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Pascal Philipp Date

Resonance asymptotics for asymptotically hyperbolic manifolds with warped-product ends

By

Pascal Philipp
Doctor of Philosophy

Mathematics

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David Borthwick, Ph.D.
Advisor

---

Michele Benzi, Ph.D.
Committee Member

---

Vladimir Oliker, Ph.D.
Committee Member

Accepted:

---

Lisa A. Tedesco, Ph.D.
Dean of the James T. Laney School of Graduate Studies

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Date

Resonance asymptotics for asymptotically hyperbolic manifolds with warped-product ends

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An abstract of
A dissertation submitted to the Faculty of the
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**Abstract**

Resonance asymptotics for asymptotically hyperbolic
manifolds with warped-product ends

By Pascal Philipp

We study the spectral theory of asymptotically hyperbolic manifolds with ends of warped-product type. Our main result is an upper bound on the resonance counting function, with a geometric constant expressed in terms of the respective Weyl constants for the core of the manifold and the base manifold defining the ends. As part of this analysis, we derive asymptotic expansions of the modified Bessel functions of complex order.

# Resonance asymptotics for asymptotically hyperbolic manifolds with warped-product ends

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Figures 2.1, 3.1, 3.2, and 6.1 are taken from [7] and have been generated by David Borthwick.

# Chapter 1

# Introduction

For the eigenvalues of the Laplacian on a compact Riemannian manifold of dimension $d$, Weyl's law states that

$$\#\{k : \sqrt{\lambda_k} \leq r\} = Wr^d + o(r^d), \tag{1.1}$$

with a constant $W$ that depends only on $d$ and the volume of the manifold. The goal of this dissertation is to find an upper bound of the form (1.1) for the counting function of the discrete spectral data of a certain type of *infinite-volume* manifolds.

We will consider *asymptotically hyperbolic manifolds with warped-product ends* $(X, g)$, with $\dim X = n + 1$, $n \geq 1$. By this we mean that $X$ admits a decomposition

$$X = K \sqcup X_0,$$

where $X_0 = (0, 1] \times \Sigma$ with $(\Sigma, h)$ a compact Riemannian manifold without boundary, and $K$ is a compact manifold with boundary $\partial K \simeq \Sigma$. The restriction of the smooth metric $g$ to $X_0$ is of the form

$$g_{|X_0} = \frac{dx^2 + h}{x^2}. \tag{1.2}$$

We allow $\Sigma$ to be disconnected, so that multiple ends can be considered without changing the notation.

The projection $x : X_0 = (0, 1] \times \Sigma \to (0, 1]$ can be continued smoothly onto K, and hence we can think of it as a function on all of $X$. The effect of the division by $x^2$ in (1.2) is illustrated in Figure 1.1, and the general concept behind (1.2) is explained in Appendix A. In particular, $(X, g)$ is a conformally compact manifold and, moreover, asymptotically hyperbolic (see Appendix A for the definitions of these terms).

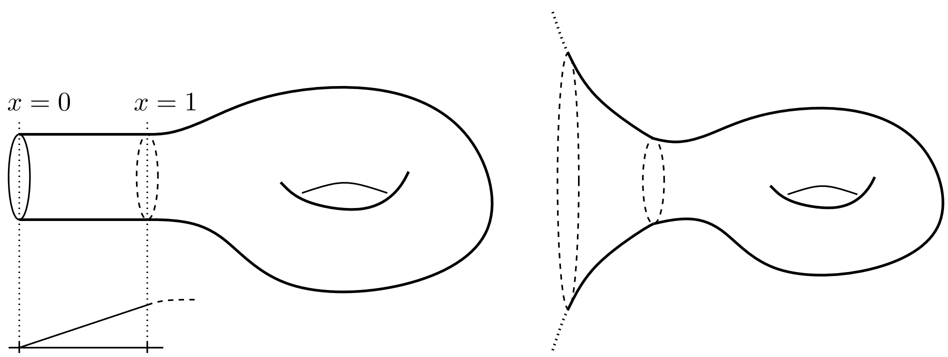


**Figure 1.1:** An example for $n = 1$ for the compact Riemannian manifold $(\overline{X}, x^2 g)$ and the boundary-defining function $x$ are on the left. After removing the boundary $\partial\overline{X}$ and dividing the metric by $x^2$, we obtain the complete infinite-area Riemannian manifold $(X, g)$.

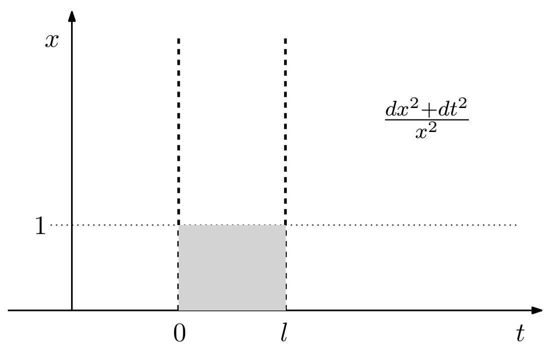


**Figure 1.2:** The hyperbolic plane with its metric. Identifying the dashed lines gives the parabolic cylinder. If $n = 1$ and $l = \operatorname{Vol}\Sigma$, then (1.2) is just the restriction of the hyperbolic metric to the shaded area.

For a general conformally compact, asymptotically hyperbolic manifold, Joshi-Sá Barreto [20] proved the existence of a product decomposition near infinity with a metric of the form (1.2) where $h = h(x, y, dy)$, meaning that $h$ could also depend on $x$. Our restrictions amount to having a fixed metric $h$ independent of $x$.

In the $n = 1$ case, $\Sigma$ is a circle, and the model $X_0$ is isometric to the flared end of the parabolic cylinder $\mathbb{H}^2/\langle z \mapsto z + 1\rangle$ (c.f. Figure 1.2). In higher dimensions $X_0$ will generally not have constant curvature.

The metric (1.2) is *even* in the sense introduced in Colin Guillarmou's Ph.D. thesis [13] (if we write points on $\overline{X}_0$ as $(x, y) \in [0, 1] \times \Sigma$, then the flow from the boundary along

$\mathrm{grad}_{x^2g}(x)$ is just the identity, and (1.2) in Definition 1.2 of [13] becomes

$$\phi^*(x^2g) = dt^2 + h,$$

where $h$, which for $g$ to be even needs to have only even powers of $t$ in its Taylor expansion at $t = 0$, is independent of $t$). Hence, the resolvent $R_g(s) := (\Delta_g - s(n-s))^{-1}$ admits a meromorphic continuation to $s \in \mathbb{C}$, with poles of finite rank, by Mazzeo-Melrose [24] and Guillarmou [13]. We define the *resonance* set $\mathcal{R}_g$ to be the set of poles of $R_g(s)$, repeated according to multiplicity. The corresponding resonance counting function is

$$N_g(t) := \#\left\{\zeta \in \mathcal{R}_g : \left|\zeta - \tfrac{n}{2}\right| \le t\right\}. \tag{1.3}$$

In the general asymptotically hyperbolic setting, we know essentially nothing of the resonance set beyond the meromorphic continuation result that allows its definition. At this level of generality, we have no bounds on $N_g(t)$ and no existence results for $\mathcal{R}_g$. The only general information we have on the distribution of resonances is a result of Guillarmou [14] that establishes exponentially thin resonance-free regions near the critical line $\mathrm{Re}\, s = \frac{n}{2}$. All of the current resonance counting results for asymptotically hyperbolic metrics assume that the sectional curvature is constant outside a compact set. Under this stronger assumption, we have $N_g(t) = O(t^{n+1})$, as well as a Poisson-type trace formula expressing the regularized wave trace as a sum over the resonance set ([17, 8, 3]). These results will now be translated to the asymptotically hyperbolic setting described above.

In order to state the first main theorem, we define the *wave 0-trace* as follows (as in [19, 3]). The *finite part* of a function of $\varepsilon$ that admits an expansion as $\varepsilon \to 0$ in terms of $\log\varepsilon$ and powers of $\varepsilon$ is

$$\underset{\varepsilon\to 0}{\mathrm{FP}}\left[\sum_{j=1}^{m} a_j\varepsilon^{-j} + a_1' \log\varepsilon + a_0 + o(1)\right] := a_0.$$

Let an operator $A$ be given and assume that its integral kernel $A(z,z')$ with respect to $dV_g$ is continuous. Then, if $A(z,z)$ has an expansion for $x \to 0$ in terms of powers of $x$, we define

$$0\text{-tr}\, A := \underset{\varepsilon\to 0}{\mathrm{FP}} \int_{\{x\ge\varepsilon\}} A(z,z)\,dV_g(z),$$

where $x$ is to be thought of as a function on all of $X$, so that $K$ is containend in $\{x \ge \varepsilon\}$ for $\varepsilon$ sufficiently small. (The definition of the 0-trace depends on the boundary-defining function, but note that our choice $\rho = x$ is somewhat canonical. Further we have that differences arising from different choices of the boundary-defining function cancel out on the left side of the relative formula in Theorem 1.1.) Now, the wave operator $\cos\left(t\sqrt{\Delta_g - \frac{n^2}{4}}\right)$ is singular across the diagonal, and the above definition cannot be applied directly. However, the regularized operator

$$\int_{-\infty}^{\infty} \varphi(t) \cos\left(t\sqrt{\Delta_g - \tfrac{n^2}{4}}\right) dt,$$

for $\varphi \in C_0^\infty(\mathbb{R})$ does have a smooth kernel, and hence the wave 0-trace below is to be understood in a distributional sense.

**Theorem 1.1.** *Assume $(X, g)$ is an asymptotically hyperbolic manifold with warped-product ends. Let $\Delta_0$ denote the Laplacian with Dirichlet boundary conditions on the model end $(X_0, g)$, and $\mathcal{R}_0$ the corresponding resonance set. The difference of the regularized wave traces satisfies*

$$\begin{aligned} &\text{0-tr}\left[\cos\left(t\sqrt{\Delta_g - \tfrac{n^2}{4}}\right)\right] - \text{0-tr}\left[\cos\left(t\sqrt{\Delta_0 - \tfrac{n^2}{4}}\right)\right] \\ &\quad = \frac{1}{2}\sum_{\zeta\in\mathcal{R}_g} e^{(\zeta-\frac{n}{2})|t|} - \frac{1}{2}\sum_{\zeta\in\mathcal{R}_0} e^{(\zeta-\frac{n}{2})|t|}, \end{aligned}$$

*in the sense of distributions on $\mathbb{R} - \{0\}$.*

We define the *scattering matrix* $S_g(s)$ of $(X, g)$ as in [20]: For $s \neq \frac{n}{2}$ with $\operatorname{Re} s = \frac{n}{2}$ and for $f \in C^\infty(\partial\overline{X})$, there exists a unique solution $u$ to $(\Delta_g - s(n-s))u = 0$ of the form

$$u = x^{n-s}F + x^s G,$$

where $F, G \in C^\infty(\overline{X})$ and $F_{|\partial\overline{X}} = f$. Then, for $s = \frac{n}{2} + iy, y \neq 0$, the scattering matrix is defined by

$$S_g(s) : f \mapsto g,$$

where $g = G_{|\partial\overline{X}}$. This is a family of pseudodifferential operators on $\Sigma = \partial\overline{X}$, and it extends meromorphically to $s \in \mathbb{C}$.

For asymptotically hyperbolic metrics, the relationship between resonances and poles of the normalized scattering matrix $\Gamma(s - \frac{n}{2})\big(\Gamma(\frac{n}{2} - s)\big)^{-1} S_g(s)$ was established in [19, 6, 14]. In particular, Guillarmou [14] showed that $\Gamma(s - \frac{n}{2})\big(\Gamma(\frac{n}{2} - s)\big)^{-1} S_g(s)$ may have 'conformal' poles $s \in \frac{n}{2} - \mathbb{N}$ which do not correspond to resonances. However, in the case of asymptotically hyperbolic metrics of warped-product type, we will see that these conformal poles are ruled out in any dimension. Hence the multiplicities of scattering poles agree with those of the resonance set, except possibly at the finitely many points $s$ where $s(n-s)$ is an eigenvalue of $\Delta_g$.

One application of the Poisson formula of Theorem 1.1 is a Weyl asymptotic for the *relative scattering phase*, which is defined as the logarithm of the Fredholm determinant of $S_g(s)S_0(s)^{-1}$ on the critical line $\operatorname{Re} s = \frac{n}{2}$. Because of the connection between resonances and scattering poles, we can follow [4] and use this asymptotic in conjunction with a contour integral involving $\det S_g(s)S_0(s)^{-1}$ to produce a precise upper bound on the resonance counting function.

To state the bound on the resonance counting function, we introduce the classical Weyl constants for the compact manifolds $K$ and $\Sigma$,

$$W_K := \frac{\operatorname{Vol}(K,g)}{(4\pi)^{\frac{n+1}{2}}\Gamma(\frac{n+3}{2})}, \quad W_\Sigma := \frac{\operatorname{Vol}(\Sigma,h)}{(4\pi)^{\frac{n}{2}}\Gamma(\frac{n+2}{2})}.$$

With these definitions, we have the asymptotic (1.1) with $W = W_K$ and $W = W_\Sigma$ for the counting functions of the eigenvalues of the Dirichlet Laplacian on $(K,g)$ and, respectively, the Laplacian on $(\Sigma,h)$. For $\arg\alpha \in [0,\frac{\pi}{2}]$, define

$$\rho(\alpha) := \sqrt{\alpha^2+1} + \alpha\log\left(\frac{i}{\alpha+\sqrt{\alpha^2+1}}\right),$$

where we are using the principal branches of the square root and logarithm. Denote by $\alpha_0 \approx 1.509$ the point on the positive real axis satisfying $\operatorname{Re}\rho(\alpha_0) = 0$, and let the curve $\gamma$ be defined as the portion of $\{\operatorname{Re}\rho(\alpha) = 0\}$ that connects $i$ and $\alpha_0$ (see Figure 6.1).

**Theorem 1.2.** *For $(X,g)$ an asymptotically hyperbolic metric with warped-product ends,*

$$(n+1)\int_0^a \frac{N_g(t)}{t}\,dt \le \Big[2W_K + c_n W_\Sigma\Big]a^{n+1} + o(a^{n+1}), \tag{1.4}$$

*where the dimensional constant is*

$$\begin{aligned} c_n := \frac{2n}{(n+1)\pi}\int_\gamma \frac{|\rho'(\alpha)|}{|\alpha|^{n+1}}\,|d\alpha| + \frac{\alpha_0^{-n}}{n+1} \\ + \frac{n(n+1)}{\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_0^\infty \frac{[-\operatorname{Re}\rho(xe^{i|\theta|})]_+}{x^{n+2}}\,dx\,d\theta. \end{aligned} \tag{1.5}$$

The integrated counting function that appears in (1.4) is common usage in applications of Jensen's formula in complex analysis. The bound (1.4) implies a corresponding bound for $N_g(t)$, at the cost of an extra factor of $e$ in the constant. However, if (1.4) is an asymptotic, i.e. if we have equality modulo $o(a^{n+1})$ in (1.4), then it is equivalent to an asymptotic for $N_g(t)$ with the *same* constant.

This thesis is organized as follows. We start by computing the spectral operators of the Dirichlet Laplacian on $X_0$ (Ch. 2). The modified Bessel functions that appear in the explicit formulas for the components of these operators will then be analyzed carefully (Ch. 3). Then we will establish a suboptimal bound on the growth of $N_g(t)$, using the Fredholm determinant method (Ch. 4). This crude estimate then allows to apply the methods used in the hyperbolic-near-infinity case in [3] to prove Theorem 1.1 (Ch. 5). For Theorem 1.2, we first show that the counting function $N_0(t)$ for $\mathcal{R}_0$ satisfies the exact asymptotic

$$N_0(t) = \left[\frac{2n}{(n+1)\pi}\int_\gamma \frac{|\rho'(\alpha)|}{|\alpha|^{n+1}}\, d\,|\alpha| + \frac{\alpha_0^{-n}}{n+1}\right] W_\Sigma\, t^{n+1} + O\big(t^{n+\frac{1}{3}}\big),$$

(§6.1). Then we bound the relative scattering determinant (which is defined in Ch. 5) in a subset of the half-plane $\{\operatorname{Re} s > \frac{n}{2}\}$ (§6.2). Lastly, we use a relative counting formula to obtain the sharper estimate for $N_g(t)$ that is stated in Theorem 1.2 (§6.3).

# Chapter 2

# The model case

The model space is $X_0 := (0, 1] \times \Sigma$, where $(\Sigma, h)$ is a compact $n$-dimensional Riemannian manifold without boundary. The metric on $X_0$ is the warped product

$$g_0 := \frac{dx^2 + h}{x^2},$$

and for the corresponding Laplacian

$$\Delta_0 = -(x\partial_x)^2 + nx\partial_x + x^2\Delta_h,$$

we impose Dirichlet boundary conditions on $\{x = 1\}$. By the scale-invariance of the $dx^2$ component of $g_0$, imposing the boundary condition at some other value $x = b$ would be equivalent to rescaling $h \rightsquigarrow b^2 h$.

The Laplacian $\Delta_h$ on the compact manifold $\Sigma$ has discrete spectrum and a complete orthonormal basis of eigenfunctions. We index these eigenfunctions with the positive square roots of the corresponding eigenvalues, that is

$$\Delta_h \phi_\lambda = \lambda^2 \phi_\lambda.$$

Eigenvalues of higher multiplicity are allowed, even though this is not reflected in the above notation. If we separate variables by setting $w = u(x)\phi_\lambda$, the equation $(\Delta_0 - s(n-s))w = 0$ translates to the coefficient equation

$$\Big[-(x\partial_x)^2 + nx\partial_x + \lambda^2 x^2 - s(n-s)\Big]u = 0. \tag{2.1}$$

This is a modified Bessel equation, with the Bessel parameter given by

$$\nu := s - \tfrac{n}{2}.$$

To simplify formulas, we will be making this identification throughout this dissertation and switch freely between $s$ and $\nu$.

The general solution to (2.1) is a linear combination of the terms $x^{\frac{n}{2}} I_{\pm\nu}(\lambda x)$ for $\lambda > 0$ and $x^{\frac{n}{2}\pm\nu}$ for $\lambda = 0$. As $x \to 0$ the Bessel function has asymptotic

$$I_\nu(\lambda x) \sim \frac{1}{\Gamma(\nu+1)} \left(\frac{\lambda x}{2}\right)^\nu, \tag{2.2}$$

for $\nu \notin -\mathbb{N}$. For future use we single out the 'outgoing' solutions

$$\begin{aligned} u_\lambda^+(s;x) &:= x^{\frac{n}{2}} I_\nu(\lambda x) \quad \text{for } \lambda > 0, \\ u_0^+(s;x) &:= x^s, \end{aligned}$$

which for $\nu \notin -\mathbb{N}$ have asymptotics proportional to $x^s$ as $x \to 0$. Solutions that satisfy the boundary condition at $x = 1$ are given by

$$\begin{aligned} u_\lambda^0(s;x) &:= \frac{\pi}{2\sin \pi\nu} x^{\frac{n}{2}} \left[ I_\nu(\lambda) I_{-\nu}(\lambda x) - I_{-\nu}(\lambda) I_\nu(\lambda x) \right] \quad \text{for } \lambda > 0, \\ u_0^0(s;x) &:= \frac{1}{2\nu} \left[ x^{n-s} - x^s \right]. \end{aligned}$$

The prefactor in $u_\lambda^0(s)$ $(\lambda > 0)$ is included to cancel zeros that would otherwise occur at $\nu \in \mathbb{Z}$, where we have the symmetry $I_{-\nu}(z) = I_\nu(z)$. In terms of the second standard solution $K_\nu(z)$ of the modified Bessel equation, which is defined by (3.11), we have

$$u_\lambda^0(s;x) = x^{\frac{n}{2}} \left[ I_\nu(\lambda) K_\nu(\lambda x) - K_\nu(\lambda) I_\nu(\lambda x) \right], \qquad \lambda > 0.$$

Similarly, the factor $(2\nu)^{-1}$ in $u_0^0(s)$ cancels the zero that would otherwise occur at $s = \frac{n}{2}$. For later use, recall Euler's reflection formula,

$$\frac{\pi}{\sin \pi\nu} = \Gamma(\nu)\Gamma(1-\nu).$$

We can now express the model resolvent, Poisson kernel, and scattering matrix in terms of the solutions $u_\lambda^+$ and $u_\lambda^0$. The construction of the resolvent or Green's function in the following section is a standard computation.

