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Silke Gehrke Date

Hamiltonicity and Pancyclicity of 4-connected, Claw- and Net-free Graphs

By

Silke Gehrke
Doctor of Philosophy

Department of Mathematics

Ronald J. Gould, Ph.D.
Advisor

Dwight Duffus, Ph.D.
Committee Member

Robert Roth, Ph.D.
Committee Member

Accepted:

Lisa A. Tedesco, Ph.D.
Dean of the Graduate School

Date

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Silke Gehrke
Master of Science, Emory University, 2008

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An abstract of
A dissertation submitted to the Faculty of the Graduate School of Emory University
in partial fulfillment of the requirements of the degree of
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Abstract

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A well-known conjecture by Manton Matthews and David Sumner states that every 4-connected $K_{1,3}$ -free graph is hamiltonian. The conjecture itself is still wide open, but several special cases have been shown so far. We will show results that support that conjecture. Especially, we will show that if a graph is 4-connected and $\{K_{1,3},\ N\}$- free, where $N = N(i, j, k)$, with $i + j + k = 5$ and $i,\ j,\ k \geq 0$, the graph is pancyclic.

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To my family,
in loving memory of my grandfather

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Chapter 1

Introduction

We assume that the reader has a basic knowledge of elementary graph theory. A good reference for undefined terms is, for example, [11].

1.1 Background

In 1859 the Irish mathematician William Rowan Hamilton introduced the so called "Icosian Game". The goal of the game was to find a route through the twenty most important cities at this time, by visiting every city precisely once and in the end returning to the city from which the route started. Translating that goal into graph theory means that one needs to find a cycle through all vertices in a graph. Such a cycle is called a hamiltonian cycle, and a graph containing a hamiltonian cycle is called a hamiltonian graph. Sufficient conditions implying hamiltonicity have been widely studied. To understand some of the major results, let us first introduce some definitions.

1.2 Basic Definitions and Notation

A graph is an ordered pair $G = (V, E)$ consisting of a set V of vertices and a set E of edges, where the edges are 2-element subsets of V. A graph H is called a **subgraph** of G if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. A graph

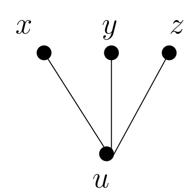
H is called an **induced subgraph** of G, if H is a subgraph of G (or is isomorphic to a subgraph of G) and two vertices in H are adjacent if, and only if, they are adjacent in G. A graph G is called **H-free**, if H is not an induced subgraph of G. More generally, if G does not contain any of the graphs H_1, $H_2, \cdots$, H_n as induced subgraphs, then G is called $\{H_1$, $H_2, \cdots$, $H_n\}$-free. A graph H is a **forbidden** (induced) subgraph of G, if G does not contain H as an (induced) subgraph. A graph is said to be **connected** if any pair of vertices in the graph is joined by a path. Otherwise a graph is called disconnected. Furthermore, we say that a graph is **k-connected** if k is the minimum number of vertices whose removal disconnects the graph or reduces it to a single vertex. Given a connected graph $G = (V, E)$, a subset S of $V(G)$ is called a **cut-set** if the removal of S leads to a disconnected graph.

For a graph whose n vertices are pairwise adjacent, we will write K_n. These graphs are called **complete** graphs or **cliques**. A bipartite graph is a graph whose vertices can be partitioned into two disjoint sets V_1 and V_2 such that every edge connects a vertex in V_1 to one in V_2. A **complete bipartite graph** $G = (V_1 \cup V_2, E)$ is a bipartite graph such that for any two vertices $v_1 \in V_1$ and $v_2 \in V_2$, $v_1v_2 \in E(G)$. The complete bipartite graph with partitions of size $|V_1| = m$ and $|V_2| = n$ is denoted by $K_{m,n}$.

There has been major interest to find graphs H_1, $H_2 \cdots$, H_l such that k-connected $\{H_1, H_2, \cdots, H_l\}$-free graphs are hamiltonian. A typical forbidden subgraph is $K_{1,3}$, known as the claw (see Figure 1.1). With $N(i, j, k)$, $i \leq j \leq k$ we denote a generalized net, that is, it is a K_3 with vertex disjoint paths of length i, j and k rooted at the vertices of the K_3 (see Figure 1.1 for an example). To indicate a possible induced claw by vertices u, x, y and z, centered at u, we will write $\langle\ u + xyz\ \rangle$.

To indicate a possible induced $N(i, j, k)$, we will write $\langle xyz+\ x_1...x_i\ + y_1...y_j\ +z_1...z_k\rangle$, where the vertices x, y and z form the K_3, $x = x_0$, $x_1, ...,$

x_i the path of length i, $y = y_0$, $y_1, ..., y_j$ the path of length j and $z = z_0$, $z_1, ..., z_k$ the path of length k. Note, that x_1 may be adjacent to any of x, y or z, that is, x_1 does not need to necessarily be adjacent to x, but might be adjacent to y or z instead.



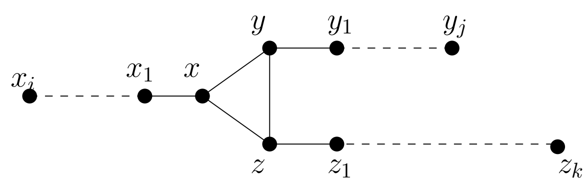


Figure 1.1:

We will denote the path on n vertices by P_n and the cycle on n vertices with C_n. For a cycle C in the graph G, we will choose one of the two possible orientations of the cycle. For $v \in V(C)$ we denote with v^+ the successor of v on C with respect to the orientation of C. Furthermore, $v^{++} = v^{+2}$ will be the successor of v^+ on C. Similarly v^- will be the predecessor of v on C and $v^{--} = v^{-2}$ will be the predecessor of v^- on C. More generally, v^{-i} will be the predecessor of $v^{-(i+1)}$ on C and v^{+i} will be the successor of $v^{+(i-1)}$ on C. We write vCw to indicate the path from $v \in V(C)$ to $w \in V(C)$ using only vertices of C and following the orientation on C. By vC^-w we denote the path from $w \in V(C)$ to $v \in V(C)$ using only vertices of C by traversing C opposite to the orientation on C. Similarly, for a path P and vertices $u \in V(P)$ and $v \in V(P)$, we denote the part of the path from u to v with uPv.

In this dissertation, we will also consider a stronger notion than hamiltonicity, that is, we will consider graphs that contain a cycle of length k, for all k with $3 \leq k \leq |V(G)|$. These graphs are called pancyclic.

1.3 Background and Outline of Results

We are now ready to present major results in hamiltonicity using forbidden subgraphs. It was shown by Ralph Faudree and Ronald Gould (see [9]) that if a 2-connected graph is P_3-free it is hamiltonian. Furthermore, it was shown that P_3 is the only single forbidden induced subgraph that implies hamiltonicity for 2-connected graphs. In the following we will consider pairs and triples of forbidden graphs that imply hamiltonicity. H. J. Broersma, J. Henk and H. J. Veldman showed in [3] that a 2-connected, $\{K_{1,3}, P_6\}$-free graph is hamiltonian. Furthermore, D. Duffus, M. Jacobson and R. Gould showed in [6] that a 2-connected $\{K_{1,3}, N(1,1,1)\}$-free graph is hamiltonian. Then in [12] M. Jacobson and R. Gould showed that a 2-connected $\{K_{1,3}, N(2,0,0)\}$-free graph is hamiltonian. Later, P. Bedrossian showed in his Ph.D. thesis [1] that 2-connected $\{K_{1,3}, N(2,1,0)\}$-free graphs are hamiltonian and R. Faudree, R. Gould, Z. Ryjáček, and I. Schiermeyer showed that a 2-connected $\{K_{1,3}, N(3,0,0)\}$-free graph of order at least ten is hamiltonian ([8]).

In [9] and [1] the following was shown:

Theorem 1.1 *Let H_1 and H_2 be connected graphs, both not equal to P_3. Let G be a 2-connected graph of order at least 10. Then G is $\{H_1, H_2\}$-free implies that G is hamiltonian if, and only if, H_1 is the claw and H_2 is one of the graphs P_6, $N(2,0,0)$, $N(3,0,0)$, $N(2,1,0)$ or $N(1,1,1)$, or a connected induced subgraph of one of these graphs.*

Furthermore, forbidden pairs of graphs implying pancyclicity were characterized.

Theorem 1.2 *[9] Let H_1 and H_2 be connected graphs, both not equal to P_3 and let G (G not equal to C_n) be a 2-connected graph of order at least 10. Then G is $\{H_1, H_2\}$-free implies that G is pancyclic if, and only if, H_1 is the claw and H_2 is one of the generalized nets $N(1,0,0)$, $N(2,0,0)$, P_4, P_5 or P_6.*

In [4], Jan Brousek characterized all minimal claw-free 2-connected non-hamiltonian graphs. Before we can state that theorem, we need to introduce a family $\mathcal{P}$ of graphs. We say that a graph belongs to the family $\mathcal{P}$, if it is obtained by taking two vertex-disjoint triangles $\langle a_1 a_2 a_3 \rangle$ and $\langle c_1 c_2 c_3 \rangle$ and joining every pair of vertices $\{a_i, c_i\}$ by a path P_r, with $r \in \{i, j, k\}$, $i, j, k \geq 3$ (see Figure 1.2 for an example) or by a triangle (see Figure 1.3).

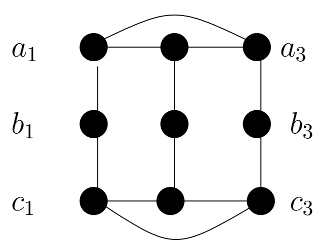


Figure 1.2: $P_{3,3,3}$

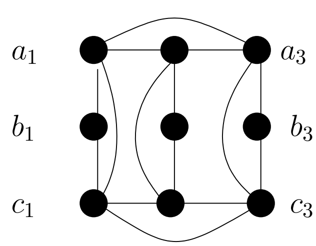


Figure 1.3: $P_{T,T,T}$

The main result in [4] is the following theorem:

Theorem 1.3 *A graph G is a minimal 2-connected non-hamiltonian claw-free graph if, and only if, G belongs to the family $\mathcal{P}$.*

In 2005, Ronald Gould, Tomasz Łuczak and Florian Pfender characterized all pairs of forbidden subgraphs that imply a 3-connected graph is pancyclic ([2]):

Theorem 1.4 *Let H_1 and H_2 be connected graphs on at least three vertices such that H_1 ; $H_2 \neq P_3$ and $H_2 \neq K_{1,3}$. Then the following statements are equivalent:*

(i) *Every 3-connected $\{H_1,\ H_2\}$-free graph G is pancyclic.*

(ii) *$H_1 = K_{1,3}$ and H_2 is a subgraph of one of the graphs from the family $F = \{P_7,\ L,\ N(4,0,0),\ N(3,1,0),\ N(2,2,0),\ N(2,1,1)\}$, where L is the graph that consists of two vertex disjoint copies of K_3 and an edge joining them.*

In conclusion, if forbidden pairs are considered that imply hamiltonicity (or pancyclicity) the claw is always one of the forbidden graphs, which was shown in [9]. But it was shown that, if on the other hand forbidden triples are considered, the claw is not necessarily one of the forbidden graphs. Ralph Faudree, Ronald Gould, Michael Jacobson and Linda Lesniak [7] found all triples of forbidden subgraphs implying hamiltonicity for sufficiently large graphs, where none of the forbidden triples contains a $K_{1,t}$, with $t \geq 3$. Furthermore, Jan Brousek gave in [5], the collection of all triples of forbidden graphs that include the claw, implying hamiltonicity for 2-connected graphs. Additionally, in [10] all remaining forbidden triples were investigated.

A well-known conjecture by Manton Matthews and David Sumner states that every 4-connected $K_{1,3}$-free graph is hamiltonian. It was shown by Zdeněk Ryjáček ([19]) that this conjecture is equivalent to an, at first sight weaker appearing earlier conjecture by Carsten Thomassen ([20]) that states that every 4-connected line graph is hamiltonian. The conjecture itself is still wide open, but several special cases have been shown so far. It was shown by

Hajo Broersma, Matthias Kriesell and Zdeněk Ryjáček in [2] that this conjecture holds for graphs that additionally do not contain induced subgraphs that are isomorphic to two triangles meeting in exactly one vertex. This result has been extended by Tomáš Kaiser, Ming Chu Li, Zdeněk Ryjáček and and Liming Xiong, (see [14]). They showed that every 4-connected claw-free graph, where every two triangles that meet in exactly one vertex contain two non-adjacent vertices with a common neighbor outside these two triangles, is hamiltonian. Furthermore, Florian Pfender showed in [18] that every 4-connected $\{K_{1,3}, T\}$-free graph is hamiltonian, where T is the line graph of the unique tree S on eight vertices with degree sequence $(3, 3, 3, 1, 1, 1, 1, 1)$. Besides these results, it has been shown by Shi Ming Zhan in [21] that every line graph of a 4-edge connected graph is hamiltonian and by Hong-Jian Lai (see [15]) that every 4-connected line graph of a planar graph is hamiltonian.

As already mentioned, the characterization of all 3-connected $\{X, Y\}$-free graphs that are pancyclic, where $X = K_{1,3}$ has been shown. Note, that if Y is a generalized net $N(i, j, k)$, then pancyclicity was shown for $i + j + k = 4$. It is the goal of this thesis to show that if a graph is 4-connected and $\{K_{1,3}, N\}$-free, where $N = N(i, j, k)$, with $i + j + k = 5$ and $i, j, k \geq 0$, then the graph is pancyclic.

Chapter 2

Hamiltonicity of 4-connected, $\{K_{1,3}, N(2,2,1)\}$-free or $\{K_{1,3}, N(3,1,1)\}$-free graphs

2.1 Preliminary Lemmas

In this section, we will give preliminary lemmas needed for showing hamiltonicity of 4-connected $\{K_{1,3}, N\}$-free graph, with $N = N(3,1,1)$ or $N = N(2,2,1)$. In [13] the result was shown for $N = N(2,2,1)$. We will show a different approach to obtain this result in this chapter. We will also show a stronger result than what is needed to obtain hamiltonicity. But we will need that stronger result later on to show pancyclicity of 4-connected $\{K_{1,3}, N(3,1,1)\}$-free graphs. Especially, it is our goal to prove the following.

Lemma 2.1 *Let $G = (V, E)$ be a 4-connected, $\{K_{1,3}, N\}$-free graph, with $N = N(3,1,1)$ or $N = N(2,2,1)$. If G is non-hamiltonian and C is a longest cycle of G, then there exists a vertex $v \in V(G) \setminus V(C)$ such that v has at least three neighbors on C.*

In the following we will consider a longest cycle C in a non-hamiltonian graph. Furthermore, we will consider a vertex v of the graph that is not contained in the cycle, such that the sum of the lengths of its three shortest paths to the cycle is minimum. We will always denote these three paths by Q_1, Q_2 and Q_3 and refer to the set $\{Q_1,\ Q_2,\ Q_3\}$ as the path system $\mathcal{Q}$. Additionally, we assume without loss of generality that $l(Q_1) \leq l(Q_2) \leq l(Q_3)$, where $l(Q_i)$ denotes the length of the path Q_i. In particular, we define the following condition.

Definition 2.2 *We say that a graph $G = (V, E)$ contains **Configuration** (C, v) if C is a longest cycle of G and $v \in V(G) \setminus V(C)$ is picked such that the sum of the length of the three shortest paths from v to C is minimum. Call the three shortest paths Q_1, Q_2, Q_3 and define $x := V(Q_1) \cap V(C)$, $y := V(Q_2) \cap V(C)$, $z := V(Q_3) \cap V(C)$. Let C be oriented such that we have the order $x\ C\ y\ C\ z\ C\ x$.*

We note that any 3-connected non-hamiltonian graph contains Configuration (C, v). Let us first note, that x, y and z do have a certain minimum distance from each other on the cycle.

Lemma 2.3 *Let $G = (V, E)$ be a 4-connected non-hamiltonian claw-free graph containing Configuration (C, v). Then $dist_C(a, b) \geq 4$ for $a \neq b$, $a, b \in \{x, y, z\}$.*

Proof. Let us suppose for the purpose of contradiction that $dist_C(x, y) < 4$. First assume that there are two intermediate vertices on the cycle between x and y, that is, there is a path $x\ x^+\ y^-\ y$ induced by the cycle. Note, that since G is claw-free and C is a longest cycle, we have the edges x^-x^+ and y^-y^+. But now we can obtain a longer cycle by inserting the paths Q_1 and Q_2 as follows: $v\ Q_1\ x\ x^+\ x^-\ C^-\ y^+\ y^-\ y\ Q_2\ v$, which leads to a contradiction (see Figure 2.1). Note, that if $dist_C(x, y) < 3$ we can obtain the same result by a similar argument. $\square$

In the following lemmas, we will show the existence and non-existence of certain edges among the vertices of C. In our hamiltonicity and pancyclicity

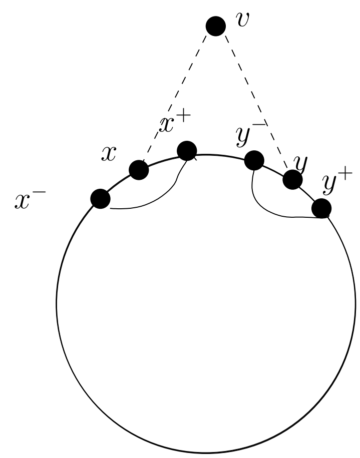


Figure 2.1:

proofs and also in the proofs of the preliminary lemmas we will make strong use of these lemmas.

Lemma 2.4 *Let $G = (V, E)$ be a 4-connected, non-hamiltonian $\{K_{1,3}\}$-free graph with Configuration (C, v). Then*

(i) $vx^- \notin E(G)$, $vx^{--} \notin E(G)$, $vx^+ \notin E(G)$, $vx^{++} \notin E(G)$, $vy^- \notin E(G)$, $vy^{--} \notin E(G)$, $vy^+ \notin E(G)$, $vy^{++} \notin E(G)$, $vz^- \notin E(G)$, $vz^{--} \notin E(G)$, $vz^+ \notin E(G)$, $vz^{++} \notin E(G)$

(ii) $x^-x^+ \in E(G)$, $y^-y^+ \in E(G)$, $z^-z^+ \in E(G)$

Proof. If $vx^- \in E(G)$, we can obtain a longer cycle than C as follows: vQ_1xCx^-v. Thus, $vx^- \notin E(G)$. Similarly, we get that $vx^+ \notin E(G)$, $vy^- \notin E(G)$, $vy^+ \notin E(G)$, $vz^- \notin E(G)$, $vz^+ \notin E(G)$. Now note, that since G is claw-free, considering $\langle\ x + x^-x^+x_{n-1}\ \rangle$, where x_{n-1} is the last vertex on

Q_1 before x, gives us the edge $x^-x^+ \in E(G)$ and similarly $y^-y^+ \in E(G)$, $z^-z^+ \in E(G)$, which shows (ii). But then if $vx^{--} \in E(G)$ we can again extend the cycle as follows: $v\ x\ x^-\ x^+\ C\ x^{--}\ v$. Similarly, the other edges mentioned in (i) can be ruled out. $\square$

Similarly, we get the following lemmas.

Lemma 2.5 *Let $G = (V, E)$ be a 4-connected, non-hamiltonian $\{K_{1,3}\}$-free graph containing Configuration (C, v). Then*

(i) $x^-y^- \notin E(G)$, $x^-z^- \notin E(G)$, $z^-y^- \notin E(G)$

(ii) $xy^- \notin E(G)$, $xy^+ \notin E(G)$, $yx^- \notin E(G)$, $yx^+ \notin E(G)$, $zx^- \notin E(G)$, $zx^+ \notin E(G)$, $zy^- \notin E(G)$, $zy^+ \notin E(G)$

Proof. To see (i), suppose $x^-y^- \in E(G)$. Then we can extend the cycle C as follows: $v\ Q_2\ y\ C\ x^-\ y^-\ C^-\ x\ Q_1\ v$, which is a contradiction to C being the longest cycle in G. Similarly, we cannot have any of the other edges mentioned in (i).

To see (ii), assume that $yx^- \in E(G)$. By considering $\langle\ y + y_{n-1}x^-y^-\ \rangle$, where y_{n-1} is the last vertex on Q_2 adjacent to y, we are forced to have $x^-y^- \in E(G)$ which is case (i). Similarly, the other edges of statement (ii) can be ruled out.

$\square$

Lemma 2.6 *Let $G = (V, E)$ be a 4-connected, non-hamiltonian $\{K_{1,3}\}$-free graph containing Configuration (C, v). Then*

(i) $x^-y^{--} \notin E(G)$, $x^{--}y^- \notin E(G)$, $x^-z^{--} \notin E(G)$, $x^{--}z^- \notin E(G)$, $y^-z^{--} \notin E(G)$, $y^{--}z^- \notin E(G)$, $x^+y^{++} \notin E(G)$, $x^{++}y^+ \notin E(G)$, $x^+z^{++} \notin E(G)$, $x^{++}z^+ \notin E(G)$, $y^+z^{++} \notin E(G)$, $y^{++}z^+ \notin E(G)$

(*ii*) $xy^{--} \notin E(G)$, $zy^{--} \notin E(G)$, $xz^{--} \notin E(G)$, $yz^{--} \notin E(G)$, $zx^{--} \notin E(G)$, $yx^{--} \notin E(G)$, $xy^{++} \notin E(G)$, $zy^{++} \notin E(G)$, $xz^{++} \notin E(G)$, $yz^{++} \notin E(G)$, $zx^{++} \notin E(G)$, $yx^{++} \notin E(G)$

(*iii*) $x^{--}y^{--} \notin E(G)$, $x^{--}z^{--} \notin E(G)$, $z^{--}y^{--} \notin E(G)$, $x^{++}y^{++} \notin E(G)$, $x^{++}z^{++} \notin E(G)$, $z^{++}y^{++} \notin E(G)$.

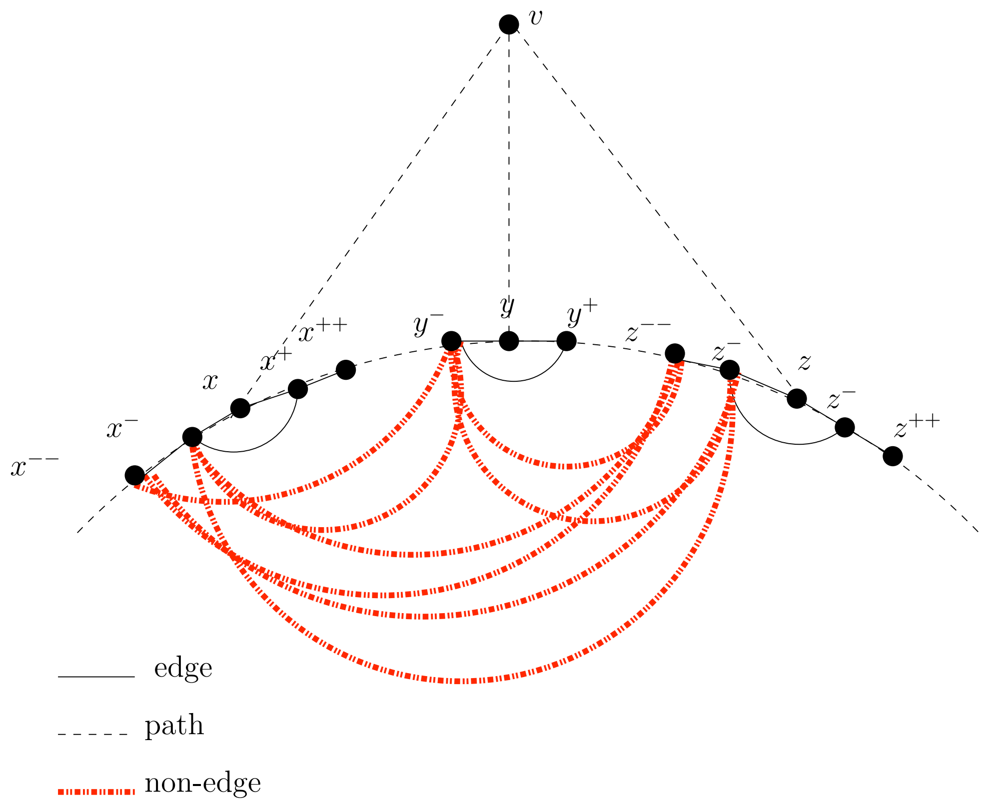


Figure 2.2: Showing some edges and non-edges of the Lemmas 2.4, 2.5, 2.6.

Proof. To see (i), assume that $x^- y^{--} \in E(G)$. We can obtain a longer cycle than C as follows: $v\ y\ y^-\ y^+\ C\ x^-\ y^{--}\ C^-\ x\ v$. Similarly, the other edges in (i) can be ruled out.

To see (ii), suppose that $xy^{--} \in E(G)$. Considering $\langle\ x + vx^- y^{--}\ \rangle$ we are forced to have $y^{--} x^- \in E(G)$ which is case (i). The other edges of this case are ruled out in a similar fashion.

To see (iii), suppose that $z^{--} y^{--} \in E(G)$. Then we can get a longer cycle than C as follows: $v\ z\ z^-\ z^+\ C\ y^{--}\ z^{--}\ C^-\ y^+\ y^-\ y\ v$. The other edges of this case are ruled out in a similar fashion.

□

Before we get to the proof of Lemma 2.1, we will prove one more helpful lemma that forbids certain edges between the path system and C.

Lemma 2.7 *Let $G = (V, E)$ be a 4-connected, non-hamiltonian claw-free graph. Suppose C is a longest cycle of G and $v \in V(G) \setminus V(C)$ such that the sum of the length of the three shortest paths from v to C is minimum. Call the three shortest paths Q_1, Q_2, Q_3 with $Q_1 := \{vx_1...x_{n-1}x\}$, $Q_2 = \{vy_1y_2...y_{m-1}y\}$, $Q_3 = \{vz_1...z_{k-1}z\}$. Then*

(i) exactly one of the edges x_1y_1, y_1z_1 or x_1z_1 does exist,

(ii) none of x_i, y_j and z_t with $1 \le i < n-1$, $1 \le j < m-1$ and $1 \le t < k-1$ is adjacent to any vertex of the cycle, unless the path Q_s with $s \in \{1,2,3\}$ has length two or less, and

(iii) none of the vertices v or a_j of the path Q_i, with $i \in \{1,2,3\}$ and $1 \le j \le l(Q_i)$, is adjacent to any other vertex in $\mathcal{Q}$, except its neighbors on Q_i and unless it is the edge mentioned in (i).

