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Connections between Classical and Umbral Moonshine

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Doctor of Philosophy

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A.B., Princeton University, 2013

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Abstract

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The classical theory of *monstrous moonshine* describes the unexpected connection between the representation theory of the monster group $\mathbb{M}$, the largest of the sporadic simple groups, and certain modular functions, called Hauptmoduln. In particular, the n-th Fourier coefficient of Klein's j-function is the dimension of the grade n part of a special infinite dimensional representation $V^\natural$ of the monster group. More generally the coefficients of Hauptmoduln are graded traces T_g of $g \in \mathbb{M}$ acting on $V^\natural$. Similar phenomena have been shown to hold for the Mathieu group M_{24}, but instead of modular functions, *mock modular forms* must be used. This has been generalized even further, to *umbral moonshine*, which associates to each of the 23 Niemeier lattices a finite group, infinite dimensional representation, and mock modular form. Both results of this dissertation involve finding unexpected connections between the classical theory of monstrous moonshine and the newer umbral moonshine. In our first result, we use *generalized Borcherds products* to associate to each pure A-type Niemeier lattice a conjugacy class g of the monster group and give rise to identities relating dimensions of representations from umbral moonshine to values of T_g. Our second result focuses on the Matheiu group M_{23}. While it inherits a moonshine from being a subgroup of M_{24}, we find a new and simpler moonshine for M_{23} such that the graded traces are, up to constant terms, identical to the monstrous moonshine Haupmoduln.

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Contents

Chapter 1

Introduction

1.1 Moonshine

Monstrous moonshine begins with the surprising connection between the coefficients of the modular function

$$J(\tau) := j(\tau) - 744 = \frac{(1+240\sum_{n=1}^{\infty}\sum_{d|n} d^3q^n)^3}{q\prod_{n=1}^{\infty}(1-q^n)^{24}} - 744 = \frac{1}{q} + 196884q + 21493760q^2 + \dots$$

and the representation theory of the monster group $\mathbb{M}$, which is the largest of the simple sporadic groups. Here $q := e^{2\pi i\tau}$ and $\tau \in \mathbb{H} := \{z \in \mathbb{C} \colon \Im z > 0\}$. McKay noticed that 196884, the q^1 coefficient of $J(\tau)$, can be expressed as a linear combination of dimensions of irreducible representations of the monster group $\mathbb{M}$. Namely,

$$196884 = 196883 + 1.$$

Thompson saw that the same was true for other Fourier coefficients of $J(\tau)$. For example,

$$21493760 = 21296876 + 196883 + 1.$$

In [Tho79b], McKay and Thompson conjectured that the n-th Fourier coefficient of $J(\tau)$ is the dimension of the grade n part of a special infinite-dimensional graded representation $V^\natural$ of $\mathbb{M}$.

This was later expanded into the full monstrous moonshine conjecture by Thompson, Conway, and Norton [CN79, Tho79a]. Since the graded dimension is just the graded trace of the identity element, they looked at the graded traces $T_g(\tau)$ of non-trivial elements g of M acting on $V^\natural$ and conjectured that they were all expansions of principal moduli, or Hauptmoduln, for certain genus zero congruence groups Γ_g commensurable with $\mathrm{SL}_2(\mathbb{Z})$. Note that these T_g are constant on each of the 194 conjugacy classes of M, and therefore are class functions, which automatically have coefficients which are $\mathbb{C}$-linear combinations of irreducible characters of M. Part of the task of proving monstrous moonshine was showing that they were in fact $\mathbb{Z}_{\geq 0}$-linear combinations.

By way of computer calculation, Atkin, Fong, and Smith [Smi85] verified the existence of a virtual representation of $\mathbb{M}$. Then using vertex-operator theory, Frenkel, Lepowsky, and Meurman [FLM84] finally constructed a representation $V^\natural$ of $\mathbb{M}$ thereby providing a beautiful algebraic explanation for the original numerical observations of McKay and Thompson. Borcherds [Bor86] further developed the theory of vertex-operator algebras, which he then used in [Bor92] to prove the full conjectures as given by Conway and Norton.

Monstrous moonshine provides an example of coefficients of modular functions enjoying distinguished properties. Moreover, their values at Heegner points have also been considered important. A *Heegner point* τ of discriminant $d < 0$ is a complex number in the upper half-plane of the form $\tau = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$ with $a, b, c \in \mathbb{Z}$, $\gcd(a, b, c) = 1$, and $d = b^2 - 4ac$. The values of principal moduli at such points are called *singular moduli*. As an example of their importance, it is a classical fact that the singular moduli of $j(\tau)$ generate Hilbert class fields of imaginary quadratic fields.

Moreover, the other McKay-Thompson series arising in monstrous moonshine satisfy analogous properties [CY96]. It is natural to ask what other interesting properties the values of the Hauptmoduln $T_g(\tau)$ could possess. We show in Theorem 1.2.2 that some of these values are related to another kind of moonshine, called *umbral moonshine.*

Recently, it was shown that phenomena similar to monstrous moonshine occur for other q-series and groups. In particular, the Mathieu group M_{24} exhibits moonshine [EOT11, Gan16], with the role of the j-invariant played by a *mock modular form* of weight 1/2, denoted $H^{(2)}(\tau)$. A mock modular form is the holomorphic part of a *harmonic weak Maass form.* Cheng, Duncan, and Harvey conjectured in [CDH14a] that this is a special case of a more general phenomenon, which they call *umbral moonshine.* For each of the 23 Niemeier lattices X they associate a vector-valued mock modular form $H^X(\tau)$, a group G^X, and an infinite-dimensional graded representation K^X of G^X such that the Fourier coefficients of H^X encode the dimensions of the graded components of K^X.

In particular, if $c^X(n, j)$ is the n-th Fourier coefficient of the j-th component of H^X, then

$$c^X(n,j) = \begin{cases} \dim_{K^X_{j,-D/4m}} & \text{if } n = -D/4m \text{ where } D \in \mathbb{Z}, D \equiv j^2 \pmod{4m}, \\ 0 & \text{otherwise,} \end{cases} \tag{1.1.1}$$

where

$$K^X = \bigoplus_{j \pmod{2m}} \bigoplus_{\substack{D\in\mathbb{Z} \\ D\equiv j^2 \pmod{4m}}} K^X_{j,-D/4m}.$$

The existence of such a K^X was recently proven by Duncan, Griffin, and Ono in [DGO15], generalizing Gannon's proof for Mathieu moonshine in [Gan16]. However, many questions still remain, including:

Question 1.1.1. *Is there a "natural" and uniform construction of K^X for all umbral*

X? Is K^X equipped with a deeper algebra structure as in the case of the monster module $V^\natural$?

Remark 1. Constructions have recently been given for a few specific cases, such as in [CD17].

1.2 Introduction of Results

In Chapter 3, we will associate a conjugacy class $g(X)$ of $\mathbb{M}$ to each pure-A type Niemeier lattice X. In Chapter 4, we associate a conjugacy class $\hat{g}$ in $\mathbb{M}$ to each conjugacy class g in the Mathieu group $M_{23} \subset M_{24}$.

In both cases, we show the first glimpses of new connections between the classical monstrous moonshine and the newer Mathieu and umbral moonshines.

Remark 2. We fully expect that they can be extended to all Niemeier lattices and to all conjugacy classes of M_{24}, but we leave that to someone else!

Remark 3. The results in Chapter 3 were joint work with Ken Ono and Larry Rolen and were published in [ORTL15].

As a convention, we will denote the names of conjugacy classes of $\mathbb{M}$ with capital letters, such as $1A$, whereas we'll use lower case letters, such as $1a$, for those of M_{23}.

1.2.1 First Result

In Chapter 3, we will use *generalized Borcherds products* (see [BO10]) to describe a connection between the mock modular forms $H^X(\tau)$ of umbral moonshine and the McKay-Thompson series $T_g(\tau)$ of monstrous moonshine. *Generalized Borcherds products* are a method to produce modular functions as infinite products of rational functions whose exponents come from the coefficients of mock modular forms, and they can be viewed as generalizations of the automorphic products in Theorem 13.3 of [Bor98].

We focus on the Niemeier lattices X whose root systems are of pure A-type according to the ADE classification. They are listed in Table 1.1, along with their Coxeter numbers $m(X)$ and the notation we will use for the mock modular form H^X.

