : c_4c_3c_2c_2', wc_1vv', c_6c_7c_8c_9\}$, $T\{c_1 : c_2c_3vv', wc_5c_4c_4', c_{10}c_9c_8c_7\}$, and $T\{c_5 : c_6c_7vv', c_4c_3c_2c_2', wc_1c_{10}c_9\}$, respectively.

We next note that if v is adjacent to c_1 and c_5 the subgraph $T\{c_5 : c_6c_7c_8c_9, c_4c_3c_2c_2', vc_1ww'\}$ is present. This implies that v must be adjacent to two vertices from the set $\{c_3, c_7, c_9\}$.

If v is adjacent to both c_3 and c_9 then $c_3c_2c_1wc_3vc_9c_8c_7c_6c_5c_4c_3$ is a dominating eulerian circuit. Note that the only vertex of C that is not included in the circuit is c_{10} and it has already been determined that c_{10} is trivial. The case where v is adjacent to both c_3 and c_7 is symmetric.

Lastly, consider when v is adjacent to both c_7 and c_9. Either v is adjacent to $s_1 = \{c_1, c_7, c_9\}$ or $s_2 = \{c_5, c_7, c_9\}$, otherwise we have the case described above. Both s_1 and s_2 are symmetric, so assume v is adjacent to s_1. In this case $c_1c_2c_3c_4c_5wc_1vc_7c_8c_9c_{10}c_1$ is the desired dominating eulerian circuit. As above, the circuit does not contain c_6, but it was determined previously that c_6 must be trivial.

Case 3b: w is adjacent to $\{c_1, c_3, c_6\}$.

By Lemma 4.10, c_2 must be nontrivial. As in the previous case, we proceed by considering the neighbors of v.

First consider v adjacent to c_3. Without any further knowledge of the adjacencies of v, we see that the subgraph $T\{c_6 : c_5c_4c_3c_3', wc_1c_2c_2', c_7c_8c_9c_{10}\}$ is present.

Next consider v adjacent to c_4. If v is adjacent to c_5 there is a triangle. If v is adjacent to c_1 there is a $T_{4,4,4}$ described by $T\{c_1 : c_2c_3ww', vc_4c_5c_6, c_{10}c_9c_8c_7\}$. Lastly, if v is adjacent to any of c_2, c_7, c_8, or c_{10} we get a longer cycle. When v is adjacent to c_2, the cycle is $c_2vc_4c_5c_6c_7c_8c_9c_{10}c_1wc_3c_2$. When v is adjacent to c_7, the cycle is $c_7vc_4c_5c_6wc_3c_2c_1c_{10}c_9c_8c_7$. When v is adjacent to c_8, the cycle is $c_8vc_4c_5c_6wc_3c_2c_1c_{10}c_9c_8$. Lastly, when v is adjacent to c_{10}, the cycle is $c_{10}vc_4c_3c_2c_1wc_6c_7c_8c_9c_{10}$. This implies that v must be adjacent to c_4, c_6, and c_9. This gives the $T_{4,4,4}$ subgraph described by $T\{c_9 : c_{10}c_1c_2c_2', c_8c_7c_6c_5, vc_4c_3w\}$. We conclude that v cannot be adjacent to c_4.

Next consider v adjacent to c_5. Since G_0 is triangle-free, v cannot be adjacent to c_6. If v is adjacent to one of c_2, c_7, or c_{10} then one of the following longer cycles is present, respectively: $c_2vc_5c_4c_3wc_6c_7c_8c_9c_{10}c_1c_2$, $c_7vc_5c_6wc_3c_2c_1c_{10}c_9c_8c_7$, or $c_{10}vc_2c_1wc_3c_4c_5c_6c_7c_8c_9c_{10}$. If v is adjacent to c_9, the subgraph $T\{c_9 : c_{10}c_1c_2c_2', c_8c_7c_6w, vc_5c_4c_3\}$ is present. This implies that v must be adjacent to $\{c_1, c_5, c_8\}$, and in this case the subgraph $T\{c_5 : c_4c_3c_2c_2', vc_1ww', c_6c_7c_8c_9\}$ is present. We conclude that, v cannot be adjacent to c_5.

Consider when v is adjacent to c_2. When v is adjacent to c_7, c_8, c_9 or c_{10} one of the longer cycles $c_7vc_2c_3c_4c_5c_6wc_1c_{10}c_9c_8c_7$, $c_2vc_8c_9c_{10}c_1wc_6c_5c_4c_3c_2$, $c_9vc_2c_1wc_3c_4c_5c_6c_7c_8c_9$, or $c_{10}vc_2c_1wc_3c_4c_5c_6c_7c_8c_9c_{10}$ is present. This implies that v must be adjacent to c_1 and c_6, which contradicts G_0 being triangle-free.