## 2.1 Resolvent

We write points on the model ends as $z = (x, w) \in (0, 1] \times \Sigma$, and, using the orthonormal basis $\{\phi_\lambda\}_{|\lambda^2 \in \sigma(\Delta_h)}$ of $L^2(\Sigma)$, our ansatz for the kernel of the model resolvent is

$$R_0(s; x, \omega, x', \omega') := \sum_\lambda a_\lambda(s; x, x')\phi_\lambda(\omega)\overline{\phi_\lambda(\omega')}. \tag{2.3}$$

In order to find this integral kernel with respect to the Riemannian volume measure of $X_0$, we note that

$$dV_g(z) = x^{-(n+1)}\, dx\, dV_h(w), \tag{2.4}$$

and that the defining property of the resolvent is

$$\big(\Delta_0 - s(n-s)\big)R_0(s) = I. \tag{2.5}$$

Applying (2.5) to $v(x)\phi_\lambda(w)$, where $v \in C_0^\infty((0, 1])$, leads to

$$\Big[-(x\partial_x)^2 + nx\partial_x + \lambda^2 x^2 - s(n-s)\Big]a_\lambda(s; x, x') = x^{n+1}\delta(x - x'). \tag{2.6}$$

The functions $a_\lambda$ will further need to satisfy the boundary conditions $a_\lambda(s; 1, x') = 0$ and, for $\nu \notin -\mathbb{N}$, $a_\lambda(s; x, x') \sim c(s, x')x^s$ as $x \to 0$. The reasoning behind the latter condition will be explained at the end of this section.

In order to conform to (2.6), we set

$$a_\lambda(s; x, x') = A_\lambda(s)\begin{cases} u_\lambda^+(s; x)u_\lambda^0(s; x') & x \le x' \\ u_\lambda^0(s; x)u_\lambda^+(s; x') & x \ge x', \end{cases} \tag{2.7}$$

where the constants $A_\lambda(s)$ are to be determined. Note that the boundary conditions for $a_\lambda$ are met. The kernels (2.7) are symmetric, continuous, and smooth away from the diagonal with

$$\Big[-(x\partial_x)^2 + nx\partial_x + \lambda^2 x^2 - s(n-s)\Big]a_\lambda(s; x, x') = 0 \qquad \text{for } x \ne x'. \tag{2.8}$$

Now fix $x' \in (0, 1]$ and define the distribution $\omega_{x'}$ on $(0, 1]$ by integration with respect to $x$ against the left hand side of (2.6). Integration by parts shows that $\omega_{x'}$ has order at most 2, and (2.8) means that it is supported on $\{x'\}$. Hence $\omega_{x'}$ is of the form

$$\omega_{x'} = c_0\delta_{x'} + c_1\delta_{x'}^{(1)} + c_2\delta_{x'}^{(2)},$$

where $\delta_{x'}^{(k)}(\psi) = \psi^{(k)}(x')$. It remains to choose $A_\lambda(s)$ so that $c_0 = (x')^{n+1}$, and to show that $c_1 = c_2 = 0$.

For $\varepsilon' > 0$ sufficiently small so that $x' - \varepsilon' > 0$, and for any test function $\psi \in C_0^\infty((0, 1+\varepsilon'])$ we have

$$\begin{aligned}\omega_{x'}(\psi) = \int_{x'-\varepsilon}^{x'+\varepsilon} \big[ -x^{n+1}\partial_x\big(x^{1-n}\partial_x a_\lambda(s;x,x')\big) \\ + \big(\lambda^2 x^2 - s(n-s)\big)a_\lambda(s;x,x')\big]\psi(x)\,dx,\end{aligned} \tag{2.9}$$

for any $\varepsilon \in (0, \varepsilon')$. (The coefficient functions $a_\lambda$ are well-defined for $x > 1$ and (2.8) is valid there as well; we can make use of this in order to deal with the case $x' = 1$ simultaneously.) Note that $\partial_x a_\lambda(s;x,x')$ is defined away from $x'$ and bounded in a neighborhood of $x'$. Hence, if $\psi_0$ is constantly equal to 1 in a neighborhood of $x'$, letting $\varepsilon \to 0$ in (2.9) gives

$$\begin{aligned}c_0 &= \lim_{\varepsilon\to 0} \int_{x'-\varepsilon}^{x'+\varepsilon} -x^{n+1}\,\partial_x\big(x^{1-n}\partial_x a_\lambda(s;x,x')\big)\,dx \\ &= -A_\lambda(s)\,(x')^2\,\mathcal{W}\left[u_\lambda^+(s;x), u_\lambda^0(s;x)\right]_{|x=x'},\end{aligned}$$

where $\mathcal{W}$ denotes the Wronskian. Using the definition of the solutions $u_\lambda^+(s)$ and $u_\lambda^0(s)$ and the formula for the Wronskian of $I_\nu(z)$ and $K_\nu(z)$ (see, for example [27, (10.28.2)]), we find

$$\begin{aligned}A_\lambda(s) &:= \frac{1}{I_\nu(\lambda)} \qquad \text{for } \lambda > 0, \\ A_0(s) &:= 1.\end{aligned} \tag{2.10}$$

Applying (2.9) to $(x - x')\psi_0$, we obtain

$$\begin{aligned}c_1 &= -\lim_{\varepsilon\to 0} \int_{x'-\varepsilon}^{x'+\varepsilon} \big(x^{n+1}(x-x')\big)\,\partial_x\big(x^{1-n}\partial_x a_\lambda(s;x,x')\big)\,dx \\ &= -\lim_{\varepsilon\to 0}\ \varepsilon\left[(x'+\varepsilon)^2\,\partial_x a_\lambda(s;x'+\varepsilon,x') + (x'-\varepsilon)^2\,\partial_x a_\lambda(s;x'-\varepsilon,x')\right] = 0,\end{aligned}$$

and $c_2 = 0$ follows similarly. Hence the definitions (2.10), (2.7) and (2.3) indeed give the resolvent on the model end $X_0$.

From the explicit formulas for $a_\lambda(s;x,x')$ we can read off the model resonance set,

$$\mathcal{R}_0 = \bigcup_{\substack{\lambda^2\in\sigma(\Delta_h)\\ \lambda\neq 0}} \left\{ s \in \mathbb{C} :\ I_{s-\frac{n}{2}}(\lambda) = 0 \right\}. \tag{2.11}$$

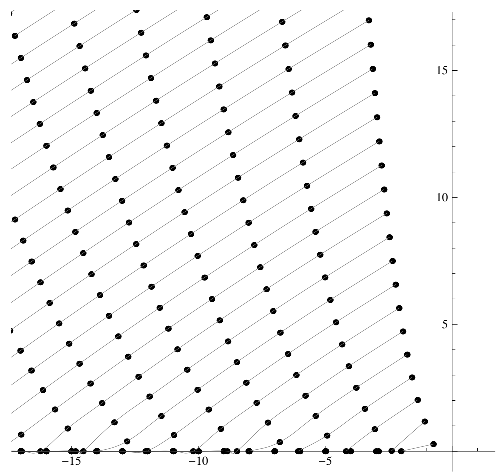


**Figure 2.1:** Resonance plot for the model case $X_0 = (0,1] \times S^2$. The thin lines indicate the spherical harmonic mode $l$, starting from $l = 1$ in the bottom right corner. The multiplicity on each line is $2l + 1$.

For any asymptotically hyperbolic metric, resonances with $\operatorname{Re} s \geq \frac{n}{2}$ could only lie in the interval $[\frac{n}{2}, n]$, by the positivity of the Laplacian and the absence of embedded eigenvalues ([23]). Since it is clear from the classical power series definition that $I_\nu(z) > 0$ for $z > 0$ and $\nu \geq 0$, the resonance set $\mathcal{R}_0$ lies completely in the half-plane $\operatorname{Re} s < \frac{n}{2}$. An example of the model resonance set is shown in Figure 2.1 (self-adjointness implies a conjugation symmetry across the real axis).

We conclude the discussion of the resolvent with a justification of our choice to use $u^0_\lambda(s)$ and $u^+_\lambda(s)$, rather than $u^0_\lambda(s)$ and $u^+_\lambda(n-s)$, for its construction. From (2.4) we derive that $x^s \in L^2(X_0)$ for $\operatorname{Re} s > \frac{n}{2}$. Consequently

$$R_0(s) : L^2(X_0) \to L^2(X_0) \qquad \text{for } \operatorname{Re} s > \tfrac{n}{2},$$

and we see that the condition $a_\lambda(s; x, x') \sim c(s, x')x^s$ at $x = 0$ amounts to choosing $\{\operatorname{Re} s > \frac{n}{2}\}$ to be the *physical* half-plane of the *modified* resolvent $R(s) = (\Delta_0 - s(n-s))^{-1}$.

## 2.2 Poisson operator

The Poisson operator $E_0(s)$ maps functions on $\Sigma$ to solutions of $(\Delta_0 - s(n-s))u = 0$ on $X_0$. These solutions are not actually eigenfunctions, since they do not lie in $L^2(X_0)$, and we call them generalized eigenfunctions. The kernel of $E_0(s)$ is obtained from the resolvent by setting

$$E_0(s;z,\omega') := \lim_{x'\to 0} x'^{-s} R_0(s;z,z'),$$

where $z' = (x',\omega')$. We can thus derive from (2.3) the decomposition

$$E_0(s;x,\omega,\omega') = \sum_{\lambda} b_\lambda(s;x)\phi_\lambda(\omega)\overline{\phi_\lambda(\omega')},$$

where

$$\begin{aligned} b_\lambda(s;x) &= \lim_{x'\to 0} x'^{-s} a_\lambda(s;x,x') \\ &= A_\lambda(s) u_\lambda^0(s;x) \lim_{x'\to 0} \left[ x'^{-s} u_\lambda^+(s;x') \right]. \end{aligned}$$

By the asymptotic (2.2) and for $\nu \notin -\mathbb{N}_0$, this reduces to

$$\begin{aligned} b_\lambda(s;x) &= \frac{1}{\Gamma(\nu+1)} \left(\frac{\lambda}{2}\right)^\nu \frac{u_\lambda^0(s;x)}{I_\nu(\lambda)} \quad \text{for } \lambda \neq 0, \\ b_0(s;x) &= \frac{1}{2\nu} \left[ x^{n-s} - x^s \right]. \end{aligned} \tag{2.12}$$

One can check that given $f \in C^\infty(\Sigma)$, we have

$$\big(\Delta_0 - s(n-s)\big) E_0(s) f = 0.$$

This corresponds to the action of the classical Poisson operator for bounded domains. However, while the classical Poisson operator produces solutions with prescribed boundary data $f$, $E_0(s)$ maps to functions with a certain asymptotic behavior as $x \to 0$. This asymptotic behavior is the subject of the next section.

## 2.3 Scattering matrix

The scattering matrix $S_0(s)$ is derived from the Poisson operator through a two-part asymptotic. For $f \in C^\infty(\Sigma)$ and $\nu \notin \mathbb{Z}$ we have

$$(2s-n)E_0(s)f \sim x^{n-s} f + x^s S_0(s) f, \tag{2.13}$$

as $x \to 0$. In the language of time-independent scattering theory, the terms with $x^{n-s}$ and $x^s$ in (2.13) are called the *incoming* and the *outgoing* part respectively. $S_0(s)$ is diagonalized by the eigenfunctions $\{\phi_\lambda\}_{|\lambda^2 \in \sigma(\Delta_h)}$, and we will use $[S_0(s)]_\lambda$ to denote the corresponding eigenvalues.

From (2.12), we find that for $\nu \notin \mathbb{Z}$ we have

$$\begin{aligned} [S_0(s)]_\lambda &= \left(\frac{\lambda}{2}\right)^{2\nu} \frac{\Gamma(-\nu)}{\Gamma(\nu)} \frac{I_{-\nu}(\lambda)}{I_\nu(\lambda)} \quad \text{for } \lambda \neq 0, \\ [S_0(s)]_0 &= -1. \end{aligned} \tag{2.14}$$

By meromorphic continuation (2.14) extends to $\nu \in \mathbb{Z}$, where the Gamma factors cause zeros and poles. The formulas (2.14) reflect the general symmetry $S(n-s) = S(s)^{-1}$ of the scattering matrix. The factors $\Gamma(-\nu)/\Gamma(\nu)$ appear in each eigenvalue $\lambda > 0$, and hence give rise to infinite-order zeros and poles of $S_0(s)$. Joshi-Sá Barreto [20] showed that $\Gamma(-\nu)/\Gamma(\nu)$ is contained as a factor in the scattering matrix of *any* asymptotically hyperbolic manifold.

The scattering poles of $(X_0, g_0)$ are defined as the poles of the normalized scattering matrix,

$$\tilde{S}_0(s) := \frac{\Gamma(s - \frac{n}{2})}{\Gamma(\frac{n}{2} - s)} S_0(s).$$

In general, the set of scattering poles can differ from the resonance set at the points $s \in \frac{n}{2} - \mathbb{N}$ and at the finite number of points $s$ so that $s(n-s) \in \sigma_p(\Delta) \subset (0, \frac{n^2}{4})$. For the model case, since $I_{-\nu}(z) \neq 0$ for $\operatorname{Re}\nu < 0$ and $z > 0$, as noted above, we see that the set of scattering poles is also given by $\mathcal{R}_0$. Hence we have seen through explicit computations that $S_0(s)$ does not have any of the 'conformal' poles mentioned in the introduction. Since their presence depends only on the structure of the metric at infinity, the same is true for the scattering matrix of $(X, g)$.

# Chapter 3

# Bessel function estimates

Our arguments rely in large part on precise estimates for the model spectral operators discussed in Chapter 2. This means we need approximations of the Bessel functions $I_\nu(z)$ and $K_\nu(z)$ when both $\nu$ and $z$ are large. The estimates given here follow from general techniques developed in Olver [28]. We need asymptotics for $\operatorname{Re}\nu \geq 0$. By the conjugation symmetry, $I_{\bar{\nu}}(z) = \overline{I_\nu(z)}$ for real $z$, it suffices to consider $\nu$ in the first quadrant. To develop the asymptotics we will use $\lambda$ as the large parameter, setting $\nu = \lambda\alpha$ and $z = \lambda x$.

The asymptotic formulas below are essentially derived from a Liouville transformation that takes the modified Bessel equation into a form similar to the Airy equation. The function $u(x) = \sqrt{\lambda x}\, I_{\pm\lambda\alpha}(\lambda x)$ satisfies the differential equation

$$\partial_x^2 u = \Big[\lambda^2 f(\alpha, x) + g(x)\Big]\, u, \tag{3.1}$$

where

$$f(\alpha, x) := \frac{\alpha^2 + x^2}{x^2}, \qquad g(x) := -\frac{1}{4x^2}.$$

In the following we consider $\lambda > 0$, $\arg\alpha \in [0, \frac{\pi}{2}]$, and $x \in \mathbb{R}_+ := (0, \infty)$.

Our goal is to understand the behavior of $u$ for large $\lambda$, and hence the zeros of $f$ are considered to be *turning points* for this equation. The turning point with $\alpha$ in the first quadrant occurs at $\alpha = ix$. To accomplish the Liouville transformation, we introduce

$$\rho(\alpha, x) := \sqrt{\alpha^2 + x^2} + \alpha \log\left(\frac{ix}{\alpha + \sqrt{\alpha^2 + x^2}}\right). \tag{3.2}$$

As noted above, we can restrict our attention to $\alpha$ in the first quadrant, so $\rho(\alpha, x)$ is well-defined using principal branches. (For the sake of comparison, we note that $\rho$ is related to

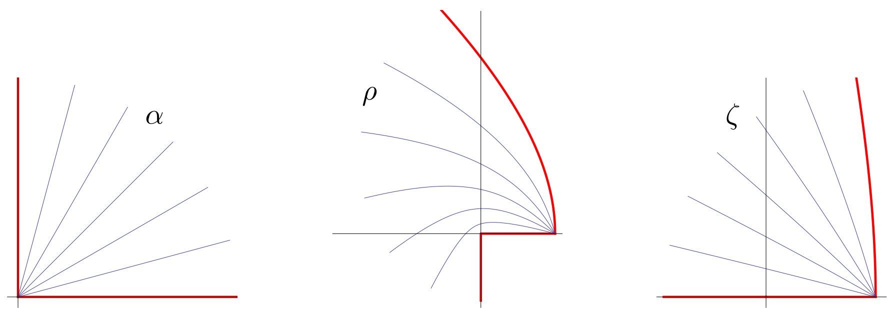


**Figure 3.1:** Illustration of the dependence of $\rho$ and $\zeta$ on $\alpha$, for $\arg\alpha \in [0, \frac{\pi}{2}]$.

the $\xi$ introduced in [28, §10.7.3] by $\rho = \alpha\xi(x/\alpha) + i\alpha\pi/2$.) We then set

$$\zeta := (\tfrac{3}{2}\rho)^{\frac{2}{3}}, \tag{3.3}$$

using the $[0, 2\pi)$ branch of log to define the power. With $\alpha$ in the first quadrant, $\rho$ and $\zeta$ occupy the sectors $\arg\rho \in [0, \frac{3\pi}{2}]$ and $\arg\zeta \in [0, \pi]$, as illustrated in Figure 3.1. The turning point at $\alpha = ix$ corresponds to the corner at $\rho = 0$. The change of variables from $\rho$ to $\zeta$ resolves this singularity.

By these definitions of $\zeta$ and $f$, we have

$$\partial_\zeta = \left(\frac{\zeta}{f}\right)^{\frac{1}{2}} \partial_x.$$

Hence, for $w = (f\zeta^{-1})^{\frac{1}{4}}u$, the equation (3.1) for $u$ transforms to

$$\partial_\zeta^2 w = \big(\lambda^2\zeta + \phi\big)w. \tag{3.4}$$

The error term is given by

$$\begin{aligned} \phi(\alpha, x) &:= \frac{\zeta g}{f} - \frac{1}{4}\,\partial_x^2\Big(\frac{\zeta}{f}\Big) + \frac{3}{16}\left(\frac{\zeta}{f}\right)^{-1}\left(\partial_x\frac{\zeta}{f}\right)^2 \\ &= \frac{5}{16}\frac{1}{\zeta^2} + \zeta\frac{4\alpha^2x^2 - x^4}{4(\alpha^2+x^2)^3}. \end{aligned} \tag{3.5}$$

Without the error term, (3.4) would be the Airy equation, and we will show that any solution of (3.4) can be represented as an Airy function plus an error we can control. This leads to the following:

**Proposition 3.1.** *There exists a constant $M > 0$ such that for $\lambda |\alpha| \geq M$ we have*

$$I_{\lambda\alpha}(\lambda x) = 2^{\frac{1}{2}} \lambda^{-\frac{1}{3}} (-i)^{\lambda\alpha} e^{-\frac{\pi i}{6}} \left( \frac{\zeta}{\alpha^2 + x^2} \right)^{\frac{1}{4}} \mathrm{Ai}\left( e^{-\frac{2\pi i}{3}} \lambda^{\frac{2}{3}} \zeta \right) \left[ 1 + O\big( (\lambda |\alpha|)^{-\frac{2}{3}} \big) \right],$$

$$K_{\lambda\alpha}(\lambda x) = 2^{\frac{1}{2}} \pi \lambda^{-\frac{1}{3}} i^{\lambda\alpha} \left( \frac{\zeta}{\alpha^2 + x^2} \right)^{\frac{1}{4}} \left[ \mathrm{Ai}\big( \lambda^{\frac{2}{3}} \zeta \big) + \langle \lambda^{\frac{2}{3}} \zeta \rangle^{-\frac{1}{4}} e^{-\lambda \operatorname{Re} \rho} O\big( (\lambda |\alpha|)^{-\frac{2}{3}} \big) \right],$$

*uniformly for $\arg \alpha \in [0, \frac{\pi}{2}]$ and $x \in \mathbb{R}_+$.*

The proof of Propositon 3.1 will be deferred to the end of this chapter. The slightly more complicated form of the asymptotic expansion of the $K$-Bessel function is due to the zeros of the Airy function on the negative real axis.

