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Ariel Keller Date

On Cycles, Chorded Cycles, and Degree Conditions

By

Ariel Keller

Doctor of Philosophy

Mathematics

Ronald J. Gould

Advisor

Vojtěch Rödl

Committee Member

Dwight Duffus

Committee Member

Hao Huang

Committee Member

Accepted:

Lisa A. Tedesco, Ph.D.

Dean of the James T. Laney School of Graduate Studies

Date

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Ariel Keller

B.S., University of Tennessee, Chattanooga, 2011

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Abstract

On Cycles, Chorded Cycles, and Degree Conditions

By Ariel Keller

Sufficient conditions to imply the existence of certain substructures in a graph are of considerable interest in extremal graph theory, and conditions that guarantee a large set of cycles or chorded cycles are a recurring theme. This dissertation explores different degree sum conditions that are sufficient for finding a large set of vertex-disjoint cycles or a large set of vertex-disjoint chorded cycles in a graph.

For an integer $t \geq 1$, let $\sigma_t(G)$ be the smallest sum of degrees of t independent vertices of G. We first prove that if a graph G has order at least $7k+1$ and degree sum condition $\sigma_4(G) \geq 8k-3$, with $k \geq 2$, then G contains k vertex-disjoint cycles. Then, we consider an equivalent condition for chorded cycles, proving that if G has order at least $11k+7$ and $\sigma_4(G) \geq 12k-3$, with $k \geq 2$, then G contains k vertex-disjoint chorded cycles. We prove that the degree sum condition in each result is sharp. Finally, we conjecture generalized degree sum conditions on $\sigma_t(G)$ for $t \geq 2$ sufficient to imply that G contains k vertex-disjoint cycles for $k \geq 2$ and k vertex-disjoint chorded cycles for $k \geq 2$. This is joint work with Ronald J. Gould and Kazuhide Hirohata.

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Contents

List of Figures

Chapter 1

Introduction

1.1 History

Extremal graph theory studies relationships between graph invariants, like the number of edges or vertices in a graph, and different graph properties. Often we are interested in how far we can push certain properties before other properties or substructures must exist in the graph. For example, we might ask what is the largest number of edges a graph of a fixed order may contain and still be acyclic. Alternatively, this tells us how many edges the graph must have to guarantee the existence of a cycle.

Over the years, many different results have been proved regarding cycles in graphs. Some such results include graph properties that guarantee a graph contains a Hamiltonian cycle, a set of cycles with specified graph elements, a large set of many different cycles, or a large set of many different chorded cycles or doubly chorded cycles.

The degree of a vertex x, $d(x)$, is defined to be the number of edges incident with x. Let $\delta(G)$ denote the minimum degree over all vertices in a graph G. Clearly, if the minimum degree is large enough relative to the number of vertices in the graph, the graph will contain a Hamiltonian cycle. In particular, Dirac's famous result [3] states that any graph G on $n \geq 3$ vertices with minimum degree $\delta(G) \geq n/2$ contains a Hamiltonian cycle (see Figure 1.1).

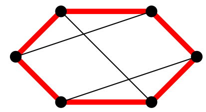


Figure 1.1. A Hamiltonian cycle in a graph G with $n = 6$ and $\delta(G) = 3$.

Ore's Theorem [13] strengthens this result, giving a weaker degree condition sufficient to imply a graph contains a Hamiltonian cycle. It states that, for a graph G on n vertices, if the degrees of any pair of nonadjacent vertices total at least n, then the graph G contains a Hamiltonian cycle. This condition allows an individual vertex to have degree less than $n/2$; hence it is possible for a graph to satisfy the condition of Ore's Theorem while not satisfying the condition of Dirac's Theorem.

In the same vein as Dirac's Theorem and Ore's Theorem for Hamiltonian cycles, density conditions can be used to force a graph to contain many disjoint cycles or chorded cycles.

Cycles are called *vertex-disjoint* if they share no vertices. Let $\delta(G)$ denote the minimum degree of G and

$$\sigma_t(G) = \min\{\sum_{x \in S} d_G(x) : S \text{ is an independent set of } G \text{ with } |S| = t\}.$$

In 1963, Corrádi and Hajnal [2] first considered a minimum degree condition that would imply a graph must contain k different vertex-disjoint cycles, proving that if $|G| \geq 3k$ and $\delta(G) \geq 2k$, then G contains k vertex-disjoint cycles. Enomoto [4] and Wang [15] independently proved a more general result, requiring a weaker condition on the degree sum of any two independent vertices: if $|G| \geq 3k$ and $\sigma_2(G) \geq 4k - 1$, then G contains k vertex-disjoint cycles. Fujita et al. [6] proved the most recent generalization of this result, showing that if $k \geq 2$, $|G| \geq 3k + 2$, and $\sigma_3(G) \geq 6k - 2$, then G contains k vertex-disjoint cycles.

In all three theorems, the degree conditions are sharp as illustrated by the graph

$G_0 = K_{2k-1} + mK_1$. The only independent vertices in G_0 are the vertices in mK_1, each of which has degree $2k-1$. It follows that for any $t \leq m$, $\sigma_t(G_0) = t(2k-1) = 2kt-t$. Any cycle in G_0 must contain two vertices of K_{2k-1} since no two vertices of mK_1 are adjacent. But then the graph G_0 cannot contain k vertex-disjoint cycles. Thus, none of the conditions $\delta(G) = 2k-1$, $\sigma_2(G) = 4k-2$, $\sigma_3(G) = 6k-3$, and in general for $t \leq m$, $\sigma_t(G) = t(2k-1) = 2kt-t$ is sufficient to imply G contains k vertex-disjoint cycles.

In Chapter 2, we consider the next value of t; that is, we show that if $\sigma_4(G) \geq 8k-3$, then G contains k vertex-disjoint cycles. We also prove that the degree sum condition is sharp, and we conjecture a sharp degree sum condition on $\sigma_t(G)$ for any fixed $t \geq 2$ to imply that a graph contains k vertex-disjoint cycles.

An extension of the study of disjoint cycles is that of disjoint chorded cycles. A *chord* of a cycle is an edge between two vertices of the cycle that is not an edge of the cycle. We say a cycle is *chorded* if it induces at least once chord and *doubly chorded* if it induces at least two chords. In 1960, Pósa [14] asked what conditions would imply a graph contains a chorded cycle. In answer to the question, Czipzer (see Lovász [12], problem 10.2) proved in 1963 that if a graph has minimum degree at least 3, it must contain a chorded cycle. More recently, the relevant literature has focused on conditions to imply a graph contains many vertex-disjoint chorded cycles. Finkel [5] extended the work of Corrádi and Hajnal by showing that if $|V(G)| \geq 4k$ and $\delta(G) \geq 3k$, then G contains k vertex-disjoint chorded cycles. Chiba et al. [1] extended this result, proving that for a graph G of order at least $3r+4s$, if $\sigma_2(G) \geq 4r+6s-1$, then G contains $r+s$ vertex-disjoint cycles, with s of them chorded. In [8], doubly chorded cycles were considered, showing that if $\sigma_2(G) \geq 6k-1$, then G contains k vertex-disjoint doubly chorded cycles.

In Chapter 3, we consider the degree condition for $t = 4$. In particular, we show that if G is a graph of order $n \geq 11k+7$, and if $\sigma_4(G) \geq 12k-3$, then G contains

k vertex-disjoint chorded cycles. Furthermore, we prove that this degree condition is sharp, and we conjecture a sharp degree condition on $\sigma_t(G)$ for any fixed $t \geq 2$ to imply the graph G contains k vertex-disjoint chorded cycles.

1.2 Definitions and Notation

We consider only simple graphs, without loops or multiedges. Let $G = (V(G), E(G))$ be a simple graph. Then $|G|$ is the order of G, $\delta(G)$ is the minimum degree of G, $comp(G)$ is the number of components of G, $\alpha(G)$ is the independence number of G. For a vertex $u \in V(G)$, the set of neighbors of u in G is denoted by $N_G(u)$, and we denote the degree of the vertex u by $d_G(u) = |N_G(u)|$. Let H be a subgraph of G. For $u \in V(G) - V(H)$, we denote the neighborhood of u in H by $N_H(u) = N_G(u) \cap V(H)$, and the degree of u in H is given by $d_H(u) = |N_H(u)|$. For $X \subseteq V(G)$, let $d_H(X) = \sum_{x \in X} d_H(x)$. For an integer $t \geq 1$, let

$$\sigma_t(G) = \min\{\sum_{v \in X} d_G(v) \,|\, X \text{ is an independent set of } G \text{ with } |X| = t.\},$$

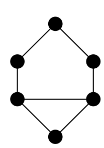
and $\sigma_t(G) = \infty$ when $\alpha(G) < t$. Note that if $t = 1$, then $\sigma_1(G) = \delta(G)$.

For a set $S \subset V(G)$, the subgraph of G induced by S is denoted by $\langle S \rangle$. If there is no fear of confusion, then we use the same symbol for a graph and its vertex set. For graphs G_1 and G_2 with $V(G_1) \cap V(G_2) = \emptyset$, $G_1 \cup G_2$ denotes the union of G_1 and G_2, $G_1 + G_2$ denotes the join of G_1 and G_2, and mG denotes the union of m disjoint copies of G, see [7].

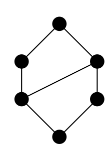
For a path (or a cycle) Q in a graph G, we write $Q = x_1, x_2, \ldots, x_t$ (, x_1), where $V(Q) = \{x_1, x_2, \ldots, x_t\}$ and $\{x_1, x_2\}, \{x_2, x_3\}, \ldots, \{x_{t-1}, x_t\}(, \{x_t, x_1\}) \in E(Q)$. If Q is a path (or a cycle), say $Q = x_1, x_2, \ldots, x_t$ (, x_1), then we assume that an orientation of Q is given from x_1 to x_t. We say that x_i *precedes* x_j, and x_j *follows* x_i, on Q if $i < j$. If $x \in V(Q)$, then x^+ denotes the first successor of x on Q and x^- denotes the first predecessor of x on Q. For $x, y \in V(Q)$, we let $Q[x, y]$ denote the path of Q from x to y (including x and y) in the given direction. The notation $Q^-[x, y]$ denotes the path from y to x in the opposite direction. We also write $Q(x, y] = Q[x^+, y]$, $Q[x, y) = Q[x, y^-]$ and $Q(x, y) = Q[x^+, y^-]$. For $u, v \in V(Q)$, we define the path $Q^\pm[u, v]$ as follows; if u precedes v on Q, then $Q^\pm[u, v] = Q[u, v]$, and if v precedes u on Q, then $Q^\pm[u, v] = Q^-[u, v]$. If T is a tree with at least one branch and $x, y \in V(T)$, where a branch vertex of a tree is a vertex of degree at least three, then we denote the path from x to y as $T[x, y]$.

For an integer $r \geq 1$ and two disjoint subgraphs A, B of G, we denote by $(d_1, d_2, \ldots, d_r)$ a degree sequence from A to B such that $d_B(v_i) \geq d_i$ and $v_i \in V(A)$ for each $1 \leq i \leq r$. Throughout this dissertation, it is sufficient to consider the case of equality in the above inequality; hence, when we write $(d_1, d_2, \ldots, d_r)$, we will assume that $d_B(v_i) = d_i$ for each $1 \leq i \leq r$. For $X, Y \subseteq V(G)$, $E(X, Y)$ denote the set of edges of G joining a vertex in X and a vertex in Y. For vertex-disjoint subgraphs H_1, H_2 of G, we simply write $E(H_1, H_2)$ instead of $E(V(H_1), V(H_2))$. A *forest* is a graph each of whose components is a tree, and a *leaf* is a vertex of a forest whose degree is at most one. A cycle of length ℓ is called an *ℓ-cycle*.

Definition 1. Any chorded six-cycle must be one of two types. Either the chord splits the cycle into a three-cycle and a five-cycle—we call this *type 1*, or the chord splits the cycle into two four-cycles—we call this *type 2*.



(a) Type 1 six-cycle.



(b) Type 2 six-cycle.

Figure 1.2. Six-cycle types.

Definition 2. We say a set $\mathscr{C} = \{C_1, \ldots C_r\}$ of r vertex-disjoint cycles in a graph G is *minimal* if $|\bigcup_{i=1}^r V(C_i)|$ is minimal over all such sets of r cycles.

Definition 3. Let $C = v_1, \ldots, v_t, v_1$ be an oriented cycle with a chord v_iv_j, $i \leq j$. We say a chord $v_kv_l \neq v_iv_j$ is *parallel* to v_iv_j if $v_k, v_l \in C[v_i, v_j]$ or $v_k, v_l \in C[v_j, v_i]$. Note that if two chords share an endpoint, they are parallel. We say two chords are *crossing* if they are not parallel.

Definition 4. Let v_iu_j and v_ku_l be two edges between two oriented paths $P_1 = v_1, \ldots, v_t$ and $P_2 = u_1, \ldots, u_s$. We say v_iu_j and v_ku_l are *parallel* if either $i \leq k$ and $j \leq l$, or $k \leq i$ and $l \leq j$. Note that if two edges between P_1 and P_2 share an endpoint, they are parallel. We say two edges between two oriented paths are *crossing* if they are not parallel.

Definition 5. Let v_iv_j and v_kv_l be two distinct edges between vertices of a path $P_1 = v_1, \ldots, v_t$, with $i < j$ and $k < l$. We say v_iv_j and v_kv_l are *nested* if either $i \leq k < l \leq j$ or $k \leq i < j \leq l$.

Definition 6. Let $P = v_1, \ldots, v_t$ be a path. We say a vertex v_i on P has a *left edge* if there exists an edge v_jv_i for any $j < i-1$. We say v_i has a *right edge* if there exists an edge v_iv_l for any $l > i+1$.

Definition 7. Let X be a set of vertices in a graph H with $|X| > 1$. We call a vertex x of X *isolated from the rest of* X if it is the only vertex of X in some component H_i of H.

For terminology and notation not defined here, see [7].

Chapter 2

Degree Conditions to Imply the Existence of Vertex-Disjoint Cycles

In this chapter, we prove a result regarding the existence of a large set of vertex-disjoint cycles in a graph. Let G be a graph such that $|G| \geq 7k+1$ and $\sigma_4(G) \geq 8k-3$ for integer $k \geq 2$. We prove that such a graph contains a set of k vertex-disjoint cycles. We also conjecture a generalized result for $\sigma_t(G)$, and we show that the degree sums in the result on $\sigma_4(G)$ and the conjecture for $\sigma_t(G)$ are sharp.

2.1 Introduction

The study of cycles in graphs is an important and rich area. One of the more interesting questions is to find conditions that insure the existence of k ($k \geq 2$) vertex-disjoint cycles. A number of such results exist. As noted in the introduction, Corrádi and Hajnal [2] proved that if a graph G has order at least $3k$ and $\delta(G) \geq 2k$, then G contains k disjoint cycles. Justesen [11] proved the same result from the condition $\sigma_2(G) \geq 4k$. Enomoto [4] and Wang [15] independently improved Justesen's bound to $\sigma_2(G) \geq 4k-1$. Fujita et al. [6] proved that if $|G| \geq 3k+2$ and $\sigma_3(G) \geq 6k-2$, then G contains k disjoint cycles. The purpose of this chapter is to further extend these results. We also conjecture the following:

Conjecture 2.1 ([10])**.** *Let G be a graph of sufficiently large order. If $\sigma_t(G) \geq 2kt - (t-1)$ for any two integers $k \geq 2$ and $t \geq 1$, then G contains k disjoint cycles.*

The cases for $t = 1, 2, 3$ have already been shown. We add to the evidence for this conjecture by showing the following:

Theorem 2.1 ([10])**.** *Let G be a graph of order $n \geq 7k+1$ for an integer $k \geq 2$. If $\sigma_4(G) \geq 8k-3$, then G contains k disjoint cycles.*

The degree sum condition conjectured above would be sharp. And in particular, the degree sum condition of Theorem 2.1 is sharp. Sharpness is given by $G = K_{2k-1} + mK_1$. The only independent vertices in G are those in mK_1. Each of these vertices has degree $2k-1$. Thus, for any t with $1 \leq t \leq m$, $\sigma_t(G) = t(2k-1) = 2kt - t$, and G fails to contain k disjoint cycles as any such cycle must contain two vertices of K_{2k-1}.

2.2 Lemmas

In the proof of Theorem 2.1, we make use of the following lemmas. Fujita, Matsumura, Tsugaki and Yamashita proved Lemmas 2.A, 2.B and 2.C in [6]. The proofs of Lemmas 2.1 and 2.5 appear after the proof of Theorem 2.1, that is, in Section 2.4.

Let $C_1, \ldots, C_r$ be r disjoint cycles of a graph G. If $C'_1, \ldots, C'_r$ are r disjoint cycles of G and $|\cup_{i=1}^r V(C'_i)| < |\cup_{i=1}^r V(C_i)|$, then we call $C'_1, \ldots, C'_r$ a *shorter (family of) cycles* than $C_1, \ldots, C_r$. We also call $\{C_1, \ldots, C_r\}$ a *minimal (family of) cycles* if G does not contain shorter r disjoint cycles than $C_1, \ldots, C_r$.

Lemma 2.A (Fujita et al. [6])**.** *Let r be a positive integer and $C_1, \ldots, C_r$ be r minimal disjoint cycles of a graph G. Then $d_{C_i}(x) \leq 3$ for any $x \in V(G) - \cup_{i=1}^r V(C_i)$ and* *for any* $1 \le i \le r$*. Furthermore,* $d_{C_i}(x) = 3$ *implies* $|C_i| = 3$*, and* $d_{C_i}(x) = 2$ *implies* $|C_i| \le 4$*.*

Lemma 2.B (Fujita et al. [6])**.** *Suppose that* F *is a forest with at least two components and* C *is a triangle. Let* x_1*,* x_2*,* x_3 *be leaves of* F *from at least two components. If* $d_C(\{x_1, x_2, x_3\}) \ge 7$*, then there exist two disjoint cycles in* $\langle F \cup C \rangle$ *or there exists a triangle* C' *in* $\langle F \cup C \rangle$ *such that* $comp(\langle F \cup C \rangle - C') < comp(F)$*.*

Lemma 2.1. *Suppose that* F *is a forest with at least two components and* C *is a triangle. Let* x_1*,* x_2*,* x_3*,* x_4 *be leaves of* F *from at least two components. If* $d_C(\{x_1, x_2, x_3, x_4\}) \ge 9$*, then there exist two disjoint cycles in* $\langle F \cup C \rangle$ *or there exists a triangle* C' *in* $\langle F \cup C \rangle$ *such that* $comp(\langle F \cup C \rangle - C') < comp(F)$*.*

Lemma 2.C (Fujita et al. [6])**.** *Let* C *be a cycle and* T *be a tree with three leaves* x_1*,* x_2*,* x_3*. If* $d_C(\{x_1, x_2, x_3\}) \ge 7$*, then there exist two disjoint cycles in* $\langle C \cup T \rangle$ *or there exists a cycle* C' *in* $\langle C \cup T \rangle$ *such that* $|C'| < |C|$*.*

Lemma 2.2. *Let* C *be a cycle and* T *be a tree with four leaves* x_1*,* x_2*,* x_3*,* x_4*. If* $d_C(\{x_1, x_2, x_3, x_4\}) \ge 9$*, then there exist two disjoint cycles in* $\langle C \cup T \rangle$ *or there exists a cycle* C' *in* $\langle C \cup T \rangle$ *such that* $|C'| < |C|$*.*

Proof. Let $X = \{x_1, x_2, x_3, x_4\}$. If $d_C(x_{i_0}) \le 2$ for some $1 \le i_0 \le 4$, then $d_C(X - \{x_{i_0}\}) \ge 7$, and we apply Lemma 2.C to $X - \{x_{i_0}\}$. Otherwise, $d_C(x_i) \ge 3$ for each $1 \le i \le 4$, and we apply Lemma 2.C to any three vertices in X. □

Lemma 2.3. *Let* G *be a graph satisfying the assumption of Theorem* 2.1*, and let* $\{C_1, \dots, C_{k-1}\}$ *be a minimal (family of)* $k-1$ *disjoint cycles of* G*. Suppose that there exists a tree* T *with at least four leaves, which is a component of* $G - \cup_{i=1}^{k-1} C_i$*. Then* G *contains* k *disjoint cycles.*

Proof. Let $\mathscr{C} = \cup_{i=1}^{k-1} C_i$, and let $X = \{x_1, x_2, x_3, x_4\}$ be a set of leaves of T. Since X is an independent set, $d_{\mathscr{C}}(X) \ge (8k-3) - 4 = 8(k-1) + 1$. Then there exists a cycle

C_i for some $1 \le i \le k-1$ such that $d_{C_i}(X) \ge 9$. Since $\{C_1, \ldots, C_{k-1}\}$ is minimal, there exist two disjoint cycles in $\langle C_i \cup T \rangle$ by Lemma 2.2. Thus G contains k disjoint cycles. □

Lemma 2.4. *Let G be a graph satisfying the assumption of Theorem 2.1, and let C_1, $\ldots$, C_{k-1} be $k-1$ minimal disjoint cycles of G. Suppose that $H = G - \cup_{i=1}^{k-1} C_i$ has at least two components at least one of which is a tree T with at least three leaves. Then there exist two disjoint cycles in $\langle C_i \cup T \rangle$ for some $1 \le i \le k-1$ or there exists a triangle C in $\langle H \cup C_i \rangle$ such that $comp(\langle H \cup C_i \rangle - C) < comp(H)$.*

Proof. Let $\mathscr{C} = \cup_{i=1}^{k-1} C_i$. Let x_1, x_2, x_3 be three leaves of the tree T, and let x_4 be a leaf from another component, and $X = \{x_1, x_2, x_3, x_4\}$. Since X is an independent set, $d_{\mathscr{C}}(X) \ge (8k-3) - 4 = 8(k-1) + 1$. Then there exists a cycle C_i for some $1 \le i \le k-1$ such that $d_{C_i}(X) \ge 9$. If $d_{C_i}(x_4) \le 2$, then $d_C(\{x_1, x_2, x_3\}) \ge 7$. By Lemma 2.C, there exist two disjoint cycles in $\langle C_i \cup T \rangle$ or there exists a cycle C in $\langle C_i \cup T \rangle$ such that $|C| < |C_i|$. Since $\{C_1, \ldots, C_{k-1}\}$ is minimal, the lemma holds. If $d_{C_i}(x_4) \ge 3$, then C_i is a triangle by Lemma 2.A. Thus the lemma holds by Lemma 2.1. □

Lemma 2.5. *Let C_1 and C_2 be two disjoint cycles such that $|C_2| \ge 6$. Suppose that C_2 contains vertices with at least one of the following degree sequences from C_2 to C_1.*

(i) $(2, 2, 2, 2, 2)$

(ii) $(5, 3)$

(iii) $(3, 1, 1, 1, 1, 1)$

(iv) $(3, 2, 1, 1)$

(v) $(3, 3, 1)$

Then $\langle C_1 \cup C_2 \rangle$ contains two disjoint cycles C_1' and C_2' such that $|C_1'| + |C_2'| < |C_1| + |C_2|$.

Lemma 2.6. *Let H be a graph with two components H_1, H_2, where $H_1 = x_1$, $\ldots$, x_s*

$(s \geq 1)$ is a path and $H_2 = y_1, \ldots, y_t$ $(t \geq 3)$ is a path. Let $W = \{x_1, y_1, y_i, y_t\}$ for any $2 \leq i \leq t-1$, and let C be a triangle. If there exists a degree sequence $(3, 3, 2, 0)$ or $(3, 3, 1, 1)$ from W to C, then $\langle H \cup C \rangle$ contains two disjoint cycles.

2.3 Proof of Theorem 2.1

For convenience, we restate our main result.