Next consider v adjacent to c_1. If v is adjacent to c_6 there is the subgraph $T\{c_1 : c_2c_3c_4c_5, vc_6ww', c_{10}c_9c_8c_7\}$. This leaves c_7, c_8, c_9, and c_{10} as possible neighbors. The only way to choose two additional neighbors from that set and keep G_0 triangle-free is to have v adjacent to c_7 and c_9. Since G_0 must have at least 14 vertices, there must be an additional vertex x not on C. By previous arguments, x can only be adjacent to c_1, c_6, c_7, c_8, c_9, or c_{10}. If x is adjacent to c_{10} there is the subgraph $T\{c_6 : c_5c_4c_3c_2, wc_1c_{10}x, c_7c_8c_9v\}$ present. If x is adjacent to c_8, there is the subgraph $T\{c_1 : c_{10}c_9c_8x, c_2c_3c_4c_5, wc_6c_7v\}$. If x is adjacent to c_6, the subgraph $T\{c_1 : c_2c_3ww', xc_6c_5c_4, c_{10}c_9c_8c_7\}$ is present. This implies x must have the same adjacencies as v. In fact, we can deduce that all additional vertices off of C must be adjacent to the same adjacencies as v and G_0 contains a spanning eulerian circuit by Lemma 4.12.

The remaining possible vertices that can be adjacent to v are c_6, c_7, c_8, c_9, and c_{10}. The only way to choose three neighbors from this set and not create a triangle is to have v adjacent to c_6, c_8, and c_{10}. Thus all neighbors off of C which are not w must be adjacent to the same subset of vertices of C, and none can be nontrivial or the endpoint of a nontrivial edge as that would give the subgraph $T\{c_6 : c_7c_8vv', c_5c_4c_3c_2, wc_1c_{10}c_9\}$. In this case, the circuit $c_6c_5c_4c_3c_2c_1wc_6c_7c_8c_9c_{10}vc_6$ is a dominating eulerian subgraph with the desired properties.

Case 3c: w is adjacent to $\{c_1, c_3, c_7\}$.

By Lemma 4.10, c_2 must be nontrivial. As in the previous cases, we proceed by considering the neighbors of v.

If v is adjacent to c_4, the subgraph $T\{c_7 : c_6c_5c_4v, wc_3c_2c_2', c_8c_9c_{10}c_1\}$ is present. The case when v is adjacent to c_{10} is symmetric.

Next consider when v is adjacent to c_6. If v is adjacent to c_2, the longer cycle $c_2vc_6c_5c_4c_3wc_7c_8c_9c_{10}c_1c_2$ is present, and if v is adjacent to c_1 the $T_{4,4,4}$ described by $T\{c_1 : c_2c_3ww', vc_6c_5c_4, c_{10}c_9c_8c_7\}$ is present. Since G_0 is triangle-free, this leaves either $s_1 = \{c_3, c_6, c_8\}$ or $s_2 = \{c_3, c_6, c_9\}$ as the adjacencies of v. In the case when the adjacencies of v are s_1, the $T_{4,4,4}$ described by $T\{c_3 : c_4c_5c_6c_7, vc_8c_9c_{10}, c_2c_1ww'\}$ is present. In the case when the adjacencies are s_2, the $T_{4,4,4}$ described by $T\{c_9 : c_{10}c_1c_2c_2', c_8c_7ww', vc_3c_4c_5\}$ is present.

Therefore, v cannot be adjacent to c_6. The case when v is adjacent to c_8 is symmetric.

Lastly, consider when v is adjacent to c_2. If v is adjacent to c_5, then the longer cycle $c_2vc_5c_4c_3wc_7c_8c_9c_{10}c_1c_2$ is present. Since G_0 is triangle-free, this implies that the set of neighbors of v must be $\{c_2, c_7, c_9\}$ and this gives the longer cycle $c_9vc_2c_1wc_3c_4c_5c_6c_7c_8c_9$ is present. Thus v cannot be adjacent to c_2.

We have now deduced that any vertex off of C must have adjacencies that are a subset of $\{c_1, c_3, c_5, c_7, c_9\}$. We note that if there is a nontrivial vertex adjacent to c_5 then $T\{c_7 : c_6c_5xx', wc_3c_2c_2', c_8c_9c_{10}c_1\}$ is present. If there is a nontrivial vertex adjacent to c_9, then the subgraph $T\{c_7 : c_8c_9xx', c_6c_5c_4c_3, wc_1c_2c_2'\}$ is present. Therefore, all nontrivial vertices have the same adjacencies as w. If there are at least two nontrivial vertices, G_0 contains a spanning eulerian subgraph by Corollary 4.13. If there is only one nontrivial vertex, there must exist at least one trivial vertex off of C, denote this vertex as v. If v is adjacent to at least two of the same vertices as w, then we can use a trail similar to the one described in the proof of Lemma 4.12 that uses an even number of vertices as the desired dominating eulerian trail. If v does not have at least two adjacencies in common with w, then it must be adjacent to both c_5 and c_9 and the subgraph $T\{c_9 : c_{10}c_1c_2c_2', c_8c_7ww', vc_5c_4c_3\}$ is present.

Case 3d: w is adjacent to $\{c_1, c_4, c_7\}$.

As in the previous cases, we consider the possible neighbors of an additional vertex v.

