Distribution Agreement

In presenting this thesis or dissertation as a partial fulfillment of the requirements for an advanced degree from Emory University, I hereby grant to Emory University and its agents the non-exclusive license to archive, make accessible, and display my thesis or dissertation in whole or in part in all forms of media, now or hereafter known, including display on the world wide web. I understand that I may select some access restrictions as part of the online submission of this thesis or dissertation. I retain all ownership rights to the copyright of the thesis or dissertation. I also retain the right to use in future works (such as articles or books) all or part of this thesis or dissertation.

Signature:

Bradley Fitzgerald Elliott | Date

Increasing paths in edge-ordered hypergraphs

By

Bradley Fitzgerald Elliott

Doctor of Philosophy

Mathematics

Vojtěch Rödl, Ph.D.

Advisor

Andrii Arman, Ph.D.

Committee Member

Dwight Duffus, Ph.D.

Committee Member

Hao Huang, Ph.D.

Committee Member

Accepted:

Lisa A. Tedesco, Ph.D.

Dean of the James T. Laney School of Graduate Studies

Date

Increasing paths in edge-ordered hypergraphs

By

Bradley Fitzgerald Elliott
B.S., University of Kentucky, 2013
M.B.A., University of Kentucky, 2014

Advisor: Vojtěch Rödl, Ph.D.

An abstract of
A dissertation submitted to the Faculty of the
James T. Laney School of Graduate Studies of Emory University
in partial fulfillment of the requirements for the degree of
Doctor of Philosophy
in Mathematics
2020

Abstract

Increasing paths in edge-ordered hypergraphs

By Bradley Fitzgerald Elliott

In this thesis, we study many variations on a classic problem of ordering the vertices or edges of a graph and determining the maximal possible length of "increasing" paths in the graph. For finite graphs, the vertex-ordering problem is completely solved, and there has been recent progress on the edge-ordering problem. Here, we prove the hypergraph version of the vertex-ordering problem: every vertex-ordered hypergraph H has an increasing path of at least $\chi(H)-1$ edges. We also provide upper bounds for the edge-ordering problem with respect to complete hypergraphs and Steiner triple systems.

For countably infinite graphs, a similar problem is studied. A result of Müller, Reiterman, and Rödl [12, 14] states that a countable graph has a subgraph with all infinite degrees if and only if any ordering of the vertices (or edges) of this graph permits an infinite increasing path. Here we study corresponding questions for hypergraphs and directed graphs. For example we show that the condition that any vertex-ordering of a simple hypergraph permits an infinite increasing path is equivalent to the condition that the hypergraph contains a subhypergraph with all infinite degrees. We prove a similar result for edge-orderings. In addition we find an equivalent condition for a graph to have the property that any vertex-ordering permits a path of arbitrary finite length. Finally we study related problems for orderings by $\mathbb{Z}$ (instead of $\mathbb{N}$). For example, we show that for every countable graph, there is an ordering of its edges by $\mathbb{Z}$ that forbids infinite increasing paths.

Increasing paths in edge-ordered hypergraphs

By

Bradley Fitzgerald Elliott

B.S., University of Kentucky, 2013

M.B.A., University of Kentucky, 2014

Advisor: Vojtěch Rödl, Ph.D.

A dissertation submitted to the Faculty of the

James T. Laney School of Graduate Studies of Emory University

in partial fulfillment of the requirements for the degree of

Doctor of Philosophy

in Mathematics

2020

Acknowledgments

This dissertation could not have been completed without the guidance and support of many people.

To my advisor Dr. Vojtěch Rödl and my collaborator Dr. Andrii Arman, thank you for your contributions to the research here. By working with you both I learned so much about not only graph theory but also writing, publication, and perseverance.

To my committee members, thank you for your feedback on this dissertation. To Dr. Hao Huang especially, I appreciate the four semesters of graph theory and combinatorics courses you taught me, which were invaluable for the completion of this work.

To all of my collaborators, thank you for improving my research abilities and my work ethic. You broadened my scope of research projects, and I had a lot of fun working with you along the way.

Finally, to my family and friends, thank you for always believing I would complete this journey, even when I didn't. To my husband Kalib especially, thank you for always listening when I described problems with my research and for letting me practice my presentations on you. I love you and can't imagine having gone through these five years without you.

Contents

1 Introduction **1**

1.1 Definitions 1

1.1.1 Graphs 1

1.1.2 Hypergraphs 2

1.1.3 Directed graphs 2

1.1.4 Paths 3

1.2 Background 3

1.2.1 Finite graphs 3

1.2.2 Countable graphs 5

1.3 Results 6

1.3.1 Finite hypergraphs 6

1.3.2 Countable graphs, hypergraphs, and digraphs 8

2 Finite Hypergraphs **16**

2.1 Tight Paths in Hypergraphs 16

2.2 Loose Paths in Steiner Triple Systems 21

2.3 Vertex-orderings in Hypergraphs 26

3 Countable Graphs, Hypergraphs, and Digraphs **29**

3.1 Countable Graphs 29

3.2 Countable Hypergraphs 33

3.2.1 Vertex-ordering 33
3.2.2 Edge-ordering 36
3.3 Countable Digraphs 40
3.4 Arbitrary Finite Paths in Countable Graphs 50
3.4.1 Paths of Arbitrary Finite Length 50
3.4.2 A Counterexample 52

A Alternative Proof of Theorem 2.1.1 **56**

Bibliography **62**

List of Figures

1.1 (a) A finite graph on 5 vertices (b) A 3-uniform hypergraph, in which each oval represents an edge (c) A directed graph, in which each arrow represents the direction of the oriented edge 2

1.2 An example of an infinite 3-branching tree, which is a 4-uniform hypergraph. Each gray polygon here represents an edge. 11

2.1 STS(9), the only Steiner triple system on 9 vertices, where each monochromatic set of line segments represents a matching. 22

3.1 Each $x \in I$ has infinite degree, each $y \in F$ has finite degree. 53

3.2 Each $x \in V_i^L$ has finite degree to U_i^R and infinite degree to V_j^R for $1 \leq j < i$. 53

Chapter 1

Introduction

My primary research interest is *extremal graph theory*. Generally, an extremal graph problem may ask, "If a graph has property X, how large or small can some other graph property Y be?" In this thesis we particularly focus on what graph properties imply the existence of long paths with certain characteristics. In this chapter we define many of the terms we will use, give a background of our topics, and describe our results.

1.1 Definitions

Here we formally define graphs, hypergraphs, and digraphs, along with many of their properties.

1.1.1 Graphs

A *graph* $G = (V, E)$ is an ordered pair, where V is called the *vertex set* of G, and $E \subseteq \binom{V}{2}$ is a family of unordered pairs of vertices called the *edge set* of G. Graphs are useful mathematical structures for modeling any system in which some pairs of objects share a certain relationship. For example, one could model Facebook's users and their friendships with a graph: the vertex set would represent the set of all Facebook users, and the edge set would represent all Facebook friendships, since a friendship is just a relationship between pairs of users. See Figure 1.1(a) for an example of a graph.

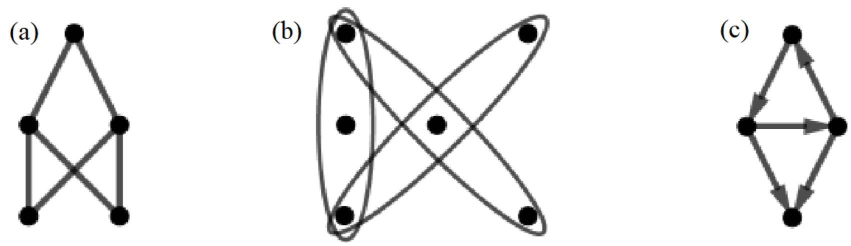

Figure 1.1: (a) A finite graph on 5 vertices
(b) A 3-uniform hypergraph, in which each oval represents an edge
(c) A directed graph, in which each arrow represents the direction of the oriented edge

The vertex set of a graph is typically finite, but we present some results for which the vertex set is countably infinite. We call a graph countable if its vertex set is countably infinite. We say a graph is complete if it contains an edge between every pair of vertices, and we denote by K_n the complete graph on n vertices.

1.1.2 Hypergraphs

For $r \geq 2$, an r-uniform *hypergraph* $H = (V, E)$ is a generalization of a graph in which each edge in E contains exactly r vertices. More formally, the edge set $E \subseteq \binom{V}{r}$. We sometimes refer to r-uniform hypergraphs as r-graphs and to elements of their edge sets as r-tuples. We denote by $K_n^{(r)}$ the complete r-graph on n vertices, which contains every possible r-tuple. See Figure 1.1(b) for an example of a 3-graph.

1.1.3 Directed graphs

A *directed graph* $D = (V, E)$ is a graph in which each edge has an orientation from one of its vertices to the other. In other words, the edges of a directed graph are ordered pairs of vertices. For $u, v \in V$, we write that $(u, v) \in E$ if there is an edge in E directed from u to v. We sometimes refer to directed graphs as digraphs and to the edges of digraphs as arcs. See Figure 1.1(c) for an example of a digraph.

1.1.4 Paths

A *path* in a graph $G = (V, E)$ is a non-repeating sequence of vertices $\{v_1, v_2, \dots, v_p\} \subset V$ such that $\{v_i, v_{i+1}\} \in E$ for all $1 \leq i \leq p-1$. Alternatively we may define a path as a sequence of distinct edges $\{e_1, \dots, e_{p-1}\} \subset E$ for which $|e_i \cap e_j| = 1$ if and only if $|i - j| = 1$. In a digraph, it is required for a path that each edge be directed from v_i to v_{i+1}; that is, $(v_i, v_{i+1}) \in E$. There are several notions of paths in k-uniform hypergraphs that we will discuss throughout this thesis. Generally, in a "tight path," two consecutive edges of the path share $k-1$ common vertices; in a "loose path," two consecutive edges share just 1 common vertex. In Figure 1.1, the graph (a) has a path containing four edges, the 3-graph (b) has a loose path containing two edges, and the digraph (c) has a directed path containing three edges.

1.2 Background

1.2.1 Finite graphs

For a finite graph $G = (V, E)$, an ordering (or labeling) of the vertices of G is a bijection from V to $[|V|]$, the set of integers from 1 to $|V|$. In a graph with vertex-ordering ϕ, an increasing path is a path $\{v_1, v_2, \dots, v_p\}$ such that $\phi(v_i) < \phi(v_{i+1})$ for all $1 \leq i \leq p-1$. Let $f_\phi(G)$ be the maximum length (i.e. maximum number of edges) of an increasing path in G with vertex-ordering ϕ, and let

$$f(G) = \min_\phi f_\phi(G),$$

so that G contains an increasing path of length $f(G)$ regardless of how its vertices are ordered. This function $f(G)$ was determined independently by Gallai, Hasse, Roy, and Vitaver [6, 9, 16, 17].

Theorem 1.2.1 (Gallai-Hasse-Roy-Vitaver Theorem). *For a graph G, $f(G) = \chi(G) - 1$, where $\chi(G)$ is the chromatic number of G.*

Similarly, for a finite graph $G = (V, E)$, an ordering of the edges of G is a bijection from E to $[|E|]$. In a graph with edge-ordering ϕ, an increasing path is a path $\{e_1, e_2, \ldots, e_p\}$ such that $\phi(e_i) < \phi(e_{i+1})$ for all $1 \leq i \leq p-1$. Let $g_\phi(G)$ be the maximum length (i.e. maximum number of edges) of an increasing path in G with edge-ordering ϕ, and let

$$g(G) = \min_{\phi} g_\phi(G).$$

Perhaps surprisingly, $g(G)$ seems much harder to determine than $f(G)$. Since Chvátal and Komlós [5] first asked what is $g(K_n)$ in 1971, significant effort has been put into bounding this function of n. Graham and Kleitman [7] showed that $\Omega(\sqrt{n}) \leq g(K_n) \leq (3/4)n$. The upper bound was subsequently improved by Rödl [15] to $(2/3 + o(1))n$, by Alspach, Heinrich, and Graham (unpublished, see [4]) to $(7/12 + o(1))n$, and by Calderbank, Chung, and Sturtevant [4] to $(1/2 + o(1))n$. After over 40 years, the lower bound was finally improved by Milans [11] to $\Omega((n/\log n)^{2/3})$. Inspired by that paper, Bucić et al. [3] raised the lower bound to

$$\frac{n}{2^{O(\sqrt{\log n \log\log n})}} = n^{1-o(1)},$$

nearly closing the gap between upper and lower bounds.

For graphs other than K_n, similar results are known. Let G be a finite graph with n vertices and average degree d. Of course $g(G) \leq d$, since G may be a disjoint union of $(d+1)$-cliques. Rödl [15] proved $g(G) = \Omega(\sqrt{d})$. For sufficiently dense graphs, Milans [11] improved this to $g(G) = \Omega(d/n^{1/3}(\log n)^{2/3})$, and Bucić et al. [3] further improved it to

$$g(G) \geq \frac{d}{2^{O(\sqrt{\log d \log \log n})}}$$

when $d \geq 2$.

1.2.2 Countable graphs

For a countable graph $G = (V, E)$ an ordering of the vertices of G by $\mathbb{N}$ is a bijection from V to $\mathbb{N} = \{0, 1, 2, \ldots\}$ and an ordering of the edges by $\mathbb{N}$ is a bijection from E to $\mathbb{N}$. Given a countable graph $G = (V, E)$ and an ordering ϕ of V by $\mathbb{N}$, we say G contains an *infinite increasing path* with respect to ϕ if there exists an infinite path of vertices $\{v_i\}_{i=1}^{\infty}$ in G such that $\phi(v_i) < \phi(v_{i+1})$ for all $i \geq 1$. Similarly, given a countable graph $G = (V, E)$ and an ordering ϕ of E by $\mathbb{N}$, we say G contains an infinite increasing path with respect to ϕ if there exists an infinite path of edges $\{e_i\}_{i=1}^{\infty}$ in G such that $\phi(e_i) < \phi(e_{i+1})$ for all $i \geq 1$. In 1982, Müller and Rödl [12] showed that a countable graph G contains an infinite increasing path with respect to any ordering of its vertices by $\mathbb{N}$ if and only if G contains a subgraph in which every vertex has infinite degree, i.e. there exists a subgraph G' of G so that for all $v \in V(G')$, $|\{u \in V(G') : \{v, u\} \in E(G')\} = \infty$. In their paper Müller and Rödl asked whether or not the condition of having a subgraph with all infinite degrees is also necessary for containing an infinite increasing path with respect to any edge-ordering of G by $\mathbb{N}$. This was confirmed to be true by Reiterman [14] in 1989. Together these two results are formulated below.

Theorem 1.2.2. *Let $G = (V, E)$ be a countable graph. Then the following are equivalent:*

(1) *G contains a subgraph G', such that any $v \in V(G')$ has infinite degree in G'.*

(2) *any ordering of V by $\mathbb{N}$ permits an infinite increasing path.*

(3) *any ordering of E by $\mathbb{N}$ permits an infinite increasing path.*

1.3 Results

In this thesis we give many results concerning increasing paths in edge-ordered hypergraphs. We also give some results on vertex-orderings, graphs, and digraphs. We consider both finite and countable graphs, and we discuss the existence of increasing paths of arbitrarily large finite length. In this section we state all of our results, whose proofs are provided in Chapters 2 and 3. This is joint work with Andrii Arman and Vojtěch Rödl, some of which appears in [2].

1.3.1 Finite hypergraphs

If $H = (V, E)$ is an r-graph, a *tight path* in H is a sequence of edges $\{e_1, e_2, \cdots, e_\ell\} \subset E$ so that there exist distinct vertices $v_1, v_2, \cdots, v_{\ell+r-1} \in V$ with

$$e_i = \{v_i, v_{i+1}, \cdots, v_{i+r-1}\}$$

for $1 \leq i \leq \ell$. A tight path is *increasing* with respect to an edge-ordering ϕ if $\phi(e_i) < \phi(e_{i+1})$ for all $1 \leq i \leq \ell - 1$. Let $g_\phi(H)$ be the maximum length of an increasing tight path in H with respect to a given edge-ordering ϕ, and let

$$g(H) = \min_{\phi} g_\phi(H),$$

so that for every edge-ordering ϕ there is an increasing tight path of length at least $g(H)$ in H.