Theorem 2.1. *Let G be a graph of order $n \geq 7k+1$ for an integer $k \geq 2$. If $\sigma_4(G) \geq 8k-3$, then G contains k disjoint cycles.*

Proof of Theorem 2.1. Suppose that the theorem does not hold. Let G be an edge-maximal counterexample. If G is a complete graph, then G contains k disjoint cycles. Thus we may assume that G is not a complete graph. Let $xy \notin E(G)$ for some $x, y \in V(G)$, and define $G' = G + xy$. Since G' is not a counterexample by the maximality of G, G' contains k disjoint cycles $C_1, \ldots, C_k$. Without loss of generality, we may assume that $xy \notin \cup_{i=1}^{k-1} E(C_i)$, that is, G contains $k-1$ disjoint cycles $C_1, \ldots, C_{k-1}$. Let $\mathscr{C} = \cup_{i=1}^{k-1} C_i$ and $H = G - \mathscr{C}$. Choose $C_1, \ldots, C_{k-1}$ such that

(1) $\sum_{i=1}^{k-1} |C_i|$ is minimal, and

(2) subject to (1), *comp*(H) is minimal.

Note that any cycle C in $\mathscr{C}$ has no chords by (1). Clearly, H is a forest, otherwise, since H contains a cycle, G contains k disjoint cycles, a contradiction. If H contains at least two components at least one of which is a tree with at least three leaves, then by Lemma 2.4, either G contains k disjoint cycles, or we contradict (2). Thus if H contains at least two components, H must be a collection of paths. If H has only one component, then it is a tree. If H is a tree with at least four leaves, then the theorem holds by Lemma 2.3. Thus if H has only one component, then H is a tree with at most three leaves.

Now, we consider two cases on $|H|$.

Case 1. $|H| \leq 7$.

Let C be a longest cycle in $\mathscr{C}$. Suppose that $|C| \leq 7$. Then $|C'| \leq 7$ for any cycle C' in $\mathscr{C}$, and $|\mathscr{C}| \leq 7(k-1)$. Since $|G| \geq 7k+1$, $|H| = |G| - |\mathscr{C}| \geq (7k+1) - 7(k-1) = 8$, contradicting the assumption of this case. Thus $|C| \geq 8$. Let $|C| = 4t + r$, $t \geq 2$ and $0 \leq r \leq 3$. Then there exist at least t disjoint independent sets in $V(C)$ each of which has four vertices. By (1) and $|C| \geq 8$, $d_C(v) \leq 1$ for any $v \in V(H)$. Thus $|E(H, C)| \leq 7$.

Suppose that $k = 2$. Then $\mathscr{C}$ has only one cycle C, and $H = G - C$. Since $|C| \geq 8$, C contains at least two independent sets each of which has four vertices. Let X_1 and X_2 be such sets. Since $d_C(X_i) = 8$ for each $i \in \{1, 2\}$, $d_H(X_i) \geq (8k - 3) - 8 = 8k - 11$. Then $d_H(X_1 \cup X_2) \geq 16k - 22 \geq 10$, since $k \geq 2$. Thus $|E(C, H)| \geq 10$, a contradiction.

Suppose that $k \geq 3$. We claim that $|E(C, C')| \geq 8t$ for some cycle C' in $\mathscr{C} - C$. Note that each of t disjoint independent sets in $V(C)$ sends at least $(8k - 3) - 8 = 8k - 11$ edges out of C. Since $|E(C, H)| \leq 7$ and $t \geq 2$, $|E(C, \mathscr{C} - C)| \geq t(8k - 11) - 7 > 8t(k - 2)$. Thus the claim holds. Since $|C| = 4t + r \leq 4t + 3$ and $|E(C, C')|/|C| \geq 8t/(4t + 3) > 8t(4t + 4) = 2t/(t + 1) > 1$, $d_{C'}(v) \geq 2$ for some $v \in V(C)$.

Suppose that $\max\{d_{C'}(v) | v \in V(C)\} = 2$. Let $X = \{v \in V(C) | d_{C'}(v) \leq 1\}$ and $Y = V(C) - X$. Then noting that $t \geq 2$ and $r \leq 3$,

$$
\begin{aligned}
8t \leq |E(C, C')| \leq |X| + 2|Y| = (|C| - |Y|) + 2|Y| = |C| + |Y| \\
\Rightarrow |Y| \geq 8t - |C| = 8t - (4t + r) = 4t - r \\
\geq 8 - 3 = 5.
\end{aligned}
$$

Thus we have the degree sequence (2,2,2,2,2) from C to C'. By Lemma 2.5(i), $\langle C \cup C' \rangle$

contains two shorter disjoint cycles, contradicting (1).

Suppose that $h = \max\{d_{C'}(v) | v \in V(C)\} \geq 3$. Let $d_{C'}(v^*) = h$ for some $v^* \in V(C)$. Since $|C'| \leq |C| = 4t + r$ by the choice of C, $d_{C'}(v^*) \leq |C'| \leq 4t + r$. Then since $t \geq 2$ and $r \leq 3$, $|E(C - v^*, C')| \geq 8t - (4t + r) = 4t - r \geq 5$. This implies that $N_{C'}(C - v^*) \neq \emptyset$. Let $Z = \{v \in V(C) | N_{C'}(v) \neq \emptyset\}$. Then $|Z| \geq 2$.

Suppose that $|Z| = 2$. Then $d_{C'}(v) \geq 5$ for any $v \in Z$ by the above observations. By Lemma 2.5(ii), $\langle C \cup C' \rangle$ contains two shorter disjoint cycles, contradicting (1).

Suppose that $|Z| \geq 3$. Since $|E(C - v^*, C')| \geq 5$, we may assume that the minimum degree sequence S from vertices of C to C' is at least one of $(h, 4, 1)$, $(h, 3, 2)$, $(h, 3, 1, 1)$, $(h, 2, 2, 1)$, $(h, 2, 1, 1, 1)$, or $(h, 1, 1, 1, 1, 1)$, where by the definition of h, if $S = (h, 4, 1)$, then $h \geq 4$, and if S is the other degree sequence, then $h \geq 3$. If $S = (h, 4, 1)$ or $(h, 3, 2)$, then by Lemma 2.5(v), $\langle C \cup C' \rangle$ contains two shorter disjoint cycles. If $S = (h, 3, 1, 1)$, $(h, 2, 2, 1)$ or $(h, 2, 1, 1, 1)$, then by Lemma 2.5(iv), $\langle C \cup C' \rangle$ contains two shorter disjoint cycles. If $S = (h, 1, 1, 1, 1, 1)$, then by Lemma 2.5(iii), $\langle C \cup C' \rangle$ contains two shorter disjoint cycles.

Case 2. $|H| \geq 8$.

Claim 1. *H is connected.*

Proof. Suppose to the contrary that H is disconnected. Then note that H is a collection of paths. Suppose that X is an independent set that consists of four leaves from at least two components in H such that $d_H(X) \leq 4$. Then $d_{\mathscr{C}}(X) \geq (8k-3)-4 = 8(k-1)+1$, and $d_{C_{i_0}}(X) \geq 9$ for some $1 \leq i_0 \leq k-1$. Thus $d_{C_{i_0}}(x) \geq 3$ for some $x \in X$, and $|C_{i_0}| = 3$ by Lemma 2.A. By Lemma 2.1 and (2), $\langle H \cup C_{i_0} \rangle$ contains two disjoint cycles, and G contains k disjoint cycles, a contradiction. Thus H does not contain such an independent set.

Now, we consider three cases on *comp*(H).

Case 1. comp$(H) \geq 4$.

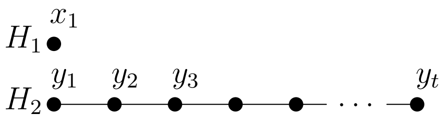


Figure 2.1. $|H_1| = 1$

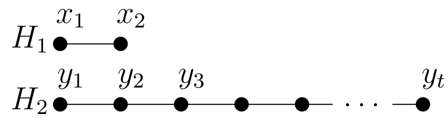


Figure 2.2. $|H_1| = 2$

We take four leaves x_1, x_2, x_3, x_4, one from each component of H. Then $X = \{x_1, x_2, x_3, x_4\}$ is an independent set such that $d_H(X) \leq 4$, a contradiction.

Case 2. comp$(H) = 3$.

We take three leaves x_1, x_2, x_3, one from each component of H. Since $|H| \geq 8$, some component of H, say H_1, has order at least 3. Now, we take the other leaf from H_1, call it x_4. Then $X = \{x_1, x_2, x_3, x_4\}$ is an independent set such that $d_H(X) \leq 4$, a contradiction.

Case 3. comp$(H) = 2$.

Let H_1, H_2 be two distinct components in H. Without loss of generality, we may assume that $|H_1| \leq |H_2|$. Suppose that $|H_1| \geq 3$. Then we take two leaves from each component of H, yielding a set X of four independent vertices such that $d_H(X) = 4$, a contradiction. Suppose that $|H_1| \in \{1, 2\}$. Since $|H| \geq 8$, $|H_2| \geq 6$. Let $H_1 = x_1$, x_s ($s \in \{1, 2\}$); $H_2 = y_1$, y_2, $\ldots$, y_t ($t \geq 6$), and let $W = \{x_1, y_1, y_3, y_t\}$ (see Figures 2.1 and 2.2). Since W is an independent set and $d_H(W) \leq 5$, $d_{\mathscr{C}}(W) \geq (8k-3) - 5 = 8(k-1)$. Then there is a cycle C_0 in $\mathscr{C}$ such that $d_{C_0}(W) \geq 8$. By Lemma 2.A, $d_{C_0}(u) \leq 3$ for any $u \in W$, and $|C_0| \leq 4$. Then the minimum possible degree sequence S from W to C_0 is (3,3,2,0), (3,3,1,1), (3,2,2,1) or (2,2,2,2).

Suppose that $|C_0| = 4$. Let $C_0 = v_1$, v_2, v_3, v_4, v_1. Then $d_{C_0}(u) \leq 2$ for any $u \in W$ by Lemma 2.A. Thus we must have degree sequence (2,2,2,2). If some $u \in W$ has consecutive neighbors in C_0, then u and these two neighbors form a 3-cycle, contradicting (1). Thus for any $u \in W$, its neighbors in C_0 are not consecutive. It follows that for any $u \in W$, either $N_{C_0}(u) = \{v_1, v_3\}$ or $N_{C_0}(u) = \{v_2, v_4\}$. Without loss of generality, we may assume that $N_{C_0}(x_1) = \{v_1, v_3\}$. If y_{i_0}, y_{j_0} with some i_0, $j_0 \in \{1, 3, t\}$ and $i_0 < j_0$ do not share neighbors in C_0 with x_1, then we can easily find two disjoint cycles, as follows. Since $N_{C_0}(y_m) = \{v_2, v_4\}$ for each $m \in \{i_0, j_0\}$, $H_2[y_{i_0}, y_{j_0}]$, v_4, y_{i_0} is a cycle, and x_1, v_3, v_2, v_1, x_1 is the other disjoint cycle (see Figure 2.3).

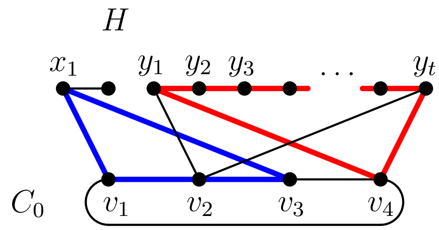


Figure 2.3. An example where $i_0 = 1$ and $j_0 = t$.

Thus at most one vertex in $\{y_1, y_3, y_t\}$ does not share neighbors in C_0 with x_1. Suppose that some vertex in $\{y_1, y_3, y_t\}$ does not share neighbors in C_0 with x_1. First, suppose that such a vertex is y_1, that is, $N_{C_0}(y_1) = \{v_2, v_4\}$. Then y_1, v_4, v_3, v_2, y_1 is a cycle. Since $v_1 \in N_{C_0}(y_i)$ for each $i \in \{3, t\}$, $H_2[y_3, y_t]$, v_1, y_3 is the other disjoint cycle. If $N_{C_0}(y_t) = \{v_2, v_4\}$, then y_t, v_4, v_3, v_2, y_t and $H_2[y_1, y_3]$, v_1, y_1 are two disjoint cycles. Suppose that $N_{C_0}(y_3) = \{v_2, v_4\}$. Then we form a 4-cycle $C_0' = y_3$, v_4, v_3, v_2, y_3. Since $v_1 \in N_{C_0}(y_i)$ for each $i \in \{1, t\}$, $\langle H \cup C_0 \rangle - C_0'$ is connected, contradicting (2) (see Figure 2.4). Thus $N_{C_0}(x_1) = N_{C_0}(y_i)$ for each $i \in \{1, 3, t\}$. Then $C_0' = H_2[y_1, y_3]$, v_1, y_1 is a 4-cycle. Since $v_3 \in N_{C_0}(u)$ for each $u \in \{x_1, y_t\}$, $\langle H \cup C_0 \rangle - C_0'$ is connected, contradicting (2). Thus if there exists a 4-cycle in $\mathscr{C}$, we get a contradiction.

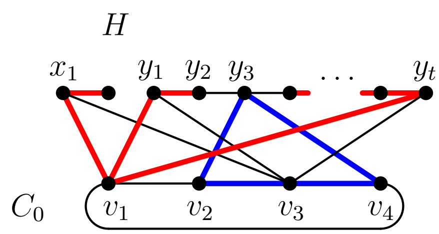


Figure 2.4. A new cycle C_0' such that $\langle H \cup C_0 \rangle - C_0'$ is connected.

Suppose that $|C_0| = 3$. Let $C_0 = v_1$, v_2, v_3, v_1.

Subcase 1. $S = (3, 3, 2, 0)$ or $S = (3, 3, 1, 1)$.

By Lemma 2.6, we can find two disjoint cycles in $\langle C_0 \cup H \rangle$, a contradiction.

Subcase 2. $S = (3, 2, 2, 1)$.

If $d_{C_0}(y_3) = 1$, then since $\{x_1, y_1, y_t\}$ satisfies the conditions of Lemma 2.B, we get a contradiction. Thus $d_{C_0}(y_3) \in \{2, 3\}$.

First, suppose that $d_{C_0}(x_1) = 1$. Let $v_1 \in N_{C_0}(x_1)$. Note that $d_{C_0}(y_i) \geq 2$ for each $i \in \{1, 3, t\}$. If $v_1 \notin N_{C_0}(y_{i_0})$ for some $i_0 \in \{1, t\}$, then $d_{C_0}(y_{i_0}) = 2$, and $C_0' = y_{i_0}$, v_3, v_2, y_{i_0} is a 3-cycle. Since $d_{C_0}(y_{i_1}) = 3$ for some $i_1 \in \{1, 3, t\} - \{i_0\}$, $v_1 \in N_{C_0}(y_{i_1})$. Then $\langle C_0 \cup H \rangle - C_0'$ is connected, contradicting (2) (see Figure 2.5). Thus $v_1 \in N_{C_0}(y_i)$ for each $i \in \{1, t\}$. Since $d_{C_0}(y_{i_2}) = 3$ for some $i_2 \in \{1, 3, t\}$, $C_0'' = y_{i_2}$, v_3, v_2, y_{i_2} is a 3-cycle. Then $\langle C_0 \cup H \rangle - C_0''$ is connected, contradicting (2).

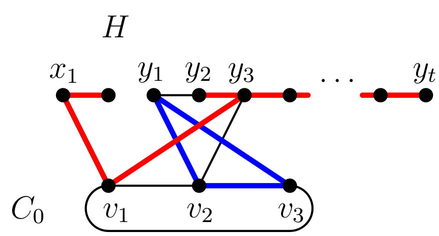


Figure 2.5. The case when $i_0 = 1$ and $i_1 = 3$.

Next, suppose that $d_{C_0}(x_1) = 2$. Without loss of generality, we may assume that v_1, $v_2 \in N_{C_0}(x_1)$. Suppose that $d_{C_0}(y_3) = 2$. Since $|C_0| = 3$, we may assume that $v_1 \in N_{C_0}(x_1) \cap N_{C_0}(y_3)$. Since $d_{C_0}(y_{j_0}) = 3$ for some $j_0 \in \{1, t\}$, $C_0' = y_{j_0}$, v_3, v_2, y_{j_0} is a 3-cycle. Then $\langle C_0 \cup H \rangle - C_0'$ is connected, contradicting (2). Suppose that $d_{C_0}(y_3) = 3$. If $v_3 \in N_{C_0}(y_{m_0})$ for some $m_0 \in \{1, t\}$, then $H_2^{\pm}[y_3, y_{m_0}]$, v_3, y_3 and x_1, v_2, v_1, x_1 are two disjoint cycles. Thus $v_3 \notin N_{C_0}(y_m)$ for each $m \in \{1, t\}$, that is, $N_{C_0}(y_m) \subseteq \{v_1, v_2\}$. Since one of y_1 and y_t has the degree 1 and the other has the degree 2, without loss of generality, we may assume that $v_1 \in N_{C_0}(y_1) \cap N_{C_0}(y_t)$. Since $d_{C_0}(y_3) = 3$, $C_0'' = y_3$, v_3, v_2, y_3 is a 3-cycle, and $\langle C_0 \cup H \rangle - C_0''$ is connected, contradicting (2) (see Figure 2.6).

Finally, suppose that $d_{C_0}(x_1) = 3$. Since $d_{C_0}(y_{i_0}) = d_{C_0}(y_{j_0}) = 2$ for some i_0, $j_0 \in \{1, 3, t\}$ with $i_0 < j_0$, we may assume that $v_1 \in N_{C_0}(y_{i_0}) \cap N_{C_0}(y_{j_0})$. Then $H_2[y_{i_0}, y_{j_0}]$, v_1, y_{i_0} is a cycle. Since $d_{C_0}(x_1) = 3$, a second disjoint cycle is given by x_1, v_3, v_2, x_1 (see Figure 2.7), a contradiction.

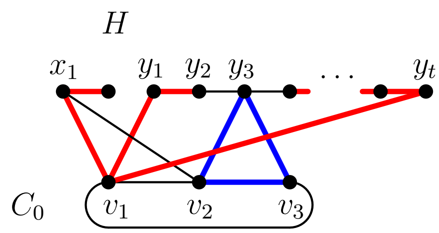


Figure 2.6. The case when $v_1 \in N_{C_0}(y_1) \cap N_{C_0}(y_t)$.

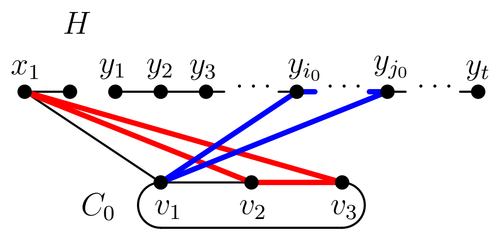


Figure 2.7. The case when $v_1 \in N_{C_0}(y_{i_0}) \cap N_{C_0}(y_{j_0})$.

Subcase 3. $S = (2, 2, 2, 2)$.

Without loss of generality, we may assume that $N_{C_0}(x_1) = \{v_1, v_2\}$. If $v_3 \in N_{C_0}(y_{i_0}) \cap N_{C_0}(y_{j_0})$ for some i_0, $j_0 \in \{1, 3, t\}$ with $i_0 < j_0$, then $H_2[y_{i_0}, y_{j_0}]$, v_3, y_{i_0} and x_1, v_2, v_1, x_1 are two disjoint cycles. Thus at most one in $\{y_1, y_3, y_t\}$ can be adjacent to v_3. Suppose that $v_3 \in N_{C_0}(y_{i_0})$ for some $i_0 \in \{1, 3, t\}$. Since $d_{C_0}(y_{i_0}) = 2$, we may assume that $v_2 \in N_{C_0}(y_{i_0})$. Then $C_0' = y_{i_0}$, v_3, v_2, y_{i_0} is a 3-cycle. For each $i \in \{1, 3, t\} - \{i_0\}$, $N_{C_0}(y_i) = \{v_1, v_2\}$. Then $\langle C_0 \cup H \rangle - C_0'$ is connected, contradicting (2). Thus $v_3 \notin N_{C_0}(y_i)$ for each $i \in \{1, 3, t\}$, that is, $N_{C_0}(y_i) = \{v_1, v_2\}$. Then $C_0'' = H_2[y_1, y_3]$, v_2, y_1 is a 3-cycle, and $\langle C_0 \cup H \rangle - C_0''$ is connected, contradicting (2). This completes the proof of Claim 1. □

Claim 2. *H is a path.*

Proof. Suppose that H is not a path. Then recall that H is a tree with one branch vertex of degree 3 in H. Then H has three leaves, say x_1, x_2, x_3. Removing the branch vertex in H, there exist three disjoint paths each of which has one vertex from $\{x_1, x_2, x_3\}$ as an endpoint. Also, some path has order at least three, say P, since there exist at least seven vertices distributed over three paths. Without loss of generality, we may assume that x_1 is one of the endpoints of P, and let the other endpoint be x_4. Let $X = \{x_1, x_2, x_3, x_4\}$ (see Figure 2.8). Then X is an independent set. Since $d_H(X) = 5$, $d_{\mathscr{C}}(X) \geq (8k-3) - 5 = 8(k-1)$. Thus there exists a cycle C_{i_0} in $\mathscr{C}$ such that $d_{C_{i_0}}(X) \geq 8$ for some $1 \leq i_0 \leq k-1$. Then $d_{C_{i_0}}(x) \geq 2$ for some $x \in X$. By Lemma 2.A, $d_{C_{i_0}}(x) \leq 3$ and $|C_{i_0}| \leq 4$.

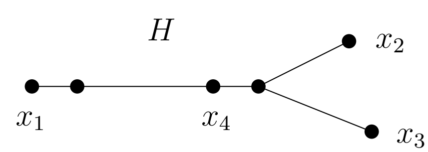


Figure 2.8. The graph H and an independent set $X = \{x_1, x_2, x_3, x_4\}$.

Case 1. $|C_{i_0}| = 3$.

Let $C_{i_0} = v_1,\ v_2,\ v_3,\ v_1$. Suppose that $d_{C_{i_0}}(x) = 2$ for each $x \in X$. Without loss of generality, let $v_1,\ v_2 \in N_{C_{i_0}}(x_1)$. Since $|C_{i_0}| = 3$, $N_{C_{i_0}}(x_2) \cap N_{C_{i_0}}(x_3) \neq \emptyset$. If $v_3 \in N_{C_{i_0}}(x_2) \cap N_{C_{i_0}}(x_3)$, then $H[x_2, x_3]$, v_3, x_2 and x_1, v_2, v_1, x_1 are two disjoint cycles (see Figure 2.9). Thus without loss of generality, we may assume that $v_1 \in N_{C_{i_0}}(x_2) \cap N_{C_{i_0}}(x_3)$. Then $H[x_2, x_3]$, v_1, x_2 is a cycle. Since $d_{C_{i_0}}(x_4) = 2$, $N_{C_{i_0}-v_1}(x_4) \neq \emptyset$. If $v_2 \in N_{C_{i_0}}(x_4)$, then $H[x_1, x_4]$, v_2, x_1 is the other disjoint cycle (see Figure 2.10), and if $v_3 \in N_{C_{i_0}}(x_4)$, then $H[x_1, x_4]$, v_3, v_2, x_1 is the other disjoint cycle. Thus there exists at least one vertex $x \in X$ such that $d_{C_{i_0}}(x) = 3$. Then the minimum possible degree sequences from X to C_{i_0} are (3,3,2,0), (3,3,1,1) or (3,2,2,1).

Subclaim 2.1. *If there exists a degree sequence at least* $(3, 3, 1, 0)$ *from* X *to* C_{i_0}, *then there exist two disjoint cycles in* $\langle H \cup C_{i_0} \rangle$*.*

First, suppose that $d_{C_{i_0}}(x_{j_0}) = 1$ for some $1 \le j_0 \le 3$. Let $v_1 \in N_{C_{i_0}}(x_{j_0})$. If $d_{C_{i_0}}(x_4) = 0$, then since $d_{C_{i_0}}(x_m) = 3$ for each $m \in \{1,2,3\} - \{j_0\}$, $H[x_{j_0}, x_m]$, v_1, x_{j_0} is a cycle. Since $d_{C_{i_0}}(x_{m'}) = 3$ for $m' \in \{1,2,3\} - \{j_0, m\}$, it follows that $x_{m'}$, v_3, v_2, $x_{m'}$ forms another cycle, vertex-disjoint from the first (see Figure 2.11). If $d_{C_{i_0}}(x_4) = 3$, then $H[x_{j_0}, x_4]$, v_1, x_{j_0} is a cycle, and since $d_{C_{i_0}}(x_{m_0}) = 3$ for some $m_0 \in \{1,2,3\} - \{j_0\}$, the other disjoint cycles is given by x_{m_0}, v_3, v_2, x_{m_0}. Next, suppose that $d_{C_{i_0}}(x_4) = 1$. Let $v_1 \in N_{C_{i_0}}(x_4)$. Then $d_{C_{i_0}}(x_{m_1}) = 3$ and $d_{C_{i_0}}(x_{m_2}) = 3$ for some $1 \le m_1 < m_2 \le 3$, and $H[x_{m_1}, x_4]$, v_1, x_{m_1} and x_{m_2}, v_3, v_2, x_{m_2} are two disjoint cycles, and Subclaim 2.1 holds.