First consider when v is adjacent to c_2. Note that the case when v is adjacent to c_6 is symmetric. If v is adjacent to any of the vertices c_5, c_6, c_8, c_9, or c_{10}, a longer cycle is present as shown in the table below. Since G_0 is triangle-free, this forces v to be adjacent to c_4 and c_7. This gives

$c_2vc_7c_8c_9c_{10}c_1wc_4c_3c_2$ as an alternate C_{10} that includes w. This gives either c_5 or c_6 as nontrivial, otherwise we have contradicted our choice of C. If c_5 is nontrivial, then $T\{w : c_7c_6c_5c_5', c_4c_3c_2v, c_1c_{10}c_9c_8\}$ is present. If c_6 is nontrivial, then $T\{w : c_4c_5c_6c_6', c_7vc_2c_3, c_1c_{10}c_9c_8\}$ is present. Therefore, v cannot be adjacent to c_2 or c_6.

Neighbor of v	Resulting Subgraph
c_5	$C_{12} = c_2vc_5c_6c_7c_8c_9c_{10}c_1wc_4c_3c_2$
c_6	$C_{11} = c_2vc_6c_5c_4wc_7c_8c_9c_{10}c_1c_2$
c_8	$C_{12} = c_2vc_8c_9c_{10}c_1wc_7c_6c_5c_4c_3c_2$
c_9	$C_{11} = c_2vc_9c_{10}c_1wc_7c_6c_5c_4c_3c_2$
c_{10}	$C_{11} = c_{10}vc_2c_1wc_4c_5c_6c_7c_8c_9c_{10}$

Next consider v adjacent to c_5. If v is adjacent to c_3 the longer cycle $c_3vc_5c_6c_7c_8c_9c_{10}c_1wc_4c_3$ is present, while v being adjacent to c_9 gives the longer cycle $c_5vc_9c_{10}c_1c_2c_3c_4wc_7c_6c_5$. If v is adjacent to c_8 or c_{10}, there is a $T_{4,4,4}$ present. This $T_{4,4,4}$ is given by $T\{c_5 : c_6c_7ww', c_4c_3c_2c_1, vc_8c_9c_{10}\}$ or $T\{c_5 : c_6c_7ww', c_4c_3c_2c_1, vc_{10}c_9c_8\}$, respectively. This leaves only c_1 and c_7 as the possible neighbors of v. Note that v cannot be nontrivial as that would give the $T_{4,4,4}$ described by $T\{c_7 : c_6c_5vv', wc_4c_3c_2, c_8c_9c_{10}c_1\}$. Therefore, as long as any dominating eulerian circuit contains the set $\{c_1, c_5, c_7\}$, then we do not need to worry further about this case. Since c_3 is symmetric to c_5, we get that every dominating circuit must also include the set $\{c_1, c_3, c_7\}$.

Next consider v adjacent to c_8. By previous arguments we know that v cannot be adjacent to c_2, c_3, c_5, or c_6, and it is clear that if v is adjacent to c_7 or c_9 there is a triangle in G_0. This implies that v is either adjacent to $s_1 = \{c_1, c_4, c_8\}$ or $s_2 = \{c_4, c_8, c_{10}\}$. First assume v adjacent to s_1 and consider an additional vertex x. If x is adjacent to c_5, we know from previous arguments that x must be adjacent to $\{c_1, c_5, c_7\}$. This gives the subgraph $T\{c_1 : c_2c_3c_4w, xc_7c_6c_5, c_{10}c_9c_8v\}$. Similarly, if x is adjacent to c_3 then it must be adjacent to $\{c_1, c_3, c_7\}$. In this case the longer cycle $c_1c_2c_3xc_7wc_4vc_8c_9c_{10}c_1$ is present. If x is adjacent to c_{10}, then the subgraph $T\{c_7 : c_8c_9c_{10}x, c_6c_5c_4v, wc_1c_2c_3\}$ is present. If x is adjacent to both c_7 and c_9, there is the longer cycle $c_7xc_9c_8vc_1c_2c_3c_4c_5c_6c_7$. This implies x must be adjacent ot either $r_1 = \{c_1, c_4, c_7\}$, $r_2 = \{c_1, c_4, c_8\}$, or $r_3 = \{c_1, c_4, c_9\}$. However, in each case x is adjacent to both c_1 and c_4, which gives the subgraph $T\{c_1 : xc_4c_3c_2, wc_7c_6c_5, vc_8c_9c_{10}\}$. Now assume that v is adjacent to s_2. This results in the alternate C_{10} given by $c_{10}vc_4c_3c_2c_1wc_7c_8c_9c_{10}$, which implies that either c_5 or c_6 must be nontrivial. If c_5 is nontrivial, then the subgraph $T\{c_8 : c_7c_6c_5c_5', c_9c_{10}c_1c_2, vc_4ww'\}$ is present. If c_6 is nontrivial, then the subgraph $T\{v : c_{10}c_1c_2c_3, c_4c_5c_6c_6', c_8c_7ww'\}$ is present. Therefore, it must be the case that v is not adjacent to c_8. The case where v is adjacent to c_{10} is symmetric.