Theorem 2.1.1. *For any $\delta > 0$ and any integer $r \geq 2$ there is an $n_0 = n_0(\delta, r)$ such that if $n \geq n_0$, then*

$$g\left(K_n^{(r)}\right) < \frac{(1+\delta)r}{r+1} n.$$

A Steiner triple system (STS) is a 3-graph in which every pair of vertices is contained in exactly one edge (or *triple*). STSs exist on n vertices if and only if $n \equiv 1$ or 3 (mod 6), and they contain no tight paths, because no pair of vertices in an STS share two edges. A *loose path* in an r-uniform hypergraph is a sequence of edges $\{e_1, \ldots, e_\ell\} \subset E$ so that there exist distinct vertices $v_1, v_2, \ldots v_{\ell(r-1)+1} \in V$ with

$$e_i = \{v_{(r-1)(i-1)+1}, v_{(r-1)(i-1)+2}, \ldots, v_{(r-1)(i-1)+r}\}.$$

Let $s(n)$ be the maximum k so that every STS on n vertices, regardless of how its edges are ordered, contains an increasing path of length k. Trivially $s(n) \leq (n-1)/2$, since each edge in a path introduces two new vertices, except the first edge which uses three. We provide an upper bound on $s(n)$ for certain n.

Theorem 2.2.1. *If $n = 9^p$ for some $p \in \mathbb{Z}$, then*

$$s(n) \leq \frac{n-1}{4}.$$

For completeness, we extend the Gallai-Hasse-Roy-Vitaver Theorem (1.2.1) to r-graphs.

Definition 1. Given an r-uniform hypergraph $H = (V, E)$ with a vertex-ordering ϕ, an *increasing loose path* of length ℓ in H is a loose path with vertex set $\{v_1, v_2, \ldots, v_{\ell(r-1)+1}\}$ and edges $\{e_1, \ldots, e_\ell\}$, where

(1) each $e_i = \{v_{(r-1)(i-1)+1}, v_{(r-1)(i-1)+2}, \ldots, v_{(r-1)(i-1)+r}\}$ for $1 \leq i \leq \ell$,

(2) all v_j-s are distinct, and

(3) $\phi(v_j) < \phi(v_{j+1})$ for $1 \leq j \leq \ell(r-1)$.

Let $f_\phi(H)$ be the maximum number of edges of an increasing loose path in H with respect to vertex-ordering ϕ, and let

$$f(H) = \min_{\phi} f_{\phi}(H).$$

Then

Theorem 2.3.1. $f(H) = \chi(H) - 1$.

1.3.2 Countable graphs, hypergraphs, and digraphs

Ordering by $\mathbb{Z}$

In [12], Müller and Rödl considered only graphs ordered by ordinals. Specifically, their Theorem 1.2.2 concerns orderings by $\mathbb{N}$. In Section 3.1 we consider countable graphs with vertices and edges ordered by $\mathbb{Z}$, which is not an ordinal. This new style of ordering allows us to consider paths that continue infinitely in both directions. Given a countable graph $G = (V, E)$ and an ordering ϕ of V by $\mathbb{Z}$, an *infinite increasing two-sided path* in G is an infinite path $\{v_i : -\infty < i < \infty\}$, with $\phi(v_i) < \phi(v_{i+1})$ for all i. For the existence of an infinite increasing two-sided path we find the following equivalent condition.

Theorem 3.1.2. *For a countable graph $G = (V, E)$ the following are equivalent.*

(1) *For every partition of V into infinite sets V_1, V_2, there exist $W_1 \subseteq V_1$, $W_2 \subseteq V_2$ so that*

(a) *for $i = 1, 2$ and for all $v \in W_i$, v has infinite degree in W_i, and*

(b) *there exists an edge $w_1w_2 \in E$ with $w_1 \in W_1$ and $w_2 \in W_2$.*

(2) *Any ordering of V by $\mathbb{Z}$ permits an infinite increasing two-sided path.*

Perhaps somewhat unexpectedly, for edge-ordering the following holds.

Proposition 3.1.3. *For every countable graph $G = (V, E)$ there exists an ordering of E by $\mathbb{Z}$ such that there is no infinite increasing path in G.*

Hypergraphs

Our main motivation for Section 3.2 is to find a hypergraph extension of the Müller-Reiterman-Rödl Theorem (1.2.2). We would like to retain the condition of "G contains a subgraph G', such that any $v \in V(G')$ has infinite degree in G'" from this theorem. For this condition to imply the existence of infinite paths, though, we need to consider "simple" (also known as "linear") hypergraphs; these are hypergraphs in which each pair of vertices share at most one edge. This is because there exist many examples of non-simple hypergraphs in which every vertex has infinite degree but which do not contain any infinite paths. For example, take the 3-graph with vertex set $V = \mathbb{N}$ and edge set $E = \{\{0, i, j\} : 1 \leq i < j\}$, which contains no infinite path because every edge contains the same vertex. For this reason, all the theorems below are for simple hypergraphs, though they still hold if "simple" is replaced by "each pair of vertices have finite co-degree". Because all hypergraphs considered here are simple, all paths in this section are loose paths.

We actually consider two hypergraph generalizations of graph paths. The first, defined in Definition 2, uses the fact that in a graph path, every two consecutive edges of the path share exactly one vertex. The second, defined in Definition 4, uses the fact that in a graph path, every vertex except the first one is contained in two edges of the path.

Definition 2. Given a k-uniform hypergraph $H = (V, E)$, an *infinite path* in H is an infinite path with vertex set $\{v_j\}_{j=1}^{\infty}$ and edges $\{e_0, \ldots, e_i, \ldots\}$, where

(1) each $e_i = \{v_{(k-1)i+1}, v_{(k-1)i+2}, \ldots, v_{(k-1)i+k}\}$ for $i \geq 0$, and

(2) all v_j-s are distinct.

Given a k-uniform hypergraph $H = (V, E)$ and an ordering ϕ of V by $\mathbb{N}$, an *infinite increasing path* in H is an infinite path with the added property that

(3) $\phi(v_j) < \phi(v_{j+1})$ for $j \geq 1$.

In the graph case, the condition of containing a subgraph with infinite degrees implies existence of an infinite increasing path under any vertex-ordering, and this condition is a natural candidate for the hypergraph case; however, the notion of a subgraph with infinite degrees has multiple interpretations in the hypergraph case.

Definition 3. Let $\ell \leq k$. We say that a k-graph $H = (V, E)$ has property C_ℓ if there is a set $V' \subseteq V$ such that $\forall v \in V'$, $|\{e \in E : v \in e \text{ and } |V' \cap e| \geq \ell\}| = \infty$. That is, if there is a set $V' \subseteq V$ so that every vertex in V' in in infinitely many edges that each have at least ℓ vertices in V'.

For a k-graph H, property C_k is perhaps the most natural hypergraph extension of the graph condition, and is equivalent to H containing an induced subhypergraph in which every vertex has infinite degree. It turns out property C_k implies more than just containing an infinite increasing path under any vertex-ordering of $V(H)$. It implies existence of what we call an infinite increasing $(k-1)$-branching tree.

Definition 4.

(1) The *infinite* $(k-1)$*-branching tree* is the unique k-uniform hypergraph so that each pair of vertices is connected by a unique path and so that all vertices have degree 2, except for a root vertex, which has degree 1. See figure 1.2.

(2) In an infinite $(k-1)$-branching tree, $t(v)$ will denote the depth of a vertex v in the tree, i.e. the number of edges on the path from v to the root.

(3) Given a k-uniform hypergraph $H = (V, E)$ and an ordering ϕ of V by $\mathbb{N}$, an *infinite increasing* $(k-1)$*-branching tree* in H is an infinite $(k-1)$-branching tree in H with the property that that for any $x, y \in V$ with $t(x) < t(y)$, $\phi(x) < \phi(y)$.

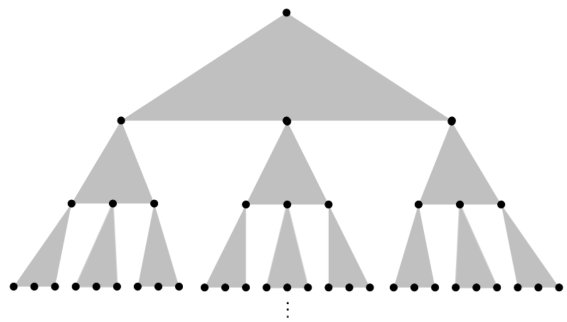

Figure 1.2: An example of an infinite 3-branching tree, which is a 4-uniform hypergraph. Each gray polygon here represents an edge.

Theorem 3.2.1. *For a countable, k-uniform hypergraph $H = (V, E)$ the following are equivalent*

(1) *H has property C_k.*

(2) *Any ordering of V by $\mathbb{N}$ permits an infinite increasing $(k-1)$-branching tree in H.*

(3) *Any ordering of V with $\mathbb{N}$ permits an infinite increasing path in H.*

For edge-orderings of k-uniform hypergraphs, infinite increasing paths and $(k-1)$-branching trees are defined similarly:

Definition 5. Given a k-uniform hypergraph $H = (V, E)$ and an ordering ϕ of E by $\mathbb{N}$, an *infinite increasing path* in H is an infinite path $\{e_1, \ldots, e_i, \ldots\}$ such that $\{\phi(e_i)\}_{i=1}^{\infty}$ is an increasing sequence.

Definition 6.

(1) In an infinite $(k-1)$-branching tree, $t(e)$ will denote the depth of an edge e in the tree, i.e. the number of edges on the path from e to the root. E.g. $t(e) = 1$ for the edge e that contains the root.

(2) Given an ordering ϕ of E by $\mathbb{N}$, an *infinite increasing* $(k-1)$*-branching tree* in H is an infinite $(k-1)$-branching tree in H with that property that for any $e_1, e_2 \in E$ with $t(e_1) < t(e_2)$, $\phi(e_1) < \phi(e_2)$.

It turns out that property C_2 is a sufficient condition for containing an infinite increasing path under any edge-ordering and a necessary condition for containing an infinite increasing $(k-1)$-branching tree under any edge-ordering.

Theorem 3.2.3. *Let $H = (V, E)$ be a simple, countable, k-uniform hypergraph. Then for the following conditions,* $(1) \Rightarrow (2) \Rightarrow (3)$.

(1) *Any ordering of E by $\mathbb{N}$ permits an infinite increasing $(k-1)$-branching tree in H.*

(2) *H has property C_2.*

(3) *Any ordering of E by $\mathbb{N}$ permits an infinite increasing path in H.*

We also notice that in the theorem above $(2) \not\Rightarrow (1)$. Consider a complete countable graph G with vertices $V = \{v_1, v_2, \ldots\}$ and complement every edge $e = \{v_i, v_j\}$ by $k-2$ new vertices $u_{i,j,1}, u_{i,j,2}, \ldots, u_{i,j,k-2}$ to form a k-uniform hypergraph $H = (V \cup U, E)$ with $E = \{\{v_i, v_j, u_{i,j,1}, u_{i,j,2}, \ldots, u_{i,j,k-2}\} : i, j \in \mathbb{N}\}$. The original vertex set V satisfies (2). On the other hand, there is no infinite $(k-1)$-branching tree in H because every hyperedge contains $k-2$ vertices of degree 1.

It remains unknown to us if (2) is equivalent to (3) or not, even for the case $k = 3$.

Directed Graphs

In Section 3.3 we consider directed graphs whose vertices or edges are ordered by $\mathbb{N}$ or $\mathbb{Z}$. These results correspond to theorems previously given about undirected graphs.

Proposition 3.3.1. *For a directed graph $D = (V, E)$, the following are equivalent.*

(1) *There exists an induced subgraph D' of D of which all vertices have infinite out-degree in D'.*

(2) *Any ordering of V by $\mathbb{N}$ permits an infinite increasing directed path in D.*

(3) *Any ordering of E by $\mathbb{N}$ permits an infinite increasing directed path in D.*

Definition 7. Given a directed graph $D = (V, E)$ and an ordering ϕ of V by $\mathbb{Z}$, we say D contains an *infinite two-sided directed path* if there exists an infinite directed path $\{v_i\}_{-\infty}^{\infty}$ in D with $\phi(v_i) < \phi(v_{i+1})$ for all i.

Proposition 3.3.8. *For a directed graph $D = (V, E)$, the following are equivalent.*

(1) *For every partition of V into infinite sets V_1, V_2 there exist $W_1 \subseteq V_1$, $W_2 \subseteq V_2$ so that*

   (a) *for all $v \in W_1$, v has infinite out-degree in $D[W_1]$,*

   (b) *for all $v \in W_2$, v has infinite in-degree in $D[W_2]$, and*

   (c) *there are vertices $w_i \in W_i$ for $i = 1, 2$, such that $(w_2, w_1) \in E$.*

(2) *Any ordering of V by $\mathbb{Z}$ permits an infinite two-sided directed path.*

Proposition 3.3.10. *For every countable directed graph $D = (V, E)$ there exists an ordering of E by $\mathbb{Z}$ such that there is no infinite increasing directed path in D.*

Paths of finite length

In [12], the authors also showed for a graph $G = (V, E)$ that for any well-ordered set $\mathcal{L}$ and any labeling of V by $\mathcal{L}$ there exists an arbitrarily long increasing path if and only if the chromatic number of G is infinite. The main theorem in Section 3.4 is about restricting $\mathcal{L}$ to $\mathbb{N}$, i.e. it is about finding an equivalent condition for a countable graph G to contain an increasing path of arbitrary finite length under any ordering of vertices by $\mathbb{N}$. Theorem 1.2.2 implies that if G has a subgraph with infinite degrees, then under any ordering of vertices by $\mathbb{N}$ we can find an infinite increasing path. Hence the only interesting case to study is when G does not have a subgraph with all infinite degrees.

Definition 8. We say that a countable graph $G = (V, E)$ has a *property FIN* if for any ordering of V by $\mathbb{N}$ and for any $k \in \mathbb{N}$ there exists an increasing path of length k in G, but there exists a vertex-ordering of G with no infinite increasing path.

Definition 9. We define $\chi^*(G)$ to be the minimum k so that $V(G)$ can be partitioned into k classes $V_1, \dots, V_k$ in such a way that for all $1 \leq j \leq k$,

(1) each V_j is an independent set, and

(2) each vertex in V_j has finite degree into $V_1, V_2, \dots, V_{j-1}$.

If no such k exists, set $\chi^*(G) = \infty$. Clearly, $\chi^*(G) \geq \chi(G)$.

The condition $\chi^*(G) = \infty$ is inspired by the condition used in Theorem 1.2.1, and this condition is sufficient to imply that for any ordering of V by $\mathbb{N}$ and for any $k \in \mathbb{N}$ there exists an increasing path of length k in G. The main result of Section 3.4 is the following.

Theorem 3.4.2. *A graph G has property FIN if and only if $\chi^*(G) = \infty$ and G does not have a subgraph in which all degrees are infinite.*

Note that the conditions of $\chi^*(G) = \infty$ and G not having a subgraph in which all degrees are infinite do not imply each other, so both are needed in the statement of Theorem 3.4.2. In particular, the complete graph on infinitely but countably many vertices has $\chi^*(G) = \infty$ but is itself a graph in which all degrees are infinite. On the other hand, any union of vertex-disjoint finite graphs has no subgraph in which all degrees are infinite and it has finite $\chi^*(G)$.

We show that χ^* cannot be changed to χ in the statement of the theorem above by constructing a bipartite graph G that has property FIN. So for property FIN, the chromatic number of a graph has insignificant impact on the existence of an infinite increasing path when $V(G)$ is ordered by $\mathbb{N}$, in contrast to the case when the labels can be elements of any well-ordered set $\mathcal{L}$.

Chapter 2

Finite Hypergraphs

In this chapter we prove theorems on finite hypergraphs of different structures, concerning both edge- and vertex-ordering.

2.1 Tight Paths in Hypergraphs

Recall the definition of $g(H)$ on page 6.

Theorem 2.1.1. *For any $\delta > 0$ and any integer $r \geq 2$ there is an $n_0 = n_0(\delta, r)$ such that if $n \geq n_0$, then*

$$g\left(K_n^{(r)}\right) < \frac{(1+\delta)r}{r+1}n.$$

We use the following claim in the proof of Theorem 2.1.1.