Thus by the claim, we have only to consider the degree sequence (3,2,2,1). If the degree 3 vertex does not lie on the path in H connecting the degree 2 vertices, then since the two vertices with degree 2 must have a common neighbor by $|C_{i_0}| = 3$, we can easily find two disjoint cycles. Thus the degree 3 vertex does lie on the path connecting the two vertices with degree 2. This implies that $d_{C_{i_0}}(x_4) = 3$, $d_{C_{i_0}}(x_1) = 2$ (see Figure 2.8), and we may assume that $d_{C_{i_0}}(x_2) = 1$ and $d_{C_{i_0}}(x_3) = 2$. Let $v_1 \in N_{C_{i_0}}(x_2)$. Since $|N_{C_{i_0}}(x_1) \cap N_{C_{i_0}}(x_4)| = 2$, there exists $v_{h_0} \in N_{C_{i_0}}(x_1) \cap N_{C_{i_0}}(x_4)$ for some $h_0 \in \{2,3\}$. Then $H[x_1, x_4]$, v_{h_0}, x_1 is a cycle. Since $d_{C_{i_0}}(x_3) = 2$, there exists $v_{h_1} \in N_{C_{i_0}}(x_3)$ for some $h_1 \in \{1,2,3\} - \{h_0\}$. If $h_1 = 1$, then $H[x_2, x_3]$, v_1, x_2 is the other disjoint cycle (see Figure 2.12), and if $h_1 \in \{2,3\}$, then $H[x_2, x_3]$, v_{h_1}, v_1, x_2 is the other disjoint cycle.

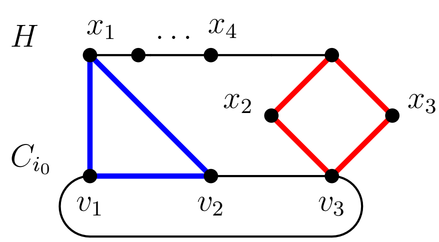


Figure 2.9. The case when $v_3 \in N_{C_{i_0}}(x_2) \cap N_{C_{i_0}}(x_3)$.

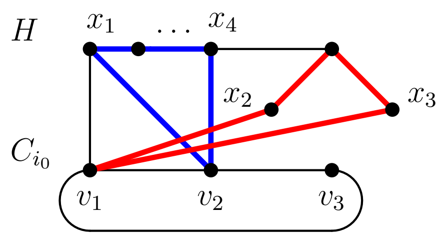


Figure 2.10. The case when $v_2 \in N_{C_{i_0}}(x_4)$.

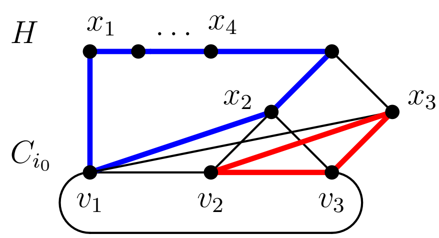


Figure 2.11. An example with $j_0 = 1, m = 2, m' = 3$.

Case 2. $|C_{i_0}| = 4$.

Let $C_{i_0} = v_1, v_2, v_3, v_4, v_1$. By Lemma 2.A, $d_{C_{i_0}}(x) \leq 2$ for each $x \in X$. Since $d_{C_{i_0}}(X) \geq 8$, $d_{C_{i_0}}(x) = 2$ for each $x \in X$. No vertex in X has consecutive neighbors in C_{i_0}, otherwise, we can immediately find a 3-cycle, contradicting (1). Thus for each $x \in X$, either $N_{C_{i_0}}(x) = \{v_1, v_3\}$ or $N_{C_{i_0}}(x) = \{v_2, v_4\}$.

Subcase 1. All four vertices in X have the same two neighbors in C_{i_0}.

We may assume that $N_{C_{i_0}}(X) = \{v_1, v_3\}$. Then $H[x_1, x_4]$, v_1, x_1 and $H[x_2, x_3]$, v_3, x_2 are two disjoint cycles.

Subcase 2. Three vertices in X have the same two neighbors in C_{i_0}.

Suppose that x_1, x_4 have the same two neighbors in C_{i_0}. Then we may assume that

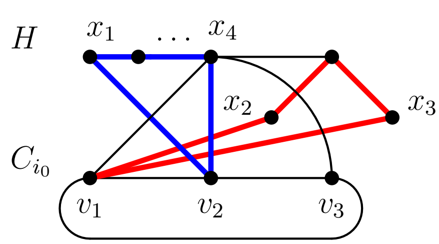


Figure 2.12. An example with $h_0 = 2$ and $h_1 = 1$.

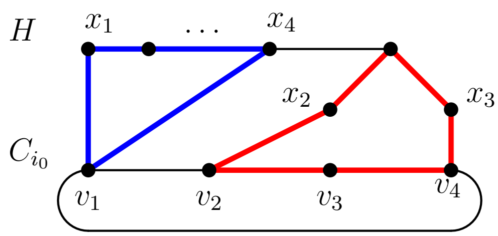


Figure 2.13. An example where $v_2 \in N_{C_{i_0}}(x_2)$ and $v_4 \in N_{C_{i_0}}(x_3)$.

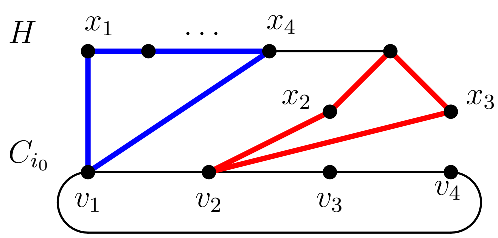


Figure 2.14. The case when x_1 and x_4 have the same neighbors in C_{i_0}.

$v_1 \in N_{C_{i_0}}(x_1) \cap N_{C_{i_0}}(x_4)$, and $H[x_1, x_4]$, v_1, x_1 is a cycle. Since $d_{C_{i_0}}(x_j) = 2$ for each $j \in \{2, 3\}$, $N_{C_{i_0} - v_1}(x_j) \neq \emptyset$. Then $\langle H[x_2, x_3] \cup (C_{i_0} - v_1)\rangle$ contains the other disjoint cycle (see Figure 2.13). Suppose that x_1, x_4 do not have the same two neighbors in C_{i_0}. Since x_2, x_3 have the same two neighbors in C_{i_0}, we repeat the above arguments, replacing x_1, x_4 with x_2, x_3.

Subcase 3. Two vertices of X have the same two neighbors in C_{i_0}, and the other two vertices of X have the same two neighbors, different from the neighbors of the first two.

Suppose that x_1, x_4 have the same two neighbors. We may assume that $v_1 \in N_{C_{i_0}}(x_1) \cap N_{C_{i_0}}(x_4)$. Then $H[x_1, x_4]$, v_1, x_1 is a cycle. Since x_2, x_3 have the same two neighbors, different from the neighbors of x_1 and x_4, $H[x_2, x_3]$, v_2, x_2 is the other disjoint cycle (see Figure 2.14). Suppose that x_1, x_4 have different neighbors. We may assume that $v_1 \in N_{C_{i_0}}(x_1)$ and $v_2 \in N_{C_{i_0}}(x_4)$. Then $H[x_1, x_4]$, v_2, v_1, x_1 is a cycle. Since x_2, x_3 have the neighbors, different from v_1, v_2, $\langle H[x_2, x_3] \cup \{v_3, v_4\}\rangle$ contains the other disjoint cycle. This completes the proof of Claim 2. □

Since H is a path by Claim 2, let $H = x_1, x_2, \ldots, x_t$ $(t \geq 8)$. Let $X =$

$\{x_1, x_3, x_5, x_t\}$. Then X is an independent set with $d_H(X) = 6$, and $d_{\mathscr{C}}(X) \geq (8k-3)-6 = 8k-9 \geq 7(k-1)$, since $k \geq 2$. Thus either $d_{C_0}(X) \geq 8$ for some cycle C_0 in $\mathscr{C}$, or $d_C(X) = 7$ for every cycle C in $\mathscr{C}$. If $d_C(X) \geq 8$ for some cycle C in $\mathscr{C}$, then we have the minimum possible degree sequences (3,3,2,0), (3,3,1,1), (3,2,2,1) or (2,2,2,2) from X to C. If $d_C(X) = 7$ for some cycle C in $\mathscr{C}$, then we have the minimum possible degree sequences (3,3,1,0), (3,2,1,1), (3,2,2,0) or (2,2,2,1) from X to C.

Claim 3. *If there exists a degree sequence at least* $(3,3,1,0)$ *from* X *to* C*, then there exist two disjoint cycles in* $\langle H \cup C \rangle$*.*

Proof. By Lemma 2.A, $|C| = 3$. Let $C = v_1, v_2, v_3, v_1$. We may assume that $d_C(x_{i_0}) = 1$ for some $i_0 \in \{1,3\}$, otherwise, $i_0 \in \{5,t\}$, and we may argue in a similar manner from the other end of the path H. Let $v_1 \in N_C(x_{i_0})$. First, suppose that $i_0 = 1$, that is, $d_C(x_1) = 1$. Then $d_C(x_{j_1}) = d_C(x_{j_2}) = 3$ for some j_1, $j_2 \in \{3,5,t\}$ with $j_1 < j_2$. Thus $H[x_1, x_{j_1}]$, v_1, x_1 and $x_{j_2}, v_3, v_2, x_{j_2}$ are two disjoint cycles. Next, suppose that $i_0 = 3$, that is, $d_C(x_3) = 1$. If $d_C(x_1) = 0$, then since $d_C(x_j) = 3$ for each $j \in \{5,t\}$, x_3, x_4, x_5, v_1, x_3 and x_t, v_3, v_2, x_t are two disjoint cycles. If $d_C(x_1) = 3$, then x_1, x_2, x_3, v_1, x_1 is a cycle, and since $d_C(x_{j_0}) = 3$ for some $j_0 \in \{5,t\}$, x_{j_0}, v_3, v_2, x_{j_0} is the other disjoint cycle. □

Claim 4. *If there exists a degree sequence at least* $(2,2,2,1)$ *from* X *to* C*, then there exist two disjoint cycles in* $\langle H \cup C \rangle$*.*

Proof. By Lemma 2.A, $|C| \leq 4$. Let $C = v_1, v_2, \ldots, v_q, v_1$, where $q = |C|$. We may assume that $d_C(x_{i_0}) = 1$ for some $i_0 \in \{5,t\}$, otherwise, $i_0 \in \{1,3\}$, and we may argue in a similar manner from the other end of the path H. Let $v_1 \in N_C(x_{i_0})$.

Case 1. $N_C(x_1) \cap N_C(x_3) \neq \emptyset$.

First, suppose that $v_{j_0} \in N_{C-v_1}(x_1) \cap N_{C-v_1}(x_3)$ for some $2 \leq j_0 \leq q$. Then x_1, x_2, x_3, v_{j_0}, x_1 is a cycle. Since $d_C(x_r) = 2$ for $r \in \{5,t\} - \{i_0\}$, $N_{C-v_{j_0}}(x_r) \neq \emptyset$.

Then $\langle H[x_5, x_t] \cup (C - v_{j_0})\rangle$ contains the other disjoint cycle. Next, suppose that $v_1 \in N_C(x_1) \cap N_C(x_3)$. Then x_1, x_2, x_3, v_1, x_1 is a cycle. Since $d_C(x_r) = 2$ for $r \in \{5, t\} - \{i_0\}$, if $v_1 \notin N_C(x_r)$, then $\langle x_r \cup (C - v_1)\rangle$ contains the other disjoint cycle. Thus we may assume that $v_1 \in N_C(x_r)$. Then $H[x_5, x_t]$, v_1, x_5 is a cycle. Since $d_C(x_i) = 2$ for each $i \in \{1, 3\}$, $N_{C-v_1}(x_i) \neq \emptyset$, and $\langle H[x_1, x_3] \cup (C - v_1)\rangle$ contains the other disjoint cycle.

Case 2. $N_C(x_1) \cap N_C(x_3) = \emptyset$.

In this case, if $|C| = 3$, then since $d_C(x_i) = 2$ for each $i \in \{1, 3\}$, $N_C(x_1) \cap N_C(x_3) \neq \emptyset$, contradicting our assumption. Thus $|C| = 4$, and either $N_C(x_1) = \{v_1, v_3\}$ and $N_C(x_3) = \{v_2, v_4\}$ or $N_C(x_1) = \{v_2, v_4\}$ and $N_C(x_3) = \{v_1, v_3\}$.

Suppose that $N_C(x_1) = \{v_1, v_3\}$ and $N_C(x_3) = \{v_2, v_4\}$. Suppose that $d_C(x_5) = 1$. Then $x_5v_1 \in E(G)$ by our earlier assumption, and $d_C(x_t) = 2$. If $x_tv_1 \in E(G)$, then $H[x_5, x_t]$, v_1, x_5 is a cycle, and x_3, v_4, v_3, v_2, x_3 is the other disjoint cycle. Thus $N_C(x_t) = \{v_2, v_4\}$. Then $H[x_3, x_t]$, v_4, x_3 and x_1, v_3, v_2, v_1, x_1 are two disjoint cycles. Suppose that $d_C(x_t) = 1$. Then we can find two disjoint cycles in $\langle H \cup C\rangle$ similar to the case where $d_C(x_5) = 1$.

Suppose that $N_C(x_1) = \{v_2, v_4\}$ and $N_C(x_3) = \{v_1, v_3\}$. Then x_1, v_4, v_3, v_2, x_1 is a cycle, and since $d_C(x_{i_0}) = 1$ for some $i_0 \in \{5, t\}$ and $x_{i_0}v_1 \in E(G)$, $H[x_3, x_{i_0}]$, v_1, x_3 is the other disjoint cycle. □

By Claims 3 and 4, if $d_C(X) \geq 8$ for some cycle C in $\mathscr{C}$, noting the minimum possible degree sequences, then $\langle H \cup C\rangle$ contains two disjoint cycles. Thus we may assume that $d_C(X) = 7$ for every cycle C in $\mathscr{C}$.

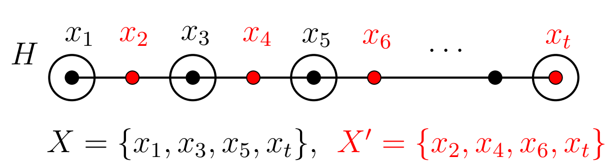
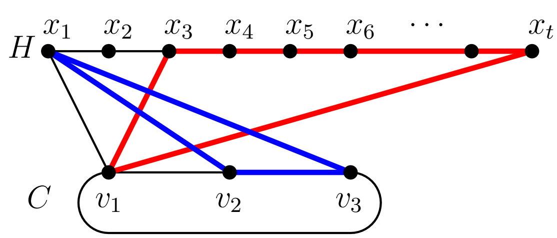


Figure 2.15. Sets X and X'.

Let $X' = \{x_2, x_4, x_6, x_t\}$ (see Figure 2.15). Then X' is an independent set with $d_H(X') = 7$, and $d_{\mathscr{C}}(X') \geq (8k-3) - 7 = 8k - 10 \geq 6(k-1)$, since $k \geq 2$. Thus we can choose some cycle C in $\mathscr{C}$ such that $d_C(X') \geq 6$. And we know that $d_C(X) = 7$, since X sends seven edges into every cycles in $\mathscr{C}$. Since $d_C(x_t) \leq 3$ by Lemma 2.A, note that $d_C(X' - \{x_t\}) \geq 6 - 3 = 3$. Now, we have only to consider degree sequences (3,2,1,1) and (3,2,2,0) from X to C by Claims 3 and 4. Since both degree sequences contain degree 3, $|C| = 3$ by Lemma 2.A. Let $C = v_1, v_2, v_3, v_1$.

Case 1. The sequence is (3,2,1,1).

Suppose that $d_C(x_1) = 3$. By the degree sequence of this case, and since $|C| = 3$, there are distinct integers $i_1, i_2 \in \{3, 5, t\}$ with $i_1 < i_2$ such that $N_C(x_{i_1}) \cap N_C(x_{i_2}) \neq \emptyset$. Without loss of generality, we may assume that $v_1 \in N_C(x_{i_1}) \cap N_C(x_{i_2})$. Then $H[x_{i_1}, x_{i_2}], v_1, x_{i_1}$ is a cycle. Since $d_C(x_1) = 3$, x_1, v_3, v_2, x_1 is the other disjoint cycle. If $d_C(x_t) = 3$, then we can find two disjoint cycles similar to the case where $d_C(x_1) = 3$. Thus we may assume that $d_C(x_{i_0}) = 3$ for some $i_0 \in \{3, 5\}$.



Suppose that $d_C(x_1) = 2$. Without loss of generality, we may assume that $v_1, v_2 \in N_C(x_1)$. First, suppose that $d_C(x_3) = 1$. Then $d_C(x_5) = 3$. If $x_3v_1 \in E(G)$, then x_1, x_2, x_3, v_1, x_1 and x_5, v_3, v_2, x_5 are two disjoint cycles. If $x_3v_2 \in E(G)$, then we can find two disjoint cycles similar to the case where $x_3v_1 \in E(G)$, replacing v_1 with v_2. If $x_3v_3 \in E(G)$, then x_3, x_4, x_5, v_3, x_3 and x_1, v_2, v_1, x_1 are two disjoint cycles. Next, suppose that $d_C(x_3) = 3$. If $x_5v_3 \in E(G)$, then x_3, x_4, x_5, v_3, x_3 and x_1, v_2, v_1, x_1 are two disjoint cycles. Thus $x_5v_{j_0} \in E(G)$ for some $j_0 \in \{1, 2\}$. If $j_0 = 1$, that is, $x_5v_1 \in E(G)$, then x_3, v_3, v_2, x_3 is a 3-cycle, and $\langle (H - x_3) \cup v_1 \rangle$ is connected and not a path. Thus we can find two disjoint cycles in $\langle H \cup C \rangle$ as in the proof of Claim 2. Similarly, we can prove the case where $j_0 = 2$.

If $d_C(x_t) = 2$, then we can find two disjoint cycles similar to the case where $d_C(x_1) = 2$. Thus we may assume that $d_C(x_{m_0}) = 2$ for some $m_0 \in \{3, 5\}$.

Then $d_C(x_i) = 1$ for each $i \in \{1, t\}$. Let $x_1v_1 \in E(G)$. Then we may assume that $d_C(x_3) = 2$ and $d_C(x_5) = 3$, otherwise, $d_C(x_3) = 3$ and $d_C(x_5) = 2$, and we may argue in a similar manner from the other end of the path H. If $x_3v_1 \in E(G)$, then $H[x_1, x_3]$, v_1, x_1 and x_5, v_3, v_2, x_5 are two disjoint cycles (see Figure 2.16). Thus $x_3v_i \in E(G)$ for each $i \in \{2, 3\}$. If $x_tv_1 \in E(G)$, then $H[x_5, x_t]$, v_1, x_5 and x_3, v_3, v_2, x_3 are two disjoint cycles. If $x_tv_2 \in E(G)$, then $H[x_5, x_t]$, v_2, x_5 and $H[x_1, x_3]$, v_3, v_1, x_1 are two disjoint cycles. If $x_tv_3 \in E(G)$, then $H[x_5, x_t]$, v_3, x_5 and $H[x_1, x_3]$, v_2, v_1, x_1 are two disjoint cycles.

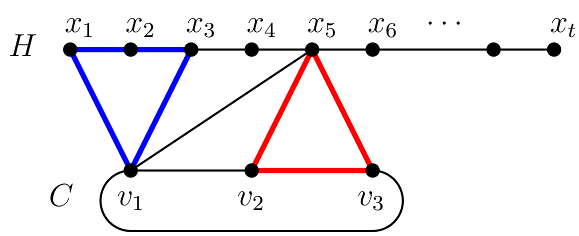


Figure 2.16. Two disjoint cycles when $x_3v_1 \in E(G)$.

Case 2. The sequence is (3,2,2,0).

We may assume that $d_C(x_{i_0}) = 0$ for some $i_0 \in \{1, 3\}$, otherwise, $i_0 \in \{5, t\}$, and we may argue in a similar manner from the other end of the path H. Let $j_0 \in \{1, 3\} - \{i_0\}$. Then $d_C(x_{j_0}) \geq 2$. Without loss of generality, we may assume that $v_1, v_2 \in N_C(x_{j_0})$.

Suppose that $d_C(x_5) = 2$. If $d_C(x_{j_0}) = 2$, then $N_C(x_{j_0}) \cap N_C(x_5) \neq \emptyset$; say $v \in N_C(x_{j_0})$, and $H[x_{j_0}, x_5]$, v, x_{j_0} is a cycle. Since $d_C(x_t) = 3$, $\langle x_t \cup (C - v) \rangle$ contains the other disjoint cycle. If $d_C(x_{j_0}) = 3$, then $d_C(x_j) = 2$ for each $j \in \{5, t\}$. Since $N_C(x_5) \cap N_C(x_t) \neq \emptyset$, say $v \in N_C(x_5) \cap N_C(x_t)$, $H[x_5, x_t]$, v, x_5 is a cycle. Since

$d_C(x_{j_0}) = 3$, $\langle x_{j_0} \cup (C - v)\rangle$ contains the other disjoint cycle.

Suppose that $d_C(x_5) = 3$. If $|N_C(x_{j_0}) \cap N_C(x_t)| = 1$, then let $v \in N_C(x_{j_0}) - N_C(x_t)$. Then $H[x_{j_0}, x_5]$, v, x_{j_0} is a cycle, and $\langle x_t \cup (C - v)\rangle$ contains the other cycle (see Figure 2.17). Thus x_{j_0}, x_t have all the same neighbors in C, say v_1, v_2. Recall that $d_C(X') \geq 6$. It follows that $d_C(X' - \{x_t\}) \geq 4$ and $d_C(X' - \{x_t\} - \{x_5\}) = d_C(\{x_4, x_6\}) \geq 1$. Suppose that $N_C(x_6) \neq \emptyset$. If $N_C(x_6) \cap N_C(x_t) \neq \emptyset$, say $v \in N_C(x_6) \cap N_C(x_t)$, then $H[x_6, x_t]$, v, x_6 is a cycle, and $\langle x_5 \cup (C - v)\rangle$ contains the other disjoint cycle. If $N_C(x_6) \cap N_C(x_t) = \emptyset$, then $x_6 v_3 \in E(G)$. Thus x_5, x_6, v_3, x_5 and x_t, v_2, v_1, x_t are two disjoint cycles.

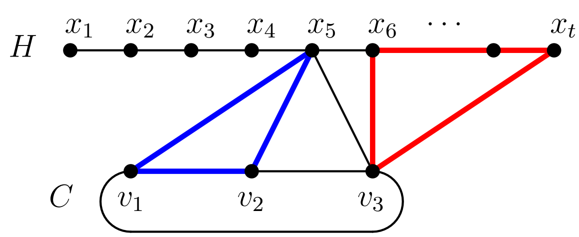


Figure 2.17. Two disjoint cycles. Example when $v = v_3$.

Suppose that $N_C(x_4) \neq \emptyset$. Then replacing x_6 in the above argument with x_4 and x_t with x_1, we can prove this case by the same arguments above. Thus $N_C(x_i) = \emptyset$ for each $i \in \{4, 6\}$. This implies that $d_C(x_2) = 3$. Then x_{j_0}, x_2, v_1, x_{j_0} and x_5, v_3, v_2, x_5 are two disjoint cycles. $\square$

2.4 Proofs of Lemmas

2.4.1 Proof of Lemma 2.1

Let $F, C, x_i\,(1 \leq i \leq 4)$ be as in Lemma 2.1. Let F_1, F_2 be two components of F, $C = v_1$, v_2, v_3, v_1, and $X = \{x_1, x_2, x_3, x_4\}$. Now, we consider two cases.

Case 1. At most two vertices of X lie in the same component of F.