Table 1.1: Pure A-type Root Systems

Root System X	Coxeter Number $m(X)$	Mock Modular Form H^X
A_1^{24}	2	$H^{(2)}(\tau)$
A_2^{12}	3	$H^{(3)}(\tau)$
A_3^{8}	4	$H^{(4)}(\tau)$
A_4^{6}	5	$H^{(5)}(\tau)$
A_6^{4}	7	$H^{(7)}(\tau)$
A_8^{3}	9	$H^{(9)}(\tau)$
A_{12}^{2}	13	$H^{(13)}(\tau)$
A_{24}^{1}	25	$H^{(25)}(\tau)$

Table 1.2 gives the monstrous moonshine dictionary for the conjugacy classes g which correspond to pure A-type cases of umbral moonshine[1]. Note that $\eta(\tau)$ is the *Dedekind eta-function*, defined by

$$\eta(\tau) := q^{1/24} \prod_{n=1}^{\infty} (1 - q^n).$$

All of our Hauptmoduln are normalized so that they have the form $q^{-1} + O(q)$, which is why all of the η-quotients in the table have a constant added to them.

There is an evident correspondence between the pure A-type lattices X in Table 1.1 and the conjugacy classes g in Table 1.2. We give this correspondence in Table 1.3.

We show that for a pure A-type Niemeier lattice X and its corresponding conju-

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[1] The case $X = A_{24}$ corresponds to $g(X) = (25Z)$, which is what Conway and Norton call a "ghost element". This means that $\Gamma_0(25)$ is the only genus zero $\Gamma_0(N)$ that does not correspond to a conjugacy class of the monster group. The parentheses are used to indicate a ghost element.

Table 1.2: The Dictionary of Monstrous Moonshine

Monster Conj. Class g	Congruence Subgroup Γ_g	McKay-Thomspon Series $T_g(\tau)$
$2B$	$\Gamma_0(2)$	$\eta(\tau)^{24}/\eta(2\tau)^{24}+24$
$3B$	$\Gamma_0(3)$	$\eta(\tau)^{12}/\eta(3\tau)^{12}+12$
$4C$	$\Gamma_0(4)$	$\eta(\tau)^{8}/\eta(4\tau)^{8}+8$
$5B$	$\Gamma_0(5)$	$\eta(\tau)^{6}/\eta(5\tau)^{6}+6$
$7B$	$\Gamma_0(7)$	$\eta(\tau)^{4}/\eta(7\tau)^{4}+4$
$9B$	$\Gamma_0(9)$	$\eta(\tau)^{3}/\eta(9\tau)^{3}+3$
$13B$	$\Gamma_0(13)$	$\eta(\tau)^{2}/\eta(13\tau)^{2}+2$
$(25Z)$	$\Gamma_0(25)$	$\eta(\tau)/\eta(25\tau)+1$

Table 1.3: Correspondence Between Umbral and Monstrous Moonshine

Root System X	Conj. Class $g(X)$
A_1^{24}	$2B$
A_2^{12}	$3B$
A_3^{8}	$4C$
A_4^{6}	$5B$
A_6^{4}	$7B$
A_8^{3}	$9B$
A_{12}^{2}	$13B$
A_{24}^{1}	$(25Z)$

gacy class $g := g(X)$, the “Galois (twisted) traces” of the CM values of the McKay-Thompson series $T_g(\tau)$ are the coefficients of the mock modular form H^X. To more precisely state this, we set up the following notation.

Let X be a pure A-type Niemeier lattice with Coxeter number $m := m(X)$ and corresponding conjugacy class $g := g(X)$. We call a pair (Δ, r) *admissible* if $\Delta \neq -3$ is a negative fundamental discriminant and $r^2 \equiv \Delta \pmod{4m}$. We also let $e(a) := e^{2\pi i a}$.

Theorem 1.2.1. *Let* $c^X(n,j)$ *be the* n*-th Fourier coefficient of the* j*-th component of* H^X*. Let* (Δ, r) *be an admissible pair for* X*. Then the twisted generalized Borcherds product*

$$\Psi_{\Delta,r}(\tau, H^X) := \prod_{n=1}^{\infty} P_\Delta(q^n)^{c^+\left(\frac{|\Delta|n^2}{4m}, \frac{rn}{2m}\right)},$$

where

$$P_\Delta(x) := \prod_{b \in \mathbb{Z}/|\Delta|\mathbb{Z}} [1 - e(b/\Delta)x]^{\left(\frac{\Delta}{b}\right)}$$

is a rational function in $T_g(\tau)$ *with a discriminant* Δ *Heegner divisor.*

Remark 4. For $\Delta = -3$, we need to replace $\Psi_{\Delta,r}(\tau, H^X)$ with $\Psi_{\Delta,r}(\tau, H^X)^3$. However, with that modification all of the theorems described in this section hold.

The next result gives a precise description of the rational functions in Theorem 1.2.1. In particular, it gives a "twisted" trace function for the values of T_g at points in the divisor and the coefficients c^+ of the mock modular forms H^X. It is often the case that coefficients of automorphic forms can be expressed in terms of singular moduli (see e.g., [BO07, BF06, DIT11, Zag02]).

Corollary 1.2.2. *By Theorem 1.2.1, we can write*

$$\Psi_{\Delta,r}(\tau, H^X) = \prod_i (T_g(\tau) - T_g(\alpha_i))^{\gamma_i}$$

for some discriminant Δ *Heegner points* α_i*. Then we have that*

$$\dim_{K^X_{r,|\Delta|/4m}} = c^X\left(\frac{|\Delta|}{4m}, \frac{r}{2m}\right) = \frac{1}{\lambda_\Delta} \sum_i \gamma_i \cdot T_g(\alpha_i),$$

where

$$\lambda_\Delta = \sum_{b \in \mathbb{Z}/|\Delta|\mathbb{Z}} e(b/\Delta) \cdot \left(\frac{\Delta}{b}\right).$$

Example. Let $X = A_1^{24}$, so $m(X) = 2$ and $g(X) = 2B$. Then the corresponding McKay-Thompson series is

$$T_g(\tau) = \frac{\eta(\tau)^{24}}{\eta(2\tau)^{24}} + 24 = \frac{1}{q} + 276q + \dots.$$

We pick the admissible pair $(\Delta, r) = (-7, 1)$. In Section 3.1, we will show that

$$\begin{aligned} \Psi_{\Delta,r}(\tau, H^X) &= \frac{(T_g(\tau) - T_g(\alpha_1))^2}{(T_g(\tau) - T_g(\alpha_2))^2} = \frac{\left(T_g(\tau) - \frac{1-45\sqrt{-7}}{2}\right)^2}{\left(T_g(\tau) - \frac{1+45\sqrt{-7}}{2}\right)^2} \\ &= 1 + 90\sqrt{-7}q + (28350 + 45\sqrt{-7})q^2 + \dots, \end{aligned}$$

where $\alpha_1 := \frac{-1+\sqrt{-7}}{4}$ and $\alpha_2 := \frac{1+\sqrt{-7}}{4}$. Note that $T_g(\alpha_1)$ and $T_g(\alpha_2)$ are algebraic integers of degree 2 which form a full set of conjugates. Their twisted trace is

$$2[T_g(\alpha_1) - T_g(\alpha_2)] = -90\sqrt{-7},$$

which matches the q^1 Fourier coefficient above. To check Corollary 1.2.2, we note that

$$\lambda_\Delta = \sum_{b \in \mathbb{Z}/7\mathbb{Z}} e(-b/7) \cdot \left(\frac{-7}{b}\right) = -\sqrt{-7}$$

and

$$\frac{1}{\lambda_\Delta} \sum_i \gamma_i T_g(\alpha_i) = 90 = c^+(7/8, 1/4) = \dim_{K^{(2)}_{1,7/8}}.$$

Example. As a second example, again consider $X = A_1^{24}$, so $m(X) = 2$ and $g(X) = 2B$. We pick the admissible pair $(\Delta, r) = (-15, 1)$. Let $\rho_1, \rho_2, \rho_3, \rho_4$ be the roots of

$$x^4 - 47x^3 + 192489x^2 - 9012848x + 122529840,$$

with ρ_1, ρ_2 having positive imaginary parts. Then

$$\Psi_{-15,1} = \frac{(T_g(\tau) - \rho_1)^2 (T_g(\tau) - \rho_2)^2}{(T_g(\tau) - \rho_3)^2 (T_g(\tau) - \rho_4)^2}.$$

We get that

$$\lambda_{-15} = \sqrt{-15},$$

and

$$\frac{1}{\lambda_\Delta} \sum_i \gamma_i T_g(\alpha_i) = 462 = c^{(2)}\left(15/8, 1/4\right) = \dim_{K^{(2)}_{1,15/8}}.$$

In view of this correspondence, it is clear that the mock modular forms of umbral moonshine have important properties. The congruence properties of their coefficients have just begun to be studied. For example, [CHM14] examines the parity of the coefficients of the McKay-Thompson series for Mathieu moonshine in relation to a certain conjecture in [CDH14b], which in our case corresponds to $X = A_1^{24}$. Congruences modulo higher primes were also considered in [MW14].