Next consider v adjacent to c_9. The other two adjacencies of v must be from the set $\{c_1, c_4, c_7\}$. If v is adjacent to c_4, then the $T_{4,4,4}$ described by $T\{c_9 : vc_4ww', c_{10}c_1c_2c_3, c_8c_7c_6c_5\}$ is present. This implies v is adjacent to $\{c_1, c_7, c_9\}$. If v is nontrivial, then by relabeling the vertices of C, we see that this case is symmetric to when w is adjacent to $\{c_1, c_3, c_5\}$. Therefore, we may assume that v is trivial.

The only vertices we have not inspected yet are c_1, c_4, and c_7, and we note that it is possible to have additional vertices adjacent to this set.

Using the same argument that we used when considering v adjacent to c_9, we see that we may assume that any nontrivial vertex must have adjacencies such that the number of vertices between consecutive adjacencies is given by $\{2, 2, 3\}$, otherwise we fall into one of the previous cases. Inspecting all of the sets of vertices that have this property, and combining this with the preceding results limiting where an additional vertex x can be present, we deduce that all non-trivial vertices are adjacent to $\{c_1, c_4, c_7\}$. If there are at least two nontrivial vertices, G_0 contains a dominating eulerian circuit with the desired properties by Corollary 4.13. If there is exactly one nontrivial vertex, since G_0 has at least 14 vertices we know there must be another vertex x. From the previous arguments, x is adjacent to one of $s_1 = \{c_1, c_4, c_7\}$, $s_2 = \{c_1, c_3, c_7\}$, $s_3 = \{c_1, c_5, c_7\}$, or $s_4 = \{c_1, c_7, c_9\}$. If x is adjacent to s_1, we use the same trail as if there had been a nontrivial vertex adjacent to that set. If x is adjacent to s_2, s_3, or s_4 we utilize the fact that x is adjacent to both c_1 and c_7 in each case and $c_1c_2c_3c_4c_5c_6c_7xc_1wc_7c_8c_9c_{10}c_1$ is the desired dominating eulerian subgraph.

5.2.4 Case 4: $c(G_0) = 9$.

Lemma 2.7 states that any graph with at most thirteen vertices is either collapsible or contractible to the Petersen graph. If G_0 is the Petersen graph, then it must be the case that every vertex is either nontrivial or the endpoint of a nontrivial edge. Label the graph as shown in Figure 5.1. Then $T\{p_1 : p_2p_7p_{10}, p_6p_8p_3, p_5p_4p_9\}$ is a $T_{3,3,3}$. Since each of p_{10}, p_3, and p_9 are nontrivial or the end vertex of a nontrivial edge, each one is adjacent to an additional vertex in G. Note that none of these vertices are adjacent, so it cannot be the case that they are both end vertices of the same nontrivial edge. Thus, this $T_{3,3,3}$ can be extended to a $T_{4,4,4}$ in G.

Now we consider when G_0 is not the Petersen graph. We know from Lemma 2.7 that G_0 must have at least 14 vertices. This implies that there are at least 5 vertices not on C. By the arguments presented in Section 4.2, there are three possible placements of the neighbors of w up to symmetry. These are $\{c_1, c_3, c_5\}$, $\{c_1, c_3, c_6\}$, and $\{c_1, c_4, c_7\}$. These configurations are shown in Figure 4.3. We proceed by considering each of these possible placements.

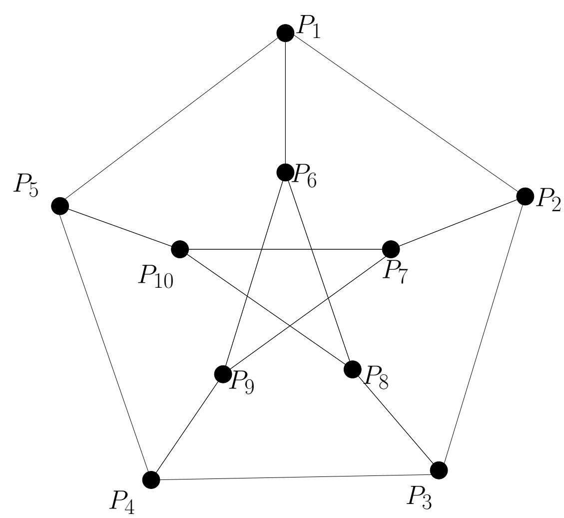

Figure 5.1: Petersen Graph

w is adjacent to $\{c_1, c_3, c_5\}$.

Recall that by Lemma 4.10 both c_2 and c_4 are nontrivial. Now consider possible neighbors of an additional vertex v that is not on C.

If v is adjacent to c_2, clearly it cannot be adjacent to c_1 or c_3 as G_0 is triangle-free. Since G_0 is 3-edge-connected, v must also be adjacent to one of the following vertices: c_4, c_6, c_7, c_8, and c_9. In each of these cases, there exists a longer cycle, which are given in the table below. Thus we conclude that v cannot be adjacent to c_2 or, by symmetry, c_4.