Proof. Part (i) follows from the fact that the graph is by assumption claw-free. To begin $\langle\ v + x_1y_1z_1\ \rangle$ implies that we have at least one of the edges

x_1y_1, y_1z_1 or x_1z_1. Suppose that x_1y_1 is an edge. Then if we suppose that y_1z_1 (or symmetrically x_1z_1) is an edge as well, we get a shorter system than $\mathcal{Q}$ by taking y_1 as the rooting vertex and the paths $Q'_1 = \{y_1x_1Q_1x\}$, $Q'_2 = \{Q_2 \setminus \{v\}\}$, $Q'_3 = \{y_1z_1Q_3z\}$ (or $Q'_1 = \{Q_1 \setminus v\}$, $Q'_2 = \{x_1y_1Q_2y\}$, $Q'_3 = \{x_1z_1Q_3z\}$), a contradiction.

It is easy to see that if (ii) of the lemma is violated we get a shorter path system. To see this, suppose that there is a vertex u of Q_1, Q_2 or Q_3 with the desired properties of the lemma. Suppose, without loss of generality, that $u \in Q_1$ and that u is adjacent to $t \in V(C)$. Then $Q'_1 := v\ Q_1\ u\ t$, $Q'_2 := Q_2$ and $Q'_3 := Q_3$ is a shorter path system, unless $t = y$ or $t = z$. If $t = y$ we set $Q'_1 := u\ Q_1\ x$, $Q'_2 := u\ y$ and $Q'_3 := u\ Q_1\ vQ_3z$, which is also a shorter path system. If $t = z$ we proceed in a similar way and also get a shorter path system.

To see (iii), suppose for the purpose of a contradiction that there is a second edge ut in the path system aside from the possible edges x_1y_1, x_1z_1 or y_1z_1. Suppose x_1y_1 is an edge and suppose $u \in Q_1$ and $t \in Q_2$. (All other cases follow from an identical argument.) Then we get a shorter path system by setting $Q'_1 := uQ_1x$, $Q'_2 := u\ tQ_2y$ and $Q'_3 := uQ_1^-vQ_3$, a contradiction. □

We are now ready to present part of the proof of Lemma 2.1. We will consider a 4-connected, non-hamiltonian $\{K_{1,3},\ N(2,2,1)\}$-free graph and will show that each of the paths Q_1, Q_2 and Q_3 has to be a single edge. To prove the lemma, we will consider a longest cycle C and show, by contradiction, that if one or more than one of the paths is not an edge, we either get a longer cycle than C, or we get an induced $N(2,2,1)$.

Lemma 2.8 *Let $G = (V, E)$ be a 4-connected, $\{K_{1,3}, N(2,2,1)\}$-free graph. If G is non-hamiltonian, then there exists a vertex $v \in V(G) \setminus V(C)$ such that v has at least three neighbors on C.*

Proof. Pick a vertex $v \in V(G) \setminus V(C)$ such that the sum of the lengths of its three shortest paths Q_1, Q_2, Q_3, to C is minimum. Call this path system $\mathcal{Q}$ and v the rooting vertex of $\mathcal{Q}$.

Case 1 *Suppose for the purpose of a contradiction that Q_1, Q_2, Q_3 are not just edges.*

Suppose $Q_1 = \{vx_1...x\}$, $Q_2 = \{vy_1y_2...y\}$, $Q_3 = \{vz_1...z\}$. Let C be oriented such that $x \leq_C y \leq_C z$. By Lemma 2.7 we can assume without loss of generality that y_1x_1 is the only other edge inside the path system (see Figure 2.3).

Consider now $N := \langle\ vx_1y_1 +\ y_2y_3 + z_1z_2\ +\ x_2\ \rangle$, where $y_3 = y^-$ if $y_2 = y$. We claim that N is an induced $N(2,2,1)$.

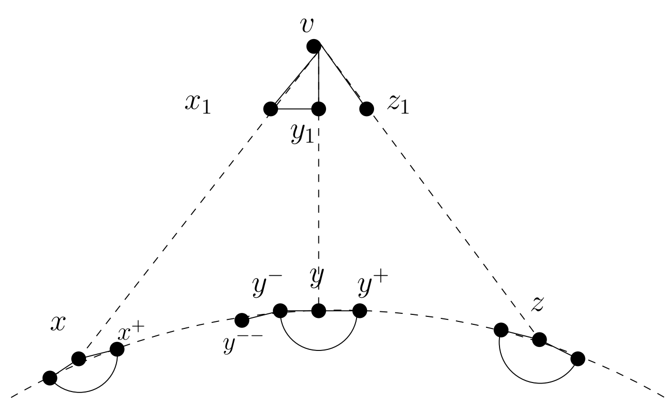


Figure 2.3:

We may assume that $y_2 \in V(C)$. Otherwise, any additional edges (that is edges from y_2 into N) would contradict Lemma 2.7.

By Lemma 2.3, we know that there exist vertices $y^- \neq x^+$, $y^{--} \neq x^+$ on the path xCy. By Lemma 2.4, v cannot have any additional edges into N, otherwise we get a longer cycle than C or a shorter system than $\mathcal{Q}$. We already argued in Lemma 2.7 that y_1, x_1 and z_1 cannot have any additional edges into N. If $z_2 \notin V(C)$, then with the same argument as before for y_1, we know that z_2 cannot have any additional neighbors in N. Thus, let us assume that $z_2 \in V(C)$. Similarly, let us suppose that $x_2 \in V(C)$, that is $x_2 = x$. Now we need to check all possible edges among the vertices $\{x, y^-, y, z\}$.

Note, that by Lemma 2.5 we can rule out all edges among these vertices except xy, zy and zy.

Subcase 1.1 *Suppose $xy \in E(G)$.*

Considering $\langle\, y + xy^-y_1 \,\rangle$, since G is claw-free, we are forced to have $xy^- \in E(G)$, which contradicts Lemma 2.5.

Subcase 1.2 *Suppose $xz \in E(G)$.*

As in Case 1.1, we are forced to have $xz^- \in E(G)$ and again we have a contradiction by Lemma 2.5.

Subcase 1.3 *Suppose $zy \in E(G)$.*

Again we are forced to have $yz^- \in E(G)$ and a contradiction by Lemma 2.5. Therefore, we obtain an induced $N(2, 2, 1)$, a contradiction.

Thus, at least one of the paths Q_1, Q_2, Q_3 has to be an edge.

Case 2 *Suppose that Q_1 is an edge, that is, $Q_1 = vx$.*

As before, we have to have one of the edges y_1x, z_1x or y_1z_1, where the cases y_1x and z_1x are symmetric.

Subcase 2.1 *Suppose $y_1z_1 \in E(G)$.*

Consider $N :=\langle\ vy_1z_1 +\ xx^- + y_2y_3\ +\ z_2\ \rangle$, where we can assume, as before, that $y_2 = y$, $z_2 = z$ and therefore $y_3 = y^-$. We already ruled out the missing edges within N in the Lemmas 2.4, 2.5 and the previous cases and thus we have an induced $N(2,2,1)$, a contradiction. Thus, $y_1z_1 \notin E(G)$.

Subcase 2.2 *Suppose, $y_1x \in E(G)$ (see Figure 2.4).*

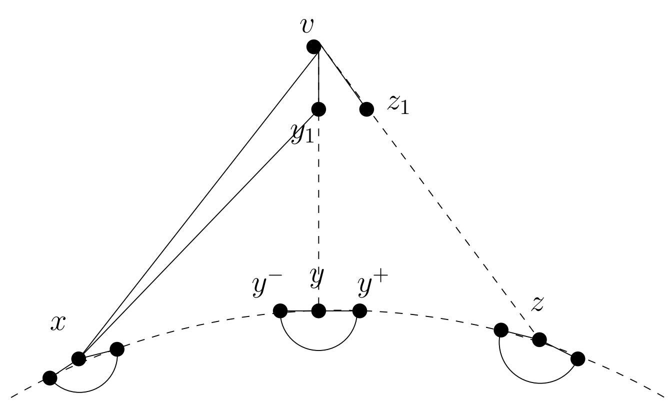


Figure 2.4:

Consider $\langle\ vy_1x +\ y_2y_3 + z_1z_2\ +\ x^-\ \rangle$, where we may assume, as before, that $y_2 = y$, $y_3 = y^-$, $z_2 = z$. We get immediately that this is an induced $N(2,2,1)$ since all additional edges have been ruled out in our previous cases and in Lemmas 2.4 and 2.5.

Hence, we can assume that at least two of Q_1, Q_2, Q_3 are single edges.

Case 3 *Suppose both Q_1 and Q_2 are edges.*

Under these conditions, choose y and z as close as possible on the cycle. By considering $\langle\ v + xyz_1\ \rangle$ we are forced to have one of $xy \in E(G)$, $xz_1 \in E(G)$ or $yz_1 \in E(G)$.

Subcase 3.1 *Suppose $yz_1 \in E(G)$. (Note, that the case $xz_1 \in E(G)$ is symmetric.)*

Consider $N := \langle\ vyz_1 +\ xx^- + z_2z_3 +\ y^-\ \rangle$, where we may assume that $z_2 = z$ and $z_3 = z^-$. Note that the edges $zy \in E(G)$ and $xz \in E(G)$ cannot be present by our earlier arguments. If we suppose that $xy \notin E(G)$, we immediately get a contradiction because we have an induced $N(2,2,1)$, since all other additional edges within N have been ruled out in Lemma 2.4 and Lemma 2.5. If we suppose that $xy \in E(G)$, then $\langle\ y\ +\ z_1y^-x\ \rangle$ implies that we get one of $z_1y^- \in E(G)$, $z_1x \in E(G)$ or $xy^- \in E(G)$. But $z_1y^- \in E(G)$ leads to a longer cycle than C, $z_1x \in E(G)$ leads to a shorter path system than $\mathcal{Q}$ and $xy^- \in E(G)$ is not possible by Lemma 2.4. Thus, $yz_1 \notin E(G)$.

Subcase 3.2 *Suppose $xy \in E(G)$.*

Consider $\langle\ xyv +\ y^+y^{++} + z_1z_2\ +\ x^+\ \rangle$. We claim that this subgraph is an induced $N(2,2,1)$.

First note that if $z_1y \in E(G)$ (or symmetrically $z_1x \in E(G)$), we get a contradiction by the same argument as in the previous case. Thus, we may suppose that $z_1y \notin E(G)$ and $z_1x \notin E(G)$.

But Lemmas 2.4, 2.5, 2.6 and the previous cases imply that this is an induced $N(2,2,1)$ unless $y^{++}y \in E(G)$. Therefore, suppose that $y^{++}y \in E(G)$. Then we get a cycle C' of the same length as C but where y and z are closer than on C which is a contradiction to our choice of C. The cycle C' can be obtained from C as follows: $y\ y^{++}\ C\ z\ C\ y^-\ y^+\ y$. Thus, we get that there has to be a vertex v that is adjacent to three vertices of C, which proves the lemma.

$\square$

To show the second part of Lemma 2.1, we will use the same approach as in Lemma 2.8. But we will show a stronger notion than what is stated in the lemma. That is, we will show that either all cycles of length 3 through $|V(G)|$

exist in the graph or there is a vertex off the cycle with three neighbors on the cycle. This stronger result will be needed in Chapter 4.

Before we show the existence of all these cycles in general, we show the existence of cycles of length three, four and five, since our argument for the general case will be based on the assumption that the graph contains at least a 5-cycle.

Lemma 2.9 *Let $G = (V, E)$ be a $\{K_{1,3}, N(3,1,1)\}$-free graph with minimum degree 4. Then G contains a 3-cycle, a 4-cycle and a 5-cycle.*

Proof. Since G is claw-free and 4-connected, the existence of triangles is immediate. Let us show that the graph also contains 4-cycles. Therefore pick an arbitrary triangle and label the vertices of that triangle with v_1, v_2 and v_3. Since G is 4-connected, each of these vertices has to have two other neighbors. Denote the neighbors of v_i with $v_{i,1}$ and $v_{i,2}$ with $i \in \{1, 2, 3\}$. If any of $v_{i,1}$ or $v_{i,2}$ is adjacent to v_j with $j \neq i$, we immediately get a 4-cycle with $v_i\ v_j\ v_{i,1}\ v_{i,2}$. Thus, we may assume that we have no such adjacencies and that all the vertices $v_{i,j}$ with $i \in \{1,2,3\}$ and $j \in \{1,2\}$ are pairwise distinct. Hence, $\langle\ v_i + v_j v_{i,1} v_{i,2}\ \rangle$ implies that $v_{i,1}$ is adjacent to $v_{i,2}$ for all i. If any of the vertices $v_{i,1}$ (or symmetrically $v_{i,2}$) is adjacent to another vertex $v_{j,1}$ or $v_{j,2}$, we get a 4-cycle. Consider the triangle spanned by the vertices v_2, $v_{2,1}$ and $v_{2,2}$. By applying the same argument to this triangle as to the triangle spanned by v_1, v_2 and v_3, we get two neighbors of $v_{2,1}$ that we denote by $v_{2,1,1}$ and $v_{2,1,2}$ and that are adjacent to each other and distinct from $v_{i,k}$ and v_i, with $i \in \{1,2,3\}$ and $k \in \{1,2\}$. Now consider the triangle spanned by $v_{2,1}$, $v_{2,1,1}$ and $v_{2,1,2}$. By 4-connectivity $v_{2,1,1}$ has to have two other neighbors except $v_{2,1}$ and $v_{2,1,2}$.

Suppose that $v_{2,1,1}$ only has neighbors within the already mentioned vertices. If $v_{2,1,1}$ is adjacent to any of v_i or $v_{2,2}$ we get a 4-cycle. If $v_{2,1,1}$ is adjacent to both $v_{1,1}$ and $v_{1,2}$, or to both $v_{3,1}$ and $v_{3,2}$ we get a 4-cycle with

v_1 $v_{1,1}$ $v_{2,1,1}$ $v_{1,2}$ or with v_3 $v_{3,1}$ $v_{2,1,1}$ $v_{3,2}$. Thus, we may suppose that $v_{2,1,1}$ is adjacent to both $v_{1,1}$ and $v_{3,1}$. But then we get a claw with $\langle\ v_{2,1,1} + v_{1,1}v_{3,1}v_{2,1}\ \rangle$ as any edge among these produces a C_4, a contradiction.

Therefore we obtain that $v_{2,1,1}$ has to have a neighbor $v_{2,1,1,1}$ that is distinct from all v_i and $v_{i,k}$ with $i \in \{1,2,3\}$ and $k \in \{1,2\}$. Then $\langle\ v_1v_2v_3+ v_{2,1}v_{2,1,1}v_{2,1,1,1}+\ v_{v_{1,1}}+v_{3,1}\ \rangle$ implies that $v_{2,1,1,1}$ is adjacent to one of v_1, v_3, $v_{1,1}$ or $v_{3,1}$, or $v_{2,1,1}$ is adjacent to one of $v_{3,1}$ or $v_{1,1}$, otherwise we have an induced $N(3,1,1)$ or a 4-cycle. We may suppose that $v_{2,1,1,1}$ is not adjacent to $v_{1,1}$ and also not adjacent to $v_{3,1}$, since if we suppose that $v_{2,1,1,1}$ is adjacent to $v_{1,1}$ (or $v_{3,1}$) we may as well consider $\langle\ v_1v_2v_3+\ v_{2,1}v_{2,1,1}v_{2,1,1,1}+ v_{v_{1,2}}+v_{3,1}\ \rangle$, respectively $\langle\ v_1v_2v_3+\ v_{2,1}v_{2,1,1}v_{2,1,1,1}+\ v_{v_{1,1}}+v_{3,2}\ \rangle$ or $\langle\ v_1v_2v_3+ v_{2,1}v_{2,1,1}v_{2,1,1,1}+\ v_{v_{1,2}}+v_{3,2}\ \rangle$. Note that we may assume that $v_{2,1,1,1}$ is not adjacent to both $v_{1,1}$ and $v_{1,2}$ (or symmetrically to both $v_{3,1}$ and $v_{3,2}$), since that leads to a 4-cycle. If $v_{2,1,1,1}$ is adjacent to v_1 (or is adjacent to v_3), $\langle\ v_1 + v_{1,1}v_{2,1,1,1}v_2\ \rangle$ (or $\langle\ v_3 + v_{3,1}v_{2,1,1,1}v_2\ \rangle$) implies that $v_{2,1,1,1}$ is adjacent to v_2 or to $v_{1,1}$ (or $v_{3,1}$) which is a contradiction to our assumption or leads to a 4-cycle. Suppose that $v_{2,1,1}$ is adjacent to $v_{1,1}$ (or $v_{3,1}$). Then $\langle\ v_{2,1,1} + v_{2,1,1,1}v_{2,1}v_{1,1}\ \rangle$ (or $\langle\ v_{2,1,1} + v_{2,1,1,1}v_{2,1}v_{3,1}\ \rangle$) implies that $v_{2,1,1,1}$ is adjacent to $v_{1,1}$ (or $v_{3,1}$), a contradiction to our assumption, or it implies that $v_{1,1}$ (or $v_{3,1}$) is adjacent to $v_{2,1}$, which leads to a 4-cycle. Therefore we conclude that the graph contains a 4-cycle.

Let us show that the graph also contains a 5-cycle. Pick a 4-cycle and label the vertices on the cycle with v_1, v_2, v_3 and v_4. Since the graph is 4-connected, it is easy to see that each vertex v_i has to have at least one other neighbor w_i off the cycle. Then $\langle\ v_1 + w_1v_2v_4\ \rangle$ implies $v_2v_4 \in E(G)$ and similarly we get that $v_1v_3 \in E(G)$, otherwise we immediately get a 5-cycle and are done. Note that the same argument shows that all w_i, w_j with $i,j \in \{1,2,3,4\}$ are pairwise distinct. Then 4-connectivity of the graph implies that w_2 has to have at least three neighbors off the cycle. Note that at least two of these three neighbors have to be adjacent to each other, otherwise we have an induced claw. If these neighbors are all adjacent to v_2, we get a 5-cycle. Thus, we may suppose that w_2 has a neighbor x_1 that is not adjacent to v_2 and that is distinct from v_i and w_i, $i \in \{1, 2, 3, 4\}$.

If x_1 is adjacent to w_1 (or symmetrically w_4 or w_3), we get a 5-cycle with $w_1\ v_1\ v_2\ w_2\ x_1\ w_1$. If x_1 is adjacent to v_1 (or symmetrically to v_4 or v_3), we get a 5-cycle with $v_1\ v_3\ v_2\ w_2\ x_1\ v_1$. Thus, we may assume that x_1 is adjacent to none of these vertices. Hence, we get from the 4-connectivity of the graph that x_1 has at least three more neighbors. Since G is claw-free, at least two of the three neighbors have to be adjacent to each other. If these neighbors are all adjacent to w_2, we get a 5-cycle with these neighbors, w_2 and x_1. Thus, we may assume that x_1 has neighbor x_2 that is not adjacent to w_2. If x_2 is adjacent to v_1 (or symmetrically to v_3 or v_4), we get the 5-cycle $v_1\ x_2\ x_1\ w_2\ v_2\ v_1$. If x_2 is adjacent to v_2, $\langle\ v_2 + w_2x_2v_1\ \rangle$ implies that x_2 is also adjacent to v_1, which leads to a 5-cycle as seen. Then $\langle\ v_1v_3v_2{+}\ w_2x_1x_2{+}\ w_1{+}w_3\ \rangle$, $\langle\ v_1v_4v_2{+}\ w_2x_1x_2{+}\ w_4{+}w_1\ \rangle$ and $\langle\ v_4v_3v_2{+}\ w_2x_1x_2{+}\ w_3{+}w_4\ \rangle$ imply that x_2 is adjacent to two of w_3, w_1 and w_4, otherwise we get an induced $N(3, 1, 1)$. Suppose that x_2 is adjacent to w_3 and w_1. Then we get the 5-cycle $w_1\ v_1\ v_3\ w_3\ x_2\ w_1$. Thus we get that the graph contains a 5-cycle. $\square$

We are now ready to present the proof that a 4-connected $\{K_{1,3}, N(3, 1, 1)\}$-free graph either contains all cycles of length three up to the order of G, or if G contains a t-cycle C but no $(t+1)$-cycle, there exists a vertex $v \in V(G)\backslash V(C)$ such that v has three neighbors on C.

Lemma 2.10 *Let $G = (V, E)$ be a 4-connected $\{K_{1,3}, N(3, 1, 1)\}$-free graph. Suppose G is not pancyclic. Let C be a cycle of length t, such that G does not contain a cycle of length $t + 1$. Then if $t \neq |V(G)|$ there exists a vertex $v \in V(G) \setminus V(C)$ such that v has three neighbors on C.*

Proof. Suppose that G does not contain a $(t+1)$-cycle, but does contain a t-cycle, with $5 \leq t \leq n-1$. Let C be a cycle of length t. Pick a vertex $v \in V(G) \setminus V(C)$ such that the sum of the lengths of its three shortest paths Q_1, Q_2, Q_3, where $l(Q_1) \leq l(Q_2) \leq l(Q_3)$, to C is minimal and under these conditions, such that $dist_C(x, y)$ is minimized and subject to that, $dist_C(y, z)$ is minimized. Call this path system $\mathcal{Q}$. Suppose that each of Q_1, Q_2, Q_3 is not just an edge, say $Q_1 = \{vx_1...x_nx\}$, $Q_2 = \{vy_1y_2...y_my\}$, $Q_3 = \{vz_1...z_kz\}$. Let C be oriented such that $x \leq_C y \leq_C z$. By Lemma 2.7 we may assume that $y_1z_1 \in E(G)$ and that there are no other edges within the path system or from the path system to the cycle. (The cases $x_1y_1 \in E(G)$ and $x_1z_1 \in E(G)$ are symmetric.) We claim that $\langle\ vy_1z_1{+}\ x_1x_2x_3{+}\ y_2{+}z_2\ \rangle$ is an induced $N(3, 1, 1)$.

Case 1 *Suppose that x and y are consecutive on the cycle.*

If Q_2 and Q_3 both have length at least three, we immediately are done (here possibly $x_2 = x$ and $x_3 = x^-$). Suppose that Q_2 has length two (see Figure 2.5).

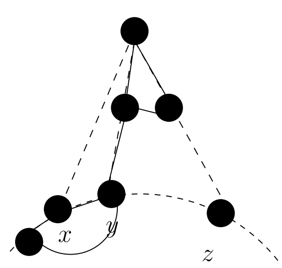


Figure 2.5:

Note that there has to be a fourth path from v to the cycle. Let the vertex on the cycle intersecting that path be w. Then we can get a $(t+1)$-cycle with w^- w^+ C x x_1 v y_1 y C z^- z^+ C w^-, with possibly $z^- = y$. Note that we need the fact shown in Lemma 2.9, that is, $t \geq 5$. Thus, Q_1 has to be an edge in that case.

Therefore, x and y must be at least distance 2 apart on the cycle if none of the paths is just an edge.

Then $\langle\ vy_1z_1{+}\ x_1x_2x_3{+}\ y_2{+}z_2\ \rangle$ is an induced $N(3,1,1)$, where we may immediately suppose that $x_2 = x$, $x_3 = x^-$ and $y_2 = y, z_2 = z$, otherwise we easily get a contradiction to Lemma 2.7. If y is adjacent to z on the cycle we obtain a $(t+1)$-cycle with y y_1 z_1 z C x^- x^+ C y. Thus, we may suppose that y and z are at least distance 2 apart on the cycle. If $xy \in E(G)$, we get a cycle of the same length, but where x and y have a shorter distance to each other, since $\langle\ y\ +\ y_1y^+x\ \rangle$ implies that x is also adjacent to y^+. Hence, x^- x^+ C y x y^+ C x^- is a $(t+1)$-cycle, contradicting our assumptions. Similarly (possibly with relabeling of the vertices), we get that y and z, and z and x are not adjacent to each other. (Note that we can relabel without hurting our minimization conditions since we may assume that $l(Q_2) = l(Q_3)$.) If $x^-y \in E(G)$, we may distinguish two cases. If we suppose that x and y have exactly one intermediate vertex on the cycle, we obtain the $(t+1)$-cycle x^- x x_1 v y_1 y C z^- z^+ C x^-. If we suppose that $x^+ \neq y^-$, consider $x_3 = x^+$, that is, consider the net $\langle\ vy_1z_1{+}\ x_1xx^+{+}\ y{+}z\ \rangle$. By the previous argument, we only need to consider the case that $x^+y \in E(G)$ (or $x^+z \in E(G)$). Note first that $xx^{++} \notin E(G)$, otherwise we get a cycle of the same length, but on which x and y have a smaller distance to each other with x x^{++} C x^- x^+ x. Then $\langle\ x^+ + x^{++}xy\ \rangle$ implies that y is adjacent to x, which is an earlier case, or to x^{++}, which once more leads to a cycle of the same length, but where x and y have shorter distance to each other: x x^+ y x^{++} C y^- y^+ C x. We can proceed in a similar manner, if we suppose that we have the edge x^+z.

Observe first that if y and z have distance two or three on the cycle, we can easily obtain a $(t+1)$-cycle by skipping the vertices on the cycle between these two and including instead all (or two) of v, y_1 and z_1. Therefore, we may suppose that y and z have at least distance four on the cycle. But then the cycle $C' := x\ x^+\ z\ x^{++}\ C\ z^-\ z^+\ C\ x$ has length t and x and y (and y and z) are closer to each other if we relabel y and z.

Hence, we may suppose that Q_1 is just an edge.

Case 2 *Suppose* $y_1z_1 \in E(G)$.

Fact 2.11 *If* Q_3 *has length at least* 5*, we get an induced* $N(3,1,1)$.

Observe that $\langle\ vy_1z_1{+}\ z_2z_3z_4{+}\ x{+}y_2\ \rangle$ is an induced $N(3,1,1)$, unless $y_2 = y$ and $xy \in E(G)$. If Q_2 has length two, x and y cannot be consecutive on the cycle, otherwise the cycle $x\ v\ y_1\ y\ C\ z^-\ z^+\ C\ x$ is a $(t+1)$-cycle. With a similar argument we get that x and y also cannot have distance two on the cycle (since we can also skip a fourth vertex $w = V(Q_4) \cap V(C)$ on the cycle). That means that x and y have at least distance three on the cycle. But then if $xy \in E(G)$, $\langle\ x + x^+yv\ \rangle$ implies $yx^+ \in E(G)$. Then $x^+\ y\ x\ C\ y^-\ y^+\ C\ x^+$ is a cycle of the same length as C, but x and y have smaller distance to each other, a contradiction.