Let $\Theta := q\frac{d}{dq} = \frac{1}{2\pi i}\frac{d}{d\tau}$. Given the product expansion of a generalized Borcherds product, it is natural to consider its logarithmic derivative. It turns out that this logarithmic derivative has nice arithmetic properties. This idea was also used in [BO10] and [Ono10].

Theorem 1.2.3. *Fix a pure A-type Niemeier lattice X with Coxeter number m. Let (Δ, r) be an admissible pair. Consider the logarithmic derivative*

$$L_{\Delta,r}(\tau) = \sqrt{\Delta} \sum a_{\Delta,r}(n) q^n := \sqrt{\Delta} \sum_n \sum_{ij=n} i c^X\left(\frac{|\Delta| i^2}{4m}, \frac{ri}{2m}\right)\left(\frac{\Delta}{j}\right) q^n$$

of $\Psi_{\Delta,r}(\tau, H^X)$. Then $L_{\Delta,r}(\tau)$ is a meromorphic weight 2 modular form.

When p is inert or ramified in $\mathbb{Q}(\sqrt{\Delta})$, it turns out that $L_{\Delta,r}(\tau)$ is more than just a meromorphic modular form; it is a p-adic modular form. Essentially, a p-adic modular form is a q-series which is congruent modulo any power of p to a holomorphic modular form; we refer the reader to Section 3.2.1 for the definition.

Theorem 1.2.4. *Let X be a pure A-type Niemeier lattice with Coxeter number m. Let (Δ, r) be admissible and suppose p is inert or ramified in $\mathbb{Q}(\sqrt{\Delta})$. Then $L_{\Delta,r}$ is a p-adic modular form of weight* 2.

We will use this result to study the p-divisibility of the coefficients $a_{\Delta,r}(n)$.

Corollary 1.2.5. *Let X, Δ, r, p be as above. Then for all $k \geq 1$ there exists $\alpha_k > 0$ such that*

$$\#\{n \leq x : a_{\Delta,r}(n) \not\equiv 0 \pmod{p^k}\} = O\left(\frac{x}{(\log x)^{\alpha_k}}\right).$$

In particular, if we let

$$\pi_{\Delta,r}(x; p^k) := \#\{n \leq x : a_{\Delta,r}(n) \equiv 0 \pmod{p^k}\},$$

then

$$\lim_{x\to\infty} \frac{\pi_{\Delta,r}(x; p^k)}{x} = 1.$$

Remark 5. Corollary 1.2.5 also applies to any constant multiple of $L_{\Delta,r}$ with integral coefficients. In the example below, we consider the coefficients of

$$\frac{L_{-7,1}(\tau)}{90\sqrt{-7}} = q + O(q^2).$$

However, it is not always the case that the analogous normalization has integral coefficients.

Example. We illustrate Corollary 1.2.5 for $X = A_1^{24}$, $\Delta = -7$, $r = 1$. Note that this is the same case considered in Example 1.2.1. The first few coefficients of the normalized logarithmic derivative are given by

$$\frac{L_{-7,1}(\tau)}{90\sqrt{-7}} =: \sum_{n\geq 1} a_{-7,1}(n)q^n = q + q^2 - 4371q^3 + q^4 + 17773755q^5 + \dots$$

The prime $p = 2$ is split in $\mathbb{Q}(\sqrt{-7})$, and so Theorem 1.2.4 and Corollary 1.2.5 do not apply. Therefore, we expect the coefficients $a_{-7,1}(n)$ to be equally distributed modulo 2, but cannot prove anything about them. The prime $p = 3$ is inert, so Corollary 1.2.5 tell us that, asymptotically, 100% of the coefficients $a_{-7,-1}(n)$ are divisible by 3. We illustrate this behavior in Table 1.4.

Table 1.4: Divisibility of $a_{-7,1}(n)$ by $p = 2, 3$

x	$\pi_2(x)/x$	$\pi_3(x)/x$
50	0.38	0.64
100	0.45	0.68
150	0.47	0.69
200	0.49	0.71
250	0.48	0.71
300	0.49	0.72
$\vdots$	$\vdots$	$\vdots$
∞	.5?	1

1.2.2 Second Result

In Chapter 4, we consider the Mathieu group M_{23}, which is the point stabilizer of the action of M_{24} on 24 points. It is a sporadic group, with about 10^7 elements in 17 conjugacy classes. M_{23} inherits a moonshine from M_{24} whose McKay-Thompson series are weight 1/2 mock modular forms. However, M_{23} exhibits another moonshine. We show that there exists a different infinite dimensional graded representation of M_{23} whose McKay-Thompson series are Hauptmoduln for monstrous genus zero congruence subgroups.

For a conjugacy class g of M_{23}, we start with the dual families of Rademacher sums (see Section 2.3 for more information about Rademacher sums):

$$\left\{ H_g^{[\mu]}(\tau) := -2q^{\mu} - 2 \sum_{\substack{\nu - \frac{1}{8} \in \mathbb{Z} \\ \nu < 0}} A_g(\mu,\nu) q^{-\nu} \mid \mu + \frac{1}{8} \in \mathbb{Z}, \mu < 0 \right\} \tag{1.2.1}$$

and

$$\left\{ F_g^{[\nu]}(\tau) := 2q^{\nu} - 2 \sum_{\substack{\mu + \frac{1}{8} \in \mathbb{Z} \\ \mu < 0}} A_g(\mu,\nu) q^{-\mu} \mid \nu - \frac{1}{8} \in \mathbb{Z}, \nu < 0 \right\}. \tag{1.2.2}$$

Here, $H_g^{[-1/8]}$ is the Mathieu moonshine McKay-Thompson series $H_g^{(2)}$. A priori, the $H_g^{[\mu]}$ are weight 1/2 mock modular forms on $\Gamma_0(n_g)$ with multiplier system ϵ^{-3}, while the $F_g^{[\nu]}$ are weight 3/2 mock modular forms on $\Gamma_0(n_g)$ with multiplier system ϵ^3. Here, ϵ is the multiplier system of $\eta(\tau)$ as described in Appendix A and n_g is the order of the elements in the conjugacy class g. However, we show in Theorem 4.1.2 that the $F_g^{[\nu]}$ are actually modular!

We then define

$$f_g(\tau) := \frac{F_g^{[-7/8]}(\tau)}{\eta^3(\tau)} = q^{-1} + \sum_{n \geq 0} c_g(n) q^n \in M_0^!(\Gamma_0(n_g)).$$

Note that $c_g(0) = 3 + \frac{A_g(-1/8,-7/8)}{2}$, which is given in Table 1.5.

Our first theorem gives a surprising connection between these functions and the McKay-Thompson series of monstrous moonshine. In particular, for each conjugacy class g of M_{23}, we associate a conjugacy class $\hat{g}$ of the monster group $\mathbb{M}$ as in Table 1.6. Then we have the following:

Theorem 1.2.1. We have that

$$f_g(\tau) = 3 + \frac{A_g(-1/8,-7/8)}{2} + T_{\hat{g}}(\tau) \in M_0^!(\Gamma_{\hat{g}}),$$

where $T_{\hat{g}}(\tau)$ is the monstrous moonshine McKay-Thompson series and $\Gamma_{\hat{g}}$ is the genus

Table 1.5: Constant Term of f_g

M_{23} Conj. Class g	Constant Term of f_g $c_g(0)$
$1a$	48
$2a$	0
$3a$	3
$4a$	4
$5a$	3
$6a$	3
$7ab$	$\frac{5}{2}$
$8a$	2
$11ab$	4
$14ab$	$\frac{7}{2}$
$15ab$	3
$23ab$	2

zero group associated to $\hat{g}$ in monstrous moonshine, as given in Table 1.6.

Remark 6. Coefficients and expressions for the monstrous moonshine McKay-Thompson series $T_g(\tau)$ can be found in [CN79]. Note that $T_{\hat{g}}(\tau)$ and $f_g(\tau)$ are differentially normalized generators of the modular functions on $\Gamma_{\hat{g}}$.

Furthermore, we show that these f_g are the McKay-Thompson series for a new moonshine on M_{23}.