Neighbors of v	Resulting Cycle
c_4	$C_{11} = c_2vc_4c_3wc_5c_6c_7c_8c_9c_1c_2$
c_6	$C_{11} = c_2vc_6c_7c_8c_9c_1wc_5c_4c_3c_2$
c_7	$C_{10} = c_2vc_7c_8c_9c_1wc_5c_4c_3c_2$
c_8	$C_{10} = c_8vc_2c_1wc_3c_4c_5c_6c_7c_8$
c_9	$C_{11} = c_9vc_2c_1wc_3c_4c_5c_6c_7c_8c_9$

Assume v is adjacent to c_1. Both c_2 and c_4 have previously been eliminated as possible neighbors of v, and v adjacent to c_9 would create a triangle. If v is adjacent to c_3, there is a $T_{4,4,4}$ described by $T\{c_1 : c_2c_3ww', vc_5c_4c_4', c_9c_8c_7c_6\}$. If v is adjacent to c_5, then the subgraph $T\{c_1 : c_2c_3ww', vc_5c_4c_4', c_9c_8c_7c_6\}$ is present. This implies that v is adjacent to two of c_6, c_7, and c_8. To keep G_0 triangle-free, it must be the case that if v is adjacent to c_1, then it is also adjacent to c_6 and c_8.

Assume v is adjacent to c_6. Both c_2 and c_4 cannot be adjacent to v, and if v is adjacent to c_5 or c_7 there is a triangle. If v is adjacent to c_3, then $c_3vc_6c_7c_8c_9c_1wc_5c_4c_3$ is a longer cycle. Thus, c_6 must be adjacent to two of c_1, c_8, and c_9. To keep G_0 triangle-free, it must be the case that v is adjacent to c_1, c_6, and c_8.

Assume v is adjacent to c_8. Once again, both c_2 and c_4 have already been eliminated as possible neighbors of v, and v being adjacent to c_9 or c_7 would create a triangle. If v is also adjacent to c_3, then the subgraph $T\{c_8 :, c_9c_1c_2c_2', vc_3ww', c_7c_6c_5c_4\}$ is a $T_{4,4,4}$ in G_0. If v is adjacent to c_5, then the other adjacency must be c_1 (to prevent a triangle.) We already know that v cannot be adjacent to both c_1 and c_5, thus v cannot be adjacent to c_5. We conclude that if v is adjacent to c_8, it must be adjacent to both c_1 and c_6 as well.

Since each of the following pairs are symmetric, $\{c_1, c_5\}$, $\{c_6, c_9\}$, and $\{c_7, c_8\}$, it can be assumed that any vertex off of C other than w is either adjacent to $\{c_1, c_6, c_8\}$ or $\{c_5, c_7, c_9\}$. First note that there cannot be a vertex v_1 adjacent to $\{c_1, c_6, c_8\}$ and a vertex v_2 adjacent to $\{c_5, c_7, c_9\}$ as that would create the longer cycle $c_1c_2c_3c_4c_5c_6v_1c_8c_7v_2c_9c_1$. Also note that a vertex v adjacent to $\{c_1, c_6, c_8\}$ cannot be nontrivial as that would create the $T_{4,4,4}$ described by $T\{c_1 : c_2c_3c_4c_4', wc_5c_6c_7, c_9c_8vv'\}$. Thus, the only nontrivial vertex not on C is w and $c_1c_2c_3c_4c_5wc_1c_9c_8c_7c_6vc_1$ is the desired dominating eulerian circuit.

w **is adjacent to** $\{c_1, c_3, c_6\}$**.**

By Lemma 4.10, c_2 must be a nontrivial vertex. As in the previous case, we proceed by considering where an additional vertex v can have its adjacencies on C.

Assume v is adjacent to c_2. Clearly, if v were adjacent to either c_1 or c_3 there would be a triangle in G_0. Since v has at least two other adjacencies on C, it must be adjacent to at least one of c_4, c_5, c_7, c_8, and c_9. Each of these produces a longer cycle, as described in the table below. Thus we conclude that v cannot be adjacent to c_2.

Neighbor of v	Resulting Cycle
c_4	$C_{11} = c_2vc_4c_5c_6c_7c_8c_9c_1wc_3c_2$
c_5	$C_{10} = c_2vc_5c_6c_7c_8c_9c_1wc_3c_2$
c_7	$C_{11} = c_2vc_7c_8c_9c_1wc_6c_5c_4c_3c_2$
c_8	$C_{10} = c_2vc_8c_9c_1wc_6c_5c_4c_3c_2$
c_9	$C_{11} = c_2vc_9c_8c_7c_6c_5c_4c_3wc_1c_2$

Assume v is adjacent to c_7. If v is adjacent to either c_6 or c_8 there is a triangle in G_0, and c_2 has already been eliminated as an adjacency for v. If v is adjacent to c_3, c_4, or c_5 there is a longer cycle (as described in the table below.) This leaves only c_1 and c_9 as the other adjacencies of v, which forces a triangle. Thus v cannot be adjacent to c_7.