## 3.1 Applications of Proposition 3.1

For the estimates below, it is convenient to rescale $\rho$ to

$$\begin{aligned} \psi(\nu, \lambda x) &:= \lambda \rho(\alpha, x) \\ &= \sqrt{\nu^2 + \lambda^2 x^2} + \nu \log \left( \frac{i\lambda x}{\nu + \sqrt{\nu^2 + \lambda^2 x^2}} \right). \end{aligned} \tag{3.6}$$

We can now combine Proposition 3.1 with the basic Airy function estimates recalled in Appendix B. In the $I$-Bessel case, (B.1) applies for all $\alpha$ in the first quadrant. For the $K$-Bessel function, we also need (B.3). This yields the following:

**Corollary 3.2.** *For $\arg \nu \in [0, \frac{\pi}{2}]$, $\lambda > 0$, and $x \in \mathbb{R}_+$, with $\nu$ sufficiently large, we have*

$$I_\nu(\lambda x) = \frac{1}{\sqrt{2\pi}} (\nu^2 + \lambda^2 x^2)^{-\frac{1}{4}} e^{-i\pi\nu/2} e^{\psi} \left[ 1 + O(\psi^{-1}) + O(\nu^{-\frac{2}{3}}) \right]. \tag{3.7}$$

*(Recall that $\psi = 0$ for $\nu = i\lambda x$.)*

*Similarly, for $\nu$ sufficiently large and $\arg(\nu - i\lambda x) \leq \frac{\pi}{2} - \varepsilon$ (which implies $\arg \zeta \in [0, \pi - \delta]$ for some $\delta$ depending on $\varepsilon$),*

$$K_\nu(\lambda x) = \sqrt{\frac{\pi}{2}} (\nu^2 + \lambda^2 x^2)^{-\frac{1}{4}} e^{i\pi\nu/2} e^{-\psi} \left[ 1 + O(\psi^{-1}) + O(\nu^{-\frac{2}{3}}) \right]. \tag{3.8}$$

*If $\arg(\nu - i\lambda x) \in [\frac{\pi}{2} - \varepsilon, \frac{\pi}{2}]$, then (3.8) is replaced by*

$$K_\nu(\lambda x) = \sqrt{\frac{\pi}{2}} (\nu^2 + \lambda^2 x^2)^{-\frac{1}{4}} e^{i\pi\nu/2} \left[ \big( e^{-\psi} + i e^{\psi} \big) \big( 1 + O(\psi^{-1}) \big) + e^{-\psi} O(\nu^{-\frac{2}{3}}) \right].$$

*(Under this condition,* $\operatorname{Re}\psi \leq 0$*, so the extra term* $ie^{\psi}$ *is* $O(1)$ *and does not affect upper bounds for* $K_\nu$*.)*

These estimates do not apply near the 'turning point' $\nu = i\lambda x$ of the transformed Bessel equation, where $\psi = 0$. We use the symbol $\asymp$ to indicate that the absolute value of the ratio of the two sides is bounded above and below by non-zero constants that do not depend on the variables (and similarly: $\preceq$ and $\succeq$).

**Corollary 3.3.** *For* $\arg\nu \in [0, \frac{\pi}{2}]$ *and* $x$ *in a compact interval of* $\mathbb{R}_+$*, suppose that* $\nu$ *is close to* $i\lambda x$ *in the sense that* $|\psi| < c^*$ *with* $c^*$ *sufficiently small. Under these conditions, for* $|\nu|$ *sufficiently large we have*

$$\begin{aligned} I_\nu(\lambda x) &\asymp (\lambda x)^{-\frac{1}{3}} e^{-i\pi\nu/2}, \\ K_\nu(\lambda x) &\asymp (\lambda x)^{-\frac{1}{3}} e^{i\pi\nu/2}. \end{aligned} \tag{3.9}$$

*Proof.* To estimate near the turning point, suppose that $\alpha = ix + \eta$ with $\operatorname{Re}\eta \geq 0$, and $\nu = \lambda\alpha$ as above. For $\eta$ sufficiently small and $x > 0$ we have

$$\rho = \frac{2\sqrt{2}}{3} x^{-\frac{1}{2}} (i\eta)^{\frac{3}{2}} (1 + O(\eta)). \tag{3.10}$$

This means $\psi \asymp \lambda x^{-\frac{1}{2}} \eta^{\frac{3}{2}}$, so that $|\psi| \leq c^*$ corresponds to $|\eta| \preceq \lambda^{-\frac{2}{3}} x^{\frac{1}{3}}$ with a constant that depends only on $c^*$. In particular, for any choice of $c^*$, (3.10) applies if $\lambda$ is sufficiently large.

Consider the estimates of Proposition 3.1, and note that the assumption $|\psi| \leq c^*$ means that $|\lambda^{\frac{2}{3}}\zeta|$ is bounded by some constant depending only on $c^*$. Hence by our choice of $c^*$ we can ensure that $\lambda^{\frac{2}{3}}\zeta$ avoids the first zero of the Airy function, so that the Airy function factors in Proposition 3.1 are bounded away from zero. We also note that

$$\frac{\zeta}{\alpha^2 + x^2} \asymp x^{-\frac{4}{3}},$$

for $\eta$ sufficiently small. The estimates then follow immediately from Proposition 3.1. □

In addition to the estimates given above for $I_\nu$, $K_\nu$, which extend to $\operatorname{Re}\nu \geq 0$ by conjugation, we will need to be able to control the ratio $I_{-\nu}/I_\nu$. This ratio appears, for example, in the scattering matrix. We can derive the necessary estimates from the results above using the identity

$$I_{-\nu}(z) = I_\nu(z) + \frac{2\sin\pi\nu}{\pi} K_\nu(z). \tag{3.11}$$

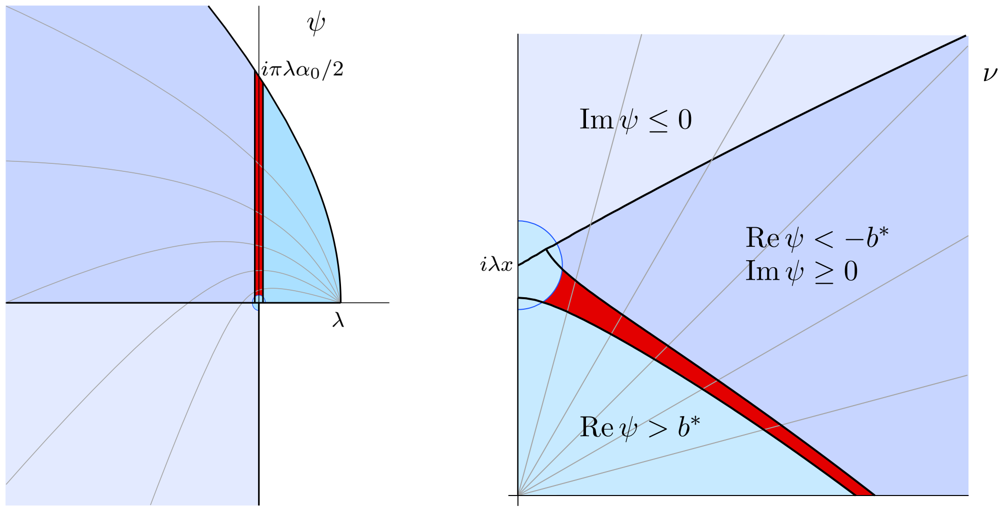


**Figure 3.2:** Regions for the estimates in Lemmas 3.4 and 3.5. The red zone contains the non-trivial zeros of $I_{-\nu}(\lambda x)$.

We first consider estimates away from the zeros of $I_{-\nu}(\lambda x)$.

**Lemma 3.4.** *Assume that* $\arg \nu \in [0, \frac{\pi}{2}]$, $\lambda > 0$, *$x$ is restricted to a compact interval of* $\mathbb{R}_+$, *and* $|\nu| \geq M$, *with $M$ sufficiently large. There exist constants $\delta > 0$ and $c^* > b^* > 0$ such that:*

*(i) For either* $\operatorname{Re}\psi > b^*$ *or* $|\psi| < c^*$,

$$\frac{I_{-\nu}(\lambda x)}{I_\nu(\lambda x)} \asymp 1. \tag{3.12}$$

*(ii) For* $\operatorname{Im}\psi \geq 0$, $\operatorname{Re}\psi < -b^*$ *and (for the lower bound)* $d(\nu, \mathbb{N}_0) \geq \delta$,

$$\frac{I_{-\nu}(\lambda x)}{I_\nu(\lambda x)} \asymp e^{-2\psi}. \tag{3.13}$$

*(iii) For* $\operatorname{Im}\psi \leq 0$ *(which occurs only when* $\operatorname{Re}\psi \leq 0$ *also),*

$$\frac{I_{-\nu}(\lambda x)}{I_\nu(\lambda x)} \asymp e^{-2\psi}. \tag{3.14}$$

*The constants in* (3.12), (3.13), *and* (3.14) *depend only on* $M$, $c^*$, $b^*$, *and* $\delta$.

*Proof.* By (3.11) we need to estimate

$$\frac{I_{-\nu}(z)}{I_\nu(z)} = 1 + \frac{2\sin\pi\nu}{\pi}\frac{K_\nu(z)}{I_\nu(z)}. \tag{3.15}$$

The approximations of Proposition 3.1 lead to

$$\frac{I_{-\nu}(\lambda x)}{I_\nu(\lambda x)} = 1 + e^{\frac{2\pi i}{3}}\big(1 - e^{2\pi i\nu}\big)\frac{\Big[\mathrm{Ai}\big((\tfrac{3}{2}\psi)^{\frac{2}{3}}\big) + \langle(\tfrac{3}{2}\psi)^{\frac{2}{3}}\rangle^{-\frac{1}{4}}e^{-\psi}\,O(\nu^{-\frac{2}{3}})\Big]}{\big(1+O(\nu^{-\frac{2}{3}})\big)\,\mathrm{Ai}\big(e^{-\frac{2\pi i}{3}}(\tfrac{3}{2}\psi)^{\frac{2}{3}}\big)}. \tag{3.16}$$

(i) For $\mathrm{Im}\,\psi \succeq \lambda$ (which is implied by $|\psi| < c^*$ and by $\mathrm{Im}\,\psi \le 0$), we can drop the term $e^{2\pi i\nu}$ in (3.16), and applying (B.2) to $\mathrm{Ai}\big((\tfrac{3}{2}\psi)^{\frac{2}{3}}\big)$ in the denominator yields

$$\frac{I_{-\nu}(\lambda x)}{I_\nu(\lambda x)} = e^{\frac{\pi i}{3}}\,\frac{\mathrm{Ai}\big(e^{-\frac{4\pi i}{3}}(\tfrac{3}{2}\psi)^{\frac{2}{3}}\big) + \langle(\tfrac{3}{2}\psi)^{\frac{2}{3}}\rangle^{-\frac{1}{4}}e^{-\psi}O(\nu^{-\frac{2}{3}})}{\mathrm{Ai}\big(e^{-\frac{2\pi i}{3}}(\tfrac{3}{2}\psi)^{\frac{2}{3}}\big)}\,\big[1+O(\nu^{-\frac{2}{3}})\big] + O(\nu^{-\frac{2}{3}}). \tag{3.17}$$

Since $\mathrm{Ai}(0) \neq 0$, the asymptotic (3.12) for $|\psi| < c^*$ follows if $c^*$ is sufficiently small.

In the case $\mathrm{Re}\,\psi > b^*$, we have for the arguments of the Airy functions in Proposition 3.1 that

$$\arg(e^{-\frac{2\pi i}{3}}\lambda^{\frac{2}{3}}\zeta) \in [-\tfrac{2\pi}{3}, -\tfrac{\pi}{3}], \qquad \arg(\lambda^{\frac{2}{3}}\zeta) \in [0, \tfrac{\pi}{3}].$$

Hence, when applying (B.1) in the derivation of (3.7) and (3.8), we can use $\varepsilon = \frac{\pi}{3}$ and $\varepsilon = \frac{2\pi}{3}$ (c.f. Appendix B) to obtain the constants 0.79 and 0.12 for the bounds on the respective $O(\psi^{-1})$ error terms. Then (3.7) and (3.8) show that for $|\psi| \ge c^*, \mathrm{Re}\,\psi > b^*$ and $|\nu| \ge M$ we have

$$\left|\frac{2\sin\pi\nu}{\pi}\frac{K_\nu(\lambda x)}{I_\nu(\lambda x)}\right| < e^{-2b^*}\left|1+O(\nu^{-\frac{2}{3}})\right|\frac{c^*+0.12}{c^*-0.79},$$

and, comparing to (3.15), the claim follows if $b^*$ is sufficiently large. Since the first zero of $\mathrm{Ai}(w)$ occurs at $w \approx -2.338$, we conclude that $b^* < c^*$ can be satisfied.

(ii) For $|\psi| \ge c^*, \mathrm{Im}\,\psi \ge 0$, $\mathrm{Re}\,\psi < -b^*$, Corollary 3.2 gives

$$\frac{2\sin\pi\nu}{\pi}\frac{K_\nu(\lambda x)}{I_\nu(\lambda x)} = i(1-e^{2\pi i\nu})e^{-2\psi}\big(1+O(\nu^{-\frac{2}{3}})\big)\big(1+O(\psi^{-1})\big),$$

where the $1 + O(\psi^{-1})$ error is the ratio of the corresponding terms in (3.8) and (3.7), and is bounded below by $(c^* - 0.79)/(c^* + 0.12)$. Hence, if $\mathrm{Re}\,\psi < -b^*$ and provided $\delta$ is not too small, the $K/I$ term dominates (3.15).

(iii) Under the assumption $\operatorname{Im}\psi \le 0$, the arguments of the Airy functions in (3.17) lie in $|\arg w| \le \frac{2\pi}{3}$. We apply (B.1) to obtain

$$\frac{I_{-\nu}(\lambda x)}{I_\nu(\lambda x)} = ie^{-2\psi}\big(1 + O(\psi^{-1}) + O(\nu^{-\frac{2}{3}})\big) + O(\nu^{-\frac{2}{3}}).$$

For $|\psi| \ge c^*$ and $M$ sufficiently large, the multiplicative error term is bounded below by $(c^* - 0.79)/(c^* + 0.12)$. This completes the proof. □

Lemma 3.4 leaves out a region where $|\psi| \ge c^*$, $\operatorname{Im}\psi \ge 0$ and $|\operatorname{Re}\psi| \le b^*$, as illustrated in Figure 3.2. In this zone lower bounds are more delicate because it contains a non-trivial portion of the zero set

$$\mathcal{Z}_{\lambda x} := \{\nu : \ I_{-\nu}(\lambda x) = 0\}.$$

**Lemma 3.5.** *Assume that* $\arg\nu \in [0, \frac{\pi}{2}]$, *$x$ is restricted to a compact interval of* $\mathbb{R}_+$, *and* $|\nu| \ge M$, *with $M$ large enough that the estimates from Proposition 3.1 apply, and that* $\operatorname{Im}\psi \ge 0$ *and* $|\operatorname{Re}\psi| \le b^*$. *Then*

$$\left|\frac{I_{-\nu}(\lambda x)}{I_\nu(\lambda x)}\right| \le C_{M,b^*}.$$

*If in addition we assume that* $d(\nu, \mathcal{Z}_{\lambda x}) \ge \langle\nu\rangle^{-\beta}$ *for some* $\beta > 0$, *then*

$$\log\left|\frac{I_{-\nu}(\lambda x)}{I_\nu(\lambda x)}\right| \ge -c_{M,\beta}\,|\nu|\log|\nu|\,.$$

*Proof.* By the estimates in Lemma 3.4, we can see that for $|\nu| \ge M$ with $M$ sufficiently large,

$$\left|\frac{I_{-\nu}(\lambda x)}{I_\nu(\lambda x)}\right| = O(1),$$

for $\nu$ on the boundary of the region in question, with constants that are independent of $\nu$ and $\lambda$. The upper bound follows immediately.

For the lower bound we apply the minimum modulus theorem in the form [21, Thm 1.11] to $f(\nu) := I_\nu(\lambda x)/I_0(\lambda x)$ (normalized so $f(0) = 1$). For $\eta > 0$ sufficiently small and $m > 0$ fixed, inside the disk $|\nu| \le m\lambda$, but excluding a set of disks whose radii sum to at most $4m\eta\lambda$, we have

$$\log\left|\frac{I_\nu(\lambda x)}{I_0(\lambda x)}\right| > -\left(3 + \log\frac{3}{2\eta}\right)\log\left(\sup_{|z|=2me\lambda}\left|\frac{I_z(\lambda x)}{I_0(\lambda x)}\right|\right). \tag{3.18}$$

Since

$$|\mathrm{Re}\,\psi(\nu,\lambda x)| = O(\lambda), \quad \text{for } |\nu| \le C\lambda,$$

we can apply Corollary 3.2 (or Corollary 3.3 in case $2me$ is close to 1) and (3.11) to deduce that for any $m > 0$,

$$\log |I_\nu(\lambda x)| \le C_m \lambda, \quad \text{for } |\nu| \le 2me\lambda,$$

for $\lambda$ sufficiently large. For the $I_0$ term the standard Bessel function asymptotic gives $I_0(\lambda x) \sim (2\pi\lambda x)^{-1/2} e^{\lambda x}$. Combining these estimates with (3.18) thus gives a lower bound

$$\log |I_\nu(\lambda x)| > -c_m(1 + \log \eta^{-1})\lambda, \tag{3.19}$$

for $|\nu| \le m\lambda$, excluding a set of disks whose radii sum to at most $4m\eta\lambda$.

Now we wish to apply the estimate to the region described in the lemma, in which $|\nu| \asymp \lambda$ and $d(\nu, \mathcal{Z}_{\lambda x}) \ge \langle \nu \rangle^{-\beta}$. We can fix $m$ independently of $\lambda$ and choose $\eta = \kappa\lambda^{-\beta-1}$. For $\kappa$ sufficiently small, the hypotheses of (3.19) will be satisfied for all $\nu, \lambda$ in the region of interest. For $\lambda$ sufficiently large, the claimed lower bound then follows from (3.19), with the extra $\log |\nu|$ coming from the variable choice of $\eta$. □

## 3.2 Proof of Proposition 3.1

We begin by making an ansatz for solutions to (3.4), in the form

$$w_\sigma := \mathrm{Ai}\big(\lambda^{\frac{2}{3}} e^{\frac{2\pi i \sigma}{3}} \zeta\big) + h_\sigma(\lambda, \alpha, x), \tag{3.20}$$

for $\sigma \in \{-1, 0, 1\}$. The differential equations for the error terms follow directly from (3.4),

$$(\partial_\zeta^2 - \lambda^2 \zeta) h_\sigma = \phi\big[h_\sigma + \mathrm{Ai}\big(\lambda^{\frac{2}{3}} e^{\frac{2\pi i \sigma}{3}} \zeta\big)\big]. \tag{3.21}$$

Our goal is to derive bounds on $h_\sigma$ from this equation.