Thus, we assume that $l(Q_3) < 5$.

Subcase 2.1 *Suppose that* x *and* y *are consecutive on the cycle.*

It is easy to see that Q_2 has to have at least length four, otherwise we get a $(t+1)$-cycle by including Q_2 and skipping z and w on the cycle, that is, by Fact 2.11, Q_2 and Q_3 have exactly length four. This implies that $\langle\ vy_1z_1{+}\ z_2z_3z{+}\ x{+}y_2\ \rangle$ is an induced $N(3,1,1)$, unless zx is an edge. Then

$\langle\ x + vzy\ \rangle$ implies $yz \in E(G)$. It is easy to see that the cycle $x\ z\ y\ C\ z^-\ z^+\ C\ x^-$ is a cycle of the same length as C, but where (after relabeling), x, y and z are closer to each other, unless z is adjacent to y on the cycle (see Figure 2.6).

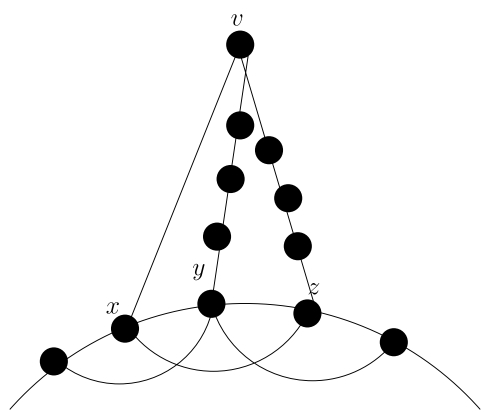


Figure 2.6:

If that is the case, we consider $\langle\ z^+zy+\ z_3z_2z_1+\ z^{++}+y_3\ \rangle$. This is an induced $N(3,1,1)$, unless y or z are adjacent to z^{++}. However, note that $\langle\ y\ +\ zz^{++}y_3\ \rangle$, respectively $\langle\ z\ +\ yz^{++}y_3\ \rangle$ implies that if one of y or z is adjacent to z^{++}, the other one is adjacent to z^{++} as well. Suppose that $t \geq 6$. Then consider $\langle\ z^{++}zy+\ z_3z_2z_1+\ z^{+3}+y_3\ \rangle$ and repeat the same argument. If we therefore assume that y and z are adjacent to z^{+3}, we get a $(t+1)$-cycle with $x\ v\ z_1\ z_2\ z_3\ z\ z^{+3}\ C\ x$, unless C is smaller than a 6-cycle to begin with. We already proved that G contains a 3-cycle, a 4-cycle and a 5-cycle. Hence, suppose that C is a 5-cycle. Then $x\ v\ z_1\ z_2\ z_3\ z\ x$ is a 6-cycle and we are done as well.

Subcase 2.2 *Suppose that x and y have distance two on the cycle.*

It is easy to see that Q_2 has to have at least length five, otherwise we get $(t+1)$-cycle by including Q_2 in the cycle and skipping z, x^+ and w on the cycle. Then by Fact 2.11 we get a contradiction.

Subcase 2.3 *Suppose that x and y have distance three on the cycle.*

If Q_2 has length three or four, we obtain a $(t+1)$-cycle by including Q_2 in the cycle and skipping x^+, x^{++} and z on the cycle. If Q_2 has length two, we get the $(t+1)$-cycle $x^-\ x^+\ x\ v\ y_1\ y\ C\ x^-$. Otherwise Q_2 has to have length at least five. By Fact 2.11 we obtain a contradiction.

Subcase 2.4 *Suppose that x and y have distance at least four on the cycle.*

Consider $\langle\ vy_1z_1{+}\ y_2y_3y_4{+}\ x{+}z_2\ \rangle$, where we may assume that $y_2 = y$, $y_3 = y^-$, $y_4 = y^{--}$ and $z_2 = z$. With the previous arguments it is easy to see that y and z have distance greater than 4 on the cycle. Hence, we get an induced $N(3,1,1)$, unless we have more edges within the vertices x, y, y^-, y^{--} and z. If xy is an edge, $\langle\ x + yx^+v\ \rangle$ implies that yx^+ is an edge as well. Then we get a cycle of the same length, but with a shorter distance between x and y with $x^-\ x\ y\ x^+\ C\ y^-\ y^+\ C\ x$, contradicting our minimality constraints. Similarly, we get a contradiction if yz or zx is an edge. If $yy^{--} \in E(G)$ we get a contradiction to our minimality constraint, since we get a cycle of the same length with $y^{--}\ y\ y^-\ y^+\ C\ y^{--}$. Then if y^- is adjacent to one of x or z, $\langle\ y^- + yy^{--}x\ \rangle$ implies $xy^{--} \in E(G)$ (respectively $\langle\ y^- + yy^{--}z\ \rangle$ implies $y^{--}z \in E(G)$). But then we can once more get a $(t+1)$-cycle, with a shorter distance between x and y (respectively between y and z) by using $x^-\ x^+\ C\ y^{--}\ x\ y^-\ C\ x^-$ (respectively using $x\ C\ y^{--}\ z\ y^-\ C\ z^-\ z^+\ C\ x$). If $xy^{--} \in E(G)$, $\langle\ y^{--} + y^-y^{-3}x\ \rangle$ implies that one of xy^{-3} or y^-y^{-3} is an edge. If the latter holds, $\langle\ x + vx^+y^{--}\ \rangle$ implies $y^{--}x^+ \in E(G)$. Then $x^-\ y^{--}\ x\ C\ y^-\ y^{-3}\ C\ x^-$ is a cycle of the same length as C, but contradicting the minimality constraint between x and y. If xy^{-3} is an edge, $x^-\ x^+\ C\ y^{-3}\ x\ y^{--}\ C\ x^-$ is a cycle of the same length as C, but contradicting the minimality constraint between x and y, unless $y^{-3} = x^+$. But then $x\ y^{--}\ C\ x^-\ x^+\ x$ is a cycle of the same length as C, contradicting the minimality constraint between x and y. If $y^{--}z$ is an edge, we can follow the same argument to obtain a contradiction.

Case 3 *Suppose $x_1y_1 \in E(G)$ (or symmetrically $x_1z_1 \in E(G)$).*

Without loss of generality assume that $x_1y_1 \in E(G)$.

Subcase 3.1 *Suppose that x and y are consecutive on the cycle.*

Observe that Q_2 must have at least length four, since otherwise we get a $(t+1)$-cycle by including Q_2 in the cycle and skipping z and w. That means that Q_3 also has to have length at least four. Then we obtain an induced $N(3,1,1)$ with $\langle\ xvy_1\!+\ z_1z_2z_3\!+\ y_2\!+\!x^-\ \rangle$, a contradiction.

Subcase 3.2 *Suppose that x and y have distance two on the cycle.*

Observe that Q_2 must have at least length five, since otherwise we can get $(t+1)$-cycle by including Q_2 in the cycle and skipping x^+, z and/ or w on the cycle. That means that Q_3 also has to have length at least five. Then $\langle\ xvy_1\!+\ z_1z_2z_3\!+\ y_2\!+\!x^-\ \rangle$ is an induced $N(3,1,1)$, a contradiction.

Subcase 3.3 *Suppose that x and y have distance three on the cycle.*

With the same argument as before, we get that Q_2 has to have at least length five, and get a contradiction.

Subcase 3.4 *Suppose that x and y have distance at least four on the cycle.*

Consider $\langle\ vxy_1\!+\ z_1z_2z_3\!+\ x^-\!+\!y_2\ \rangle$. We may assume that $y_2 = y$, $z_2 = z$ and $z_3 = z^-$. Note that we get with the same argument as previously that y and z have distance at least 4 on the cycle. Then we have an induced $N(3,1,1)$, unless the vertices z^-, z, x^-, x and y have more edges amongst each other. If $xy \in E(G)$, $\langle\ x\ +\ vx^+y\ \rangle$ implies that yx^+ is an edge as well. Then the cycle $x^-\ x\ y\ x^+\ C\ y^-\ y^+\ C\ x^-$ is a cycle of the same length as C, but with a shorter distance between x and y. Similarly, we get a contradiction if yz or xz is an edge, where in the latter case we also need to relabel the vertices. If $x^-y \in E(G)$ we get a cycle of length $t+1$ with $x^-\ y\ y_1\ v\ x\ C\ y^-\ y^+\ C\ z^-\ z^+\ C\ x^-$. Similarly, we may suppose that we do not have the edge x^-z. If y is adjacent to z^-, we obtain the $(t+1)$-cycle $v\ y_1\ y\ z^-\ C^-\ y^+\ y^-\ C^-\ x^+\ x^-\ C^-\ w^+\ w^-\ C^-\ z^+\ z\ z_1\ v$, where $w \in V(C)$ is the vertex a fourth path of v leads to (see Figure 2.7).

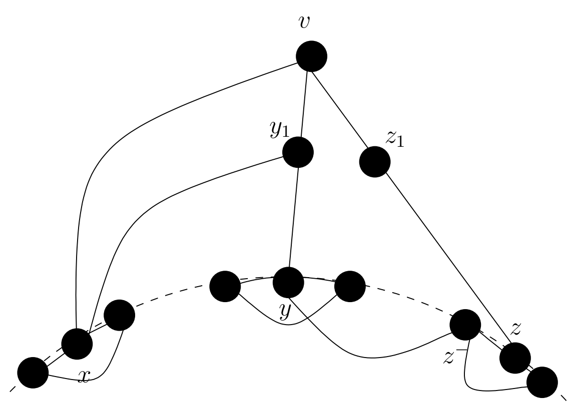


Figure 2.7:

If x^- is adjacent to z^-, we distinguish between two cases. If z is adjacent to z^{--}, we get a contradiction, since the cycle $z^{--}\ z\ z^-\ z^+\ C\ z^{--}$ is a cycle of the same length, but with a smaller distance between y and z than in C. Therefore we may assume that z is not adjacent to z^{--}. Then $\langle\ z^- + zz^{--}x^-\ \rangle$ implies with the previous arguments that x^- is also adjacent to z^{--}. But then we get a $(t+1)$-cycle with $v\ x\ C\ z^{--}\ x^-\ C^-\ z^+\ z\ z_1\ v$, that is we included v and z_1 to the cycle, but skipped z^-. If $xz^- \in E(G)$, we get a $(t+1)$-cycle with $v\ x\ z^-\ C^-\ y^+\ y^-\ C^-\ x^+\ x^-\ C^-\ z\ z_1\ v$, that is, we included

v and z_1 in the cycle, but skipped y.

Thus, by the above cases, we obtain that v has to be adjacent to at least two vertices on the cycle. We may therefore assume that Q_1 and Q_2 are only edges.

Case 4 *Suppose $xy \in E(G)$.*

Subcase 4.1 *Suppose that x and y have distance at most three on the cycle.*

Then we immediately get a $(t+1)$-cycle with $x\ v\ y\ C\ x$, $x^-\ x^+\ x\ v\ y\ C\ x^-$ or with $x^-\ x^+\ x\ v\ y\ y^-\ y^+\ C\ x^-$.

Subcase 4.2 *Suppose that x and y are at distance at least four on the cycle.*

We claim that $\langle\ vxy+\ z_1z_2z_3+\ x^-+y^-\ \rangle$ is an induced $N(3,1,1)$. Assume first that $z_2 = z$, $z_3 = z^-$. Using previous arguments we obtain that y and z have distance at least 4 on the cycle. From Lemma 2.4 and Lemma 2.5 we get that all of the edges x^-y^-, x^-y, x^-z^-, x^-z, xz^-, y^-z^-, y^-z and yz^- lead to a longer cycle than C. We also get that the longer cycle is a $(t+1)$-cycle if Q_1 and Q_2 are used or a $(t+2)$-cycle if Q_3 is used. If the latter is the case, we can always modify the resulting $(t+2)$-cycle to a $(t+1)$-cycle by skipping x or y on the cycle. Thus we only need to consider the case that xz or yz is an edge (see Figure 2.8).

But then $\langle\ z\ +\ z_1z^-x\ \rangle$, respectively $\langle\ z\ +\ z_1z^-y\ \rangle$, implies that z^-x, respectively z^-y, is an edge and we are in a previous case. Hence, we get a contradiction in this case. If $l(Q_3) \geq 4$ we get immediately a contradiction and if $l(Q_3) = 3$ we obtain, using previous arguments, that y and z have distance at least 4 on the cycle and get a contradiction as in the case of $l(Q_3) = 2$.

Case 5 *Suppose $xz_1 \in E(G)$ (or symmetrically $yz_1 \in E(G)$).*

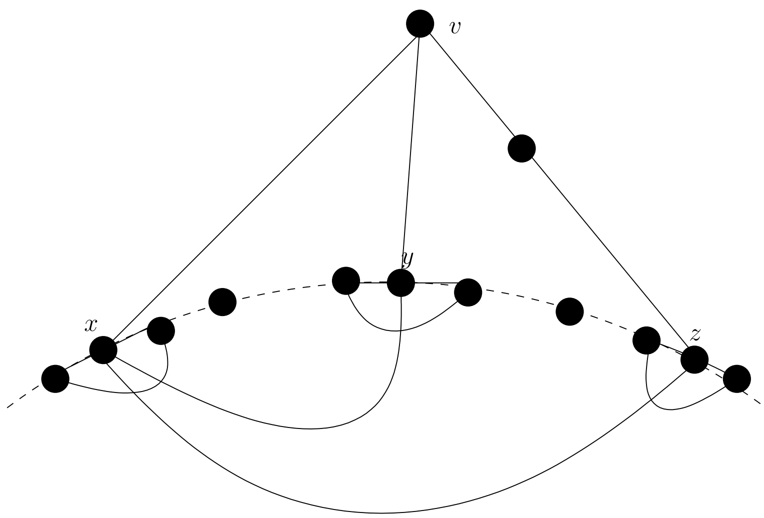


Figure 2.8:

Suppose, without loss of generality, $xz_1 \in E(G)$.

Subcase 5.1 *Suppose that x and y have distance at most three on the cycle.*

If they are consecutive we get a $(t+1)$-cycle with $x\ v\ y\ C\ x$, if they have distance two on the cycle we get a $(t+1)$-cycle with $x^-\ x^+\ x\ v\ y\ C\ x^-$ and if they have distance three on the cycle, we get a $(t+1)$-cycle with $x^-\ x^+\ x$ $v\ y\ y^-\ y^+\ C\ x^-$.

Subcase 5.2 *Suppose that x and y are at distance at least four on the cycle.*

We claim that $\langle\ vxz_1{+}\ z_2z_3z_4{+}\ y{+}x^-\ \rangle$ is an induced $N(3,1,1)$. Suppose that $l(Q_3) = 2$, that is $z_2 = z$, $z_3 = z^-$ and $z_4 = z^{--}$. By previous arguments we get that y and z have distance at least 4 on the cycle. From Lemma 2.4, Lemma 2.5, Lemma 2.6 and Case 4 we get that all edges except the edge

zz^{--} between the vertices z, z^-, z^{--}, y, x and x^- lead to a longer cycle, that is to a cycle of length $t+1$ or $t+2$. But if the latter is the case, we may skip y or x on the cycle, to obtain a $(t+1)$-cycle. Note furthermore, that if zz^{--} is an edge, we get a cycle of the same length as C, but where y and z have a shorter distance on the cycle to each other than on C, by using y C z^{--} z z^- z^+ C y.

If $l(Q_3) = 3$, we claim that $\langle\, vxz_1 + z_2 zz^- + y + x^- \,\rangle$ is an induced $N(3,1,1)$. But then using the same arguments as in the case $l(Q_3) = 2$ we get a contradiction. Similarly, we obtain a contradiction if $l(Q_3) > 3$.

Therefore, v is adjacent to three vertices on the cycle. $\square$

From Lemma 2.10 and Lemma 2.8 we obtain Lemma 2.1, stated in the beginning of this section.

2.2 Hamiltonicity of 4-connected, $\{K_{1,3},\ N\}$-free graphs, with $N = N(2,2,1)$ or $N = N(3,1,1)$

Using Lemma 2.1 we will show hamiltonicity of 4-connected $\{K_{1,3},\ N\}$-free graphs, with $N = N(2,2,1)$ or $N = N(3,1,1)$. The approach in the proofs will be by contradiction. That is, we will assume that the graph is not hamiltonian and then consider a longest cycle C in the graph. By the lemmas of the previous section we know there is a vertex off the cycle that is adjacent to at least three vertices on the cycle. We will then show that the graph has to contain an induced $N(3,1,1)$ or an induced $N(2,2,1)$, contradicting our assumptions.

Theorem 2.12 *Let $G = (V, E)$ be a 4-connected $\{K_{1,3}, N(2,2,1)\}$-free graph. Then G is hamiltonian.*

Proof. Suppose, for the purpose of contradiction, that G is not hamiltonian. If C is a longest cycle in G, by Lemma 2.8 we see there exists a vertex $v \in V(G) \setminus V(C)$ such that v has three neighbors on C. Say these neighbors are x, y and z and suppose that the orientation of C is such that $x <_C y <_C z$. By Lemma 2.3 we obtain $dist_C(a,b) \geq 4$ for $a \neq b$, a, $b \in \{x,\ y,\ z\}$. Under these conditions choose C and v such that $dist_C(x,y)$ is minimized and with respect to that, $dist_C(y,z)$ is minimized. Then $\langle\ v + xyz\ \rangle$ implies that we have an edge within the vertices x, y and z. Suppose, without loss of generality, that $yz \in E(G)$.

We claim that $N := \langle\ vyz+\ xx^{-}+z^{-}z^{--} + y^{-}\ \rangle$ is an induced $N(2,2,1)$. Note, that there cannot be any additional edges from v to any of the vertices in N, since otherwise we could extend the cycle immediately. Furthermore, by the Lemmas 2.4, 2.5 and 2.6, we have $x^{-}z^{-} \notin E(G)$, $x^{-}z^{--} \notin E(G)$,

$x^-z \notin E(G)$, $x^-y^- \notin E(G)$, $x^-y \notin E(G)$, $xz^- \notin E(G)$, $xz^{--} \notin E(G)$, $xy^- \notin E(G)$, $y^-z^- \notin E(G)$ and $y^-z^{--} \notin E(G)$.

Thus, all that is left is to check the cases $zz^{--} \in E(G)$, $xz \in E(G)$ and $xy \in E(G)$.

Case 1 *Suppose $zz^{--} \in E(G)$.*

By using the cycle $C' := z\ z^{--}\ \overleftarrow{C}\ z^+\ z^-\ z$ we find a cycle of the same length as C but with $dist_{C'}(y,z) < dist_C(y,z)$, a contradiction.

Case 2 *Suppose $xy \in E(G)$ (or symmetrically $xz \in E(G)$).*

By considering $\langle\ y + y^-xz\ \rangle$ and by recalling that we already ruled out the edges xy^- and zy^-, we get $xz \in E(G)$ (see Figure 2.9). We claim that

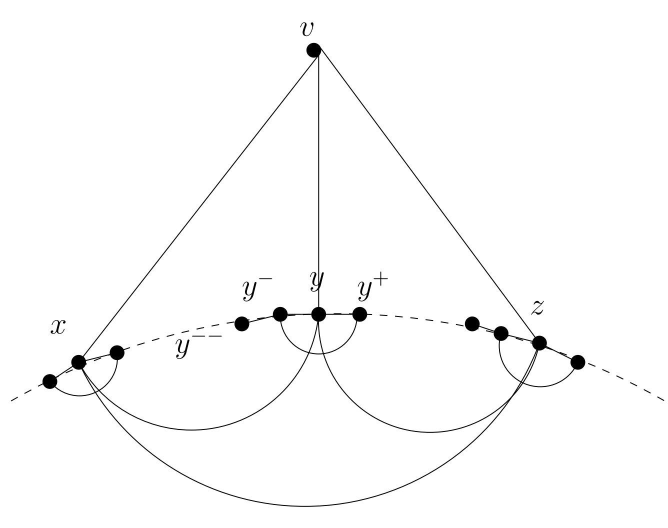


Figure 2.9:

$\langle\ xyz +\ y^-y^{--} + z^-z^{--}\ +\ x^-\ \rangle$ is an induced $N(2,2,1)$. Note, that we already considered most of the additional edges that could destroy the chosen net. Thus, we only need to check the edges involving y^{--}. But all additional edges from y^{--} into $\{x, z, z^-, z^{--}, x^-\}$ cannot exist by Lemma 2.5 and Lemma 2.6. Hence, we just need to check the case $yy^{--} \in E(G)$.

Suppose $y^{--}y \in E(G)$. As in Case 1 we could form a new cycle $C' := y\ y^{--}Cy^+\ y^-\ y$ where $dist_{C'}(x, y) < dist_C(x, y)$, which contradicts our choice of C.

Thus, we have an induced $N(2, 2, 1)$, and this final contradiction implies G is hamiltonian.

□

To show hamiltonicity if $N = N(3, 1, 1)$, we will use the same main approach in the proof. But for this theorem, we will also take more advantage of the fact that the graph is 4-connected. That is, we will need the fact that the vertex off the longest cycle in the non-hamiltonian graph G has to have a fourth path to the cycle. Using that path and other constraints we will then show that the graph contains an induced $N(3, 1, 1)$.

Theorem 2.13 *Let $G = (V, E)$ be a 4-connected $\{K_{1,3}, N(3, 1, 1)\}$-free graph. Then G is hamiltonian.*

Proof. Suppose, for the purpose of contradiction, that G is not hamiltonian and let C be a longest cycle in G. Then by Lemma 2.10 there exists a vertex $v \in V(G)\backslash V(C)$ that is adjacent to at least three vertices x, y and z on the cycle. Under these conditions choose C and v such that $dist_C(x, y)$ is minimized and with respect to that, $dist_C(y, z)$ is minimized. Let C be oriented so that $x <_C y <_C z$. Lemmas 2.4, 2.5 and 2.6 imply x^+x^-, $y^-y^+, z^-z^+ \in E(G)$. Furthermore, $\langle\ v + xyz\ \rangle$ implies xy, xz or $yz \in E(G)$.

Case 1 *Suppose, without loss of generality, that $xy \in E(G)$ and $yz \notin E(G)$, $xz \notin E(G)$.*

Then $\langle\ vxy+\ zz^-z^{--}+\ x^-+y^-\ \rangle$ is an induced $N(3, 1, 1)$ and we have an immediately contradiction, since all other possible additional edges would lead to a cycle longer than C by Lemma 2.4, Lemma 2.5 and Lemma 2.6, or would lead to a cycle of the same length as C but with a shorter distance between y and z.

Case 2 *Suppose $xy \in E(G)$ and without loss of generality we may assume that $yz \in E(G)$ as well.*

Then from $\langle\ y + y^{-}xz\ \rangle$ we see that $xz \in E(G)$. Note that, by 4-connectivity, v has another (shortest) path Q to C, which is disjoint from vx, vy and vz.

Subcase 2.1 *Q has length at least 2 (see Figure 2.10).*

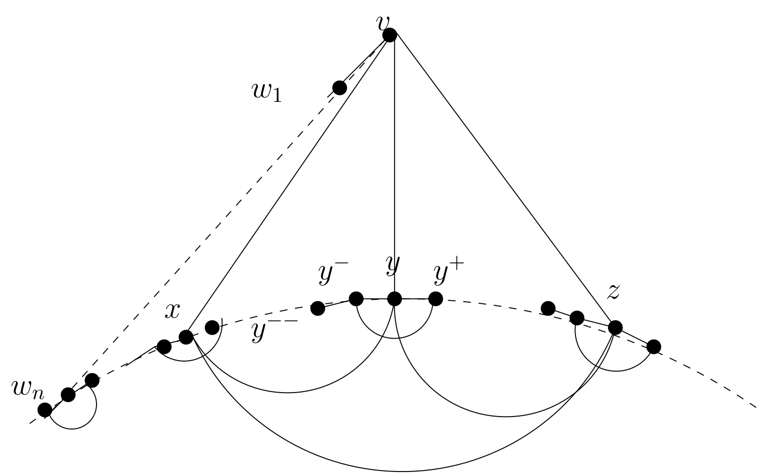


Figure 2.10:

Let $Q = \{vw_1w_2....w_n\}$ for $n \geq 2$, where $w_n \in V(C)$ and $w_i \notin V(C)$ for all $i \neq n$. Then we claim that $\langle\ xyv+\ w_1w_2w_3+\ x^{+}+y^{+}\ \rangle$ is an induced $N(3,1,1)$. Note $w_3 = w_2^{+}$ if $w_2 \in V(C)$. Then if w_ix and $w_iy \notin E(G)$, this is an induced $N(3,1,1)$. Now suppose that w_{n-1} is adjacent to one of $\{x,y,z\}$. Suppose without loss of generality that $w_{n-1}x \in E(G)$. Then $\langle\ x + w_{n-1}yx^+\ \rangle$ implies that $yw_{n-1} \in E(G)$. Similarly, we get that $w_{n-1}z \in E(G)$, that is, w_{n-1} is adjacent to all of x, y and z. Note furthermore, if w_n is adjacent to one of $\{x,y,z\}$, say to x, then $\langle\ w_n + xw_{n-1}w_n^+\ \rangle$ implies $w_{n-1}x \in E(G)$ and we can repeat the above argument to get that w_{n-1} is adjacent to all of x, y and z. Observe that we can do this argument for any $w_i \in V(Q)\backslash\ w_n$, $i \in \{1, 2, ..., n-1\}$ and thus we may assume that none of the vertices on Q is adjacent to any of $\{x,y,z\}$.

Then if w_n is not adjacent to two vertices of $\{x,y,z\}$, say x and y, then we have that $\langle\ xyv+\ w_1w_2w_3+\ x^+{+}y^+\ \rangle$ is an induced $N(3,1,1)$, respectively $\langle\ xyv+\ w_1w_nw_+ +\ x^+{+}y^+\ \rangle$ if $n=2$ is an induced $N(3,1,1)$. Thus, suppose that w_n is adjacent to at least two vertices of $\{x,y,z\}$, and, thus as argued before is adjacent to all of them.

Subcase 2.2 *Assume that the path Q is just an edge.*

Suppose that $Q = \{vw\}$ and suppose without loss of generality that $w <_C x$. Under all these possibilities and with respect to all our conditions pick w such that w is closest to x. Then we immediately observe (else we are done) that wx, xy, $xz \in E(G)$ (see Figure 2.11).