Claim 2.1.2. *For the complete r-uniform hypergraph $K_{2r}^{(r)}$ on $2r$ vertices,*

$$g\left(K_{2r}^{(r)}\right) < r+1.$$

Proof. For any edge $e \in E\left(K_{2r}^{(r)}\right)$ there is exactly one other edge $e' \in E\left(K_{2r}^{(r)}\right)$ for which $e \cap e' = \emptyset$. Let ϕ be an edge-ordering of $K_{2r}^{(r)}$ so that for any such pair of edges $e, e' \in E\left(K_{2r}^{(r)}\right)$ where $e \cap e' = \emptyset$, $\phi(e)$ and $\phi(e')$ are consecutive integers. The statement of the claim follows from the observation that for a tight path of length $r+1$ in $K_{2r}^{(r)}$, the initial and terminal edges are disjoint. $\square$

The proof of Theorem 2.1.1 presented relies on the existence of certain block designs. A set $\mathcal{B}$ of q-tuples $B \in \binom{[n]}{q}$ is called a *design* with parameters (n, q, r, λ) if every r-tuple of $[n]$ belongs to exactly λ elements of $\mathcal{B}$, i.e. to exactly λ "blocks" B. For the existence of such a design, it is necessary that $\binom{q-i}{r-i} \mid \lambda\binom{n-i}{r-i}$ for all $0 \le i \le r-1$. (To see why, consider any i-tuple $S \in \binom{[n]}{i}$. There are $\binom{n-i}{r-i}$ r-tuples containing S, and each appears in λ blocks of $\mathcal{B}$. Each block covers $\binom{q-i}{r-i}$ of these r-tuples containing S.) In a celebrated paper, Keevash [10] showed that this necessary condition is also sufficient for the existence of a (n, q, r, λ)-design, provided n is sufficiently large. Specifically, Keevash's result gives us the following.

Theorem 2.1.3 ([10]). *If n is sufficiently large and $\binom{2r-i}{r-i} \mid \binom{n-i}{r-i}$ for all $0 \le i \le r-1$, then there exists a $(n, 2r, r, 1)$-design $\mathcal{B}$.*

Since [10] is not yet published, we also offer another proof of Theorem 2.1.1 in Appendix A that relies on an approximate version of [10].

Proof of Theorem 2.1.1. While n, the order of $K_n^{(r)}$, may not satisfy the divisibility conditions of Theorem 2.1.3, we claim that there exists some n' with $n \ge n' \ge n-(2r)!$ such that $\binom{2r-i}{r-i} \mid \binom{n'-i}{r-i}$ for all $0 \le i \le r-1$. Indeed, let $n' = \ell(2r)! + r - 1$ where $\ell \in \mathbb{N}$ is maximized with the constaint that $n' \le n$. For the divisibility conditions to be satisfied, it is needed that

$$(2r-i)(2r-i-1)\cdots(2r-i-r+1)\Big|(n'-i)(n'-i-1)\cdots(n'-i-r+1)$$

for all $0 \le i \le r-1$. Since the right-hand side $(n'-i)(n'-i-1)\cdots(n'-i-r+1)$ contains a factor of $n'-r+1$ for all i, it always contains a factor of $(2r)!$. Since the left-hand side $(2r-i)(2r-i-1)\cdots(2r-i-r+1)$ is always a divisor of $(2r)!$, the divisibility condition is met for all i. Note that for large n, $\frac{n'}{n} \to 1$.

Label the vertices of $K_n^{(r)}$ by $[n]$, let $K_{n'}^{(r)}$ be the induced subhypergraph on the first n' vertices of $K_n^{(r)}$. Let $\mathcal{B}$ be a $(n', 2r, r, 1)$-design on $K_{n'}^{(r)}$, the existence of which is guaranteed by Theorem 2.1.3. The design $\mathcal{B}$ has the following properties.

(1) Each edge $e \in E\left(K_{n'}^{(r)}\right) = \binom{[n']}{r}$ appears in exactly one block $B \in \mathcal{B}$.

(2) Each $(r-1)$-subset of $[n']$ appears in exactly $\frac{n'-(r-1)}{r+1}$ blocks of $\mathcal{B}$.

(3) For any two distinct $(r-1)$-subsets I and J of $[n']$, $|I \cup J| \geq r$. By (1) above, $I \cup J$ appears in at most one block of $\mathcal{B}$.

We introduce the following claim, whose proof is postponed until after the proof of Theorem 2.1.1.

Claim 2.1.4. *For $k = \left\lfloor (1+\delta)\frac{n'-(r-1)}{r+1} \right\rfloor$ there exists a partition of the blocks of $\mathcal{B}$ into classes $\mathcal{F}_1, \ldots, \mathcal{F}_k$ so that for all $i = 1, \ldots, k$, and all distinct $B, B' \in \mathcal{F}_i$,*

$$|B \cap B'| \leq r-2.$$

Let our design $\mathcal{B}$ be partitioned into $\mathcal{F}_1, \cdots, \mathcal{F}_k$ as described by Claim 2.1.4. We define ϕ to be an edge-ordering of $K_{n'}^{(r)}$ satisfying the following.

i) The ordering ϕ labels the r-subsets of each $B \in \mathcal{B}$ as in Claim 2.1.2.

ii) For any $B \in \mathcal{F}_i$, $B' \in \mathcal{F}_j$ with $i < j$, make $\phi(e) < \phi(e')$ for all r-subsets $e \subset B$, $e' \subset B'$.

This ordering is well-defined, by (1) above. Note that for distinct blocks $B, B' \in \mathcal{F}_i$, no increasing tight path in $K_{n'}^{(r)}$ uses both an edge $e \subset B$ and an edge $e' \subset B'$, since the intersection of the blocks has order at most $r-2$ while two consecutive edges of a tight path must have intersection of order exactly $r-1$. Along with Claim 2.1.2, this implies that an increasing tight path with respect to ϕ in $K_{n'}^{(r)}$ uses at most r edges from each partition class $\mathcal{F}_i$. Fixing any $\delta' < \delta$, we have

$$g\left(K_{n'}^{(r)}\right) \leq rk \leq r(1+\delta')\frac{n'-(r-1)}{r+1} < \frac{(1+\delta')r}{r+1}n'.$$

To determine $g\left(K_n^{(r)}\right)$, we extend the edge-ordering ϕ of $K_{n'}^{(r)}$ to an edge-ordering ψ of $K_n^{(r)}$. For $e \in E\left(K_{n'}^{(r)}\right)$, we let $\psi(e) = \phi(e)$. Then we label the edges from $E\left(K_n^{(r)}\right) \setminus E\left(K_{n'}^{(r)}\right)$ arbitrarily with the integers $\{\binom{n'}{r}+1, \binom{n'}{r}+2, \ldots, \binom{n}{r}\}$. Any increasing path with respect to ψ in $K_n^{(r)}$ can start by using at most $g\left(K_{n'}^{(r)}\right)$ edges of $K_{n'}^{(r)}$. Then it must use edges from $E\left(K_n^{(r)}\right) \setminus E\left(K_{n'}^{(r)}\right)$, meaning at least one out of every r vertices must be from $V\left(K_n^{(r)}\right) \setminus V\left(K_{n'}^{(r)}\right)$, since r consecutive vertices from $K_{n'}^{(r)}$ would mean an edge of $K_{n'}^{(r)}$ had been used. Since there are $n - n' \leq (2r)!$ such vertices in $V\left(K_n^{(r)}\right) \setminus V\left(K_{n'}^{(r)}\right)$, the original increasing path from $K_{n'}^{(r)}$ can be extended by at most $r(2r)!$ edges. So

$$g\left(K_n^{(r)}\right) \leq \frac{(1+\delta')r}{r+1}n' + r(2r)! \leq \frac{(1+\delta)r}{r+1}n$$

□

Now it remains only to prove Claim 2.1.4. The proof uses the following result of Pippenger and Spencer [13]. Let $D(\mathcal{A})$ be the maximum degree of $\mathcal{A}$, $d(\mathcal{A})$ be the minimum degree of $\mathcal{A}$, and $C(\mathcal{A})$ be the maximum co-degree of $\mathcal{A}$ (i.e., the maximum number of edges of $\mathcal{A}$ all containing the same pair of vertices).

Theorem 2.1.5 ([13]). *For every $\ell \geq 2$ and $\delta > 0$, there exists $\delta' > 0$ and n_0 such that if $\mathcal{A}$ is an ℓ-uniform hypergraph on $v(\mathcal{A}) \geq n_0$ vertices satisfying*

$$d(\mathcal{A}) \geq (1-\delta')D(\mathcal{A})$$

and

$$C(\mathcal{A}) \leq \delta' D(\mathcal{A}),$$

then the edges of $\mathcal{A}$ can be partitioned into $(1+\delta)D(\mathcal{A})$ matchings.

Proof of Claim 2.1.4. Define an auxiliary hypergraph $\mathcal{A}$ with

$$V(\mathcal{A}) = \binom{[n']}{r-1}, \text{ and}$$

$$E(\mathcal{A}) = \left\{ \binom{B}{r-1} : B \in \mathcal{B} \right\}.$$

The hypergraph $\mathcal{A}$ has the following properties.

(a) Each edge of $\mathcal{A}$ corresponds to a distinct block of $\mathcal{B}$.

(b) Each vertex of $\mathcal{A}$ has degree exactly $\frac{n'-(r-1)}{r+1}$. This follows from (2) on page 18.

(c) For any pair of vertices in $\mathcal{A}$, they share at most one edge of $\mathcal{A}$. This follows from (3) on page 18.

(d) Two edges of $\mathcal{A}$ are disjoint if and only if the corresponding blocks of $\mathcal{B}$ share at most $r-2$ vertices.

By (d), a matching in $\mathcal{A}$ corresponds to a set of blocks of $\mathcal{B}$ in which each pair of blocks shares at most $r-2$ vertices. Any partition of $E(\mathcal{A})$ into k disjoint matchings (for some k) gives a partition of $\mathcal{B}$ into classes $\mathcal{F}_1, \mathcal{F}_2, \ldots, \mathcal{F}_k$ so that for all $i = 1, \ldots k$, and all distinct $B, B' \in \mathcal{F}_i$,

$$|B \cap B'| \leq r-2.$$

In order to keep the number k of disjoint matchings in $\mathcal{A}$ small, we use [13]. By (b), $d(\mathcal{A}) = D(\mathcal{A}) = \frac{n'-(r-1)}{r+1}$, and by (c), $C(\mathcal{A}) \leq 1$. Then by Theorem 2.1.5, the edges of $\mathcal{A}$ can be partitioned into $(1+\delta)\frac{n'-(r-1)}{r+1}$ matchings, so the number k of partition classes $\mathcal{F}_1, \cdots, \mathcal{F}_k$ of $\mathcal{B}$ is at most $\left\lfloor (1+\delta)\frac{n'-(r-1)}{r+1} \right\rfloor$. $\square$

2.2 Loose Paths in Steiner Triple Systems

Recall the definitions of a Steiner triple system and of $s(n)$ on page 7.

Theorem 2.2.1. *If $n = 9^p$ for some $p \in \mathbb{Z}$, then*

$$s(n) \leq \frac{n-1}{4}.$$

The foundation of this theorem is that the edges of an STS on 9 vertices can be ordered so all increasing paths have length at most 2. We then construct larger STSs of order 9^p and order their edges based on this initial ordering of STS(9). If one wanted to lower the factor of 1/4 in the statement of the theorem, one could consider an STS on 15 vertices. If its edges could be ordered so every increasing path had length at most 3, then the 1/4 factor could be lowered to 3/14 for $n = 15^p$.

Proof. We prove the theorem by induction on p. First, if $p = 1$, then $n = 9$, and there is only one Steiner triple system on 9 vertices, namely the affine plane of order 3 (see Figure 2.1), denoted STS(9). The edges of STS(9) can be partitioned into 4 perfect matchings of 3 edges apiece, and note that any pair of edges from two different matchings must intersect at exactly one vertex.

Order the edges of STS(9) so that the edges within each matching receive consecutive integer labels. Any increasing path under this ordering uses at most one edge from each matching. An increasing path cannot use edges from 3 different matchings, since the first and last edge (being from different matchings) would intersect. Therefore the longest increasing path under this edge-ordering of STS(9) uses 2 edges, and $s(9) = 2$.

We start the proof of the inductive case with the following fact and a labeling of a graph based on that fact.

Fact 1. There exists an affine plane of order 9. That is, there exists a 9-graph with

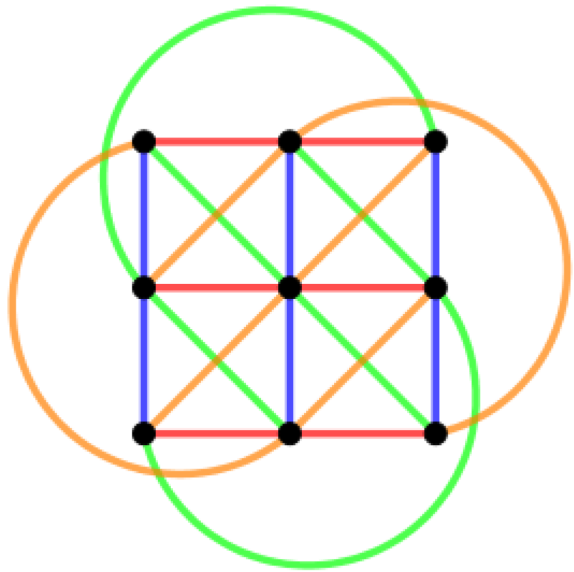

Figure 2.1: STS(9), the only Steiner triple system on 9 vertices, where each monochromatic set of line segments represents a matching.

81 vertices and 90 edges whose edges can be partitioned into ten matchings with the property that any pair of edges from two different matchings share exactly one vertex.

We will call such graph P—for explicit constructions of this graph, see e.g. [8]. If we label $V(P)$ by $\mathbb{F}_9^2$, then $E(P)$ can be partitioned into the following sets E_1, E_2, in which every $a_{j,i} \in [0,8]$:

$$E_1 = \{\left\{\begin{array}{l}(0, a_{0,i}),\\ (1, a_{1,i}),\\ \vdots,\\ (8, a_{8,i})\end{array}\right\} : 1 \leq i \leq 81\}, \qquad E_2 = \{\left\{\begin{array}{l}(i, 0),\\ (i, 1),\\ \vdots,\\ (i, 8)\end{array}\right\} : 0 \leq i \leq 8\}$$

Since E_2 is itself a matching, let $M_0, M_1, \ldots, M_8 \subset E_1$ represent the other nine disjoint matchings within E.

The idea of the proof of the inductive case is to build a Steiner triple system H on 9^p vertices by starting with nine Steiner triple systems $(B_0, B_1, \ldots, B_8)$, each on 9^{p-1} vertices, and adding edges between the B_i-s. To determine the structure of the new edges we add, we construct a 9-graph called F whose edges use one vertex from each B_i. The edges of F are based on the edges of P, which we've just defined. We build $E(F)$ in such a way that it too can be partitioned into perfect matchings, and so every pair of edges in F share at most one vertex. For each edge in F, we will place a copy of STS(9) in our graph on $B_0 \cup \cdots \cup B_8$, which results in a Steiner triple system on 9^p vertices. We then order the edges of this Steiner triple system (exploiting the matchings in F and the edge-ordering of the base case) to prevent long increasing paths.

Now assume the theorem is true for $p-1$ and we prove it for p. Let STS(9^{p-1}) represent the Steiner triple system on 9^{p-1} vertices from the inductive hypothesis. There exists an edge-ordering of STS(9^{p-1}) with no increasing path longer than $\frac{9^{p-1}-1}{4}$. Take nine distinct copies of STS(9^{p-1}) and call them $B_0, B_1, \ldots, B_8$. We will construct a Steiner triple system on $B_0 \cup \cdots \cup B_8$ and assign it an edge-ordering.

Label the vertices of each B_i with the elements of $\{i\} \times \mathbb{F}_9^{p-1}$, so each vertex of $B_0 \cup \cdots \cup B_8$ is labeled by a distinct p-tuple in $\mathbb{F}_9^p$. We define a 9-graph F with vertex set $V(F) = V(B_0) \cup \cdots \cup V(B_8)$ and edge set

$$E(F) = \{ \left\{ \begin{array}{l} (0, a_{0,i_1}, a_{0,i_2}, \ldots, a_{0,i_{p-1}}), \\ (1, a_{1,i_1}, a_{1,i_2}, \ldots, a_{1,i_{p-1}}), \\ \vdots, \\ (8, a_{8,i_1}, a_{8,i_2}, \ldots, a_{8,i_{p-1}}) \end{array} \right\} : 1 \leq i_1, i_2, \ldots, i_{p-1} \leq 81\},$$

where these are the same $a_{j,i}$-s as in P.