Neighbor of v	Resulting Cycle
c_3	$C_{10} = c_7vc_3c_4c_5c_6wc_1c_9c_8c_7$
c_4	$C_{11} = c_7vc_4c_5c_6wc_3c_2c_1c_9c_8c_7$
c_5	$C_{10} = c_7vc_5c_6wc_3c_2c_1c_9c_8c_7$

Assume v is adjacent to c_9. The vertex v cannot be adjacent to c_1 or c_8 as that would create a triangle. If v is also adjacent to c_4 or c_5 then we get the longer cycle $c_9vc_4c_3c_2c_1wc_6c_7c_8c_9$ or $c_9vc_5c_4c_3c_2c_1wc_6c_7c_8c_9$, respectively. This implies that v is adjacent to c_3, c_6, and c_9. There are at least 3 more vertices off of C, so consider x to be one of these vertices. If x is adjacent to c_3, c_6, or c_9 we get the following $T_{4,4,4}$ subgraphs respectively: $T\{c_6 : c_7c_8c_9v, c_5c_4c_3x, wc_1c_2c_2'\}$, $T\{c_3 : wc_1c_2c_2', vc_9c_8c_7, c_4c_5c_6x\}$, or $T\{c_6 : c_7c_8c_9x, vc_3c_4c_5, wc_1c_2c_2'\}$. This implies x is either adjacent to $\{c_1, c_4, c_8\}$ or $\{c_1, c_5, c_8\}$. These cases produce the longer cycles $c_8vc_4c_5c_6wc_3c_2c_1c_9c_8$ and $c_1vc_5c_4c_3wc_6c_7c_8c_9c_1$, respectively. Thus, v cannot be adjacent to c_9.

Assume v is adjacent to c_3. The only adjacencies of v that don't create triangles and haven't been previously eliminated are c_1, c_5, c_6, and c_8. If v is adjacent to c_8 there is the $T_{4,4,4}$ described by $T\{c_8 : c_9c_1c_2c_2', vc_3ww', c_7c_6c_5c_4\}$. Thus, to prevent a triangle, v is either adjacent to $\{c_1, c_3, c_5\}$ or $\{c_1, c_3, c_6\}$. If v is adjacent to the former, then $c_1vc_5c_4c_3wc_6c_7c_8c_9c_1$ is a longer cycle. If v is adjacent to the latter, v is actually symmetric to w and must be nontrivial by Lemma 4.10.

Assume v is adjacent to c_5. If v is also adjacent to c_1, we get the longer cycle described in the previous paragraph. This leaves c_8 as the only possible adjacency which does not create either a triangle, longer cycle, or $T_{4,4,4}$ as described in earlier paragraphs. Since v needs at least three neighbors, we conclude that it is not adjacent to c_5.

At this point, the only vertices that v can be adjacent to are c_1, c_3, c_4, c_6, and c_8. The only ways that we can choose three neighbors from among those without creating a triangle or a configuration that we have previously eliminated are as follows: $\{c_1, c_3, c_6\}$, $\{c_1, c_4, c_6\}$, and $\{c_1, c_6, c_8\}$. Recall that any vertex adjacent to $\{c_1, c_3, c_6\}$ must be nontrivial. Any vertex v adjacent to $\{c_1, c_4, c_6\}$ cannot be nontrivial as that would give $T\{c_6 : c_7c_8c_9c_1, c_5c_4vv', wc_3c_2c_2'\}$. Assume there is a vertex adjacent to $\{c_1, c_6, c_8\}$ which is nontrivial. That would give the following two alternate cycles of length 9: $c_1c_2c_3c_4c_5c_6vc_8c_9c_1$ and $c_1c_2c_3c_4c_5c_6c_7c_8vc_1$. The first of these alternate cycles forces c_7 to be nontrivial, while the second forces c_9 to be nontrivial. This implies that the $T_{4,4,4}$ described by $T\{c_6 : vc_8c_7c_7', c_5c_4c_3c_2, wv_1c_9c_9'\}$ is a subgraph of G_0. Therefore the only nontrivial vertices not on C must be adjacent to $\{c_1, c_3, c_6\}$. If there are at least two nontrivial vertices not on C, then G_0 contains a dominating eulerian circuit by Corollary 4.13. If there is exactly one nontrivial vertex w, then there must be a trivial vertex v not on C. If v is adjacent to $\{c_1, c_3, c_6\}$, we can treat it like a nontrivial vertex and use the dominating eulerian circuit described in the proof of Corollary 4.13. If v is adjacent to $\{c_1, c_4, c_6\}$, then the dominating trail is $c_1c_2c_3wc_1xc_4c_5c_6c_7c_8c_9c_1$. If v is adjacent to $\{c_1, c_6, c_8\}$, then the dominating trail is $c_1c_2c_3c_4c_5c_6wc_1c_9c_8c_7c_6vc_1$.

w is adjacent to $\{c_1, c_4, c_7\}$.

By Lemma 4.11, we may assume that any vertex v that is not on C has at least two adajencies from the set $\{c_1, c_4, c_7\}$.