*Step 1.* To determine the appropriate boundary conditions for (3.21), we need to identify the modified Bessel functions $I$ and $K$ with the $w_\sigma$ in the ansatz (3.20). As $|z| \to \infty$, the Airy function $\mathrm{Ai}(z)$ decreases exponentially in the sector $|\arg z| < \frac{\pi}{3}$, and it increases exponentially for $|\arg z| \in (\frac{\pi}{3}, \pi)$. Note that $\rho \sim x$ for $x \to \infty$, and consequently $|\zeta| \to \infty$ with $\arg \zeta \to 0$ in this limit. $K_{\lambda\alpha}(\lambda x)$ decays exponentially for $x \to \infty$. Hence the equation

$$\big(f\zeta^{-1}\big)^{\frac{1}{4}} \sqrt{x}\, K_{\lambda\alpha}(\lambda x) = c_K(\lambda, \alpha)\, w_\sigma(\lambda, \alpha, x), \tag{3.22}$$

implies $\sigma = 0$ as well as the boundary condition

$$h_0 = O(x^{-2}) \qquad \text{for } x \to \infty.$$

On the other hand, if $\operatorname{Re}\alpha > 0$, then $I_{\lambda\alpha}(\lambda x) \to 0$ as $x \to 0$, and for $\operatorname{Re}\alpha = 0$ we have $I_{\lambda\alpha}(\lambda x) = O(1)$. For $x \to 0$ we have $\rho \sim \alpha \log x$, and consequently $x \to 0$ takes $\zeta \to \infty$ in the sector $\left[\frac{2\pi}{3}, \pi\right]$. This shows that we need $\sigma = -1$ in (3.20) for an approximation of the $I$-Bessel function.

As the two cases are quite similar, in the following we will present the detailed arguments only for the case $\sigma = 0$. We then indicate briefly how to handle the $\sigma = -1$ case at the end.

*Step 2.* The next step is to transform the equation (3.21) with $\sigma = 0$ into an integral equation. Note that for the homogeneous version of (3.21) we can use the independent solutions

$$\operatorname{Ai}\bigl(\lambda^{\frac{2}{3}}\zeta\bigr), \qquad \operatorname{Ai}\bigl(\lambda^{\frac{2}{3}}e^{-\frac{2\pi i}{3}}\zeta\bigr).$$

From the well-known formula for the Wronskian of Airy functions (see, e.g. [27, (9.2.8)]), we have

$$\mathcal{W}\left[\operatorname{Ai}(z), \operatorname{Ai}(e^{\pm\frac{2\pi i}{3}}z)\right] = \frac{1}{2\pi}e^{\mp\frac{i\pi}{6}}.$$

If we apply the method of variation of parameters to (3.21), treating the entire right-hand side as the source term, and taking into account the boundary condition $h_0(x) = O(x^{-2})$ as $x \to \infty$, then the result is a recursive integral equation,

$$h_0(\lambda,\alpha,x) = \frac{-2\pi}{e^{\frac{i\pi}{6}}\lambda^{\frac{2}{3}}}\int_x^\infty K(x,y)\phi(y)\bigl[h_0(\lambda,\alpha,y) + \operatorname{Ai}\bigl(\lambda^{\frac{2}{3}}\zeta(y)\bigr)\bigr]\sqrt{\frac{f(y)}{\zeta(y)}}\,dy, \tag{3.23}$$

where

$$K(x,y) := \operatorname{Ai}\bigl(\lambda^{\frac{2}{3}}\zeta(y)\bigr)\operatorname{Ai}\bigl(\lambda^{\frac{2}{3}}e^{-\frac{2\pi i}{3}}\zeta(x)\bigr) - \operatorname{Ai}\bigl(\lambda^{\frac{2}{3}}e^{-\frac{2\pi i}{3}}\zeta(y)\bigr)\operatorname{Ai}\bigl(\lambda^{\frac{2}{3}}\zeta(x)\bigr).$$

*Step 3.* We next want to estimate the kernel appearing in the integral equation (3.23), so that we can apply the method of successive approximations as outlined in Appendix C. Note that (B.1) and (B.3) imply a global upper bound,

$$|\operatorname{Ai}(w)| \le C\langle w\rangle^{-\frac{1}{4}}\exp\bigl(-\operatorname{Re}\left[\tfrac{2}{3}w^{\frac{3}{2}}\right]\bigr),$$

valid for all $w \in \mathbb{C}$. This estimate and the monotonicity of $\operatorname{Re}\rho(\alpha, .\,)$ imply that for $x \le y$

we have

$$|K(x,y)| \le p(x)q(y),$$

where

$$p(x) = p(\lambda,\alpha,x) = \langle \lambda^{\frac{2}{3}}\zeta(\alpha,x)\rangle^{-\frac{1}{4}} e^{-\lambda \operatorname{Re}\rho(\alpha,x)},$$
$$q(y) = q(\lambda,\alpha,y) = c\langle \lambda^{\frac{2}{3}}\zeta(\alpha,y)\rangle^{-\frac{1}{4}} e^{\lambda \operatorname{Re}\rho(\alpha,y)}.$$

The supremum

$$\kappa := \sup_{x\in\mathbb{R}_+} p(x)q(x),$$

is bounded independently of $\lambda$ and $\alpha$.

Applying the method of successive approximations from Appendix C to the integral equation (3.23) leads to an estimate of the error $h_0$ in the ansatz (3.20). The supremum

$$\tilde\kappa := \sup_{y\in\mathbb{R}_+} q(y)\left|\operatorname{Ai}\bigl(\lambda^{2/3}\zeta(y)\bigr)\right|,$$

is bounded independently of $\lambda$ and $\alpha$, and hence we have

$$|h_0(x)| \le p(x)\frac{\tilde\kappa}{\kappa}\bigl[\exp(\kappa\Phi(x)) - 1\bigr], \tag{3.24}$$

where

$$\Phi(x) := c\lambda^{-\frac{2}{3}}\int_x^\infty \left|\bigl(\phi f^{\frac{1}{2}}\zeta^{-\frac{1}{2}}\bigr)(\alpha,t)\right|\,dt,$$

with $\phi$ given by (3.5).

*Step 4.* From (3.24) we see that for a uniform error bound it suffices to estimate the integral

$$M(\alpha) := \int_0^\infty \left|\bigl(\phi f^{\frac{1}{2}}\zeta^{-\frac{1}{2}}\bigr)(\alpha,x)\right|\,dx. \tag{3.25}$$

Before showing that the integral converges, we analyze the dependence of the integrand on $|\alpha|$. First observe that $\zeta(\alpha,x) = \alpha^{\frac{2}{3}}\zeta(1,\frac{x}{\alpha})$, so that from (3.5) we can see that $\phi$ scales as

$$\phi(\alpha,x) = \alpha^{-\frac{4}{3}}\phi(1,\tfrac{x}{\alpha}).$$

Substituting $z = \frac{x}{\alpha}$, we obtain

$$\int_0^\infty \left|\bigl(\phi f^{\frac{1}{2}}\zeta^{-\frac{1}{2}}\bigr)(\alpha,x)\right|\,dx = |\alpha|^{-\frac{2}{3}}\int_{\{\arg z = -\arg\alpha\}} \left|\bigl(\phi f^{\frac{1}{2}}\zeta^{-\frac{1}{2}}\bigr)(1,z)\right|\,d|z|\,. \tag{3.26}$$

Recall the definition (3.2) of $\rho = \frac{2}{3}\zeta^{\frac{3}{2}}$. For $\delta > 0$ sufficiently small we have

$$\rho(1,z) \asymp \begin{cases} z & |z| \geq \frac{1}{2}, |z+i| \geq \delta \\ -\log z & |z| \leq \frac{1}{2} \\ |z+i|^{\frac{3}{2}} & |z+i| \leq \delta, \end{cases} \tag{3.27}$$

where $z$ lies in the fourth quadrant of the complex plane, $\arg z \in [-\frac{\pi}{2}, 0]$. The behavior close to $z = -i$ is found via a Taylor expansion. We now consider the contributions of the regions in (3.27) to the integral on the right hand side of (3.26) separately. Let $\theta = -\arg\alpha$.

Denote by $J_1$ the set $\{z \in \mathbb{C} : \arg z = \theta, |z| \geq \frac{1}{2}, |z+i| \geq \delta\}$. With the definitions of $\zeta$ and $f$, and the formula (3.5) for $\phi$, we find

$$\int_{J_1} \left| (\phi f^{\frac{1}{2}} \zeta^{-\frac{1}{2}})(1,z) \right| \, d|z| \leq c \int_{\frac{1}{2}}^{\infty} |z|^{-\frac{5}{3}} \, d|z| < \infty,$$

independently of $\theta$.

The contribution of $J_2 := \{\arg z = \theta, |z| \leq \frac{1}{2}\}$ to the integral is bounded by

$$c_1 \int_0^{\frac{1}{2}} |z|^{-1} (-\log|z|)^{-\frac{5}{3}} \, d|z| + c_2 \int_0^{\frac{1}{2}} |z| \left(\log^2|z| + \tfrac{\pi^2}{4}\right)^{\frac{1}{6}} d|z| < \infty.$$

The third and final region to consider is $J_3 := \{\arg z = \theta, |z+i| \leq \delta\}$ The integrand of (3.26) is bounded near $z = -i$, and hence the contribution from $J_3$ is uniformly bounded as well.

We conclude from these estimates that

$$M(\alpha) = O\big(|\alpha|^{-\frac{2}{3}}\big). \tag{3.28}$$

By (3.24) we find that for $\lambda|\alpha|$ sufficiently large,

$$|h_0(x)| \leq c(\lambda|\alpha|)^{-\frac{2}{3}} p(x).$$

After comparing to the ansatz (3.22), we have now established that

$$K_{\lambda\alpha}(\lambda x) = c_K(\lambda,\alpha)\Big(\frac{\zeta}{\alpha^2+x^2}\Big)^{\frac{1}{4}} \Big[\mathrm{Ai}\big(\lambda^{\frac{2}{3}}\zeta\big) + p(x)\, O\big((\lambda|\alpha|)^{-\frac{2}{3}}\big)\Big]. \tag{3.29}$$

*Step 5.* It remains to fix the constant $c_K$ from (3.22), i.e. to compare the behavior for

$x \to \infty$ on the left and on the right of this equation. As $x \to \infty$,

$$\rho(\alpha, x) = x + \alpha \log i + O(x^{-1}),$$

so that

$$\left(\frac{\zeta}{\alpha^2 + x^2}\right)^{\frac{1}{4}} \mathrm{Ai}\left(\lambda^{\frac{2}{3}}\zeta\right) \sim \frac{x^{-\frac{1}{2}}}{2\pi^{\frac{1}{2}}} \lambda^{-\frac{1}{6}} e^{-\lambda(x + \alpha \log i)},$$

as $x \to \infty$. On the other hand,

$$K_{\lambda\alpha}(\lambda x) \sim \left(\frac{\pi}{2\lambda x}\right)^{\frac{1}{2}} e^{-\lambda x},$$

as $x \to \infty$. We conclude that

$$c_K(\lambda, \alpha) = 2^{\frac{1}{2}} \pi \lambda^{-\frac{1}{3}} e^{\lambda\alpha \log i}. \tag{3.30}$$

The combination of (3.29) and (3.30) completes the proof of the $K$-Bessel asymptotic in Propostion 3.1. As we noted above, the $I$-Bessel estimate follows from very similar arguments with $\sigma = -1$. In this case the integral for the error term is

$$h_{-1}(\lambda, \alpha, x) = \frac{2\pi}{e^{\frac{i\pi}{6}} \lambda^{\frac{2}{3}}} \int_0^x K(x, y)\phi(y) \big[h_{-1}(\lambda, \alpha, y) + \mathrm{Ai}\big(\lambda^{\frac{2}{3}} e^{-\frac{2\pi i}{3}} \zeta(y)\big)\big] \sqrt{\frac{f(y)}{\zeta(y)}}\, dy.$$

The key difference is that range of integration now starts from 0, but the method of successive approximations applies in the same way. From (3.28) we can derive

$$I_{\lambda\alpha}(\lambda x) = c_I(\lambda, \alpha) \left(\frac{\zeta}{\alpha^2 + x^2}\right)^{\frac{1}{4}} \mathrm{Ai}\left(e^{-\frac{2\pi i}{3}} \lambda^{\frac{2}{3}} \zeta\right) \left[1 + O\big((\lambda\, |\alpha|)^{-\frac{2}{3}}\big)\right]. \tag{3.31}$$

To compute $c_I(\lambda, \alpha)$, we note that as $x \to 0$,

$$\rho(\alpha, x) = \alpha \log x + \alpha + \alpha \log \frac{i}{2\alpha} + O(x^2).$$

With $\alpha$ in the first quadrant, $x \to 0$ takes $e^{-\frac{2\pi i}{3}} \zeta \to \infty$ in the sector $(0, \frac{\pi}{3})$. In this limit,

$$\left(\frac{\zeta}{\alpha^2 + x^2}\right)^{\frac{1}{4}} \mathrm{Ai}\left(e^{-\frac{2\pi i}{3}} \lambda^{\frac{2}{3}} \zeta\right) \sim \frac{\alpha^{-\frac{1}{2}}}{2\pi^{\frac{1}{2}}} e^{\frac{\pi i}{6}} \lambda^{-\frac{1}{6}} e^{\lambda[\alpha \log x + \alpha + \alpha \log \frac{i}{2\alpha}]}.$$

Comparing this to the asymptotic

$$I_{\lambda\alpha}(\lambda x) \sim \frac{1}{\Gamma(\lambda\alpha+1)} \left(\frac{\lambda x}{2}\right)^{\lambda\alpha},$$

as $x \to 0$, we find that

$$c_I(\lambda, \alpha) = \frac{2\pi^{\frac{1}{2}}}{\Gamma(\lambda\alpha+1)} \left(\frac{\lambda}{2}\right)^{\lambda\alpha} \alpha^{\frac{1}{2}} e^{-\frac{\pi i}{6}} \lambda^{\frac{1}{6}} e^{-\lambda[\alpha+\alpha\log\frac{i}{2\alpha}]}.$$

In conjunction with (3.31) and Stirling's formula,

$$\Gamma(\nu+1) = \sqrt{2\pi}\, e^{(\nu+\frac{1}{2})\log\nu-\nu}(1+O(\nu^{-1})), \quad \text{for } |\arg\nu| \leq \pi-\varepsilon, \tag{3.32}$$

this completes the proof for the $I$-Bessel case.

# Chapter 4

# Resonance order of growth

For an asymptotically hyperbolic manifold $(X, g)$ with warped-product ends, the model estimates of the previous chapters lead to a growth estimate on the resonance counting function $N_g(t)$. The basic technique is the Fredholm determinant method of Melrose [25, 26], as adapted to the hyperbolic setting by Guillopé-Zworski [18]. Indeed, the only real difference in our proof from that of [18] lies in the model estimates proven in Chapter 2.

Let $R_0(s)$ denote the resolvent for the model end $X_0 = (0, 1] \times \Sigma$, as studied in Chapter 2. The resonance set $\mathcal{R}_0$ was identified explicitly in (2.11), and we let $N_0(t)$ denote the corresponding counting function. In Proposition 6.3 we will show that

$$N_0(t) \sim c\, t^{n+1}, \tag{4.1}$$

and compute the constant explicitly. The main goal of this chapter is to prove the following:

**Proposition 4.1.** *Let $(X, g)$ be a conformally compact manifold with asymptotically hyperbolic warped-product ends. Then the resonance counting function satisfies*

$$N_g(t) = O((t \log t)^{n+1}).$$

The bound in Proposition 4.1 is not optimal and will be refined later in Chapter 6.

## 4.1 Spectral operator estimates

Our first step is to apply the estimates from Chapter 3 to the formulas for the model resolvent, Poisson operator, and scattering matrix from Chapter 2. For the resolvent, we only need estimates in the physical half-plane, $\operatorname{Re} s \geq \frac{n}{2}$. Throughout this section we will make use of functions in $C_0^\infty(0,1)$ as cutoff functions on $X_0$ that depend only on the $x$ variable.

**Proposition 4.2.** *Suppose $\chi_1, \chi_2 \in C_0^\infty(0,1)$ are cutoff functions with disjoint supports and $\sigma \geq 0$. Then for $\operatorname{Re}(s - \frac{n}{2}) \geq \varepsilon$, we have*

$$\|\chi_1 R_0(s) \chi_2\|_{\mathcal{L}(H^0, H^\sigma)} \leq C_{\varepsilon,\sigma} \langle s \rangle^{-1+\sigma}.$$

*For $0 \leq \operatorname{Re}(s - \frac{n}{2}) \leq \varepsilon$, with $\left|s - \frac{n}{2}\right| \geq \varepsilon$, we have*

$$\|\chi_1 R_0(s) \chi_2\|_{\mathcal{L}(H^0, H^\sigma)} \leq C_{\varepsilon,\sigma} \langle s \rangle^{-\frac{2}{3}+\sigma}.$$

*Proof.* By a standard argument involving resolvent identities, it suffices to prove the estimates for $\sigma = 0$ (see, e.g. [2, Lemma 9.8]).

The first bound depends only on the location of the spectrum. Since there is no discrete spectrum by (2.11) and the remark following, we have $\sigma(\Delta_0) = [\frac{n^2}{4}, \infty)$. From the spectral theorem and the fact that

$$d(s(n-s), \sigma(\Delta_0)) = \begin{cases} \left|s - \frac{n}{2}\right|^2 & \operatorname{Re}(s - \frac{n}{2}) \geq |\operatorname{Im} s| \\ 2\left|\operatorname{Re}(s - \frac{n}{2}) \operatorname{Im} s\right| & \operatorname{Re}(s - \frac{n}{2}) \leq |\operatorname{Im} s|, \end{cases}$$

we find that

$$\|R_0(s)\| \leq C_\varepsilon \langle s \rangle^{-1},$$

for $\operatorname{Re} s \geq \frac{n}{2} + \varepsilon$.

For the bound near the critical line we turn to the decomposition (2.3). Since the cutoffs yield a smoothing operator with compactly supported coefficients, it suffices to obtain pointwise estimates of the coefficients $a_\lambda$. By symmetry we need only consider $x_1 < x_2$, in which case we have

$$a_\lambda(s; x_1, x_2) = \frac{\pi}{2 \sin \pi\nu} (x_1 x_2)^{\frac{n}{2}} I_\nu(\lambda x_1) \left[ I_{-\nu}(\lambda x_2) - \frac{I_{-\nu}(\lambda)}{I_\nu(\lambda)} I_\nu(\lambda x_2) \right], \tag{4.2}$$

or, using (3.11),

$$a_\lambda(s; x_1, x_2) = (x_1 x_2)^{\frac{n}{2}} I_\nu(\lambda x_1) \left[ K_\nu(\lambda x_2) - \frac{K_\nu(\lambda)}{I_\nu(\lambda)} I_\nu(\lambda x_2) \right]. \tag{4.3}$$

Here $\nu := s - \frac{n}{2}$ as always.