Claim 2.2.2. *For any two vertices*

$$(x, a_{x,i_1}, a_{x,i_2}, \ldots, a_{x,i_{p-1}}), (y, a_{y,i_1}, a_{y,i_2}, \ldots, a_{y,i_{p-1}}) \in E(F)$$

with $x \neq y$, there exists exactly one edge in $E(F)$ containing both vertices.

Proof. For each $k \in [1, p-1]$, Fact 1 implies there is exactly one edge $e_k \in E(P)$ containing both vertices (x, a_{x,i_k}) and (y, a_{y,i_k}). Therefore for every other $z \in [0, 8]$, $z \neq x, y$, the vertex (z, a_{z,i_k}) in e_k is uniquely determined. There is therefore exactly one edge in $E(F)$ that also uses these values of a_{z,i_k}. □

Claim 2.2.3. *The edges of F can be partitioned into 9^{p-1} perfect matchings, each of order 9^{p-1}.*

Proof. Recall that $M_0, \ldots, M_8 \subset E_1$ are disjoint matchings in P. For each $j = (j_1, j_2, \ldots, j_{p-1}) \in \mathbb{F}_9^{p-1}$, define $N_j \subset E(F)$ as

$$N_j = \{ \left\{ \begin{array}{l} (0, a_{0,i_1}, a_{0,i_2}, \ldots, a_{0,i_{p-1}}), \\ (1, a_{1,i_1}, a_{1,i_2}, \ldots, a_{1,i_{p-1}}), \\ \vdots, \\ (8, a_{8,i_1}, a_{8,i_2}, \ldots, a_{8,i_{p-1}}) \end{array} \right\} : \text{ for each } k \in [1, p-1], \left\{ \begin{array}{l} (0, a_{0,i_k}), \\ (1, a_{1,i_k}), \\ \vdots, \\ (8, a_{8,i_k}) \end{array} \right\} \in M_{j_k} \}.$$

First we show that for all j, N_j is a matching. To see this, suppose for some j that there exist edges $e, e' \in N_j$ with $|e \cap e'| \geq 1$, and we will see that indeed $e = e'$. Let

$$(x, a_{x,i_1}, a_{x,i_2}, \ldots, a_{x,i_{p-1}}) \in e \cap e'$$

for some $0 \leq x \leq 8$. For each $k \in [1, p-1]$, there is only a single edge $m \in M_{j_k}$ that has (x, a_{x,i_k}) as one of its vertices (since M_{j_k} is itself a matching). This implies that for every other $y \in [0, 8]$, $y \neq x$, the vertex (y, a_{y,i_k}) in m is uniquely determined. There is therefore exactly one edge in $E(F)$ that also uses theres values of a_{y,i_k}, so $e = e'$.

There are clearly 9^{p-1} of these matchings N_j based on our choice of j. Because the edges in each N_j are constructed based on choosing one edge from each matching $M_{j_1}, \ldots, M_{j_{p-1}}$, and because each $|M_{j_k}| = 9$, each matching N_j contains exactly 9^{p-1} edges. Because F is a 9-graph with 9^p vertices, each matching N_j must be a perfect matching. □

Construct a hypergraph H by starting with the vertices and edges of $B_0 \cup \cdots \cup B_8$, and for all $e \in E(F)$, place a copy of STS(9) on $V(e)$ in H; that is, for the nine vertices in H corresponding to $V(e)$, add 12 edges to H in the same configuration as the edges of STS(9). We claim that H is a Steiner triple system. To verify this, we must verify that every pair of vertices in H share exactly one edge. If $u, v \in V(H)$ are both from the same B_i, then they are never in any edge of F together, so we know by induction that they share exactly one edge in H (because B_i is itself an STS(9^{p-1})). Now suppose $u \in B_i$ and $v \in B_j$ with $i \neq j$. Then by Claim 2.2.2, these two vertices share exactly one edge $e \in E(F)$. By placing an STS(9) on $V(e)$ in H, we have ensured that u and v share exactly one edge in our hypergraph H. Therefore H is a Steiner triple system, and we will denote it by STS(9^p).

Now we define an edge-ordering of STS(9^p). Let $N_0, N_1, \ldots, N_{9^{p-1}-1}$ be the matchings described by Claim 2.2.3. For $0, \leq j \leq 9^{p-1} - 1$, let

$$E_j = \{e \in E(H) : \exists n \in N_j \text{ with } e \subset n\},$$

so that E_j is the set of edges in H resulting from the edges of N_j. Let

$$E_{9^{p-1}} = \bigcup_{i=0}^{8} E(B_i),$$

so that $E_{9^{p-1}}$ is the set of edges inherited from the nine couples of STS(9^{p-1}). Then $E_0, E_1, \ldots, E_{9^{p-1}}$ partition the edges of STS(9^p). Order the edges of STS(9^p) by $\mathbb{N}$ with the following scheme:

1. The edges within each E_j are labeled with consecutive integers.

2. For $0 \leq j \leq 9^{p-1} - 1$, within each E_j, the edges of each copy of STS(9) are labeled with consecutive integers, in the same way that the edges of STS(9)

were labeled in the base case.

3. For $E_{9^{p-1}}$, the edges of each copy of B_i are labeled by the inductive assumption, and those labels are then shifted upward until all edge-labels of $E(H)$ are unique.

An increasing path in STS(9^p) under this edge-ordering can use at most 2 edges from each E_j, $0 \leq j \leq 9^{p-1} - 1$, because the copies of STS(9) in each E_j are vertex-disjoint and only 2 edges can be used from any STS(9) (as in the base case). Within $E_{9^{p-1}}$, an increasing path can use edges only from a single B_i, since the subgraphs $B_0, \ldots, B_8$ are vertex-disjoint in $E_{9^{p-1}}$. Therefore, using the inductive assumption,

$$s(9^p) \leq 2(9^{p-1}) + s(9^{p-1}) \leq 2(9^{p-1}) + \frac{9^{p-1} - 1}{4} = \frac{9^p - 1}{4}.$$

□

It is worth noting that a similar argument to that used in [3] can be applied to Steiner triple systems, giving

$$s(n) \geq n^{1-o(1)}.$$

Indeed, such an argument applies to any simple hypergraph.

2.3 Vertex-orderings in Hypergraphs

Recall the definitions of an increasing loose path and of $f(H)$ on page 7.

Theorem 2.3.1. $f(H) = \chi(H) - 1$.

Proof. First, we show $f(H) \leq \chi(H) - 1$. With $k = \chi(H)$, let $\chi : V \to [k]$ be any proper k-coloring of H. Let ϕ be any ordering of V such that for any $u, v \in V$, $\phi(u) < \phi(v)$ if $\chi(u) < \chi(v)$. That is, ϕ should map all vertices of the same color class onto a single interval. Let $P = \big\{\{v_{(r-1)(i-1)+1}, v_{(r-1)(i-1)+2}, \ldots, v_{(r-1)(i-1)+r}\}\big\}_{i=1}^{\ell}$ be an increasing loose path in H consisting of ℓ edges. This implies that

$$\chi(v_{(r-1)(i-1)+1}) \leq \chi(v_{(r-1)(i-1)+2}) \leq \cdots \leq \chi(v_{(r-1)(i-1)+r})$$

for all $1 \leq i \leq \ell$. Because χ is a proper coloring, it cannot be that $\chi(v_{(r-1)(i-1)+1}) = \chi(v_{(r-1)(i-1)+2}) = \cdots = \chi(v_{(r-1)(i-1)+r})$ for any $1 \leq i \leq \ell$, so we have that $\chi(v_{(r-1)(i-1)+1}) < \chi(v_{(r-1)(i-1)+r})$. Since χ is a k-coloring and by the definition of ϕ, ℓ can be at most $k-1$. Therefore $f(H) \leq \chi(H) - 1$.

Second, we show $f(H) \geq \chi(H) - 1$. Fix some ordering ϕ of V so that the longest increasing loose path in H has $f(H)$ edges. We define a partition of V iteratively in the following way. Let $H_1 = H$, and let

$$V_1 = \Big\{v \in V(H_1) : \forall e \in E(H_1) \text{ with } v \in e, \exists u \in e \text{ with } \phi(u) < \phi(v)\Big\},$$

i.e. V_1 consists of vertices of H that are never the least in any edge. Let $H_2 = H[V \setminus V_1]$, the induced subhypergraph on $V \setminus V_1$, and let

$$V_2 = \Big\{v \in V(H_2) : \forall e \in E(H_2) \text{ with } v \in e, \exists u \in e \text{ with } \phi(u) < \phi(v)\Big\},$$

i.e. V_2 consists of vertices of H_2 that are never the least in any edge of H_2. Note that for every vertex $v \in V_2$ there is an edge $\{vu_1u_2 \cdots u_{r-1}\} \in E(H)$ with $\phi(v) < \phi(u_j)$ for all $1 \leq j \leq r-1$, or else v would have been in V_1. In general, let

$$H_i = H\Big[V \setminus \bigcup_{j=1}^{i-1} V_j\Big]$$

and let

$$V_i = \Big\{v \in V(H_i) : \forall e \in E(H_i) \text{ with } v \in e, \exists u \in e \text{ with } \phi(u) < \phi(v)\Big\}.$$

Let m be the largest integer so that V_m is non-empty, so that $\{V_i\}_{i=1}^m$ is a partition of V. For every vertex $v \in V_i$, $2 \leq i \leq m$, there is an edge $\{v, u_1, u_2, \ldots, u_{r-1}\} \in E(H)$ with $\phi(v) < \phi(u_j)$ for all $1 \leq j \leq r-1$ and with $u_j \in V_{k_j}$ for some $k_j \geq i-1$, or else v would have been in some V_k, $k \leq i-1$.

An increasing loose path can be formed in H by first taking any vertex $v_1 \in V_m$. Select any edge $\{v_1, v_2, \ldots, v_r\}$ for which $\phi(v_1) < \phi(v_2) < \cdots < \phi(v_r)$. The vertex v_r is either in V_{m-1} or V_m. Continue by selecting any edge $\{v_{(r-1)+1}, v_{(r-1)+2}, \ldots, v_{(r-1)+r}\}$ for which $\phi(v_{(r-1)+1}) < \phi(v_{(r-1)+2}) < \cdots < \phi(v_{(r-1)+r})$. We know that vertices $v_{(r-1)+2}, \ldots, v_{(r-1)+r}$ have not already appeared in the path because we are choosing vertices whose labels are increasing, so we do indeed construct a loose path, not a walk. The vertex $v_{(r-1)+r}$ will be in some V_j for $j \geq m-2$. Continue in this way, forming an increasing loose path by taking the vertex $v_{(r-1)(i-1)+1}$ and selecting any edge

$$\{v_{(r-1)(i-1)+1}, v_{(r-1)(i-1)+2}, \ldots, v_{(r-1)(i-1)+r}\}$$

for which

$$\phi(v_{(r-1)(i-1)+1}) < \phi(v_{(r-1)(i-1)+2}) < \cdots < \phi(v_{(r-1)(i-1)+r}).$$

The vertex $v_{(r-1)(i-1)+r}$ will be in some V_j for $j \geq m-i$. In this way an increasing loose path of at least $m-1$ edges can be formed, so $f(H) \geq m-1$. Define a coloring $\chi : V \to [m]$ so that $\chi(v) = i$ if $v \in V_i$. Note that no edge in H has all of its vertices in a single partition set V_i, so χ gives a proper coloring of H by m colors. Therefore $\chi(H) \leq m \leq f(H) + 1$, completing the proof. $\square$

Chapter 3

Countable Graphs, Hypergraphs, and Digraphs

In this chapter we prove several theorems on countably infinite graphs, hypergraphs, and digraphs.

3.1 Countable Graphs

Whereas Theorem 1.2.2 considers vertex-orderings of a graph by the natural numbers $\mathbb{N}$, we start this chapter by considering orderings by all integers $\mathbb{Z}$. Given a countable graph $G = (V, E)$ and an ordering ϕ of V by $\mathbb{Z}$, we say G contains an *infinite increasing path* if there exists an infinite path of vertices $\{v_i\}_{i=1}^{\infty}$ in G such that $\phi(x_i) < \phi(x_{i+1})$ for all $i \geq 1$.

Proposition 3.1.1. *For a countable graph $G = (V, E)$ the following are equivalent.*

(1) *For every subset $V_1 \subseteq V$ where both V_1 and $V \setminus V_1$ are infinite, there exists $W_1 \subseteq V_1$ such that all $v \in W_1$ have infinite degree in W_1.*

(2) *Any ordering of V by $\mathbb{Z}$ permits an infinite increasing path in G.*

Proof. One implication is clear. Suppose (1) and let ϕ be any ordering of V by $\mathbb{Z}$. We let

$$V_1 = \{v \in V : \phi(v) > 0\}$$

$$V_2 = \{v \in V : \phi(v) \leq 0\}.$$

By (1) there exists a set $W_1 \subseteq V_1$ so that for all $v \in W_1$, v has infinite degree in $G[W_1]$. Then by Theorem 1.2.2, there exists an infinite increasing path in $G[W_1]$ and hence in G.

Now suppose (1) does not hold. Then there exists a partition of V into infinite sets V_1, V_2 so every subset $W_1 \subseteq V_1$ has a vertex of finite degree in $G[W_1]$. Then by Theorem 1.2.2, there exists a vertex-ordering of $G[V_1]$ by $\mathbb{N}$ containing no infinite increasing path. Order the vertices of V_2 arbitrarily by $\mathbb{Z} \setminus \mathbb{N}$. Since all vertices with positive label are contained in $G[V_1]$, and since $G[V_1]$ contains no infinite increasing path, neither does G. □

Ordering using $\mathbb{Z}$ instead of $\mathbb{N}$ allows us to consider paths that continue infinitely in both directions. Recall the definition of an infinite increasing two-sided path on page 8.

Theorem 3.1.2. *For a countable graph $G = (V, E)$ the following are equivalent.*

(1) *For every partition of V into infinite sets V_1, V_2, there exist $W_1 \subseteq V_1$, $W_2 \subseteq V_2$ so that*

   (a) for $i = 1, 2$ and for all $v \in W_i$, v has infinite degree in W_i, and

   (b) there exists an edge $w_1 w_2 \in E$ with $w_1 \in W_1$ and $w_2 \in W_2$.

(2) *Any ordering of V by $\mathbb{Z}$ permits an infinite increasing two-sided path.*

Proof. One implication is clear. Suppose (1) and let ϕ be any ordering of V by $\mathbb{Z}$. We let

$$V_1 = \{v \in V : \phi(v) \geq 0\},$$

$$V_2 = \{v \in V : \phi(v) < 0\}.$$

Let W_1 and W_2 be the subsets of V_1 and V_2 respectively that are ensured by (1), and let $w_1 \in W_1$ and $w_2 \in W_2$ be the adjacent vertices. Let $v_1 = w_1$, and for $i \geq 1$, let v_{i+1} be a neighbor of v_i in W_1 so that $\phi(v_{i+1}) > \phi(v_i)$. Since each v_i has infinitely many such neighbors, $\{v_i\}_{i=1}^{\infty}$ forms an infinite increasing path in W_1. Similarly let $v_0 = w_2$ and for $i \leq 0$, let v_{i-1} be a neighbor of v_i in W_2 so that $\phi(v_{i-1}) < \phi(v_i)$. Since v_1 and v_0 are connected by an edge in G, the path formed by $\{v_i : -\infty < i < \infty\}$ is an infinite increasing two-sided path in G.

Now suppose (1) does not hold. Then there are two possible cases.

Case I - there exists a partition of V into infinite sets V_1, V_2 such that for $i = 1$ or $i = 2$ no subset $W_i \subseteq V_i$ induces a subgraph with all infinite degrees. Then by Proposition 3.1.1, there is an ordering of V by $\mathbb{Z}$ with no infinite increasing path. Such an ordering clearly forbids an infinite increasing two-sided path.

Case II - there exists a partition of V into infinite sets V_1, V_2 so that for any subsets $W_1 \subseteq V_1$, $W_2 \subseteq V_2$ with all vertices of W_i having infinite degree in W_i, there is no edge between vertices of W_1 and W_2. For $i = 1, 2$, consider the family $\mathcal{W}_i$ of such sets W_i, where $\mathcal{W}_i$ is partially ordered by inclusion, and observe that each $\mathcal{W}_i$ contains a maximal element. Fix vertex-maximal subsets $W_1 \in \mathcal{W}_i$ and $W_2 \in \mathcal{W}_i$. With $U_i = V_i \setminus W_i$ for $i = 1, 2$, the vertex-maximality of W_i implies

(i) every nonempty subset $U \subseteq U_i$ contains a vertex $v \in U$ with finite degree in $G[U]$,

(ii) every $v \in U_i$ has finite neighborhood in W_i.

