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An Exposition of the Functional Equation for the Riemann Zeta Function

and its Values at Integers

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An abstract of
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Abstract

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This paper is an exposition of the development of the zeta function, as well as proving and deriving the essential elements which lead to the functional equation of the Riemann zeta function. We start from the historical background and motivation of defining the zeta function. As Euler first defined this function for the real numbers, he utilized it to prove that there exist infinitely many primes. In addition, a proof for $\zeta(2)$, the solution to Basel Problem, was also included in this paper. Then we move on to the Riemann zeta function and its analytic continuation on the whole complex plane. Finally, with an objective to evaluate the zeta function for all the positive even integers, we examine the Bernoulli numbers and their connection with the zeta function.

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1 Introduction and Statement of the Formula

When it was first introduced, the zeta function was merely used to prove the infinitude of prime numbers. The argument of the function was defined first for real numbers s. As Riemann later extended the function to complex variables, the Riemann zeta function becomes a pivotal element in analytic number theory, especially in proving the Prime Number Theorem. Over the past two centuries, the applications of Riemann Zeta Function have extended to physics, probability theory, and applied statistics.

1.1 Preliminary

In order to present the theorems and results, the following definitions are necessary.

Definition 1.1.1. *[Riemann]* For complex number s with $\mathrm{Re}(s) > 1$, the zeta function is defined by

$$\zeta(s) := \frac{1}{1^s} + \frac{1}{2^s} + \frac{1}{3^s} + \cdots = \sum_{n=1}^{\infty} n^{-s}$$

Definition 1.1.2. For complex s with s with $\mathrm{Re}(s) > 0$, the Gamma function is defined by

$$\Gamma(s) = \int_0^\infty e^{-t} t^{s-1} dt$$

1.2 Formula

Using the previous two definitions, the functional equation for Riemann zeta function is expressed as following.

Theorem 1.2.1. [Functional Equation for $\zeta(s)$] *For all $s \in \mathbb{C}$, we have that*

$$\pi^{-\frac{1}{2}s}\zeta(s)\Gamma\left(\frac{1}{2}s\right) = \pi^{-\frac{1}{2}(1+s)}\Gamma\left[\frac{1}{2}(1-s)\right]\zeta(1-s).$$

The left hand side is often defined as the Λ-function. In other words, we have that

$$\Lambda(s) = \pi^{-\frac{1}{2}s}\zeta(s)\Gamma\left(\frac{1}{2}s\right),$$

and so, the functional equation is more succinctly stated as $\Lambda(s) = \Lambda(1-s)$.

2 Historical Background

While Euclid gave a proof for the infinitude of prime numbers in around 300 BC, by using proof by contradiction. Mathematicians had no idea about how to quantify this infinitude until Riemann's famous Memoir on the ζ-function.

2.1 Zeta of the Real

When Leonhard Euler first introduced the zeta function, complex analysis did not even come into being yet. Thus, in the first half of the eighteenth century, Euler only defined the zeta function on the real numbers. However, this did not limit Euler's ability to utilize the zeta function taking only real arguments. The case of $\zeta(1)$ was used efficiently to prove the infinitude of prime numbers, which will be shown in later section. This also linked the zeta function to the prime numbers for the first time. In addition, Euler also found the value of the $\zeta(2)$, which provides the solution to the Basel problem. In addition, Euler determined the zeta function has rational values at negative integers.

2.2 Riemann Zeta Function

If Euler gave birth to the zeta function, then Bernhard Riemann fully developed the function using complex analysis. Without doubt, Riemann's only paper in the field of number theory, "On the Number of Primes Less Than a Given Magnitude," published in 1859, is probably the most influential and fundamental work for the zeta function. Not only did Riemann extend the Euler definition of the function to a complex variable, but he also proved the analytical continuation of the function, thereby obtaining a functional equation for the zeta function.

After studying the zeros of the zeta function extensively, Riemann proved that for $\sigma < 0$ the function has trivial zeros at the points $s = -2, -4, -6, \cdots$, and conjectured that the infinitely many zeros that lie within the critical strip of $0 \le \sigma \le 1$ are all on the line of $s = \frac{1}{2}$. Here we use the assumption that $s = \sigma + it$, where $\sigma, t \in \mathbb{R}$. Finally, with the help of complex analysis, Riemann established a relation between the zeros of the zeta function and the distribution of prime numbers.

3 Applications of the Zeta Function by Euler

Before introducing the Riemann zeta function and the details of deriving the functional equation for the zeta function, we will take a look at two major applications Euler used the zeta function that he defined for. Above all, he used the zeta function defined for a real variable to rigorously prove the infinitude of prime numbers. Namely, this is the major application of $\zeta(1)$. Secondly, the answer to the Basel Problem happens to be exactly $\zeta(2)$. Euler first found the exact result to be $\frac{\pi^2}{6}$ and provided a solution to the problem at an age of twenty-eight. Proofs for these results will be presented in this section.

3.1 Proof of the Infinitude of Prime Numbers

Theorem 3.1.1. [Euler] *There exist infinitely many primes.*

Proof. For $s > 1$, and let p be primes, then Euler's identity states that

$$\zeta(s) = \prod_{p} (1 - p^{-s})^{-1}.$$

This is based on the fact that every natural number can be factorized into prime numbers uniquely, according to the Fundamental Theorem of Arithmetic. Taking logarithm on both sides of the equation, we obtain that

$$\log \zeta(s) = -\sum_p \log(1 - p^{-s}) = \sum_p \sum_{n=1}^{\infty} \frac{(p^{-s})^n}{n},$$

by using the Taylor series expansion.