If all vertices off of C have the adjacencies $\{c_1, c_4, c_7\}$, then G_0 contains a spanning eulerian circuit by Lemma 4.12. Therefore, we can assume that there exists at least one vertex not on C that does not have this set of adjacencies as w.

Since there are at least four additional vertices off of C, there must be at least two of these vertices, say v_1 and v_2, with two neighbors in common. Without loss of generality, we can assume that these vertices are both adjacent to c_1 and c_4. The third adjacency for each of these vertices must be one of $\{c_6, c_7, c_8\}$.

Assume that v_1 is adjacent to c_6 and v_2 is adjacent to c_8. The cycle $c_1v_1c_6c_5c_4v_2c_8c_7wc_1$ is an alternate C_9 that includes w. This implies that one of c_2, c_3, and c_9 is nontrivial. In each case, a $T_{4,4,4}$ exists. These are summarized below.

Nontrivial Vertex	Resulting Subgraph
c_2	$T\{v_1 : c_1c_9c_8v_2, c_4c_3c_2c_2', c_6c_7ww'\}$
c_3	$T\{v_2 : c_1c_2c_3c_3', c_4c_5c_6v_1, c_8c_7ww'\}$
c_9	$T\{c_6 : c_5c_4ww', v_1c_1c_2c_3, c_7c_8c_9c_9'\}$

Now assume that both v_1 and v_2 are adjacent to c_6 (the case when they are both adjacent to c_8 is symmetric.) Then $c_1c_9c_8c_7wc_4c_5c_6v_1c_1$ is an alternate C_9 containing w. This implies that either c_2 or c_3 is nontrivial.

First consider the case when c_2 is nontrivial. The vertex c_8 must have degree at least three, so there must be either an adjacency off of C or a chord incident with c_8. If c_8 has an adjacency, say x, off of C, then $T\{v_1 : c_4c_3c_2c_2', c_6c_7ww', c_1c_9c_8x\}$ is a $T_{4,4,4}$. Therefore, it must be the case that there must be a chord incident with c_8. Clearly, both c_1c_8 and c_6c_8 create triangles. The chords c_2c_8 and c_5c_8 create the longer cycles $c_8c_2c_3c_4c_5c_6c_7wc_1c_9c_8$ and $c_8c_5c_6c_7wc_4c_3c_2c_1c_9c_8$, respectively. The chord c_3c_8 gives the $T_{4,4,4}$ described by $T\{w : c_4c_3c_2c_2', c_7c_8c_9c_9', c_1v_1c_6c_5\}$. This forces the chord c_4c_8 to be present. The vertex c_5 must also have degree at least three. If c_5 is adjacent to a vertex x, then $T\{c_8 : c_7c_6c_5x, c_4c_3c_2c_2', c_9c_1ww'\}$ is a $T_{4,4,4}$. So, it must be the case that there is a chord incident with c_5. Both c_3c_5 and c_5c_7 create triangles, and each of c_2c_5, c_5c_8, and c_5c_9 create longer cycles. The C_{10} present when we have the edge c_2c_5 is $c_5c_2c_3c_4wc_1c_9c_8c_7c_6c_5$, the C_{10} present when we have the edge c_5c_9 is $c_5c_9c_8c_7c_6vc_1c_2c_3c_4c_5$, and the C_{10} created when the edge c_2c_8 is present was described when discussing chords incident with c_8. Therefore, the chord c_1c_5 must be present. Lastly, we consider the vertex c_9. It too must have either an adjacency or a chord. If it has an adjacency x, then $T\{w : c_4c_3c_2c_2', c_1c_5c_6v, c_7c_8c_9x\}$ is a $T_{4,4,4}$. The chords c_2c_9 and c_7c_9 clearly create triangles. The chords c_4c_9 and c_5c_9 also create triangles since we know that the chords c_4c_8 and c_1c_5 must also be present. Lastly, the chords c_3c_9 and c_6c_9 create the longer cycles $c_9c_3c_2c_1wc_4c_5c_6c_7c_8c_9$ and $c_9c_8c_7wc_1c_2c_3c_4c_5c_6c_9$, respectively. Thus, we conclude that c_2 must be trivial.

The other case to consider is when c_3 is nontrivial. The vertex c_9 needs either an adjacency off of C or a chord incident to it. If there is a vertex x adjacent to c_9, then $T\{w : c_1c_2c_3c_3', c_4c_5c_6v, c_7c_8c_9x\}$ is a $T_{4,4,4}$. Therefore there must be a chord at c_9. The chords c_2c_9 and c_7c_9 create triangles. The chord c_4c_9 creates the $T_{4,4,4}$ described by $T\{c_9 : c_1c_2c_3c_3', c_4c_5c_6v, c_8c_7ww'\}$. Lastly, the chords c_3c_9, c_5c_9, and c_6c_9 create the longer cycles $c_9c_3c_2c_1wc_4c_5c_6c_7c_8c_9$, $c_9c_5c_6vc_1c_2c_3c_4wc_7c_8c_9$, and $c_9c_6c_5c_4c_3c_2c_1wc_7c_8c_9$, respectively. Thus, c_3 also cannot be nontrivial. This proves that v_1 and v_2 could not have both had c_6 (or both had c_8) as their third adjacency.