Since $d_C(X) \geq 9$, $d_C(x_{i_0}) \geq 3$ for some $1 \leq i_0 \leq 4$. By $|C| = 3$, $d_C(x_i) \leq 3$

for each $1 \leq i \leq 4$. Thus $d_C(x_{i_0}) = 3$. Without loss of generality, we may assume that $i_0 = 1$, that is, $d_C(x_1) = 3$. Then $d_C(\{x_2, x_3, x_4\}) \geq 6$. Also, we may assume that $d_C(x_2) \geq d_C(x_3) \geq d_C(x_4)$. Now, we claim that $d_C(\{x_2, x_3\}) \geq 4$. Otherwise, if $d_C(\{x_2, x_3\}) \leq 3$, then $d_C(x_{j_0}) \leq 1$ for some $j_0 \in \{2, 3\}$. That implies that $d_C(x_4) \leq 1$, since $d_C(x_4)$ is the smallest degree in $\{x_2, x_3, x_4\}$. Then $d_C(\{x_2, x_3, x_4\}) \leq 3+1 = 4$, a contradiction. Thus the claim holds. Noting our assumption of this case, $\{x_1, x_2, x_3\}$ is a set of leaves from at least two components of F. Since $d_C(\{x_1, x_2, x_3\}) \geq 3+4 = 7$, Lemma 2.B applies, completing this case.

Case 2. Three vertices of X lie in the same component of F.

Without loss of generality, we may assume that x_1, x_2, $x_3 \in V(F_1)$, $x_4 \in V(F_2)$, and $d_C(x_1) \geq d_C(x_2) \geq d_C(x_3)$. Recall that $d_C(X) \geq 9$. It follows that the minimum possible degree sequence S from X to C is (3,3,3,0), (3,3,2,1) or (3,2,2,2).

Subcase 1. $S = (3, 3, 3, 0)$.

If $d_C(x_{i_0}) = 0$ for some $1 \leq i_0 \leq 3$, then $i_0 = 3$, that is, $d_C(x_3) = 0$. Now, we take $\{x_1, x_2, x_4\}$ that is a set of leaves from at least two components of F. Since $d_C(\{x_1, x_2, x_4\}) = 9$, Lemma 2.B applies. If $d_C(x_4) = 0$, then $d_C(x_i) = 3$ for each $1 \leq i \leq 3$. Since all the x_is are leaves, x_3 does not lie on the path in F_1 connecting x_1 and x_2. Then $F_1[x_1, x_2]$, v_1, x_1 and x_3, v_3, v_2, x_3 are two disjoint cycles in $\langle F \cup C \rangle$.

Subcase 2. $S = (3, 3, 2, 1)$.

Take $\{x_1, x_2, x_4\}$. If $d_C(x_4) \in \{1, 2\}$, then $d_C(\{x_1, x_2\}) \geq 6$. If $d_C(x_4) = 3$, then $d_C(\{x_1, x_2\}) \geq 5$. Since $d_C(\{x_1, x_2, x_4\}) \geq 7$ for all cases, Lemma 2.B applies.

Subcase 3. $S = (3, 2, 2, 2)$.

Take $\{x_1, x_2, x_4\}$. If $d_C(x_4) = 2$, then $d_C(\{x_1, x_2\}) \geq 5$. If $d_C(x_4) = 3$, then $d_C(\{x_1, x_2\}) \geq 4$. Since $d_C(\{x_1, x_2, x_4\}) \geq 7$ for all cases, Lemma 2.B applies. □

2.4.2 Proof of Lemma 2.5

Proof of (i). Let v_1, v_2, v_3, v_4, v_5 be the vertices such that $d_{C_1}(v_i) = 2$ for each $1 \leq i \leq 5$, appearing in this order on C_2. Let $w_1, w_2 \in N_{C_1}(v_1)$ appear in this order on C_1. The neighbors of v_1 partition C_1 into two intervals $C_1(w_1, w_2]$ and $C_1(w_2, w_1]$. We claim that each of v_2, v_3, v_4, v_5 has one neighbor in different interval of C_1.

First, suppose that $v_{i_1}, v_{i_2}, v_{i_3}$ for some $2 \leq i_1 < i_2 < i_3 \leq 5$ have both their neighbors in a common interval of C_1, say $C_1(w_1, w_2]$. We may assume that at least one of their neighbors is not w_2. Let $z_{i_1} \in N_{C_1(w_1,w_2)}(v_{i_1})$ and $z_{i_2} \in N_{C_1(w_1,w_2)}(v_{i_2})$. Then $C_1^{\pm}[z_{i_1}, z_{i_2}]$, $C_2^{-}[v_{i_2}, v_{i_1}]$, z_{i_1} and $C_1[w_2, w_1]$, v_1, w_2 form a shorter pair of disjoint cycles, since v_{i_3} is not used (see Figure 2.18).

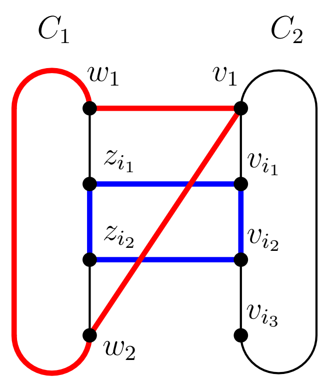


Figure 2.18. Shorter cycles in $\langle C_1 \cup C_2 \rangle$.

Next, suppose that v_{i_1}, v_{i_2} for some $2 \leq i_1 < i_2 \leq 5$ have both their neighbors in a common interval of C_1, say $C_1(w_1, w_2]$. Then we may assume that $i_1 = 2$ and $i_2 = 5$, otherwise, we can prove the other pairs of i_1 and i_2 by the same arguments above. Let $z_{i_1} \in N_{C_1(w_1,w_2)}(v_2)$ and $z_{i_2} \in N_{C_1(w_1,w_2)}(v_5)$. If $N_{C_1(w_1,w_2)}(v_{j_0}) \neq \emptyset$ for some $j_0 \in \{3, 4\}$, then there exist shorter two disjoint cycles. Thus $N_{C_1(w_1,w_2)}(v_j) = \emptyset$ for each $j \in \{3, 4\}$. Since $d_{C_1}(v_j) = 2$ for each $j \in \{3, 4\}$, $N_{C_1(w_2,w_1]}(v_j) \neq \emptyset$. Let $z_{i_3} \in N_{C_1(w_2,w_1]}(v_3)$ and $z_{i_4} \in N_{C_1(w_2,w_1]}(v_4)$. Then $C_1^{\pm}[z_{i_3}, z_{i_4}]$, $C_2^{-}[v_4, v_3]$, z_{i_3} and $C_1^{\pm}[z_{i_1}, z_{i_2}]$, $C_2[v_5, v_2]$, z_{i_1} are shorter two disjoint cycles, since w_2 is not used (see Figure 2.19).

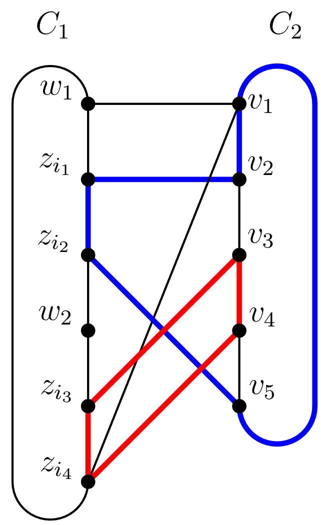


Figure 2.19. Shorter cycles in $\langle C_1 \cup C_2 \rangle$.

Finally, suppose that v_{i_0} for some $2 \leq i_0 \leq 5$ has both the neighbors in an interval of C_1, say $C_1(w_1, w_2]$. Then we have only to consider $i_0 = 2$ or $i_0 = 3$, otherwise, we take a cycle from v_1 in the opposite direction. First, suppose that $i_0 = 2$. Let $x_1, x_2 \in N_{C_1(w_1,w_2]}(v_2)$, appearing in this order on C_1. If $x_2 \neq w_2$, then $C_1[x_1, x_2]$, v_2, x_1 and $C_1[w_2, w_1]$, v_1, w_2 are shorter two disjoint cycles, since v_3 is not used. Thus $x_2 = w_2$. Let $y_1, y_2 \in N_{C_1}(v_3)$, appearing in this order on C_1. Suppose that $y_1 \in C_1(w_1, w_2)$. Then $C_1^{\pm}[x_1, y_1]$, $C_2^{-}[v_3, v_2]$, x_1 and $C_1[w_2, w_1]$, v_1, w_2 are shorter two disjoint cycles, since v_4 is not used. Thus $y_1 \notin C_1(w_1, w_2)$, that is, $y_1 \in C_1[w_2, w_1]$. Note that $y_2 \in C_1(w_2, w_1]$. If $y_1 \neq w_2$, then $C_1[x_1, w_2]$, v_2, x_1 and $C_1[y_1, y_2]$, v_3, y_1 are shorter two disjoint cycles, since v_1 is not used. Thus $y_1 = w_2$. If $y_2 \neq w_1$, then $C_1[w_2, y_2]$, v_3, w_2 and $C_1[w_1, x_1]$, $C_2^{-}[v_2, v_1]$, w_1 are shorter two disjoint cycles, since v_4 is not used. Thus $y_2 = w_1$. Let $z_1, z_2 \in N_{C_1}(v_4)$, appearing in this order on C_1. Suppose that $z_1 \in C_1[w_1, w_2)$. Then $C_1[w_1, z_1]$, $C_2^{-}[v_4, v_3]$, w_1 and $C_2[v_1, v_2]$, w_2, v_1 are shorter two disjoint cycles, since v_5 is not used. Suppose that $z_1 \in C_1[w_2, w_1)$. Then $C_1[w_1, x_1]$, $C_2^{-}[v_2, v_1]$, w_1 and $C_1[w_2, z_1]$, $C_2^{-}[v_4, v_3]$, w_2 are shorter two disjoint cycles, since v_5 is not used. Next, suppose that $i_0 = 3$. Then, by the same arguments as the case where $i_0 = 2$, we have shorter two disjoint cycles, replacing v_2 with v_3.

Thus each of v_2, v_3, v_4, v_5 has one neighbor in each interval of C_1. Let $x \in N_{C_1(w_1,w_2]}(v_2), y \in N_{C_1(w_1,w_2]}(v_3), z \in N_{C_1(w_2,w_1]}(v_4), u \in N_{C_1(w_2,w_1]}(v_5)$. Then $C_1^{\pm}[x,y]$, $C_2^{-}[v_3,v_2]$, x and $C_1^{\pm}[z,u]$, $C_2^{-}[v_5,v_4]$, z are shorter two disjoint cycles, since v_1 is not used. □

Proof of (ii). Let $v_1, v_2 \in V(C_2)$ such that $d_{C_1}(v_1) = 5$ and $d_{C_1}(v_2) = 3$, appearing in this order on C_2. Let w_1, w_2, w_3, w_4, $w_5 \in N_{C_1}(v_1)$, appearing in this order on C_1, and let u_1, u_2, $u_3 \in N_{C_1}(v_2)$, appearing in this order on C_1. The neighbors of v_1 partition C_1 into five intervals $C_1(w_i, w_{i+1}], 1 \leq i \leq 5 \pmod 5$. Suppose that u_{i_0}, $u_{j_0} \in C_1(w_{m_0}, w_{m_0+1}] \pmod 5$ for some $1 \leq i_0 < j_0 \leq 3$ and for some $1 \leq m_0 \leq 5$. Without loss of generality, we may assume that $i_0 = 1$, $j_0 = 2$ and $m_0 = 1$. Then $C_1[u_1,u_2]$, v_2, u_1 and $C_1[w_3,w_4]$, v_1, w_3 are shorter two disjoint cycles, since w_1 is not used. Thus neighbors of v_2 are contained in different intervals. Since C_1 is partitioned into five intervals, some two neighbors of v_2 lie in neighboring intervals, say $u_1 \in (w_1, w_2]$ and $u_2 \in C_1(w_2, w_3]$. Then $C_1[u_1,u_2]$, v_2, u_1 and $C_1[w_4,w_5]$, v_1, w_4 are shorter two disjoint cycles, since w_1 is not used. □

Proof of (iii). Let v_1, v_2, v_3, v_4, v_5, v_6 be the vertices on C_2 with the degree sequence (3,1,1,1,1,1), appearing in this order on C_2. Without loss of generality, we may assume that $d_{C_1}(v_1) = 3$ and $d_{C_1}(v_i) = 1$ for each $2 \leq i \leq 6$. Let $w_1, w_2, w_3 \in N_{C_1}(v_1)$, appearing in this order on C_1. The neighbors of v_1 partition C_1 into three intervals: $C_1(w_1,w_2]$, $C_1(w_2,w_3]$, $C_1(w_3,w_1]$. Then there exist some integer $1 \leq i_0 \leq 3$ and distinct integers $2 \leq j_1 < j_2 \leq 5$ such that $N_{C_1(w_{i_0},w_{i_0+1}]}(v_{j_1}) \neq \emptyset$ and $N_{C_1(w_{i_0},w_{i_0+1}]}(v_{j_2}) \neq \emptyset$. Without loss of generality, we may assume that $i_0 = 1$. Let $u_1 \in N_{C_1(w_1,w_2]}(v_{j_1})$ and $u_2 \in N_{C_1(w_1,w_2]}(v_{j_2})$. Then $C_1^{\pm}[u_1,u_2]$, $C_2^{-}[v_{j_2},v_{j_1}]$, u_1 and $C_1[w_3,w_1]$, v_1, w_3 are shorter two disjoint cycles, since v_6 is not used. □

Proof of (iv). Let v_1, v_2, v_3, v_4 be the vertices on C_2 with the degree sequence

(3,2,1,1), say $d_{C_1}(v_1) = 3$, $d_{C_1}(v_2) = 2$ and $d_{C_1}(v_i) = 1$ for each $i \in \{3, 4\}$. Suppose that v_1, v_2 are in this order on C_2. Let w_1, w_2, $w_3 \in N_{C_1}(v_1)$ be in this order on C_1, and let u_1, $u_2 \in N_{C_1}(v_2)$ be in this order on C_1. Let v_3, v_4 be in this order on C_2. Let $z_1 \in N_{C_1}(v_3)$, and let $z_2 \in N_{C_1}(v_4)$. The neighbors of v_1 partition C_1 into three intervals: $C_1(w_1, w_2]$, $C_1(w_2, w_3]$, $C_1(w_3, w_1]$. If v_2 has both its neighbors in the same interval in C_1, then we can find shorter two disjoint cycles. If the neighbors of v_2 are into two different intervals of C_1 and neither is in $\{w_1, w_2, w_3\}$, then we can also find shorter two disjoint cycles. Thus the neighbors of v_2 are into two different intervals of C_1 and at least one of them is at an endpoint of these intervals. Without loss of generality, we may assume that $u_1 \in C_1(w_1, w_2]$ and $u_2 \in C_1(w_2, w_3]$. Now, we consider two cases.

Case 1. v_3, $v_4 \in C_2(v_1, v_2)$ or v_3, $v_4 \in C_2(v_2, v_1)$.

Without loss of generality, we may assume that v_3, $v_4 \in C_2(v_1, v_2)$. If $z_2 \in C_1(w_1, w_3)$, then $C_1^{\pm}[u_1, z_2]$, $C_2[v_4, v_2]$, u_1 and $C_1[w_3, w_1]$, v_1, w_3 are shorter two disjoint cycles, since v_3 is not used. If $z_2 \in C_1[w_3, w_1)$, then $C_1[u_2, z_2]$, $C_2[v_4, v_2]$, u_2 and $C_1[w_1, w_2]$, v_1, w_1 are shorter two disjoint cycles, since v_3 is not used. Thus $z_2 = w_1$.

If $u_2 \in C_1(w_2, w_3)$, then $C_1[u_1, u_2]$, v_2, u_1 and $C_2[w_3, w_1]$, v_1, w_3 are shorter two disjoint cycles, since v_3 is not used. Thus $u_2 = w_3$.

If $z_1 \in C_1(w_3, u_1)$, then $C_1^{\pm}[z_1, w_1]$, $C_2[v_1, v_3]$, z_1 and $C_1[u_1, w_3]$, v_2, u_1 are shorter two disjoint cycles, since v_4 is not used. Thus $z_1 \in C_1[u_1, w_3]$.

Suppose that $u_1 \in C_1(w_1, w_2)$. If $z_1 \in C_1[u_1, w_2)$, then $C_1[w_1, z_1]$, $C_2[v_3, v_4]$, w_1 and $C_1[w_2, w_3]$, v_1, w_2 are shorter two disjoint cycles, since v_2 is not used. If $z_1 = w_2$, then $C_2[v_1, v_3]$, w_2, v_1 and $C_1[w_1, u_1]$, $C_2^-[v_2, v_4]$, w_1 are shorter two disjoint cycles, since w_3 is not used. If $z_1 \in C_1(w_2, w_3]$, then $C_1[z_1, w_3]$, $C_2[v_1, v_3]$, z_1 and $C_1[w_1, u_1]$, $C_2^-[v_2, v_4]$, w_1 are shorter two disjoint cycles, since w_2 is not used. Thus $u_1 = w_2$.

Now, we consider two disjoint cycles $C' = w_1$, $C_2[v_1, v_4]$, w_1 and $C'' = C_1[w_2, w_3]$, v_2, w_2. Note that $|C_2| \geq 6$. If $C_2(v_4, v_2) \neq \emptyset$ or $C_2(v_2, v_1) \neq \emptyset$, then C' and C''

are shorter two disjoint cycles. Thus $C_2(v_4, v_2) = \emptyset$ and $C_2(v_2, v_1) = \emptyset$. First, suppose that $z_1 \in C_1[w_2, w_3)$. If $C_2(v_1, v_3) \neq \emptyset$, then $C_1[w_3, w_1]$, v_1, w_3 and $C_2[v_3, v_2]$, $C_1[w_2, z_1]$, v_3 are shorter two disjoint cycles. If $C_2(v_3, v_4) \neq \emptyset$, then $C_1[w_2, z_1]$, $C_2^-[v_3, v_1]$, w_2 and $C_1[w_3, w_1]$, $C_2[v_4, v_2]$, w_3 are shorter two disjoint cycles. Next, suppose that $z_1 = w_3$. If $C_2(v_1, v_3) \neq \emptyset$, then $C_1[w_1, w_2]$, v_1, w_1 and $C_2[v_3, v_2]$, w_3, v_3 are shorter two disjoint cycles. If $C_2(v_3, v_4) \neq \emptyset$, then $C_2[v_1, v_3]$, w_3, v_1 and $C_1[w_1, w_2]$, $C_2^-[v_2, v_4]$, w_1 are shorter two disjoint cycles.

Case 2. $v_3 \in C_2(v_1, v_2)$ and $v_4 \in C_2(v_2, v_1)$.

If $z_1 \in C_1(w_1, w_3)$, then $C_1^{\pm}[u_1, z_1]$, $C_2[v_3, v_2]$, u_1 and $C_1[w_3, w_1]$, v_1, w_3 are shorter two disjoint cycles, since v_4 is not used. If $z_1 \in C_1[w_3, w_1)$, then $C_1[u_2, z_1]$, $C_2[v_3, v_2]$, u_2 and $C_1[w_1, w_2]$, v_1, w_1 are shorter two disjoint cycles, since v_4 is not used. Thus $z_1 = w_1$. Then $C_2[v_1, v_3]$, w_1, v_1 and $C_1[u_1, u_2]$, v_2, u_1 are shorter two disjoint cycles, since v_4 is not used. □

Proof of (v). Let v_1, v_2, v_3 be the vertices on C_2 with the degree sequence (3,3,1). Suppose that v_1, v_2, v_3 exist in this order on C_2. Without loss of generality, we may assume that $d_{C_1}(v_i) = 3$ each $i \in \{1, 2\}$ and $d_{C_1}(v_3) = 1$. Suppose that w_1, w_2, $w_3 \in N_{C_1}(v_1)$ exist in this order on C_1. Let $W = \{w_1, w_2, w_3\}$. These neighbors of v_1 partition C_1 into three intervals: $C_1(w_1, w_2]$, $C_1(w_2, w_3]$, $C_1(w_3, w_1]$. Let u_1, u_2, $u_3 \in N_{C_1}(v_2)$, and suppose that u_1, u_2, u_3 are in this order on C_1.

Case 1. Some two neighbors of v_2 are in the same interval of C_1.

Without loss of generality, we may assume that u_1, $u_2 \in C_1(w_1, w_2]$. Then $C_1[u_1, u_2]$, v_2, u_1 and $C_1[w_3, w_1]$, v_1, w_3 are shorter two disjoint cycles, since v_3 is not used.

Case 2. No two neighbors of v_2 are in the same interval of C_1.

Then $u_1 \in C_1(w_1, w_2]$, $u_2 \in C_1(w_2, w_3]$, and $u_3 \in C_1(w_3, w_1]$. First, suppose that

$u_{i_0}, u_{j_0} \notin W$ for some $1 \leq i_0 < j_0 \leq 3$. Without loss of generality, we may assume that $i_0 = 1$ and $j_0 = 2$, that is, $u_1 \in C_1(w_1, w_2)$ and $u_2 \in C_1(w_2, w_3)$. Then $C_1[u_1, u_2]$, v_2, u_1 and $C_1[w_3, w_1]$, v_1, w_3 are shorter two disjoint cycles, since v_3 is not used.

Next, suppose that $u_{i_0} \notin W$ for only some $1 \leq i_0 \leq 3$. Without loss of generality, we may assume that $i_0 = 1$, that is, $u_1 \in C_1(w_1, w_2)$. Then note that $u_3 = w_1$, $C_1[w_1, u_1]$, v_2, w_1 and $C_1[w_2, w_3]$, v_1, w_2 are shorter two disjoint cycles, since v_3 is not used.

Finally, suppose that $u_i = w_{i+1}$ (mod 3) for each $1 \leq i \leq 3$. Without loss of generality, we may assume that $v_3 z_1 \in E(G)$ for $z_1 \in (w_2, w_3]$. Now, we have two choices for constructing shorter two disjoint cycles. We may construct $C_1[w_1, w_2]$, v_2, w_1 and $C_1[z_1, w_3]$, $C_2^-[v_1, v_3]$, z_1, or $C_1[w_1, w_2]$, v_1, w_1 and $C_1[z_1, w_3]$, $C_2[v_2, v_3]$, z_1. Since $|C_2| \geq 6$, one of these two choices must leave out a vertex of C_2, and hence we may form shorter two disjoint cycles. □

2.4.3 Proof of Lemma 2.6

Let $C = v_1, v_2, v_3, v_1$.

Case 1. The sequence is (3,3,2,0).

Suppose that $d_C(x_1) = 0$. Then $d_C(y_{i_0}) = 3$ for some $i_0 \in \{1, i, t\}$, and we may assume that $i_0 = 1$, that is, $d_C(y_1) = 3$. Since $d_C(y_r) \geq 2$ for each $r \in \{i, t\}$ and $|C| = 3$, $v_{m_0} \in N_C(y_i) \cap N_C(y_t)$ for some $1 \leq m_0 \leq 3$. Without loss of generality, we may assume that $m_0 = 1$. Then $H_2[y_i, y_t]$, v_1, y_i and y_1, v_3, v_2, y_1 are two disjoint cycles.

Suppose that $d_C(x_1) = 2$. Without loss of generality, we may assume that $v_1, v_2 \in N_C(x_1)$. Then x_1, v_2, v_1, x_1 is a cycle. Since $d_C(y_{i_0}) = d_C(y_{j_0}) = 3$ for some $i_0, j_0 \in \{1, i, t\}$ with $i_0 < j_0$ and $|C| = 3$, $v_3 \in N_C(y_{i_0}) \cap N_C(y_{j_0})$. Then $H_2[y_{i_0}, y_{j_0}]$, v_3, y_{i_0} is the other disjoint cycle.

Suppose that $d_C(x_1) = 3$. Since $d_C(y_{i_0}) \geq 2$ and $d_C(y_{j_0}) \geq 2$ for some $i_0, j_0 \in \{1, i, t\}$ with $i_0 < j_0$ and $|C| = 3$, $v_{m_0} \in N_C(y_{i_0}) \cap N_C(y_{j_0})$ for some $1 \leq m_0 \leq 3$. Without loss of generality, we may assume that $m_0 = 1$. Then $H_2[y_{i_0}, y_{j_0}]$, v_1, y_{i_0} and x_1, v_3, v_2, x_1 are two disjoint cycles.