Assume that there are two vertices of $\{x,y,z,w\}$ that have distance 4 along C to each other. (Note, they all have to have distance at least 4 due to Lemma 2.3.) Suppose without loss of generality that $dist_C(x,y) = 4$. Then $\langle\ wyz+\ y^-y^{--}y^{-3}+\ z^-{+}w^-\ \rangle = \langle\ wyz+\ y^-y^{--}x^+ +\ z^-{+}w^-\ \rangle$ is an induced $N(3,1,1)$, since we cannot have any additional edges. Note, that we cannot have any of wy^-, wz^-, $wy^{-3} = wx^+$, wy^{--}, $yy^{-3} = yx^+$, yz^-, yw^-, zy^-, zy^{--}, $zy^{-3} = zx^+$, zw^-, since any of these would immediately imply that C is not a longest cycle. Note, if $yy^{--} \in E(G)$, then we obtain a cycle of the

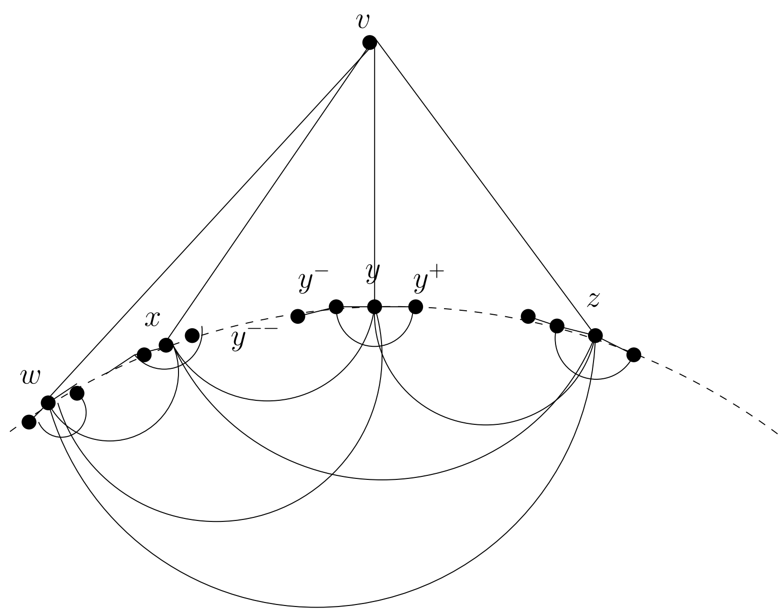


Figure 2.11:

same length as C, but where x and y have a smaller distance by using $y\ y^{--}$ $x^+\ x\ C\ y^+\ y^-\ y$.

If $z^-x^+ \in E(G)$, then $\langle\ x^+ + xy^{--}z^-\ \rangle$ is either an induced claw or we get one of xy^{--}, xz^- or $y^{--}z^- \in E(G)$, which all lead to a longer cycle than C. Similarly, we cannot have $w^-x^+ \in E(G)$. If $x^+y^- \in E(G)$, then $\langle\ wyz+$ $y^-y^{-3}x^-+\ z^-+w^-\ \rangle$ is an induced $N(3,1,1)$, since we already considered all possible additional edges. Thus, we get an induced $N(3,1,1)$.

Let us now suppose that all of $\{w, x, y, z\}$ have distance greater than 4 to each other.

Subcase 2.2.1 *Assume that* y^- *is not adjacent to any vertex other than* y^{--}

and y in xCy.

Then consider $\langle\ wyz{+}\ y^-y^{--}y^{-3}{+}\ z^-{+}w^-\ \rangle$. This is either an induced $N(3,1,1)$ or y^{-3} has more edges into that structure (since all other edges would immediately lead to a longer cycle than C or, as in the case of $yy^{--} \in E(G)$, to a cycle C^* of the same length as C, but with $dist_C(x,y) > dist_{C^*}(x,y)$). Thus, we check all possible edges from y^{-3} into that structure.

Subcase 2.2.2 *Suppose $yy^{-3} \in E(G)$.*

Then $\langle\ y + y^{-3}vy^-\ \rangle$ implies $y^-y^{-3} \in E(G)$, which is a contradiction to our assumption.

Subcase 2.2.3 *Suppose $wy^{-3} \in E(G)$.*

Then $\langle\ w\ +\ vw^+y^{-3}\ \rangle$ and $\langle\ w\ +\ vw^-y^{-3}\ \rangle$ imply $w^+y^{-3} \in E(G)$ and $w^-y^{-3} \in E(G)$. Then $\langle\ y^{-3} + y^{-2}y^{-4}w\ \rangle$ implies that at least one of $y^{-4}w$, $y^{-2}w$ or $y^{-2}y^{-4}$ are edges. Note, that $y^{-2}w \in E(G)$ would immediately lead to a longer cycle than C. If $y^{-2}y^{-4} \in E(G)$, then by using $y\ y^-\ y^{--}\ y^{-4}\ C^-$ $w\ y^{-3}\ w^-\ C^-\ y$ we get a cycle of the same length as C, but where the distance between the vertices x and y on this cycle is shorter. If $y^{-4}w \in E(G)$, then by using $y\ C^-\ y^{-3}\ w\ y^{-4}\ C^-\ x\ C^-\ w^+\ w^-\ C^-\ y$ we get a cycle of the same length as C, but by relabeling the vertices w, x and y, we get a cycle where x and y have a shorter distance on the cycle to each other. Thus, $wy^{-3} \notin E(G)$.

Subcase 2.2.4 *Suppose $zy^{-3} \in E(G)$.*

We can follow the same argument as in the previous case to obtain a contradiction.

Subcase 2.2.5 *Suppose $y^-y^{-3} \in E(G)$.*

This is not possible by assumption.

Subcase 2.2.6 *Suppose $w^-y^{-3} \in E(G)$.*

Then let us consider $\langle\ xyz+\ y^-y^{--}y^{-3}+\ x^-+z^-\ \rangle$. By following the same argument as above we get that this is either an induced $N(3,1,1)$ or we have one of $x^-y^{-3} \in E(G)$ or z^-y^{-3}. Suppose, without loss of generality, that $x^-y^{-3} \in E(G)$. Then $\langle\ y^{-3} + x^-w^-y^{-2}\ \rangle$ is an induced claw and we get a contradiction.

Subcase 2.2.7 *Suppose $z^-y^{-3} \in E(G)$.*

With the same argument as in the previous case we get a contradiction.

Subcase 2.3 *Suppose y^-has neighbors other than y^{--} and y in xCy.*

Pick $t \in V(xCy), ty^- \in E(G)$ such that $dist_C(y,t)$ is as large as possible (Figure 2.12).

Subcase 2.3.1 *Suppose that $t \neq x^+$, $t \neq y^{-3}$.*

Consider $\langle\ wyz+\ y^-tt^-+\ w^-+z^-\ \rangle$. We claim that this is an induced $N(3,1,1)$. To show this, let us consider any additional edges within that structure. Note, that we only need to consider the edges involving t and t^-, since all others lead to a longer cycle than C.

Subcase 2.3.1.1 *Suppose that $yt \in E(G)$.*

Then $\langle\ y\ +\ y^+vt\ \rangle$ implies $ty^+ \in E(G)$ and $\langle\ t\ +\ t^+t^-y^-\ \rangle$ implies $t^+y^- \in E(G)$ or $t^+t^- \in E(G)$, since $t^-y^- \in E(G)$ would contradict our choice of t. If $t^+t^- \in E(G)$ we can shorten the distance between x and y on C by using $y^+\ t\ y\ y^-\ C^-\ t^+\ t^-\ C^-\ x$, thus we cannot have $t^+t^- \in E(G)$. If we suppose that $t^+y^- \in E(G)$ then considering $\langle\ t\ +\ t^+t^-y\ \rangle$ implies $t^+y \in E(G)$, since $t^-y \in E(G)$ implies using $\langle\ y\ +\ vy^-t^-\ \rangle$ that $t^-y^- \in E(G)$, contradicting our choice of t. But then we can again shorten $dist_C(x,y)$ by using $y^-\ y^+\ C$ $t^+\ y\ t\ t^-\ C\ x$. Thus, $yt \notin E(G)$.

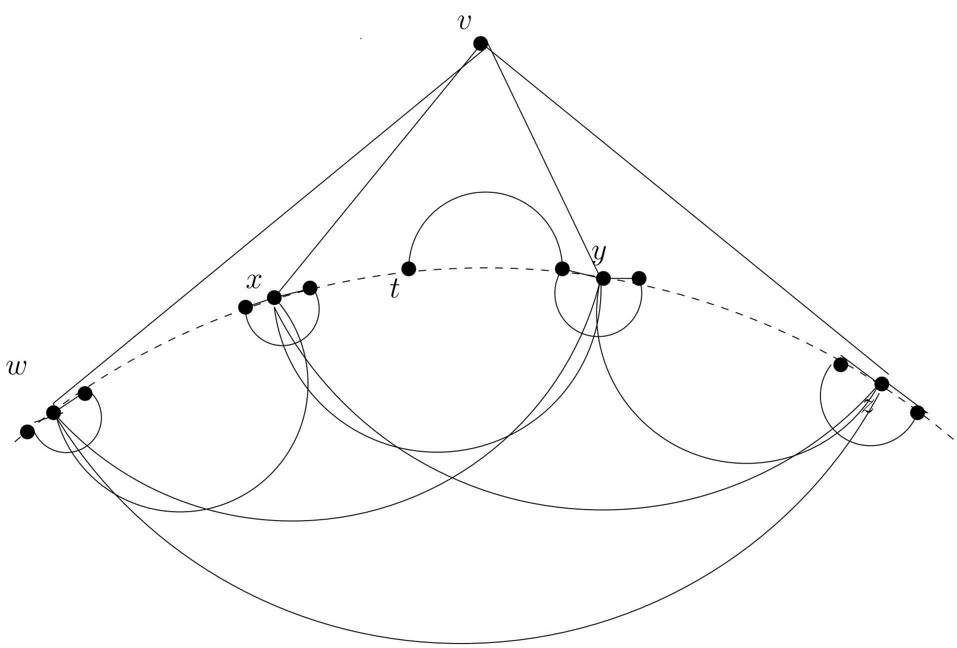


Figure 2.12:

Subcase 2.3.1.2 *Suppose that $wt \in E(G)$.*

By consideration of $\langle\ w + vw^{+}t\ \rangle$ and $\langle\ w + vw^{-}t\ \rangle$ show that $w^{+}t \in E(G)$ and $w^{-}t \in E(G)$. Additionally, $\langle\ t + t^{-}y^{-}w\ \rangle$ implies $t^{-}w \in E(G)$. But then by using $y\ C^{-}\ t\ w\ t^{-}\ C^{-}\ x\ C^{-}\ w^{+}\ w^{-}\ C^{-}\ y$ and by relabeling w as y, we obtain a cycle with a shorter distance between x and y. Thus, $wt \notin E(G)$.

Subcase 2.3.1.3 *Suppose that $zt \in E(G)$.*

With the same argument as in the case $wt \in E(G)$, we obtain a contradiction and hence, $zt \notin E(G)$.

Subcase 2.3.1.4 *Suppose that $w^-t \in E(G)$.*

Then $\langle\ xyz{+}\ y^-tw^-{+}\ z^-{+}x^-\ \rangle$ is an induced $N(3,1,1)$, since any additional edges would lead to a longer cycle or an induced claw. (For example the edge tx would lead to $\langle\ t + xy^-w^-\ \rangle$.) Thus, $w^-t \notin E(G)$.

Subcase 2.3.1.5 *Suppose that $z^-t \in E(G)$.*

Similar, to the previous case, we obtain a contradiction.

Subcase 2.3.1.6 *Suppose that $wt^- \in E(G)$.*

Cosidering $\langle\ w + vt^-w^+\ \rangle$ and $\langle\ w + vt^-w^-\ \rangle$ implies w^+t^-, $w^-t^- \in E(G)$. Then considering $\langle\ t^- + tt^{--}w\ \rangle$ implies $tt^{--} \in E(G)$ or $tw \in E(G)$ or $t^{--}w \in E(G)$. If $tt^{--} \in E(G)$ then using $y\ C^-\ t\ t^{--}\ C^-\ x\ C^-\ w^+\ t^-\ w\ C^-\ y$ we can shorten $dist_C(x, y)$, which is a contradiction. Since we already considered the case $wt \in E(G)$, we just need to consider if $wt^{--} \in E(G)$. But then by using $x\ C\ t^{--}\ w\ t^-\ C\ y\ C\ w^+\ w^-\ C$ and by relabeling we can again get a cycle with shorter distance between x and y and thus get a contradiction.

Subcase 2.3.1.7 *Suppose that $zt^- \in E(G)$.*

Similar to the case $wt^- \in E(G)$, we can get a contradiction.

Subcase 2.3.1.8 *Suppose that $w^-t^- \in E(G)$.*

Note first, that we cannot have $t^-z^- \in E(G)$, otherwise $\langle\ t^- + w^-z^-t\ \rangle$ is an induced claw. Consider $\langle\ xyz{+}\ y^-tt^-{+}\ x^-{+}z^-\ \rangle$. We claim that this is an induced $N(3,1,1)$. Note, that the only critical edges (the rest have already been considered) are the edges xt, x^-t, xt^-, x^-t^-. If we suppose that $xt \in E(G)$, then we can get a longer cycle by using $v\ x\ t\ C\ w^-\ t^-\ C^-$

$x^+ \ x^- \ C^- \ w \ v$, which is a contradiction. If we assume that $x^-t \in E(G)$, then $\langle\ wyz\text{+}\ y^-tx^-\text{+}\ z^-\text{+}w^-\ \rangle$ is an induced $N(3,1,1)$. If we suppose that $t^-x^- \in E(G)$, then $\langle\ t^- + tx^-w^-\ \rangle$ is an induced claw and if we suppose that $t^-x \in E(G)$, then $\langle\ x + x^-t^-v\ \rangle$ implies $x^-t^- \in E(G)$, respectively we get an induced claw. Thus, we cannot have $w^-t^- \in E(G)$.

Subcase 2.3.1.9 *Suppose that $z^-t^- \in E(G)$.*

Similar to the case when $w^-t^- \in E(G)$, we can get a contradiction.

Subcase 2.3.2 *Suppose that $t = x^+$.*

Then consider $\langle\ wyz\text{+}\ y^-x^+x^-\text{+}\ w^-\text{+}z^-\ \rangle$. This is either an induced $N(3,1,1)$ or we have one of $w^-x^+ \in E(G)$ or $z^-x^+ \in E(G)$. But both lead to an induced claw with $\langle\ x^+ + y^-x^-w^-\ \rangle$ respectively $\langle\ x^+ + y^-x^-z^-\ \rangle$. Thus, we get a contradiction in that case.

Subcase 2.3.3 *Suppose that $t = y^{-3}$.*

We claim that $\langle\ wyz\text{+}\ y^-y^{-3}y^{-4}\text{+}\ w^-\text{+}z^-\ \rangle$ is an induced $N(3,3,1)$. To see this, we consider all additional possible edges from y^{-3} and y^{-4} into that structure. Note, that $y^{-4}y^- \in E(G)$ would be an immediate contradiction to our assumption or the choice of t. Thus, $yy^{-4} \notin E(G)$ because otherwise $\langle\ y + vy^-y^{-4}\ \rangle$ implies again $y^{-4}y^- \in E(G)$, which is a contradiction. If $yy^{-3} \in E(G)$, then $\langle\ y + vy^+y^{-3}\ \rangle$ implies $y^{-3}y^+ \in E(G)$. If we then consider $\langle\ y^{-3} + y^{-2}y^{-4}y\ \rangle$ we get that $yy^{-2} \in E(G)$ or $y^{-2}y^{-4} \in E(G)$. If the first one holds, we can again get a cycle of same length as C but with a distance of x and y which is shorter on that cycle than on C, contradicting our choice of C. But if we assume that $y^{-2}y^{-4} \in E(G)$, we can also shorten the distance between x and y by using $y^+ \ y^{-3} \ y \ y^- \ y^{--} \ y^{-4} \ C \ x \ C \ y^+$. Thus, $yy^{-3} \notin E(G)$ as well.

But now for all other edges, that is for $y^{-3}w$, $y^{-3}z$, $y^{-3}w^{-}$, $y^{-3}z^{-}$, $y^{-4}w$, $y^{-4}z$, $y^{-4}w^{-}$ and $y^{-4}z^{-}$ we can use the same argumentation as for the case 2.3.1 by replacing in the argument t with y^{-3} to get a contradiction.

Thus, the theorem follows.

$\square$

Chapter 3

Hamiltonicity of 4-connected, $\{K_{1,3}, N(3,2,0)\}$-free, $\{K_{1,3}, N(4,1,0)\}$-free and $\{K_{1,3}, N(5,0,0)\}$-free graphs

In this section, we will show that 4-connected, $\{K_{1,3},\ N\}$-free graphs are hamiltonian, where $N = N(3,2,0)$, $N = N(4,1,0)$ or $N = N(5,0,0)$. To show that, we will use a result of Łuczak and Pfender ([16]). They showed the following theorem:

Theorem 3.1 *Every 3-connected $\{K_{1,3},\ P_{11}\}$-free graph is hamiltonian.*

In all the proofs of this section we will assume, for the purpose of a contradiction that the graph is not hamiltonian and therefore contains an induced P_{11}. We will then consider the neighborhood of the vertices of that induced path and from there we will obtain a contradiction. To show that a 4-connected, $\{K_{1,3},\ N(3,2,0)\}$-free graph is hamiltonian, we will follow an identical argument as Gould, Łuczak and Pfender used in [13], to show that a 3-connected, $\{K_{1,3},\ N(3,1,0)\}$-free graph is hamiltonian. That is, we will consider a smallest non-hamiltonian $\{K_{1,3}, N(3,2,0)\}$-free graph. By considering the neighborhood of the induced P_{11} we will be able to construct a smaller non-hamiltonian $\{K_{1,3}, N(3,2,0)\}$-free graph and therefore obtain a contradiction.

Theorem 3.2 *Let $G = (V, E)$ be a 4-connected graph that is $\{K_{1,3}$, $N(3,2,0)\}$-free. Then G is hamiltonian.*

Proof. Suppose that G is a non-hamiltonian $\{K_{1,3}, N(3,2,0)\}$-free graph. Let G be such a graph with the minimum number of vertices. Then by [16] we know that G contains an induced P_{11}. Let $P = v_1v_2...v_{11}$ be an induced P_{11}. Since G is claw-free, every vertex $w \in V(G) \setminus V(P)$ that is adjacent to v_i, $\quad i \in \{2, 3, ..., 10\}$, is adjacent to one of v_{i-1} or v_{i+1}. Since G contains no induced copy of $N(3,2,0)$, we get that $|N(w) \cap V(P)| \geq 3$, unless $N(w) \cap V(P)$ is one of $\{1,2\}, \{2,3\}, \{9,10\}$ or $\{10,11\}$. If w is adjacent to three non-consecutive vertices of $\{v_2, v_3, ..., v_{10}\}$, then since G is claw-free $|N(w) \cap V(P)| \geq 4$.

Claim 3.2.1 *If w is adjacent to 3 non-consecutive vertices of $\{v_3, v_4, ..., v_9\}$ we get an induced copy of $N(3,2,0)$.*

Proof of Claim 3.2.1: Since G is clawfree we immediately have that w is adjacent to precisely 4 vertices on the path. If these 4 vertices are consecutive, it is easy to see that no matter where on the path they are, we can always obtain an induced $N(3,2,0)$. Suppose that w is adjacent to the vertices v_i, v_j, v_k and v_l on P. Observe that if v_i, v_j, v_k, v_l are not all consecutive on P, then we can, without loss of generality, assume that v_i and v_j are neighbors on P and also that v_k and v_l are neighbors on P. If these two pairs are separated on the path by one, two or three vertices, it is again easy to see that we get an induced $N(3,2,0)$. But also if these two pairs are separated by more than three vertices we get an induced $N(3,2,0)$ which proves the claim. $\square_{Claim\ 3.2.1}$

Thus, each vertex $w \in V(G) \setminus V(P)$ which is adjacent to one of $\{v_4$, v_5,

v_6, v_7, $v_8\}$ has precisely three neighbors on P namely v_{i-1}, v_i, v_{i+1} for some $i \in \{3, 4, ..., 9\}$. Let us now define $V_i := \{v_i\} \cup \{w \in V(G) \setminus V(P) | N(w) \cap V(P) = \{v_{i-1}, v_i, v_{i+1}\}\}$ for $i \in \{4, 5, 6, 7, 8\}$.

Claim 3.2.2 (*i*) *The path* $v_1.....v_{i-1}v_i'v_{i+1}.....v_{11}$ *is induced for all* $i \in \{4,5,...,8\}$, $v_i' \in V_i$.

(*ii*) *Every two vertices of* $V_i, i \in \{4, 5, 6, 7, 8\}$ *are adjacent.*

(*iii*) *All vertices of* V_i *and* V_{i+1}, $i \in \{4, 5, 6, 7\}$ *are adjacent.*

(*iv*) $N(V_i) = V_{i-1} \cup V_{i+1}$ *for* $i \in \{5, 6, 7\}$.

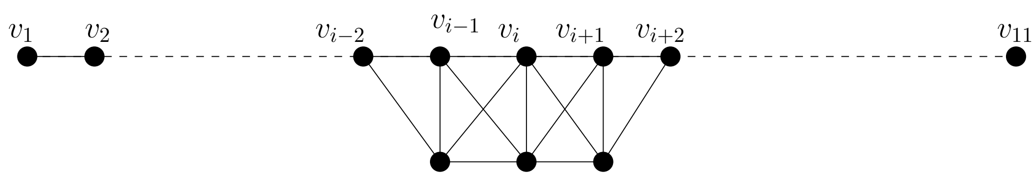


Figure 3.1:

Proof of Claim 3.2.2: As noted before, we have that each $v_i' \in V_i \setminus \{v_i\}$ has exactly three neighbors on P which are v_{i-1}, v_i, v_{i+1} for $i \in \{4, 5, 6, 7, 8\}$. Therefore (i) follows immediately. To see (ii), let us consider v_i' and $v_i'' \in V_i$. Considering $\langle\ v_{i+1} + v_{i+2}v_i'v_i''\ \rangle$ and recalling that G is clawfree and (i) holds, we obtain that $v_i'v_i'' \in E(G)$, which shows (ii). To see (iii), let $v_i' \in V_i$, $v_j' \in V_j \setminus \{v_j\}$ for $4 \leq i < j \leq 8$. By (i) we know that $v_1...v_{i-1}$ v_i' $v_{i+1}...$ v_{11} is an induced P_{11}, thus v_j must have precisely three consecutive neighbors on that path. Hence, we get that $v_i'v_j' \in E(G)$ if $j = i + 1$, otherwise they are not adjacent, which shows (iii). To see (iv), observe that if $v_i' \in V_i$ for $i \in \{5, 6, 7\}$ has a neighbor $w \in V(G) \setminus V(P)$.

Then due to $\langle\ v_i' + wv_{i-1}v_{i+1}\ \rangle$, we get that w must have a neighbor on P. Thus, $w \in V_{i-1} \cup V_i \cup V_{i+1}$. $\square_{Claim\ 3.2.2}$

Let us construct a new graph G' with fewer vertices than G which is also not hamiltonian and $\{K_{1,3}, N(3,2,0)\}$-free. To do this we delete all vertices from G that are in V_6 and connect all vertices from V_5 with all vertices from V_7. Then the graph G' obtained by this process is again claw-free and also does not contain an induced copy of $N(3,2,0)$, which can be seen by taking Claim 3.2.2 into consideration. By assumption we had that G was a non-hamiltonian $\{K_{1,3}, N(3,2,0)\}$-free graph with the smallest number of vertices, thus we get that G' has to be hamiltonian. But each hamiltonian cycle of G' can easily be extended to a hamiltonian cycle of G (again, by taking Claim 3.2.2 into consideration) which gives us the desired contradiction.

$\square$

To obtain hamiltonicity of a 4-connected $\{K_{1,3},\ N(4,1,0)\}$-free graph, we will once more assume non-hamiltonicity of the graph and then follow the same construction as in Theorem 3.2.

Theorem 3.3 *Let $G = (V, E)$ be a 4-connected graph that is $\{K_{1,3}$, $N(4,1,0)\}$-free. Then G is hamiltonian.*

Proof. Suppose that G is a non-hamiltonian $\{K_{1,3}, N(4,1,0)\}$-free graph. Let G be such a graph with the minimum number of vertices. Then by [16] we know that G contains an induced P_{11}. Let $P = v_1v_2...v_{11}$ be an induced P_{11}. Since G is claw-free, every vertex $w \in V(G) \setminus V(P)$ that is adjacent to $v_i,\quad i \in \{2, 3, ..., 10\}$ is adjacent to one of v_{i-1} or v_{i+1}. Since G contains no induced copy of $N(4,1,0)$, we get that $|N(w) \cap V(P)| \geq 3$, unless $N(w) \cap V(P)$ is one of $\{1,2\}$ or $\{10,11\}$. If w is adjacent to three non-consecutive vertices of $\{v_2,\ v_3,\ ...\ ,\ v_{10}\}$, then since G is claw-free $|N(w) \cap V(P)| \geq 4$.

Claim 3.3.1 *If w is adjacent to* 3 *non-consecutive vertices of* $\{v_2, v_4, ..., v_{10}\}$ *we get an induced copy of* $N(4,1,0)$.

Proof of Claim 3.3.1: Since G is clawfree we immediately have that $w \in V(G) \setminus V(P)$ is adjacent to precisely 4 vertices on the path. If these 4 vertices are consecutive, it is easy to see that no matter where on the path they are, we can always obtain an induced $N(4,1,0)$. Suppose that w is adjacent to the vertices v_i, v_j, v_k and v_l on P. Observe that if v_i, v_j, v_k, v_l are not all consecutive on P, then we can, without loss of generality, assume that v_i and v_j are neighbors on P and also that v_k and v_l are neighbors on P. If these two pairs are separated on the path by one, two, three or more than three vertices, it is again easy to see that we get an induced $N(4,1,0)$. That proves the claim. $\square_{Claim\ 3.3.1}$

Thus, each vertex $w \in V(G) \setminus V(P)$ which is adjacent to one of $\{v_3$, v_5, v_6, v_7, $v_9\}$ has precisely three neighbors on P namely v_{i-1}, v_i, v_{i+1} for some $i \in$ {3,4,....,9}. Let us now define $V_i := \{v_i\} \cup \{w \in V(G)\setminus V(P) | N(w) \cap V(P) = \{v_{i-1}, v_i, v_{i+1}\}\}$ for $i \in \{3, 5, 6, 7, 9\}$. Since the proof of Claim 3.2.2 in Theorem 3.2 only needed the 4-connectivity and claw-freeness of the graph, we obtain that Claim 3.2.2 holds here as well. By following the same argument as in Theorem 3.2 we obtain the theorem.