We may assume that $|\nu| \geq M$ such that the estimates of Proposition 3.1 apply. First we consider the case away from the turning point. That is, we assume $|\psi| \geq c^*$ for any of the values $x = 1$, $x_1$ or $x_2$. By the analysis used in the proof of Corollary 3.3, this corresponds to an assumption that $|\nu - i\lambda x| \geq c\lambda^{\frac{1}{3}}$, for $x = 1$, $x_1$, and $x_2$. Then we can apply (3.7) and (3.8) directly in (4.3), giving the estimate

$$\begin{aligned} |a_\lambda(s; x_1, x_2)| \leq C \left|\nu^2 + (\lambda x_1)^2\right|^{-\frac{1}{4}} \left|\nu^2 + (\lambda x_2)^2\right|^{-\frac{1}{4}} \\ \times \left(e^{\operatorname{Re}[\psi(\nu,\lambda x_1) - \psi(\nu,\lambda x_2)]} + e^{\operatorname{Re}[\psi(\nu,\lambda x_1) + \psi(\nu,\lambda x_2) - 2\psi(\nu,\lambda)]}\right). \end{aligned} \tag{4.4}$$

Since

$$\partial_x \psi(\nu, \lambda x) = \frac{\sqrt{\nu^2 + \lambda^2 x^2}}{x},$$

we observe that $\operatorname{Re}\psi$ is an increasing function of $x$ for $\operatorname{Re}\nu \geq 0$. Thus the final expression in (4.4) is $O(1)$. The worst-case scenario for the estimate of the prefactors is $|\nu - i\lambda x_j| \asymp c\lambda^{\frac{1}{3}}$, in which case $\left|\nu^2 + (\lambda x_j)^2\right| \asymp |\nu|^{\frac{4}{3}}$. Under these assumptions we conclude that

$$|a_\lambda(s; x_1, x_2)| = O(\langle s\rangle^{-\frac{2}{3}}),$$

uniformly in $\lambda$.

If $\nu$ is near the turning point with respect to any of $x = 1$, $x_1$, or $x_2$, then we use the corresponding estimates from (3.9) for those terms. As an example, suppose $\nu$ lies near the turning point for $x_2$ but not for $x_1$ or 1. Then (4.4) becomes

$$|a_\lambda(s; x_1, x_2)| \leq C \left|\nu^2 + (\lambda x_1)^2\right|^{-\frac{1}{4}} |\lambda x_2|^{-\frac{1}{3}} \left(e^{\operatorname{Re}[\psi(\nu,\lambda x_1)]} + e^{\operatorname{Re}[\psi(\nu,\lambda x_1) - 2\psi(\nu,\lambda)]}\right).$$

For the first term in brackets we note that $\nu = i\lambda x_2 + O(\lambda^{\frac{1}{3}})$ and the fact that $|i\lambda x_1 - i\lambda x_2| \asymp \lambda$ imply $\operatorname{Im}\nu > \lambda x_1$ for $\nu$ sufficiently large. This gives $\operatorname{Re}\psi(\nu, \lambda x_1) \leq 0$ (see Figure 4.1) and similarly we find $\operatorname{Re}\psi(\nu, \lambda) > 0$. Hence the bracketed term is $O(1)$ and the claim follows for $\nu$ near the turning point for $x_2$. The cases where $\nu$ is close to $i\lambda$ and $i\lambda x_1$ are very similar. □

We turn next to estimates of the Poisson operator, which is quite straightforward in the physical half-plane.

**Proposition 4.3.** *For* $\chi \in C_0^\infty(0, 1)$ *and* $\operatorname{Re} s \geq \frac{n}{2}$,

$$\mu_k(\chi E_0(s)) \leq C e^{c_1\langle s\rangle - c_2 k^{1/n}},$$

*where* $\mu_k$ *denotes the* $k$*-th singular value. The same estimate holds if* $\chi$ *is replaced by a*

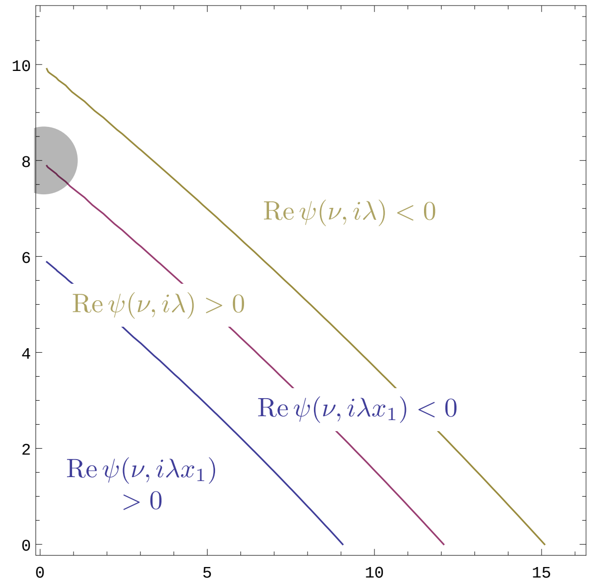


**Figure 4.1:** The lines $\operatorname{Re}\psi(\nu, i\lambda x) = 0$ for $x = x_1, x_2, 1$ (here: $\lambda = 10, x_1 = 0.6, x_2 = 0.8$). Comparing to Figure 3.2, we see that $\nu$ close to $i\lambda x_2$ gives $\operatorname{Re}\psi(\nu, i\lambda) > 0$ and $\operatorname{Re}\psi(\nu, i\lambda x_1) < 0$.

*radial differential operator with coefficients in* $C_0^\infty(0,1)$.

*Proof.* Since the cutoff depends only on $x$, the operator $(\chi E_0(s))^* \chi E_0(s)$ is diagonal with respect to the eigenfunctions $\phi_\lambda$, with eigenvalues given by

$$\int |\chi(x) b_\lambda(s;x)|^2 \, \frac{dx}{x^{n+1}}, \tag{4.5}$$

for $\lambda^2 \in \sigma(\Delta_h)$. Up to a possible change of ordering, these values correspond with the set of values of $\mu_k(\chi E_0(s))^2$.

To analyze the asymptotics, we set $\nu = s - \frac{n}{2}$ and use the conjugation symmetry to restrict our attention to $\operatorname{Im}\nu \geq 0$. We assume that $|\nu| \geq M$ with $M$ large enough that Proposition 3.1 applies. From (2.12) and (3.11) we can write

$$b_\lambda(s;x) = \frac{(\lambda/2)^\nu}{\Gamma(\nu+1)} x^{\frac{n}{2}} \left[ K_\nu(\lambda x) - \frac{K_\nu(\lambda)}{I_\nu(\lambda)} I_\nu(\lambda x) \right], \qquad (\lambda > 0). \tag{4.6}$$

Assuming $M$ is sufficiently large, Corollary 3.2 (along with Corollary 3.3 if either $\psi(\nu, \lambda x)$ or $\psi(\nu, \lambda)$ is close to zero) shows that the $K_\nu(\lambda x)$ term dominates in (4.6). The key point is that $x < 1$ and $\operatorname{Re}\psi(\nu, \lambda x)$ is a increasing function of $x$. Thus for $|\nu| \geq M$ and $\lambda > 0$ we have

$$|b_\lambda(s;x)| \leq C \left| \frac{(i\lambda/2)^\nu}{\Gamma(\nu+1)} \right| e^{-\operatorname{Re}\psi(\nu,\lambda x)}. \tag{4.7}$$

Applying Stirling's formula, (3.32), then yields

$$\log |b_\lambda(s;x)| \leq \operatorname{Re}\left[ -\nu \log\left( \frac{2x\nu}{\nu + \sqrt{\nu^2+\lambda^2 x^2}} \right) + \nu - \sqrt{\nu^2+\lambda^2x^2} - \tfrac{1}{2}\log\nu \right] + O(1). \tag{4.8}$$

If $\lambda x \gg |\nu|$ then this estimate reduces to

$$\log |b_\lambda(s;x)| \leq -\lambda x + O\left( |\nu| \log \frac{\lambda}{|\nu|} \right).$$

Hence, for $\lambda \geq m\,|\nu|$ with $m$ sufficiently large, we have

$$\log |b_\lambda(s;x)| \leq -c\lambda.$$

On the other hand, for $\lambda < m\,|\nu|$, (4.8) clearly shows that

$$\log |b_\lambda(s;x)| = O(\langle s \rangle).$$

The result follows from the formula (4.5) for the eigenvalues of $(\chi E_0(s))^* \chi E_0(s)$ and the Weyl asymptotic for the values of $\lambda^2 \in \sigma(\Delta_h)$.

To extend the estimates to include radial derivatives is a straightforward exercise using (4.6) and the identities

$$\begin{aligned}
\partial_x I_\nu(\lambda x) &= \lambda I_{\nu+1}(\lambda x) + \frac{\nu}{x} I_\nu(\lambda x), \\
\partial_x K_\nu(\lambda x) &= -\lambda K_{\nu+1}(\lambda x) + \frac{\nu}{x} K_\nu(\lambda x).
\end{aligned}$$

□

The extension of Proposition 4.3 to the non-physical plane is complicated by the presence of poles at the resonances. For this purpose it is most convenient to use the scattering matrix, because the scattering matrix is already diagonalized.

**Proposition 4.4.** *For* $\operatorname{Re} s \leq \frac{n}{2}$, $d(s, \mathcal{R}_0) \geq \langle s \rangle^{-\beta}$, *and* $d(s, \frac{n}{2} - \mathbb{N}_0) \geq \delta$, *with* $\beta, \delta > 0$, *we have*

$$\|S_0(s)\| \leq e^{C\langle s \rangle \log \langle s \rangle}.$$

*Proof.* Since our Bessel asymptotics are restricted to $\operatorname{Re}\nu \geq 0$, it is convenient to produce a lower bound of $S_0(s)$ in the region $\operatorname{Re} s \geq \frac{n}{2}$ and then exploit the symmetry $S_0(n-s) = S_0(s)^{-1}$. By the conjugation symmetry, $S_0(\overline{s}) = \overline{S_0(s)}$, we are free to restrict our attention to the quadrant $\arg \nu \in [0, \frac{\pi}{2}]$, and we can further assume $|\nu| \geq M$.

Consider the eigenvalue computed in (2.14),

$$[S_0(n-s)]_\lambda = \left(\frac{\lambda}{2}\right)^{-2\nu} \frac{\Gamma(\nu)}{\Gamma(-\nu)} \frac{I_\nu(\lambda)}{I_{-\nu}(\lambda)}. \tag{4.9}$$

Applying Lemmas 3.4 and 3.5 to (4.9), we find that for $|\nu| \geq M$, $\arg \nu \in [0, \frac{\pi}{2}]$, $d(\nu, \frac{n}{2} - \mathcal{R}_0) \geq \langle \nu \rangle^{-\beta}$, and $d(\nu, \mathbb{N}_0) \geq \delta$ we have

$$\left| \frac{I_\nu(\lambda)}{I_{-\nu}(\lambda)} \right| \preceq \begin{cases} e^{C\langle \nu \rangle \log \langle \nu \rangle} & |\operatorname{Re} \psi| \leq b, |\psi| \geq c \\ 1 & \text{otherwise,} \end{cases} \tag{4.10}$$

with constants that depend only on $b, c, \beta$, and $\delta$. Using Stirling's formula and the Euler reflection formula, we find that

$$\log \frac{\Gamma(\nu)}{\Gamma(-\nu)} = 2\nu \log \nu - (2 + i\pi)\nu + O(1),$$

for $\arg \nu \in [0, \frac{\pi}{2}]$ with $d(\nu, \mathbb{N}_0) \geq \delta$. The claimed estimate follows by applying these estimates to (4.9). □

Using the standard identity

$$E_0(s) = -E_0(n-s) S_0(s), \tag{4.11}$$

we can estimate

$$\mu_k(\chi E_0(s)) \leq \mu_k(\chi E_0(n-s)) \, \|S_0(s)\| \, .$$

Hence Propositions 4.3 and 4.4 together give us the:

**Corollary 4.5.** *For $\chi \in C_0^\infty(0,1)$, $\operatorname{Re} s \leq \frac{n}{2}$ with $d(s, \mathcal{R}_0) \geq \langle s \rangle^{-\beta}$, and $d(s, \frac{n}{2} - \mathbb{N}_0) \geq \delta$,*

$$\mu_k(\chi E_0(s)) \leq C e^{c_1 \langle s \rangle \log \langle s \rangle - c_2 k^{1/n}}.$$

*The same estimate holds if $\chi$ is replaced by a radial differential operator with coefficients in $C_0^\infty(0,1)$.*

## 4.2 Resonance counting estimate

Choose smooth cutoff functions $\chi_k \in C_0^\infty(X)$, such that

$$\chi_k = \begin{cases} 1 & \text{on } K \\ 1 & \text{in } X_0 \text{ when } x \geq 2^{-k} \\ 0 & \text{in } X_0 \text{ when } x \leq 2^{-(k+1)}. \end{cases}$$

(In other words, the cutoffs are nested and their derivatives have disjoint supports.) For some fixed $s_0$ with $\operatorname{Re} s_0$ large we define the parametrix,

$$M(s) := \chi_2 R_g(s_0)\chi_1 + (1-\chi_0)R_0(s)(1-\chi_1),$$

as a meromorphic function of $s \in \mathbb{C}$. This satisfies

$$(\Delta_g - s(n-s))M(s) = 1 - L(s), \tag{4.12}$$

with the error term

$$\begin{aligned} L(s) := &-[\Delta, \chi_2]R_g(s_0)\chi_1 + [s(n-s) - s_0(n-s_0)]\chi_2 R_g(s_0)\chi_1 \\ &+ [\Delta, \chi_0]R_0(s)(1-\chi_1). \end{aligned} \tag{4.13}$$

Note that $\chi_3 L(s) = L(s)$. Using this and applying the resolvent to (4.12), we can write

$$M(s)\chi_3 = R_g(s)\chi_3(1 - L(s)\chi_3).$$

Because the resolvent is a pseudodifferential operator of order $-2$, $L(s)\chi_3$ is a pseudodifferential operator of order $-1$ with compactly supported coefficients. In dimension $n+1$, a pseudodifferential operator of order $-m$ will be trace class for $m > n+1$ (see e.g. [2, Prop. A.26]). Thus, $(L(s)\chi_3)^{n+2}$ is a trace class operator and we can define the Fredholm determinant

$$D(s) := \det\left[1 - (L(s)\chi_3)^{n+2}\right]. \tag{4.14}$$

From Vodev [31, Appendix], we obtain the following:

**Lemma 4.6.** *The resonance set $\mathcal{R}_g$ (counted with multiplicities) is contained within the union of the set of zeros of $D(s)$ and $n+2$ copies of the set $\mathcal{R}_0$.*

The proof of the Lemma is essentially identical to that of [2, Cor. 9.3].

**Lemma 4.7.** *For $\beta, \delta > 0$, suppose that $d(s, \mathcal{R}_0) \geq \langle s\rangle^{-\beta}$ and $d(s, \frac{n}{2} - \mathbb{N}_0) \geq \delta$. Then for $\varepsilon > 0$ sufficiently small we have*

$$\log|D(s)| \leq \begin{cases} C\langle s\rangle^{n+1} & \text{for } \operatorname{Re} s - \frac{n}{2} \geq \varepsilon \\ C(\langle s\rangle \log\langle s\rangle)^{n+1} & \text{for } \operatorname{Re} s - \frac{n}{2} \leq -\varepsilon \\ C(\langle s\rangle \log\langle s\rangle)^{n+\frac{4}{3}} & \text{for } \left|\operatorname{Re} s - \frac{n}{2}\right| \leq \varepsilon. \end{cases}$$

*Proof.* To estimate the growth of $D(s)$, we separate

$$L(s)\chi_3 = T_0 + T_1(s) + T_2(s),$$

corresponding to the three terms on the right-hand side of (4.13). All terms are compactly supported, and $T_2(s)$ is smoothing and therefore trace class.

To break up the determinant estimates we can use [18, Lemma 6.1], which says that for compact operators $A, B$ in the $p$-th Schatten class,

$$|\det(1 + (A+B)^p)| \leq \det(1 + 2^{p-1}|A|^p)^{2p} \det(1 + 2^{p-1}|B|^p)^{2p}. \tag{4.15}$$

This estimate is based on the Weyl inequality for determinants in terms of singular values,

$$\log|\det(1+A)| \leq \sum_{k=1}^{\infty} \mu_k(A), \tag{4.16}$$

which we will also need to make use of below. By applying (4.15) twice to the formula for $D(s)$, with $p = n+2$, we deduce the bound

$$\begin{aligned} |D(s)| \leq \det\bigl(1 + C_0 |T_0|^{n+2}\bigr)^{2n+4} \\ \times \det\bigl(1 + C_1 |T_1(s)|^{n+2}\bigr)^{(2n+4)^2} \det\bigl(1 + C_2 |T_2(s)|\bigr)^{(2n+4)^2}. \end{aligned} \tag{4.17}$$

The high powers here don't trouble us, because the estimate we are seeking is exponential.

The first term on the right in (4.17) is just a constant. To estimate the second term, we note that $T_1(s)$ is quadratic in $s$,

$$T_1(s) = [s(n-s) - s_0(n-s_0)]\chi_2 R_g(s_0)\chi_1.$$

Since $R_g(s_0)$ has order $-2$, $T_1(s)^{n+2}$ is a pseudodifferential operator of order $-2(n+2)$, with compactly supported coefficients. We therefore have a bound on the singular values (see e.g. [2, Prop. A.26]),

$$\mu_k(T_1(s)^{n+2}) \leq C\langle s\rangle^{2(n+2)} k^{-\frac{2(n+2)}{n+1}}.$$

Thus by (4.16),

$$\det\bigl(1 + C_1 \,|T_1(s)|^{n+2}\bigr) \leq \prod_{k=1}^{\infty} \left(1 + C\Bigl(\frac{\langle s\rangle^{n+1}}{k}\Bigr)^{\gamma}\right),$$

where $\gamma := 2(n+2)/(n+1)$. We can then estimate

$$\log\det\bigl(1 + C_1 \,|T_1(s)|^{n+2}\bigr) \leq C \int_1^{\infty} \log\left(1 + C\Bigl(\frac{\langle s\rangle^{n+1}}{x}\Bigr)^{\gamma}\right)\, dx = O(\langle s\rangle^{n+1}).$$

Finally, the proof comes down to a growth estimate on

$$\det\bigl(1 + C_2 \,|T_2(s)|\bigr),$$

where

$$T_2(s) := [\Delta, \chi_0] R_0(s)(\chi_3 - \chi_1).$$

From Proposition 4.2 we can use comparison to eigenvalues of the Laplacian on a compact domain (see [2, §9.4] for details) to derive the bound

$$\mu_k(T_2(s)) \leq C \min\left\{k^{-2}\langle s\rangle^{2(n+1)}, 1\right\},$$

for $\mathrm{Re}(s - \frac{n}{2}) \geq \varepsilon$. We can then apply the Weyl determinant estimate (4.16) to deduce

$$\log\det\bigl(1 + C_2 \,|T_2(s)|\bigr) = O(\langle s\rangle^{n+1}). \tag{4.18}$$

(See e.g. the proof of [2, Lemma 9.12].) Similarly, for $0 \leq \mathrm{Re}(s - \frac{n}{2}) \leq \varepsilon$ (assuming $\varepsilon < 1/6$), Proposition 4.2 yields

$$\log\det\bigl(1 + C_2 \,|T_2(s)|\bigr) \leq C\langle s\rangle^{n+\frac{4}{3}}. \tag{4.19}$$

To obtain bounds for $\mathrm{Re}(s - \frac{n}{2}) \leq 0$, we appeal again to the estimate (4.15) to write

$$\det(1 + 2\,|T_2(n-s)|) \leq \det(1 + 2\,|T_2(s)|)^2 \det(1 + 2\,|T_2(s) - T_2(n-s)|)^2. \tag{4.20}$$

The first determinant on the right has already been dealt with. As for the second, we can use the identity

$$R_0(s) - R_0(n-s) = (2s-n)E_0(s)E_0(n-s)^t, \tag{4.21}$$

(c.f. [2, eq. (7.33)]) to reduce this to a determinant involving

$$T_2(s) - T_2(n-s) = (2s-n)[\Delta, \chi_0] E_0(s) E_0(n-s)^t (\chi_3 - \chi_1).$$