As $s \to 1$, the zeta function becomes a harmonic series. Since harmonic series are divergent, $\zeta(s) \to \infty$, as $s \to 1$. Also, we have

$$\sum_p \sum_{n=1}^{\infty} \frac{(p^{-s})^n}{n} = \sum_p p^{-s} + \sum_p \sum_{n=2}^{\infty} \frac{p^{-ns}}{n}.$$

For the second term, there is the inequality that

$$\begin{aligned}\sum_p \sum_{n=2}^{\infty} \frac{p^{-ns}}{n} < \sum_p \sum_{n=2}^{\infty} p^{-n} &= \sum_p \left[\sum_{n=0}^{\infty} p^{-n} - 1 - p^{-1}\right] \\ &= \sum_p \frac{1}{1-\frac{1}{p}} - 1 - \frac{1}{p} \\ &= \sum_p \frac{1}{p^2 - p} < 1,\end{aligned}$$

by using Taylor series expansion again. Applying integral test at the last step for the series confirms that the term is less than 1.

However, as $s \to 1$, $\zeta(s) \to \infty$, $\log \zeta(s) \to \infty$ as well. Thus, we have proved that

$$\sum_p p^{-s} \to \infty, \text{as } s \to 1.$$

This proves the infinitude of primes, and further shows that the series of reciprocals of primes diverges. □

In other words, $\zeta(1) = \infty$, which of course can be shown by using the integral convergence test.

3.2 Basel Problem

The Basel Problem was first posed by Pietro Mengoli in 1644 and solved by Euler in 1735. The problem asks for the precise summation of the reciprocals of the squares of the natural numbers,

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \lim_{n\to\infty} \left(\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots + \frac{1}{n^2} \right).$$

The series is approximately equal to 1.644934. The Basel problem asks for the exact sum of this series and the proof. The problem is named after Basel, the Swiss town where Euler was born and the home to the Bernoulli Family, for its effort in attacking the problem. Presented below is an elementary proof due to Cauchy.

Theorem 3.2.1. [Cauchy]

$$\zeta(2) = \frac{\pi^2}{6}.$$

Proof. By De Moivre's Theorem, we have

$$e^{inx} = \cos(nx) + i\sin(nx) = (\cos x + i \sin x)^n,$$

so that we get

$$\frac{\cos(nx) + i\sin(nx)}{(\sin x)^n} = \frac{(\cos x + i\sin x)^n}{(\sin x)^n} = \left(\frac{\cos x + i \sin x}{\sin x}\right)^n = (\cot x + i)^n.$$

Using the Binomial Theorem, we can write

$$\begin{aligned}(\cot x + i)^n &= \binom{n}{0}\cot^n x + \binom{n}{1}(\cot^{n-1} x)i + \cdots + \binom{n}{n-1}(\cot x)i^{n-1} + \binom{n}{n} i^n \\ &= \left[\binom{n}{0}\cot^n x - \binom{n}{2}\cot^{n-2} x \pm \cdots\right] + i\left[\binom{n}{1}\cot^{n-1} x - \binom{n}{3}\cot^{n-3} x \pm \cdots\right].\end{aligned}$$

Combining the two previous equations together, their imaginary parts must be the same, i.e.

$$\frac{\sin(nx)}{(\sin x)^n} = \binom{n}{1}\cot^{n-1} x - \binom{n}{3}\cot^{n-3} x \pm \cdots .$$

Based on this property, fix a positive integer m, set $n = 2m+1$ and consider $x_r = \frac{r\pi}{2m+1}$ for $r = 1, 2, \cdots, m$. Then naturally nx_r is a multiple of π, which is a zero for the sine function. Thus, we have

$$0 = \binom{2m+1}{1}\cot^{2m} x_r - \binom{2m+1}{3}\cot^{2m-2} x_r + \binom{2m+1}{5}\cot^{2m-4} x_r \pm \cdots (-1)^m \binom{2m+1}{2m+1},$$

for $r = 1, 2, \cdots, m$. On the interval of $(0, \frac{\pi}{2})$, the values $x_1, x_2, \cdots, x_m$ are distinct. Since the function $\cot^2 x$ is injective on this interval, the numbers $t_r = \cot^2 x_r$ are distinct for $r = 1, 2, \cdots, m$. In fact, these m numbers are the roots of the m-th degree polynomial,

$$p(t) := \binom{2m+1}{1} t^m - \binom{2m+1}{3} t^{m-1} \pm \cdots + (-1)^m \binom{2m+1}{2m+1}.$$

We know the sum of the roots has the relationship with the coefficients of the polynomial,

$$\cot^2 x_1 + \cot^2 x_2 + \cdots + \cot^2 x_m = \frac{\binom{2m+1}{3}}{\binom{2m+1}{1}} = \frac{2m(2m-1)}{6}.$$

Substituting this with the trigonometric identity $\csc^2 x = \cot^2 x + 1$, we have

$$\csc^2 x_1 + \csc^2 x_2 + \cdots + \csc^2 x_m = \frac{\binom{2m+1}{3}}{\binom{2m+1}{1}} + m = \frac{2m(2m+2)}{6}.$$

Recalling the Taylor series representations of the cotangent and cosecant functions, we can derive the inequality that $\cot^2 x < \frac{1}{x^2} < \csc^2 x$. After substituting back