Case 2. The sequence is (3,3,1,1).

Suppose that $d_C(x_1) = 1$. Then $d_C(y_{i_0}) = 3$ for some $i_0 \in \{1, i, t\}$, and we may assume that $i_0 = 1$, that is, $d_C(y_1) = 3$. Since one of y_i and y_t has degree 3 to C and the other one of them has degree 1 to C, noting that $|C| = 3$, $v_{m_0} \in N_C(y_i) \cap N_C(y_t)$ for some $1 \leq m_0 \leq 3$. Without loss of generality, we may assume that $m_0 = 1$. Then $H_2[y_i, y_t]$, v_1, y_i and y_1, v_3, v_2, y_1 are two disjoint cycles.

Suppose that $d_C(x_1) = 3$. Since one of y_1, y_i, y_t has degree 3 to C and the others of them have degree 1 to C, $d_C(y_{i_0}) = 3$ and $d_C(y_{j_0}) = 1$ for some distinct $i_0, j_0 \in \{1, i, t\}$. Then note that either $i_0 < j_0$ or $i_0 > j_0$. Since $|C| = 3$, $v_{m_0} \in N_C(y_{i_0}) \cap N_C(y_{j_0})$ for some $1 \leq m_0 \leq 3$. Without loss of generality, we may assume that $m_0 = 1$. Then $H_2^{\pm}[y_{i_0}, y_{j_0}]$, v_1, y_{i_0} and x_1, v_3, v_2, x_1 are two disjoint cycles. □

Chapter 3

Degree Conditions to Imply the Existence of Vertex-Disjoint Chorded Cycles

In this chapter, we extend our work on vertex-disjoint cycles to vertex-disjoint chorded cycles. In particular, we consider the existence of a large set of vertex-disjoint chorded cycles in a graph. Let G be a graph such that $|G| \geq 11k + 7$ and $\sigma_4(G) \geq 12k - 3$ for integer $k \geq 2$. We prove that such a graph contains a set of k vertex-disjoint cycles. We also conjecture a generalized result for $\sigma_t(G)$. And we show that the degree sums in the result on $\sigma_4(G)$ and the conjecture for $\sigma_t(G)$ are sharp.

3.1 Introduction

An extension of the study of disjoint cycles is that of disjoint chorded cycles. A chord of a cycle is an edge between two vertices of the cycle that is not an edge of the cycle. We say a cycle is *chorded* if it induces at least once chord and *doubly chorded* if it induces at least two chords. As noted in the introduction, interest in ensuring a chorded cycle as a subgraph dates back to 1960, when Pósa first asked what conditions would imply the existence of a chorded cycle in a graph. In 1963, Czipzer (see Lovász [12], problem 10.2) provided an answer to the question by proving that if a graph has minimum degree at least 3, it must contain a chorded cycle. In the years since, results have focused on guaranteeing the existence of a set of k disjoint chorded cycles. Finkel [5] proved a Corrádi-Hajnal type result for chorded cycles, showing that if $|V(G)| \geq 4k$ and $\delta(G) \geq 3k$, then G contains k vertex-disjoint chorded cycles. Chiba et al. [1] extended this result, proving that for a graph G of order at least $3r + 4s$, if $\sigma_2(G) \geq 4r + 6s - 1$, then G contains $r + s$ vertex-disjoint cycles, with s of them chorded. The following corollary is a direct consequence of this theorem of Chiba et al. [1]:

Corollary 1. *Suppose that $|G| \geq 4k$ and $\sigma_2(G) \geq 6k - 1$. Then G contains k vertex-disjoint chorded cycles.*

Both Corollary 1 and Finkel's result are sharp as evidenced by the graph $G_0 = K_{3k-1,\ n-3k+1}$. For this graph, $\delta(G_0) = 3k - 1, \sigma_2(G_0) = 6k - 2$ and $\sigma_t(G_0) = 3kt - t$. But G_0 cannot contain k vertex-disjoint chorded cycles, as any chorded cycle must contain 3 vertices from the $3k - 1$ partite set. Hence, in general, at least $\sigma_t(G) \geq 3kt - t + 1$ is necessary to imply G contains k vertex-disjoint chorded cycles. This pattern uncovered in the sharpness example for Corollary 1 and Finkel's result motivated Conjecture 3.1.

Conjecture 3.1 ([9]). *Let G be a graph of sufficiently large order. If $\sigma_t(G) \geq 3kt - t + 1$ for any two integers $k \geq 2$ and $t \geq 1$, then G contains k vertex-disjoint chorded cycles.*

Note that the conjectured degree sum condition would be sharp by the same example. The purpose of this chapter is to further extend the known results on chorded cycles and to add to the evidence for Conjecture 3.1 by proving the case when $t = 4$. We show the following:

Theorem 3.1 ([9]). *If G is a graph of order $n \geq 11k + 7$ and if $\sigma_4(G) \geq 12k - 3$, then G contains k vertex-disjoint chorded cycles.*

It follows from the graph G_0 described above that Theorem 3.1 is sharp with respect to the degree sum condition $\sigma_4(G) \geq 12k - 3$.

The proof of Theorem 3.1 in Section 3.3 proceeds by contradiction using an edge-maximal counterexample. An edge-maximal counterexample G does not contain k chorded cycles, but if any edge is added, the resulting graph does contain k chorded cycles. Thus, G must contain a set $\mathscr{C}$ of $k - 1$ vertex-disjoint chorded cycles. We let $H = G \setminus \bigcup_{i=1}^{k-1} V(C_i)$; that is, H is what is left in G after the chorded cycles are removed. We first prove that the order of H must be large enough. Then we show that H must contain a large connected component, and in this connected component, we find a set X of four independent vertices having small degree in H. Finally, we use the σ_4 condition to find many edges between the set X and some cycle C in the set $\mathscr{C}$. We get a contradiction by constructing two vertex-disjoint chorded cycles in $\langle H \cup C \rangle$.

3.2 Preliminaries

In the proof of Theorem 3.1, we make use of the following Lemmas, as well as Theorem 3.2 due to Czipzer (Lovász [12], problem 10.2), and Theorem 3.3, a direct consequence of Chiba et al. [1].

Theorem 3.2. (Czipzer (see [12], problem 10.2)) *Suppose $|G| \geq 4$ and $\delta(G) \geq 3$. Then G contains a chorded cycle.*

Theorem 3.3. (Chiba, Fujita, Gao, Li [1]) *Suppose that $|G| \geq 4k$ and $\sigma_2(G) \geq 6k-1$. Then G contains k vertex-disjoint chorded cycles.*

Lemma 3.1. *Let $\mathscr{C} = \{C_1, C_2, \ldots, C_r\}$ be a minimal set of r vertex-disjoint cycles in a graph G. For any i, $1 \leq i \leq r$, the cycle C_i cannot have two parallel chords.*

Proof. This follows easily from the minimality of $\mathscr{C}$. □

Lemma 3.2. *Let $\mathscr{C} = \{C_1, C_2, \ldots, C_r\}$ be a minimal set of r vertex-disjoint cycles in a graph G. If $|C_i| \geq 7$ for some $1 \leq i \leq r$, then C_i has at most two chords. Furthermore, if it has two chords, these chords must be crossing.*

Proof. Suppose C_i contains at least three chords. By Lemma 3.1, no two of them can be parallel. Thus they are all mutually crossing. Label the endpoints of three of these chords $v_1, v_2, \ldots v_6$ in that order. Because the chords are mutually crossing, the three chords are given by v_1v_4, v_2v_5, v_3v_6. These six endpoints partition the vertex set of C_i into six path segments: $C_i[v_1, v_2)$, $C_i[v_2, v_3)$, $\ldots$, $C_i[v_6, v_1)$. Since $|C_i| \geq 7$, some segment contains at least one vertex of C_i which is not an endpoint of one of the three chords. Without loss of generality, say $C_i[v_1, v_2)$ contains some vertex of C_i other than v_1. Then, v_2, $C_i[v_5, v_1]$, $C_i^-[v_4, v_2]$ is a smaller chorded cycle. (See Figure 3.1.) Thus, C_i contains at most two chords, and by Lemma 3.1 they must cross. □

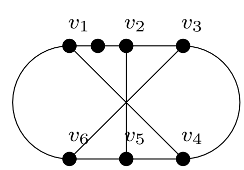


Figure 3.1. A smaller chorded cycle.

Lemma 3.3. *Let r be a positive integer and $\mathscr{C} = \{C_1, \ldots, C_r\}$ be a set of r minimal vertex-disjoint chorded cycles of a graph G such that the number of K_4s in $\mathscr{C}$ is maximal. And suppose G does not contain $r+1$ vertex-disjoint chorded cycles. Then, $d_{C_i}(x) \leq 4$ for any $x \in V(G) - \cup_{j=1}^{r} V(C_j)$ and any $i, 1 \leq i \leq r$. Furthermore, if $C \in \mathscr{C}$ and $x \in V(G) - \cup_{j=1}^{r} V(C_j)$ such that $d_C(x) = 4$, then $C = K_4$ and if $d_C(x) = 3$, then $|C| \leq 5$ or C is a type 2 chorded six-cycle (see Definition 1).*

Proof. Suppose we have a chorded cycle C and a vertex $x \in V(G) - \cup_{j=1}^{r} V(C_j)$ such that $d_C(x) \geq 4$.

Claim 5. *If $d_C(x) \geq 4$, then cycle C is a 4-cycle, and hence also $d_C(x) = 4$.*

Proof. Suppose to the contrary $|C| \geq 5$. Consider four neighbors of x on C, say $\{v_1, v_2, v_3, v_4\} = X \subseteq N_C(x)$, in that order. These neighbors define five intervals $C[v_i, v_{i+1})$ on C, where $i = 1, \ldots 4$, and for $i = 4, i+1 = 1$. Since $|C| \geq 5$, by the Pigeonhole Principle, a vertex of $C - X$ lies in one of the intervals $C[v_i, v_{i+1})$. Without loss of generality, say there is a vertex of $C - X$ in $C[v_1, v_2)$. Then $\langle C[v_2, v_4] \cup x \rangle$ induces a shorter chorded cycle in $\langle C \cup x \rangle$, contradicting the minimality of $\mathscr{C}$. Thus, $d_C(x) \geq 4$ implies $|C| = 4$, which in turn implies $d_C(x) = 4$. Hence, for any $x \in V(G) - \cup_{j=1}^{r} V(C_j)$ and for any $i, 1 \leq i \leq r$, we know that $d_C(x) \leq 4$. □

Claim 6. *If $|C| = 4$, then $C = K_4$.*

Proof. Suppose $C \neq K_4$. Then, $C = K_4 - e$. Label the vertices of C with v_1, v_2, v_3, v_4, in that order, such that the chord is given by v_1v_3. Then, $\langle \{v_1, v_2, v_3\} \cup x \rangle = K_4$. This contradicts the fact that the number of K_4s in $\mathscr{C}$ was maximal. □

Now suppose $d_C(x) = 3$.

Claim 7. *Either $|C| \leq 5$ or C is a type 2 chorded six-cycle.*

Proof. Let $X = \{v_1, v_2, v_3\}$ be neighbors of x in C in that order on the cycle. If $|C| \geq 7$, then some interval defined by two consecutive neighbors of x contains at least two vertices of $C - X$. Without loss of generality, say $C[v_1, v_2)$ contains at least two vertices of $C - X$. Then $\langle C[v_2, v_1] \cup x \rangle$ induces a smaller chorded cycle, contradicting the minimality of $\mathscr{C}$. Thus, $|C| < 7$.

Suppose C is a type 1 chorded six- cycle. Label the vertices of C with x_1, x_2, $\ldots$, x_6 in order such that the three-cycle is given by x_1, x_2, x_3, x_1 and the five-cycle is given by x_1, x_3, x_4, x_5, x_6, x_1.

If x has two neighbors in the three-cycle, then $\langle C[x_1, x_3] \cup x \rangle$ contains a chorded four-cycle. On the other hand, if x is adjacent to all three of the vertices outside of the three-cycle, that is, x_4, x_5, x_6, we get a chorded four-cycle from $\langle C[x_4, x_6] \cup x \rangle$. Thus, x must be adjacent to one vertex in the three-cycle and two vertices outside the three-cycle. Let x be adjacent to one of $\{x_1, x_2, x_3\}$ and any two of $\{x_4, x_5, x_6\}$.

If x is adjacent to x_1, then $\langle x \cup x_1 \cup C[x_4, x_6] \rangle$ contains a chorded five-cycle if x is adjacent to x_4, or contains a chorded four-cycle if x is not adjacent to x_4. A similar argument applies if x is adjacent to x_3. Suppose x is adjacent to x_2. Then, if x is adjacent to x_4, $\langle x \cup C[x_1, x_4] \rangle$ induces a chorded five-cycle x_1, x_3, x_4, x, x_2, x_1 with edge x_2x_3 as a chord. Otherwise, if x is not adjacent to x_4, it must be adjacent to x_6, and $\langle x \cup C[x_1, x_3] \cup x_6 \rangle$ induces a chorded five-cycle x_1, x_3, x_2, x, x_6, x_1 with edge x_1x_2 as a chord. In all cases we can find a smaller chorded cycle, contradicting the minimality of $\mathscr{C}$. Hence, if $d_C(x) = 3$, the cycle C cannot be a type 1 chorded six-cycle. And since $|C| < 7$, it follows that either C is a type 2 chorded six-cycle, or $|C| \leq 5$. Thus, the claim holds. □

This completes the proof of Lemma 3.3. □

Lemma 3.4. *Suppose we have three edges either all mutually parallel or all mutually crossing, connecting two paths, P_1, P_2. Then there is a chorded cycle in $\langle P_1 \cup P_2 \rangle$*

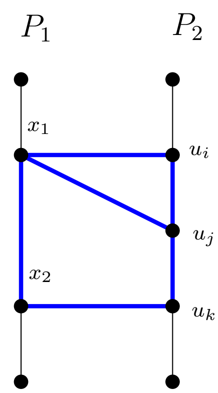
Proof. Say the edges are x_1y_1, x_2y_2, x_3y_3. Without loss of generality, let x_1, x_2, and x_3 appear in that order in P_1. If the edges are mutually crossing, the endpoints y_1, y_2, y_3 must appear in the order y_3, y_2, y_1 on P_2. Else, the edges are all mutually parallel, and the endpoints y_1, y_2, y_3 must appear in that order in P_2. In either case, $P_1[x_1, x_3]$, y_3, $P_2^{\pm}(y_3, y_1]$, x_1 is a chorded cycle with x_2y_2 as a chord. □

Lemma 3.5. *Suppose we have at least five edges connecting two paths P_1 and P_2. Then we can form a chorded cycle in $\langle P_1 \cup P_2 \rangle$ which leaves out at least one vertex from P_1 or P_2.*

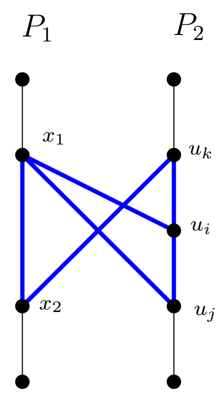
Proof. Any two edges between P_1 and P_2 are either parallel or crossing. Since there are five edges between P_1 and P_2, by the Pigeonhole Principle there must be either three mutually parallel edges or three mutually crossing edges. Then, by Lemma 3.4, we can form a chorded cycle in $\langle P_1 \cup P_2 \rangle$. Suppose this chorded cycle uses every vertex of P_1 and P_2. Then the cycle has at least three chords, and by Lemma 3.2, a shorter chorded cycle exists in $\langle P_1 \cup P_2 \rangle$. □

Lemma 3.6. *Let x_1, x_2 be two vertices on a path P_1, each having degree two to another path P_2. Then we can form a chorded cycle in $\langle P_1[x_1, x_2] \cup P_2 \rangle$.*

Proof. Let u_i, u_j, $i < j$, be x_1's neighbors on $P_2 = u_1, \ldots, u_s$. If x_2 has a neighbor that lies in $P_2[u_j, u_s]$ or $P_2[u_1, u_i]$, then we can easily form a chorded cycle in $\langle P_1[x_1, x_2] \cup P_2 \rangle$. (See Figure 3.2.)



(a) Note that it is possible $u_j = u_k$.



(b) Note that it is possible $u_k = u_i$.

Figure 3.2. A chorded cycle in $\langle P_1[x_1, x_2] \cup P_2 \rangle$.

Thus, both of x_2's neighbors in P_2 must lie in $P_2(u_i, u_j)$, call them u_k, u_l wtih $k < l$. So the neighbors of x_1 and x_2 lie in the order u_i, u_k, u_l, u_j on P_2. (See Figure 3.3.) Then, $P_1[x_1, x_2]$, u_k, $P_2(u_k, u_j]$, x_1 forms a chorded cycle, with chord x_2u_l. □

Lemma 3.7. *Let x_1, x_2, x_3 be three vertices which lie either in order x_1, x_2, x_3 or in order x_3, x_2, x_1 on a path P_1, with x_1 having degree two and x_2, x_3 each having degree 1 to another path P_2. Then we can form a chorded cycle in $\langle P_1[x_1, x_3] \cup P_2 \rangle$.*

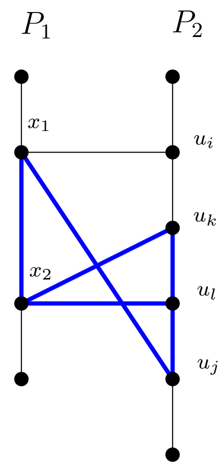


Figure 3.3. A chorded cycle in $\langle P_1[x_1, x_2] \cup P_2 \rangle$.

Proof. We may assume x_1, x_2, x_3 lie in that order, else we can reverse the order of the path. Let w_1, w_2 be x_1's neighbors in P_2. As in the previous lemma, if either x_2 or x_3 has a neighbor that lies beyond w_2 or prior to w_1 in P_2, then we can easily form a chorded cycle in $\langle P_1 \cup P_2 \rangle$. Thus, the neighbor of each of x_2, x_3 lies in $P_2(w_1, w_2)$. Call x_2's neighbor w_3 and x_3's neighbor w_4. If w_3 appears before w_4 in $P_2(w_1, w_2)$, then we have three parallel edges between P_1 and P_2, one from each of the w_i's. Else, w_3 appears in $P_2(w_4, w_2)$, and we have three mutually crossing edges between P_1 and P_2, one from each of the w_i's. In either case, a chorded cycle exists by Lemma 3.4. □

Lemma 3.8. *Let H be a graph containing a path P. If there exist nested edges between vertices of P in $E(G) - E(P)$, then H contains a chorded cycle.*

Proof. The proof is obvious. (See Figure 5.) □

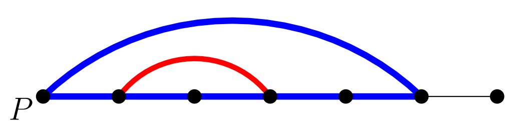


Figure 3.4. Nested edges in a path.

Lemma 3.9. *Let H be a graph containing a path $P = v_1, v_2, \cdots, v_n$ and v_i, v_{i+1} be* *neighboring vertices on the path. If v_i has a right edge v_iv_j and v_{i+1} has a left edge $v_{i+1}v_k$ then H contains a chorded cycle.*

Proof. Clearly, $P[v_k, v_i]$, v_j, $P^-(v_j, v_{i+1}]$, v_k is a cycle with edge v_iv_{i+1} as a chord. □

Lemma 3.10. *Let H be a graph containing a path $P = v_1, v_2, \cdots, v_n$ and v_i, v_{i+1} be neighboring vertices on the path. Then v_i and v_{i+1} cannot both have degree at least 4 to P.*

Proof. Suppose $d_P(v_i) \geq 4$ and $d_P(v_{i+1}) \geq 4$. Then v_i has two neighbors in $P[v_1, v_{i-2}] \cup P[v_{i+2}, v_n]$, and v_{i+1} has two neighbors in $P[v_1, v_{i-1}] \cup P[v_{i+3}, v_n]$. If v_i has a neighbor in $P[v_{i+2}, v_n]$ and v_{i+1} has a neighbor in $P[v_1, v_{i-1}]$, then H contains a chorded cycle by Lemma 3.9. Thus, either v_i must have two neighbors in $P[v_1, v_{i-2}]$ or v_{i+1} has two neighbors in $P[v_{i+3}, v_n]$. In either case, nested edges exist and H contains a chorded cycle by Lemma 3.8. □

Lemma 3.11. *Let H be a graph containing a path $P_1 = v_1, \ldots, v_t$, $t \geq 12$, and not containing a chorded cycle. If $v_iv_t \in E(H)$ for any $i \leq t-2$, then $d_{P_1}(v_k) \leq 3$ for any $k > i$ and $d_{P_1}(v_{i+1}) = 2$. And if $v_1v_j \in E(H)$ for any $j \geq 3$, then $d_{P_1}(v_l) \leq 3$ for any $l < j$ and $d_{P_1}(v_{j-1}) = 2$.*

Proof. Suppose $v_iv_t \in E(H)$ for some $i \leq t-2$. No vertex v_k with $k > i$ has a right edge, otherwise that edge nests with v_iv_t, and by Lemma 3.8, H contains a chorded cycle, a contradiction. Thus, $d_{P_1}(v_k) \leq 3$ for any $k > i$. Furthermore, vertex v_{i+1} cannot have a left edge by Lemma 3.9. Thus, $d_{P_1}(v_{i+1}) = 2$

By symmetry, the same proof shows that if $v_1v_j \in E(H)$ for some $j \geq 3$, then $d_{P_1}(v_l) \leq 3$ for any $l < j$ and $d_{P_1}(v_{j-1}) = 2$. □

Lemma 3.12. *Let H be a graph containing a path $P_1 = v_1, \ldots, v_t$, $t \geq 12$, and not containing a chorded cycle. If $d_{P_1}(v_1) = 1$, then one of v_3, v_4, v_5 has degree two in $\langle P_1 \rangle$. Or if $v_1v_3 \in E(H)$, then one of v_4, v_5, v_6 has degree two in $\langle P_1 \rangle$.*

Proof. Let either $v_1v_3 \in E(H)$ or $d_{P_1}(v_1) = 1$. If $v_1v_3 \in E(H)$, we let $i = 4$, and if $d_{P_1}(v_1) = 1$, we let $i = 3$. Vertex v_i cannot have a left edge, else in the first case we get a chorded cycle, and in the second case we have $d_{P_1}(v_1) = 2$; hence, we have a contradiction in either case. If vertex v_i has degree 2 in P_1, we are done. Thus v_i must have a right edge, say v_iv_j. If $j = i + 2$, then vertex v_{i+1} cannot have a left edge or a right edge and must have degree 2, else we get a chorded cycle. Thus, $j > i + 2$. By Lemma 3.9, v_{i+1} cannot have a left edge. If v_{i+1} has degree 2 we are done. Thus, v_{i+1} has a right edge, say $v_{i+1}v_k$. If $k \leq j$, then we have nested edges and a chorded cycle by Lemma 3.8, a contradiction. Thus, $k > j$. By the same argument as for v_{i+1}, vertex v_{i+2} either has degree 2, or has a right edge $v_{i+1}v_l$ such that $l > k$. In the later case, edges v_iv_j, $v_{i+1}v_k$, $v_{i+2}v_l$ are three parallel edges between the subpaths v_i, v_{i+1}, v_{i+2} and $v_j, \ldots, v_l$, and hence a chorded cycle exists by Lemma 3.4, a contradiction. Thus, vertex v_{i+2} must have degree 2 in P_1, and we are done. □

Lemma 3.13. *Let H be a graph containing a path $P_1 = v_1, \ldots, v_t$, $t \geq 12$ and not containing a chorded cycle. If $d_{P_1}(v_t) = 1$, then one of v_{t-4}, v_{t-3}, v_{t-2} has degree two in $\langle P_1 \rangle$. Or if $v_tv_{t-2} \in E(H)$, then one of v_{t-5}, v_{t-4}, v_{t-3} has degree two in $\langle P_1 \rangle$.*

Proof. The lemma follows from the proof of Lemma 3.12 by symmetry. □

Lemma 3.14. *Let $H = \langle P_1 \cup P_2 \rangle$, where $P_1 = v_1, \ldots, v_t$, $P_2 = u_1, \ldots, u_s$, such that H does not contain a chorded cycle. If a vertex $v_i \in P_1$ is adjacent to an endpoint of P_2 and a vertex $v_j \in P_1$ with $j \geq i + 2$ is adjacent to an endpoint of P_2, then one of v_{i+1}, v_{j-1} has degree 2 in $\langle P_1 \cup P_2 \rangle$.*

Proof. Let $H = \langle P_1 \cup P_2 \rangle$ such that H does not contain a chorded cycle. Let vertex $v_i \in P_1$ be adjacent to an endpoint of P_2, without loss of generality say u_1, and let vertex $v_j \in P_1$ be adjacent an endpoint of P_2, for some $j \geq i + 2$, without loss of generality say u_t. (If instead v_i, v_j are both adjacent to u_1 or u_t, in the cycles following replace u_1, $P_2(u_1, u_t]$ and u_t, $P_2^-(u_t, u_1]$ with just u_1 or u_t as necessary.)