By Corollary 4.5, assuming $d(s, \mathcal{R}_0) \geq \langle s \rangle^{-\beta}$, and $d(s, \frac{n}{2} - \mathbb{N}_0) \geq \delta$, we find that

$$\mu_k(T_2(s) - T_2(n-s)) \leq C e^{c_1 \langle s \rangle \log\langle s \rangle - c_2 k^{1/n}}.$$

In particular,

$$\|T_2(s) - T_2(n-s)\| = \mu_0(T_2(s) - T_2(n-s)) \leq C e^{c_1 \langle s \rangle \log\langle s \rangle}.$$

This time we can exploit the Weyl determinant estimate (4.16) in the form

$$|\det(1+A)| \leq (1 + \|A\|)^m \exp\left( \sum_{k=m+1}^{\infty} \mu_k(A) \right),$$

with $m = (c_1 \langle s \rangle \log\langle s \rangle / c_2)^n$. This yields

$$\log\det(1 + 2\,|T_2(s) - T_2(n-s)|) = O((\langle s \rangle \log\langle s \rangle)^{n+1}).$$

By applying this estimate to the second factor in (4.20), and using (4.18) and (4.19) for the first factor, we can thereby deduce that (4.19) holds for $-\varepsilon \leq \mathrm{Re}(s - \frac{n}{2}) \leq 0$ and (4.18) holds for $\mathrm{Re}(s - \frac{n}{2}) \leq -\varepsilon$ with $d(s, \mathcal{R}_0) \geq \langle s \rangle^{-\beta}$, and $d(s, \frac{n}{2} - \mathbb{N}_0) \geq \delta$. □

*Proof of Proposition 4.1.* To complete the argument, let $\mathcal{R}_0$ denote the set of resonances of $X_0$. By the asymptotic (4.1), we can form the Weierstrass product,

$$H_0(s) := \prod_{\zeta \in \mathcal{R}_0} \left(1 - \frac{s}{\zeta}\right) \exp\left[\frac{s}{\zeta} + \cdots + \frac{1}{n+1}\left(\frac{s}{\zeta}\right)^{n+1}\right].$$

Lindelöf's Theorem (see e.g. [1, Thm. 2.10.1]) shows that the associated entire function

$$g_0(s) = H_0(s) H_0(e^{i\pi/(n+1)} s),$$

is of finite type, so that

$$\log|g_0(s)| \leq C \langle s \rangle^{n+1}. \tag{4.22}$$

From (4.13) we can see the poles of $D(s)$ are contained within some finite number of copies of $\mathcal{R}_0$. Hence, for some $N > 0$, the function $h(s) := g_0(s)^N D(s)$ will be entire. Using (4.22) we can apply the bounds from Lemma 4.7 to $h(s)$. And since $h(s)$ is entire, we can use the maximum modulus theorem to fill in the missing disks around $\mathcal{R}_0$ and $\frac{n}{2} - \mathbb{N}_0$, and the Phragmén-Lindelöf theorem to extend the stronger bound into the strip at $\operatorname{Re} s = \frac{n}{2}$. The result is that

$$\log |h(s)| \leq C(\langle s \rangle \log \langle s \rangle)^{n+1},$$

for all $s \in \mathbb{C}$. Since, by Lemma 4.6, the zero set of $h(s)$ contains $\mathcal{R}_g$, the claimed counting estimate follows from Jensen's formula. □

# Chapter 5

# Poisson formula

To establish the Poisson formula for resonances, we need to introduce the relative scattering determinant. Let $S_g(s)$ and $S_0(s)$ denote the scattering matrices associated to $(X, g)$ and the background manifold $(X_0, g_0)$, respectively. By (4.12) we have the relation

$$M(s) = R_g(s) - R_g(s)L(s), \tag{5.1}$$

from which we can derive, by taking boundary limits on the right and left, that

$$S_0(s) = S_g(s) - (2s - n)E_g(s)^t[\Delta, \chi_0]E_0(s). \tag{5.2}$$

This shows in particular that $S_g(s)S_0(s)^{-1} - 1$ is smoothing and hence trace class on $\Sigma$. Thus we can define the relative scattering determinant,

$$\tau(s) := \det S_0(s)^{-1}S_g(s).$$

By the order bound of Proposition 4.1, we can define the Weierstrass product,

$$H_g(s) := \prod_{\zeta \in \mathcal{R}_g} \left(1 - \frac{s}{\zeta}\right) \exp\left[\frac{s}{\zeta} + \cdots + \frac{1}{n+1}\left(\frac{s}{\zeta}\right)^{n+1}\right],$$

and we recall that $H_0(s)$ was defined as the corresponding product over $\mathcal{R}_0$.

**Proposition 5.1.** *The relative scattering determinant admits a Hadamard factorization of the form*

$$\tau(s) = e^{q(s)} \frac{H_g(n-s)}{H_g(s)} \frac{H_0(s)}{H_0(n-s)}, \tag{5.3}$$

*with $q(s)$ a polynomial of order at most $n+1$.*

*Proof.* To work out the divisor of $\tau(s)$, we can appeal to the theory developed by Gohberg-Sigal [11, §4–5] to deduce that

$$\operatorname{Res}_\zeta \frac{\tau'}{\tau}(s) = \operatorname{tr}\operatorname{Res}_\zeta \left[S_g'(s)S_g(s)^{-1}\right] - \operatorname{tr}\operatorname{Res}_\zeta \left[S_0'(s)S_0(s)^{-1}\right].$$

Letting $m_g(\zeta)$ denote the multiplicity of a resonance at $\zeta$, we have the relation

$$-\operatorname{tr}\operatorname{Res}_\zeta \left[S_g'(s)S_g(s)^{-1}\right] = m_g(\zeta) - m_g(n-\zeta) + \sum_{k\in\mathbb{N}} \Big(\mathbb{1}_{n/2-k}(\zeta) - \mathbb{1}_{n/2+k}(\zeta)\Big) d_k, \tag{5.4}$$

where $d_k$ is the dimension of the kernel of the $k$-th conformal Laplacian on $(\Sigma, h)$. This result is due to Guillarmou [14] (with earlier partial results by [6, 12, 19], and with a restriction that was later removed in [16]).

Since the $d_k$ cancel between the $S_g(s)$ and $S_0(s)$ terms, we obtain

$$\operatorname{Res}_\zeta \frac{\tau'}{\tau}(s) = m_g(n-\zeta) - m_g(\zeta) + m_0(\zeta) - m_0(n-\zeta).$$

This proves the claimed formula with $q(s)$ an entire function. It remains to show that $q(s)$ is a polynomial and bound its order.

Using the parametrix formula (4.12) and the fact that $\chi_3 L(s) = L(s)$ we can rewrite the identity (5.1) as

$$M(s) = R_g(s) - M(s)(1 - L(s)\chi_3)^{-1} L(s).$$

The corresponding scattering matrix identity is

$$S_0(s) = S_g(s) - (2s-n)E_0(s)^t(1-\chi_1)(1-L(s)\chi_3)^{-1}[\Delta, \chi_0]E_0(s).$$

Using the identity (4.11), the relative scattering determinant is thus given by

$$\tau(s) = \det\Big(1 - (2s-n)E_0(n-s)^t(1-\chi_1)(1-L(s)\chi_3)^{-1}[\Delta,\chi_0]E_0(s)\Big). \tag{5.5}$$

The $L(s)\chi_3$ term we write as

$$(1 - L(s)\chi_3)^{-1} = \left(1 + L(s)\chi_3 + \cdots + (L(s)\chi_3)^{n+1}\right)\left(1 - (L(s)\chi_3)^{n+2}\right)^{-1}.$$

Using Proposition 4.2, the identity (4.21), and Corollary 4.5, we have

$$\left\| 1 + \cdots + (L(s)\chi_3)^{n+1} \right\| = O(e^{C\langle s\rangle}).$$

Since $(L(s)\chi_3)^{n+2}$ is trace class we can use a resolvent estimate from Gohberg-Krein [10] (see also [2, Thm. A.23]) to obtain the estimate

$$\left\| (1 - (L(s)\chi_3)^{n+2})^{-1} \right\| \leq \frac{\det(1 + |L(s)\chi_3|^{n+2})}{D(s)},$$

where $D(s)$ is the determinant (4.14). Lemma 4.7 gives the upper bound

$$\log |D(s)| = O\left( (\langle s\rangle \log\langle s\rangle)^{n+\frac{4}{3}} \right),$$

for $d(s, \mathcal{R}_0) \geq \langle s\rangle^{-\beta}$ and $d(s, \frac{n}{2} - \mathbb{N}_0) \geq \delta$. We can clearly derive the corresponding estimate for $\log\det(1 + |L(s)\chi_3|^{n+2})$ by the same argument. The minimum modulus theorem [30, 8.7.1] shows that if we assume that $\beta > n + 4/3$, then the upper bound for $D(s)$ implies the lower bound

$$-\log |D(s)| = O(\langle s\rangle^{n+\frac{4}{3}+\varepsilon}),$$

for $\varepsilon > 0$, $d(s, \mathcal{R}_0) \geq \langle s\rangle^{-\beta}$ and $d(s, \frac{n}{2} - \mathbb{N}_0) \geq \delta$. So our estimate becomes

$$\left\| (1 - L(s)\chi_3)^{-1} \right\| \leq e^{C\langle s\rangle^m}, \tag{5.6}$$

for $m > n + \frac{4}{3}$ and $d(s, \mathcal{R}_0) \geq \langle s\rangle^{-\beta}$ and $d(s, \frac{n}{2} - \mathbb{N}_0) \geq \delta$.

Returning to (5.5), after combining (5.6) with the singular values estimates for the $E_0(s)$ terms from Corollary 4.5, we can use the Weyl determinant estimate to deduce that

$$\log |\tau(s)| = O(\langle s\rangle^{(n+1)m}),$$

for $m > n + \frac{4}{3}$ and $d(s, \mathcal{R}_0) \geq \langle s\rangle^{-\beta}$ and $d(s, \frac{n}{2} - \mathbb{N}_0) \geq \delta$. This implies at least that $q(s)$ is polynomial, although with an order possibly much higher than claimed.

Once $q(s)$ is known to be polynomial, the fact that its maximal order is $n + 1$ follows by examining the asymptotics of the relative heat trace as in [3, Prop. 7.2]. We will not include those details here, because in Proposition 6.5 in the next chapter we will establish much sharper estimates on the growth of $\log\tau(s)$ for $\left|\arg(s - \frac{n}{2})\right| \leq \frac{\pi}{2} - \varepsilon$. Those estimates imply in particular that $q(s)$ has order at most $n + 1$. □

The Poisson formula follows from Proposition 5.1, by the same analysis developed for the surface case by Guillopé-Zworski [19]. In the case considered here we could follow exactly the argument in Borthwick [3, §8]. The factorization of Proposition 5.1 which is our starting point here was given in that paper by [3, Prop. 7.2]. (See also the detailed expository account of this same argument in [2, §11].) The crucial step is a Birman-Krein type formula that relates the derivative of the scattering determinant to the 0-traces of the spectral measures,

$$-\partial_s \log \tau(s) = (2s-n)\Big(\text{0-tr}[R_g(s) - R_g(n-s)] - \text{0-tr}[R_0(s) - R_0(n-s)]\Big). \tag{5.7}$$

In the present context this follows immediately from a result of Guillarmou [15, Thm. 3.10], which shows that each 0-trace on the right is given by the Kontsevich-Vishik trace of the logarithmic derivative of the corresponding scattering matrix. When we take the difference of these two formal traces, we recover the actual trace of the logarithmic derivative of the relative scattering matrix.

The traces on the right in (5.7) are the Fourier transforms of regularized wave traces. Proposition 5.1 gives an explicit formula for the left side and shows that it is a tempered distribution. Taking the Fourier transform of (5.7) (as in [3, Thm. 1.2], for example), yields the proof of the Poisson formula stated in Theorem 1.1.

Finally we consider the asymptotics of the scattering phase,

$$\sigma(t) := \frac{i}{2\pi} \log \tau(\tfrac{n}{2} + it), \tag{5.8}$$

with branches chosen so that $\sigma(t)$ is continuous. By the properties of the scattering matrix, $\sigma(t)$ is a real-valued odd function of $t \in \mathbb{R}$. Using the analysis of the big singularity of the wave traces at $t = 0$, developed in the asymptotically hyperbolic case by Joshi-Sá Barreto [20], and the method from Guilopé-Zworski [19, Thm. 1.5], we can derive the:

**Corollary 5.2.** *Assume $(X, g)$ is asymptotically hyperbolic metric with warped-product ends, with core $K$. As $t \to +\infty$,*

$$\sigma(t) = W_K t^{n+1} + O(t^n),$$

*where $W_K$ is the Weyl constant*

$$W_K := \frac{(4\pi)^{-\frac{n+1}{2}}}{\Gamma(\frac{n+3}{2})} \operatorname{Vol}(K, g).$$

As a final remark, we note that because the Poisson formula of Theorem 1.1 includes the resonances of the background metric $g_0$, it does not lead to a lower bound for resonances along the lines of [19] or [3]. The technique used in those arguments, based on the big singularity of the wave trace at $t = 0$, would produce a lower bound only for the sum $N_g(t) + N_{g_0}(t)$, as in [5, Cor. 3.2]. The results of §6.1 will show that $N_{g_0}(t)$ saturates the resonance bound, and so the joint lower bound yields no information on $N_g(t)$.

# Chapter 6

# Sharp upper bounds

In this chapter we will refine the crude counting estimate of Proposition 4.1 into the proof of Theorem 1.2. The first step is to compute the asymptotic constant of the counting function for the model space $(X_0, g)$. This amounts to counting zeros of Bessel functions, a similar argument to a calculation of Stefanov [29].

Proposition 5.1 shows how the divisor of the relative scattering determinant $\tau(s)$ is determined by the resonance sets $\mathcal{R}_g$ and $\mathcal{R}_0$. Using a contour integral as in [4, Prop. 3.2], we obtain the formula (which is due to Froese [9]):

**Proposition 6.1.** *As $a \to \infty$,*

$$\int_0^a \frac{N_g(t) - N_0(t)}{t}\,dt = 2\int_0^a \frac{\sigma(t)}{t}\,dt + \frac{1}{2\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \log\left|\tau(\tfrac{n}{2} + ae^{i\theta})\right|\,d\theta + O(\log a).$$

The asymptotic for the scattering phase $\sigma(t)$ was given in Corollary 5.2. Hence for the application of Proposition 6.1 we must establish the asymptotic for $N_0(t)$ and estimate $|\tau(s)|$ for $\operatorname{Re} s \geq \frac{n}{2}$.

## 6.1 Asymptotic counting for the model space

The resonances of the model space were identified explicitly in (2.11) as zeros of $I_\nu(\lambda)$, where $\nu := s - \frac{n}{2}$ and $\lambda^2 \in \sigma(\Delta_h)$. In this section we will use the Bessel function asymptotics from Chapter 3 to work out the constant in the asymptotic that we claimed for the model space counting function $N_0(r)$ in (4.1).

Since our Bessel function asymptotics assume that $\operatorname{Re}\nu \geq 0$, we will study the zeros through the reflection formula,

$$I_{-\nu}(\lambda) = I_\nu(\lambda) + \frac{2\sin\pi\nu}{\pi} K_\nu(\lambda). \tag{6.1}$$

There are two distinct sources of zeros of $I_{-\nu}(\lambda)$. For $|\nu| \succeq \lambda$, the $K_\nu(\lambda)$ term is dominant in (6.1). Thus $I_{-\nu}(\lambda)$ has some zeros which are perturbations of the integer points where $\sin\pi\nu = 0$. We refer to these as 'trivial' zeros, as they are quite easy to count. Note that because the trivial zeros are perturbations of simple zeros, and the zero set of $I_{-\nu}(\lambda)$ has a conjugation symmetry, the trivial zeros must remain on the real axis. They can never occur precisely at an integer, however, since $I_{-k}(z) = I_k(z)$ for $k \in \mathbb{Z}$, and $I_k(z) > 0$ for $z > 0$.

The 'non-trivial' zeros of $I_{-\nu}(\lambda)$ occur within the highlighted zone shown in Figure 3.2 (and its reflection by conjugation, of course). Within this zone and away from the real axis, the approximation (3.17) is valid, and the zeros are approximately given by solutions of the equation

$$\mathrm{Ai}\left(e^{\frac{2\pi i}{3}}(\tfrac{3}{2}\psi)^{2/3}\right) = 0, \tag{6.2}$$

where $\psi = \psi(\nu, \lambda)$, as defined in (3.6). Since we fix $x = 1$ for the applications in this chapter, we define $\rho(\alpha) := \rho(\alpha, 1)$ and recall the definition,

$$\rho(\alpha) := \sqrt{\alpha^2+1} + \alpha \log\left(\frac{i}{\alpha + \sqrt{\alpha^2+1}}\right), \tag{6.3}$$

and the relationship $\psi(\nu, \lambda) = \lambda\rho(\alpha)$, where $\nu = \lambda\alpha$. Within the zone that contains the non-trivial zeros, the corresponding values of $\psi$ are close to the positive imaginary axis. Hence we can apply the approximation (B.3) to reduce (6.2) to a simpler equation

$$\cosh\!\big(\lambda\rho(\alpha) - i\tfrac{\pi}{4}\big) = 0. \tag{6.4}$$

Our strategy will be to count the solutions of (6.4) and then control the distances between these solutions and the true zeros, for which $I_{-\lambda\alpha}(\lambda) = 0$.

We will count the solutions of (6.4) that lie on

$$\gamma := \big\{\alpha : \operatorname{Re}\rho(\alpha) = 0, \arg\alpha \in [0, \tfrac{\pi}{2}]\big\} - \{iy : y > 1\},$$

and then relate the corresponding counting function to $N_0(r)$. Let $\alpha_0$ denote the real solution to $\operatorname{Re}\rho(\alpha) = 0$. The curve $\gamma$ is shown in Figure 6.1; it corresponds to the center of the highlighted zone in Figure 3.2. Note that the actual resonance lines in Figure 2.1 are

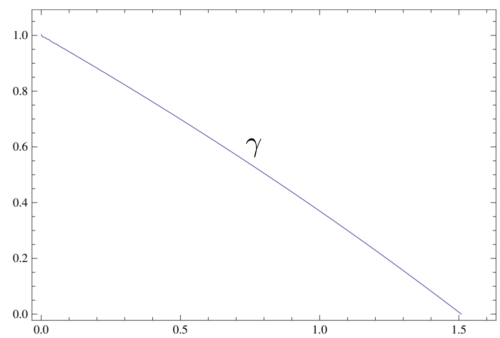


**Figure 6.1:** The curve $\gamma$ containing solutions of (6.4).

well approximated by the reflections of $\gamma$ across the imaginary axis, scaled by the square roots $\lambda$ of the eigenvalues.

Let $W_\Sigma$ denote the Weyl constant for the compact manifold $(\Sigma, h)$,

$$W_\Sigma := \frac{\mathrm{Vol}(\Sigma, h)}{\Gamma(\frac{n}{2}+1)(4\pi)^{n/2}},$$

defined by the asymptotic,

$$\#\big\{\lambda^2 \in \sigma(\Delta_h) : \lambda \le r\big\} = W_\Sigma \cdot r^n + O(r^{n-1}). \tag{6.5}$$

If we index the square roots of the eigenvalues of $\Delta_h$ as $0 = \lambda_0 < \lambda_1 \le \lambda_2 \le \dots$, we can define an integer-valued function $m = m(r)$ on $\mathbb{R}_+$ by demanding

$$\lambda_m \le r \qquad \text{and} \qquad \lambda_{m+1} > r.$$

Then, since $m + 1 = \#\{k : \lambda_k \le r\}$, the Weyl law (6.5) leads to

$$\left(\frac{m}{W_\Sigma}\right)^{\frac{1}{n}} = \left[r^n + O(r^{n-1})\right]^{\frac{1}{n}} = r + O(1).$$

This gives the asymptotic formula

$$\lambda_k = W_\Sigma^{-\frac{1}{n}}\, k^{\frac{1}{n}} + O(1), \tag{6.6}$$

for the eigenvalues of $\Delta_h$.