Theorem 1.2.2. There exists a graded M_{23}-module

$$V = \bigoplus_{n=-1}^{\infty} V_n$$

Table 1.6: Correspondance Between Conj. Classes of M_{23} and the Monster Group $\mathbb{M}$

M_{23} Conj. Class g	Monster Conj. Class $\hat{g}$	Congruence Subgroup $\Gamma_{\hat{g}}$
$1a$	$1A$	$\Gamma_0(1)$
$2a$	$2B$	$\Gamma_0(2)$
$3a$	$3B$	$\Gamma_0(3)$
$4a$	$4C$	$\Gamma_0(4)$
$5a$	$5B$	$\Gamma_0(5)$
$6a$	$6E$	$\Gamma_0(6)$
$7ab$	$7B$	$\Gamma_0(7)$
$8a$	$8E$	$\Gamma_0(8)$
$11ab$	$11A$	$\Gamma_0(11)+11$
$14ab$	$14C$	$\Gamma_0(14)+14$
$15ab$	$15C$	$\Gamma_0(15)+15$
$23ab$	$23AB$	$\Gamma_0(23)+23$

such that the graded trace of g on M_{23} is $2f_g(\tau)$, i.e.

$$2f_g(\tau) = \sum_{n=-1}^{\infty} \operatorname{tr}(g \mid V_n) q^n.$$

Remark 7. While the order of M_{23} divides the order of the monster group $\mathbb{M}$, [NW02] showed that M_{23} is not a subgroup of $\mathcal{M}$. This rules out the possibility that our moonshine for M_{23} comes directly from monstrous moonshine via restriction.

Remark 8. The reason we use $2f_g$ is that a few of the constant terms of f_g are half-integral, as can be seen in Table 1.5.

Example. Let $g = 1a$. Then

$$F_{1a}^{[-7/8]}(\tau) = 2q^{-7/8} + 90q^{1/8} + 393480q^{9/8} + O(q^{17/8}) = 2(J(\tau)+48) * \eta^3(\tau)$$

and

$$f_{1a}(\tau) = q^{-1}+48+196884q+21493760q^2+21493760q^3+O(q^4) = 48+J(\tau) = 48+T^{\natural}_{1A}(\tau)$$

In monstrous moonshine we famously have

$$196884 = 196883 + 1,$$

where 196883 and 1 are dimensions of representations of the monstrous group $\mathbb{M}$. However, in our new moonshine for M_{23} we have that 196884 equals

$$9\cdot\mathbf{1}+2\cdot\mathbf{22}+8\cdot\mathbf{45}+8\cdot\mathbf{230}+15\cdot\mathbf{231}+6\cdot\mathbf{253}+28\cdot\mathbf{770}+32\cdot\mathbf{896}+36\cdot\mathbf{990}+24\cdot\mathbf{1035}+39\cdot\mathbf{2024},$$

where 1, 22, 45, 230, 231, 253, 770, 896, 990, 1035, and 2024 are **all** of the dimensions of the irreducible representations of M_{23}, as can be seen in Appendix B. Therefore, it would be extremely hard to notice this moonshine by comparing coefficients of Hauptmoduln to the degrees of irreducible characters!

Chapter 2

Background

2.1 Vector-Valued Modular Forms

In this section, we follow [BO10] in giving the needed background on vector-valued modular forms, though we state results in less generality. Also see [BFOR17].

2.1.1 A Lattice Related to $\Gamma_0(m)$

We will define a lattice L and a dual lattice L' related to $\Gamma_0(m)$ such that the components of our vector-valued modular forms are labeled by the elements of L'/L.

We consider the quadratic space

$$V := \{X \in \mathrm{Mat}_2(\mathbb{Q}) : \mathrm{tr}(X) = 0\}$$

with the quadratic form $P(X) := m\det(X)$.[1] The corresponding bilinear form is then $(X, Y) := -m\,\mathrm{tr}(XY)$. Let L be the lattice

$$L := \left\{ \begin{pmatrix} b & -a/m \\ c & -b \end{pmatrix}; \quad a, b, c \in \mathbb{Z} \right\}.$$

---

[1] Note that this corrects a typo in [BO10].

The dual lattice is then given by

$$L' := \left\{ \begin{pmatrix} b/2m & -a/m \\ c & -b/2m \end{pmatrix} ; \quad a, b, c \in \mathbb{Z} \right\}.$$

We will switch between viewing elements of L' as matrices and as quadratic forms, with the matrix

$$X = \begin{pmatrix} b/2m & -a/m \\ c & -b/2m \end{pmatrix}$$

corresponding to the integral binary quadratic form

$$Q = [mc, b, a] = mcx^2 + bxy + cy^2.$$

Note that then $P(X) = -\operatorname{Disc}(Q)/4m$.

We identify L'/L with $(\frac{1}{2m}\mathbb{Z})/\mathbb{Z}$, and the quadratic form P with the quadratic form $\frac{j}{2m} \mapsto \frac{-j^2}{4m}$ on $\mathbb{Q}/\mathbb{Z}$. We will also occasionally identify $\frac{j}{2m} \in \mathbb{Q}/\mathbb{Z}$ with $j \in \mathbb{Z}/2m\mathbb{Z}$.

For a fundamental discriminant Δ and $r/2m \in L'/L$ with $r^2 \equiv \Delta \pmod{4m}$, let

$$Q_{\Delta,r} := \{Q = [mc, b, a] : a, b, c \in \mathbb{Z}, \operatorname{Disc}(Q) = \Delta, b \equiv r \pmod{2m}\}. \tag{2.1.1}$$

The action of $\Gamma_0(m)$ on this set is given by the usual action of congruence subgroups on binary quadratic forms. We will later be working with $Q_{\Delta,r}/\Gamma_0(m)$.

2.1.2 The Weil Representation

By $\mathrm{Mp}_2(\mathbb{Z})$ we denote the integral metaplectic group. It consists of pairs (γ, ϕ), where $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$ and $\phi : \mathbb{H} \to \mathbb{C}$ is a holomorphic function with $\phi^2(\tau) = c\tau + d$. The group $\widetilde{\Gamma} := \mathrm{Mp}_2(\mathbb{Z})$ is generated by $S := (\left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right), \sqrt{\tau})$ and $T := (\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right), 1)$.

We consider the *Weil representation* ρ_L of $\mathrm{Mp}_2(\mathbb{Z})$ corresponding to the discriminant form L'/L. We denote the standard basis elements of $\mathbb{C}[L'/L]$ by $\mathfrak{e}_j$, $j/2m \in L'/L$. Then the Weil representation ρ_L associated with the discriminant form L'/L is the unitary representation of $\widetilde{\Gamma}$ on $\mathbb{C}[L'/L]$ defined by

$$\rho_L(T)\mathfrak{e}_j = e(j^2/4m)\mathfrak{e}_j,$$

and

$$\rho_L(S)\mathfrak{e}_j = \frac{e(-1/8)}{\sqrt{2m}} \sum_{i\in\mathbb{Z}/2m\mathbb{Z}} e(ij/2m)\mathfrak{e}_i.$$

2.1.3 Harmonic Weak Maass Forms

If $f\colon \mathbb{H} \to \mathbb{C}[L'/L]$ is a function, we write

$$f = \sum_{j\in\mathbb{Z}/2m\mathbb{Z}} f_j\mathfrak{e}_j$$

for its decomposition into components. For $k \in \frac{1}{2}\mathbb{Z}$, let $M^!_{k,\rho_L}$ denote the space of $\mathbb{C}[L'/L]$ valued weakly holomorphic modular forms of weight k and type ρ_L for the group $\widetilde{\Gamma}$. The subspaces of holomorphic modular forms (resp. cusp forms) are denoted by M_{k,ρ_L} (resp. S_{k,ρ_L}). Now, assume that $k \leq 1$. A twice continuously differentiable function $f : \mathbb{H} \to \mathbb{C}[L'/L]$ is called a *harmonic weak Maass form* (of weight k with respect to $\widetilde{\Gamma}$ and ρ_L) if it satisfies:

1. $f(M\tau) = \phi(\tau)^{2k}\rho_L(M,\phi)f(\tau)$ for all $(M,\phi) \in \widetilde{\Gamma}$;
2. $\Delta_k f = 0$;
3. There is a polynomial

$$P_f(\tau) = \sum_{j\in\mathbb{Z}/2m\mathbb{Z}} \sum_{\substack{n\in\mathbb{Z}-\frac{j^2}{4m},\\ -\infty<<n\leq 0}} c^+(n,h)e(n\tau)\mathfrak{e}_j$$

such that

$$f(\tau) - P_f = O(e^{-\epsilon v})$$

for some $\epsilon > 0$ as $v \to +\infty$.