$x_r = \frac{r\pi}{2m+1}$ and summing up the inequalities, we obtain

$$\frac{2m(2m-1)}{6} < \left(\frac{2m+1}{\pi}\right)^2 + \left(\frac{2m+1}{2\pi}\right)^2 + \cdots + \left(\frac{2m+1}{m\pi}\right)^2 < \frac{2m(2m+2)}{6}.$$

Thus, we have that

$$\frac{\pi^2}{6}\left(\frac{2m}{2m+1}\right)\left(\frac{2m-1}{2m+1}\right) < \frac{1}{1^2} + \frac{1}{2^2} + \cdots + \frac{1}{m^2} < \frac{\pi^2}{6}\left(\frac{2m}{2m+1}\right)\left(\frac{2m+2}{2m+1}\right).$$

As m approaches infinity, both the lower bound and the upper bound approach $\frac{\pi^2}{6}$. Therefore,

$$\zeta(2) = \sum_{n=1}^{\infty} \frac{1}{n^2} = \lim_{m\to\infty}\left(\frac{1}{1^2} + \frac{1}{2^2} + \cdots + \frac{1}{m^2}\right) = \frac{\pi^2}{6}.$$

□

4 Riemann Zeta Function

As we mentioned before, Riemann took the zeta function defined by Euler only for the real variable to a domain of complex variable. In order to achieve this goal, Riemann worked on the analytical continuation of the zeta function. The functional equation for the zeta function is the key to this analytic continuation. To start from scratch, we will need to understand the Gamma function and the Poisson Summation Formula. Thus, this section will start the discussion with these two topics.

4.1 Gamma Function and its Properties

As defined previously, we have

$$\Gamma(s) = \int_0^{\infty} e^{-t} t^{s-1} dt, s \in \mathbb{C}, \text{for } \operatorname{Re}(s) > 0.$$

The Gamma function also has the property $\Gamma(s+1) = s\Gamma(s)$. This can be shown by using integration by parts,

$$\Gamma(s+1) = \int_0^\infty e^{-t}t^s dt = (-e^{-t}t^s)|_{t=0}^\infty + s\int_0^\infty e^{-t}t^{s-1}dt = s\Gamma(s).$$

Notice that the fact that $\mathrm{Re}(s) > 0$ guarantees the term $(-e^{-t}t^s)|_{t=0}^\infty$ goes to zero. This ensures the property $\Gamma(s+1) = s\Gamma(s)$ to be true. This is why we first define $\Gamma(s)$ for $\mathrm{Re}(s) > 0$.

However, we want to take our Gamma function to the whole complex plane. This then requires the analytic continuation. As we wish to extend the definition of our Gamma function to the left half plane, we would like the nice property to hold as well. Therefore, we begin by defining $\Gamma(s) = \frac{\Gamma(s+1)}{s}$ on the left half plane. We know that $\Gamma(1) = 1$, simply by computing the integral. This can also be verified from a probability prospective, using the Gamma distribution. The cumulative probability of the Gamma distribution from zero to infinity has to equal to 1. However, according to this recurrence relationship, $\lim_{s\to 0}\Gamma(s) = \infty$. Therefore, Gamma function has an asymptote at $s = 0$.

We now extend the function to the strip corresponding to $(-1, 0)$, i.e. $\mathrm{Re}(s) \in (-1, 0)$. We define $\Gamma(s) = \frac{\Gamma(s+1)}{s}$, for $\mathrm{Re}(s) \in (-1, 0)$. Firstly, $\Gamma(s+1)$ is analytic for $\mathrm{Re}(s) \in (-1, 0)$, since $s+1$ will be on the right half of the plane. Secondly, $\frac{1}{s}$ is also analytic for $\mathrm{Re}(s) \in (-1, 0)$. Thus, $\Gamma(s) = \frac{\Gamma(s+1)}{s}$ is analytic for $\mathrm{Re}(s) \in (-1, 0)$. By doing so, we have extended the analyticity of the Gamma function to the region $\mathrm{Re}(s) > -1$. Similarly, the Gamma function also has an asymptote at $s = -1$, due to its recurrence relationship to $s = 0$.

By continuing this process, we can extend the Gamma function to the whole complex plane, with asymptotes at every non-positive integers. Furthermore, from the recurrence relationship $\Gamma(s) = \frac{\Gamma(s+1)}{s}$, we know that every non-positive integer is in fact a simple pole. This gives us the meromorphic Gamma function, defined on the whole complex plane.

4.2 From Gamma to Zeta Function

As we have now achieved the analytic continuation of the Gamma function, it is then natural to try to relate it to the zeta function. In fact, with some careful manipulation, we can see the two are closely related.

Starting from the classic definition of the Gamma function, we have

$$\Gamma\left(\frac{1}{2}s\right) = \int_0^\infty e^{-t} t^{\frac{1}{2}s-1} dt,$$

for $\mathrm{Re}(s) > 0$. Let $t = n^2\pi x$, then $dt = n^2\pi dx$, substituting this back into the definition gives us,

$$\Gamma\left(\frac{1}{2}s\right) = n^2\pi \int_0^\infty e^{-n^2\pi x}(n^2\pi x)^{\frac{1}{2}s-1} dx = n^s \pi^{\frac{1}{2}s} \int_0^\infty e^{-n^2\pi x} x^{\frac{1}{2}s-1} dx.$$