By (i), Theorem 1.2.2 implies that there is an ordering ϕ_U of U_1 by $\mathbb{N}$ that forbids infinite increasing paths in U_1. By (ii) we can form an ordering ϕ of V_1 by positive integers that preserves ϕ_U (i.e. $\phi(u_1) < \phi(u_2)$ iff $\phi_U(u_1) < \phi_U(u_2)$ for all $u_1, u_2 \in U_1$) and where $\phi(u) > \phi(w)$ for all $u \in U_1$, $w \in W_1$ with u adjacent to w. Since ϕ

preserves ϕ_U, U_1 contains no infinite increasing path with respect to ϕ.

Now follow a similar procedure for V_2 and W_2, with $U_2 = V_2 \setminus W_2$. Here we order V_2 by *negative* integers to prevent any infinite *decreasing* path. Construct an ordering ψ of V_2 by negative integers so that for any $u \in U_2$, $w \in W_2$ with u adjacent to w, $\psi(u) < \psi(w)$, and so that U_2 contains no infinite decreasing path with respect to ψ.

Finally, for all $v \in V(G)$, let

$$\gamma(v) = \begin{cases} \phi(v) & v \in V_1 \\ \psi(v) + 1 & v \in V_2, \end{cases}$$

where we add 1 to $\psi(v)$ so that 0 is used as a label. Note that γ is an ordering of $V(G)$ by $\mathbb{Z}$ and $\gamma(V_1) = \mathbb{N}, \gamma(V_2) = \mathbb{Z} \setminus \mathbb{N}$. Observe the following with respect to γ:

$$(3) \begin{cases} \text{there are no infinite increasing paths in } U_1, \\ \text{there are no infinite decreasing paths in } U_2, \\ \text{there are no edges from } W_1 \text{ to } W_2. \end{cases}$$

We claim that there is no infinite increasing two-sided path in G with respect to γ. Suppose $\{v_i : -\infty < i < \infty\}$ is such a path, and without loss of generality assume $v_0 \in V_2$, $v_1 \in V_1$. If $v_1 \in U_1$, then all vertices $v_2, v_3, \ldots$ must be in U_1, and so $v_1, v_2, \ldots$ is an infinite increasing path in U_1, contradicting (3). Similarly if $v_0 \in U_2$, then $v_{-1}, v_{-2}, \ldots \in U_2$, so there is an infinite decreasing path in U_2, contradicting (3). Consequently, $v_1 \in W_1, v_0 \in W_2$, which contradicts (3). Therefore G contains no infinite increasing two-sided path.

□

Perhaps interestingly, when ordering edges by $\mathbb{Z}$, the result is entirely different from the vertex-ordering case, as the next result shows.

Proposition 3.1.3. *For every countable graph $G = (V, E)$ there exists an ordering of E by $\mathbb{Z}$ such that there in no infinite increasing path in G.*

Proof. Suppose G contains a matching M with infinitely many edges. Define an ordering ϕ of E by $\mathbb{Z}$ so that $\phi(e) > 0$ only if $e \in M$. Then there is no infinite increasing path in G, since no two edges with positive labels are incident.

Now suppose G contains no infinite matching. Then G does not contain an infinite path, regardless of how the edges are ordered, since any infinite path must contain an infinite matching as a subgraph. □

Proposition 3.1.3 immediately implies

Corollary 3.1.4. *For every countable graph $G = (V, E)$ there exists an ordering of E by $\mathbb{Z}$ such that there is no infinite increasing two-sided path in G.*

3.2 Countable Hypergraphs

3.2.1 Vertex-ordering

We start this section by proving Theorem 3.2.1. Recall Definitions 2, 3, and 4.

Theorem 3.2.1. *For a countable, k-uniform hypergraph $H = (V, E)$ the following are equivalent.*

(1) *H has property C_k.*

(2) *Any ordering of V by $\mathbb{N}$ permits an infinite increasing $(k-1)$-branching tree in H.*

(3) *Any ordering of V with $\mathbb{N}$ permits an infinite increasing path in H.*

In the proof of this theorem we use Theorem 1.2.2.

Proof of Theorem 3.2.1. **(1)** $\Rightarrow$ **(2)**. Let V' be the subset of V with the property described by (1). Then for any ordering of V with $\mathbb{N}$ any increasing $(k-1)$-branching tree of depth ℓ that uses vertices of V' can be extended to an increasing $(k-1)$-branching tree of depth $\ell+1$ using vertices of V'. Consequently there exists an infinite increasing $(k-1)$-branching tree.

(2) $\Rightarrow$ **(3)**. This is obvious, since an infinite increasing path can be found in any infinite increasing $(k-1)$-branching tree along some branch of the tree.

(3) $\Rightarrow$ **(1)**. In a hope to derive a contradiction suppose (3) holds but (1) does not. Then there is a well-ordering $\prec$ of V so that for every $v \in V$, the set

$$\{e \in E : v \in e \text{ and } \forall u \in e \text{ with } u \neq v, v \prec u\}$$

is finite. For any edge $e \in E$ let v_e be the minimal vertex of e, i.e. $v_e \prec u$ for all $u \in e$, $u \neq v_e$). Let

$$S_e = \Big\{\{v_e, u\} : u \in e, u \neq v_e\Big\}$$

be a (2-uniform) edge set of a star with center in v_e. Consider a graph G defined by

$$V(G) = V(H),$$

$$E(G) = \bigcup_{e \in E(H)} S_e.$$

Observe that for any $v \in V(G)$, the set

$$\Big\{u : \{v, u\} \in E(G), v \prec u\Big\}$$

remains finite. Therefore there is no subset $V' \subseteq V(G)$ with $\deg_{G[V']}(v) = \infty$ for every $v \in V'$. Consequently by Theorem 1.2.2, there is a vertex-ordering ϕ of $V(G)$ by $\mathbb{N}$ so that G contains no infinite increasing path.

By our assumption of (3) let $P = e_0, \dots, e_i, \dots$ be an infinite increasing path in H with respect to ϕ, and let $V(P) = \{v_1, v_2, \dots\}$ where $\phi(v_i) < \phi(v_{i+1})$ for $i \geq 1$. In particular $e_i = \{v_{(k-1)i+1}, v_{(k-1)i+2}, \dots, v_{(k-1)i+k}\}$ for $i \geq 0$. For each i let $u_i = v_{(k-1)i+1}$ and $w_i = v_{(k-1)i+k}$ be the minimal and maximal vertex of e_i with respect to ϕ. Note that u_i is not necessarily equal to v_{e_i}, which is the minimal vertex of e_i with respect to the well-ordering $\prec$.

In order to arrive at a contradiction we construct an infinite increasing path P' in G as a union of edges and 2-paths as follows. For each $i = 0, 1, \dots$ define a path P_i to be

- $u_i w_i$ if $v_{e_i} \in \{u_i, w_i\}$.
- $u_i v_{e_i} w_i$ if $v_{e_i} \notin \{u_i, w_i\}$.

Finally, set P' to be a path that is the concatenation of P_i-s with increasing labels on vertices, which contradicts our claim that G contains no infinite increasing path. $\square$

For a k-uniform hypergraph, property C_ℓ clearly implies property $C_{\ell-1}$, for $\ell \leq k$. In order to see that property $C_{\ell-1}$ does not imply property C_ℓ, consider a k-uniform hypergraph H obtained from an infinite complete $(\ell-1)$-uniform hypergraph, the edges of which are extended by pairwise disjoint $(k-\ell+1)$-tuples. H clearly has property $C_{\ell-1}$ but not property C_ℓ.

However for property C_2 we still get a result analogous to Theorem 3.2.1.

Definition 10. Let $H = (V, E)$ be a k-uniform hypergraph. For an ordering ϕ of V by $\mathbb{N}$ we say that infinite loose path $e_0, \dots, e_i, \dots$ is *skip-increasing* if there exist vertices $v_0, v_1, v_2, \dots$ so that for every $i \geq 0$, $\phi(v_i) < \phi(v_{i+1})$ and $\{v_i, v_{i+1}\} \subset e_i$.

By mimicking the proof of Theorem 3.2.1 one can get the following:

Proposition 3.2.2. *For a countable, k-uniform hypergraph $H = (V, E)$ the following are equivalent.*

(1) *H has property C_2.*

(2) *Any ordering of V by $\mathbb{N}$ permits an infinite skip-increasing path in H.*

3.2.2 Edge-ordering

Now, we consider edge-orderings of hypergraphs. Unfortunately we were not able to find sufficient and necessary conditions. Recall Definitions 5 and 6.

Theorem 3.2.3. *Let $H = (V, E)$ be a simple, countable, k-uniform hypergraph. Then for the following conditions,* (1) $\Rightarrow$ (2) $\Rightarrow$ (3).

(1) *Any ordering of E by $\mathbb{N}$ permits an infinite increasing $(k-1)$-branching tree in H.*

(2) *H has property C_2.*

(3) *Any ordering of E by $\mathbb{N}$ permits an infinite increasing path in H.*

Proof. We start with showing the easier implication (2) $\Rightarrow$ (3). Let H have property C_2, that is, there is a set $V' \subseteq V$ such that

$$\forall v \in V',\ |\{e \in E : v \in e \text{ and } |V' \cap e| \geq 2\}| = \infty.$$

Let ϕ be any ordering of E by $\mathbb{N}$ and let $e_1 \in E$ be such that $|V' \cap e| \geq 2$. Assume we have constructed an increasing path $e_1, e_2, \ldots, e_j$ and assume there exists some vertex $v \in e_j \setminus e_{j-1}$ with $v \in V'$. Since H is simple and v has infinite degree, there are infinitely many edges e containing v with $e_i \cap (e \setminus \{v\}) = \emptyset$ for all $i = 1, \ldots, j$ and with $\phi(e) > \phi(e_j)$. Choose any one of these edges to be e_{j+1}, and observe that there exists some vertex $u \in e_{j+1} \setminus e_j$ with $u \in V'$. In this inductive way we form the infinite increasing path $e_1, e_2, e_3, \ldots$.

Now, we show (1) $\Rightarrow$ (2). We will show that if H does not satisfy (2) it does not satisfy (1) as well. Due to our assumption of not (2), there exists a well-ordering $\prec$ of V so that for every $v \in V$, the set

$$\{\{v, u_1, \ldots, u_{k-1}\} \in E : v \prec u_i \text{ for some } i\}$$

is finite. Consequently we have

Fact 2. For any vertex $v \in V$, there are finitely many edges in E in which v is not the maximal vertex with respect to $\prec$.

Consider also an arbitrary ordering of V by $\mathbb{N}$, where order is denoted by $<$. If $e = \{v_1, v_2, \ldots, v_k\}$ with $v_1 < v_2 < \cdots < v_k$, we say $\ell(e) = v_1$ and $s(e) = v_2$, for the least and second-least vertices. Divide the edges of E into Type I edges (E_I) and Type II edges (E_{II}) so that

$$E_I = \{e : \ell(e) \text{ is } \textit{not} \text{ the maximum vertex of } e \text{ with respect to } \prec\}$$

$$E_{II} = E \setminus E_I.$$

Due to Fact 2 and the fact that every natural number has only finitely many predecessors, we infer that

Proposition 3.2.4. *Any vertex $v \in V$ can be in only finitely many Type I edges.*

We will construct separate orderings of E_I and E_{II}, each of which forbids an infinite increasing $(k-1)$-branching tree. Let $H_I = (V, E_I)$. Note that each vertex of H_I has finite degree.

Claim 3.2.5. *There is an edge-ordering ϕ by $\mathbb{N}$ of H_I with no infinite increasing path.*

Proof of Claim. The edge set E_I can be partitioned into finite sets $E_1, E_2, \ldots$ so that for all edges $e \in E_i$, any edge $f \in E_I$ incident to e is in E_j, $j \leq i+1$. (For instance, we may set E_i to be the edges that are distance $i-1$ away from some fixed edge e.) Give to E_I an ordering ϕ by $\mathbb{N}$ so that for all $i \in \mathbb{N}$,

$$\phi(e) < \phi(f) \text{ for all } e \in E_{2i-1}, f \in E_{2i},$$

$$\phi(f) > \phi(e) \text{ for all } f \in E_{2i}, e \in E_{2i+1}.$$

If an increasing path in E_I uses an edge from any E_{2i-1} or E_{2i}, then the path cannot later use any edges from E_j for any $j \geq 2i+1$. Such a path must be finite, so H_I can contain no infinite increasing path. □

Of course Claim 3.2.5 implies there is an edge-ordering ϕ by $\mathbb{N}$ of H_I with no infinite increasing $(k-1)$-branching tree.

Let $H_{II} = (V, E_{II})$. We now construct an ordering ψ by $\mathbb{N}$ on E_{II} in the following way. Suppose $e_1, e_2 \in H_{II}$ and recall the definitions of $\ell(e)$ and $s(e)$ from page 37.

1. If $s(e_1) < s(e_2)$, then $\psi(e_1) < \psi(e_2)$.

2. If $s(e_2) = s(e_1)$ and $\ell(e_1) \prec \ell(e_2)$, then $\psi(e_1) > \psi(e_2)$.

Recall that we want to show that H_{II} does not contain an infinite increasing $(k-1)$-branching tree with respect to ψ. Assume the contrary, that is that H_{II} contains such a tree T. We now recursively construct a branch of T that is a path $e_1, e_2, e_3, \ldots$ (with $t(e_i) = i$) satisfying $\ell(e_1) \succeq \ell(e_2) \succeq \ell(e_3) \succeq \cdots$. Assume we have constructed a path $e_1, \ldots, e_r$ with $\ell(e_1) \succeq \ell(e_2) \succeq \cdots \succeq \ell(e_r)$ for some $r \geq 1$. We choose to extend the path through either $\ell(e_r)$ or $s(e_r)$. At least one of these vertices is not in e_{r-1} (or not a root in case $r = 1$), and that vertex is incident to an edge $e_{r+1} \in E(T)$ with $\psi(e_{r+1}) > \psi(e_r)$. Recall that $\ell(e) \succ v$ for all $v \in e$, $v \neq \ell(e)$ for any Type II edge e.

Let $v = e_r \cap e_{r+1}$ and consider the following exhaustive cases, of which only three are possible.

1. If $v = \ell(e_r) = \ell(e_{r+1})$, then of course $\ell(e_r) \succeq \ell(e_{r+1})$.

2. It is impossible that $v = \ell(e_r)$ and $v \neq \ell(e_{r+1})$. If so then $v \geq s(e_{r+1})$, implying $s(e_{r+1}) \leq v = \ell(e_r) < s(e_r)$. This contradicts that $\psi(e_{r+1}) > \psi(e_r)$.

3. If $v = s(e_r)$ and $v = \ell(e_{r+1})$, and since $\ell(e_r) \succ s(e_r)$ for all Type II edges, $\ell(e_r) \succ v = \ell(e_{r+1})$.

4. If $v = s(e_r) = s(e_{r+1})$, then since $\psi(e_{r+1}) > \psi(e_r)$, by the definition of ψ, it must be that $\ell(e_r) \succ \ell(e_{r+1})$.

5. It is impossible that $v = s(e_r)$ and $v > s(e_{r+1})$. If so then $s(e_{r+1}) < s(e_r)$, which contradicts to $\psi(e_{r+1}) > \psi(e_r)$.

So for all possible ways in which e_{r+1} intersects e_r, we have $\ell(e_r) \succeq \ell(e_{r+1})$, and so $\ell(e_1) \succeq \ell(e_2) \succeq \ell(e_3) \succeq \cdots$. There are no consecutive equalities (otherwise three edges on a loose path would intersect at a single vertex). But this is a contradiction, since $\succ$ is a well-ordering and so every decreasing sequence must be finite. Hence, H_{II} has no infinite increasing $(k-1)$-branching tree with respect to ψ.

Having shown that for both H_I and H_{II} there exist edge-orderings ϕ and ψ by $\mathbb{N}$ that forbid any infinite increasing $(k-1)$-branching trees, we will construct an edge-ordering γ of H forbidding any infinite increasing $(k-1)$-branching trees.