$\square$

To obtain hamiltonicity of a 4-connected $\{K_{1,3}, N(5,0,0)\}$-free graph, we will once more assume non-hamiltonicity of the graph. Then we will consider an induced P_{11} and by considering the neighborhood of that path we will show that there exists a cut-set of size three in the graph.

Theorem 3.4 *Let $G = (V, E)$ be a 4-connected $\{K_{1,3},\ N(5,0,0)\}$-free graph. Then G is hamiltonian.*

Proof. Suppose G is a non-hamiltonian $\{K_{1,3}, N(5,0,0)\}$-free graph. Then by Theorem 3.1 we know that G contains an induced P_{11}, say $P = v_1v_2...v_{11}$. Since G is 4-connected, v_6 has to have a neighbor off the path, say v. Then because of $\langle\ v_6 + v_5v_7v\ \rangle$, v has to be adjacent to one of v_5 or v_7.

Suppose, without loss of generality, that v is adjacent to v_5. Note that v cannot be adjacent to a vertex in each of $\{v_1,\ v_2,\ v_3\}$ and $\{v_7,\ v_8,\ v_9,\ v_{10},\ v_{11}\}$ because otherwise we an induced claw is centered at v. Note, that if v is not adjacent to any of $\{v_7,\ v_8,\ v_9,\ v_{10},\ v_{11}\}$ we have an induced $N(5,0,0)$. Let i be the smallest index in the set $\{7,8,9,10,11\}$ such that $vv_i \in E(G)$. Suppose at first that $i \geq 8$ and $i \neq 11$. Then, to avoid a claw, v has to be adjacent to v_{i+1}. Then we either have an induced $N(5,0,0)$ or we additionally get that $vv_4 \in E(G)$. But then we get an induced claw centered at v: $\langle\ v + v_4v_6v_i\ \rangle$. Thus, v is not adjacent to any of $\{v_8, v_9, v_{10}\}$. We now want to show that v cannot be adjacent to v_{11} and thus can only be adjacent to v_7.

Let us suppose that $vv_{11} \in E(G)$. This implies that v is not adjacent to any other vertices except v_5, v_6 and v_{11} on P. By connectivity, v_1 has to have another neighbor off the path. Call such a neighbor w (see Figure 3.2). Then

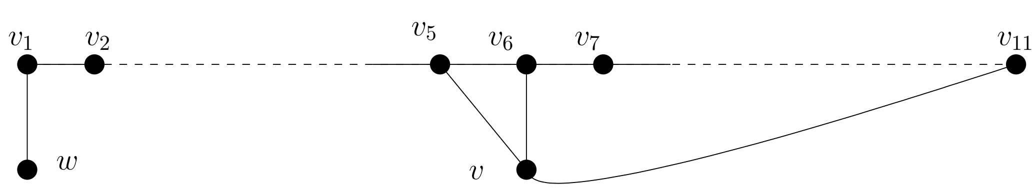


Figure 3.2:

we get that $wv \in E(G)$ or $wv_6 \in E(G)$ and $wv_7 \in E(G)$, since otherwise

$\langle\ vv_5v_6+\ v_4v_3v_2v_1w\ \rangle$ is an induced $N(5,0,0)$. (Note, that if w is adjacent to any of $\{v_2,\ v_3,\ v_4,\ v_5\}$, then we get, since G is clawfree that w also has to be adjacent to one of the neighbors of that vertex. Then either we get an $N(5,0,0)$ or a claw centered at w.)

Claim 3.4.1 *If $vv_{11} \in E(G)$, then $wv \notin E(G)$.*

Proof of Claim 3.4.1: Suppose $wv \in E(G)$. Since G is 4-connected, v_{11} has to have another adjacency off the path. Call that neighbor t. Due to $\langle\ v_{11} + v_{10}tv\ \rangle$ we need to have one of $tv_{10} \in E(G)$ or $tv \in E(G)$.

Claim 3.4.2 *$tv_{10} \notin E(G)$.*

Proof of Claim 3.4.2:

Suppose that $tv_{10} \in E(G)$. Then due to $\langle\ tv_{10}v_{11}+\ v_9v_8v_7v_6v_5\ \rangle$, t has to have more adjacencies in $\{v_9,\ v_8,\ v_7,\ v_6,\ v_5\}$. Then if t is adjacent to any z in $\{v_9,\ v_8,\ v_7\}$, then t also has to be adjacent to one of the neighbors of that vertex on the path P, since we always can get a triangle with t, z and one of the neighbors of z and this triangle together with the path P to v_1 gave a $N(5,0,0)$. Otherwise we clearly get a claw centered at t and thus a contradiction. If $tv_6 \in E(G)$ then either we get an induced $N(5,0,0)$ with $\langle\ tv_6v_7+\ v_5v_4v_3v_2v_1\ \rangle$ which is a contradiction or $tv_5 \in E(G)$ as well. But then because of $\langle\ tv_{10}v_{11}+\ v_5v_4v_3v_2v_1\ \rangle$ we get that t has to have another adjacency in $\{v_1,\ v_2,\ v_3,\ v_4\}$ which leads to a claw centered ad t. Thus, $tv_6 \notin E(G)$. Hence, we can suppose that $tv_5 \in E(G)$ and $tv_6 \notin E(G)$, which leads immediately to $tv_4 \in E(G)$ (and t cannot have any other neighbors on P, since that gives an induced claw centered at t). Then considering $N = \langle\ tv_{10}v_{11}+\ v_4v_3v_2v_1w\ \rangle$ leads to the conclusion that w has to have more neighbors in N.

Case 1 *Suppose that $wv_4 \in E(G)$ or $wv_3 \in E(G)$.*

Then we get a claw centered at w with $\langle\ w + v_1v_4v\ \rangle$, respectively with $\langle\ w + v_1v_3v\ \rangle$, and thus a contradiction.

Case 2 *Suppose that* $wt \in E(G)$.

Then $\langle\ t + wv_{10}v_5\ \rangle$ and $\langle\ t + wv_{11}v_5\ \rangle$ force the edges wv_{10} and wv_{11}. Note, that $wv_5 \notin E(G)$ has been already observed. But then we get an induced $N(5, 0, 0)$ with $\langle\ wv_{10}v_{11}+\ v_1v_2v_3v_4v_5\ \rangle$. Note, that we could destroy that particular net, if we have either another edge from w into $\{v_2, v_3, v_4, v_5\}$, which gives an induced claw centered at w or only the edge wv_2 which then leads to the net $\langle\ wv_{10}v_{11}+\ v_2v_3v_4v_5v_6\ \rangle$, which then leads to another edge from w into $\{v_3,\ v_4,\ v_5,\ v_6\}$, and thus gives an induced claw centered at w. Thus, we get a contradiction.

Case 3 *Suppose that* $wv_2 \in E(G)$

Then we either get an induced $N(5, 0, 0)$ with $\langle\ wv_1v_2+\ v_3v_4v_5v_6v_7\ \rangle$ or have additionally the edge wv_7, since all other edges have already been ruled out or can be ruled out with the same argument as before. But $\langle\ w + v_1v_7v\ \rangle$ induces a claw and thus leads again to a contradiction.

Case 4 *Suppose that* $wv_{10} \in E(G)$.

Them $\langle\ w + v_1vv_{10}\ \rangle$ is an induced claw in the graph, a contradiction.

Case 5 *Suppose that* $wv_{11} \in E(G)$.

Note that, due to connectivity, v_1 has to have a neighbor u other than v_2 and w in G. (Clearly, $u \neq v, u \neq t$, since otherwise we have a claw centered at t, respectively v.) Then if u is also not adjacent to v_6 and v_7 the same argument as before, shows that u has the same adjacencies (and non-adjacencies) as w. Then considering $\langle\ v_1 + v_2uw\ \rangle$ we get (since then $uv_2 \notin E(G)$ by the same argument as for w) that $uw \in E(G)$. But then $\langle\ uwv_1{+}\ v_2v_3v_4v_5v_6\ \rangle$ is an induced $N(5,0,0)$ or it leads to an induced claw at u or w. Now assume that $uv_6 \in E(G)$ and also $uv_7 \in E(G)$. Then $\langle\ uv_6v_7{+}\ v_8v_9v_{10}v_{11}w\ \rangle$ implies that $uw \in E(G)$, since all other additional edges lead to a claw or have already been considered. Now consider $\langle\ wuv_1{+}\ vv_5tv_{10}v_9\ \rangle$. Note first, that we cannot have any more edges from w into that structure, because either those would lead to an immediate claw or have been considered earlier. Also, any further edge from u into the path would lead to an induced claw. Thus, this structure is an induced $N(5,0,0)$, unless we have one of the following edges: wt, ut or uv, where we don't have $uv \in E(G)$ by assumption, respectively if we had that edge, then $\langle\ u + vv_1v_7\ \rangle$ is an induced claw. (Then we had the first case.) Note that $wt \in E(G)$ was already considered in case 2, and we got that $wt \notin E(G)$ holds. If, on the other hand, $ut \in E(G)$ then $\langle\ t + v_5v_{10}u\ \rangle$ is an induced claw.

Therefore, we cannot have the edge tv_{10} if $wv \in E(G)$ and vv_5, vv_6, $vv_{11} \in E(G)$. $\square_{Claim\ 3.4.2}$

Thus, we can now assume that $tv \in E(G)$ (Figure 3.3).

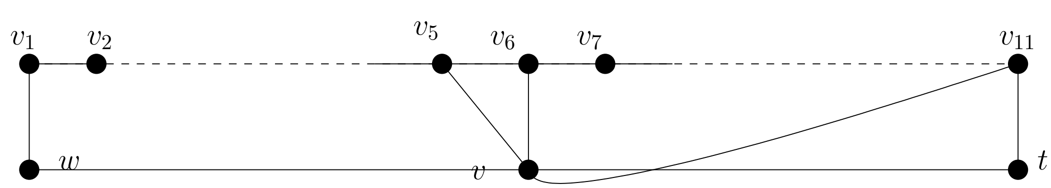


Figure 3.3:

Then $\langle\ v_{11}tv{+}\ v_5v_4v_3v_2v_1\ \rangle$ implies that t has to be adjacent to one of $\{v_5, v_4, v_3, v_2, v_1\}$.

Case 6 *Suppose $tv_2 \in E(G)$.*

Then $\langle\ v_2 + v_1v_3t\ \rangle$ implies that $tv_1 \in E(G)$ or $tv_3 \in E(G)$. But this triangle along with a 5-path along P ($v_3\ v_4\ v_5\ v_6\ v_7$ or $v_4\ v_5\ v_6\ v_7\ v_8$) forces t to have more adjacencies on P (else we get an induced $N(5, 0, 0)$) which then leads to an induced claw centered at t and thus a contradiction.

Case 7 *Suppose $tv_3 \in E(G)$ or $tv_4 \in E(G)$.*

Similar to the case $tv_2 \in E(G)$, we can get a contradiction.

Case 8 *Suppose $tv_1 \in E(G)$*

Then $\langle\ v_1\ +\ v_2wt\ \rangle$ implies that we have one of $tw \in E(G)$, $tv_2 \in E(G)$ or $wv_2 \in E(G)$.

Subcase 8.1 *Suppose $tw \in E(G)$.*

Then $\langle\ v_1wt{+}\ v_{11}v_{10}v_9v_8v_7\ \rangle$ implies that w or t has to have additional neighbors in $\{v_7,\ v_8,\ v_9,\ v_{10},\ v_{11}\}$. If t is adjacent to any of $\{v_7,\ v_8,\ v_9\}$, we immediately get an induced claw centered at t and $tv_{10} \notin E(G)$ by Claim 3.4.2. Thus, t cannot have more adjacencies into $\{v_7,\ v_8,\ v_9,\ v_{10}\}$ and w has to have adjacencies into $\{v_7,\ v_8,\ v_9,\ v_{10},\ v_{11}\}$.

If $wv_{10} \in E(G)$, then also $wv_9 \in E(G)$ or $wv_{11} \in E(G)$. But then $\langle\ wv_9v_{10}{+}\ v_8v_7v_6v_5v_4\ \rangle$, respectively $\langle\ wv_{11}v_{10}{+}\ v_9v_8v_7v_6v_5\ \rangle$ is an induced $N(5, 0, 0)$, and any additional edge leads to an induced claw centered at w. Similarly, there cannot be the edges wv_8 or wv_9. Also, $wv_7 \notin E(G)$, since otherwise $\langle\ w\ +\ v_7v_1v\ \rangle$ is an induced claw. Thus, assume that $wv_{11} \in E(G)$. Now $\langle\ wvt{+}\ v_6v_7v_8v_9v_{10}\ \rangle$ is an induced $N(5, 0, 0)$ unless one of w or t is adjacent to at least one of $\{v_6, v_7, v_8, v_9, v_{10}\}$. Note that we already ruled out all the edges from w into that set, except wv_6 and wv_7, which we do not have by our initial assumption. (Observe that it is not possible to have both $wv \in E(G)$ and $wv_6, wv_7 \in E(G)$, since $\langle\ w\ +\ v_1vv_7\ \rangle$ is an induced claw and otherwise we get in addition $wv_5 \in E(G)$ which leads by previous arguments to an induced $N(5,0,0)$ or an induced claw.) But we also cannot have any further edges from t into that set, since all would lead to an induced claw or have been considered earlier (i.e. $tv_{10} \notin E(G)$). Therefore, we obtain a contradiction and $tw \notin E(G)$.

Subcase 8.2 *Suppose* $wv_2 \in E(G)$.

Then considering $\langle\ wv_1v_2+\ v_3v_4v_5v_6v_7\ \rangle$ forces at least one additional edge from w into that structure. Note that for all edges except wv_7 we can modify the structure above to get either an induced $N(5,0,0)$ or get more edges from w to P which will then lead to an induced claw centered at w. But if $wv_7 \in E(G)$ we get an induced claw centered at w with $\langle\ w\ +\ v_1vv_7\ \rangle$. Hence, $wv_2 \notin E(G)$.

Subcase 8.3 *Suppose* $tv_2 \in E(G)$.

This case has already been considered in Case 6 and leads to a contradiction.

Thus we obtain that $tv_1 \notin E(G)$.

Case 9 *Suppose* $tv_5 \in E(G)$.

Then $\langle\ v_5\ +\ v_4v_6t\ \rangle$ implies $tv_4 \in E(G)$ or $tv_6 \in E(G)$. If $tv_6 \notin E(G)$, then $\langle\ v_4v_5t+\ v_6v_7v_8v_9v_{10}\ \rangle$ is an induced $N(5,0,0)$ or forces more edges from t into that structure which all lead to a contradiction as considered previously. Thus, we can immediately assume that $tv_6 \in E(G)$ and also get, by the previous cases that t is not adjacent to any of $\{v_1,\, v_2,\, v_3,\, v_4,\, v_7,\, v_8,\, v_9,\, v_{10}\}$ (otherwise we obtain a case which we already treated or an induced claw centered at t). Hence, we obtain that t has the same neighborhood on the path P as v (they are "the same"), that is if $tv_5 \in E(G)$ then t and v have the same neighborhood on P under the assumption that $vw \in E(G)$.

Also note that if $wv \in E(G)$, then $wt \in E(G)$ as well, because otherwise by an earlier argument, w would be adjacent to v_6 and v_7 and that leads to a claw centered at w. But then $\langle\ wtv{+}\ v_6v_7v_8v_9v_{10}\ \rangle$ is an induced $N(5,0,0)$, which is a contradiction.

Thus, if $vv_{11} \in E(G)$ we get that $wv \notin E(G)$.

$\square_{\textit{Claim } 3.4.1}$

Therefore, assume $wv_6 \in E(G)$ and $wv_7 \in E(G)$. Again, v_{11} has to have a neighbor t off the path, with $t \neq v, t \neq w$. Then $\langle\ v_{11} + v_{10}tv\ \rangle$ implies an analogue to the first case that $vt \in E(G)$ or $v_{10}t \in E(G)$ has to hold.

Case 10 *Suppose* $vt \in E(G)$*.*

Considering $\langle\ v_{11}vt{+}\ v_6wv_1v_2v_3\ \rangle$ implies that we need to have one of the edges wv_2, tw, tv_1, tv_2 or tv_3, since otherwise we have an induced $N(5,0,0)$. But if $wv_2 \in E(G)$, $\langle\ wv_1v_2{+}\ v_7v_8v_9v_{10}v_{11}\ \rangle$ is an induced $N(5,0,0)$. If we suppose that $tw \in E(G)$, then we are precisely in the symmetric case of Claim 3.4.1 with w playing the role of v and t the role of w. Thus, we get that w cannot be adjacent to t. If we suppose that $tv_1 \in E(G)$, then $\langle\ v_1 + v_2tw\ \rangle$ implies (as the only possibility that has not been checked yet) that $v_2t \in E(G)$. But then $\langle\ v_1v_2t{+}\ v_3v_4v_5v_6v_7\ \rangle$ is an induced $N(5,0,0)$. Similarly, we cannot have $tv_2 \in E(G)$ or $tv_3 \in E(G)$. Thus, $vt \notin E(G)$.

Case 11 *Suppose* $v_{10}t \in E(G)$*.*

Then $\langle\ v_{10}v_{11}t{+}\ v_9v_8v_7v_6v_5\ \rangle$ implies that t has to be adjacent to one of $\{v_5$, v_6, v_7, v_8, $v_9\}$. If $tv_5 \notin E(G)$, then we can always find a new $N(5,0,0)$ in the graph. (Suppose $tv_i \in E(G)$, $i \in \{6,7,8,9\}$. Then t has to be adjacent to one of v_{i-1} or v_{i+1}. Take two adjacencies of t as described with i as small as possible in the indexset above, then t, these vertices and v_iPv_1 form a $N(5,0,0)$.) Hence, assume that $tv_5 \in E(G)$. Then one of tv_4 or tv_6 needs to be an edge. If $tv_4 \notin E(G)$, $\langle\ tv_{10}v_{11}+\ v_5v_4v_3v_2v_1\ \rangle$ is an induced $N(5,0,0)$ or forces an induced claw centered at t. Therefore, we can suppose that tv_5 and tv_4 are edges. But then $\langle\ v_{10}v_{11}t+\ v_9v_8v_7wv_1\ \rangle$ is an induced $N(5,0,0)$ and the only additional edge within that structure we have not yet considered is tw. But $\langle\ t\ +\ v_{10}wv_4\ \rangle$ is then an induced claw, which again gives us a contradiction.

Thus, $v_{10}t \notin E(G)$ and therefore we obtain that if there is a vertex v that is adjacent to v_5, v_6 and v_{11}, there cannot be a neighbor of v_1 off the path that is adjacent to any vertex of $\{v,\ v_5,\ v_6,\ v_4,\ v_3,\ v_2\}$. Thus we obtain an induced $N(5,0,0)$ with $\langle\ vv_5v_6+\ v_4v_3v_2v_1w\ \rangle$, which is a contradiction.

Therefore we can only have vertices v that are adjacent to the vertices v_5, v_6 and v_7, and thus to no other vertices on the path P except these. Since G is 4-connected, v has to have another neighbor off the path, say x is such a neighbor. Then $\langle\ v\ +\ xv_5v_7\ \rangle$ implies that $xv_5 \in E(G)$ or $xv_7 \in E(G)$. Let us suppose, without loss of generality that $xv_5 \in E(G)$. Note that the vertices $\{v_1,\ v_2,\ v_3,\ v_4,\ v_5,\ v,\ v_7,\ v_8,\ v_9,\ v_{10},\ v_{11}\}$ form an induced P_{11}. Thus, we can apply our previous argument and obtain that $xv_7 \in E(G)$.

Then $\langle\ v_5 + v_4v_6x\ \rangle$ implies that $xv_6 \in E(G)$ or $xv_4 \in E(G)$. If $xv_4 \in E(G)$, $\langle\ v_4v_5x+\ v_7v_8v_9v_{10}v_{11}\ \rangle$ implies that x has to have neighbors in $\{v_8, v_9, v_{10}, v_{11}\}$ which then leads to an induced claw centered at x whith this neighbor, v and v_4. Thus, $xv_6 \in E(G)$ and therefore all neighbors of v have the same adjacencies as v to the path P. But then $\{v_5, v_6, v_7\}$ is a 3-cut of G and we have a contradiction. Thus, G is hamiltonian.

$\square$

Chapter 4

Pancyclicity

4.1 Preliminary Lemmas

In this section we will prove all preliminary lemmas needed to show pancyclicity of 4-connected $\{K_{1,3},\ N(i,j,k)\}$-free graphs, with $i, j, k \geq 0$ and $i + j + k = 5$. At first, we will consider cycles that do contain chords and afterwards we will consider induced cycles. To show Lemma 4.2 we will follow the basic idea that was used in [13] to show that a 3-connected $\{T\}$-free graph, where T is one of $N(4,0,0)$, $N(3,1,0)$ or P_7, is pancyclic. In particular, we use this lemma from [13] to prove the theorem that follows.

Lemma 4.1 *Let $G = (V, E)$ be a claw-free graph with minimum degree three, and let C be a cycle of length t without edges between vertices of distance two on the cycle, for some $t \geq 5$. Set $X = \{v \in V(C)|\ v$ has no chord$\}$, and suppose for some chord xy of C we have $|X \cap V(xCy)| \leq 2$. Then G contains cycles C' and C'' of lengths $t-1$ and $t-2$, respectively.*

Theorem 4.2 *Let $G = (V, E)$ be a graph with n vertices. Let G be $\{K_{1,3}\}$-free and suppose G contains a cycle of length at least t with at least one chord and no cycles of length $t-1$, where $5 \leq t \leq n-1$. Then G contains an induced copy of $N(4,1,0)$, $N(5,0,0)$ and $N(3,2,0)$.*

Proof. Let C be a cycle of length t in G and let C have at least one chord and suppose there exist no cycles of length $t-1$ in G. Define the set $X := \{v \in V(C) | v \text{ has no chord}\}$. Pick y and $x \in V(C)$ such that $|V(xCy) \cap X|$ is minimal, and for no other chord $x'y'$ such that $x' \in V(x^+Cy^-)$, $y' \in V(y^+Cx^-)$, and $|V(xCy) \cap X| = |V(x'Cy') \cap X|$, we have $|V(x'Cy')| < |V(xCy)|$. Since, by assumption, G has no cycle of length $t-1$ we may assume that C contains no hops, that is no edges between vertices of the cycle that have distance two on the cycle. By Lemma 4.1 we know that if a claw-free graph with minimum degree 3 has a cycle of length t, with $t \geq 5$, and if for some chord xy of the cycle we have $|X \cap V(xCy)| \leq 2$, then G contains cycles of length $t-1$ and $t-2$. Therefore we can assume that $|X \cap V(xCy)| \geq 3$.

Claim 4.2.1 *The chord xy can be chosen such that $|V(xCy) \setminus \{x, y\}| \geq 5$.*

Proof of Claim 4.2.1: Suppose that this is not the case and let xy be a chord which minimizes $|V(xCy) \cap X|$ and suppose that $|V(xCy) \setminus \{x, y\}| < 5$.

Consider the cycle $C \setminus \{V(xCy) \setminus \{x, y\}\} := C'$ and find a chord uw in C' such that $|V(uCw) \cap X|$ is minimal and under those conditions $|V(uCw)|$ minimal. Note, that C' has to have a chord. Since considering $\langle\, x + x^-x^+y \,\rangle$ and recalling that G is clawfree forces the edge x^-y (since the other possible edges are either hops or shorter chords than xy). If $|V(uCw) \setminus \{u, w\}| \geq 5$ we are done immediately. Since any chord in C' is also a chord in C, we only need treat the cases that $|V(uCw) \setminus \{u, w\}| = 4$ or $|V(uCw) \setminus \{u, w\}| = 3$. We also have that $V(uCw) \cap X = \{v_1, v_2, v_3\}$ or $V(uCw) \cap X = \{v_1, v_2, v_3, v_4\}$. Since $\{v_1, v_2, v_3\} \in X$ and G is 4-connected, $\{v_1,\ v_2,\ v_3\}$, respectively $\{v_1,$ $v_2,\ v_3,\ v_4\}$ have to have neighbors in $V(G) \setminus V(C)$.

Case 1 *Let us suppose that $V(uCw) \cap X = \{v_1, v_2, v_3\}$ and let us assume that $|V(uCw) \setminus \{u, w\}| = 3$.*

It was shown in [13] that if $|V(xCy)| = 3$, $|V(uCw)| = 3$ is not possible, if G is a 3-connected claw-free graph which, for some $5 \leq t \leq |V(G)|$, contains a cycle of length t with at least one chord but contains no cycles of length $t-1$. Thus, let us suppose that $|V(xCy)| = 4$.

Subcase 1.1 *Assume that* $\{v_1, v_2, v_3\}$ *have* (*at least*) 3 *neighbors in* $V(G) \setminus V(C)$.

Call 3 of these neighbors w_1, w_2 and w_3. We may assume that $v_i w_i \in E(G)$. We will show that we can include the vertices $\{w_1, w_2, w_3\}$ in C' and thus obtain a cycle of length $t-1$, contradicting our assumption. Observe, that $\langle\, v_2 + v_1 v_3 w_2 \,\rangle$ forces us to get an additional edge $v_1 w_2$ or $v_3 w_2$. Without loss of generality assume that $v_1 w_2 \in E(G)$. The case $v_3 w_2 \in E(G)$ is symmetric and follows by the same argument (see Figure 4.1).