Putting the two coefficients onto the other side,

$$\Gamma\left(\frac{1}{2}s\right) \pi^{-\frac{1}{2}s} n^{-s} = \int_0^\infty e^{-n^2\pi x} x^{\frac{1}{2}s-1} dx.$$

For $\mathrm{Re}(s) > 1$, we can sum up n from 1 to ∞,

$$\sum_{n=1}^{\infty} \Gamma\left(\frac{1}{2}s\right) \pi^{-\frac{1}{2}s} n^{-s} = \sum_{n=1}^{\infty} \int_0^\infty e^{-n^2\pi x} x^{\frac{1}{2}s-1} dx.$$

Since $\zeta(s)$ is also only defined for $\mathrm{Re}(s) > 1$ at this point, we can rewrite the left hand side with $\zeta(s)$,

$$\pi^{-\frac{1}{2}s}\Gamma\left(\frac{1}{2}s\right)\zeta(s) = \int_0^\infty x^{\frac{1}{2}s-1} \sum_{n=1}^{\infty} e^{-n^2\pi x} dx. \tag{4.2.1}$$

It is legitimate to inverse the order on the right hand side, since

$$\sum_{n=1}^{\infty}\int_0^{\infty} e^{-n^2\pi x}x^{\frac{1}{2}s-1}dx$$

is convergent.

Define $\omega(x)=\sum_{n=1}^{\infty}e^{-n^2\pi x}$, then (4.2.1) can be written as

$$\pi^{-\frac{1}{2}s}\Gamma\left(\frac{1}{2}s\right)\zeta(s)=\int_0^{\infty}x^{\frac{1}{2}s-1}\omega(x)dx. \tag{4.2.2}$$

4.3 Poisson's Summation Formula

Before we derive further a functional equation for $\omega(x)$, we will have to derive the transformation law for theta function first. In order to obtain that, we will need Poisson's Summation Formula as our starting point. Thus, we will first prove the Poisson's Summation Formula in this section.

Theorem 4.3.1. *If $f(x)$ is a real, continuous, and monotonic function, then*

$$\sum_{n=A}^{B}{}'f(n)=\sum_{v=-\infty}^{\infty}\int_A^B f(x)e^{2\pi ivx}dx,$$

where $\sum'$ denotes that the end terms of the sum are replaced by $\frac{1}{2}f(A)$ and $\frac{1}{2}f(B)$.

The proof for this formula heavily involves the Fourier series expansion of a function. Thus, it is worthwhile to review the Fourier series expansion for a given function. For a periodic function $f(x)$, integrable on $[0,1]$, then

$$f(x)=\lim_{N\to\infty}\left[\frac{a_0}{2}+\sum_{v=1}^{N}a_v\cos(2\pi vx)+b_v\sin(2\pi vx)\right],$$

$$a_v=2\int_0^1 f(x)\cos 2\pi vxdx, b_v=2\int_0^1 f(x)\sin 2\pi vxdx.$$

This is the Fourier series of $f(x)$.

Proof. We divide this proof into two cases. The first case is the simple one, where $A = 0$ and $B = 1$. The second one is the general case.

Case 1: Since $A = 0$ and $B = 1$, then

$$\sum_{n=A}^{B}{}' f(n) = \frac{1}{2}f(0) + \frac{1}{2}f(1).$$

Let $f_1(x)$ coincide with $f(x)$ for $0 \leq x < 1$, and be a periodic function of period 1. Thus, $f_1(x)$ has Fourier series representation,

$$\frac{1}{2}a_0 + \sum_{v=1}^{\infty}(a_v \cos 2\pi vx + b_v \sin 2\pi vx),$$

$$a_v = 2\int_0^1 f(x)\cos 2\pi vxdx, b_v = 2\int_0^1 f(x)\sin 2\pi vxdx.$$

If x is a continuous point, then this series converges to $f_1(x)$. However, if x is an ordinary discontinuity, then this series converges to the mean of the left and right limit of $f_1(x)$. It is easy to see that $x = 0$ is in fact an ordinary discontinuity of $f_1(x)$. For $x = 0$, we have

$$\lim_{x\to 0^-} f_1(x) = f_1(1) = f(1), \text{ and } \lim_{x\to 0^+} f_1(x) = f_1(0) = f(0).$$

Thus, this gives us

$$\begin{aligned}
\frac{1}{2}\left[f(1) + f(0)\right] &= \frac{1}{2}a_0 + \sum_{v=1}^{\infty} a_v \\
&= \frac{1}{2}a_0 + \frac{1}{2}\left[\sum_{v=1}^{\infty} a_v + \sum_{v=-\infty}^{-1} a_v\right] \\
&= \sum_{v=-\infty}^{\infty}\int_0^1 f(x)\cos 2\pi vxdx.
\end{aligned}$$

This proves the basic case of $A = 0$ and $B = 1$.