If vertex v_{i+1} has a left edge, say $v_{i+1}v_k$, with $k < i$, then $P_1[v_k, v_i]$, u_1, $P_2(u_1, u_t]$, v_j, $P_1^-(v_j, v_{i+1}]$, v_k forms a chorded cycle with edge v_iv_{i+1} as a chord. By symmetry, vertex v_{j-1} cannot have a right edge, else a chorded cycle exists with the edge $v_{j-1}v_j$ as a chord.

Thus, either v_{i+1} or v_{j-1} has degree 2 in $\langle P_1 \cup P_2 \rangle$ and we are done, or vertex v_{i+1} has a right edge, and vertex v_{j-1} has a left edge.

No vertex in $P_1[v_i, v_j]$ can have an edge that does not lie on P_1 to some other vertex in $P_1[v_i, v_j]$, else this edge is a chord of the cycle $P_1[v_i, v_j]$, u_t, $P_2^-(u_t, u_1]$, v_i.

Thus, we have edges $v_{i+1}v_k$, with $k > j$, and $v_{j-1}v_l$, with $l < i$. But then, $P_1[v_l, v_i]$, u_1, $P_2(u_1, u_s]$, v_j, $P_1(v_j, v_k]$, v_{i+1}, $P_1(v_{i+1},\ v_{j-1}]$, v_l forms a chorded cycle with edges v_iv_{i+1} and $v_{j-1}v_j$ as chords.

Thus, one of v_{i+1}, v_{j-1} has degree 2 in H, and hence is also independent from v_1, v_t, u_1, u_s. □

Lemma 3.15. *Let $H = \langle P_1 \cup P_2 \rangle$, where $P_1 = v_1, \ldots, v_t$, $P_2 = u_1, \ldots, u_s$, such that P_1, P_2 is a maximal pair of paths, with P_1 as long as possible. Suppose H does not contain a chorded cycle or a Hamiltonian path. Finally, suppose $d_{P_1}(\{u_1, u_s\}) \geq 1$. If v_1 has a neighbor v_i in $P_1[v_4, v_t]$, then $d_H(v_{i-1}) = 2$. If v_t has a neighbor v_j in $P_1[v_1, v_{t-3}]$, then $d_H(v_{j+1}) = 2$.*

Proof. Suppose v_1 is adjacent to a vertex in $P_1[v_4, v_t]$. If v_1 is adjacent to v_t, then H contains a Hamiltonian path, a contradiction. Thus, v_1 has a neighbor v_i in $P_1[v_4, v_{t-1}]$. Note that vertex v_{i-1} cannot be adjacent to any vertex in P_2, else either H contains a Hamiltonian path or there exists a maximal pair of paths P_1', P_2' such that $|P_1'| > |P_1|$, a contradiction. By Lemma 3.11, v_{i-1} has degree 2 in P_1. Hence, $d_H(v_{i-1}) = 2$.

By symmetry, a similar argument shows that if v_t has a neighbor v_j in $P_1[v_1, v_{t-3}]$, then $d_H(v_{j+1}) = 2$. □

3.3 Proof of Theorem 3.1

For convenience, we restate our main result.

Theorem 3.1. *Let $k \geq 2$ be a positive integer. If G is a graph of order $n \geq 11k + 7$ with $\sigma_4(G) \geq 12k - 3$, then G contains k vertex-disjoint chorded cycles.*

Proof of Theorem 3.1. Let G be an edge-maximal counterexample. That is, G fails to have k vertex-disjoint chorded cycles, but for any new edge e, $G + e$ does have k vertex-disjoint chorded cycles. This implies there exists a collection of $k - 1$ vertex-disjoint chorded cycles in G. Over all such collections, choose one, say $\mathscr{C}$, such that:

(1) $\mathscr{C}$ is minimal.

(2) Subject to (1), the number of components in $H = G - \cup_{i=1}^{k-1} V(C_i)$ is minimal.

(3) Subject to (1) and (2), the number of K_4s in $\mathscr{C}$ is maximal.

Claim 1. $|H| \geq 18$.

Proof. Suppose to the contrary that $|H| \leq 17$. First suppose $|V(C_i)| \leq 11$ for all i, $1 \leq i \leq k - 1$. Since by assumption $|G| \geq 11k + 7$, it follows that $|H| \geq (11k + 7) - 11(k - 1) = 18$, a contradiction. Thus, $|V(C_i)| \geq 12$ for some i.

Let C be a largest cycle in $\mathscr{C}$. By Lemma 3.2, $|C| \geq 12$ implies that C contains at most two chords and these chords must be crossing. Let $|C| = 4t + r$ where $t \geq 3$ and $0 \leq r \leq 3$.

Subclaim 1.1. *The cycle C contains t different sets $X_1, \ldots, X_t$ of four independent vertices each, such that $d_C(X_1 \cup X_2 \cup \cdots \cup X_t) \leq 8t + 4$*

Proof. Cycle C has at most two chords, and if it has two chords, they must be crossing. For any $4t$ vertices of C, their degree sum in C is at most $4t \times 2 + 4$, since C has at most 2 chords. Thus it only remains to show that C contains t sets of four independent vertices each.

Recall that $|C| = 4t + r \geq 4t$. Start anywhere on C and label the first $4t$ vertices of C with labels 1 through t in order, starting over again with 1 after using label t. If $r \geq 1$, label the remaining r vertices of C with the labels $t + 1, \ldots, t + r$. (See Figure 3.5.) The labeling above yields t sets of 4 vertices each, where all the vertices labeled with 1 are one set, all the vertices labeled with 2 are another set, and so on. Given this labeling, since $t \geq 3$, any vertex in C has a different label than the vertex that preceeds it on C and the vertex that succeeds it on C. Let $\tilde{C}$ be cycle C minus its chords, if it has any. Then, the vertices in each of the sets are independent in $\tilde{C}$. Thus, the only way vertices in the same set are dependent in C is if the endpoints of a chord of C were given the same label. Note that any vertex labeled i is distance at least 3 in $\tilde{C}$ from any other vertex labeled i. Thus, if a vertex and the neighbor preceeding it on C or the neighbor succeeding it on C have their labels swapped, the vertices in each of the classes are still independent in $\tilde{C}$.

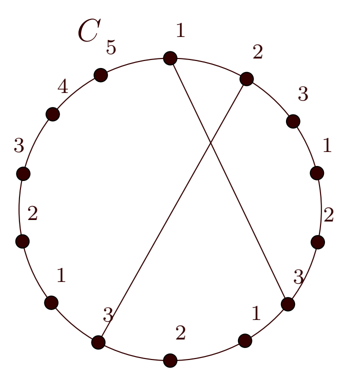


Figure 3.5. An example where $t = 3$ and $r = 2$.

Case 1.1.1. Suppose no chord of C has endpoints with the same label. Then, we have found t sets of 4 independent vertices in C, and we are done.

Case 1.1.2. Suppose one chord of C has endpoints with the same label. Because C contains at most two chords and those chords must be crossing, each chord has an endpoint with a neighbor that is not an endpoint of a chord. Pick such an endpoint of the chord whose endpoints were assigned the same label, and swap the label of this vertex with its non-endpoint neighbor. The vertices in each of the resulting classes are still independent in $\tilde{C}$, and now no chord of C has endpoints with the same label. Thus, we have found t sets of four independent vertices each in C.

Case 1.1.3. Suppose two chords of C each have endpoints with the same label.

Subcase 1. If an endpoint of one chord of C is adjacent to an endpoint of the other chord, swap the labels of these adjacent endpoints. Then, the vertices in each of the resulting classes are still independent in $\tilde{C}$, and now no chord of C has endpoints with the same label. Thus, we have found t sets of four independent vertices each in C.

Subcase 2. If no endpoint of the first chord in C is adjacent to an endpoint of the second chord, then swap the labels of an endpoint of the first chord, call it e_1 and one of its neighbors in $\tilde{C}$. The vertices in each of the resulting classes are still independent in $\tilde{C}$. Now pick an endpoint of the second chord that is not adjacent to a vertex that has had its label swapped, call it e_2. Then, pick a neighbor in $\tilde{C}$ of e_2 that is of maximal distance in $\tilde{C}$ from e_1. This neighbor is not adjacent to any vertex which has had its color swapped. Thus, we may swap the labels of e_2 and its selected neighbor, and the vertices in each of the resulting classes are still independent in $\tilde{C}$. Furthermore, now no chord of C has endpoints with the same label, and thus we have found our sets.

In all cases, we were able to construct t different sets of four independent vertices each in C. Thus, Subclaim 1.1 holds. □

Since $|C| \geq 12$, $d_C(v) \leq 2$ for any vertex $v \in V(H)$; otherwise, we could form a chorded cycle shorter than C in $\langle C \cup H \rangle$, contradicting (1). Because $|H| \leq 17$ and each vertex of H has at most two neighbors in C, it follows that $|E(H, C)| \leq 34$.

Each set of four independent vertices in C has at least $12k - 3$ edges in G, since $\sigma_4(G) \geq 12k - 3$. Thus, $X_1 \cup X_2 \cup \cdots \cup X_t$ has total degree at least $t(12k - 3)$ in G.

Suppose that $k = 2$. Then $\mathscr{C}$ has only one cycle C, and $H = G - C$. By Subclaim 1.1, C contains t independent sets $X_i, 1 \leq i \leq t$ each of which has four vertices and such that $d_C(X_1 \cup \cdots \cup X_t) \leq 8t + 4$. Then, $d_H(X_1 \cup \cdots \cup X_t) \geq t(12k - 3) - (8t + 4)$ $= 12kt - 11t - 4 \geq 24t - 11t - 4 = 13t - 4 \geq 35$, since $t \geq 3$. Thus, $|E(C, H)| \geq 35$, a contradiction.

Suppose that $k \geq 3$. We bound the order of $E(C, \mathscr{C} - C)$ from below.

$$|E(C, \mathscr{C} - C)| \geq |E(X_1 \cup \cdots \cup X_t, \mathscr{C} - C)|$$

Subtracting from $d_G(X_1 \cup \cdots \cup X_t)$ both $d_C(X_1 \cup \cdots \cup X_t)$ and $d_H(C)$, we get:

$$\begin{aligned} |E(X_1 \cup \cdots \cup X_t, \mathscr{C} - C)| &\geq t(12k - 3) - (8t + 4) - 34 \\ &= 12kt - 3t - 8t - 4 - 34 \\ &= 12kt - 11t - 38. \end{aligned}$$

And since $t \geq 3$,

$$\begin{aligned} 12kt - 11t - 38 &\geq 12kt - 12t - 35 = 12t(k - 1) - 35 \\ &> 12t(k - 1) - 12t = 12t(k - 2). \end{aligned}$$

Thus, $|E(C, C')| \geq |E(X_1 \cup \cdots \cup X_t, C')| \geq 12t$ for some cycle C' in $\mathscr{C} - C$, since $\mathscr{C} - C$ contains $k - 2$ cycles. Because $|C| = 4t + r \leq 4t + 3$, it follows that the average degree to C' of the vertices of $X_1 \cup \cdots \cup X_t$ is greater than 2; that is,

$$|E(X_1 \cup \cdots \cup X_t, C')|/|C| \geq \frac{12t}{4t+3} \geq \frac{3t}{t+1} > 2.$$

It follows that $d_{C'}(v) \geq 3$ for some vertex $v \in X_1 \cup \cdots \cup X_t$.

Let $h = \max\{d_{C'}(v) | v \in X_1 \cup \cdots \cup X_t\}$. Let v^* be a vertex of C such that $d_{C'}(v^*) = h$, and let v^{**} be a vertex of $C - v^*$ having maximal degree to C'. Certainly $d_{C'}(v^{**}) \leq h$. By the maximality of C, we know that $|C'| \leq |C| = 4t + r$. It follows that $h = d_{C'}(v^*) \leq |C'| \leq 4t + r$. Recall that $t \geq 3$ and $r \leq 3$.

$$\begin{aligned} \text{Then, } |E((X_1 \cup \cdots \cup X_t) - v^*, C')| &\geq 12t - d_{C'}(v^*) \\ &\geq 12t - (4t + r) = 8t - r \geq 21. \end{aligned} \tag{3.4}$$

$$\begin{aligned} \text{Futher, } |E((X_1 \cup \cdots \cup X_t) - v^* - v^{**}, C')| &\geq 12t - d_{C'}(v^*) - d_{C'}(v^{**}) \\ &\geq 12t - (4t + r) - (4t + r) = 4t - 2r \geq 6. \end{aligned} \tag{3.5}$$

Case 1.1. Suppose that $h = 3$.

Then because we have $4t$ vertices in $X_1 \cup \cdots \cup X_t$ sending a sum of at least $12t$ edges to C', it follows that every vertex of $X_1 \cup \cdots \cup X_t$ sends 3 edges to C'. Thus, there are at least 12 vertices in C having degree 3 to C'.

Let $W = \{w_1, w_2, \ldots, w_{12}\}$ be a set of 12 vertices of C, each having degree 3 to C'. Let v_1, v_2, v_3 be w_1's neighbors in C'. They partition C' into three intervals: $C'[v_1, v_2)$, $C'[v_2, v_3)$, $C'[v_3, v_1)$. Denote $W - \{w_1\}$ by W'.

Claim 1.1.1. No three vertices in W' all have three neighbors to the same single interval in C'.

Proof. Suppose three different vertices in W', say w_i, w_j, w_l, $2 \leq i < j < k \leq 12$, all have three neighbors to the same single interval in C', without loss of generality say $C'[v_1, v_2)$. Then each of w_i, w_j, w_l has at least two neighbors in $C(v_1, v_2)$. So there exist 6 edges between $C[w_i, w_l]$ and $C'(v_1, v_2)$. By Lemma 3.5, a chorded cycle exists in $\langle C[w_i, w_l] \cup C'(v_1, v_2)\rangle$ that leaves out at least one vertex. And $\langle w_1 \cup C'[v_2, v_1]\rangle$ forms a second chorded cycle, vertex-disjoint from the first. Thus, we have constructed a shorter pair of vertex-disjoint chorded cycles in $\langle C \cup C'\rangle$, contradicting (1). Thus, the claim holds. □

Claim 1.1.2. No vertex $w_i, 2 \leq i \leq 12$ has three or more neighbors in a single interval of C'.

Proof. Suppose w_i has three neighbors in a single interval of C', without loss of generality say $C'[v_1, v_2)$. Then by Lemma 3.4, a chorded cycle exists in $\langle w_i \cup C'[v_1, v_2)\rangle$. By Claim 1.1.1, at most one other vertex in $\{w_2, \ldots, w_{12}\}$, call it w_j, has at least three neighbors in $C'[v_1, v_2)$. Thus, every vertex in $\{w_2, \ldots, w_{12}\} - \{w_i, w_j\}$ has edges into $C'[v_2, v_1)$. And therefore, by Lemma 3.5, there exists a chorded cycle in $\langle C - w_i, C'[v_2, v_1)\rangle$ which leaves out at least one vertex. Together with the chorded cycle in $\langle w_i \cup C'[v_1, v_2)\rangle$, we have a shorter pair of vertex-disjoint chorded cycles in $\langle C \cup C'\rangle$, contradicting (1). Thus, the claim holds. □

Thus, every vertex in $W - w_1$ sends edges into at least 2 intervals.

Note that the set of vertices $\{w_7, \ldots, w_{12}\}$ sends 18 edges to C'. It follows that some interval in C' gets at least 6 edges from $\{w_7, \ldots, w_{12}\}$, say $C'[v_1, v_2)$. Then there exists a chorded cycle in $\langle C[w_7, w_{12}] \cup C'[v_1, v_2)\rangle$ which leaves out at least one vertex, by Lemma 3.5. Also, because every vertex sends edges to at least 2 intervals, each of $w_2, \ldots, w_5$ has an edge into $C'[v_2, v_1)$. This implies that $|E(C[w_1, w_5], C'[v_2, v_1))| \geq 6$. Hence by Lemma 3.5, there exists a chorded cycle in $\langle C[w_1, w_5] \cup C'[v_2, v_1)\rangle$. Thus, we have formed a shorter pair of vertex-disjoint chorded cycles, contradicting (1). This completes Case 1.1.

Case 1.2. Suppose that $h \geq 4$.

Recall that $|E((X_1 \cup \cdots \cup X_t) - v^*, C')| \geq 21$ and $|E((X_1 \cup \cdots \cup X_t) - v^* - v^{**}, C')| \geq 6$, by (3.4) and (3.5). Thus, $N_{C'}(C - v^* - v^{**}) \neq \emptyset$, and letting $W = \{v \in V(C) | N_{C'}(v) \neq \emptyset\}$, it follows that $|W| \geq 3$; that is, at least three vertices in C have neighbors in C'.

Subcase 1. Suppose that $|W| = 3$. Let $W = \{w_1, w_2, w_3\}$ where $d_{C'}(w_1) \geq d_{C'}(w_2) \geq d_{C'}(w_3)$.

Then, $|E(\{w_2, w_3\}, C')| \geq 21$, and $|E(\{w_3\}, C')| \geq 6$. Since $d_{C'}(w_1) \geq d_{C'}(w_2) \geq d_{C'}(w_3)$, it follows that $d_{C'}(w) \geq 6$ for any $w \in W$. Since $|E(\{w_2, w_3\}, C')| \geq 21$ and $d_{C'}(w_2) \geq d_{C'}(w_3)$, it follows that $d_{C'}(w_2) \geq 11$. Thus, we have degree sequence at least $(11, 11, 6)$ from W to C'.

Let $v_1, v_2, \ldots, v_6$ denote w_3's neighbors in C', appearing in that order on C'. The neighbors of w_3 partition C' into six intervals, $C'[v_i, v_{i+1})$, for all $1 \leq i \leq 6$ (where $i + 1 = 1$ for $i = 6$). Because $\{w_1, w_2\}$ sends at least 22 edges total into C', some interval in C' receives at least 4 edges from $\{w_1, w_2\}$, without loss of generality say $C'[v_1, v_2)$. And either every interval receives at least one edge from $\{w_1, w_2\}$ or some interval receives at least five edges from $\{w_1, w_2\}$.

If every interval receives at least one edge, then taking the interval with at least 4 edges and a neighboring interval, some pair of neighboring intervals receives at least five edges total from $\{w_1, w_2\}$, without loss of generality say intervals $C'[v_1, v_2)$ and $C'[v_2, v_3)$. There exist five edges between $C[w_1, w_2]$ and $C'[v_1, v_3)$. Thus, by Lemma 3.5, there exists a chorded cycle in $\langle C[w_1, w_2] \cup C'[v_1, v_3) \rangle$ which leaves out at least one vertex of $\langle C[w_1, w_2] \cup C'[v_1, v_3) \rangle$. And $\langle w_3 \cup C'[v_3, v_5] \rangle$ forms a second chorded cycle in $\langle C \cup C' \rangle$, vertex-disjoint from the first, contradicting (1).

Thus, some interval in C' receives at least five edges from $\{w_1, w_2\}$, without loss of generality say $[v_1, v_2)$. By Lemma 3.5, there exists a chorded cycle in $\langle P_1 \cup C'[v_1, v_2) \rangle$ which leaves out at least one vertex of $\langle P_1 \cup C'[v_1, v_2) \rangle$. And $\langle w_3 \cup C'[v_3, v_5] \rangle$ forms a second chorded cycle in $\langle C \cup C' \rangle$, vertex-disjoint from the first, contradicting (1).

Subcase 2. Suppose that $|W| \geq 4$.

Recall that vertex v^* has at least four neighbors in C'. Let v_1, v_2, v_3, v_4 be neighbors of v^* in C'. Note that $v_1, \ldots, v_4$ partition C' into four intervals, $C'[v_i, v_{i+1})$ (where $i + 1 = 1$ for $i = 4$). By (4), there are at least 21 more edges into C' from $C - v^*$. By the Pigeonhole Principle, some interval $C'[v_i, v_{i+1})$ contains six of these additional edges. Without loss of generality, say this interval is $C'[v_4, v_1)$. Then by Lemma 3.5, $\langle C - v^* \cup C'[v_4, v_1) \rangle$ contains a chorded cycle which leaves out at least one vertex of $\langle C - v^* \cup C'[v_4, v_1) \rangle$. Note that $C_1 = v^*$, $C'[v_1, v_3]$, v^* forms a chorded cycle with the edge v^*v_2 as a chord, and it uses no vertices from $C'[v_4, v_1)$. Thus we have a pair of shorter vertex-disjoint chorded cycles in $\langle C \cup C' \rangle$, contradicting (1)

This completes the proof of Claim 1. Thus, $|H| \geq 18$. □

Claim 2. *Every component H_i of H that has a vertex x with $d_{H_i}(x) \leq 2$ either contains two independent vertices each with degree at most two in H_i, or contains a vertex with degree at most two in H_i that is not a cut-vertex.*

Proof. Suppose not. It follows that H_i fails to contain two independent vertices each with degree at most two in H_i. Furthermore, H_i contains a vertex v such that $d_{H_i}(v) \leq 2$ and v is a cut-vertex. Since v is a cut-vertex, $d_{H_i}(v) = 2$. Let a and b be the neighbors of v in H_i. Let H_i' be the component of $H_i - \{v\}$ containing a and H_i'' be the component of $H_i - \{v\}$ containing b. Either $d_{H_i}(a) \geq 3$ or $d_{H_i}(b) \geq 3$, otherwise a, b are two independent vertices in H_i such that their degree sum in H_i is at most 4. Say $d_{H_i}(b) \geq 3$. (See Figure 3.6.)

If $|H_i''| < 4$, then there exists a vertex v_2 in H_i with degree at most two in H_i independent from v, a contradiction. Thus, $|H_i''| \geq 4$. Then, Theorem 3.3 implies that $\sigma_2(H_i'') < 5$. This implies that there exist two vertices $x_1, x_2 \in H_i''$ such that $d_{H_i''}(\{x_1, x_2\}) \leq 4$. Thus, either each of x_1, x_2 has degree in H_i'' at most 2, or one of

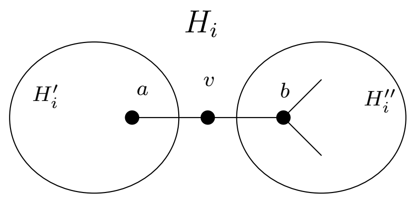


Figure 3.6. The case when $d_{H_i}(b) \geq 3$.

then has degree one in H_i''. Vertex b has degree at least 2 in H_i'', so it is possible that one of these two vertices is b, say $b = x_1$, but then the other vertex, x_2, would still have degree at most 2 in H_i''. Thus, there must be some vertex in H_i'', other than vertex b, having degree at most 2 in H_i''. But this vertex is independent from v, a contradiction. Thus, the claim holds. □

Claim 3. *H is either connected, or H has two components, one of which has order less than 4.*

Proof. Suppose not. Then H is disconnected, and if it has two components, both of them have order at least 4.

Subclaim 3.1. *H contains a set X of four independent vertices from at least two components of H such that $d_H(X) \leq 8$.*

Proof. The number of components of H, $comp(H)$, is at least 2. Label the components of H with $H_1, H_2, \ldots, H_{comp(H)}$. We will consider three cases: $comp(H) \geq 4$, $comp(H) = 3$, $comp(H) = 2$.

Case 3.1.1. Suppose $comp(H) \geq 4$.

Then, there exists $x_i \in H_i$ for $1 \leq i \leq 4$ such that $d_{H_i}(x_i) \leq 2$. Otherwise, by Theorem 3.2, H_i would contain a chorded cycle, yielding a contradiction. Then the set $X = \{x_1, x_2, x_3, x_4\}$ is a set of four independent vertices from four different components in H, and $d_H(X) \leq 8$.