Let γ be an ordering of E by $\mathbb{N}$ so that

$$\text{for all } e, f \in E_I, \gamma(e) < \gamma(f) \text{ iff } \phi(e) < \phi(f),$$

$$\text{for all } e, f \in E_{II}, \gamma(e) < \gamma(f) \text{ iff } \psi(e) < \psi(f), \text{ and}$$

$$\text{for all } e \in E_I, f \in E_{II} \text{ with } e \text{ incident to } f, \gamma(e) < \gamma(f).$$

This third restriction on γ is possible because each vertex in H has finite degree in H_I, so f can only intersect finitely man edges in E_I. The label of f can be chosen such that it's larger than the labels of all its incident edges in E_I.

The ordering γ inherits the ordering of E_I by ϕ and the ordering of E_{II} by ψ. Neither of these edge sets contains an infinite increasing $(k-1)$-branching tree with their respective ordering. Suppose H contains an infinite increasing $(k-1)$-branching tree T by γ. At least one edge $e \in E_{II}$ must be used, and then every edge following e in T must also be from E_{II}, by the definition of γ. This would require that E_{II} contains an infinite increasing $(k-1)$-branching tree, a contradiction. So there exists an ordering of E by $\mathbb{N}$ containing no infinite increasing $(k-1)$-branching tree.

$\square$

3.3 Countable Digraphs

Let $D = (V, E)$ be a directed graph where an edge $(u, v) \in E$ is oriented from u to v.

Proposition 3.3.1. *For a directed graph $D = (V, E)$, the following are equivalent.*

(1) *There exists an induced subgraph D' of D of which all vertices have infinite out-degree in D'.*

(2) *Any ordering of V by $\mathbb{N}$ permits an infinite increasing directed path in D.*

(3) *Any ordering of E by $\mathbb{N}$ permits an infinite increasing directed path in D.*

The proof of Proposition 3.3.1 mimics the proof of Theorem 1.2.2, and so we start by proving the following lemma. We say D is in $\mathcal{V}_{inf}$ if for any ordering of V by $\mathbb{N}$, there exists an infinite increasing directed path in D.

Lemma 3.3.2. *Let $D = (V, E)$ be a directed graph and let the sets V_i, $i \in \mathbb{N}$, form a partition of V such that the digraphs $D_i = D[V_i]$ do not belong to $\mathcal{V}_{inf}$. If for all*

$i \geq 1$ and each $x \in V_i$ the set

$$\Big\{(x,y) \in E : y \in \bigcup_{j=i+1}^{\infty} V_j\Big\}$$

is finite, then D does not belong to $\mathcal{V}_{inf}$.

Proof. Let $V = v_1, v_2, \ldots$. For each $v \in V$, we say $h(v) = i$ if $v \in V_i$. Let $\leq_i$ be an ordering of V_i so that there is no infinite increasing directed path in D_i with respect to $\leq_i$. For any set $M \subseteq V$, define

$$R(M) = M \cup \Big\{y \in V : \exists x \in M \text{ with } (x,y) \in E, h(x) < h(y)\Big\},$$

$$R|_k(M) = \{x \in R(M) : h(x) \leq k\}, \text{ and}$$

$$U(M) = \Big\{y \in V : \exists x \in M \text{ with } h(y) = h(x) \text{ and } y \leq_{h(y)} x\Big\}.$$

We say that $(U \circ R|_k)^1 M = U(R|_k(M))$ and for all $i \geq 1$,

$$(U \circ R|_k)^{i+1} M = U(R|_k((U \circ R|_k)^i M)).$$

We define sets $Q_n, P_n \subset V$ and positive integers k_n inductively. Let $Q_0 = \emptyset$. Now assume Q_n is already given. Let $X_{n+1} = Q_n \cup \{v_{n+1}\}$ and

$$k_{n+1} = \max\big\{h(x) : x \in \ R(X_{n+1})\big\}.$$

We define

$$P_{n+1} = \Big\{y \in (U \circ R|_{k_{n+1}})^{k_{n+1}} X_{n+1}\Big\}.$$

Finally, let

$$Q_{n+1} = U(R(P_{n+1})).$$

Observe that $U(R|_{k_{n+1}}(P_{n+1})) = P_{n+1}$. It is clear that Q_i and P_i are finite for all i, the sequence $\{k_i\}$ is non-decreasing, and that $Q_0 \subset Q_1 \subset \cdots$ and $\bigcup_{i=0}^{\infty} Q_i = V$.

Note 1. If $x \in Q_i$ and $h(x) \leq k_i$ for some i, then $x \in P_i$.

We can then define an ordering $\prec$ of V in the following way: we say that $x \prec y$ iff

either $x \in Q_i$, $y \notin Q_i$ for some i,

or $x, y \in Q_{i+1} \setminus Q_i$ for some i and $h(x) > h(y)$,

or $x, y \in Q_{i+1} \setminus Q_i$ for some i and $h(x) = h(y)$ and $x \leq_{h(x)} y$.

Claim 3.3.3. *If $x \in Q_i \setminus Q_{i-1}$ for some i, $y \in Q_i$, and $h(x) < h(y)$, then $x \succ y$.*

Indeed, either $y \in Q_{i-1}$ or $y \in Q_i \setminus Q_{i-1}$, both cases of which are clear from the definition of $\prec$. The following lemma proves a useful property about the ordering $\prec$.

Lemma 3.3.4. *For an edge $(x, y) \in E$, if $h(x) \leq k_n < h(y)$ for some n, then $x \succ y$.*

Proof. Let $x \in Q_j \setminus Q_{j-1}$ for some j. Then $y \in P_{j+1}$, meaning $h(y) \leq k_{j+1}$, and $k_n < k_{j+1}$. Thus $h(x) \leq k_j$, and Note 1 implies $x \in P_j$. So $y \in Q_j$, and Claim 3.3.3 implies $x \succ y$. $\square$

Suppose now that G contains an infinite increasing directed path $\{(x_i, x_{i+1})\}_{i=1}^{\infty}$ with respect to the ordering $\prec$. Choose j such that $h(x_1) \leq k_j$. Then by Lemma 3.3.4 we know $h(x_i) \leq k_j$ for all i. Since Q_j is finite, we can fix an index i_0 so that $x_i \notin Q_j$ for all $i \geq i_0$. We prove that $h(x_{i+1}) \leq h(x_i)$ for all $i \geq i_0$. Suppose on the contrary that $h(x_{i+1}) > h(x_i)$ for some $i \geq i_0$. Let $x_i \in Q_r \setminus Q_{r-1}$ for some $r > j$, then $h(x_i) \leq k_j < k_r$. By Note 1, $x_i \in P_r$ and so $x_{i+1} \in Q_r$. But then, by Claim 3.3.3, $x_{i+1} \prec x_i$, a contradiction.

Therefore, for some t, $h(x_t) = h(x_{t+1}) = h(x_{t+2}) = \cdots$. Notice that the restriction of the ordering $\prec$ to the set V_i is just the ordering $\leq_i$. From the properties of $\leq_i$, it follows that every set V_i contains only a finite number of vertices of the sequence $x_1, x_2, x_3, \ldots$. Thus the increasing directed path cannot be infinite, and $D \notin \mathcal{V}_{inf}$. $\square$

Corollary 3.3.5. *Let D be a countable directed graph with* $\deg(v) < \infty$ *for all* $v \in V(D)$*. Then* $D \notin \mathcal{V}_{inf}$*.*

Proof. Let $D_i = \{v_i\}$ so that $|D_i| = 1$ for all $i \geq 1$ and $\bigcup_{i=1}^{\infty} D_i = V(D)$. □

Proof of Proposition 3.3.1. Clearly (1) ⇒ (2) and (1) ⇒ (3).

To show (2) ⇒ (1), we define the sets M_α, V_α where α is a countable ordinal. We let

$$M_1 = \{v \in V : \deg_D^{out}(v) < \infty\},$$

$$V_\alpha = V \setminus \bigcup_{\beta<\alpha} M_\beta$$

$$M_\alpha = \{v \in V_\alpha : \deg_{D[V_\alpha]}^{out}(v) < \infty\}.$$

Set $s(D) = \min\{\alpha : M_\alpha = \emptyset\}$, and let ω_1 denote the first uncountable ordinal. First we prove $\bigcup_{\alpha<\omega_1} M_\alpha = V$. Suppose not, and let

$$V^* = V \setminus \bigcup_{\alpha<\omega_1} M_\alpha,$$

$$D^* = D[V^*],$$

and $\alpha_0 = s(D)$. For any $v \in V^*$, if $\deg_{D^*}^{out}(v) < \infty$ then $v \in M_\alpha$ for some $\alpha < \omega_1$, a contradiction. So $\deg_{D^*}^{out}(v) = \infty$ for every $v \in V^*$, which contradicts (1).

Now we prove by transfinite induction on $s(D)$ that there exists an ordering of $V(D)$ by $\mathbb{N}$ not containing an infinite increasing path. Lemma 3.3.2 and Corollary 3.3.5 prove the statement for $s(D) = \omega_0$ with $V_i = M_i$.

Assume that we proved this statement for all ordinals smaller than $s(D)$. If $s(D)$ is a successor, then $V = V_{s(D)} \bigcup_{\alpha<s(D)} M_\alpha$ is a partition of V into two sets, each of which has a labelling without an infinite increasing path. By Lemma 3.3.2, there is an ordering of $V(D)$ with the same property.

If $s(D)$ is a limit ordinal, then $s(D) = \lim_{n\to\infty} \alpha_n$, where $\{\alpha_i\}_1^\infty$ is an increasing sequence. In this case $V = \bigcup_{n=1}^{\infty} \left(\bigcup_{\alpha_{n-1} < \alpha \leq \alpha_n} M_\alpha \right)$ is a partition of V into sets $U_n = \bigcup_{\alpha_{n-1} < \alpha \leq \alpha_n} M_\alpha$, each of which has an ordering without an infinite increasing path by the induction hypothesis. Again, by Lemma 3.3.2, there is an ordering of $V(D)$ with the same property.

To show (3) $\Rightarrow$ (1), we mimic Reiterman's proof of Theorem 1.2.2. Assume not (1), so there exists a well-ordering $\prec$ of V such that for all $v \in V$, $\{(v,u) \in E : v \prec u\}$ is finite. We may assume $V = \mathbb{N}$ with the usual ordering $\leq$. For an edge $e \in E$, let $\ell(e)$ be the vertex of e that has the smaller label in the ordering $\leq$. Conversely, $r(e)$ represents the vertex with larger label. Let

$$L = \big\{e \in E : \ell(e) \prec r(e)\big\}, \text{ and}$$

$$L_u = \big\{(u,v) \in L\big\},$$

so that the two orderings $\leq$ and $\prec$ agree on the edges L. Note that for all $u \in V$, there are finitely many $(u,v) \in E$ with $v \leq u$, and there are finitely many $(u,v) \in E$ with $u \prec v$. So each set L_u is finite, and the out-degree of u in L is finite.

Claim 3.3.6. *The subgraph $D' = (V, L)$ is not in $\mathcal{V}_{inf}$, so there is an ordering ϕ of D' with no infinite increasing directed path.*

Proof of Claim 3.3.6. Since the out-degree of every vertex in V is finite in L, L can be partitioned into finite sets $K_1, K_2, \ldots$ so that for all edges $(v,u) \in K_i$, any edge (u,w) is in K_j for some $j \leq i+1$. Give to L an ordering ϕ by $\mathbb{N}$ so that for all $i \in \mathbb{N}$,

$$\phi(e) < \phi(f) \text{ for all } e \in K_{2i-1}, f \in K_{2i},$$

$$\phi(f) > \phi(e) \text{ for all } f \in K_{2i}, e \in K_{2i+1}.$$

If an increasing directed path in L uses an edge from any K_{2i-1} or K_{2i}, then the path can use no edges from K_j for any $j \geq 2i+1$. Such a path must be finite. $\square$

Now we give an ordering ψ to $E \setminus L$, where the orderings $\leq$ and $\prec$ disagree. Let

$$K_i = \big\{(v,i),(i,v) \in E \setminus L : v \leq i\big\}.$$

Note that sets K_i are finite and $K_1, K_2, \dots$ forms a partition of $E \setminus L$. Suppose ψ is an ordering defined on $\bigcup_{j<i} K_j$, and proceed by induction. Define ψ on K_i so that

$$\text{for all } e \in K_i \text{ and all } f \in \bigcup_{j<i} K_j, \psi(e) > \psi(f),$$

$$\text{for all } e, f \in K_i, \psi(e) > \psi(f) \text{ iff } \ell(e) \prec \ell(f).$$

Finally, let γ be an ordering of E by $\mathbb{N}$ so that

$$\text{for all } e, f \in L, \gamma(e) < \gamma(f) \text{ iff } \phi(e) < \phi(f),$$

$$\text{for all } e, f \in E \setminus L, \gamma(e) < \gamma(f) \text{ iff } \psi(e) < \psi(f), \text{ and}$$

$$\text{for all } (u,v) \in E \setminus L, (v,w) \in L_v, \gamma((v,w)) < \gamma((u,v)),$$

which is possible because L_v is a finite set.

Observe that if an edge $(u,v) \in E \setminus L$, then for all $(v,w) \in E$ with $\gamma((u,v)) < \gamma((v,w))$, $(v,w) \in E \setminus L$. Indeed, for all $(v,w) \in L$, $(v,w) \in L_v$, so $\gamma((v,w)) < \gamma((u,v))$. Hence an increasing directed path in D with respect to γ cannot ever use an edge of L after using an edge of $E \setminus L$.

Claim 3.3.7. *If $(u,v),(v,w) \in E \setminus L$ and $\gamma((u,v)) < \gamma((v,w))$, then $\ell((u,v)) \succeq \ell((v,w))$.*

Proof. If $u < v < w$, then $u \succ v$ because $(u,v) \in E \setminus L$, so $\ell((u,v)) \succeq \ell((v,w))$. If

$u < v$ and $w < v$, then $(u, v), (v, w) \in K_v$, so the claim follows from the definition of γ and ψ. If $v < u$ and $v < w$, then $\ell((u, v)) = \ell((v, w))$. Finally, it is not possible to have $w < v < u$ because this would imply, by the definition of ψ, that $\psi((u, v)) > \psi((v, w))$ and hence $\gamma((u, v)) > \gamma((v, w))$. □

We claim that D does not contain an infinite increasing directed path with respect to γ. Suppose that $\{(v_i, v_{i+1})\}_{i=1}^{\infty}$ is such a path. This path cannot contain edges only from L, because γ extends ϕ, so there exists a j such that for all $i \geq j$ edges $(v_i, v_{i+1}) \in E \setminus L$. Claim 3.3.7 gives that $\ell((v_i, v_{i+1})) \succeq \ell((v_{i+1}, v_{i+2})) \succeq \cdots$ for all $i \geq j$. Since $\prec$ is a well-ordering, the set $\{\ell((v_i, v_{i+1}))\}_j^{\infty}$ contains a minimal element, so for some k, $\ell((v_k, v_{k+1})) = \ell((v_{k+1}, v_{k+2})) = \cdots$. But this is impossible since no more than two edges of a path can use the same vertex, so γ is an edge-ordering of D that forbids any infinite increasing path, implying not (3). □

We now considering edge-orderings of directed graphs by $\mathbb{Z}$.

Proposition 3.3.8. *For a directed graph $D = (V, E)$, the following are equivalent.*

(1) *For every partition of V into infinite sets V_1, V_2 there exist $W_1 \subseteq V_1$, $W_2 \subseteq V_2$ so that*

   (a) for all $v \in W_1$, v has infinite out-degree in $D[W_1]$,

   (b) for all $v \in W_2$, v has infinite in-degree in $D[W_2]$, and

   (c) there are vertices $w_i \in W_i$ for $i = 1, 2$, such that $(w_2, w_1) \in E$.

(2) *Any ordering of V by $\mathbb{Z}$ permits an infinite two-sided directed path.*

Before proving Proposition 3.3.8, we most prove the following.

Proposition 3.3.9. *For a directed graph $D = (V, E)$, the following are equivalent.*

(1) *For every infinite subset $V' \subseteq V$, there exists $W \subseteq V'$ so that for all $v \in W$, v has infinite out-degree in $D[W]$.*

(2) *Any ordering of V by $\mathbb{Z}$ permits an infinite increasing directed path.*

Proof of Proposition 3.3.9. Suppose (1), and let ϕ be any ordering of V by $\mathbb{Z}$. Set

$$V' = \{v \in V : \phi(v) > 0\}.$$

By (1) there exists a set $W \subseteq V'$ so that for all $v \in W$, v has infinite out-degree in $D[W]$. Then by Theorem 3.3.1, there exists an infinite increasing directed path in $D[W]$ and hence in D.