Note here that

$$\Delta_k := -v^2\left(\frac{\partial^2}{\partial u^2} + \frac{\partial^2}{\partial v^2}\right) + ikv\left(\frac{\partial}{\partial u} + i\frac{\partial}{\partial v}\right)$$

is the usual weight k hyperbolic Laplace operator, and that $\tau = u+iv$. We denote the vector space of these harmonic weak Maass forms by $\mathcal{H}_{k,\rho_L}$. The Fourier expansion of any $f \in \mathcal{H}_{k,\rho_L}$ gives a unique decomposition $f = f^+ + f^-$, where

$$\begin{aligned} f^+(\tau) &= \sum_{j\in\mathbb{Z}/2m\mathbb{Z}} \sum_{\substack{n\in\mathbb{Z}-\frac{j^2}{4m},\\ -\infty<<n}} c^+(n,j)e(n\tau)\mathfrak{e}_j, & (2.1.2)\\ f^-(\tau) &= \sum_{j\in L'/L} \sum_{\substack{n\in\mathbb{Q},\\ n<0}} c^-(n,j)W(2\pi nv)e(n\tau)\mathfrak{e}_j, & (2.1.3) \end{aligned}$$

and $W(x) := \int_{-2x}^{\infty} e^{-t}t^{-k}dt = \Gamma(1-k, 2|x|)$ for $x < 0$. Then f^+ is called the *holomorphic part* and f^- the *nonholomorphic part* of f. The polynomial P_f is also uniquely determined by f and is called its *principal part.* We define a *mock modular form* of weight k to be the holomorphic part f^+ of a harmonic weak Maass form f of weight k which has $f^- \neq 0$. Its weight is just the weight of the harmonic weak Maass form.

Recall that there is an antilinear differential operator defined by

$$\xi_k : \mathcal{H}_{k,\overline{\rho}_L} \to S_{2-k,\rho_L}, \quad f(\tau) \mapsto \xi_k(f)(\tau) := 2iy^k\overline{\frac{\partial}{\partial\overline{\tau}}},$$

where $\overline{\rho}_L$ is the complex conjugate representation. The Fourier expansion of $\xi_k(f)$ is given by

$$\xi_k(f) = -\sum_{j\in\mathbb{Z}/2m\mathbb{Z}} \sum_{n\in\mathbb{Q},n>0} (4\pi n)^{1-k}\overline{c^-(-n,j)}q^n\mathfrak{e}_j.$$

The kernel of ξ_k is equal to $M^!_{k,\bar{\rho}_L}$, and we have the following exact sequence:

$$0 \to M^!_{k,\bar{\rho}_L} \to \mathcal{H}_{k,\bar{\rho}_L} \to S_{2-k,\rho_L} \to 0.$$

We call $\xi_k(f)$ the *shadow* of f. Note that $\xi_k(f)$ uniquely determines f^-, but the f^+ is only determined up to the addition of a weakly holomorphic modular form.

2.2 Umbral Moonshine

In this section, we summarize the main objects and conjectures of umbral moonshine. However, we first briefly describe Mathieu moonshine, which umbral moonshine generalized.

2.2.1 Mathieu Moonshine

In 2010, the study of a new form of moonshine commenced, called Mathieu moonshine. Let $\mu(z,\tau) := \mu(z,z,\tau)$ be Zwegers' famous function from his thesis [Zwe02], which is defined in the appendix. Let $H^{(2)}(\tau)$ be the q-series

$$H^{(2)}(\tau) := -8 \sum_{\omega \in \{\frac{1}{2}, \frac{1+\tau}{2}, \frac{\tau}{2}\}} \mu(\omega,\tau) = 2q^{-1/8}(-1 + 45q + 231q^2 + \dots), \tag{2.2.1}$$

which occurs in the decomposition of the elliptic genus of a K3 surface into irreducible characters of the $N = 4$ superconformal algebra. This is a mock-modular form, and plays the role of $J(\tau)$ in Mathieu moonshine. Eguchi, Ooguri, and Tachikawa conjectured that the Fourier coefficients encode dimensions of irreducible representations of the Mathieu group M_{24} [EOT11]. This was extended to the full Mathieu moonshine conjecture by [Che10,EH11,GHV10a,GHV10b], which included providing mock modular forms $H^{(2)}_g$ for every $g \in M_{24}$. The existence of an infinite dimensional M_{24} module underlying the mock modular forms was shown by Gannon in 2012 [Gan16].

In the context of umbral moonshine, $H^{(2)}(\tau)$ is viewed as vector-valued with components $H_r^{(2)}(\tau)$ for $r \in \mathbb{Z}/4\mathbb{Z}$. However, since $H_0^{(2)} = H_2^{(2)} = 0$ and $H_3^{(2)} = -H_1^{(2)}$, in practice we often just focus on the component $H_1^{(2)}$. That's what's given in (2.2.1).

2.2.2 The Objects of Umbral Moonshine

Cheng, Duncan, and Harvey generalized even further - conjecturing that Mathieu moonshine is but one example of a more general phenomenon which they call umbral moonshine [CDH14a].

For each of the 23 Niemeier root systems X, which are unions of irreducible simply-laced root systems with the same Coxeter number, they associate many objects, including a group G^X (playing the role of M), a mock modular form $H^X(\tau)$ (playing the role of $j(\tau)$), and an infinite dimensional graded G^X module K^X (playing the role of the M-module $V^\natural$) Table 2.1 gives a more complete list of the associated objects.

The ADE classification of simply laced Dynkin diagrams allows us to classify the irreducible components of the Niemeier root systems X. We will focus on the simplest cases - the root systems of pure A-type, i.e. $X = A_{m-1}^{24/(m-1)}$, where $(m-1) \mid 24$. In these cases, the lambency ℓ is an integer and equals m, and $\Gamma^X = \Gamma_0(m)$. The case $X = A_1^{24}$ corresponds to Mathieu moonshine, with $G^X = M_{24}$ and $H^X = H^{(2)}$, as defined above. We will generally refer to H^X , S^X, ψ^X, and T^X as $H^{(m)}$, $S^{(m)}$, $\psi^{(m)}$, and j_m respectively. These are the main quantities from Table 2.1 that we will work with, and we will only define them for pure A-type. This is done in Section 2.2.4.

Table 2.1: Objects Associated to a Niemeier Root System X

L^X	The Niemeier lattice corresponding to X.
m	The Coxeter number of all irreducible components of X.
W^X	The Weyl group of X.
$G^X := \mathrm{Aut}(L^X)/W^X$	The umbral group corresponding to X.
π^X	The (formal) product of Frame shapes of Coxeter elements of irreducible components of X.
Γ^X	The genus zero subgroup attached to X.
T^X	The normalized Hauptmodul of Γ^X, whose eta-product expansion corresponds to π^X.
ℓ	The lambency. A symbol that encodes the genus zero group Γ^X. Sometimes used instead of X to denote which case of umbral moonshine is being considered.
ψ^X	The unique meromorphic Jacobi form of weight 1 and index m satisfying certain conditions.
H^X	The vector-valued mock modular form of weight $1/2$ whose $2m$ components furnish the theta expansion of the finite part of ψ^X. Called the umbral mock modular form.
S^X	The vector-valued cusp form of weight $3/2$ which is the shadow of H^X. Called the umbral shadow.
H_g^X	The umbral McKay-Thompson series attached to $g \subset G^X$. It is a vector-valued mock modular form of weight $1/2$, and equals H^X when g is the identity.
S_g^X	The vector-valued cusp form *conjectured* to be the shadow of H_g^X.
K^X	The *conjectural* infinite dimensional graded G^X-module whose graded super-dimension is encoded by H^X.

2.2.3 The Conjectures and Proof Strategy of Umbral Moonshine

The main conjectures of umbral moonshine are as follows:

1. The mock modular form H^X encodes the graded super-dimension of a certain infinite-dimensional, $\mathbb{Z}/2m\mathbb{Z} \times \mathbb{Q}$-graded G^X-module K^X.

2. The graded super-characters H_g^X arising from the action of G^X on K^X are vector-valued mock modular forms with concretely specified shadows S_g^X.

Remark 9. Originally, it was thought that H_g^X was the unique, up to scale, mock modular form of weight $1/2$ for $\Gamma_0(n)$ with *optimal growth*, for suitably chosen n, multiplier system, and shadow. However, this was shown in [CDH18] to be false in a few cases. An alternate analogy of the genus zero property from monstrous moonshine was given and proven in [CDH18]. It uses Rademacher sums and will be discussed in Section 2.3.4.