Case 2: Let $f_1(x+n)$ coincide with $f(x+n)$, for $0 \leq x < 1$, and $n = A, A+1, \cdots, B-1$. Consider when $n = A$, let $x = 0$, we have

$$\lim_{x \to 0^-} f_1(x+A) = f(1+A), \text{ and } \lim_{x \to 0^+} f_1(x+A) = f(A).$$

The Fourier series representation is

$$\frac{1}{2}a_0 + \sum_{v=1}^{\infty} \left[a_v \cos 2\pi v(x+A) + b_v \sin 2\pi v(x+A)\right],$$

$$a_v = 2\int_A^{A+1} f(x)\cos 2\pi vx dx, b_v = 2\int_A^{A+1} f(x)\sin 2\pi vx dx.$$

Again, $x = 0$ is an ordinary discontinuity. Thus, this gives us

$$\begin{aligned}\frac{1}{2}\left[f(A+1) + f(A)\right] &= \frac{1}{2}a_0 + \sum_{v=1}^{\infty}\left[a_v \cos 2\pi vA + b_v \sin 2\pi vA\right] \\ &= \sum_{v=-\infty}^{\infty} \int_A^{A+1} f(x)\cos 2\pi vx dx.\end{aligned}$$

Similarly, when $n = A+1$, we have

$$\frac{1}{2}\left[f(A+2) + f(A+1)\right] = \sum_{v=-\infty}^{\infty} \int_{A+1}^{A+2} f(x)\cos 2\pi vx dx.$$

Continuing this process, when $n = B-1$, we have

$$\frac{1}{2}\left[f(B) + f(B-1)\right] = \sum_{v=-\infty}^{\infty} \int_{B-1}^{B} f(x)\cos 2\pi vx dx.$$

Therefore, combining all these terms together, we obtain the desired result that

$$\frac{1}{2}f(A) + f(A+1) + \cdots + f(B-1) + \frac{1}{2}f(B) = \sum_{n=A}^{B}{}' f(n) = \sum_{v=-\infty}^{\infty} \int_A^B f(x)e^{2\pi ivx} dx.$$

□

4.4 Transformation Law for Theta Function

Definition 4.4.1. *[Jacobi]* Define functions $\omega(x)$ and $\theta(x)$ by

$$\omega(x) = \sum_{n=1}^{\infty} e^{-n^2\pi x}, \text{ and } \theta(x) = \sum_{n=-\infty}^{\infty} e^{-n^2\pi x}.$$

Naturally, one can see the relationship between $\omega(x)$ and $\theta(x)$ as

$$2\omega(x) = \theta(x) - 1.$$

Theorem 4.4.1. [Transformation Law for $\theta(x)$] *we have that*

$$\sum_{n=-\infty}^{\infty} e^{-\frac{(n+\alpha)^2\pi}{x}} = x^{\frac{1}{2}} \sum_{n=-\infty}^{\infty} e^{-n^2\pi x + 2\pi i n\alpha}$$

This is called the transformation law for theta function. As $\alpha = 0$ in the case for $\theta(x)$ as defined above, the theta function satisfies the relationship

$$\theta\left(\frac{1}{x}\right) = x^{\frac{1}{2}}\theta(x).$$

Proof. By Poisson's Summation Formula, we have

$$\sum_{n=-N}^{N}{}' e^{\frac{-(n+\alpha)^2\pi}{x}} = \sum_{v=-\infty}^{\infty} \int_{-N}^{N} e^{-\frac{(t+\alpha)^2\pi}{x}} e^{2\pi i v t} dt,$$

for the function $f(n) = e^{\frac{-(n+\alpha)^2\pi}{x}}$ is real, continuous, and monotonic.

In order to take $N \to \infty$, we need to calculate

$$\int_{N}^{\infty} e^{-\frac{(t+\alpha)^2\pi}{x} + 2\pi i v t} dt = \int_{N}^{\infty} e^{-\frac{(t+\alpha)^2\pi}{x}} \left[\cos(2\pi v t) + i\sin(2\pi v t)\right] dt$$

to verify whether the equation will still hold. Using integration by parts, we evaluate

$$\int_N^\infty e^{-\frac{(t+\alpha)^2\pi}{x}} \cos 2\pi vtdt = \frac{1}{2\pi v} \sin 2\pi vte^{-\frac{(t+\alpha)^2\pi}{x}}|_{t=N}^\infty - \frac{1}{2\pi v}\int_N^\infty \sin 2\pi vtd\left[e^{-\frac{(t+\alpha)^2\pi}{x}}\right].$$

Since the first positive term vanishes as $N \to \infty$, we can bound this integral

$$|\int_N^\infty e^{-\frac{(t+\alpha)^2\pi}{x}} \cos 2\pi vtdt| < \frac{1}{2\pi v}\int_N^\infty d\left[e^{-\frac{(t+\alpha)^2\pi}{x}}\right] = \frac{1}{2\pi v} e^{-\frac{(N+\alpha)^2\pi}{x}}.$$

Thus, we have that

$$|\sum_{v\neq 0}\int_N^\infty e^{-\frac{(t+\alpha)^2\pi}{x}} \cos 2\pi vtdt| < Ce^{-\frac{(N+\alpha)^2\pi}{x}},$$

where C is a constant. This vanishes as we take $N \to \infty$. Thus, the limit operation is justified. The equality holds, as the remainder eventually approaches 0.

Therefore, we have acquired the following equation,

$$\sum_{n=-\infty}^{\infty} e^{\frac{-(n+\alpha)^2\pi}{x}} = \sum_{v=-\infty}^{\infty}\int_{-\infty}^\infty e^{-\frac{(t+\alpha)^2\pi}{x}+2\pi ivt}dt. \tag{4.4.1}$$

Now we let $t + \alpha = xu$, then one can substitute it in,

$$e^{-\frac{(t+\alpha)^2\pi}{x}+2\pi ivt} = e^{-xu^2\pi+2\pi iv(xu-\alpha)}, \text{ and } dt = xdu.$$