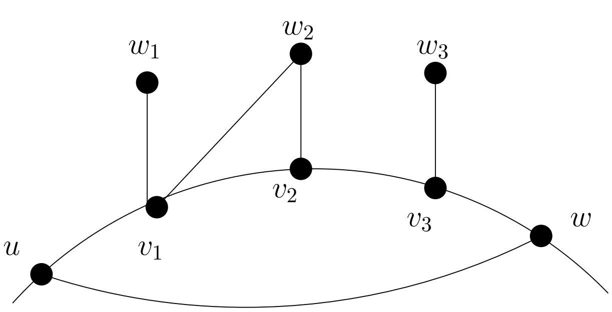


Figure 4.1:

Note that $\langle\, v_3 + w w_3 v_2 \,\rangle$ forces one of $v_2 w_3 \in E(G)$ or $w w_3 \in E(G)$. In both cases we can easily include the vertex w_3 to the cycle C'. Without loss of generality assume that we have the edge $v_2 w_3 \in E(G)$. Considering $\langle\, v_1 + w_1 u v_2 \,\rangle$ we get that we have the edge $u w_1$ or $v_2 w_1$. If $u w_1 \in E(G)$, we can easily include w_1, w_2 and w_3 to the cycle and are done, since $\langle C - V(x^+ C y^-) +$

$\{w_1, w_2, w_3\}\rangle$ is a cycle of length $t-1$. If $v_2w_1 \in E(G)$, we consider $\langle\ v_1 + w_1w_2u\ \rangle$. We therefore obtain that we have one of $uw_1 \in E(G)$, $uw_2 \in E(G)$ or $w_1w_2 \in E(G)$. If $uw_1 \in E(G)$ we are in the just treated case and if $uw_2 \in E(G)$ we can relabel w_1 and w_2 and are again in that same case. If $w_1w_2 \in E(G)$ we can include w_1, w_2 and w_3 as follows: $u\ v_1\ w_1\ w_2\ v_2\ w_3\ v_3\ C'\ u$ and get again a $(t-1)$-cycle. Thus, we have a contradiction, completing this case.

Subcase 1.2 *Assume that* $\{v_1, v_2, v_3\}$ *has* 2 *neighbors in* $V(G) \setminus V(C)$.

Call these neighbors w_1 and w_2. By connectivity of G we get that all of v_1, v_2 and v_3 have to be connected to both w_1 and w_2. Since G is 4-connected, the graph $G \setminus \{u, w\}$ is at least 2-connected. Thus, there are at least 2 paths from $\{v_1, v_2, v_3, w_1, w_2\}$ to $V(C)$. Since $\{v_1, v_2, v_3\} \in X$ we get that at least one of these paths has to use w_1. Consider the neighbor of w_1 on that path and call it z, $z \neq v_i, i \in \{1, 2, 3\}$ (see Figure 4.2).

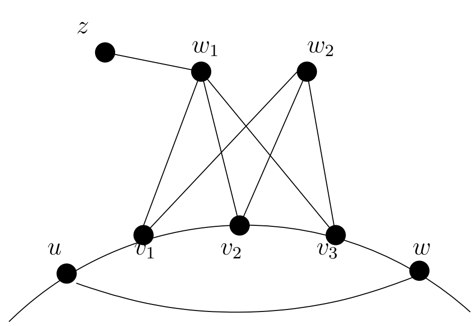


Figure 4.2:

Note, that $\langle\ w_1 + zv_1v_3\ \rangle$ forces one of $zv_1 \in E(G), zv_3 \in E(G)$ or $v_1v_3 \in E(G)$, otherwise we get an induced claw. Since C is a cycle without hops, we cannot have $v_1v_3 \in E(G)$. Furthermore, if $z \in V(C)$ and one of $zv_1 \in E(G)$

or $zv_3 \in E(G)$, we get a chord which contradicts that $\{v_1, v_2, v_3\} \in X$. If on the other hand $z \notin V(C)$, we get that $\{v_1, v_2, v_3\}$ have more than 2 neighbors off the cycle, which again gives a contradiction to our assumption.

Thus, we are done in the case that $|V(uCw)| = 3$.

Case 2 *Suppose that* $|V(uCw) \setminus \{u, w\}| = 4$ *and* $|V(uCw) \cap X| = 3$. *Assume furthermore that* $|V(xCy) \setminus \{x, y\}| = 4$.

Let us suppose that $V(uCw) \cap X = \{v_1, v_2, v_3\}$.

Subcase 2.1 *Assume that* $\{v_1, v_2, v_3\}$ *have (at least) 3 neighbors in* $V(G) \setminus V(C)$.

Call 3 of these neighbors w_1, w_2 and w_3, where we may assume that $v_i w_i \in E(G)$.

Subcase 2.1.1 *Let us suppose that* $V(uCw) = \{v_1 v_2 v_3 a\}$.

Observe, that $\langle\ v_2 + v_1 v_3 w_2\ \rangle$ forces us to get an additional edge $v_1 w_2$ or $v_3 w_2$. Without loss of generality, assume that we have the edge $v_1 w_2 \in E(G)$. The case $v_3 w_2 \in E(G)$ is symmetric and follows by an identical argument. Note, that $\langle\ v_1 + w_1 w_2 u\ \rangle$ is also a claw unless one of $uw_1 \in E(G)$ or $w_1 v_2 \in E(G)$, and $\langle\ v_3 + v_2 w_2 a\ \rangle$ is a claw unless $v_2 w_3 \in E(G)$ or $aw_3 \in E(G)$.

Subcase 2.1.1.1 *Assume that* $uw_1 \in E(G)$.

If $v_2 w_3 \in E(G)$ we can get a cycle of length $t - 1$ by inserting w_1, w_2 and w_3 into C and instead leaving out the vertices $V(xCy) \setminus \{x, y\}$. But if $aw_3 \in E(G)$ we can get a $(t-1)$-cycle in the same way.

Subcase 2.1.1.2 *Assume that* $w_1 v_2 \in E(G)$.

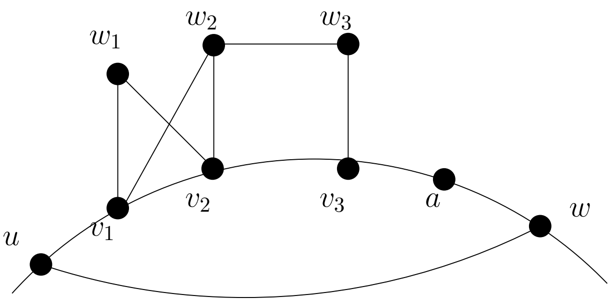


Figure 4.3:

If $w_3v_2 \in E(G)$ then $\langle\ v_2 + w_1w_2w_3\ \rangle$ forces at least one of w_1w_2, w_2w_3 or w_1w_3 to be an edge. If we have $w_2w_3 \in E(G)$ (see Figure 4.3) we can get a $(t-1)$-cycle as follows: $x\ C\ v_1\ w_1\ v_2\ w_2\ w_3\ v_3\ C\ y\ x$. If w_1w_2, we can get a $(t-1)$-cycle similarly. If w_1w_3, a $(t-1)$-cycle can again be obtained: $y\ C\ v_3\ w_3\ w_1\ v_2\ w_2\ v_1\ C\ x\ y$.

If on the other hand $aw_3 \in E(G)$, then first note that since G is 4-connected and $v_3 \in X$, v_3 has another neighbor in $V(G) \setminus V(C)$. If $w_2v_3 \in E(G)$, we obtain a cycle of length $t-1$ by inserting $v_1\ w_1\ v_2\ w_2\ v_3\ w_3\ a$ and leaving out the vertices of $xCy \setminus \{x, y\}$. Similarly, if $v_3w_1 \in E(G)$, we can insert $v_1\ w_2\ v_2\ w_1\ v_3\ w_3\ a$ and leave out the same vertices as above. Therefore suppose that there is another vertex $t \notin V(C)$ such that $v_3t \in E(G)$. Considering $\langle\ v_3 + tw_3v_2\ \rangle$ forces us to have one of $v_2t \in E(G)$, $w_3t \in E(G)$ or $v_2w_3 \in E(G)$. Since we already considered the case $v_2w_3 \in E(G)$ and since if $v_2t \in E(G)$ holds, we can just rename t as w_3 and have therefore also treated this case. The only case to consider is $w_3t \in E(G)$.

Suppose $w_3t \in E(G)$, then we can include into C the vertices w_2, w_3 and t as follows: $v_1\ w_2\ v_2\ v_3\ t\ w_3\ a$ and then by excluding the vertices of $xCy \setminus \{x, y\}$ as previously, we again obtain a cycle of length $t-1$, which is the contradiction we needed.

Subcase 2.1.2 *Suppose that $V(uCw) = \{v_1, a, v_2, v_3\}$, that is, the vertex a has neighbors v_i and v_j within the path uCw and without loss of generality we may assume that these neighbors are v_1 and v_2.*

By considering $\langle\ v_1 + uaw_1\ \rangle$, we are forced to get $uw_1 \in E(G)$ or $aw_1 \in E(G)$. By considering $\langle\ v_2 + av_3w_2\ \rangle$ we are forced to have one of $aw_2 \in E(G)$ or $v_3w_2 \in E(G)$ and by considering $\langle\ v_3 + wv_2w_3\ \rangle$ we are forced to have one of $v_2w_3 \in E(G)$ or $ww_3 \in E(G)$, otherwise we get an induced claw. If $aw_2 \in E(G)$ then in all cases $v_2w_3 \in E(G)$ or $ww_3 \in E(G)$ and $uw_1 \in E(G)$ or $aw_1 \in E(G)$, we can include the vertices w_1, w_2, w_3 in C and exclude the vertices of $V(xCy) \setminus \{x, y\}$, to obtain a cycle of length $t-1$. Thus, suppose that $w_2v_3 \in E(G)$. Note first that with both edges $uw_1 \in E(G)$ or $aw_1 \in E(G)$, we can include w_1 in the cycle. If $w_3w \in E(G)$, then we can include the vertices w_1, w_2, w_3 on the cycle and obtain, as before, a cycle of length $t-1$. Thus, suppose that $v_2w_3 \in E(G)$.

Since G is 4-connected, v_1 has to have another neighbor in $V(G) \setminus V(C)$ other than w_1.

Subcase 2.1.2.1 *Suppose that $v_1w_3 \in E(G)$ or $v_1w_2 \in E(G)$.*

If the latter is the case, we can get a $(t-1)$-cycle by using $x\ C\ v_1\ w_2\ v_3\ C\ x$. If the former is the case, we can get a $(t-1)$-cycle by using $x\ C\ v_1\ w_3\ v_3\ C\ x$.

Subcase 2.1.2.2 *Suppose $v_1t \in E(G)$, where $t \in \{V(G) \setminus \{V(C) \cup \{w_1, w_2, w_3\}\}\}$*

Note first, that $\langle\ v_1 + uat\ \rangle$ forces $ut \in E(G)$ or $at \in E(G)$, and $\langle\ v_1 + uaw_1\ \rangle$ forces $uw_1 \in E(G)$ or $aw_1 \in E(G)$, otherwise we get an induced claw. Suppose $ut \in E(G)$. If then $aw_1 \in E(G)$, we can include t, w_1 and one of w_2

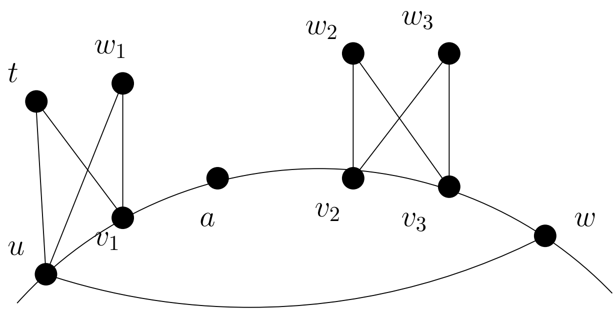


Figure 4.4:

or w_3 in the cycle and obtain with the previous arguments a $(t-1)$-cycle. Thus, suppose $uw_1 \in E(G)$ and $aw_1 \notin E(G)$ (see Figure 4.4). Considering $\langle\ v_1 + aw_1t\ \rangle$, we are forced to get one of $w_1t \in E(G)$ or $at \in E(G)$. If $w_1t \in E(G)$ we can again include t and w_1 and one of w_2 or w_3 to the cycle. If $at \in E(G)$ we can proceed similarly and obtain a $(t-1)$-cycle. (To include t and w_1 use $x\ C\ u\ w_1\ v_1\ t\ a\ w_2\ C\ y\ x$.) Hence, we are done in the case that $ut \in E(G)$.

Assume that $at \in E(G)$. If then $uw_1 \in E(G)$, we immediately can include t, w_1, w_2 in the cycle and proceed as before to get a $(t-1)$-cycle. Thus, suppose that $aw_1 \in E(G)$ and $uw_1 \notin E(G)$. Considering $\langle\ v_1 + uw_1t\ \rangle$ forces one of $ut \in E(G)$ or $tw_1 \in E(G)$. But in both cases we can include t and w_1 and one of w_2 or w_3 and then, as before, get a cycle of length $t-1$.

Thus we obtain by all of these cases that $\{v_1, v_2, v_3\}$ cannot have three neighbors in $V(G) \setminus V(C)$.

Subcase 2.2 *Suppose that* $\{v_1, v_2, v_3\}$ *have only two neighbors in* $V(G) \setminus V(C)$. *Call these neighbors* w_1 *and* w_2.

Since G is 4-connected we get immediately that $v_iw_j \in E(G)$ for $i \in \{1,2,3\}$ and $j \in \{1,2\}$. Since G is 4-connected, there have to be at least two paths from $\{v_1,\, v_2,\, v_3,\, a,\, w_1,\, w_2\}$ to $V(C)$, and there has to be a path that does not use the vertex a in $G \setminus \{u,w\}$. Since $\{v_1,\, v_2,\, v_3\} \in X$ we get that at least one of these paths has to use w_1 or w_2. Suppose without loss of generality that w_1 is the vertex which is used for that purpose. Then by following the exact same argument as in Case 1.2 we get a contradiction.

Case 3 *Suppose that* $|V(uCw) \setminus \{u,w\}| = 4$ *and* $|V(uCw) \cap X| = 3$*. Assume furthermore that* $|V(xCy) \setminus \{x,y\}| = 3$*.*

Suppose that $V(uCw) \cap X = \{v_1,\, v_2,\, v_3\}$. By the connectivity of G, $\{v_1,\, v_2,\, v_3\}$ must have at least two neighbors w_1 and w_2 off the cycle. Then it is easy to see from the previous case that we can obtain a cycle of length $t-1$, since we can (as described before) include w_1 and w_2 to the cycle C'.

Case 4 *Suppose that* $|V(uCw) \setminus \{u,w\}| = 4$ *and* $|V(uCw) \cap X| = 4$*. Assume furthermore that* $|V(xCy) \setminus \{x,y\}| = 4$*.*

Subcase 4.1 *Assume that* $\{v_1, v_2, v_3, v_4\}$ *has at least three neighbors in* $V(G) \setminus V(C)$*.*

Call three of these neighbors w_1, w_2 and w_3 where we may assume that $v_iw_i \in E(G)$. But now we can follow the same argument as in Case 1.1, where v_4 plays the role of w and again get a contradiction.

Subcase 4.2 *Assume that* $\{v_1,\, v_2,\, v_3,\, v_4\}$ *has two neighbors in* $V(G) \setminus V(C)$*.*

Again, we can follow the same argument as in Case 1.2 and get a contradiction.

Case 5 *Suppose that* $|V(uCw) \setminus \{u,w\}| = 4$ *and* $|V(uCw) \cap X| = 4$*. Assume furthermore that* $|V(xCy) \setminus \{x,y\}| = 3$*.*

Suppose that $V(uCw) \cap X = \{v_1, v_2, v_3, v_4\}$. By connectivity, $\{v_1, v_2, v_3, v_4\}$ must have at least two neighbors w_1 and w_2 off the cycle. Then it is easy to see from the previous case that we can obtain a cycle of length $t-1$, since we can easily include w_1 and w_2 in the cycle C'.

Therefore, we have shown that x and y can be chosen such that
$|V(xCy) \setminus \{x, y\}| \geq 5$. $\square_{Claim\ 4.2.1}$

Now consider $\langle\ x + x^-x^+y\ \rangle$. Since C has no hops and xy is a minimal chord, we get that $x^-x^+ \notin E(G)$ and $x^+y \notin E(G)$. Thus, $x^-y \in E(G)$, otherwise we have an induced claw. Similarly, considering $\langle\ y + xy^-y^+\ \rangle$ forces $y^+x \in E(G)$ (see Figure 4.5).

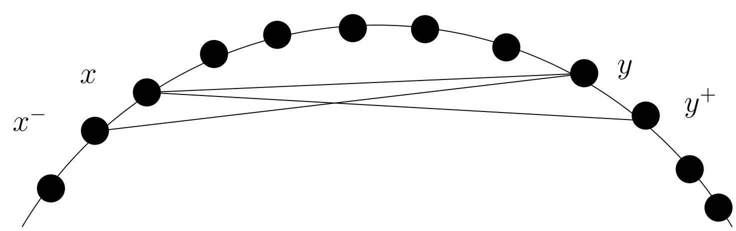


Figure 4.5:

Claim 4.2.2 *G contains an induced $N(4, 1, 0)$.*

Proof of Claim 4.2.2: We claim that $\langle\ yx^-x+\ y^-y^{--}y^{-3}y^{-4}+\ x^{--}\ \rangle$ forms an induced $N(4, 1, 0)$. If $x^{--}y$ is an edge, we can get a cycle of length $t-1$ by leaving out x^- as follows: $x^{--}\ y\ C\ x\ y^+\ C^-\ x^{--}$. Similarly, we do not have any of the edges $x^{--}y^-$, $x^{--}y^{--}$ (we could skip the vertex y^-), x^-y^- (we could skip the vertex y) and x^-y^{--} (we could skip the vertex y^-). Note that the edge xy^-, is a a shorter chord than xy which is a contradiction to our choice of xy. Similarly, we do not have any of the edges xy^{-4}, xy^{-3}, xy^{--}, x^-y^{-3}, x^-y^{-4}, $x^{--}x$, $x^{--}y^{-3}$ and $x^{--}y^{-4}$.

Hence, G contains an induced $N(4,1,0)$. $\square_{\textit{Claim 4.2.2}}$

Claim 4.2.3 *G contains an induced $N(5,0,0)$.*

Proof of Claim 4.2.3: We claim that $\langle\ xx^-y^+ +\ x^+x^{++}x^{+3}x^{+4}x^{+5}\ \rangle$ is an induced $N(5,0,0)$. All of x^-x^+, x^-x^{++}, x^-x^{+3}, x^-x^{+4}, xx^{++}, xx^{+3}, xx^{+4}, xx^{+5}, x^+x^{+3}, x^+x^{+4}, x^+x^{+5}, $x^{++}x^{+4}$, $x^{++}x^{+5}$, $x^{+3}x^{+5}$, $x^{+4}y^+$, $x^{+5}y^+$ would lead to a shorter chord than xy, contradicting our choice of that chord. Also, the edges x^-x^{+5} is either also a shorter chord than xy or $x^-x^{+5} = x^-y^-$, which leads to a $(t-1)$-cycle using $x^-\ y^-\ C^-\ x\ y^+\ C^-\ x^-$. Thus, we have an induced $N(5,0,0)$.

$\square_{\textit{Claim 4.2.3}}$

Claim 4.2.4 *G contains an induced copy of $N(3,2,0)$.*

Proof of Claim 4.2.4: Observe that if $|V(xCy) \setminus \{x,y\}| > 5$, we get an induced copy of $N(3,2,0)$ with $\langle\ yx^-x+\ y^-y^{--}y^{-3}+x^+x^{++}\ \rangle$. Thus suppose there is no short chord like that, that is, we have that $|V(xCy) \setminus \{x,y\}| = 5$. Consider $N := \langle\ yx^-x+\ y^-y^{--}y^{-3}+x^{--}x^{-3}\ \rangle$. By using Claim 4.2.2 we only need check edges involving x^{-3} to see if these vertices give an induced $N(3,2,0)$. If $x^{-3}y^- \in E(G)$, we can get a $(t-1)$-cycle, contradicting our assumption, by leaving out y as follows: $x^{-3}\ C^-\ y\ x^-\ C\ y^-\ x^{-3}$. Similarly, if $x^{-3}y \in E(G)$, we can get a cycle of length $t-1$ by leaving out x^{--} with $x^{-3}\ C\ y^+\ x^-\ C^-\ y\ x^{-3}$. Furthermore, the edges $x^{-3}x$ and $x^{-3}x^-$ are shorter chords in C, than the chord xy, a contradiction.

Thus, the only edges that could destroy the potential net are $x^{-3}y^{-3}$ and $x^{-3}y^{--}$. But $\langle\ y^{-3} + x^{-3}y^{--}y^{-4}\ \rangle$ is an induced claw, unless $x^{-3}y^{--} \in E(G)$, since the edges $x^{-3}y^{-4}$ and $y^{--}y^{-4}$ are shorter chords than the xy-chord. Similarly, $\langle\ y^{--} + y^{-}y^{-3}x^{-3}\ \rangle$ implies $x^{-3}y^{-3} \in E(G)$.

Considering $\langle\ x^{-3} + x^{--}x^{-4}y^{-3}\ \rangle$ and $\langle\ x^{-3} + x^{--}x^{-4}y^{-2}\ \rangle$ we are additionally forced to have $x^{-4}y^{--} \in E(G)$ and $x^{-4}y^{-3} \in E(G)$, otherwise we get an induced claw and all other possibilities are hops, lead to shorter chords or a $(t-1)$-cycle as shown before. Now we consider symmetrically to our initial net N, the net induced by $\langle\ xyy^{+}+\ x^{+}x^{++}x^{+3}+y^{++}y^{+3}\ \rangle$. By the same argument as for N, either these vertices form a $N(3,2,0)$ and we are done, or we have the edges $y^{+3}y^{-3}$, $y^{+3}y^{-4}$, $y^{+4}y^{-3}$ and $y^{+4}y^{-4}$. Let us now consider $N' := \langle\ xyy^{+}+\ y^{-}y^{--}x^{-4}+y^{++}y^{+3}\ \rangle$. We claim that N' is an induced copy of $N(3,2,0)$. Note that we only need check the edges involving x^{-4} since all the other possibilities have already been ruled out in this or a symmetric case. If $x^{-4}x$ is an edge, we have a shorter chord than xy, a contradiction. If $x^{-4}y^{-}$ is an edge, we can obtain a cycle of length $t-1$ by excluding y^{--} with x^{-4} y^{-} C y^{+3} y^{-3} x^{-3} C y^{-4} y^{+4} C x^{-4}. Similarly, if $x^{-4}y \in E(G)$, we can get a cycle of length $t-1$ by excluding the vertex y^{-} with x^{-4} y C^{-} y^{+3} y^{-3} y^{--} x^{-3} C y^{-4} y^{+4} C x^{-4}, and if $x^{-4}y^{+3} \in E(G)$, we can obtain a cycle of length $(t-1)$ (excludes y^{-3}) with x^{-4} y^{+3} C^{-} y^{-2} x^{-3} C y^{-4} y^{+4} C x^{-4}. Suppose that $x^{-4}y^{+} \in E(G)$. Then $\langle\ y^{+} + x^{-4}yy^{++}\ \rangle$ forces $x^{-4}y^{++} \in E(G)$, as yy^{++} is a hop and $x^{-4}y$ leads to a $(t-1)$-cycle as shown. But then we can again obtain a $(t-1)$-cycle by skipping y^{-3} with x^{-4} y^{++} C^{-} y^{--} x^{-3} C y^{-4} y^{+3} C x^{-4}. Finally, if we assume that $x^{-4}y^{++} \in E(G)$, then $\langle\ y^{++} + y^{+}y^{+3}x^{-4}\ \rangle$ implies that we have $x^{-4}y^{+} \in E(G)$ or $x^{-4}y^{+3} \in E(G)$, otherwise we have an induced claw. But both if these edges lead to a $(t-1)$-cycle as seen earlier. Thus, N' is an induced copy of $N(3,2,0)$ and the claim follows (see Figure 4.6). $\square_{\mathit{Claim}\ 4.2.4}$

$\square$

From that theorem we obtain immediately the following corollary.

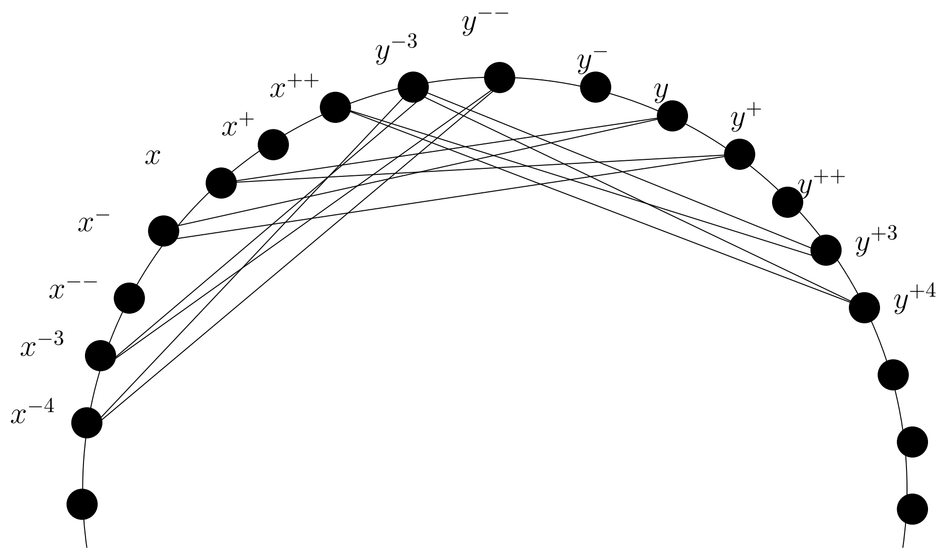


Figure 4.6:

Corollary 4.3 *If $G = (V, E)$ is a 4-connected $\{N,\ K_{1,3}\}$-free graph, with $N = N(4, 1, 0)$, $N = N(5, 0, 0)$ or $N = N(3, 2, 0)$, G is pancyclic, provided all cycles contain chords.*

In order to show the existence of a $(t-1)$-cycle if C_t is an induced cycle, we will use the following lemmas in the next section.

Lemma 4.4 *Let C_t, $t \geq 4$, be a cycle in a graph $G = (V, E)$. If there exists a vertex $w \notin V(C_t)$ such that w is adjacent to at least 4 consecutive vertices of C_t, then a C_{t-1} exists.*

Proof. Say $C_t = v_1\ v_2\ ....\ v_t$. Suppose, without loss of generality, that w is adjacent to v_1, v_2, v_3 and v_4. Then $v_1\ w\ v_4 C_t v_1$ is a $(t-1)$-cycle. $\square$

Lemma 4.5 *Let $G = (V, E)$ be a 4-connected, claw-free graph. Let C_t, $4 \leq t \leq 11$, be an induced cycle in G. Let w be a vertex off the cycle with four non-consecutive adjacencies on the cycle. Then G contains a C_{t-1}.*

Proof. Enumerate C_t with $v_1\ v_2\ ...\ v_t$. Without loss of generality, assume w is adjacent to v_1 and v_2. Furthermore assume that w is adjacent to v_i and v_{i+1}, $2 < i < t$. If $t \leq 9$ it is easy to see that two vertices of v_1, v_2, v_i and v_{i+1} must have distance at most three on the cycle. Suppose, without loss of generality, that these vertices are v_1 and v_4. Then $v_1\ w\ v_4 C_t v_1$ is a C_{t-1}.