**Lemma 6.2.** *Let $M(r; \theta_1, \theta_2)$ denote the number of zeros of (6.4) for $|\nu| \le r$, $\arg \nu \in$*

*$[\theta_1,\theta_2)$, and $\lambda^2 \in \sigma(\Delta_h) - \{0\}$. For $0 \le \theta_1 < \theta_2 \le \frac{\pi}{2}$ this count satisfies the asymptotic*

$$M(r;\theta_1,\theta_2) = \frac{nW_\Sigma}{(n+1)\pi}\, r^{n+1} \int_{\gamma_{|[\theta_1,\theta_2]}} \frac{|\rho'(\alpha)|}{|\alpha|^{n+1}}\, |d\alpha| + O(r^n), \tag{6.7}$$

*where $\gamma$ is parametrized by $\theta = \arg\alpha$.*

*Proof.* For any $\lambda$, the zeros of (6.4) with $\operatorname{Re}\alpha > 0$ lie on the curve $\gamma$. Note that the zeros of (6.4) with $\operatorname{Re}\alpha = 0$ are not included in the count $M(r;\theta_1,\frac{\pi}{2})$. As an alternative parametrization of $\gamma$, define $\tilde\gamma(t)$ implicitly by

$$\rho(\tilde\gamma(t)) = i\pi t.$$

The constant $\alpha_0 \approx 1.509$ is the value of $\alpha$ at which the curve $\gamma$ intersects the real axis. For any real $\alpha$, $\rho(\alpha)$ has imaginary part $\frac{\pi}{2}\alpha$. Hence we have $\rho(\alpha_0) = \frac{i\pi}{2}\alpha_0$, and the domain of $\tilde\gamma$ is seen to be $t \in [0,\frac{\alpha_0}{2}]$.

For $0 \le \theta_1 < \theta_2 \le \frac{\pi}{2}$, let $t_1$ and $t_2$ be the corresponding parameters so that $\gamma(\theta_j) = \tilde\gamma(t_j)$. For fixed $\lambda$, the number of zeros of (6.4) with $\arg\alpha \in [\theta_1,\theta_2)$ is given exactly by the number of points in $\lambda(t_2,t_1] \cap (\mathbb{N} - \frac{1}{4})$. We can thus estimate the number of zeros in this range as

$$\lambda(t_1 - t_2) + O(1), \tag{6.8}$$

where the error term is bounded by $\pm 1$.

Now consider the full count, summed over $\lambda$. The number of $\lambda$'s for which $\gamma$ intersects $\{|\alpha| \le r/\lambda\}$ is $O(r^n)$ by the Weyl law, so that by applying (6.8) for each $\lambda$ and summing the errors we obtain

$$M(r;\theta_1,\theta_2) = \tilde M(r;\theta_1,\theta_2) + O(r^n), \tag{6.9}$$

where

$$\tilde M(r;\theta_1,\theta_2) := \sum_\lambda \lambda\, \ell\Big(\tilde\gamma^{-1}\Big[\gamma_{|[\theta_1,\theta_2]} \cap \{|\alpha| \le \tfrac{r}{\lambda}\}\Big]\Big).$$

For some fixed $\theta$ and small $\Delta\theta$, we define $t$ and $\Delta t$ by $\tilde\gamma(t) = \gamma(\theta)$ and $\tilde\gamma(t - \Delta t) = \gamma(\theta + \Delta\theta)$. Then we can estimate

$$\left| \tilde M(r;\theta,\theta+\Delta\theta) - \sum_{\lambda|\gamma(\theta)| \le r} \lambda\Delta t \right| \le \sum_{r/|\gamma|_{\max} \le \lambda \le r/|\gamma|_{\min}} \lambda\, \Delta t,$$

where the extrema of $|\gamma|$ are taken over the sector $\arg\alpha \in [\theta,\theta+\Delta\theta]$. Since $|\gamma|_{\max} - |\gamma|_{\min}$

is bounded by $c\Delta\theta$, the Weyl law implies that the number of terms in the sum on the right hand side of the above inequality is $O(r^n\Delta\theta)$. Hence we can write

$$\tilde{M}(r;\theta,\theta+\Delta\theta) = \sum_{\lambda\leq r/|\gamma(\theta)|} \lambda\Delta t \quad + O(r^{n+1}\Delta t\Delta\theta).$$

Using the eigenvalue asymptotics (6.6), we find

$$\sum_{\lambda\leq r/|\gamma(\theta)|} \lambda = \frac{nW_\Sigma}{n+1}\left(\frac{r}{|\gamma(\theta)|}\right)^{n+1} + O(r^n).$$

Since $|\Delta t| \leq c\,|\Delta\theta|$, we conclude

$$\tilde{M}(r;\theta,\theta+\Delta\theta) = \frac{nW_\Sigma}{n+1}\left(\frac{r}{|\gamma(\theta)|}\right)^{n+1}\Delta t + O(r^n\Delta t) + O(r^{n+1}(\Delta\theta)^2). \tag{6.10}$$

We now choose a partition of the interval $[\theta_1,\theta_2]$ and sum up the corresponding equations (6.10). Letting $|\Delta\theta| \to 0$, the first term of (6.10) becomes an integral with respect to $t$. The second term becomes $O(r^n)$ after summing, the constant for its bound changing by a factor of $|t_1 - t_2| \leq \frac{\alpha_0}{2}$. The rightmost term in (6.10) vanishes. Then (6.9) gives

$$M(r;\theta_1,\theta_2) = \frac{nW_\Sigma}{n+1}\, r^{n+1}\int_{t_2}^{t_1} \frac{1}{|\tilde{\gamma}|^{n+1}}\,dt + O(r^n).$$

The final step is to note that $(\rho\circ\tilde{\gamma})'(t) = i\pi$, so that the change of variables from $t$ to arclength is accounted for by introducing a factor of $|\rho'|/\pi$. □

**Proposition 6.3.** *The resonance counting function $N_0$ for the model space $X_0$ satisfies the asymptotic*

$$N_0(r) = \left[\frac{2nW_\Sigma}{(n+1)\pi}\int_\gamma \frac{|\rho'(\alpha)|}{|\alpha|^{n+1}}\,d\,|\alpha| + \frac{W_\Sigma}{n+1}\alpha_0^{-n}\right] r^{n+1} + O\big(r^{n+\frac{1}{3}}\big), \tag{6.11}$$

*where $W_\Sigma$ is the Weyl constant for $(\Sigma, h)$, $\gamma = \{\alpha :\ \operatorname{Re}\rho(\alpha) = 0\} - \{iy\ :\ y > 1\}$, and $\alpha_0$ is the real solution to $\operatorname{Re}\rho(\alpha) = 0$.*

*Proof.* From Lemma 3.4 we know that the non-trivial zeros of $I_{-\nu}(\lambda)$ in $\operatorname{Im}\nu \geq 0$, $|\nu| \geq M$, where $M$ is sufficiently large, are contained in the region

$$S_\lambda := \{\arg\nu \in [0, \tfrac{\pi}{2}] :\ \operatorname{Im}\psi \geq 0, |\operatorname{Re}\psi| \leq b^*\} \cap \{|\nu| \geq M\}.$$

(The strip $\{\operatorname{Im}\psi(\,.\,,\lambda) \geq 0, |\operatorname{Re}\psi(\,.\,,\lambda)| \leq b^*\}$ contains the highlighted zone of Figure 3.2, its center is $\lambda\gamma$; $S_\lambda$ is empty for $\lambda\alpha_0(1+\varepsilon) < M$ and $S_\lambda$ is not affected by the intersection with $\{|\nu| \geq M\}$ for $\lambda$ sufficiently large.) From Corollary 3.2 and (3.11) we deduce that for $\psi$ and $\nu$ sufficiently large we have

$$\frac{I_{-\nu}(\lambda)}{I_\nu(\lambda)} - (1 + ie^{-2\psi}) = ie^{-2\psi}\big[O(\psi^{-1}) + O(\nu^{-\frac{2}{3}}) - e^{2i\pi\nu}\big], \tag{6.12}$$

for $\nu \in S_\lambda$. Note that the zeros of the function

$$1 + ie^{-2\psi} = 2e^{-\psi + i\frac{\pi}{4}} \cosh\big(\psi - i\tfrac{\pi}{4}\big),$$

in (6.12) correspond precisely to the solutions of (6.4). The width of the strip $S_\lambda$ in the $\nu$-plane, where by 'width' we mean the lengths of intersections with curves $\operatorname{Im}\psi = \text{const}$, is $O(\lambda^{\frac{1}{3}})$ close to the turning point $\nu = i\lambda$, and $O(1)$ close to $\lambda\alpha_0$.

We will now prepare to apply Rouché's theorem to the functions on the left hand side of (6.12). Let $\nu_{\lambda,m}$ denote the solution of (6.4) for which

$$\psi(\nu_{\lambda,m}) = i\pi(m - \tfrac{1}{4}),$$

with $m \in \mathbb{N} \cap [\frac{1}{4}, \frac{1}{4} + \frac{\lambda\alpha_0}{2}]$. Define $\gamma_{\lambda,m}$ to be the contour obtained from the lines

$$\{\operatorname{Re}\psi = b\}, \quad \{\operatorname{Re}\psi = -b\}, \quad \{\operatorname{Im}\psi = \pi(m - \tfrac{3}{4})\}, \quad \{\operatorname{Im}\psi = \pi(m + \tfrac{1}{4})\},$$

in the $\nu$-plane. Then each $\nu_{\lambda,m}$ lies within $\gamma_{\lambda,m}$, and we have that on $\gamma_{\lambda,m}$

$$\left|e^{-2\psi}\right| \leq \beta \left|1 + ie^{-2\psi}\right|,$$

where the constant $\beta$ depends only on $b^*$.

In order to control the right hand side of (6.12) we define for $\sigma, \tau > 0$ the region

$$\Gamma_{\sigma,\tau} := \left\{\nu :\ \operatorname{Im}\nu \geq \tau,\ \operatorname{Re}\nu \geq \sigma(\operatorname{Im}\nu)^{1/3}\right\}.$$

Recall from the proof of Corollary 3.3 that for some small $\delta > 0$ we have

$$\psi \asymp \lambda^{-\frac{1}{2}}(\nu - i\lambda)^{\frac{3}{2}} \qquad \text{for } |\nu - i\lambda| < \delta\lambda.$$

Outside that half-disc, $|\psi|$ is bounded from below by the values on its boundary, i.e. by $c\lambda$

(see Figure 3.2 for the mapping properties of $\psi$). For $\nu \in \{|\nu - i\lambda| < \delta\lambda\} \cap \Gamma_{\sigma,\tau}$ we have

$$|\psi| \succeq \sigma^{\frac{3}{2}}.$$

Hence, by letting both $\sigma$ and $\tau$ be large enough, (6.12) yields

$$\left| \frac{I_{-\nu}(\lambda)}{I_\nu(\lambda)} - (1 + ie^{-2\psi}) \right| < \frac{1}{\beta} \left| e^{-2\psi} \right|,$$

on $S_\lambda \cap \Gamma_{\sigma,\tau}$.

Rouché's theorem now implies that for $\lambda$ sufficiently large, $I_{-\nu}(\lambda)$ has exactly one zero within every $\gamma_{\lambda,m}$ that is contained in $\Gamma_{\sigma,\tau}$. After increasing $\sigma$ and $\tau$ we have that for $\lambda$ sufficiently large, $I_{-\nu}(\lambda)$ has exactly one zero within every $\gamma_{\lambda,m}$ that intersects $\Gamma_{\sigma,\tau}$. Moreover, there are no other zeros in $\Gamma_{\sigma,\tau}$ since the contours $\gamma_{\lambda,m}$ cover the regions $S_\lambda$. The diameters of the $\gamma_{\lambda,m}$ are $O(\lambda^{1/3})$ with a constant that depends only on $b^*$. Consequently,

$$\begin{aligned} \#\left\{\zeta \in \mathcal{R}_0 : \left|\zeta - \tfrac{n}{2}\right| \le r, \ -\zeta \in \Gamma_{\sigma,\tau}\right\} &\le M(r + \mu r^{1/3}; 0, \tfrac{\pi}{2}) + O(r^n) \\ &= M(r; 0, \tfrac{\pi}{2}) + O\big(r^{n+\frac{1}{3}}\big). \end{aligned} \tag{6.13}$$

for some constant $\mu$, where the $O(r^n)$ error term in the middle expression is caused by contours $\gamma_{\lambda,m}$ that intersect $\partial\Gamma_{\sigma,\tau}$. We now claim that for $\lambda$ sufficiently large, the number of zeros of $I_{-\nu}(\lambda)$ in $S_\lambda - \Gamma_{\sigma,\tau}$ is uniformly bounded by a constant depending only on $\sigma$ and on $\tau$. We postpone the proof of this detail to the next paragraph and continue the counting argument first. After noting that $M(r - \mu r^{1/3}; \theta_1, \theta_2)$ provides an asymptotic lower bound for (6.13) for all $0 < \theta_1 < \theta_2 < \frac{\pi}{2}$, we conclude

$$\#\left\{\zeta \in \mathcal{R}_0 : \left|\zeta - \tfrac{n}{2}\right| \le r, \ \operatorname{Im}\zeta \ne 0\right\} = 2M(r; 0, \tfrac{\pi}{2}) + O\big(r^{n+\frac{1}{3}}\big).$$

For fixed $\lambda$ sufficiently large, the zeros of $I_{-\nu}(\lambda)$ outside $\Gamma_{\sigma,\tau}$ are contained in two components of $S_\lambda$. We skip the easier case, the component that is near $\lambda\alpha_0$, and concentrate on $S_\lambda - \Gamma_{\sigma,0}$. Let $c^*$ be the constant from Lemma 3.4. Since we have no zeros for $\nu$ with $|\psi| < c^*$, we consider the region

$$R = \{\psi \in \mathbb{C} : |\psi| \ge c_1, \operatorname{Im}\psi \le c_2, -b^* \le \operatorname{Re}\psi \le b^*\},$$

in the $\psi$-plane, where $c_1 \in (b^*, c^*)$ and $c_2 = m - \frac{3}{4}, m \in \mathbb{N}$, is chosen so that the preimage of $R$ under $\psi(\,.\,,\lambda)$ overlaps with $\Gamma_{\sigma,0}$. Note that both $\operatorname{Ai}\big((\frac{3}{2}\psi)^{\frac{2}{3}}\big)$ and $\operatorname{Ai}\big(e^{-\frac{2\pi i}{3}}(\frac{3}{2}\psi)^{\frac{2}{3}}\big)$ are

$\asymp 1$ on $R$. Applying Proposition 3.1 and Stirling's formula to the reflection formula (6.1), we obtain

$$\frac{I_{-\nu}(\lambda)}{I_\nu(\lambda)} = 1 + e^{\frac{2\pi i}{3}} \frac{\mathrm{Ai}\big((\frac{3}{2}\psi)^{\frac{2}{3}}\big)}{\mathrm{Ai}\big(e^{-\frac{2\pi i}{3}}(\frac{3}{2}\psi)^{\frac{2}{3}}\big)} \Big[1 + O\big(\nu^{-\frac{2}{3}}\big)\Big],$$

where the error term is an analytic function in a neighborhood of $R$. The function

$$1 + e^{\frac{2\pi i}{3}} \frac{\mathrm{Ai}\big((\frac{3}{2}\psi)^{\frac{2}{3}}\big)}{\mathrm{Ai}\big(e^{-\frac{2\pi i}{3}}(\frac{3}{2}\psi)^{\frac{2}{3}}\big)},$$

on $R$ has a limited number of zeros and does not depend on $\lambda$. Hence, by Rouché's theorem, the number of zeros of $I_{-\nu}(\lambda)$ outside $\Gamma_{\sigma,0}$ is bounded uniformly in $\lambda$.

It remains to show that the contribution of the trivial zeros to the counting function is given by the second term in the constant claimed in (6.11). By factoring out $\frac{2}{\pi}K_\nu(\lambda)$ on the right hand side of (6.1), and applying the asymptotics of Corollary 3.2, we obtain

$$I_{-\nu}(\lambda) = \frac{2}{\pi}K_\nu(\lambda)\Big[\sin(\pi\nu) - \frac{1}{2}e^{i2\pi\nu}e^{2\psi}\big(1 + O(\lambda^{-1})\big)\Big].$$

There exist positive constants $c$ and $\varepsilon$ such that

$$\operatorname{Re}\rho(\alpha) \le -c(\alpha - \alpha_0),$$

for real $\alpha \ge (1-\varepsilon)\alpha_0$. Hence for real $\nu$ sufficiently large with $\nu \ge \lambda(1-\varepsilon)\alpha_0$, we have

$$I_{-\nu}(\lambda) = \frac{2}{\pi}K_\nu(\lambda)\Big(\sin\pi\nu + O\big(e^{-2c(\nu-\lambda\alpha_0)}\big)\Big).$$

This gives

$$\#\left\{\zeta \in \mathcal{R}_0 :\ \left|\zeta - \tfrac{n}{2}\right| \le r,\ \operatorname{Im}\zeta = 0\right\} = \sum_{m=1}^{\lfloor r \rfloor} \#\{\lambda\alpha_0 \le m\} + O(r^n).$$

Estimating the sum using Weyl's law (6.5) completes the proof. □

From the proof of Proposition 6.3 we observe that in the model case we have a resonance-free region with boundary given by a cube-root: for some small $\sigma$,

$$\mathcal{R}_0 \cap \left\{s \in \mathbb{C} :\ \operatorname{Re}(s - \tfrac{n}{2}) \ge -\sigma\,|\operatorname{Im} s|^{1/3}\right\} = \emptyset.$$

## 6.2 Estimate of the scattering determinant

The goal of this section is to find an upper bound for $\log|\tau(s)|$ with $s$ in a sufficiently big subset of $\{\operatorname{Re} s > \frac{n}{2}, \operatorname{Im} s \geq 0\}$.

For some small $\eta > 0$, let $x_j = 1 - \eta j$ for $j = 0, 1, 2, 3$. We choose cutoff functions $\chi_j \in C^\infty((0,1])$ so that $\chi_j(x) = 1$ for $x \geq x_j$ and $\chi_j(x) = 0$ for $x \leq x_{j+1}$. With the model Poisson operator $E_0(s)$ defined as in §2.2, we can express the relative scattering determinant as

$$\tau(s) = \det\left(1 + (2s-n)E_0(s)^t[\Delta_0, \chi_2]R(s)[\Delta_0, \chi_1]E_0(n-s)\right).$$

Since the derivations of this identity and of Lemma 6.4 follow [4, Lemmas 4.1 and 5.2] closely, we omit the proofs.

**Lemma 6.4.** *For* $\operatorname{Re} s \geq \frac{n}{2}$ *with* $d(s(n-s), \sigma(\Delta_g)) \geq \varepsilon$, *the relative scattering determinant can be estimated by*

$$\log|\tau(s)| \leq \sum_{\lambda^2 \in \sigma(\Delta_h)} \log\bigl(1 + C\kappa_\lambda(s)\bigr), \tag{6.14}$$

*where*

$$\kappa_\lambda^2(s) = |2s-n|^2 \int_{x_2}^{x_1} x^{-(n+1)} |b_\lambda(n-s,x)|^2\, dx \int_{x_3}^{x_2} x^{-(n+1)} |b_\lambda(s,x)|^2\, dx,$$

*with the coefficients* $b_\lambda(s;x)$ *as defined in* (2.12), *and where the constant* $C$ *depends only on* $\eta$ *and* $\varepsilon$.