Substituting these back into 4.4.1,

$$\begin{aligned}(4.4.1) &= x\sum_{v=-\infty}^{\infty} e^{-2\pi iv\alpha}\int_{-\infty}^\infty e^{-\pi xu^2+2\pi ivxu}du \\ &= x\sum_{v=-\infty}^{\infty} e^{-2\pi iv\alpha}\int_{-\infty}^\infty e^{-\pi x(u+iv)^2-\pi xv^2}du \\ &= Ax^{-\frac{1}{2}}x\sum_{v=-\infty}^{\infty} e^{-\pi xv^2-2\pi iv\alpha}.\end{aligned}$$

Since we have

$$\int_{-\infty}^{\infty} e^{-\pi x z^2} dz = Ax^{-\frac{1}{2}},$$

for some constant A. This can be verified from the integral of normal distribution, manipulating the exponents. We let $\alpha = 0$, this gives us

$$\sum_{n=-\infty}^{\infty} e^{-\frac{n^2\pi}{x}} = Ax^{\frac{1}{2}} \sum_{v=-\infty}^{\infty} e^{-\pi x v^2}. \tag{4.4.2}$$

Applying the relationship (4.4.2) to the right hand side of (4.4.2) itself again, we can solve for A,

$$\begin{aligned}
\sum_{n=-\infty}^{\infty} e^{-\frac{n^2\pi}{x}} &= Ax^{\frac{1}{2}} \sum_{v=-\infty}^{\infty} e^{-\pi x v^2} \\
&= Ax^{\frac{1}{2}} Ax^{-\frac{1}{2}} \sum_{v=-\infty}^{\infty} e^{-\frac{v^2\pi}{x}} \\
&= A^2 \sum_{v=-\infty}^{\infty} e^{-\frac{v^2\pi}{x}}.
\end{aligned}$$

Therefore, we may conclude that $A = 1$, since A is the constant coefficient of a integral with integrand larger than 0. We have then achieved the desired expression,

$$\sum_{n=-\infty}^{\infty} e^{-\frac{(n+\alpha)^2\pi}{x}} = x^{\frac{1}{2}} \sum_{v=-\infty}^{\infty} e^{-v^2\pi x + 2\pi i v \alpha},$$

by replacing v with $-v$ on the right hand side of the equation.

$\square$

Finally, if we let $\alpha = 0$, we have

$$\sum_{n=-\infty}^{\infty} e^{-\frac{n^2\pi}{x}} = x^{\frac{1}{2}} \sum_{v=-\infty}^{\infty} e^{-v^2\pi x}.$$

This gives us the functional equation for the theta function,

$$\theta(x^{-1}) = x^{\frac{1}{2}}\theta(x).$$

Also, the functional equation for $\omega(x)$,

$$\omega(x) = x^{-\frac{1}{2}}\omega(x^{-1}) + \frac{1}{2}x^{-\frac{1}{2}} - \frac{1}{2}.$$

4.5 Functional Equation of Riemann Zeta Function

Picking off from where (4.2.2) we left off for deriving the functional equation of the Riemann zeta function,

$$\begin{aligned}
\pi^{-\frac{1}{2}s}\Gamma\left(\frac{1}{2}s\right)\zeta(s) &= \int_0^\infty x^{\frac{1}{2}s-1}\omega(x)dx \\
&= \int_0^1 x^{\frac{1}{2}s-1}\omega(x)dx + \int_1^\infty x^{\frac{1}{2}s-1}\omega(x)dx \\
&= \int_\infty^1 -x^{-\frac{1}{2}s-1}\omega(x^{-1})dx + \int_1^\infty x^{\frac{1}{2}s-1}\omega(x)dx \\
&= \int_1^\infty x^{-\frac{1}{2}s-1}\omega(x^{-1})dx + \int_1^\infty x^{\frac{1}{2}s-1}\omega(x)dx.
\end{aligned}$$

The purpose of transforming the upper and lower bound of the limit on the integral is to create integrals that will later converge, in order to save us the trouble in evaluating the integrals later. Substituting the functional equation we derived for $\omega(x)$, we have

$$\begin{aligned}
\pi^{-\frac{1}{2}s}\Gamma\left(\frac{1}{2}s\right)\zeta(s) &= \int_1^\infty x^{\frac{1}{2}s-1}\omega(x)dx + \int_1^\infty x^{-\frac{1}{2}s-1}\left(-\frac{1}{2} + \frac{1}{2}x^{\frac{1}{2}} + x^{\frac{1}{2}}\omega(x)\right)dx \\
&= \int_1^\infty \left(x^{\frac{1}{2}s-1} + x^{-\frac{1}{2}s-\frac{1}{2}}\right)\omega(x)dx - \frac{1}{s} + \frac{1}{s-1} \\
&= \int_1^\infty \left(x^{\frac{1}{2}s-1} + x^{-\frac{1}{2}s-\frac{1}{2}}\right)\omega(x)dx + \frac{1}{s(s-1)}.
\end{aligned}$$

Although this only holds for $\mathrm{Re}(s) > 1$, the right hand side of the equation converges for all s on the complex plane. The term $\frac{1}{s(s-1)}$ is a finite, and the integral is also finite, because $\omega(x) = O(e^{-\pi x})$.

Therefore, we have obtained the analytic continuation for the zeta function. In addition, if we replace s with $1-s$, we realize that the right hand side of the equation stays the same. Thus, we have achieved a functional equation for the Riemann zeta function,

$$\pi^{-\frac{1}{2}s}\zeta(s)\Gamma\left(\frac{1}{2}s\right) = \pi^{-\frac{1}{2}(1+s)}\Gamma\left[\frac{1}{2}(1-s)\right]\zeta(1-s).$$

5 Riemann Zeta Function and Bernoulli Numbers

As we have already derived the functional equation for the Riemann zeta function, we will now evaluate the zeta function at some points, namely all the integer points.