Case 3.1.2. Suppose $comp(H) = 3$.

Then some component of H, say H_1, has order at least four, since $|H| \geq 18$. Then, there exist at least two independent vertices in H_1. Otherwise, any two vertices in H_1 are adjacent, and hence H_1 contains a K_4, contradicting the fact that H contains no chorded cycles. Thus, H_1 contains at least two independent vertices. It follows from Theorem 3.3 that there exist two independent vertices in H_1, call them x_1, x_4, such that $d_{H_1}(\{x_1, x_4\}) \leq 4$. Otherwise, $\sigma_2(H_1) \geq 5$, and H_1 contains a chorded cycle. As in Case 1, by Theorem 3.2 there must exist $x_2 \in H_2$ and $x_3 \in H_3$ such that $d_{H_2}(x_2) \leq 2$ and $d_{H_3}(x_3) \leq 2$. Then the set $X = \{x_1, x_2, x_3, x_4\}$ is a set of four independent vertices from two components of H with $d_H(X) \leq 8$.

Case 3.1.3. Suppose $comp(H) = 2$.

Since we supposed Claim 3 does not hold, by assumption $|H_1| \geq 4$ and $|H_2| \geq 4$. Then, as in component H_1 in Case 2, there must exist $x_1, x_2 \in H_1$ and $x_3, x_4 \in H_2$ such that x_1, x_2 and x_3, x_4 are independent and $d_{H_1}(\{x_1, x_2\}) \leq 4$, $d_{H_2}(\{x_3, x_4\}) \leq 4$. Otherwise, if one of the components of H does not contain any two independent vertices, it must contain a K_4, a contradiction; or if, for any two independent vertices in the component, their degree sum in the component is at least 5, then by Theorem 3.3, the component contains a chorded cycle, a contradiction. Thus, $X = \{x_1, x_2, x_3, x_4\}$ is a set of four independent vertices from two components of H with $d_H(X) \leq 8$.

Therefore, in all cases, Subclaim 3.1 holds. □

In the above construction of X, if $comp(H) = 2$, then exactly two vertices of X are from one component of H and exactly two are from the other component of H. Thus either $comp(H) \geq 3$, or no $x \in X$ is isolated from the rest of X. Also, according to the construction of X above, if any x_j in H_i is isolated from the rest of X, then we know $d_H(x_j) = d_{H_i}(x_j) \leq 2$. And if x_j is a cut-vertex, by Claim 2, there exists a second vertex x_t in H_i, not adjacent to x_j, with $d_{H_i}(x_t) \leq 2$. Thus, we can remove from X some other vertex x_l which was isolated from the rest of X and add x_t to X. Then $d_H(X) \leq 8$ still, and x_j is no longer isolated from the rest of X. Thus, without loss of generality, we may assume that if a vertex x is isolated from the rest of X, it is not a cut-vertex.

Since $d_H(X) \leq 8$, it follows that $d_{\mathscr{C}}(X) \geq 12k - 3 - 8 = 12k - 11 > 12(k-1)$. Thus, there is some cycle $C \in \mathscr{C}$ such that $d_C(X) \geq 13$. Note that if we have only two components, x_1 lies in the same component as some other x_i.

Also, by Lemma 3.3, for any $x_i \in X$, $d_C(x_i) \leq 4$. It follows that the possible degree sequences are: $(4,4,4,1)$, $(4,4,3,2)$, $(4,3,3,3)$. Hence, by Lemma 3.3, $C = K_4$, since in all cases there exists $x_i \in X$ such that $d_C(x_i) = 4$. Let $C = v_1, v_2, v_3, v_4, v_1$.

We consider two cases based on the number of components of H.

Case 3.1. Suppose $comp(H) = 2$.

Then each component of H contains two vertices of X. Let x_1, x_2 be in one component of H, call it H_1 and x_3, x_4 in the other, call it H_2.

Without loss of generality, let x_4 be the vertex of X with smallest degree to C. If we have degree sequence $(4,4,4,1)$ or $(4,4,3,2)$, it immediately follows that either x_1 or x_2 has degree 4 to C, say x_1. If instead we have degree sequence $(4,3,3,3)$, then we can label $x_1, \ldots, x_4$ so that x_1 has degree 4, x_1, x_2 are in one component of H, and x_3, x_4 are in the other.

Thus, we may assume without loss of generality that x_4 is the vertex of X with smallest degree to C and that x_1 has degree 4 to C. It follows that x_2, x_3 have degree at least 3 to C.

Let P_1 be a path in H_1 connecting x_1 and x_2, and let P_2 be a path in H_2 connecting x_3 and x_4.

Vertices x_3 and x_4 must share a neighbor in C, say v_1. Take a second neighbor of x_3 in C, say v_2. Then $v_1, v_2, x_3, P_2(x_3, x_4], v_1$ is a chorded cycle in $\langle H \cup C \rangle$ with x_3v_1 as a chord. Since x_2 has three neighbors in C, it is adjacent to at least one of the remaining vertices of C, say v_3. Vertex x_1 is adjacent to v_3 and v_4. Thus, x_2, v_3, v_4, x_1, $P_1(x_1, x_2]$, v_3 is a second chorded cycle in $\langle H \cup C\rangle$ with x_1v_3 as a chord, vertex-disjoint from the first. (See Figure 3.7.)

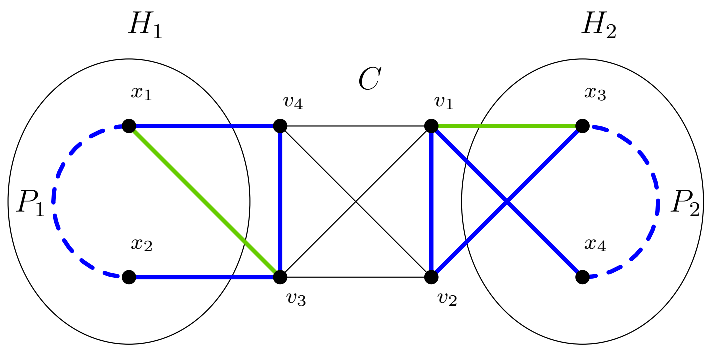


Figure 3.7. Two vertex-disjoint chorded cycles in $\langle H \cup C\rangle$.

Therefore, if $comp(H) = 2$, we get two vertex-disjoint chorded cycles in $\langle H \cup C\rangle$, a contradiction.

Case 3.2. Suppose $comp(H) \geq 3$.

Recall that we have one of the following degree sequences from X to C: $(4, 4, 4, 1)$, $(4, 4, 3, 2)$, $(4, 3, 3, 3)$. Label the vertices of X with $x_i, 1 \leq i \leq 4$ such that $d_C(x_1) \geq d_C(x_2) \geq d_C(x_3) \geq d_C(x_4)$.

Since $|C| = 4$, for each possible degree sequence, x_2, x_3, x_4 must all have a common neighbor in C, say v_1. And vertex x_1 has degree 4 to C. Thus, $C' = x_1$, v_2, v_3, v_4, x_1 is a chorded cycle in $\langle H \cup C\rangle$ with chord x_1v_3.

Recall that, by the construction of X in Subclaim 3.1, if $comp(H) = 2$, no vertex $x \in X$ is isolated from the rest of X. Hence, if x_1 is the only vertex of X in its component H_i of H, then $comp(H) \geq 3$, x_1 it is not a cut-vertex, and $comp(H_i - \{x_1\}) = 1$. Then, replacing C in $\mathscr{C}$ by C', the remaining H has fewer components, a contradiction.

Otherwise, some other vertex x_j of X is also in H_i. Since $d_{H_i}(x_1) \leq 2$, $comp(H_i -$

$\{x_1\}) \leq 2$. Further, the new H formed by replacing C in $\mathscr{C}$ with C' has fewer components, since one of the two components of $H_i - \{x_1\}$ contains x_j for some $2 \leq j \leq 4$, and x_2, x_3, x_4 are all connected in the new H. Again we have a contradiction. (See Figure 3.8.) Thus, in all cases the claim holds. □

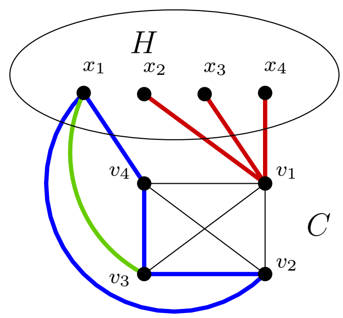


Figure 3.8. Fewer components in H.

Now by Claim 1, $|H| \geq 18$, and by Claim 3, H is either connected or has only two components, one of which has order at most 3. Thus, H is either connected or has a component H_i such that $|H_i| \geq 15$. Let $\tilde{H}$ be the largest component of H.

Claim 4. *$\tilde{H}$ contains a set X of four independent vertices such that $d_{\tilde{H}}(X) \leq 8$.*

Proof.

Subclaim 4.1. *If $\tilde{H}$ contains a Hamiltonian path, we can find the desired set X.*

Proof. Suppose $\tilde{H}$ contains a Hamiltonian path. Then $\tilde{H} = \langle P_1 \rangle$, where $P_1 = v_1, \ldots, v_t$, $t \geq 15$. Without loss of generality, let $d_{\tilde{H}}(v_1) \leq d_H(v_t)$, otherwise we relabel the path.

If $v_1 v_t \in E(\tilde{H})$, then every vertex of $\tilde{H}$ has degree two by Lemma 3.11. The set $X = \{v_1, v_3, v_5, v_7\}$ forms a set of four independent vertices with degree 8 in $\tilde{H}$, and we are done.

Thus, $v_1 v_t \notin E(\tilde{H})$. It follows that v_1 and v_t are independent. Also, $d_{\tilde{H}}(v_1) \leq 2$ and $d_{\tilde{H}}(v_t) \leq 2$ else a chorded cycle exists in $\tilde{H}$, a contradiction.

Suppose $d_{\tilde{H}}(v_1) = 1$ and $d_{\tilde{H}}(v_t) = 1$. By Lemma 3.12 one of v_3, v_4, v_5 has degree 2 in $\tilde{H}$, call it v_i, and one of v_{t-4}, v_{t-3}, v_{t-2} has degree 2 in $\tilde{H}$, call it v_j. Then, choose $X = \{v_1, v_i, v_j, v_t\}$, and we are done.

Suppose $d_{\tilde{H}}(v_1) = 1$ and $d_{\tilde{H}}(v_t) = 2$ with $v_t v_j \in E(\tilde{H})$. Suppose $j \leq t - 5$. Then vertices v_{j+1} and v_{j+3} are independent from v_t. By Lemma 3.11, vertex v_{j+1} has degree 2 in $\tilde{H}$, and vertex v_{j+3} has degree at most 3 in H. Choose $X = \{v_1, v_{j+1}, v_{j+3}, v_t\}$, and we are done.

So, $j > t - 5$. By Lemma 3.12, one of v_3, v_4, v_5 has degree 2 in $\tilde{H}$, say v_i. If $j \leq t-3$, then v_{j+1} is still independent from v_t and has degree 2 by Lemma 3.11. So, $X = \{v_1, v_i, v_{j+1}, v_t\}$ is the desired set. Thus, $j = t-2$. By Lemma 3.13, one of v_{t-5}, v_{t-4}, v_{t-3} has degree two in $\tilde{H}$, call it v_j. Since $t \geq 15$, v_i and v_j are independent, and $X = \{v_1, v_i, v_j, v_t\}$ is the desired set.

Thus, $d_{\tilde{H}}(v_1) = 2$ and $d_{\tilde{H}}(v_t) = 2$.

Suppose we have either $v_1 v_3$ or $v_t v_{t-2}$ in $E(\tilde{H})$. Without loss of generality, say $v_1 v_3$. Then, one of v_4, v_5, v_6 has degree 2 in $E(H)$ by Lemma 3.12, say v_i. If $v_t v_{t-2} \in E(\tilde{H})$, then one of $v_{t-5}, v_{t-4}, v_{t-3}$ has degree two in $\tilde{H}$, call it v_j, and $X = \{v_1, v_i, v_j, v_t\}$ is the desired set.

If $v_t v_{t-2} \notin E(\tilde{H})$, then $v_t v_s \in E(\tilde{H})$ for some $s < t-2$. Hence, vertex v_{s+1} has degree 2 by Lemma 3.11 and is independent from v_t. Clearly, $s \geq 3$, else we have a chorded cycle. If $v_{s+1} \notin \{v_{i-1}, v_i, v_{i+1}\}$, then $X = \{v_1, v_i, v_{s+1}, v_t\}$ is the desired set.

Thus, $v_{s+1} \in \{v_{i-1}, v_i, v_{i+1}\}$. This implies that $v_s \in \{v_{i-2}, v_{i-1}, v_i\}$. Clearly, $v_s \neq v_i$, since $v_s v_t \in E(\tilde{H})$, and vertex v_i has degree two in $E(\tilde{H})$. So, $v_s = v_{i-2}$ or $v_s = v_{i-1}$. Since $v_i \in \{v_4, v_5, v_6\}$ and $s \geq 3$, we know that $v_s \in \{v_3, v_4, v_5\}$. Then, if one of v_{s+4} or v_{s+5} has degree 2, $X = \{v_1, v_i, v_{s+4}, v_t\}$, or $X = \{v_1, v_i, v_{s+5}, v_t\}$, and we are done. Thus, both v_{s+4} or v_{s+5} have degree at least 3 in $\tilde{H}$. Furthermore, neither v_{s+4} nor v_{s+5} has a right edge, else this edge nests with $v_s v_t$, and we have a chorded cycle by Lemma 3.8. Thus, both v_{s+4} or v_{s+5} have left edges. It follows that

$v_{s+4}v_k, v_{s+5}v_l \in E(\tilde{H})$, and $k < l < s$ else we have nested edges and a chorded cycle by Lemma 3.8. But then, $v_k, P_1, v_s, v_t, P_1^-, v_{s+4}, v_k$ is a chorded cycle with edge v_lv_{s+5} as a chord.

Thus, neither v_1v_3 or $v_tv_{t_2}$ is in $E(\tilde{H})$. It follows that $v_1v_i, v_tv_j \in E(\tilde{H})$ for some $i > 3$, $j < t-2$. And $d_{\tilde{H}}(v_{i-1}) = 2$, $d_{\tilde{H}}(v_{j+1}) = 2$. Then, $X = \{v_1, v_{i-1}, v_{j+1}, v_t\}$, unless $i - 1 \in \{j, j+1, j+2\}$.

Thus $i - 1 \in \{j, j+1, j+2\}$. And hence, $i > j$. Claim: $d_{\tilde{H}}(v_3) = 2$. We know v_3 cannot have a left edge, else we have nested edges. And if v_3 has a right edge v_3v_k with $k \leq i$, we have nested edges and hence a chorded cycle by Lemma 3.8. If v_3 has a right edge v_3v_k with $k > i$, since $i > j$, we again get a chorded cycle, $v_1, \tilde{H}, v_j, v_t, \tilde{H}^-, v_i, v_1$ with edge v_3, v_k as a chord. Thus, $d_{\tilde{H}}(v_3) = 2$. Claim: $d_{\tilde{H}}(v_{t-2}) = 2$. We know v_{k-2} cannot have a right edge, else we have nested edges. And if v_{k-2} has a left edge $v_{k-2}v_l$ with $l \geq j$, we have nested edges and hence a chorded cycle by Lemma 3.8. If v_{t-2} has a left edge $v_{t-2}v_l$ with $l < j$, since $i > j$, we again get a chorded cycle, $v_1, \tilde{H}, v_j, v_t, \tilde{H}^-, v_i, v_1$ with edge v_lv_{t-2} as the chord.

In all cases, Subclaim 4.1 holds. □

Thus, we may assume the component $\tilde{H}$ does not contain a Hamiltonian path. Choose two paths P_1 and P_2 in H such that:

(A) P_1 and P_2 are a maximal pair of paths; that is, the sum of the lengths of P_1 and P_2 is maximal.

(B) Subject to (A), path P_1 is as long as possible.

Let $P_1 = v_1, \ldots, v_t$ and $P_2 = u_1, \ldots, u_s$.

Subclaim 4.2. *No endpoint of P_1 or P_2 has a neighbor in $\tilde{H} - \langle P_1 \cup P_2 \rangle$. No endpoint of P_1 has a neighbor in P_2. Hence, $d_{\tilde{H}}(v_1) = d_{P_1}(v_1)$ and $d_{\tilde{H}}(v_t) = d_{P_1}(v_t)$. No endpoint p of a path P_i or vertex p in $\tilde{H} - \langle P_i \rangle$ can have degree $d_{P_i}(p) > 2$. Furthermore, $d_{\tilde{H}}(v_1) \leq 2$, $d_{\tilde{H}}(v_t) \leq 2$, and $d_{P_1}(\{u_1, u_s\}) \leq 3$.*

Proof. Clearly, none of v_1, v_t, u_1, u_s has a neighbor outside $\langle P_1 \cup P_2 \rangle$, else P_1, P_2 is not a maximal pair of paths. Furthermore, neither v_1 nor v_t can have a neighbor in P_2, else we can choose a maximal pair of paths P_1', P_2' such that P_1' is longer than P_1, contradicting (2). And no endpoint p of a path P_i or vertex p in $\tilde{H} - \langle P_i \rangle$ can have degree $d_{P_i}(p) > 2$, else $\tilde{H}$ contains a chorded cycle. So, $d_{\tilde{H}}(v_1) \leq 2$ and $d_{\tilde{H}}(v_t) \leq 2$.

Suppose $d_{P_1}(\{u_1, u_s\}) \geq 4$. Clearly, $d_{P_1}(u_1) = 2$ and $d_{P_1}(u_s) = 2$, else we have a chorded cycle. But then by Lemma 3.6, we again have a chorded cycle. Hence, $d_{P_1}(\{u_1, u_s\}) \leq 3$. □

Subclaim 4.3. *If $|P_2| \leq 3$, then we may assume $\tilde{H} = \langle P_1 \cup P_2 \rangle$.*

Proof. Suppose $|P_2| \leq 3$. Without loss of generality, we may assume $d_{P_1}(u_1) \leq d_{P_1}(u_s)$. It follows from Subclaim 4.2 that $d_{P_1}(u_1) \leq 1$ and $d_H(u_1) \leq 2$.

Claim: No vertex of P_2 has a neighbor outside $\langle P_1 \cup P_2 \rangle$.

By Subclaim 4.2, no endpoint or P_2 has a neighbor in $\tilde{H} - \langle P_1 \cup P_2 \rangle$. Hence, if $|P_2| \leq 2$, no vertex of P_2 has a neighbor in $\tilde{H} - \langle P_1 \cup P_2 \rangle$. Thus $|P_2| = 3$. Suppose $v_1 v_t \in E(\tilde{H})$. Then any vertex of P_1 can be regarded as an endpoint of the path, and hence by Subclaim 4.2, no vertex of P_1 has a neighbor in $\tilde{H} - \langle P_1 \rangle$. Furthermore, for any i, j with $i < j$ and $j \neq i + 1$, we know that $v_i v_j \notin E(\tilde{H})$; otherwise, we have nested edges in P_1, and by Lemma 3.8, a chorded cycle exists in $\langle P_1 \rangle$. Now, since $|\tilde{H}| \geq 15$, it follows that $|P_1| \geq 12$, and $X = \{v_1, v_3, v_5, v_7\}$ forms the desired set. Thus, we may assume $v_1 v_t \notin E(\tilde{H})$.

If $u_1 u_3 \in E(\tilde{H})$, then no vertex of P_2 has a neighbor outside $\langle P_1 \cup P_2 \rangle$, else we can form a longer path P_2', contradicting (A). Thus, $u_1 u_3 \notin E(\tilde{H})$, and hence $d_{P_1}(u_1) \leq 1$, $d_{P_1}(u_3) \leq 2$ and $d_{\tilde{H}}(u_1) \leq 2$, $d_{\tilde{H}}(u_3) \leq 3$.

Suppose a vertex on P_2 has a neighbor w_1 in $\tilde{H} - \langle P_1 \cup P_2 \rangle$. By Subclaim 4.2, clearly $u_1 w_1$, $u_3 w_1 \notin E(\tilde{H})$. So $u_2 w_1 \in E(\tilde{H})$. If $d_{\tilde{H}}(\{u_1, u_3\}) \leq 4$, then $X = \{v_1, v_t, u_1, u_3\}$ forms the desired set. Thus, we may assume $d_{\tilde{H}}(u_1) = 2$ and $d_{\tilde{H}}(u_3) =$

3. Hence, $d_{P_1}(u_1) = 1$ and $d_{P_1}(u_3) = 2$. Clearly, w_1 has no neighbor in $\tilde{H} - \langle P_1 \cup P_2 \rangle$, else we can form a longer path P_2' and hence a longer pair of paths P_1, P_2', contradicting (A). If $d_{\tilde{H}}(w_1) \leq 2$, then $X = \{v_1, v_t, w_1, u_1\}$ forms the desired set. Thus, w_1 has two neighbors on P_1. Note that the vertices w_1 and u_3 lie on a path $P = w_1$, u_2, u_3, and w_1, u_3 send two edges each to P_1. By Lemma 3.6, there exists a chorded cycle in $\langle P_1 \cup P \rangle$, a contradiction. Thus, we may assume no vertex on P_2 has a neighbor in $\tilde{H} - \langle P_1 \cup P_2 \rangle$, and the claim holds.

Claim: No vertex of P_1 has a neighbor in $\tilde{H} - \langle P_1 \cup P_2 \rangle$.

Suppose there exists a vertex v_i in P_1 with a neighbor w_1 in $\tilde{H} - \langle P_1 \cup P_2 \rangle$. If $d_{\tilde{H}}(w_1) \leq 2$, then $X = \{v_1, v_t, u_1, w_1\}$ forms the desired set and we are done. Thus, $d_{\tilde{H}}(w_1) \geq 3$. Hence we have one of the following cases:

1. Vertex w_1 has 3 neighbors in P_1, but then $\tilde{H}$ contains a chorded cycle by Lemma 3.4.

2. Vertex w_1 has 2 neighbors in P_1 and one neighbor in $\tilde{H} - \langle P_1 \cup P_2 \rangle$.

3. Vertex w_1 has 2 neighbors in $\tilde{H} - \langle P_1 \cup P_2 \rangle$ and one neighbor in P_1.

Case 4.3.2. Suppose w_1 lies in case 2.

Then, vertex w_1 has two neighbors in P_1, say v_i, v_j, and one neighbor in $\tilde{H} - \langle P_1 \cup P_2 \rangle$, say w_2. If $d_{\tilde{H}}(w_2) \leq 2$, then $X = \{v_1, v_t, u_1, w_2\}$ forms the desired set, and we are done. Thus, $d_{\tilde{H}}(w_2) \geq 3$, and one of the following cases must occur:

(a) Vertex w_2 has 1 neighbor in $\tilde{H} - \langle P_1 \cup P_2 \rangle$ and 2 neighbors in P_1.

(b) Vertex w_2 has 2 neighbors in $\tilde{H} - \langle P_1 \cup P_2 \rangle$ and 1 neighbor in P_1.

(c) Vertex w_2 has 3 neighbors in $\tilde{H} - \langle P_1 \cup P_2 \rangle$.

If w_2 lies in case (a), we have two vertices on a path w_1, w_2, each sending two edges to another path P_1, and by Lemma 3.6, a chorded cycle exists, a contradiction.

If w_2 lies in case (b), let w_3 be the additional neighbor of w_2 in $\tilde{H} - \langle P_1 \cup P_2 \rangle$. If $d_{\tilde{H}}(w_3) \leq 2$, $X = \{v_1, v_t, u_1, w_3\}$ is the desired set, and we are done. Thus, $d_{\tilde{H}}(w_3) \geq 3$, and hence w_3 sends two edges to P_1, else a path P_2' longer than P_2 exists in $\tilde{H} - \langle P_1 \cup P_2 \rangle$, contradicting the maximality of P_1, P_2. But then the path w_1, w_2, w_3 sends at least 5 edges to P_1, and a chorded cycle exists by Lemma 3.5, a contradiction.