We will now describe the general strategy used by Gannon in [Gan16] and Duncan-Griffin-Ono in [DGO15] to prove the umbral moonshine conjectures.

In order to prove moonshine for a group G with proposed McKay-Thompson series $T_g(\tau)$, one approach is to study the series T_χ where $\chi \in \hat{G}$, defined by

$$T_\chi(\tau) := \frac{1}{|G|} \sum_{g \in G} \chi(g) T_g(\tau), \tag{2.2.2}$$

where the sum is over all elements of G. The idea is that if a G-module V exists for which the $T_g(\tau)$ are the graded traces, then we have the following. First, there are nonnegative integers $m_\chi(n)$ such that $V = \bigoplus_n V_n$ with $V_n = \bigoplus_\chi V_\chi^{m_\chi(n)}$. Secondly, we'll have that

$$T_g(\tau) = \sum_n \sum_\chi m_\chi(n) \chi(g) q^n. \tag{2.2.3}$$

So in order to prove that there exists a G-module V for which the $T_g(\tau)$ are the graded traces, it's enough to prove that the coefficients $m_\chi(n)$ in (2.2.3) are nonnegative integers. Then we can use them to construct V out of irreducibles. Note that this is not a completely satisfying conclusion, as we hope for moonshine modules to have "natural" constructions equipped with deeper algebraic structure, like the monster module $V^\natural$.

Starting with (2.2.2) and (2.2.3), the orthogonality of characters implies that

$$T_g(\tau) = \sum_\chi \chi(g) T_\chi(\tau).$$

This in turn gives us that

$$T_\chi(\tau) = \sum_n m_\chi(n) q^n.$$

So the goal is then to show that the coefficients of the $T_\chi(\tau)$ are nonnegative integers. This can be broken into two steps. First, showing that they're integers, and next showing that they're nonnegative. Note that Atkin, Fong, and Smith used this strategy on monstrous moonshine in [Smi85], but didn't quite show that the $m_\chi(n)$ were nonnegative.

2.2.4 Defining the Umbral Mock Modular Forms

In this section we define the mock modular forms H^X from umbral moonshine, as well as their shadows S^X and non-holomorphic parts. Note that we only give definitions for the pure A-type cases - see [CDH14a] for a more detailed and general definition. We also refer the reader to Appendix A for definitions of $\varphi_1^{(m)}(\tau, z)$, $\mu_{m,0}(\tau, z)$, $\theta_{m,j}(\tau, z)$, and $R(u; \tau)$.

For a pure A-type Niemeier lattice X with Coxeter number m, define the Jacobi form ψ^X by

$$\psi^X(\tau, z) := c_m \varphi_1^{(m)}(\tau, z) \mu_{1,0}(\tau, z),$$

where $c_m = 2$ for $m = 2, 3, 4, 5, 7, 13$ and $c_m = 1$ for $m = 9, 25$. We can break up ψ^X into a finite part ψ_F^X and a polar part ψ_P^X. The polar part is given by

$$\psi_P^X(\tau, z) = \frac{24}{m-1}\mu_{m,0}(\tau, z).$$

Then the mock modular form H^X is defined by

$$\psi_F^X(\tau, z) = \psi^X(\tau, z) - \psi_P^X(\tau, z) = \sum_{j \in \mathbb{Z}/2m\mathbb{Z}} H_j^X(\tau)\theta_{m,j}(\tau, z), \tag{2.2.4}$$

where

$$\theta_{m,j}(\tau, z) := \sum_{n \equiv j \pmod{2m}} q^{n^2/4m} y^n.$$

We also define the shadow $S^X(\tau)$, the non-holomorphic part $F^X(\tau)$, and the harmonic weak Maass form $\widehat{H}^X(\tau)$ corresponding to the mock modular form H^X via their components:

$$\begin{aligned}
S_j^X(\tau) &:= \sum_{n \equiv j \pmod{2m}} n q^{n^2/4m}, && (2.2.5)\\
F_j^X(\tau) &:= \int_{-\overline{\tau}}^{i\infty} \frac{S_j^X(z)}{\sqrt{-i(z+\tau)}} dz && (2.2.6)\\
&= -2mq^{-(j-m)^2/4m} R\left(\frac{j-m}{2m}(2m\tau) + \frac{1}{2}; 2m\tau\right), \text{ and} \\
\widehat{H}_j^X(\tau) &:= H_j^X(\tau) + F_j^X(\tau) && (2.2.7)
\end{aligned}$$

Note that by definition, $S_j^X(\tau) = -S_{-j}^X(\tau)$. Therefore, $S_0^X = S_m^X = 0$. The same is true of H_j^X. We can write this in terms of Shimura's theta functions as $S_j^X(\tau) = \theta(\tau; j, 2m, 2m, x)$ [Shi73]. Then using the transformation laws for his θ-

functions, we get that S^X transforms as follows:

$$\begin{aligned} S_j^X(\tau+1) &= e(j^2/4m)S_j^X(\tau), \text{ and} \\ S_j^X(-1/\tau) &= \tau^{3/2}\frac{e(-1/8)}{\sqrt{2m}}\sum_{i \pmod{2m}} e(ij/2m)S_k^X(\tau). \end{aligned}$$

Thus, we have

$$\begin{aligned} S^X(\tau+1) &= \rho_L(T)S^X(\tau), \text{ and} \\ S^X(-1/\tau) &= \tau^{3/2}\rho_L(S)S^X(\tau). \end{aligned}$$

From these transformations, we see that $S^X(\tau) : \mathbb{H} \to \mathbb{C}[L'/L]$ is a weight 3/2 vector-valued modular form transforming under the Weil representation ρ_L, i.e. an element of the space $M_{3/2,\rho_L}$. From [CDH14a], we know that $H^{(m)}$ is a mock modular form with shadow S^X. This gives us the following theorem.

Theorem 2.2.1. *We have that $\widehat{H}^X(\tau) : \mathbb{H} \to \mathbb{C}[L'/L]$ is a weight 1/2 vector-valued harmonic weak Maass form transforming under the Weil representation $\overline{\rho}_L$, i.e., it is an element of $\mathcal{H}_{1/2,\overline{\rho}_L}$. Moreover, it has shadow $S^X(\tau)$, non-holomorphic part F^X, and principal part $P(\tau) = -2q^{-1/4m}(\mathfrak{e}_1 - \mathfrak{e}_{2m-1})$.*

2.3 Rademacher Sums

In this section, we will discuss a method of building modular forms that will be important in Chapter 4. For more details, see [CD14].

2.3.1 Introduction To Rademacher Sums

The general idea is as follows: If you want to construct a symmetric function from a non-symmetric one, you can simply sum its images under the desired group of symmetries. Of course, when the group of symmetries is infinite, issues of convergence come up.

To address this problem, Poincaré (see [Poi11]) started with a function that was already invariant under a large enough group of symmetries and then restricted the summation to representatives of the cosets of the subgroup fixing f. So if we let $f(\tau) = e(m\tau)$, where $m \in \mathbb{Z}$, then f is invariant under the subgroup of upper triangular matrices, denoted Γ_∞. Therefore, we can consider

$$\widetilde{f}(\tau) := \sum_{M \in \Gamma_\infty \backslash \Gamma} f(M\tau) \frac{1}{(c\tau + d)^w}$$

where $M = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$.

For $w \geq 4$, this sum converges absolutely, locally uniformly in τ, and so gives a holomorphic function on the upper half plane. When $m \geq 0$, it's bounded at $i\infty$ and so $\widetilde{f}(\tau)$ is a modular form of weight w on $\mathrm{SL}_2(\mathbb{Z})$. This also works for more general congruence subgroups Γ and multipliers.

For $w \leq 2$, more work is required. Rademacher (see [Rad39]) came up with a solution for $w = 0$. He showed that

$$J(\tau) + 12 = e(-\tau) + \lim_{K \to \infty} \sum_{\substack{M \in \Gamma_\infty \backslash \Gamma \\ 0 < c < K \\ -K^2 < d < K^2}} e(-M\tau) - e(-a/c) \tag{2.3.1}$$

where again $M = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$.

This sum is conditionally convergent, and the right hand side is a modification of the Poincaré sum with $w = 0$ and $m = -1$. This idea has been successfully generalized to other groups and (some) weights, but the modularity doesn't usually completely survive the regularization procedure - it instead yields mock modular forms.