5.1 Values of the Riemann Zeta Function

Before examining the value of the zeta function at any other integer values, we will take a look at $\zeta(1)$ more closely. After all, this specific value motivated Euler to define this function and was later used to prove the infinitude of prime numbers. We begin reviewing this value from the most classic definition of the zeta function.

$$\begin{aligned}
\zeta(s) = \sum_{n=1}^{\infty} n^{-s} &= 1^{-s} + 2^{-s} + \cdots + n^{-s} + \cdots \\
&= (1^{-s} - 2^{-s}) + 2(2^{-s} - 3^{-s}) + 3(3^{-s} - 4^{-s}) + \cdots \\
&= \sum_{n=1}^{\infty} n\left[n^{-s} - (n+1)^{-s}\right] \\
&= \sum_{n=1}^{\infty} ns\int_{n}^{n+1} x^{-s-1}dx \\
&= s\int_{1}^{\infty} [x]x^{-s-1}dx \\
&= s\int_{1}^{\infty} x^{-s}dx - s\int_{1}^{\infty} (x)x^{-s-1}dx,
\end{aligned}$$

where $[x]$ denotes the largest integer less than x, and (x) denotes the fractional part of x, i.e. $[x] = x - (x)$.

After evaluating the first integral, we have the following equation

$$\zeta(s) = \frac{s}{s-1} - s\int_1^\infty (x)x^{-s-1}dx,$$

whereas the second integral converges for $\mathrm{Re}(s) > 0$. Therefore, $\zeta(s)$ has only one simple pole at $s = 1$, and the residue at 1 is 1.

Next, we will take a look at $\zeta(0)$. This time we follow the functional equation we have derived. We will start from the property for $\Gamma(s)$ we previously mentioned, $\Gamma(s+1) = s\Gamma(s)$. By taking $s \to 0$, we notice that

$$\lim_{s\to 0} \frac{1}{2}s\Gamma\left(\frac{1}{2}s\right) = \lim_{s\to 0} \Gamma\left(\frac{1}{2}s + 1\right) = \Gamma(1) = 1.$$

Thus, it is clear that

$$\lim_{s\to 0} \frac{\Gamma\left(\frac{1}{2}s\right)}{\left(\frac{1}{2}s\right)^{-1}} = 1.$$

If we now go back to the expression for zeta that we just derived and take limit operation on both sides,

$$\lim_{s\to 0} \pi^{-\frac{1}{2}s}\zeta(s)\Gamma\left(\frac{1}{2}s\right) = \lim_{s\to 0} \frac{s}{s-1} - s\int_1^\infty (x)x^{-s-1}dx.$$

We can now replace the term $\lim_{s\to 0}\Gamma\left(\frac{1}{2}s\right)$ with $\left(\frac{1}{2}s\right)^{-1}$, this turns out to be

$$\lim_{s\to 0} \pi^{-\frac{1}{2}s}\zeta(s)\left(\frac{1}{2}s\right)^{-1} = \lim_{s\to 0} \frac{s}{s-1} - s\int_1^\infty (x)x^{-s-1}dx.$$

This eventually gives us the final result that

$$\lim_{s\to 0} \zeta(0) = -\frac{1}{2}.$$

We finally conclude that $\zeta(0) = -\frac{1}{2}$.

5.2 Bernoulli Numbers

Bernoulli Numbers are named after Jacob Bernoulli, who set a goal for himself to find a formula for the finite sum of powers of consecutive positive integers,

$$S_k(n) = 1^k + 2^k + 3^k + \cdots + (n-1)^k,$$

where $k = 1, 2, \cdots, n$ and $n = 2, 3, \cdots$. Notice that $S_k(n)$ shares a similar terminal goal with the zeta function, when it is evaluated at integers. Except zeta function aims for an infinite sum, whereas here the $S_k(n)$ function is a finite sum.

The formula for first few n's are quite well known. For example,

$$\begin{aligned}
S_1(n) &= \frac{1}{2}n^2 - \frac{1}{2}n \\
S_2(n) &= \frac{1}{3}n^3 - \frac{1}{2}n^2 + \frac{1}{6}n \\
S_3(n) &= \frac{1}{4}n^4 - \frac{1}{2}n^3 + \frac{1}{4}n^2 \\
S_4(n) &= \frac{1}{5}n^5 - \frac{1}{2}n^4 + \frac{1}{3}n^3 - \frac{1}{30}n \\
S_5(n) &= \frac{1}{6}n^6 - \frac{1}{2}n^5 + \frac{5}{12}n^4 - \frac{1}{12}n^2 \\
&\cdots
\end{aligned}$$

The formula for $S_k(n)$ was then found,

$$\begin{aligned}
S_k(n) &= \frac{1}{k+1}n^{k+1} - \frac{1}{2}n^k + \frac{k}{12}n^{k-1} + 0n^{k-2} + \frac{k(k-1)(k-2)}{720}n^{k-3} + \cdots \\
&= \frac{1}{k+1}\left[B_0 n^{k+1} + \binom{k+1}{1}B_1 n^k + \cdots + \binom{k+1}{k}B_k n\right] \\
&= \frac{1}{k+1}\sum_{j=0}^{k}\binom{k+1}{j}B_j n^{k+1-j}.
\end{aligned}$$