If $t = 10$ and w is furthermore adjacent to two (consecutive) vertices of v_3, v_4, v_5 and v_6, or to two (consecutive) vertices of v_7, v_8, v_9 and v_{10}, we get a C_9. Thus, suppose w is adjacent to v_6 and v_7. By connectivity v_9 has to have another neighbor z off the cycle. Since G is claw-free, z is adjacent to v_8 or v_{10} as well. Then $v_2\ w\ v_6\ v_7\ v_8\ z\ v_9\ v_{10}\ v_1\ v_2$, respectively $v_2\ w\ v_6\ v_7\ v_8\ v_9\ z\ v_{10}\ v_1\ v_2$, is a C_9.

If $t = 11$ and w is adjacent to any two (consecutive) vertices of v_3, v_4, v_5 and v_6, or to two (consecutive) vertices of v_8, v_9, v_{10} and v_{11}, we get by the previous argument a C_{10}. Thus, suppose w is adjacent to v_6 and v_7 or to v_7 and v_8 (see Figure 4.7).

These two cases are symmetric and by the same argument as in the case $t = 10$ the result follows.

$\square$

Lemma 4.6 *Let C_t, $12 \leq t \leq |V(G)|$ be an induced cycle in a $\{K_{1,3}, N(5,0,0)\}$-free, 4-connected graph $G = (V, E)$. Suppose w is a vertex off the cycle with 4 neighbors on the cycle. Then G contains a cycle of length $t - 1$.*

Proof. Say $C_t = v_1\ v_2\ ...\ v_t$. Suppose, without loss of generality, that w is adjacent to v_1 and v_2. Then $\langle\ v_1 v_2 w +\ v_3 v_4 v_5 v_6 v_7\ \rangle$ and $\langle\ v_1 v_2 w +$

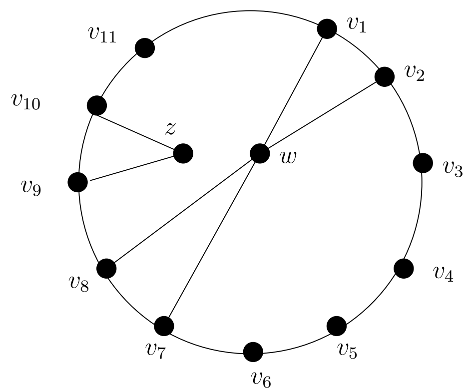


Figure 4.7:

$v_t v_{t-1} v_{t-2} v_{t-3} v_{t-4}$ $\rangle$ imply that w is adjacent to at least one vertex of v_3, v_4, v_5, v_6, v_7 and at least one vertex of v_t, v_{t-1}, v_{t-2}, v_{t-3}, v_{t-4}. This leads to an induced claw centered at w or a C_{t-1} by Lemma 4.4, unless $t = 12$ and w is adjacent to v_6 and v_7. Suppose that is the case. Since G is 4-connected, v_9 has to have another neighbor off the cycle, say z. Since G is claw-free, z has to be adjacent to v_{10} or v_8. Without loss of generality, suppose $zv_8 \in E(G)$. Then v_2 w $v_6 C v_8$ z $v_9 C v_2$ is a C_{11}. □

Lemma 4.7 *Let C_t, $12 \leq t \leq |V(G)|$ be an induced cycle in a $\{K_{1,3}$, $N(4,1,0)\}$-free, 4-connected graph $G = (V, E)$. Suppose w is a vertex off the cycle with 4 neighbors on the cycle. Then G contains a cycle of length $t - 1$.*

Proof. Say $C_t = v_1\ v_2\ ...\ v_t$. Suppose, without loss of generality, that w is adjacent to v_1 and v_2. Then $\langle$ $v_1 v_2 w+$ $v_3 v_4 v_5 v_6 +$ v_t $\rangle$ implies that w is adjacent to one of v_3, v_4 or v_5, otherwise w is adjacent to 4 consecutive neighbors on the cycle and we are done by Lemma 4.4. If w is adjacent to

v_4, we obtain a $(t-1)$-cycle with $v_1\ w\ v_4Cv_1$. Thus, we may assume that w is adjacent to v_5 and since G is claw-free to v_6. Then $\langle\ v_5v_6w+\ v_1v_tv_{t-1}v_{t-2}+\ v_4\ \rangle$ is an induced $N(4,1,0)$ since any additional edge leads to an induced claw, a contradiction. Therefore, G contain a cycle of length $t-1$. □

Lemma 4.8 *Let C_t, $12 \leq t \leq |V(G)|$ be an induced cycle in a $\{K_{1,3}$, $N(3,2,0)\}$-free, 4-connected graph $G=(V,E)$. Suppose w is a vertex off the cycle with 4 neighbors on the cycle. Then G contains a cycle of length $t-1$.*

Proof. Say $C_t = v_1\ v_2\ ...\ v_t$. Suppose, without loss of generality, that w is adjacent to v_1 and v_2. Then $\langle\ wv_1v_2+\ v_3v_4v_5+v_tv_{t-1}\ \rangle$ implies that w is adjacent to one of v_{t-1}, v_4 or v_5, otherwise we are done by Lemma 4.4. If w is adjacent to v_4 (or symmetrically to v_{t-1}), we obtain a C_{t-1} with $v_1\ w\ v_4Cv_1$. Thus, we may assume that w is adjacent to v_5 and v_6. Then $\langle\ wv_5v_6+\ v_1v_tv_{t-1}+v_4v_3\ \rangle$ is an induced $N(3,2,0)$ since any additional edge leads to an induced claw, a contradiction. Therefore, G contain a cycle of length $t-1$.

□

Lemma 4.9 *Let C_t, $t \geq 8$, be an induced cycle in a graph $G=(V,E)$. If there exists a vertex $w \notin V(C_t)$ such that w is adjacent to exactly 2 vertices of C_t, then $N(3,2,0)$, $N(4,1,0)$ and $N(5,0,0)$ exist.*

Proof. Say $C_t = v_1\ v_2\ ....\ v_t$. Suppose, without loss of generality, w is adjacent to v_1 and v_2. Then $\langle\ wv_1v_2+\ v_3v_4v_5+v_tv_{t-1}\ \rangle$ is an induced $N(3,2,0)$, $\langle\ wv_1v_2+\ v_3v_4v_5v_6+\ v_t\ \rangle$ is an induced $N(4,1,0)$ and $\langle\ wv_1v_2+\ v_3v_4v_5v_6v_7\ \rangle$ is an induced $N(5,0,0)$. □

Lemma 4.10 *Let C_t, $t \geq 6$, be an induced cycle in a claw-free, 4-connected graph $G=(V,E)$. Suppose that all vertices off the cycle with neighbors on the cycle have exactly three neighbors on the cycle. Then G contains a cycle of length $t-1$.*

Proof. If a vertex off the cycle has exactly three adjacencies on the cycle, these vertices have to be consecutive vertices on the cycle, otherwise we obtain an induced claw. Let C_t be $v_1\ v_2\ ....\ v_t\ v_1$. Denote the set of vertices that are adjacent to v_{i-1}, v_i and v_{i+1} with V_i (here subscripts are taken mod t). Observe, that all sets V_i form a clique. Since consider two vertices v and w of V_i. Then $\langle\ v_{i-1} + v_{i-2}vw\ \rangle$ implies $vw \in E(G)$, since otherwise this is an induced claw.

Claim 4.10.1 *For $w_i \in V_i$ we have $N(w_i) = \{v_{i-1}\} \cup \{v_i\} \cup \{v_{i+1}\} \cup V_{i-1} \cup V_i \cup V_{i+1}$ or we immediately obtain a cycle of length $t-1$.*

Proof of Claim 4.10.1: Let z be a neighbor of v, $v \in V_i$. Suppose that z is not in the set $\{v_{i-1}\} \cup \{v_i\} \cup \{v_{i+1}\} \cup V_{i-1} \cup V_i \cup V_{i+1}$. Observe that $\langle\ v + v_{i-1}v_{i+1}z\ \rangle$ implies that z is adjacent to v_{i-1} or to v_{i+1}. Without loss of generality, suppose that z is adjacent to v_{i+1}. Since $z \notin \{v_{i-1}\} \cup \{v_i\} \cup \{v_{i+1}\} \cup V_{i-1} \cup V_i \cup V_{i+1}$ and since by assumption z has to be adjacent to three consecutive vertices of C, z has to be in V_{i+2}. Then $v_{i-1}\ v\ z\ v_{i+3}Cv_{i-1}$ is a $(t-1)$-cycle. $\Box_{Claim\ 4.10.1}$

Claim 4.10.2 *There are at most two $V_i = \emptyset$, $V_j = \emptyset$, $i, j \in \{1,\ 2,\ ....,t\}$ and if there are two such sets then $j = i + 1$.*

Proof of Claim 4.10.2: Let us suppose that there is $V_i = \emptyset$ and $V_j = \emptyset$, $i < j$. Suppose that $j \neq i + 1$. Then v_i, v_j is a 2-cut-set by Claim 4.10.1 or we immediately have a $(t-1)$-cycle, since there has to be at least one vertex in $v_{i+1}Cv_{j-1}$. Thus, $j = i + 1$ and there can be at most two sets $V_i = \emptyset$, $V_j = \emptyset$, $i, j \in \{1,\ 2,\ ....,t\}$. $\Box_{Claim\ 4.10.2}$

By Claim 4.10.1 and Claim 4.10.2 we obtain that there are at least $t-2$ non-empty sets V_i. Therefore, we may suppose, without loss of generality, that $V_s \neq \emptyset$, $1 \leq s \leq t-2$. Let k be the smallest number such that

$$\sum_{i=1}^{k} |V_i \cup \{v_i\}| \geq t-1$$

. Then $v_1\ V_1\ V_2\ ...\ V_k^*\ v_k\ v_{k-1}\ ...\ v_1$ is a $(t-1)$-cycle, where $V_k^* \subseteq V_k$, with $|V_k^*| = t-1- \left(\sum_{i=1}^{k-1} |V_i \cup v_i| + 1\right)$. For an example see Figure 4.8. $\square$

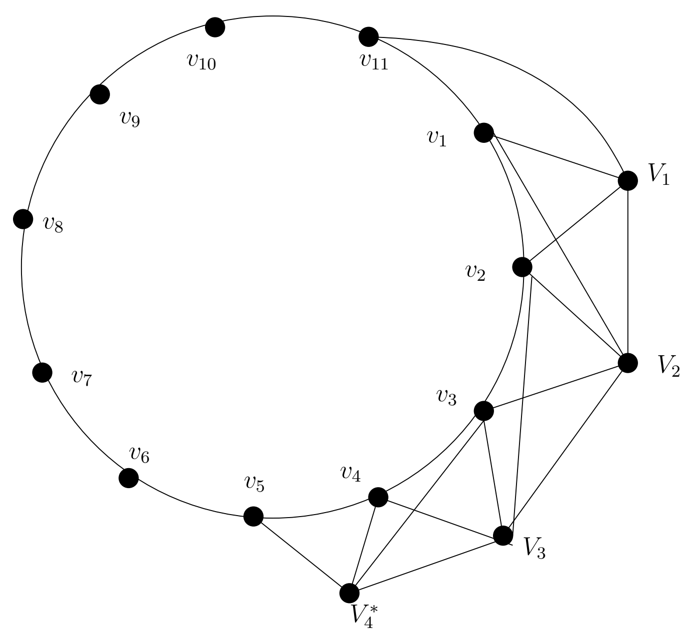


Figure 4.8:

From these Lemmas, we easily obtain the following corollary.

Corollary 4.11 *If $G = (V, E)$ is a 4-connected $\{N,\ K_{1,3}\}$-free graph, with $N = N(3, 2, 0)$, $N = N(4, 1, 0)$ or $N = N(5, 0, 0)$, G is pancyclic, provided it contains cycles of length three, four, five and six.*

Proof. By Theorem 3.2, 3.3 and 3.4, G is hamiltonian. By connectivity, $C_{|V(G)|}$ is not induced. By Theorem 4.2, G contains a $C_{|V(G)|-1}$. Let C_t now be a cycle of G, $8 \leq t \leq |V(G)| - 1$. If C_t is not induced, G contains a cycle of length $t-1$ by Theorem 4.2. If C_t is induced, G contains a cycle of length $t-1$ by Lemma 4.4, Lemma 4.5, Lemma 4.6, Lemma 4.7, Lemma 4.8, Lemma 4.9 and Lemma 4.10, since it is easy to see that any vertex off the cycle can only have two, three (consecutive) or four neighbors on the cycle, otherwise we get an induced claw. Therefore we obtain inductively that G contains cycles of length seven through $|V(G)|$, and the result follows. □

4.2 Pancyclicity of 4-connected, $\{K_{1,3}, N\}$-free graphs

In this section we will extend our earlier results by showing pancyclicity of all the graphs shown to be hamiltonian in Chapters 2 and 3. The first main result of this Chapter follows easily from a result shown by Gould, Łuczak and Pfender in [13].

Theorem 4.12 *If $G = (V, E)$ is a 4-connected $\{N(2,2,1), K_{1,3}\}$-free graph, G is pancyclic.*

Proof. It was shown in [13] that every 3-connected $\{N(2,2,1), K_{1,3}\}$-free graph, is either pancyclic or isomorphic to a graph G_1, where G_1 is obtained from the graph $L(S(K_4))$, where $L(S(K_4))$ is the line graph of the graph obtained by subdivision of every edge of the graph K_4. By contracting two edges x_1x_2 and x_3x_4 of $L(S(K_4))$, with x_1x_2 and x_3x_4 are such that $N(x_i) \cap N(x_j) = \emptyset$ for $1 \leq i < j \leq 4$ (see Figure 4.9) we obtain the graph G_1.

But it is easy to see that this graph is not 4-connected. Hence, we get pancyclicity. □

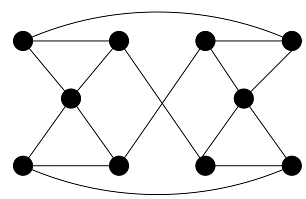


Figure 4.9:

To show pancyclicity for 4-connected, $\{K_{1,3}, N\}$-free graphs, if $N = N(3,2,0)$, $N = N(4,1,0)$ or $N = N(5,0,0)$, we will use Corollary 4.3 and Corollary 4.11 of the previous section.

Theorem 4.13 *If $G = (V,E)$ is a 4-connected $\{N(5,0,0),\ K_{1,3}\}$-free graph, G is pancyclic.*

Proof. By Corollary 4.3 and Corollary 4.11 we only need to show the existence of cycles of length t, $3 \le t \le 6$. Especially, we need to consider an induced C_{t+1} and show the existence of a C_t. Say $C_{t+1} = v_1\ v_2\ ...\ v_{t+1}\ v_1$.

Claim 4.13.1 *The graph contains a cycle of length six.*

Proof of Claim 4.13.1: Suppose we have an induced C_7. If there exists a vertex v off the cycle that is adjacent to 4 vertices on the cycle, we are done by Lemmas 4.4 and 4.5.

Let us assume now that there exists a vertex v off the cycle that has exactly 2 neighbors on the cycle. Assume that v is adjacent to v_1 and v_2. Since G is 4-connected, v has to have at least two more neighbors off the cycle, label two of these neighbors with x and y. Suppose at first that x and y are adjacent to each other. Then $\langle\ vxy{+}\ v_2v_3v_4v_5v_6\ \rangle$ implies that x or y have to be adjacent to one of v_2, v_3, v_4, v_5 or v_6. If x or y are adjacent to one of v_4, v_5 or v_6, we can get a 6-cycle by including v and x (or y) and leaving out v_7, v_6 and

v_5, or v_1, v_7 and v_6, respectively v_3, v_4 and v_5. Thus, suppose that y (or x) is adjacent to v_3. Then we get a 6-cycle with $v_1\ v_2\ v_3\ y\ x\ v\ v_1$ (see Figure 4.10).

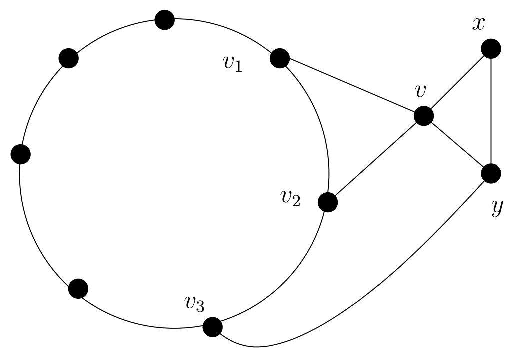


Figure 4.10:

If y (or x) is not adjacent to v_3, we may assume that y (or x) is adjacent to v_2 and v_1. Then $\langle\ vyv_2 +\ v_3v_4v_5v_6v_7\ \rangle$ implies that y is adjacent to v_7 and we can proceed with a similar (symmetric argument) as before to obtain a 6-cycle. Let us now suppose that x and y are not adjacent to each other. Then $\langle\ v + xyv_1\ \rangle$ and $\langle\ v + xyv_2\ \rangle$ imply that one of x or y is adjacent to v_1 and one of x or y is adjacent to v_2.

Case 1 *Suppose that x is adjacent to v_1 and y is adjacent to v_2 (see Figure 4.11).*

Then $\langle\ vyv_2 +\ v_3v_4v_5v_6v_7\ \rangle$ implies that y has to be adjacent to one of v_3, v_4, v_5, v_6 or v_7. If y is adjacent to v_3, we can obtain a cycle of length 6 with $y\ v\ x\ v_1\ v_2\ v_3\ y$. If y is adjacent to v_7, we can get 6-cycle with $v_2\ v\ x\ v_1\ v_7\ y\ v_2$. If $yv_4 \in E(G)$, we get 6-cycle with $y\ v\ v_1\ v_2\ v_3\ v_4\ y$. If $yv_5 \in E(G)$, we get a 6-cycle with $v_2\ y\ v_5\ v_6\ v_7\ v_1\ v_2$. If yv_3 we get a 6-cycle with $v_3\ y\ v\ x\ v_1\ v_2\ v_3$, and if $yv_6 \in E(G)$, we get a 6-cycle with $v_2\ y\ v_6\ v_5\ v_4\ v_3\ v_2$.

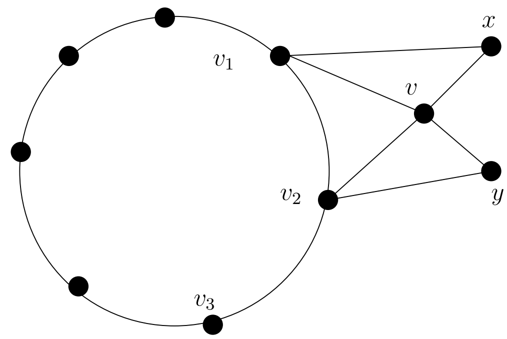


Figure 4.11:

Case 2 *Suppose that x is adjacent to v_1 and v_2 (see Figure 4.12).*

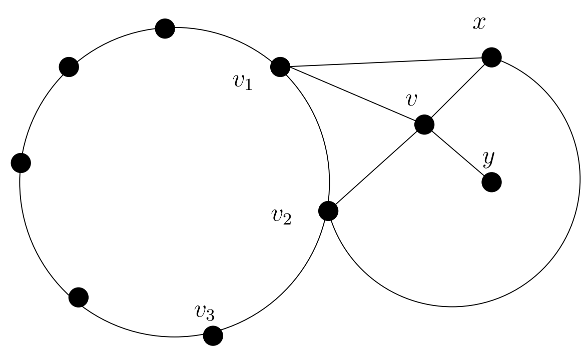


Figure 4.12:

Then $\langle\ vxv_2 +\ v_3v_4v_5v_6v_7\ \rangle$ implies that x is adjacent to v_3 or v_7, since if x is adjacent to one of v_4, v_5 or v_6, we can immediately get a 6-cycle by including x and leaving out 2 vertices on the 7-cycle. Suppose without loss of generality that x is adjacent to v_3. If $v_3y \in E(G)$ we obtain a 6-cycle with

$v_1\ x\ v\ y\ v_3\ v_2\ v_1$. Therefore, we may suppose that v_3 is adjacent to a vertex z off the cycle, with $z \neq v$ (since v has by assumption exactly two neighbors on the cycle), $z \neq y$, $z \neq x$. Then $\langle\ v_3 + v_2v_4z\ \rangle$ implies that z is adjacent to v_2 or v_4. But if z is adjacent to v_2, $v_3\ z\ v_2\ v_1\ v\ x\ v_3$ is 6-cycle. Thus, we may assume that z is adjacent to v_4, and not to v_2. By our previous observations we know that if z has another neighbor on the cycle, this neighbor has to be v_5. Then by considering $\langle\ zv_3v_4 +\ v_5v_6v_7v_1v\ \rangle$ or $\langle\ zv_5v_4 +\ v_6v_7v_1vy\ \rangle$ we get that z has to be adjacent to v or y, or y has to be adjacent to two of v_4, v_5, v_6, v_7. But in all of these cases, we obtain a cycle of length 6: If $zv \in E(G)$, $v_3\ z\ v\ x\ v_1\ v_2\ v_3$ is a 6-cycle, if $zy \in E(G)$, $z\ y\ v\ v_1\ v_2\ v_3\ z$ is a 6-cycle, and if $yv_4 \in E(G)$, $v_1\ v\ y\ v_4\ v_3\ v_2\ v_1$. Similarly, we get a 6-cycle if $yv_5 \in E(G)$, $yv_6 \in E(G)$ or $yv_7 \in E(G)$.

Thus, we may now suppose that all vertices in the graph that have adjacencies on the induced 7-cycle, do have precisely 3 consecutive neighbors on the cycle. By Lemma 4.10 the claim follows.

$\square_{Claim\ 4.13.1}$

Claim 4.13.2 *The graph G contains a cycle of length five.*

Proof of Claim 4.13.2: Suppose that we have an induced 6-cycle. If there is a vertex off the cycle that has 4 or more neighbors on the cycle, we are done by Lemmas 4.4 and 4.5. Suppose that there exists a vertex v off the cycle that has three neighbors on the cycle. Since G is claw-free it is easy to see that all three neighbors of v have to be consecutive. Suppose these neighbors are v_1, v_2 and v_3. Because of 4-connectivity of the graph, v has to have another neighbor off the cycle, say x. Then $\langle\ v + v_1v_3x\ \rangle$ implies that x is adjacent to v_1 or v_3. Suppose, without loss of generality, that x is adjacent to v_3. Then we get a 5-cycle with $v_1\ v_2\ v_3\ v\ x\ v_1$. Hence, all vertices off the cycle that have neighbors on the cycle have precisely 2 neighbors on the cycle. Let v be such a vertex and suppose that v is adjacent to v_1 and v_2. By 4-connectivity we have that v has to be adjacent to at least 2 more vertices off the cycle, say x and y. Then $\langle\, v + xyv_1 \,\rangle$ implies that $xy \in E(G)$, or one of x or y has to be adjacent to v_1. If x and y have distance 2 to the cycle, then $xy \in E(G)$. Then $\langle\, vxy + v_2v_3v_4v_5v_6 \,\rangle$ is an induced $N(5,0,0)$ and we have a contradiction.

Thus, at least one of x or y has to be adjacent to the cycle. We assume that x is adjacent to the cycle. If x is adjacent to v_6 and v_1, $v_6\ x\ v\ v_2\ v_1\ v_6$ is a 5-cycle. Similarly, if x is adjacent to v_6 and v_5 we get the same 5-cycle. With a symmetric argument we obtain a 5-cycle if x is adjacent to v_2 and v_3 or v_3 and v_4. If x is adjacent to v_4 and v_5 then $v_1\ v\ x\ v_5\ v_6\ v_1$ is a 5-cycle. It is left to consider the case that x is adjacent to v_1 and v_2.

Case 3 *Suppose there exist no vertices of distance two to the cycle.*

Then let V_i be the set of the vertices that are adjacent to v_i and v_{i+1}. Note, that $\langle\, v_i + v_{i-1}ab \,\rangle$, with a and b in V_i implies that a and b are adjacent. That means that each V_i is a clique. Then if there does not exist a vertex w_i in V_i that is adjacent to a vertex in some V_j, $j \neq i$, the vertices v_i and v_{i+1} form a cut-set. Suppose that $i = 1$ for now and consider $w \in V_1$ that has a neighbor z in some V_j, $j \neq 1$. If $z \in V_2$, we can get a 5-cycle with $v_1\ v_2\ v_3\ z\ w\ v_1$ (see Figure 4.13).

Similarly, we get a 5-cycle if $z \in V_6$. If $z \in V_3$ we get a 5-cycle with $w\ v_2\ v_3\ v_4\ z\ w$. Symmetrically, we can get a 5-cycle if $z \in V_5$. If $z \in V_4$ we get a 5-cycle with $w\ v_2\ v_3\ v_4\ z\ w$. Thus, we are done if there exist no vertices of distance two to the cycle.

Case 4 *Suppose there exists a vertex y that has distance two to the cycle.*

Suppose that y has distance 2 to the cycle through $v \in V_1$. If one of V_3 or V_5 is not empty, we obtain an induced $N(5,0,0)$ if y is not adjacent to all of

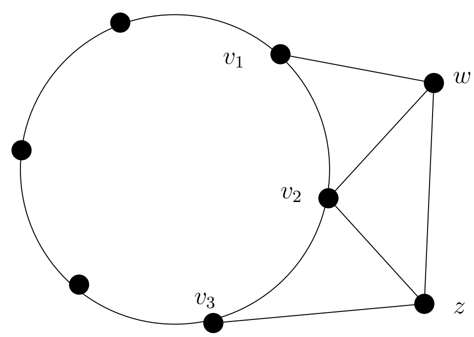


Figure 4.13:

V_3, respectively V_5. Suppose that $w \in V_3$, then $\langle\ wv_3v_4{+}\ v_5v_6v_1vy\ \rangle$ implies $yw \in E(G)$. But then $v_2\ v\ y\ w\ v_3\ v_2$ is a 5-cycle. If $V_3 = \emptyset$, then v_3 has at least two neighbors off the cycle that are both also adjacent to v_2 and adjacent to each other. Call these neighbors s and t. Then
$\langle\ stv_3{+}\ v_4v_5v_6v_1v\ \rangle$ implies that v is adjacent to s (or t) which leads to a 5-cycle with $v_1\ v_2\ v_3\ s$ (or t) $v\ v_1$.