Let Γ be a subgroup of $\mathrm{SL}_2(\mathbb{R})$ that is commensurable with $\mathrm{SL}_2(\mathbb{Z})$ and contains $-I$. Say it has width h at the cusp $i\infty$. Let ψ be a multiplier of weight $w \in \mathbb{R}$ and

$0 \leq \alpha < 1$ be such that $\psi(T^h) = e(\alpha)$. Then for any index μ such that $h\mu + \alpha \in \mathbb{Z}$, we can define the Rademacher sum $R^{[\mu]}_{\Gamma,\psi,w}$, which is a mock modular form on Γ with weight w and multiplier ψ. We will not give the general definition here (see [CD14]), but it is similar in structure to that of $J(\tau)$ in 2.3.1.

2.3.2 Rademacher Series and Zagier Duality

In practice, it's often more useful to write Rademacher sums is in terms of their Fourier expansion. We expect (and can prove in many cases) that

$$R_{\Gamma,\psi,w}(\tau) = q^u + \sum_{\substack{h\nu+\alpha\in\mathbb{Z} \\ \nu\geq 0}} c_{\Gamma,\psi,w}(\mu,\nu)q^\nu,$$

where the Fourier coefficients are called Rademacher series and are given in terms of Kloosterman sums $K_{\gamma,\psi}$ and Bessel functions $B_{\gamma,w}$.

These Rademacher series exhibit a Zagier duality, which generalizes that in [Zag02]. In particular, we have that

$$c_{\Gamma,\overline{\psi},2-w}(-\nu,-\mu) = c_{\Gamma,\psi,w}(\mu,\nu)$$

when $\mu,\nu \in \frac{1}{h}(\mathbb{Z}-\alpha)$. This comes from a symmetry in the Bessel functions and Kloosterman sums that define these series. Therefore, we expect (and can prove in many cases) dual families of Rademacher sums whose coefficients lie on a grid:

$$\{R^{[\mu]}_{\Gamma,\psi,w} \mid h\mu+\alpha\in\mathbb{Z}, \mu<0\},\ \{R^{[\nu]}_{\Gamma,\overline{\psi},2-w} \mid h\nu-\alpha\in\mathbb{Z}, \nu<0\}.$$

2.3.3 Monstrous Moonshine Functions as Rademacher Sums

For monstrous moonshine, we look at $\Gamma = \Gamma_g$ for $g \in \mathbb{M}$, $\psi = 1$, $w = 0$, and $\mu = -1$. We have that

$$R^{[-1]}_{\Gamma_g,1,0} = q^{-1} + \sum_{k\geq 0} c_{\Gamma_g,1,0}(-1,k) q^k$$

where

$$c_{\Gamma_g,1,0}(-1,k) = \frac{2\pi}{\sqrt{k}} \sum_{b>0} \frac{1}{|g|b} I_1\left(\frac{4\pi\sqrt{k}}{|g|b}\right) K(k,1,|g|b).$$

Here, K is a Kloosterman sum and I is an I-Bessel function. Both are defined in Appendix A.

We have that T_g matches the Rademacher sum $R^{[-1]}_{\Gamma_g,1,0}$ up to the constant term:

$$T_g(\tau) = R^{[-1]}_{\Gamma_g,1,0}(\tau) - c_{\Gamma_g,1,0}(-1,0).$$

Furthermore, we have that $R^{[-1]}_{\Gamma,1,0}$ is modular exactly when Γ has genus zero. See [DF11] for proofs of all this, starting with the convergence and Fourier expansion of $R^{[\mu]}_{\Gamma,1,0}$.

2.3.4 Mathieu Moonshine Functions as Rademacher Sums

Let $g \in M_{24}$. Then we define a character ρ_g as follows. Define n_g to be the order of g and h_g be the minimal length among cycles in the cycle shape of g when g is regarded as a permutation in the unique non-trivial permutation action of M_{24} on 24 points. Then ρ_g is given by

$$\rho_g(\gamma) = e\left(-\frac{cd}{n_g h_g}\right) \tag{2.3.2}$$

for $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_0(n_g)$. The fact that (2.3.2) defines a morphism of groups $\Gamma_0(n_g) \to \mathbb{C}^\times$ relies on the fact that h_g is always a divisor of 24. Note that h_g is also a divisor of n_g, and for $g \in M_{23}$, $h_g = 1$. Finally, let $\psi_g = \rho_g \epsilon^{-3}$, where ϵ is the multiplier system of $\eta(\tau)$ as given in Appendix A.

Then for $\mu, \nu < 0$ satisfying $\mu \in \mathbb{Z} - 1/8, \nu \in \mathbb{Z} + 1/8$, we have the following theorems. They are proven in [CD12] for $\mu = -1/8$, but the same methods work more generally. For ease of notation, we let $\psi = \psi_g$ and $\Gamma = \Gamma_0(n_g)$.

Theorem 2.3.1. The Rademacher sums $R^{[\mu]}_{\Gamma,\psi,1/2}$ and $R^{[\nu]}_{\Gamma,\overline{\psi},3/2}$ converge, locally uniformly for $\tau \in \mathbb{H}$ and are weak mock modular forms bounded at all cusps besides $i\infty$.

Theorem 2.3.2. The Rademacher series $c_{\Gamma,\psi,1/2}(\mu, -\nu)$ and $c_{\Gamma,\overline{\psi},3/2}(\nu, -\mu)$ converge and are equal. Moreover, they are the coefficients of the corresponding Rademacher sums:

$$R^{[\mu]}_{\Gamma,\psi,1/2} = q^{\mu} + \sum_{\substack{\nu<0 \\ \nu\in\mathbb{Z}+1/8}} c_{\Gamma,\psi,1/2}(\mu, -\nu) q^{-\nu}$$

and

$$R^{[\nu]}_{\Gamma,\overline{\psi},3/2} = q^{\nu} + \sum_{\substack{\mu<0 \\ \mu\in\mathbb{Z}-1/8}} c_{\Gamma,\overline{\psi},3/2}(\nu, -\mu) q^{-\mu}.$$

Remark 10. Usually, ν is defined be positive and in $\mathbb{Z} - 1/8$, so that the coefficients of $R^{[\mu]}_{\Gamma,psi,1/2}$ are $c_{\Gamma,\psi,1/2}(\mu, \nu)$. However, since we will be working extensively with the dual family we have defined ν to be what is usually $-\nu$.

The reason that [CD12] focused on the case where $\mu = -1/8$ is that those Rademacher sums are the ones that appear in Mathieu moonshine. In particular, they proved the following:

Theorem 2.3.3. We have that $H^{(2)}_g(\tau) = -2R^{[\mu]}_{\Gamma,\psi,1/2}$.

More generally, almost all of the umbral moonshine McKay-Thompson series H^X_g are equal to the appropriate vector-valued Rademacher sums. The only exceptions are when $X = A_8^3$ and the order of g is a multiple of 3, in which case a vector-valued theta series must be added to the Rademacher sum. See [CDH18] for more details.

2.4 Replicability of Monstrous T_g

In this section, we give an overview of the *Theory of Replicability* as developed by Conway, Norton, and others in [CN79], [Nor84], and [ACMS92], and proved by Borcherds in [Bor92].

In Conway and Norton's first paper [CN79] on moonshine, they found what they called *replication formulas*. For example, if the nth coefficient of $T_g(\tau)$ is $c_g(n)$, then the triplication formula can be written as:

$$\frac{1}{3}(T_g^3(\tau) - T_{g^3}(3\tau)) = (c_g(3)q + c_g(6)q^2 + \cdots) + c_g(1)T_g(\tau) + c_g(2).$$

Considering the coefficient of q^2 on both sides gives that

$$2c_g(1)c_g(2) + c_g(4) = c_g(6) + c_g(1)c_g(2),$$

and hence allows us to recursively compute $c_g(6)$ in terms of $c_g(1)$, $c_g(2)$, and $c_g(4)$ using

$$c_g(6) = c_g(4) + c_g(1)c_g(2).$$

The function $T_{g^3}(\tau)$ is called the 3rd replicate of $T_g(\tau)$, and for more general functions f we can work backwards to define the 3rd replicate $f^{(3)}$ using the triplication formula.

In [Nor84], Norton expanded upon his previous work with Conway to develop a general definition and framework for replicable functions. Let $f(\tau) = q^{-1} + \sum c_n q^n$, and define

$$F(\sigma,\tau) = \log(f(p) - f(q)) = \log(p^{-1} - q^{-1}) - \sum_{m,n=1}^{\infty} c_{m,n} p^m q^n,$$

where $p = e(\sigma)$, $q = e(\tau)$, and $\sigma, \tau \in \mathbb{H}$. Then f is *replicable* if $c_{a_1,b_1} = c_{a_2,b_2}$ whenever $a_1b_1 = a_2b_2$ and $(a_1, b_1) = (a_2, b_2)$. This condition is necessary and sufficient to generate replication formulas like those for the monstrous moonshine functions.