The coefficients B_j are called the Bernoulli numbers. There also exist recurrent relations for B_j, namely

$$B_0 = 1,$$
$$B_0 + 2B_1 = 0,$$
$$B_0 + 3B_1 + 3B_2 = 0,$$
$$B_0 + 4B_1 + 6B_2 + 4B_3 = 0,$$
$$\dots$$
$$\sum_{j=0}^{k} \binom{k+1}{j} B_j = 0, k \geq 1.$$

We could also define the Bernoulli numbers analytically. Consider the function

$$f(z) = \frac{z}{e^z - 1},$$

it then has the power series expansion

$$\frac{z}{e^z - 1} = 1 - \frac{1}{2}z + \frac{1}{2!}\frac{1}{6}z^2 + \frac{1}{4!}\left(-\frac{1}{30}\right)z^4 + \cdots = \sum_{n=0}^{\infty} B_n \frac{z^n}{n!}.$$

It seems that the coefficients B_n here are exactly the same Bernoulli numbers but defined in another way. Below is a list of some Bernoulli numbers

n	0	1	2	3	4	5	6	7	8	9	10
B_n	1	$-\frac{1}{2}$	$\frac{1}{6}$	0	$-\frac{1}{30}$	0	$\frac{1}{42}$	0	$-\frac{1}{30}$	0	$\frac{5}{66}$

5.3 Bernoulli Numbers and the Zeta Function

Surprisingly, it turns out that the Bernoulli numbers will give us the values of the zeta function at positive even integers. In order to derive the formula for calculating these values, we start from the infinite product of the sine function due to Euler.

For $z \in \mathbb{C}$, we have the following product representation

$$\sin z = z \prod_{n=1}^{\infty} \left(1 - \frac{z^2}{n^2 \pi^2}\right).$$

Taking the logarithmic derivative, we have

$$\begin{aligned}\frac{d}{dz}(\log \sin z) &= \cot z \\ &= \frac{1}{z} - 2 \sum_{n=1}^{\infty} \frac{z}{n^2\pi^2 - z^2}\end{aligned}$$

Thus, this gives us

$$\begin{aligned}z \cot z &= 1 - 2 \sum_{n=1}^{\infty} \frac{z^2}{n^2 \pi^2 - z^2} \\ &= 1 - 2 \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} \frac{1}{n^{2k}} \frac{z^{2k}}{\pi^{2k}} \\ &= 1 - 2 \sum_{k=1}^{\infty} \frac{z^{2k}}{\pi^{2k}} \zeta(2k).\end{aligned}$$

On the other hand, we can also calculate $z \cot z$ from another approach,

$$\begin{aligned}z \cot z = z \frac{\cos z}{\sin z} &= iz \frac{e^{iz} + e^{-iz}}{e^{iz} - e^{-iz}} \\ &= iz \frac{e^{2iz} + 1}{e^{2iz} - 1} \\ &= iz + \frac{2iz}{e^{2iz} - 1}\end{aligned}$$

Using the analytic definition of the Bernoulli numbers, we conclude that

$$\frac{z}{e^z - 1} = \sum_{n=0}^{\infty} B_n \frac{z^n}{n!},$$

evaluating at $2iz$, we have

$$\frac{2iz}{e^{2iz}-1} = \sum_{n=0}^{\infty} B_n \frac{(2iz)^n}{n!} = 1 - iz + \sum_{n=2}^{\infty} B_n \frac{(2iz)^n}{n!}.$$

Therefore, we have another expression for $z \cot z$,

$$z \cot z = 1 + \sum_{n=2}^{\infty} B_n \frac{(2iz)^n}{n!}.$$

Comparing the coefficients for the terms of z^n from the two expressions of $z \cot z$, we obtain the expression for $\zeta(2n)$,

$$\zeta(2n) = (-1)^{n+1} \frac{(2\pi)^{2n}}{2(2n)!} B_{2n}, \tag{5.3.1}$$

for n being positive integers.

6 Conclusion

The formula (5.3.1) will give us the values of the zeta function at any positive even integer. Reviewing what we have already derived, we have found that for s being negative even integers, $\zeta(s) = 0$. The zeta function has a simple pole at $s = 1$, and has a value of $-\frac{1}{2}$ at $s = 0$. We also know that the values of the zeta function at positive even integers can be expressed by Bernoulli numbers. Furthermore, by manipulating the functional equation of the zeta function, we can calculate the zeta function at any negative odd integers from the positive even integers.

However, there are still many details of the zeta function which remain as mysteries today. The famous Riemann Hypothesis and the values of the zeta function at odd positive integers serve as the best examples among the mysteries.

The Riemann Hypothesis conjectures that all the zeros of the $\zeta(s)$ in the critical strip, $0 < \text{Re}(s) < 1$, lie on the line $\text{Re}(s) = \frac{1}{2}$. So far, we only know that there are infinitely many zeros on the line of $\text{Re}(s) = \frac{1}{2}$, and at least a positive proportionate of all the zeros lie on the line.

Regarding the values of the zeta function values at positive odd integers, there is the conjecture that $\zeta(2n+1)$ is irrational, if $n > 1$ is odd. The only we know for sure is $\zeta(3)$, which was proven to be irrational by the Greek-French mathematician Roger Apéry in 1979. The nature of the values of the zeta function at other positive odd integers remain unknown.

7 Appendix

References

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[3] Sury, B., *Bernoulli Numbers and the Riemann Zeta Function*, 2003