By the above arguments we see that at least one of v_1 and v_2 must be adjacent to c_7. Without loss of generality, assume v_1 is adjacent to c_6 and v_2 is adjacent to c_7 (the case where one is adjacent to c_8 and the other c_7 is symmetric.) If there is a vertex x adjacent to c_9, it must be the case that x is adjacent to the set $\{c_4, c_7, c_9\}$ since x must have at least two neighbors in common with w. This gives the longer cycle $c_1c_2c_3c_4xc_9c_8c_7c_6vc_1$. Therefore, there must be a chord at c_9. The chords c_2c_9 and c_7c_9 create triangles, and the chords c_3c_9 and c_6c_9 create the longer cycles $c_9c_3c_2c_1wc_4c_5c_6c_7c_8c_9$ and $c_9c_6c_5c_4c_3c_2c_1wc_7c_8c_9$, respectively. When the chord c_5c_9 is present, $c_9c_5c_6c_7wc_4c_3c_2c_1c_9$ is an alternate cycle of length nine that includes w. Since C was chosen to contain the largest number of nontrivial vertices, this implies that c_8 must be nontrivial. This gives the $T_{4,4,4}$ described by $T\{c_4 : c_3c_2c_1v_2, v_1c_6c_7w, c_5c_9c_8c_8'\}$. Therefore, it must be the case that the chord c_4c_9 is present. Next consider the vertex c_5. If c_5 has an adjacency, x, not on C then there is a $T_{4,4,4}$ described by $T\{c_1 : c_2c_3c_4w, v_1c_6c_5x, c_9c_8c_7v_2\}$. Therefore there must be a chord at c_5. The chords c_3c_5 and c_5c_7 create triangles. The chords c_2c_5 and c_5c_8 create the longer cycles $c_5c_2c_3c_4wc_1c_9c_8c_7c_6c_5$ and $c_8c_5c_6c_7wc_4c_3c_2c_1c_9c_8$, respectively. The chord c_5c_9 was discussed previously, which leaves only c_1c_5 as a possibility. When both c_1c_5 and c_4c_9 are present, then $c_1c_5c_6v_1c_4c_9c_8c_7wc_1$ is an alternate C_9 that includes w. This implies that either c_2 or c_3 is nontrivial. If c_2 is the nontrivial vertex, there is a $T_{4,4,4}$ described by $T\{c_9 : c_4c_3c_2c_2', c_8c_7ww', c_1vc_6c_5\}$. If c_3 is the nontrivial vertex, there is a $T_{4,4,4}$ described by $T\{c_9 : c_1c_2c_3c_3', c_4c_5c_6v_1, c_8c_7ww'\}$. This implies that it cannot be the case that v_1 is adjacent to c_6 and v_2 is adjacent to c_7.

The above arguments imply that it must be the case that both v_1 and v_2 are adjacent to c_7. Since c_1, c_4, and c_7 are all symmetric, it ends up that all vertices off of C must have the same adjacencies. Therefore, we are done since this case was handled previously.

With this, we conclude the proof of Theorem 1.12.

$\square$

Chapter 6

Future Work

The results shown in Chapter 3 greatly reduce the possibilities for pairs $\{X, Y\}$ such that a 3-connected graph being $\{X, Y\}$-free implies the graph is hamiltonian. Paired with the previous results discussed in Chapter 1 and the new results in Chapters 3, 4, and 5, the only pairs for which it is unknown whether or not a 3-connected, $\{X, Y\}$-free graph is hamiltonian are $\{K_{1,3}, Ł_3\}$ and $\{K_{1,3}, Ł_5\}$. Determining whether these forbidden pairs imply hamiltonicity is a natural next question, as that would complete the classification of all forbidden pairs that imply a 3-connected graph is hamiltonian.

One of the things to note about the above problem is that the method of proof used in this dissertation to show that 3-connected, $K_{1,3}, N_{i,j,k}$-free graphs are hamiltonian will not work. The proofs presented here utilized the fact that generalized nets are stable under the Ryjáček closure operation, i.e. if a graph G is net-free then the Ryjáček closure of G is also net-free. It is well-known that the $Ł_k$ graphs are not stable under this closure operation (see [3]). It would be of interest to try to develop new closure operations under which this particular graph is stable.

There are several other forbidden subgraph problems that imply properties such as pancyclicity, hamiltonian-connectedness, and existence of two-factors. One possibility that is of interest to me is to work on forbidden pairs that imply a 3-connected graph is hamiltonian-connected. It is already known that any forbidden pair must contain the claw [6]. There are still several generalized nets and $\L_k$ graphs for which it is unknown if they can be included in a forbidden pair that implies hamiltonian connectedness.

Lastly, it would be remiss to not mention the open-problem that has fueled the area of forbidden subgraphs for almost 30 years- the Matthews-Sumner Conjecture (Conjecture 1.1). While this problem is not in my immediate scope of future work, it is my hope to continue to solve various subproblems related to this famous conjecture.

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