Thus, w_2 lies in case (c). Let w_3 and w_4 be neighbors of w_2 in $\tilde{H} - \langle P_1 \cup P_2 \rangle$. If either w_3 or w_4 has degree at most 2 in $\tilde{H}$, we can find the desired set X and we are done. If either w_3 or w_4 has another neighbor in $\tilde{H} - \langle P_1 \cup P_2 \rangle$, then we can find a path P_2' in $\tilde{H} - \langle P_1 \cup P_2 \rangle$ longer than P_2 (since $|P_2| \leq 3$), a contradiction. Thus, w_3 and w_4 must each have two neighbors in P_1. But then, by Lemma 3.6, a chorded cycle exists, a contradiction.

Case 4.3.3. Suppose w_1 lies in case 3.

Let w_2, w_3 be the neighbors of w_1 in $\tilde{H} - \langle P_1 \cup P_2 \rangle$. If $d_{\tilde{H}}(w_2) = 2$ or $d_{\tilde{H}}(w_3) = 2$, then $X = \{v_1, v_t, u_1, w_2\}$ or $\{v_1, v_t, u_1, w_3\}$ is the desired set and we are done. Thus, $d_{\tilde{H}}(w_2) \geq 3$ and $d_{\tilde{H}}(w_3) \geq 3$. For each of w_2 and w_3 one of the following cases must occur:

(a) The vertex has 1 neighbor in $\tilde{H} - \langle P_1 \cup P_2 \rangle$ and 2 neighbors in P_1.

(b) The vertex has 2 neighbors in $\tilde{H} - \langle P_1 \cup P_2 \rangle$ and 1 neighbor in P_1.

(c) The vertex has 3 neighbors in $\tilde{H} - \langle P_1 \cup P_2 \rangle$.

Suppose either w_2 or w_3 is in case (c), without loss of generality say w_2. Then w_2 has a neighbor w_4 in $\tilde{H} - \langle P_1 \cup P_2 \rangle$ distinct from w_3, and hence w_3, w_1, w_2, w_4 forms a path P_2' longer than P_2 (since $|P_2| \leq 3$), a contradiction. Thus, each of w_2, w_3 have at least one neighbor in P_1.

Suppose either w_2 or w_3 is in case (b), without loss of generality say w_2. Then, either w_2 has a neighbor w_4 in $\tilde{H} - \langle P_1 \cup P_2 \rangle$ distinct from w_3 and we get a contradiction as before, or w_2 is adjacent to w_3. Let v_j be the neighbor of w_2 on P_1, and let v_l be the neighbor of w_3 on P_1. Then, v_j, P, v_l, w_3, w_1, w_2, v_j forms a chorded cycle with the edge w_2w_3 as a chord.

It follows that both w_2 and w_3 must lie in case (a). Then, we have five edges between the paths w_2, w_1, w_3 and P_1, and by Lemma 3.5, a chorded cycle exists, a contradiction.

Thus, if any vertex in P_1 or P_2 has a neighbor outside $\langle P_1 \cup P_2 \rangle$, then we can either find the desired set, or we get a contradiction. Hence no vertex in P_1 or P_2 has a neighbor in $\tilde{H} - \langle P_1 \cup P_2 \rangle$. And because $\tilde{H}$ is connected, it follows that $\tilde{H} = \langle P_1 \cup P_2 \rangle$, and Subclaim 4.3 holds. □

Subclaim 4.4. *For the endpoints u_1, u_s of P_2, we must have $d_{P_1}(\{u_1, u_s\}) \geq 1$.*

Suppose, to the contrary, that $d_{P_1}(\{u_1, u_s\}) = 0$.

If $v_1v_t \notin E(H)$ and $u_1u_s \notin E(H)$, then v_1, v_t, u_1, u_s are all independent and each have degree at most 2 in H, hence $X = \{v_1, v_t, u_1, u_s\}$ is the desired set and we are done. Thus, either $v_1v_t \in E(H)$ or $u_1u_s \in E(H)$.

Case 4.4.1. Suppose $|P_2| \leq 3$.

Then, by Subclaim 4.3, $\tilde{H} = \langle P_1 \cup P_2 \rangle$. If $v_1v_t \in E(H)$, then every vertex of P_1 can be regarded as an endpoint, and no vertex of P_1 has a neighbor in P_2. Hence, every vertex v of P_1 has $d_{P_1}(v) = d_H(v) = 2$, otherwise we have nested edges and a chorded cycle by Lemma 3.8. We know $|P_1| \geq 8$ since $\langle P_1 \cup P_2 \rangle = \tilde{H} \geq 15$. Thus, v_1, v_3, v_5, v_7 are all independent, $X = \{v_1, v_3, v_5, v_7\}$ is the desired set, and we are done.

Thus, $v_1v_t \notin E(H)$, and hence $u_1u_s \in E(H)$. Suppose that at least one of v_1, v_t has degree 1 in P_1, or that either v_1v_3 or $v_{t-2}v_t$ is in $E(H)$. Then by Lemmas 3.12 and 3.13, one of v_2, v_3, v_4, v_5, v_{t-5}, v_{t-4}, v_{t-3}, v_{t-2} has degree 2 in P_1, call it v_i, and hence is also independent from v_1, v_t. Thus, $X = \{v_1, v_t, u_1, v_i\}$ is the desired set, and we are done. So, $v_1v_j \in E(H)$ for some $j \geq 3$ and $v_iv_t \in E(H)$ for some $i \leq t-3$. Then the path P_1 could be rewritten with vertex v_{i+1} as an endpoint, and hence $d_H(v_{i+1}) = d_{P_1}(v_{i+1})$. By Lemma 3.11, vertex v_{i+1} has degree 2 in P_1, and hence $X = \{v_1, v_{i+1}, v_t, u_1\}$ is the desired set, and we are done.

Case 4.4.2. Suppose $|P_2| \geq 4$.

Proof. If $v_1v_t \in E(\tilde{H})$ and $u_1u_s \in E(\tilde{H})$, then every vertex of P_1 and every vertex of P_2 can be regarded as an endpoint, and no vertex of P_1 or P_2 has a neighbor in $\tilde{H} - \langle P_1 \cup P_2 \rangle$. Hence, every vertex v of P_1 or vertex u of P_2 has $d_{P_1}(v) = d_H(v) = 2 = d_{P_2}(u) = d_{\tilde{H}}(u)$, otherwise we have nested edges and a chorded cycle by Lemma 3.8. We know $|P_1| \geq |P_2| \geq 4$. Thus, v_1, v_3, u_1, u_3 are all independent, $X = \{v_1, v_3, u_1, u_3\}$ is the desired set, and we are done.

If $v_1v_t \in E(\tilde{H})$ and $u_1u_s \notin E(\tilde{H})$, then again for any vertex $v \in P_1$, $d_{P_1}(v) = d_{\tilde{H}}(v) = 2$. Also u_1, u_s are independent. And because $d_{P_1}(u_1) = 0 = d_{P_2}(u_s)$, we know that $d_{\tilde{H}}(u_1) \leq 2$ and $d_{\tilde{H}}(u_s) \leq 2$. Hence, $X = \{v_1, v_2, u_1, u_s\}$ is the desired set, and we are done.

Thus, $v_1v_t \notin E(H)$ and $u_1u_s \in E(H)$. Suppose that at least one of v_1, v_t has degree 1 in P_1, or that either v_1v_3 or $v_{t-2}v_t$ is in $E(H)$. Then by Lemmas 3.12 and 3.13, one of v_2, v_3, v_4, v_5, v_{t-5}, v_{t-4}, v_{t-3}, v_{t-2} has degree 2 in P_1, call it v_i, and hence is also independent from v_1, v_t. Thus, $X = \{v_1, v_t, u_1, v_i\}$ is the desired set, and we are done. So, $v_1v_j \in E(H)$ for some $j \geq 3$ and $v_iv_t \in E(H)$ for some $i \leq t-3$. Then the path P_1 could be rewritten with vertex v_{i+1} as an endpoint, and hence $d_H(v_{i+1}) = d_{P_1}(v_{i+1})$. By Lemma 3.11, vertex v_{i+1} has degree 2 in P_1, and hence $X = \{v_1, v_{i+1}, v_t, u_1\}$ is the desired set, and we are done.

Thus, $d_{P_1}(\{u_1, u_s\}) \geq 1$, and Subclaim 4.4 holds. □

Case 4.1. Suppose that $|P_2| = 1$.

Then $P_2 = u_1$. By Subclaim 4.3, $\tilde{H} = \langle P_1 \cup P_2 \rangle$. Hence, $|P_1| \geq 14$. And by Subclaim 4.2, $d_{\tilde{H}}(u_1) \leq 2$.

Subcase 1. Suppose $d_{\tilde{H}}(u_1) = 2$.

Let v_i, v_j, $i < j$ be u_1's neighbors on P_1. If v_i, v_j are consecutive on P_1, then $\tilde{H}$ contains a Hamiltonian path, and we are done by Subclaim 4.1. Thus, $j \geq i + 2$. Furthermore, neither of v_i, v_j is an endpoint of P_1 by Subclaim 4.2. By Lemma 3.14, one of v_{i+1}, v_{j-1} has degree 2 in $\tilde{H}$, say v_{i+1}. Then, $X = \{v_1, v_t, u_1, v_{i+1}\}$ is the desired set.

Subcase 2. Suppose $d_{\tilde{H}}(u_1) = 1$.

At most one vertex in $P_1[v_3, v_{12}]$ is adjacent to u_1. It follows that there exists in $P_1[v_3, v_{12}]$ a group of at least 4 consecutive vertices all nonadjacent to u_1 and another distinct group of at least 5 consecutive vertices all nonadjacent to u_1, say $v_i, \ldots, v_{i+3}$ and $v_j, \ldots, v_{j+4}$, or there exists a group of 6 consecutive vertices all nonadjacent to u_1, say $v_i, \ldots, v_{i+5}$. Thus, there exist at least three distinct pairs of two consecutive vertices all nonadjacent to u_1: either $\{v_i, v_{i+1}\}$, $\{v_{i+2}, v_{i+3}\}$, and $\{v_j, v_{j+1}\}$; or $\{v_i, v_{i+1}\}$, $\{v_{i+2}, v_{i+3}\}$, and $\{v_{i+4}, v_{i+5}\}$.

By Lemma 3.10, at least one vertex from each of the three pairs has degree at most three to P_1.

Recall that since v_1, v_t are endpoints of the path, by Subclaim 4.2, $d_{P_1}(v_1) \leq 2$ and $d_{P_1}(v_t) \leq 2$. Thus, vertex v_1 has at most one neighbor in $P_1[v_3, v_{12}]$ and vertex v_t has at most one neighbor in $P_1[v_3, v_{12}]$. Thus, at least one of the three vertices above, all nonadjacent to u_1 and having degree at most three to P_1, is also independent from v_1 and v_t, call it v_k. Then, $X = \{v_1, v_t, u_1, v_k\}$ is the desired set, and we are done.

Case 4.2. Suppose that $|P_2| = 2$.

Recall by Subclaim 4.2, $d_{P_1}(\{u_1, u_2\}) \leq 3$ and $d_{P_1}(u_1) \leq d_{P_1}(u_2)$. So, $d_{P_1}(u_1) \leq 1$ and $d_{P_1}(u_2) \leq 2$.

Subcase 1. Suppose $\{u_1, u_2\}$ has 2 or more distinct neighbors on P_1.

Say these neighbors are v_i and v_j with $i < j$. We know that j must be at least $i+2$. Otherwise $j = i+1$ and we can form either a Hamiltonian path, if each of u_1, u_s has an endpoint to P_1, in which case we are done by Subclaim 4.1, or a maximal pair of paths P_1', P_2' with $|P_1'| > |P_1|$, a contradiction.

But now, by Lemma 3.14, one of v_{i+1}, v_{j-1}, call it v_l has degree 2 in $\tilde{H}$. Hence, $X = \{v_1, v_t, u_1, v_l\}$ forms the desired set, and we are done.

Subcase 2. Suppose $\{u_1, u_2\}$ has one distinct neighbor in P_1.

Since $d_{P_1}(u_1) < d_{P_1}(u_2)$, either $d_{P_1}(u_1) = 0$ or $d_{P_1}(u_1) = 1 = d_{P_1}(u_2)$ and u_1, u_2 have the same neighbor in P_1. Thus, $d_{P_1}(u_1) \leq 1$ and $d_{\tilde{H}}(u_1) \leq 2$.

If $d_{\tilde{H}}(v_1) = 1$, $d_{\tilde{H}}(v_t) = 1$, or either v_1v_3 or $v_{t-2}v_t \in E(\tilde{H})$, by Lemmas 3.12 and 3.13, one of v_3, v_4, v_5, v_6, v_{t-5}, v_{t-4}, v_{t-3}, or v_{t-2} has degree two in $\tilde{H}$, call it v_l. Then, $X = \{v_1, v_t, u_1, v_l\}$ forms the desired set, and we are done.

Thus, v_1 must have a neighbor v_i in $P_1[v_4, v_t]$ and v_t must have a neighbor v_j in $P_1[v_1, v_{t-2}]$. Then, by Lemma 3.15, $d_{\tilde{H}}(v_{i-1}) = 2$ and $d_{\tilde{H}}(v_{j+1}) = 2$. Thus, $X = \{v_1, v_t, v_{i-1}, v_{j+2}\}$ forms the desired set and we are done. This completes Case 4.2.

Case 4.3. Suppose that $|P_2| = 3$.

We know $\tilde{H} = \langle P_1 \cup P_2 \rangle$ by Subclaim 4.3. Recall, by Subclaim 4.2, that $3 \geq d_{P_1}(\{u_1, u_3\} \geq 1$. If $u_1u_3 \in E(\tilde{H})$, then there is at most one edge between P_1 and P_2, else a chorded cycle exists. It follows that $d_{\tilde{H}}(u_1) \leq 2$. By Lemmas 3.12 and 3.13, if $d_{\tilde{H}}(v_1) = 1$, $d_{\tilde{H}}(v_t) = 1$, or either v_1v_3 or $v_{t-2}v_t \in E(\tilde{H})$, then one of v_3, v_4, v_5, v_6, v_{t-5}, v_{t-4}, v_{t-3}, or v_{t-2} has degree two in $\tilde{H}$, call it v_l. Then, $X = \{v_1, v_t, u_1, v_l\}$ forms the desired set, and we are done.

Thus, v_1 must have a neighbor in $P_1[v_4, v_t]$ and v_t must have a neighbor in $P_1[v_1, v_{t-2}]$. By Lemma 3.15, if v_1 has a neighbor v_i in $P_1[v_4, v_t]$ or v_t has a neighbor

v_j in $P_1[v_1, v_{t-3}]$, then either $X = \{v_1, v_t, v_{i-1}, u_1\}$ or $X = \{v_1, v_t, v_{j+1}, u_1\}$ forms the desired set, and we are done.

Case 4.4. Suppose that $|P_2| = s \geq 4$.

Suppose both u_1 and u_s have an edge into P_1. Then $d_{P_2}(u_1) = 1$ and $d_{P_2}(u_s) = 1$, else a chorded cycle exists. Hence, by Subclaim 4.2, $d_{\tilde{H}}(u_1) \leq 2$. Then if $d_{P_1}(u_1) = 1$ and $d_{P_1}(u_s) = 1$, we see that $X = \{v_1, v_t, u_1, u_s\}$ is the desired set. Thus, $d_{P_1}(u_s) \geq 2$. Let v_i, v_j be neighbors of u_s on P_1. Consider vertex u_{s-1}; if it has degree at most 2 in $\tilde{H}$, then $\{v_1, v_t, u_1, u_{s-1}\}$ is the desired set, and we are done. Hence, u_{s-1} must have degree 3 or more. If u_{s-1} has degree 3 in P_2, a chorded cycle exists, a contradiction. Thus u_{s-1} has a neighbor in P_1 or in $\tilde{H} - \langle P_1 \cup P_2 \rangle$. If u_1 or u_{s-1} has an edge to the left or the right of both v_i and v_j, we have three parallel edges between P_1 and P_2 and hence a chorded cycle exists by Lemma 3.4. Thus, the neighbors on P_1 of u_1 and u_{s-1} must lie in $P_1[v_i, v_j]$. But then we again get three parallel chords, or three crossing chords, and hence a chorded cycle by Lemma 3.4. Thus, u_{s-1} must have a neighbor w_1 in $\tilde{H} - \langle P_1 \cup P_2 \rangle$.

If $d_{\tilde{H}}(w_1) \leq 2$, then $\{v_1, v_t, u_1, w_1\}$ is the desired set, and we are done. Thus, $d_{\tilde{H}}(w_1) \geq 3$. Vertex w_1 cannot have a neighbor in $\tilde{H} - \langle P_1 \cup P_2 \rangle$, else we can form a longer pair of paths P_1, P_2', a contradiction. Furthermore, vertex w_1 cannot have two neighbors in P_1, else by Lemma 4 we have a chorded cycle, since u_s has two neighbors in P_1. Thus, vertex w_1 has two neighbors in P_2 and one neighbor in P_1.

Let v_l be the neighbor of u_1 in P_1 and v_m be the neighbor of w_1 in P_1. Vertex w_1 is not adjacent to u_1 or u_s, hence w_1's second neighbor u_i in P_2 lies in $P_2[u_2, u_{s-1})$. Then w_1, $P_2^-[u_{s-1}, u_1]$, v_i, $P_1^{\pm}(v_i, v_j]$, v_m, w_1 forms a chorded cycle with $w_1 u_i$ as a chord, a contradiction.

Thus, in all cases, Claim 4 holds. □

Thus, $\tilde{H}$ is connected with $|\tilde{H}| \geq 15$, and there exists a set X in $\tilde{H}$ containing

4 independent vertices such that $d_{\tilde{H}}(X) = d_H(X) \leq 8$. It follows that $d_{\mathscr{C}}(X) \geq 12k - 3 - 8 = 12k - 11 > 12(k-1)$. And hence there exists $C \in \mathscr{C}$ such that $d_C(X) \geq 13$. By Lemma 3.3, for any $x_i \in X$, $d_C(x_i) \leq 4$. It follows that the possible degree sequences are: $(4,4,4,1), (4,4,3,2), (4,3,3,3)$. Hence, by Lemma 3.3, $C = K_4$ since in all cases there exists $x_i \in X$ such that $d_C(x_i) = 4$. Let $C = v_1, v_2, v_3, v_4, v_1$.

Case 1. Suppose we have sequence $(4,4,4,1)$.

Let x_4 have degree 1 to C and let the vertices x_1, x_2, x_3 have degree 4 to C. Without loss of generality, say x_4 is adjacent to v_1.

Since $\tilde{H}$ is connected, there is a path from x_4 to some other $x_i \in X$ disjoint from $X - \{x_4, x_i\}$. Without loss of generality say there is such a path P connecting x_4 and x_3. (See Figure 3.9.)

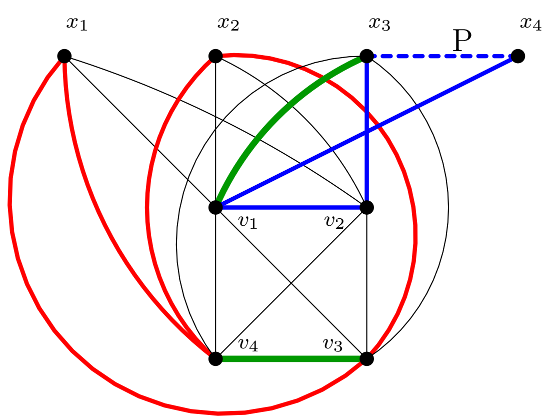


Figure 3.9. A path P connecting x_3 and x_4.

Then, x_4, v_1, v_2, x_3, $P(x_3, x_4]$ is a chorded cycle with v_1x_3 as a chord, and x_1, v_3, x_2, v_4, x_1 is a chorded cycle with v_3v_4 as a chord. Thus, we have two chorded cycles in $\langle \tilde{H} \cup C \rangle$, a contradiction.

Case 2. Suppose we have sequence $(4,4,3,2)$.

Label the vertices of X with x_1, x_2, x_3, x_4 such that $d_C(x_1) = 4$, $d_C(x_2) = 4$, $d_C(x_3) = 3$, $d_C(x_4) = 2$. Without loss of generality, say x_4 is adjacent to v_1 and v_2.

Since $\tilde{H}$ is connected, there is a path P from x_4 to some other $x_i \in X$ disjoint from $X - \{x_4, x_i\}$.

Subcase 1. Suppose path P connects x_4 and the vertex of X with degree 3 to C, that is x_3.

Vertices x_3 and x_4 have a common neighbor in C, say it's v_1. Then v_1, v_2, $P[x_4, x_3]$, v_1 forms a chorded cycle with edge v_1x_4 as a chord. (See Figure 3.10.) Vertices x_1 and x_2 both have degree 4 to C, hence they are both adjacent to v_3 and v_4. Then, x_1, v_3, x_2, v_4, x_1 forms a second chorded cycle with edge v_3v_4 as a chord. (See Figure 3.10.) Thus, we have two chorded cycles in $\langle \tilde{H} \cup C \rangle$, a contradiction.

Subcase 2. Suppose path P connects x_4 and a vertex of X with degree 4 to C. Without loss of generality, say P connects x_4 and x_1.

Vertices x_2 and x_3 have three common neighbors in C, at least one of which is not also a neighbor of x_4. Say v_3 is one of these common neighbors, and call the other one v_i. Then x_2, v_i, x_3, v_3, x_2 is a chorded cycle with chord v_iv_3. At least one of x_4's neighbors in C has not yet been used, say v_1. Let v_j be the last remaining vertex of C. Vertex x_4 may or may not be adjacent to v_j, but certainly x_1 is adjacent to both v_1 and v_j. Thus, x_1, P, x_4, v_1, v_j, x_1 forms a second chorded cycle with chord v_1x_1. (See Figure 3.11.) Again, we have two chorded cycles in $\langle \tilde{H} \cup C \rangle$, a contradiction.

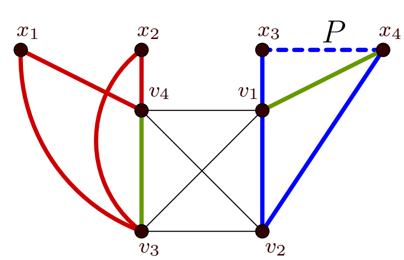


Figure 3.10. A chorded cycle.

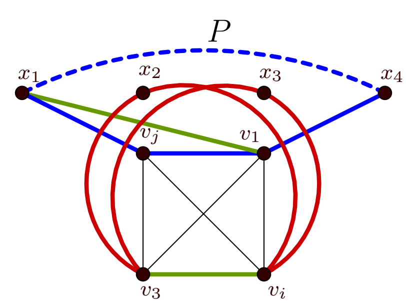


Figure 3.11. A chorded cycle.

Case 3. Suppose we have sequence $(4, 3, 3, 3)$.

Label the vertices of X with x_1, x_2, x_3, x_4 such that that $d_C(x_1) = 4$, $d_C(x_2) = 3$, $d_C(x_3) = 3$, $d_C(x_4) = 3$. Since $\tilde{H}$ is connected, there is a path from x_1 to some other $x_i \in X$ disjoint from $X - \{x_1, x_i\}$. Without loss of generality, say there is such a path P connecting x_1 and x_2. Vertices x_3 and x_4 share two neighbors in C, say v_1, v_2. Then x_3, v_1, x_4, v_2, x_3 is a chorded cycle with v_1v_2 as a chord. Vertex x_2 has degree 3 to C; therefore, it has some remaining neighbor in C, say v_4. Vertex x_1 is adjacent to both v_3 and v_4. Then, $P[x_1, x_2]$, v_4, v_3, x_1 is a second chorded cycle with x_1v_4 as a chord. (See Figure 3.12.) Thus, we have two chorded cycles in $\left\langle \tilde{H} \cup C \right\rangle$, a contradiction.

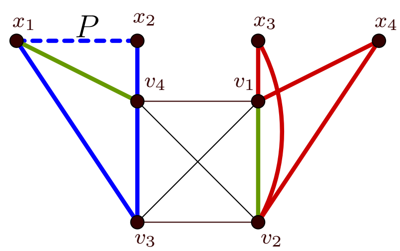


Figure 3.12. Two chorded cycles in $\left\langle \tilde{H} \cup C \right\rangle$.

In all cases we get a contradiction. Thus, there cannot be an edge-maximal counterexample and the proof is complete.

□

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