Using the identity

$$E_0(s) = -E_0(n-s)S_0(s),$$

we find

$$\kappa_\lambda(s) \leq \left|\frac{I_\nu(\lambda)}{I_{-\nu}(\lambda)}\right| \left|\nu\left(\frac{\lambda}{2}\right)^{-2\nu} \frac{\Gamma(\nu)}{\Gamma(-\nu)}\right| \int_{x_3}^{x_1} x^{-(n+1)}\, |b_\lambda(s,x)|^2\, dx, \tag{6.15}$$

for $k > 0$. Define the following set of radii $a$, for which the corresponding circles stay away from the zeros of the scattering matrix in the sense of Proposition 4.4.

$$\Lambda := \left\{a \in \mathbb{R}_+ : \min_\theta d(ae^{i\theta}, \tfrac{n}{2} - \mathcal{R}_0) \geq \langle a\rangle^{-\beta}, d(a, \mathbb{N}_0) \geq \delta\right\},$$

where $\beta > n+1$ and $\delta > 0$. Then, for $|\nu| \in \Lambda$, we have control of $I_\nu/I_{-\nu}(\lambda)$ by (4.10). The requirement for Lemma 6.4, that $d(s(n-s), \sigma(\Delta_g)) \geq \varepsilon$, will be satisfied if $|\theta| \leq \frac{\pi}{2} - \varepsilon a^{-2}$ for $\nu = ae^{i\theta}$ with $a$ sufficiently large. With these two restrictions we obtain:

**Proposition 6.5.** *For $a \in \Lambda$ we have*

$$\log \left|\tau(\tfrac{n}{2} + ae^{i\theta})\right| \leq B(\theta)a^{n+1} + o(a^{n+1}),$$

*uniformly for $|\theta| \leq \frac{\pi}{2} - \varepsilon a^{-2}$, with*

$$B(\theta) := 2nW_\Sigma \int_0^\infty \frac{[-\operatorname{Re}\rho(xe^{i|\theta|})]_+}{x^{n+2}}\, dx,$$

*where $W_\Sigma$ is the Weyl constant for $\Delta_h$ and $[\cdot]_+$ denotes the positive part.*

*Proof.* We let $\nu = s - \frac{n}{2}$ as always and we use the conjugation symmetry to restrict our attention to $\operatorname{Im}\nu \geq 0$. Recall that $\operatorname{Re}\rho(\alpha, x)$ is monotonically increasing in $x$. For $|\nu| \geq M$, where $M$ is the constant from Proposition 3.1, (4.7) and (6.15) give

$$\kappa_\lambda(s) \leq C\, e^{-2\lambda \operatorname{Re}\rho(\alpha, x_3)} g_\lambda(\nu), \tag{6.16}$$

for $\lambda > 0$, $\alpha = \frac{\nu}{\lambda}$, and

$$g_\lambda(\nu) := \left| \frac{I_\nu(\lambda)}{I_{-\nu}(\lambda)} \right|.$$

Given $\nu = ae^{i\theta}$, we split the sum (6.14) according to the sign of $\operatorname{Re}\rho(\frac{\nu}{\lambda}, x_3)$. The sum over $\lambda$ with $\operatorname{Re}\rho(\alpha, x_3) < 0$ is finite and we further divide it into contributions from the Poisson kernel and from the scattering matrix. Since the $\lambda = 0$ term in the sum (6.14) is $O(a)$, we can write

$$\log|\tau(s)| \leq \Sigma_L + \Sigma_P + \Sigma_S + O(a),$$

where

$$\begin{aligned}
\Sigma_L &:= \sum_{\operatorname{Re}\rho(\alpha,x_3)\geq 0} \log\bigl(1 + Ce^{-2\lambda \operatorname{Re}\rho(\alpha,x_3)} g_\lambda(\nu)\bigr), \\
\Sigma_P &:= \sum_{\operatorname{Re}\rho(\alpha,x_3)< 0} 2\lambda \operatorname{Re}[-\rho(\alpha, x_3)], \\
\Sigma_S &:= \sum_{\operatorname{Re}\rho(\alpha,x_3)< 0} \log(1 + Cg_\lambda(\nu)).
\end{aligned}$$

Let us now index the spectrum $\{\lambda > 0\}$ by $\lambda_k$, $k \in \mathbb{N}$. Define the constant $\omega = W_\Sigma^{-1/n}$, so that $\lambda_k \sim \omega k^{1/n}$ by Weyl's law. Also define the function $A(\theta)$ and constants $q, Q > 0$ by

$$\operatorname{Re}\rho(A(\theta)e^{i\theta}, x_3) = 0, \qquad q\omega k^{1/n} \leq \lambda_k \leq Q\omega k^{1/n}.$$

First we show that $\Sigma_L$ contributes only of lower order. For $a$ sufficiently large, the factors $g_\lambda(\nu)$ in $\Sigma_L$ are bounded by Lemma 3.4, and therefore we have

$$\Sigma_L \le C \sum \exp\bigl(-2\lambda_k[\operatorname{Re}\rho(\alpha, x_3)]_+\bigr),$$

where the sum is for $k$ from $\lfloor(\frac{a}{Q\omega A(\theta)})^n\rfloor$ to $\infty$. The monotonicity in $x$ of $\operatorname{Re}\rho(\alpha, x)$ yields

$$\Sigma_L \le C \sum \exp\bigl(-2[\operatorname{Re}\rho(ae^{i\theta}, q\omega k^{1/n} x_3)]_+\bigr).$$

Switching to an integral over $k$ and substituting $x = \frac{a}{\omega k^{1/n}}$, we find

$$\Sigma_L \le Ca^n \int_0^{QA(\theta)} \frac{1}{x^{n+1}} \exp\Bigl(-\frac{2a}{x}[\operatorname{Re}\rho\bigl(xe^{i\theta}, qx_3\bigr)]_+\Bigr)\,dx + O(1).$$

Since $\rho(0, qx_3) > 0$, the integral exists, and the fact that the integrand is decreasing in $a$ shows that $\Sigma_L = O(a^n)$.

For an estimate of $\Sigma_P$, define numbers $\mu_k$ with $\lambda_k = (1+\mu_k)\omega k^{1/n}$ for all k. Then

$$2\lambda_k[-\operatorname{Re}\rho(\alpha, x_3)]_+ = 2\omega k^{1/n}\bigl[-\operatorname{Re}\rho\bigl(\tfrac{a}{\omega k^{1/n}}e^{i\theta}, (1+\mu_k)x_3\bigr)\bigr]_+.$$

The number of $\mu_k$ with absolute value greater than $\eta/x_3$ is finite and independent of $a$. The corresponding terms in the sum $\Sigma_P$ are $O(a)$. For all other $k$ we have $(1+\mu_k)x_3 \ge x_3 - \eta =: x_4$ and hence, by monotonicity of $\operatorname{Re}\rho$ and letting $x = \frac{a}{\omega k^{1/n}}$ as above,

$$\Sigma_P \le \sum 2\omega k^{1/n}[-\operatorname{Re}\rho(xe^{i\theta}, x_4)]_+ + O(a),$$

where the sum is for $k$ from 1 to $\lceil(\frac{a}{q\omega A(\theta)})^n\rceil$. Switching to the corresponding integral over $x$, we find

$$\Sigma_P \le \frac{2n}{\omega^n}\int_0^\infty \frac{[-\operatorname{Re}\rho(xe^{i\theta}, x_4)]_+}{x^{n+2}}\,dx \cdot a^{n+1} + O(a).$$

With a simple change of variables we can scale the $x_4$ out of the integral, yielding

$$\Sigma_P \le x_4^{-n} B(\theta) a^{n+1} + O(a). \tag{6.17}$$

The number of terms in the sum $\Sigma_S$ is, as for $\Sigma_P$, less than $\lceil(\frac{a}{q\omega A(\theta)})^n\rceil$. Most of them are bounded by Lemma 3.4, and Lemma 3.5 states that the remaining terms, those for which

$\nu$ is in the highlighted zone of Figure 3.2, are $O(a\log a)$ for $a \in \Lambda$. More precisely:

$$\Sigma_S \leq O(a^n) + K(\nu)\, c\, a \log a, \qquad\qquad (a \in \Lambda), \tag{6.18}$$

where, with $b^*$ being the constant from Lemma 3.4,

$$K(\nu) = \#\big\{k > 0 :\ \operatorname{Im}\psi(\nu, \lambda_k) \geq 0,\ |\operatorname{Re}\psi(\nu, \lambda_k)| \leq b^*\big\}. \tag{6.19}$$

Now suppose that for fixed $\nu$ we have $\mu_1, \mu_2$ with

$$\operatorname{Re}\psi(\nu, \mu_1) = -b^*, \qquad \operatorname{Re}\psi(\nu, \mu_2) = +b^*,$$

where the condition on $\mu_1$ is replaced by $\operatorname{Im}\psi(\nu, \mu_1) = 0$ for $\arg\nu$ close to $\frac{\pi}{2}$. We observe that $K(\nu)$ is given by the number of eigenvalues $\lambda_k$ between $\mu_1$ and $\mu_2$. Since the width of the region

$$\{\nu : |\operatorname{Re}\psi(\nu, \mu)| \leq b^*,\ \operatorname{Im}\psi(\nu, \mu) \geq 0\},$$

in the $\nu$-plane is $O(\nu^{1/3})$ uniformly in $\theta = \arg\nu$, we can estimate

$$K(\nu) \leq W_\Sigma\left(\left(\frac{a}{|\gamma(\theta)|} + ca^{1/3}\right)^n - \left(\frac{a}{|\gamma(\theta)|} - ca^{1/3}\right)^n\right) + O(a^{n-1}).$$

This shows $K(\nu) = O(a^{n-2/3})$, and from (6.18) we obtain $\Sigma_S = O(a^{n+1/3}\log a)$.

We conclude

$$\log\left|\tau(\tfrac{n}{2} + ae^{i\theta})\right| \leq x_4^{-n} B(\theta) a^{n+1} + C_{\varepsilon,\eta}\, a^{n+1/3} \log a,$$

where the constant might blow up as $\eta \to 0$. This gives

$$\limsup_{a\to\infty}\left[\frac{\log\left|\tau(\frac{n}{2} + ae^{i\theta})\right|}{a^{n+1}} - B(\theta)\right] \leq (x_4^{-n} - 1)B(\theta),$$

which, by letting $x_4 \to 1$, completes the proof. $\square$

## 6.3 Completing the sharp estimate

*Proof of Theorem 1.2.* With the asymptotics of the scattering phase, as stated in Corollary 5.2, the relative counting formula from Proposition 6.1 becomes

$$(n+1)\int_0^a \frac{N_g(t)-N_0(t)}{t}\,dt = 2W_K a^{n+1} + \frac{n+1}{2\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \log\left|\tau(\tfrac{n}{2}+ae^{i\theta})\right|\,d\theta + o(a^{n+1}).$$

We can then apply the asymptotic for $N_0(t)$ from Proposition 6.3 and the scattering determinant estimate from Proposition 6.5. Comparing the result to (1.5) shows that for Theorem 1.2 it remains to show that the contribution of

$$\int_{\frac{\pi}{2}-\varepsilon a^{-2}\le|\theta|\le\frac{\pi}{2}} \log\left|\tau(\tfrac{n}{2}+ae^{i\theta})\right|\,d\theta,$$

is of lower order. If we assume $a\in\Lambda$ then by the Hadamard factorization (5.3) of $\tau$ and the Minimum Modulus Theorem [30, Thm. 8.71], we have the estimate

$$\left|\tau(\tfrac{n}{2}+ae^{i\theta})\right| \le C_\epsilon \exp\left(a^{n+1+\epsilon}\right),$$

for any $\epsilon>0$, provided $\beta>n+1$ in the definition of $\Lambda$. This implies

$$\int_{\frac{\pi}{2}-\varepsilon a^{-2}\le|\theta|\le\frac{\pi}{2}} \log\left|\tau(\tfrac{n}{2}+ae^{i\theta})\right|\,d\theta = O(a^{n-1+\epsilon}),$$

which suffices to complete the proof. □

# Appendix A

# Asymptotics of the sectional curvatures

Let $(N, h)$ be a compact Riemannian manifold with non-empty boundary and denote by $M$ the interior of $N$. The metric on the cotangent bundle of $(N, h)$ is

$$|df|_h^2 := f_i h^{ij} f_j = |\mathrm{grad}\, f|_h^2 .$$

We call a smooth and non-negative function $\rho$ on $N$ a *boundary-defining function* if

$$\rho^{-1}(0) = \partial N \qquad \text{and} \qquad d\rho \neq 0 \text{ on } \partial N.$$

Given such a function $\rho$, we can define a non-compact Riemannian manifold $(M, g)$ by setting

$$g := \rho^{-2}\, h. \tag{A.1}$$

The measurement of angles is not affected by (A.1), and hence we call (A.1) a *conformal change* and the manifold $(M, g)$ is called *conformally compact.* The following standard fact about the curvatures of such manifolds has first been noted in Rafe Mazzeo's dissertation [22].

**Proposition A.1.** *The manifold $(M, g)$ as defined above is complete and its sectional curvatures have the asymptotic behavior*

$$\sec_g \to -\,|d\rho|_h^2 \qquad \text{as } \rho \to 0.$$

For instance, let $\mathbb{D}$ stand for the open unit disk in $\mathbb{R}^2$. Then $\mathbb{D}$ with the metric

$$\frac{4\left(dx^2+dy^2\right)}{(1-x^2-y^2)^2},$$

is the Poincaré disk model of the hyperbolic plane $\mathbb{H}^2$. Multiplying this metric with the square of

$$\varphi := \tfrac{1}{2}\big(1-x^2-y^2\big), \tag{A.2}$$

we obtain the standard Euclidean metric, which extends to a smooth metric on the closed unit disk. Since $\varphi$ is a boundary-defining function for $(\overline{\mathbb{D}}, dx^2+dy^2)$, we conclude that $\mathbb{H}^2$ is conformally compact. Proposition A.1 further states that the sectional curvatures of $\mathbb{H}^2$ approach $-1$ at infinity.

In general, $|d\rho|_h$ can vary along the boundary $\partial N$. Hence, by Proposition A.1, the limits of the sectional curvatures of $(M,g)$ lie in some compact interval of $(-\infty,0)$. If those limits are the same and equal to $-1$ for all possible paths escaping to infinity, then the conformally compact manifold is called *asymptotically hyperbolic*.

We end this review of conformal compactness with the construction of a class of boundary-defining functions that always yield asymptotically hyperbolic manifolds. Given a compact manifold $(N,h)$ with non-empty boundary, there exists $\varepsilon>0$ such that the mapping

$$\begin{aligned} \{p\in N \,:\, d_h(p,\partial N)\le\varepsilon\} \;&\to\; [0,\varepsilon]\times\partial N \\ p \;&\mapsto\; \big(d_h(p,\partial N), q\big), \end{aligned}$$

where $q$ is the unique point on $\partial N$ with $d_h(p,q)=d_h(p,\partial N)$, is well-defined and a diffeomorphism. Hence one possible choice for $\rho$ would be to define

$$\rho(p) := d_h(p,\partial N),$$

in a neighborhood of $\partial N$, and then continue this function smoothly and positively onto the remainder of $N$. In this case, $\rho$ is a Riemannian submersion in a neighborhood of the boundary, and (A.1) gives rise to an asymptotically hyperbolic manifold. Note that the boundary-defining function $x$ in (1.2) is of that form (while (A.2) is not).

# Appendix B

# Asymptotic behavior of the Airy function

The Airy function has zeros only on the negative real axis, with the first at $w \approx -2.338$. For large arguments we can derive the Airy function asymptotics (following [28, §4.4.1]) from the integral representation

$$\mathrm{Ai}(w) = \frac{1}{2\pi} e^{-\xi} \int_0^\infty e^{-\sqrt{w}t} \cos(\tfrac{1}{3}t^{\frac{3}{2}})\, t^{-\frac{1}{2}}\, dt,$$

for $|\arg w| < \pi$, where

$$\xi := \tfrac{2}{3} w^{\frac{3}{2}}.$$

Hence, if we pull out the leading term by setting

$$\mathrm{Ai}(w) = \frac{1}{2\pi^{\frac{1}{2}}} w^{-\frac{1}{4}} e^{-\xi} \big[1 + R(w)\big],$$

the remainder term is given exactly by

$$R(w) = \frac{w^{\frac{1}{4}}}{\pi^{\frac{1}{2}}} \int_0^\infty e^{-\sqrt{w}t} \left[\cos(\tfrac{1}{3}t^{\frac{3}{2}}) - 1\right] t^{-\frac{1}{2}}\, dt.$$

This is now easy to estimate by applying Taylor's theorem to the cosine term at $t = 0$. The result is that for $w = re^{i\theta}$, with $r > 0$ and $|\theta| < \pi$,

$$|R(w)| \leq \frac{r^{\frac{1}{4}}}{\pi^{\frac{1}{2}}} \int_0^\infty e^{-\sqrt{r}\cos(\theta/2)t} \frac{1}{18} t^{\frac{5}{2}}\, dt$$
$$= \frac{5}{48} \frac{r^{-3/2}}{[\cos(\theta/2)]^{7/2}}.$$

For $|\arg w| \leq \pi - \varepsilon$, this gives the uniform estimate

$$\mathrm{Ai}(w) = \frac{1}{2\pi^{\frac{1}{2}}} w^{-\frac{1}{4}} e^{-\xi} \big[1 + O(|\xi|^{-1})\big], \tag{B.1}$$

with the constant in the error term bounded by $\frac{5}{72}(\sin\varepsilon/2)^{-\frac{7}{2}}$.

We can also develop asymptotics near the negative real axis using the identity

$$\mathrm{Ai}(w) = e^{\frac{\pi i}{3}} \mathrm{Ai}(e^{-\frac{2\pi i}{3}} w) + e^{-\frac{\pi i}{3}} \mathrm{Ai}(e^{-\frac{4\pi i}{3}} w). \tag{B.2}$$

From (B.1) this yields

$$\mathrm{Ai}(w) = \frac{1}{2\pi^{\frac{1}{2}}} w^{-\frac{1}{4}} \left(e^{-\xi} + ie^{\xi}\right) \big[1 + O(|w|^{-\frac{3}{2}})\big], \tag{B.3}$$

uniformly for $\arg w \geq \frac{\pi}{3} + \varepsilon$.

# Appendix C

# The method of successive approximations

Consider the recursive integral equation

$$h(x) = \int_x^\infty K(x,y)\phi(y)\big[h(y) + f(y)\big]\,dy, \tag{C.1}$$

where all functions appearing are complex-valued and measurable on $\mathbb{R}_+$. We obtain a formal solution for (C.1) in the form

$$h(x) := \sum_{j=1}^{\infty} h_j(x), \tag{C.2}$$

by setting

$$h_j(x) := \int_x^\infty K(x,y)\phi(y)h_{j-1}(y)\,dy, \qquad j \in \mathbb{N}, \tag{C.3}$$

where $h_0 := f$.

Suppose that there are positive functions $p$ and $q$ such that

$$|K(x,y)| \le p(x)q(y),$$

for $x \le y$, and that

$$\tilde{\kappa} := \sup_{y\in\mathbb{R}_+} q(y)\,|f(y)| < \infty,$$

$$\kappa := \sup_{y\in\mathbb{R}_+} p(y)q(y) < \infty.$$

Suppose further that the integral

$$\Phi(x) := \int_x^\infty |\phi(y)|\ dy,$$

converges for all $x > 0$.

Then from (C.3) we obtain the estimates

$$|h_j(x)| \le p(x)\frac{\tilde{\kappa}}{\kappa}\frac{\big(\kappa\,\Phi(x)\big)^j}{j!},$$

for $j \in \mathbb{N}$. Consequently, the formal series (C.2) converges absolutely under these hypotheses and satisfies

$$|h(x)| \le p(x)\frac{\tilde{\kappa}}{\kappa}\big[\exp\big(\kappa\,\Phi(x)\big) - 1\big].$$

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