Now suppose (1) does not hold. Then there exists an infinite subset $V' \subseteq V$ so every subset $W \subseteq V'$ has a vertex of finite out-degree in $D[W]$. Then by Theorem 3.3.1, there exists a vertex-ordering of $D[V']$ by $\mathbb{N}$ containing no infinite increasing directed path. Order the vertices of $V \setminus V'$ arbitrarily by $\mathbb{Z} \setminus \mathbb{N}$; if $V \setminus V'$, then partition V' into two infinite subsets, ordering one by $\mathbb{N}$ as above and ordering the other along with $V \setminus V'$ by $\mathbb{Z}$. Since all vertices with positive label are contained in $D[V]$, and since $D[V]$ contains no infinite increasing directed path, neither does D. $\square$

Now we are ready to prove Proposition 3.3.8.

Proof of Proposition 3.3.8. The proof is analogous to the proof of of Theorem 3.1.2. One implication is clear. Suppose (1) and let ϕ be any ordering of V by $\mathbb{Z}$. We let

$$V_1 = \{v \in V : \phi(v) \geq 0\},$$

$$V_2 = \{v \in V : \phi(v) < 0\}.$$

Let W_1 and W_2 be the subsets of V_1 and V_2 respectively that are ensured by (1), and let $w_1 \in W_1$ and $w_2 \in W_2$ be vertices of some edge (w_2, w_1). Let $v_1 = w_1$, and for $i \geq 1$, let v_{i+1} be some vertex in W_1 so that $(v_i, v_{i+1}) \in E$ and $\phi(v_{i+1}) > \phi(v_i)$. Since each v_i has infinitely many such neighbors, $\{v_i\}_{i=1}^{\infty}$ forms an infinite increasing directed path in W_1. Similarly let $v_0 = w_2$ and for $i \leq 0$, let v_{i-1} be a vertex in W_2 so that $(v_{i-1}, v_i) \in E$ and $\phi(v_{i-1}) < \phi(v_i)$. Since $(v_0, v_1) \in E$, the path formed by $\{v_i : -\infty < i < \infty\}$ is an infinite increasing two-sided directed path in D.

Now suppose (1) does not hold. Then there are two possible cases.

Case I - there exists a partition of V into infinite sets V_1, V_2 such that either no subset $W_1 \subseteq V_1$ induces a subgraph with all infinite out-degrees, or no subset $W_2 \subseteq V_2$ induces a subgraph with all infinite in-degrees. In the former case, by Proposition 3.3.9, there is an ordering of V by $\mathbb{Z}$ with no infinite increasing directed path. Such an ordering clearly forbids an infinite increasing two-sided directed path. In the latter case, let $\overline{D}$ be a directed graph defined by

$$V(\overline{D}) = V(D)$$

$$E(\overline{D}) = \{(u, v) : (v, u) \in E(D)\}$$

Then by Proposition 3.3.9, there is an ordering ϕ of $V(\overline{D})$ by $\mathbb{Z}$ with no infinite increasing directed path. Define the ordering ψ of $V(D)$ by $\mathbb{Z}$ such that

$$\psi(v) = -\phi(v)$$

for all $v \in V(D)$. Clearly D contains no *infinite decreasing anti-directed path* (that is, no path $\{(v_{i-1}, v_i) : \psi(v_{i-1}) < \psi(v_i),\ i \leq -1\}$) with respect to ψ, so it contains no infinite increasing two-sided directed path.

Case II - there exists a partition of V into infinite sets V_1, V_2 so that for any subsets $W_1 \subseteq V_1$ with all vertices having infinite out-degree in W_1 and $W_2 \subseteq V_2$ with all vertices having infinite in-degree in W_2, there is no edge between vertices of W_1 and W_2. For $i = 1, 2$, consider the family $\mathcal{W}_i$ of such sets W_i, where $\mathcal{W}_i$ is partially ordered by inclusion, and observe that each $\mathcal{W}_i$ contains a maximal element.

Fix maximal subsets $W_1 \in \mathcal{W}_i$ and $W_2 \in \mathcal{W}_i$. With $U_i = V_i \setminus W_i$ for $i = 1, 2$, the maximality of W_i implies

(i) every nonempty subset $U \subseteq U_1$ contains a vertex $v \in U$ with finite out-degree in $G[U]$,

(i) every nonempty subset $U \subseteq U_2$ contains a vertex $v \in U$ with finite in-degree in $G[U]$,

(iii) for every $v \in U_1$, there are finitely many $u \in W_1$ with $(v, u) \in E$,

(iv) for every $v \in U_2$, there are finitely many $u \in W_2$ with $(u, v) \in E$.

By (i), Theorem 3.3.1 implies that there is an ordering ϕ_U of U_1 by $\mathbb{N}$ that forbids infinite increasing paths in U_1. By (iii) we can form an ordering ϕ of V_1 by positive integers that preserves ϕ_U (i.e. $\phi(u_1) < \phi(u_2)$ iff $\phi_U(u_1) < \phi_U(u_2)$ for all $u_1, u_2 \in U_1$) and where $\phi(u) > \phi(w)$ for all $u \in U_1$, $w \in W_1$ with $(u, w) \in E$. Since ϕ preserves ϕ_U, U_1 contains no infinite increasing directed path with respect to ϕ.

Now follow a similar procedure for V_2 and W_2, with $U_2 = V_2 \setminus W_2$. Here we order V_2 by *negative* integers to prevent any infinite decreasing anti-directed path (recall definition on page 48). Construct an ordering ψ of V_2 by negative integers so that for any $u \in U_2$, $w \in W_2$ with $(wu) \in E$, $\psi(u) < \psi(w)$, and so that U_2 contains no infinite decreasing anti-directed path with respect to ψ.

Finally, for all $v \in V(G)$, let

$$\gamma(v) = \begin{cases} \phi(v) & v \in V_1 \\ \psi(v) + 1 & v \in V_2, \end{cases}$$

where we add 1 to $\psi(v)$ so that 0 is used as a label. Note that γ is an ordering of

$V(G)$ by $\mathbb{Z}$ and $\gamma(V_1) = \mathbb{N}, \gamma(V_2) = \mathbb{Z} \setminus \mathbb{N}$. Observe the following with respect to γ:

$$(3) \begin{cases} \text{there are no infinite increasing directed paths in } U_1, \\ \text{there are no infinite decreasing anti-directed paths in } U_2, \\ \text{there are no directed edges from } W_2 \text{ to } W_1. \end{cases}$$

We claim that there is no infinite two-sided directed path in G with respect to γ. Suppose $\{(v_i, v_{i+1}) : i \in \mathbb{N}\}$ is such a path, and without loss of generality assume $v_0 \in V_2$, $v_1 \in V_1$. If $v_1 \in U_1$, then all vertices $v_2, v_3, \ldots$ must be in U_1, because $\gamma(v) > \gamma(w)$ for all $v \in E_1$ and $w \in W_1$, and so $v_1, v_2, \ldots$ is an infinite increasing directed path in U_1, contradicting (3). Similarly if $v_0 \in U_2$, then $v_{-1}, v_{-2}, \ldots \in U_2$, so there is an infinite decreasing anti-directed path in U_2, contradicting (3). Therefore, $v_1 \in W_1, v_0 \in W_2$, which contradicts (3) once again. Therefore G contains no infinite increasing two-sided directed path.

□

Proposition 3.3.10. *For every countable directed graph $D = (V, E)$ there exists an ordering of E by $\mathbb{Z}$ such that there is no infinite increasing directed path in D.*

Proof. This proposition follows directly from Proposition 3.1.3. □

3.4 Arbitrary Finite Paths in Countable Graphs

3.4.1 Paths of Arbitrary Finite Length

We first prove the following claim concerning χ^*. Recall the definitions of property FIN (Definition 8) and $\chi^\star(G)$ (Definition 9).

Definition 11. We say that for an integer k a graph G has property FIN_k if for any ordering of vertices of G by $\mathbb{N}$ there exists an increasing path with k vertices, but there exists an ordering with no increasing path of $k+1$ vertices. If no such k exists, then G has property FIN.

The following is a simple modification of the well-known Gallai-Hasse-Roy-Vitaver Theorem (1.2.1).

Claim 3.4.1. *A graph G has property FIN$_k$ if and only if $\chi^*(G) = k$.*

Proof. Let $G = (V, E)$ have property FIN$_k$ and consider an ordering of V by $\mathbb{N}$ so that G does not contain any increasing path of more than k vertices. Let V_k be the set of vertices that are maximal (i.e. are not adjacent to any vertex of larger label). Note that V_k cannot be empty, or else G would contain an increasing path of infinite length. Delete V_k from V to obtain the set $U_{k-1} = V - V_k$ and consider G_{k-1}, the subgraph of G induced on U_{k-1}. Let V_{k-1} be the set of all vertices that are maximal within G_{k-1}. We delete V_{k-1} from U_{k-1} to obtain U_{k-2} and let G_{k-2} be induced on U_{k-2}. We continue this way until we exhaust all vertices of G. Note that for all $j \in [k-1]$ and every vertex $v \in V_j$, there is a vertex $u \in V_{j+1}$ adjacent to v and with larger label (otherwise, v itself should belong to V_{j+1}, not V_j). Now we show that V_0 is empty. If there were some vertex v_0 in V_0, we would have an increasing path of length $k+1$ beginning at v_0. Consequently $V_1, V_2, \ldots, V_k$ constitutes a partition of V.

Further, observe that each set V_i, $1 \leq i \leq k$, is an independent set, since two adjacent vertices in some graph cannot both be maximal. We say that vertex w dominates v if $\{w, v\} \in E$ and label of w is larger than of v. Since vertices are ordered by integers, each vertex $v \in V_i$ can dominate only finitely many vertices of $V_1, V_2, \ldots, V_{i-1}$ and cannot be dominated by any vertices in those sets, so v has finite degree into $V_1, V_2, \ldots, V_{i-1}$. So, if G has property FIN$_k$, then $\chi^*(G) \leq k$.

On the other hand if $\chi^*(G) = k$ with partition $V_1, \ldots, V_k$, we can label vertices of V_1 arbitrarily and then label vertices of $V_2, \ldots, V_k$ in succession, always making sure that the label of any vertex $v \in V_j$ is higher than that of its neighbors from $V_1, \ldots, V_{j-1}$. This is possible because v has finite degree into $V_1, \ldots, V_{j-1}$ by $\chi^*(G)$. This ordering prevents an increasing path of $k+1$ vertices. Therefore G has property FIN_j for some $j \leq k$.

Finally, if $\chi^*(G) = k$, then we have that G has property FIN_j for some $j \leq k$ and then $k = \chi^*(G) \leq j \leq k$, which implies that G has property FIN_k. Also, if G has property FIN_k, then $\chi^*(G) = k$. □

Now we are ready to prove Theorem 3.4.2.

Theorem 3.4.2. *A graph G has property FIN if and only if $\chi^*(G) = \infty$ and G does not have a subgraph in which all degrees are infinite.*

Proof. First, suppose G has property FIN. Since for every ordering of $V(G)$ by $\mathbb{N}$ there is an increasing path of arbitrary finite length, G does not have property FIN_k for any k, and hence by Claim 3.4.1, $\chi^\star(G) = \infty$. Also, G cannot contain a subgraph in which all degrees are infinite, for otherwise Theorem 1.2.2 implies that every ordering of $V(G)$ by $\mathbb{N}$ would contain an infinite increasing path, which would contradict FIN.

Now suppose G does not have property FIN. Then either there exists an ordering of $V(G)$ by $\mathbb{N}$ forbidding increasing paths of length k for some k, or for every ordering of $V(G)$ there is an infinite increasing path. In the former case, G has property FIN_j for some $j < k$, implying by Claim 3.4.1 that $\chi^\star(G) = j < \infty$. In the latter case, Theorem 1.2.2 implies G has a subgraph in which all degrees are infinite. □

3.4.2 A Counterexample

Clearly, if there is an ordering of $V(G)$ by $\mathbb{N}$ that prevents an increasing path of $k+1$ vertices, then $\chi(G) \leq k$. Unfortunately, the converse is not true.

Proposition 3.4.3. *There exists a bipartite graph H that has property FIN.*

Hence, we cannot replace condition $\chi^*(G) = \infty$ with $\chi(G) = \infty$ in the statement of Theorem 3.4.2. To prove Proposition 3.4.3, we first define a "half-graph" and then construct graph H.

Definition of a half-graph

Start with I and F – two copies of $\mathbb{N}$. Let $G = (V, E)$ be a graph with $V(G) = I \cup F$ and

$$E = \{\{x, y\} \ : \ x \in I, \ y \in F, \ x \leq y\}.$$

Graph G defined on $I \cup F$ in such a way is called the *half-graph* and is denoted by $G[I, F]$. Note that all vertices of I have infinite degree in G, while vertices of F have finite degree. See Figure 3.1 for an illustration.

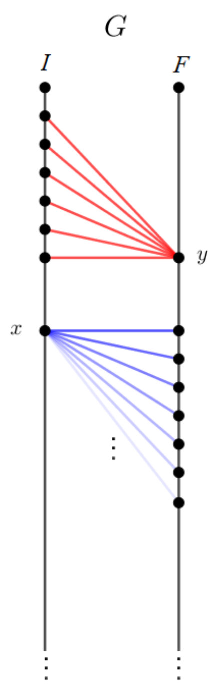

Figure 3.1: Each $x \in I$ has infinite degree, each $y \in F$ has finite degree.

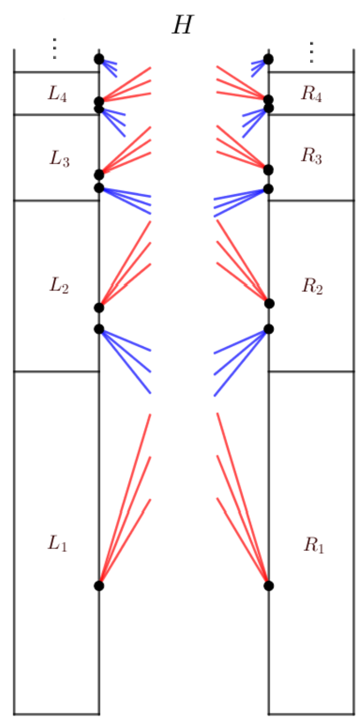

Figure 3.2: Each $x \in V_i^L$ has finite degree to U_i^R and infinite degree to V_j^R for $1 \leq j < i$.

Construction of H

Let L and R be again two copies of $\mathbb{N}$. Then set $V(H) = L \cup R$. To define $E(H)$ we need some preliminary definitions. For each $i \geq 1$,

$$L_i = \{n \in L \ : n = 2^{i-1}(2j-1),\ j \in \mathbb{N}\},$$

$$R_i = \{n \in R \ : n = 2^{i-1}(2j-1),\ j \in \mathbb{N}\},$$

Note, that from this definition we have that $L = \bigcup_{i=1}^{\infty} L_i$ and $R = \bigcup_{i=1}^{\infty} R_i$. Notice that we can view each L_i (or R_i) as a copy of $\mathbb{N}$ with the order inherited from L (or R).

We define a graph H_1 on $V(H)$ where, for each i, there is a copy of G between sets L_i and R_j for each $j > i$ (L_i is the set with finite degrees in G). More precisely,

$$E(H_1) = \bigcup_{i=1}^{\infty} \left(\bigcup_{j=i+1}^{\infty} G[R_j, L_i] \right).$$

Each vertex in L has finite degree in H_1, and each vertex in R has infinite degree in H_1. Similarly we define a graph H_2 on $V(H)$ where, for each i, there is a copy of G between sets R_i and L_j for each $j > i$ (R_i is the set with finite degrees in G). More precisely,

$$E(H_2) = \bigcup_{i=1}^{\infty} \left(\bigcup_{j=i+1}^{\infty} G[L_j, R_i] \right).$$

Each vertex in R has finite degree in H_2, and each vertex in L has infinite degree in H_2. Also note that $E(H_1) \cap E(H_2) = \emptyset$. Then

$$E(H) = E(H_1) \cup E(H_2).$$

Notice that, by a construction, H is bipartite. See Figure 3.2 for an illustration.

Proof of Proposition 3.4.3.

Claim 3.4.4. *H does not contain a subgraph with infinite degrees.*

Proof. Assume that H does contain a subgraph H' with all degrees infinite. Let $x \in L_t$ be a vertex of H' with the smallest possible t.