Thus we obtain a 5-cycle in all cases. $\square_{\mathit{Claim}\ 4.13.2}$

Claim 4.13.3 *The graph contains a cycle of length four.*

Proof of Claim 4.13.3: Assume now that we have an induced 5-cycle. We want to show that we also have a 4-cycle. If there exists a vertex off the cycle that has three or more neighbors on the cycle, we immediately obtain a 4-cycle. Thus, all vertices off the cycle that have neighbors on the cycle have precisely 2 neighbors on the cycle. Furthermore, by connectivity every vertex on the cycle has to have at least two neighbors off the cycle. By counting we obtain that there is a vertex $v_i \in V(C)$ with neighbors v and u off the cycle and neighbors v_{i-1} and v_{i+1} on the cycle, such that v is also adjacent to v_{i-1} and u is also adjacent to v_{i+1}. Suppose that $i = 2$, thus v is adjacent to v_1 and v_2, and u is adjacent to v_2 and v_3 (see Figure 4.14).

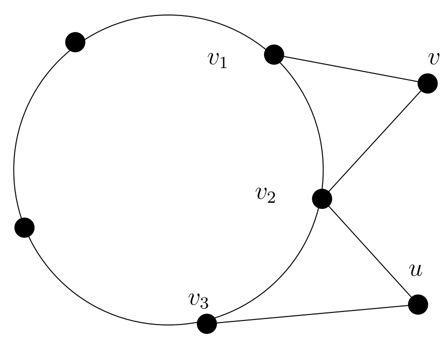


Figure 4.14:

Then v has to have two more neighbors x and y. First suppose that one of them has distance one to the cycle, say x. If x is adjacent to v_1 and v_2, we get a 4-cycle with $v_1\ v_2\ x\ v\ v_1$. If x is adjacent to v_2 and v_3 (that is $x = u$), we obtain a 4-cycle with $v_2\ v\ x\ v_3\ v_2$. With a symmetric argument we get a 4-cycle if x is adjacent to v_1 and v_5. If x is adjacent to v_4 (and v_5 or v_3), we get a 4-cycle with $v_5\ x\ v\ v_1\ v_5$, respectively with $v_3\ x\ v\ v_2\ v_3$.

Suppose now, that x and y both have distance two to the induced 5-cycle. Then $\langle\ v + xyv_1\ \rangle$ implies that $xy \in E(G)$. Now $\langle\ v_2v_3u{+}\ v_4v_5v_1vx\ \rangle$ implies that u is adjacent to v or x. But in both cases we get a 4-cycle (with $z\ v\ v_1\ v_2\ z$, respectively $z\ x\ v\ v_2\ z$). $\square_{Claim\ 4.13.3}$

Since G is claw-free and 4-connected by assumption, we are guaranteed that G contains triangles and thus G is pancyclic. $\square$

In a similar manner we will show that 4-connected, $\{N(4,1,0),\ K_{1,3}\}$-free and a 4-connected, $\{N(3,2,0),\ K_{1,3}\}$-free graph is pancyclic. That is, we will show the existence of cycles of length t, $3 \leq t \leq 6$. To show this, we will use an argument similar to the previous proof.

Theorem 4.14 *If $G = (V, E)$ is a 4-connected $\{N(4,1,0),\ K_{1,3}\}$-free graph, G is pancyclic.*

Proof. By Corollary 4.3 and Corollary 4.11 we only need to show the existence of cycles of length t, $3 \leq t \leq 6$. Especially, we need to consider an induced C_{t+1} and show the existence of a C_t. Say $C_{t+1} = v_1\ v_2\ ...\ v_{t+1}\ v_1$.

Claim 4.14.1 *The graph contains a cycle of length six.*

Proof of Claim 4.14.1: Suppose now, that we have an induced 7-cycle. If there exists a vertex w off the cycle with four neighbors on the cycle, we are done by Lemmas 4.4 and 4.5. Suppose now that there exists a vertex w that is adjacent to precisely two vertices on the cycle and suppose that these vertices are v_1 and v_2. Since G is 4-connected, w has to have another neighbor z_1 off the cycle. Then $\langle\ v_1v_2w+\ v_3v_4v_5v_6+\ z_1\ \rangle$ implies that z_1 has adjacencies on the cycle as well. If z_1 is adjacent to v_5 (or to v_6) we obtain a 6-cycle by skipping v_2, v_3 and v_4 (or v_3, v_4 and v_5) and including instead w and z_1 to the cycle. Assume that z_1 is adjacent to v_3. Due to connectivity v_3 has to have another neighbor x off the cycle. If x is also adjacent to v_2 we get a 6-cycle with $v_1\ w\ z_1\ v_3\ x\ v_2\ v_1$. Thus, we may assume that x is adjacent to v_4. Furthermore, $\langle\ v_3\ +\ z_1xv_2\ \rangle$ implies that z_1 is adjacent to v_2 as well. (Since if $xz_1 \in E(G)$ we obtain immediately a 6-cycle.) Observe that w has to have another neighbor z_2 off the cycle. Then if z_1 and z_2 are adjacent, we get a 6-cycle with $v_1\ w\ z_1\ z_2\ v_3\ v_2\ v_1$. So suppose that they are not adjacent (see Figure 4.15).

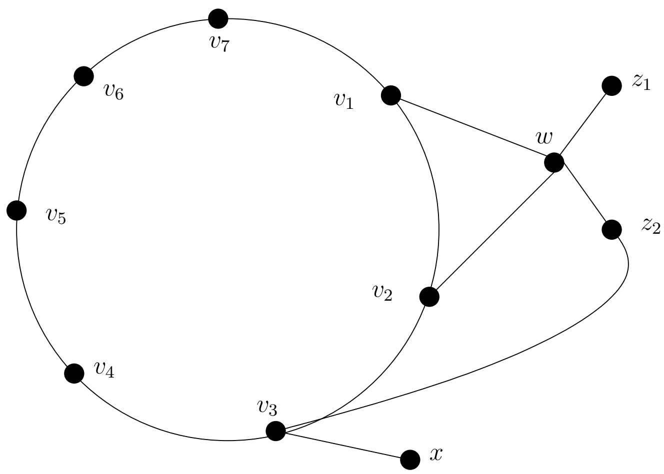


Figure 4.15:

Then $\langle\ w + z_1z_2v_1\ \rangle$ implies that one of z_1 or z_2 has to be adjacent to v_1. If z_2 is adjacent to v_1, we obtain a 6-cycle with $v_1\ z_2\ w\ z_1\ v_3\ v_2\ v_1$. So suppose now that z_1 and v_1 are adjacent and z_2 is therefore not adjacent to v_1. Then $\langle\ v_1wz_1+\ v_7v_6v_5v_4+\ z_2\ \rangle$ implies that z_2 has to be adjacent to one of v_7, v_6, v_5 or v_4 (since w does not have more adjacencies to the cycle by assumption and z_1 is already adjacent to three vertices on the cycle). But in all of these cases it it easy to see that we can obtain a 6-cycle then. Thus, let us assume that z_1 is not adjacent to v_5, v_6, v_3, v_4 (since z_1 is not adjacent to v_3 and v_5) and to v_7 (since that is symmetric to the case that z_1 is adjacent to v_3). Thus, z_1 is adjacent to v_1 and v_2 and no other vertex on the cycle. Then z_1 has to have another neighbor off the cycle, say y. Then $\langle\ v_2wz_1+\ v_3v_4v_5v_6+\ y\ \rangle$ implies that y has to be adjacent to at least one of v_6, v_5, v_4, v_3, v_2 or w. If y is adjacent to v_6 or v_5 we can obtain a 6-cycle by including z_1 and y and skipping v_2, v_3 and v_4, respectively v_3, v_4 and v_5. If y is adjacent to v_3, we get a 6-cycle with $v_1\ w\ z_1\ y\ v_3\ v_2\ v_1$. If y is adjacent to v_4, $\langle\ v_4 + v_3v_5y\ \rangle$ implies that y is adjacent to v_5 or v_3, which we already considered. If $yv_2 \in E(G)$, $\langle\ v_2 + v_1v_3y\ \rangle$ implies with the already considered cases that $yv_1 \in E(G)$ as well. Then $\langle\ v_1 + v_7wy_1\ \rangle$ implies that $yv_7 \in E(G)$ or $yw \in E(G)$. If the former holds we get the 6-cycle $v_7\ y\ z_1\ v_2\ v_1\ v_7$, and if the latter holds, y behaves precisely like z_1 which we already considered. But that means that the vertices v_1 and v_2 are a cut-set of size two, a contradiction. Since we can repeat this argumentation with any neighbor of w, we are done in that case. Let us therefore assume now that all vertices, that are adjacent to the cycle, have precisely three adjacencies to the cycle. Then the claim follows by Lemma 4.10. $\square_{Claim\ 4.14.1}$

Claim 4.14.2 *The graph contains a cycle of length five.*

Proof of Claim 4.14.2: Suppose now that we have an induced 6-cycle. If there is a vertex off the cycle that has 4 or more neighbors on the cycle, it is easy to see that we can get a 5-cycle. Suppose that there exists a vertex v off the cycle that has three neighbors on the cycle. Since G is claw-free, all three neighbors of v have to be consecutive. Suppose these neighbors are v_1, v_2 and v_3. Because of 4-connectivity of the graph, v has to have another neighbor off the cycle, say x. Then $\langle\ v + v_1v_3x\ \rangle$ implies that x is adjacent to v_1 or v_3. Suppose that x is adjacent to v_3. Then we get a 5-cycle with $v_1\ v_2\ v_3\ v\ x\ v_1$. Hence, all vertices off the cycle that have neighbors on the cycle have precisely 2 neighbors on the cycle. Let V_i denote the set of all vertices off the cycle that are adjacent to v_i and v_{i+1}. It is easy to see that each V_i forms a clique. Without loss of generality we may assume that $V_1 \neq \emptyset$. Let us first consider the case that there is an i such that $V_i = \emptyset$. Suppose $i = 2$. Since v_2 and v_3 have to have at least two neighbors off the cycle, V_1 and V_3 have size at least two. Also because of connectivity there has to be a path from V_1 to the cycle not using v_1 and v_2. Let the first vertex on such a path, that is not in V_1 anymore, be t, and let t be adjacent to $q_2 \in V_1$. Then $\langle\ v_2 q_1 q_2 +\ v_3 v_4 v_5 v_5 +\ t\ \rangle$ with $q_1 \in V_1$, implies that t has to be adjacent to v_3, v_4, v_5, v_6 or q_1. If t is adjacent to q_1, we get a 5-cycle with v_1 v_2 q_2 t q_1 v_1 (see Figure 4.16), if t is adjacent to v_3 (or v_5), we get a 5-cycle with v_2

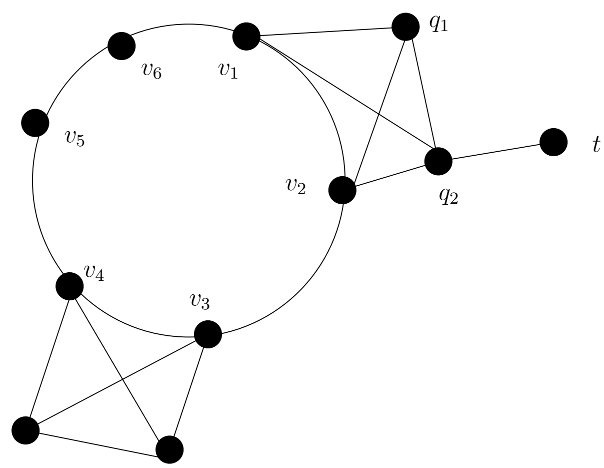


Figure 4.16:

q_1 q_2 t v_3 v_2 (or with v_1 q_2 t v_5 v_6 v_1), similarly if t is adjacent to v_6 we get a 5-cycle. If t is adjacent to v_4, t also has to be adjacent to one of v_5 or v_3 and we therefore obtain in all cases a 5-cycle. Thus, we can now assume that for all i, $V_i \neq \emptyset$. If there is some V_i that contains more than two vertices, we are immediately done. Furthermore, if there is some V_i that contains two vertices, we can use the same argument as above and get a 5-cycle. Thus, we may assume that each V_i consists of precisely one vertex w_i. Then if any of the w_i are adjacent to each other, we get immediately a 5-cycle. Since there has to be another path from w_1 to the cycle, w_1 has a neighbor t_1, $t_1 \neq w_i$,

$t \neq v_i$, $\forall i$. Then $\langle\ v_1v_2w_1+\ v_3v_4v_5w_5+\ t_1\ \rangle$ implies that t_1 has to be adjacent to w_5, since if t was adjacent to any vertices on the cycle, t_1 would be some w_i and we got immediately a 5-cycle. But if t_1 is adjacent to w_5, we also obtain a 5-cycle with $v_6\ v_1\ w_1\ t_1\ w_5\ v_6$. Thus, we always obtain a 5-cycle. $\square_{Claim\ 4.14.2}$

Claim 4.14.3 *The graph contains a cycle of length four.*

Proof of Claim 4.14.3: Suppose now that we have an induced 5-cycle. If there exists a vertex off the cycle that has three or more neighbors on the cycle, we immediately obtain a 4-cycle. Thus, we may assume that all vertices with neighbors on the cycle do have precisely two neighbors on the cycle. Let us define the sets V_i as in Claim 4.14.2. If there is some V_i with at least two vertices, we immediately obtain a 4-cycle. Therefore, all V_i are non-empty and contain precisely one vertex w_i. If any of the w_i are adjacent to each other, a 4-cycle results. We may therefore assume that they are all non-adjacent. Due to connectivity, each w_i has to have at least two more neighbors off the cycle. Note, that all of these neighbors form again a clique. Since consider t_1, t_2 neighbors of w_1 off the cycle. Then $\langle\ w_1 + v_1t_1t_2\ \rangle$ implies $t_1t_2 \in E(G)$. That means, that each w_i has precisely two neighbors off the cycle, since otherwise we immediately obtain a 4-cycle. Also note that we may assume that the neighbors off the cycle of each w_i are not adjacent to any vertices on the cycle. Otherwise they would be a vertex w_j and we can obtain immediately a 4-cycle. Call the neighbors of w_4 that are off the cycle q_1 and q_2. Let us first suppose that $q_i \neq t_k$, $i, k \in \{1, 2\}$. Then $\langle\ v_1v_2w_1+\ v_3v_4w_4q_i+\ t_k\ \rangle$ implies that t_k (or q_i, that is a symmetric case) is adjacent to w_4 or q_i. If $t_kw_4 \in E(G)$, $\langle\ w_4 + q_it_kv_4\ \rangle$ implies $t_kq_i \in E(G)$ as well. But then we obtain a 4-cycle with $t_1\ q_1\ q_2\ w_4\ t_1$ (see Figure 4.17).

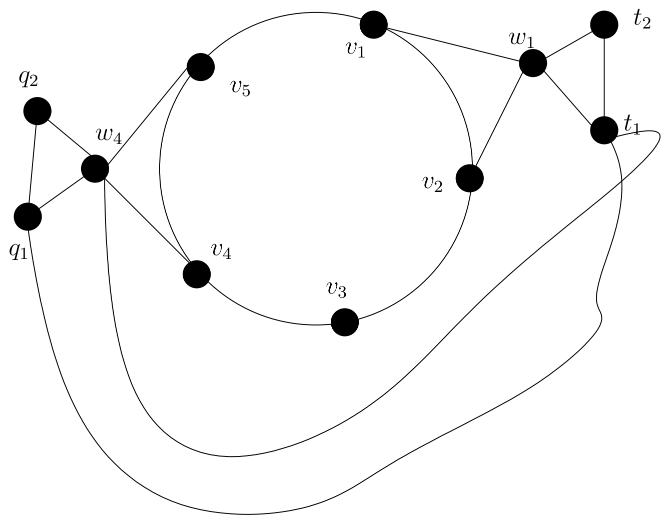


Figure 4.17:

If we suppose that $t_1 = q_1$ and $t_2 = q_2$ we get a 4-cycle with $w_1\ t_1\ w_4\ t_2\ w_1$. If we suppose that t_1 is adjacent to q_2 (or symmetrically t_1 is adjacent to q_1, or t_2 is adjacent to q_1 or to q_2), we get an induced $N(4,1,0)$ with $\langle\ q_1q_2w_4 +\ v_4v_3v_2v_1 +\ t_1\ \rangle$, respectively we obtain a 4-cycle. Suppose now that one of the t_k is the same vertex as one of the q_i. Suppose without loss of generality that $t_1 = q_1$. But then we obtain an induced $N(4,1,0)$ with $\langle\ t_1w_1t_2 +\ v_2v_3v_4v_5 +\ q_2\ \rangle$, or t_2 and q_2 are adjacent, which gives us the 4-cycle $w_1\ t_2\ q_2\ q_1\ w_1$. Therefore, we do have a 4-cycle in the graph. $\square_{Claim\ 4.14.3}$

Due to the claw-freeness it is obvious that we have triangles in the graph. Thus we obtain pancyclicity of the graph. $\square$

Theorem 4.15 *If* $G = (V, E)$ *is a* 4*-connected* $\{N(3,2,0),\ K_{1,3}\}$*-free graph,*

G is pancyclic.

Proof. By Corollary 4.3 and Corollary 4.11 we only need to show the existence of cycles of length t, $3 \leq t \leq 6$. Especially, we need to consider an induced C_{t+1} and show the existence of a C_t. Say $C_{t+1} = v_1\ v_2\ ...\ v_{t+1}\ v_1$.

Claim 4.15.1 *The graph contains a cycle of length six.*

Proof of Claim 4.15.1: Suppose, we have an induced C_7. Clearly, no vertex can have four neighbors on the cycle, else we are done immediately. Let us suppose that there exists a vertex w with two neighbors on the cycle. Suppose that w is adjacent to v_1 and v_2. Since G is 4-connected, v_4 has to have another neighbor t off the cycle. Then $\langle\ v_1v_2w+\ v_3v_4t+v_7v_6\ \rangle$ implies that t is adjacent to v_3 or to v_6. (If t is adjacent to w, we get a 6-cycle with $v_1\ v_2\ v_3\ v_4\ t\ w\ v_1$. Similarly, we obtain a 6-cycle if t is adjacent to any other vertex.)

Case 1 *Suppose that t is adjacent to v_3.*

Then $\langle\ v_1v_2w+\ v_7v_6v_5+v_3t\ \rangle$ implies that t is also adjacent to v_5. Connectivity implies that t has to have another neighbor q off the cycle. $\langle\ t\ +\ qv_3v_5\ \rangle$ implies that q is adjacent to v_3 or v_5. Suppose at first that q is not adjacent to v_3, but adjacent to v_5. Then $\langle\ v_1v_2w+\ v_3tq+v_7v_6\ \rangle$ implies that q is also adjacent to v_6 or w. In both of these cases we get immediately a 6-cycle. Let us therefore suppose that q is adjacent to v_3 (see Figure 4.18).

Then $\langle\ tqv_3+\ v_5v_6v_7+v_2w\ \rangle$ implies that q is adjacent to v_2, which immediately leads to a 6-cycle, or q is adjacent to v_5 and therefore to v_4 as well. But then q has the exact same adjacencies as t on the cycle and we can repeat the previous argumentation. That is, we either obtain a 6-cycle or a contradiction because we can find a 3-cut-set or an induced $N(3, 2, 0)$.

Case 2 *Suppose that t is adjacent to v_6.*

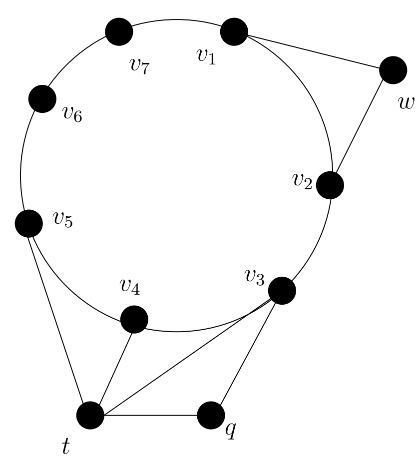


Figure 4.18:

In particular that means that t is adjacent to v_4, v_5 and v_6. As in the previous case, t has to have another neighbor q off the cycle. Then $\langle\, t + qv_6v_4 \,\rangle$ implies that q is adjacent to v_4 or v_6. Since these possibilities are symmetric, suppose that q is adjacent to v_4. But now we can follow an identical argument as before with t and obtain therefore a 6-cycle (if we get to Case 4.2) or a contradiction, since if all neighbors off the cycle of t behave like t, v_4, v_5 and v_6 form a 3-cut-set. Hence, we may suppose that all vertices off the cycle that have neighbors on the cycle, have precisely three consecutive neighbors on the cycle. Then the claim follows by Lemma 4.10. $\square_{Claim\ 4.15.1}$

Claim 4.15.2 *The graph contains a cycle of length five.*

Proof of Claim 4.15.2:

Suppose now that we have an induced 6-cycle. If there is a vertex off the cycle that has four neighbors on the cycle, we immediately obtain a 5-cycle. Suppose now that there exists a vertex v that is adjacent to three vertices on the cycle. Suppose that v is adjacent to v_6 v_1 and v_2. Then v has to have another neighbor t off the cycle and because of $\langle\ v + tv_6v_2\ \rangle$ we may assume that t is adjacent to v_6. This gives us immediately the 5-cycle $v_6\ t\ v\ v_2\ v_1\ v_6$. Thus, we may assume that all vertices off the cycle, that have neighbors on the cycle, do have exactly two neighbors on the cycle. Let us partition these vertices into sets W_i, with $v \in W_i$ if, and only if, $v \notin V(C)$ and $vv_i \in E(G)$ and $vv_{i+1} \in E(G)$. It is easy to see that every W_i forms a clique. If some W_i contains more than two vertices, we obtain immediately a 5-cycle. Note, that there have to exist some W_i and W_{i+2} (possibly modulo 6) that are both non-empty. Suppose that W_3 and W_1 are both non-empty. Because of connectivity, there has to be another path from W_1 to the cycle, not using v_1 and v_2. Let the first vertex on such a path not contained in W_1 be s and let $w_1 \in W_1$ be a vertex adjacent to s. Let w_3 be a vertex in W_3. It is easy to see that if $s \in W_i$ for some $i \neq 1$, we can immediately get a 5-cycle. But then $\langle\ v_4v_3w_3+\ v_2w_1s+v_5v_6\ \rangle$ is an induced $N(3, 2, 0)$, contradiction. Thus, $s \in W_i$ for some $i \neq 1$ and we obtain a 5-cycle. $\square_{Claim\ 4.15.2}$

Claim 4.15.3 *The graph contains a cycle of length four.*

Proof of Claim 4.15.3: Suppose, we have an induced 5-cycle. Immediately, we can suppose that all vertices off the cycle with neighbors on the cycle do have precisely two neighbors on the cycle, since we are otherwise immediately done. Let us partition these vertices into cliques W_i as in Claim 4.15.2. It is easy to see that there have to exists non-empty cliques W_i, W_{i+1} and W_{i+3} (all modulo 5), let us therefore suppose without loss of generality that there exists $w_1 \in W_1$, $w_2 \in W_2$ and $w_4 \in W_4$. Due to connectivity, w_1

has to have one more neighbors off the cycle that does not belong to W_1. Let t be such a neighbor. It is also easy to see that if $t \in W_i$ for some $i \neq 1$, we can get a 4-cycle. Thus, we may suppose that t is not adjacent to the cycle. Then $\langle\ v_4v_5w_4 +\ v_1w_1t + v_3w_2\ \rangle$ is an induced $N(3,2,0)$ or any additional edges between w_4, w_2, w_1 and t result in a 4-cycle. Therefore, we have also a 4-cycle on the graph. $\square_{Claim\ 4.15.3}$

Since G is claw-free, we also have 3-cycles in the graph, which proves the theorem.

$\square$

Recall, that we showed in Chapter two in Lemma 2.10 that a 4-connected, $\{N(3,1,1),\ K_{1,3}\}$-free graph that contains a cycle of length t, but no cycle of length $t+1$ with $3 \leq t \leq |V(G)|-1$ has a vertex off the t-cycle that has three neighbors on the cycle. Then in Theorem 2.13 we used that, to show hamiltonicity of such a graph. But it is easy to see that we can do the same argument as in Theorem 2.13 to obtain as well pancyclicity of the graph. Thus, we get the following theorem:

Theorem 4.16 *If $G=(V,E)$ is a 4-connected $\{N(3,1,1),\ K_{1,3}\}$-free graph, G is pancyclic.*

4.3 Open questions and summary

In this dissertation we have obtained results that support the conjecture of Manton Matthews and David Sumner.

Conjecture 4.17 *[17] Every 4-connected $K_{1,3}$-free graph is hamiltonian.*

That is, by including one more assumption we showed hamiltonicity and an even stronger result than that. Especially, we showed pancyclicity for 4-connected $\{K_{1,3}, N\}$-free graphs, where N is a generalized net, such that the sum of lengths of the paths of the generalized net is equal to five. Thus, if the conjecture holds, it is obvious that our result concerning hamiltonicity is not sharp. Naturally, there is still work to do in this area. We believe that it is possible to generalize our result as follows:

Conjecture 4.18 *Every k-connected with $k \geq 3$, $\{K_{1,3},\ N(i,j,t)\}$-free graph with $i, j, t \geq 0$ and $i + j + t = k + 1$ is pancyclic.*

In this dissertation, we showed this Conjecture for $k = 4$. Florian Pfender, Ronald J. Gould and Tomasz Łuczak showed in [13] that the conjecture holds for $k = 3$. In [9] it was shown that if H_1 and H_2 are connected graphs, both not equal to P_3 and if G ($G \neq C_n$) be a 2-connected graph of order at least 10, then G is $\{H_1,\ H_2\}$-free implies that G is pancyclic if, and only if, H_1 is the claw and H_2 is the generalized net $N(1,0,0)$. That is, if $k = 2$ our conjecture needs stronger assumptions to hold, especially forbidding the claw and $N(i,j,t)$ with $i, j, t \geq 0$ and $i+j+t = 3$ is not enough, we need to forbid the claw and $N(1,0,0)$.

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