Notice that x has a finite degree into each $\bigcup_{i=t+1}^{\infty} R_i$, since there are only finitely many vertices in R less than or equal to x. Also, x has no neighbours in R_t. So in order for x to have infinite degree in H', H' has to contain infinitely many vertices of $\bigcup_{i=1}^{t-1} R_i$. Let y be the smallest vertex in $V(H') \cap \bigcup_{i=1}^{t-1} R_i$ such that $xy \in E(H')$. Assume that $y \in R_s$ for some $s < t$. Now, y has finite degree in $\bigcup_{i=s+1}^{\infty} L_i$ and none of the vertices of H' are in $\bigcup_{i=1}^{s-1} L_i$ (by minimality of t). Hence, y has a finite degree in H', a contradiction. $\square$

Hence by Theorem 1.2.2 there is an ordering of $V(H)$ by $\mathbb{N}$ with no infinite increasing path.

Claim 3.4.5. *Any ordering of H by $\mathbb{N}$ contains an increasing path of arbitrary length.*

Proof. Notice that any vertex v of L_{t+1} has infinite neighborhood in R_t for any $t \geq 2$. Let $V(H)$ be ordered by $\mathbb{N}$ and let k be an integer. Then for any $v_1 \in L_{k+1}$ there is $v_2 \in R_k$ adjacent to v_1 and with larger label. Similarly for $v_2 \in R_k$ there is $v_3 \in L_{k-1}$ adjacent to v_2 and with larger label. Continuing this way we obtain an increasing path of length k. $\square$

Therefore, H has property FIN. $\square$

Appendix A

Alternative Proof of Theorem 2.1.1

Here we give another proof of Theorem 2.1.1 that avoids using the unpublished paper of Keevash [10]. Instead we use the following theorem of Alon and Yuster [1], which gives an approximate version of Keevash's result. For hypergraph $H = (V, E)$ let $\delta(H)$, $\Delta(H)$, and $\Delta_2(H)$ denote the minimum degree, maximum degree, and maximum co-degree of H, respectively, and let $g(H) = \Delta(H)/\Delta_2(H)$. If $\mathcal{F} \subset 2^V$, we say a matching M of H is $(\alpha, \mathcal{F})$-perfect if for each $F \in \mathcal{F}$, at least $\alpha|F|$ vertices of F are covered by M. Let $s(\mathcal{F}) = \min_{F\in\mathcal{F}} |F|$.

Theorem A.0.1 ([1]). *For any integer $r \geq 2$, a real $C > 1$, and a real $\epsilon > 0$ there exists a real $\mu = \mu(r, C, \epsilon)$ and a real $K = K(r, C, \epsilon)$ so that the following holds: If the r-uniform hypergraph $H = (V, E)$ on N vertices satisfies:*

(i) $\delta(H) \geq (1 - \mu)\Delta(H)$,

(ii) $g(H) > \max\{1/\mu, K(\ln N)^6\}$,

then for every $\mathcal{F} \subset 2^V$ with $|\mathcal{F}| \leq C^{g(H)^{1/(3r-3)}}$ and with $s(\mathcal{F}) \geq 5g(H)^{1/(3r-3)} \ln(|\mathcal{F}|g(H))$ there is a $(1 - \epsilon, \mathcal{F})$-perfect matching in H.

To replicate our original proof above of Theorem 2.1.1, ideally we would pack $K_n^{(r)}$ with exactly $\binom{n}{r}/\binom{2r}{r}$ copies of $K_{2r}^{(r)}$. The following claim, which is a generalization of

Theorem 3.1 of [1], allows us to find a "nearly-perfect" packing of $K_{2r}^{(r)}$ in $K_n^{(r)}$. The proof of A.0.2 is similar to that of Theorem 3.1 of [1], though we consider ℓ-tuples L and the corresponding edge-sets F_L where the original proof used just vertices $v \in V$ and the edge-sets F_v.

Claim A.0.2. *Let ℓ, k, t be integers with $1 \le \ell < k < t$. For n sufficiently large there is a $K_t^{(k)}$-packing of $K_n^{(k)}$ with order at least $(1-o(1))\binom{n}{k}/\binom{t}{k}$ so that each ℓ-tuple of vertices is contained in at most $o(n^{k-l})$ k-tuples not covered by a copy of $K_t^{(k)}$.*

Proof of Claim A.0.2. We apply Theorem A.0.1 with some small $\epsilon > 0$, $r = \binom{t}{k}$, and, say, $C = 1.1$. Given $K_n^{(k)}$ we create an auxiliary hypergraph $H = H(n,k)$ as follows. The vertices of H are the edge of $K_n^{(k)}$. The edges of H are the copies of $K_t^{(k)}$ in $K_n^{(k)}$. Notices that H is r-uniform and has $N = \binom{n}{k}$ vertices. Also, $\Delta(H) = \delta(H) = \binom{n-k}{t-k}$. Notice that any two edges of $K_n^{(k)}$ appear together in at most $\binom{n-k-1}{t-k-1}$ copies of $K_t^{(k)}$. Thus $\Delta_2(H) \le \binom{n-k-1}{t-k-1}$ and $g(H) \ge (n-k)/(t-k)$. For each ℓ-tuple $L \in \binom{[n]}{\ell}$ let

$$F_L = \{e \in E\left(K_n^{(k)}\right) : L \subset e\}.$$

Note that F_L is a subset of the vertices of H and $|F_L| = \binom{n-\ell}{k-\ell}$. Let

$$\mathcal{F} = \{F_L : L \in \binom{[n]}{\ell}\}.$$

Thus $|\mathcal{F}| = \binom{n}{\ell}$ and $s(\mathcal{F}) = \binom{n-\ell}{k-\ell}$. Let K and μ be the constants from Theorem A.0.1. For n (and thus N) sufficiently large, the conditions of Theorem A.0.1 are satisfied. Therefore H has a $(1-\epsilon, \mathcal{F})$-perfect matching. This in turn implies there is a $K_t^{(k)}$-packing of $K_n^{(k)}$ such that each ℓ-tuple of vertices is contained in at most $\epsilon\binom{n-\ell}{k-\ell}$ uncovered k-tuples. Such a packing has order at least $(1-\epsilon)\binom{n}{k}/\binom{t}{k}$. Letting $\epsilon \to 0$ gives the result. □

Alternative proof of Theorem 2.1.1. Apply Claim A.0.2 to $K_n^{(r)}$ with $k = r$, $t = 2r$, and $\ell = r-1$, and let $\mathcal{B}$ represent the family of copies of $K_{2r}^{(r)}$ packing $K_n^{(r)}$. Let $K(n,\mathcal{B})$ be the subgraph of $K_n^{(r)}$ with edge set $\{e \in E\left(K_n^{(r)}\right) : \exists B \in \mathcal{B} \text{ with } e \subset B\}$. We first show how to order the edges of $K(n,\mathcal{B})$. Then we show how to order the remaining edges of $K_n^{(r)}$. Similar to the above proof of Theorem 2.1.1, $\mathcal{B}$ has the following properties:

(1) Each edge $e \in E\left(K_n^{(r)}\right) = \binom{[n']}{r}$ appears in at most one set $B \in \mathcal{B}$.

(2) Each $(r-1)$-subset of $[n]$ appears in at most $\frac{n-(r-1)}{r+1}$ sets of $\mathcal{B}$. Also each $(r-1)$-subset of $[n]$ appears in at least $\frac{(n-o(n))-(r-1)}{r+1}$ sets of $\mathcal{B}$, since as a result of Claim A.0.2 each $(r-1)$-subset is contained in at most $o(n)$ r-tuples not covered by an element of $\mathcal{B}$.

(3) For any two distinct $(r-1)$-subsets I and J of $[n]$, $|I \cup J| \geq r$. By (1) above, $I \cup J$ appears in at most one set of $\mathcal{B}$.

The following claim and its proof are nearly identical to Claim 2.1.4, but we include them for completeness.

Claim A.0.3. *For $k = \left\lfloor (1+\delta)\frac{n-(r-1)}{r+1} \right\rfloor$ there exists a partition of the sets of $\mathcal{B}$ into classes $\mathcal{F}_1, \ldots, \mathcal{F}_k$ so that for all $i = 1, \ldots, k$, and all distinct $B, B' \in \mathcal{F}_i$,*

$$|B \cap B'| \leq r-2.$$

Let our family $\mathcal{B}$ be partitioned into $\mathcal{F}_1, \cdots, \mathcal{F}_k$ as described by Claim 2.1.4. We define ϕ to be an edge-ordering of $K(n,\mathcal{B})$ satisfying the following.

i) The ordering ϕ labels the r-subsets of each $B \in \mathcal{B}$ as in Claim 2.1.2.

ii) For any $B \in \mathcal{F}_i$, $B' \in \mathcal{F}_j$ with $i < j$, make $\phi(e) < \phi(e')$ for all r-subsets $e \subset B$, $e' \subset B'$.

This ordering is well-defined, by (1) above. As in the proof of Theorem 2.1.1, note that for distinct sets $B, B' \in \mathcal{F}_i$, no increasing tight path in $K(n, \mathcal{B})$ uses both an edge $e \subset B$ and an edge $e' \subset B'$, since the intersection of the blocks has order at most $r-2$. Along with Claim 2.1.2, this implies that an increasing tight path with respect to ϕ in $K(n, \mathcal{B})$ uses at most r edges from each partition class $\mathcal{F}_i$, giving

$$g\left(K(n, \mathcal{B})\right) \leq rk \leq r(1+\delta)\frac{n-(r-1)}{r+1} < \frac{(1+\delta)r}{r+1}n.$$

Now we consider the set of r-tuples not covered by some $K_{2r}^{(r)} \in \mathcal{B}$. Let

$$\mathcal{U} = E\left(K_n^{(r)}\right) \setminus E\left(K(n, \mathcal{B})\right).$$

As a result of Claim A.0.2, each $(r-1)$-tuple of vertices in $K_n^{(r)}$ is contained in at most $o(n)$ sets in $\mathcal{U}$. Therefore, for each $e \in \mathcal{U}$, the number of other edges $e' \in \mathcal{U}$ for each $|e \cap e'| = r-1$ is at most $\binom{r}{r-1}o(n) = o(n)$. We can then partition $\mathcal{U}$ into at most $o(n)$ sets $\mathcal{U}_i$ so that for any two distinct $e, e' \in \mathcal{U}_i$, $|e \cap e'| \leq r-2$.

To determine $g\left(K_n^{(r)}\right)$, we extend the edge-ordering ϕ of $K(n, \mathcal{B})$ to an edge-ordering ψ of $K_n^{(r)}$. For $e \in E\left(K(n, \mathcal{B})\right)$, we let $\psi(e) = \phi(e)$. Then for each set $\mathcal{U}_i$, have ψ map its edges to consecutive integers. Note that no two edges from the same set $\mathcal{U}_i$ can be used in an increasing path, since their intersection has order at most $r-2$. Any increasing path with respect to ψ in $K_n^{(r)}$ can start by using at most $g\left(K(n, \mathcal{B})\right)$ edges of $K(n, \mathcal{B})$. Then it must use edges from $\mathcal{U}$. Since there are $o(n)$ partition classes $\mathcal{U}_i$ of $\mathcal{U}$, each of which can contribute at most one edge to an increasing class, we have

$$g\left(K_n^{(r)}\right) \leq \frac{(1+\delta)r}{r+1}n + o(n) \leq \frac{(1+2\delta)r}{r+1}n$$

$\square$

Proof of Lemma A.0.3. Define an auxiliary hypergraph $\mathcal{A}$ with

$$V(\mathcal{A}) = \binom{[n]}{r-1}, \text{ and}$$

$$E(\mathcal{A}) = \left\{ \binom{B}{r-1} : B \in \mathcal{B} \right\}.$$

The hypergraph $\mathcal{A}$ has the following properties.

(a) Each edge of $\mathcal{A}$ corresponds to a distinct set of $\mathcal{B}$.

(b) Each vertex of $\mathcal{A}$ has degree at most $\frac{n-(r-1)}{r+1}$ and degree at least $\frac{(n-o(n))-(r-1)}{r+1}$. This follows from (2) above.

(c) For any pair of vertices in $\mathcal{A}$, they share at most one edge of $\mathcal{A}$. This follows from (3) above.

(d) Two edges of $\mathcal{A}$ are disjoint if and only if the corresponding sets of $\mathcal{B}$ share at most $r-2$ vertices.

By (d), a matching in $\mathcal{A}$ corresponds to a family of sets of $\mathcal{B}$ in which each pair of sets shares at most $r-2$ vertices. Any partition of $E(\mathcal{A})$ into k disjoint matchings (for some k) gives a partition of $\mathcal{B}$ into classes $\mathcal{F}_1, \mathcal{F}_2, \ldots, \mathcal{F}_k$ so that for all $i = 1, \ldots k$, and all distinct $B, B' \in \mathcal{F}_i$,

$$|B \cap B'| \leq r-2.$$

In order to keep the number k of disjoint matchings in $\mathcal{A}$ small (and thus the number of partition classes of $\mathcal{B}$ small), we again use Theorem 2.1.5 above. By (b),

$$d(\mathcal{A}) \geq \frac{(n-o(n))-(r-1)}{r+1} \geq (1-\delta')\frac{n-(r-1)}{r+1} \geq (1-\delta')D(\mathcal{A}),$$

and by (c),

$$C(\mathcal{A}) \leq 1 \leq \delta' D(\mathcal{A}).$$

Then by Theorem 2.1.5, the edges of $\mathcal{A}$ can be partitioned into $(1+\delta)\frac{n-(r-1)}{r+1}$ matchings, so the number k of partition classes $\mathcal{F}_1, \cdots, \mathcal{F}_k$ of $\mathcal{B}$ is at most $\left\lfloor (1+\delta)\frac{n-(r-1)}{r+1} \right\rfloor$.

$\square$

Bibliography

[1] N. Alon and R. Yuster. On a hypergraph matching problem. *Graphs Combin.*, 21(4):377–384, 2005.

[2] A. Arman, B. Elliott, and V. Rödl. Increasing paths in countable graphs. *Submitted. arXiv:1905.03612.*

[3] M. Bucić, M. Kwan, A. Pokrovskiy, B. Sudakov, T. Tran, and A. Wagner. Nearly-linear monotone paths in edge-ordered graphs. *Submitted. arXiv:1809.01468.*

[4] A. R. Calderbank, F. R. K. Chung, and D. G. Sturtevant. Increasing sequences with nonzero block sums and increasing paths in edge-ordered graphs. *Discrete Math.*, 50(1):15–28, 1984.

[5] V. Chvátal and J. Komlós. Some combinatorial theorems on monotonicity. *Canad. Math. Bull.*, 14:151–157, 1971.

[6] T. Gallai. On directed paths and circuits. In *Theory of Graphs (Proc. Colloq., Tihany, 1966)*, pages 115–118. Academic Press, New York, 1968.

[7] R. L. Graham and D. J. Kleitman. Increasing paths in edge ordered graphs. *Period. Math. Hungar.*, 3:141–148, 1973.

[8] M. Hall, Jr., J. D. Swift, and R. Killgrove. On projective planes of order nine. *Math. Tables Aids Comput.*, 13:233–246, 1959.

[9] M. Hasse. Zur algebraischen Begründung der Graphentheorie. II. *Math. nachr.*, 30:155–164, 1965.

[10] P. Keevash. The existence of designs. *Submitted. arXiv:1401.3665.*

[11] K. G. Milans. Monotone paths in dense edge-ordered graphs. *J. Comb.*, 8(3):423–437, 2017.

[12] V. Müller and V. Rödl. Monotone paths in ordered graphs. *Combinatorica*, 2(2):193–201, 1982.

[13] N. Pippenger and J. Spencer. Asymptotic behavior of the chromatic index for hypergraphs. *J. Combin. Theory Ser. A*, 51(1):24–42, 1989.

[14] J. Reiterman. A note on monotone paths in labeled graphs. *Combinatorica*, 9(2):231–232, 1989.

[15] V. Rödl. *Master's Thesis, Charles University*, 1973.

[16] B. Roy. Nombre chromatique et plus longs chemins d'un graphe. *Rev. Française Informat. Recherche Opérationnelle*, 1(5):129–132, 1967.

[17] L. M. Vitaver. Determination of minimal coloring of vertices of a graph by means of Boolean powers of the incidence matrix. *Dokl. Akad. Nauk SSSR*, 147:758–759, 1962.