In proving the monstrous moonshine conjecture, Borcherds studied the Lie algebra $\mathfrak{m}$ associated to the monster group $\mathbb{M}$, and showed that it admits the denominator identity

$$p^{-1} \prod_{\substack{m,n\in\mathbb{Z} \\ m>0}} (1-p^m q^n)^{c(nm)} = J(\sigma) - J(\tau),$$

where σ, τ, p, and q are as before and $c(n)$ is the nth coefficient of J. This is known as the *Koike-Norton-Zagier formula*, and gives many recursive formulas which can be used to calculate the coefficients $c(n)$.

For example, if we look at the coefficient of p^3q on both sides, we get that

$$p^{-1}(-c(4)p^4q) + p^{-1}(-pq^{-1})(-c(1)pq)(-c(2)p^2q) + p^{-1}(-pq^{-1})(-c(6)p^3q^2) = 0,$$

which gives the familiar

$$c(6) = c(4) + c(1)c(2).$$

Put together, the recursive formulas allow us to compute the coefficients of $J(\tau)$ given just the values of $c(1)$, $c(2)$, $c(3)$, and $c(5)$ to start with.

The same is true of coefficients of the other McKay-Thompson series T_g, where $c_g(n)$ is the nth coefficient of T_g:

$$p^{-1} \exp\left(-\sum_{k>0} \sum_{\substack{m,n\in\mathbb{Z} \\ m>0}} \frac{1}{k} c_{g^k}(nm) p^{mk} q^{nm} \right) = T_g(\sigma) - T_g(\tau).$$

This gives us the following recursive formulas:

$$
\begin{aligned}
c_g(4k) &= c_g(2k+1) + \frac{c_g(k)^2 - c_{g^2}(k)}{2} + \sum_{1 \le j < k} c_g(j) c_g(2k-j), \\
c_g(4k+1) &= c_g(2k+3) - c_g(2) c_g(2k) + \frac{c_g(2k)^2 + c_{g^2}(2k)}{2} \\
&\quad + \frac{c_g(k+1)^2 - c_{g^2}(k+1)}{2} + \sum_{1 \le j \le k} c_g(j) c_g(2k-j+2) \\
&\quad + \sum_{1 \le j < k} c_{g^2}(j) c_g(4k-4j) + \sum_{1 \le j < 2k} (-1)^j c_g(j) c_g(4k-j), \\
c_g(4k+2) &= c_g(2k+2) + \sum_{1 \le j \le k} c_g(j) c_g(2k-j+1), \\
c_g(4k+3) &= c_g(2k+4) - c_g(2) c_g(2k+1) - \frac{c_g(2k+1)^2 - c_{g^2}(2k+1)}{2} \\
&\quad + \sum_{1 \le j \le k+1} c_g(j) c_g(2k-j+3) + \sum_{1 \le j \le k} c_{g^2}(j) c_g(4k-4j+2) \\
&\quad + \sum_{1 \le j \le 2k} (-1)^j c_g(j) c_g(4k-j+2).
\end{aligned}
$$

We will use these recursive formulas in Section 4.2.1.

Chapter 3

Proof of First Result

3.1 Relating Umbral and Monstrous Moonshine

In this section, we explain the relationship between the mock modular forms H^X from umbral moonshine and the Hauptmoduln T_g from monstrous moonshine.

3.1.1 Twisted Generalized Borcherds Products

Let $c^X(n,j)$ be the n-th Fourier coefficient of H_j^X. Let (Δ, r) be an admissible pair, so that $\Delta \neq -3$ is a negative fundamental discriminant and $r^2 \equiv \Delta \pmod{4m}$. Let $\Psi_{\Delta,r}^X := \Psi_{\Delta,r}(\tau, \widehat{H}^X)$ be the generalized twisted Borcherds product defined in Theorem 1.2.1.

To understand the statement of the next theorem, we need to define the twisted Heegner divisor $Z_{\Delta,r}^X$ associated to $\widehat{H}^X$. First, let

$$Z_{\Delta,r}\left(\frac{-1}{4m}, \frac{j}{2m}\right) := \sum_{Q \in \mathcal{Q}_{\Delta,jr}/\Gamma_0(m)} \frac{\chi_\Delta(Q)}{w(Q)} \alpha_Q,$$

where $w(Q)$ is the order of the stabilizer of the quadratic form Q in $\Gamma_0(m)$, χ_Δ is the generalized genus character defined in [GKZ87], and α_Q is the unique root of $Q(x,1)$

in $\mathbb{H}$. Then define

$$Z^X_{\Delta,r} := \sum_{j\in\mathbb{Z}/2m\mathbb{Z}} \sum_{n<0} c^X(n,j) Z_{\Delta,r}(n,j) = 2Z_{\Delta,r}\left(\frac{-1}{4m}, \frac{-1}{2m}\right) - 2Z_{\Delta,r}\left(\frac{-1}{4m}, \frac{1}{2m}\right),$$

since the principal part of $\widehat{H}^X$ is $-2q^{-1/4m}(\mathfrak{e}_1 - \mathfrak{e}_{2m-1})$.

Theorem 3.1.1. *We have that* $\Psi^X_{\Delta,r}$ *is a modular function for* $\Gamma_0(m)$ *with divisor* $Z^{(m)}_{\Delta,r}$.

Proof: From Theorem 6.1 and 6.2 of [BO10], we know that $\Psi_{\Delta,r}$ is a modular function for $\Gamma_0(m)$ with finite order unitary character σ and divisor $Z^X_{\Delta,r}$. It remains to show that σ is trivial.

Since $\Delta \neq -3$, we know that $w(Q) = 2, 4$ for all $Q \in Q_{\Delta,\pm r}$. Moreover, note that $\{Q : Q \in Q_{\Delta,-r}\} = \{-Q : Q \in Q_{\Delta,r}\}$ and that

$$\frac{\chi_\Delta(-Q)}{w(-Q)}\alpha_{-Q} = -\frac{\chi_\Delta(Q)}{w(Q)}\alpha_Q,$$

so

$$Z^X_{\Delta,r} = \sum_{Q\in\mathbb{Q}_{\Delta,r}/\Gamma_0(m)} -4\frac{\chi_\Delta(Q)}{w(Q)}\alpha_Q.$$

Therefore, $Z^X_{\Delta,r}$ is an integral degree zero divisor.

Since $\Gamma_0(m)$ has genus zero, $Z^X_{\Delta,r}$ is a principal divisor on $X_0(m)$ and we may consider a meromorphic function f on $X_0(m)$ with associated divisor $Z^X_{\Delta,r}$. The expression $\left|\Psi^X_{\Delta,r}/f\right|$ defines a harmonic function on $X_0(m)$ with no singularities, and therefore must be constant. So $\Psi^X_{\Delta,r}/f$ is a holomorphic function on $\mathbb{H}$ with constant modulus, and must therefore also be constant. So σ is trivial. $\square$

3.1.2 Proofs of Theorem 1.2.1 and Corollary 1.2.2

Proof of Theorem 1.2.1: Since $\Gamma_0(m)$ has genus zero, Theorem 3.1.1 implies that $\Psi^X_{\Delta,r}$ is a rational function in the Hauptmodul for $\Gamma_0(m)$. The normalized Hauptmodul, which we call $j_m(\tau)$, is defined by

$$j_m(\tau) := \frac{\eta(\tau)^{24/(m-1)}}{\eta(m\tau)^{24/(m-1)}} + \frac{24}{m-1}. \tag{3.1.1}$$

But using Table 1.1, we see that $j_m(\tau)$ is equal to $T_{g(X)}(\tau)$, the graded trace of $g(X) \in M$ on V. $\square$

Proof of Corollary 1.2.2: From Theorem 1.2.1, we have that

$$\prod_{n=1}^{\infty} P_\Delta(q^n)^{c^+\left(\frac{|\Delta|n^2}{4m}, \frac{rn}{2m}\right)} = \prod_i (T_g(\tau) - T_g(\alpha_i))^{\gamma_i}.$$

We equate the q^1 Fourier coefficients of each side, using Table 1.2 to get the Fourier expansion

$$T_g(\tau) = \frac{1}{q} + O(q).$$

$\square$

3.1.3 Examples

For each pure A-type case X with Coxeter number m, we illustrate how to write $\Psi^X_{\Delta,r}$ as a rational function in j_m. Note that here $\Delta < 0$ is a fundamental discriminant and $r \in \mathbb{Z}$ is such that $\Delta \equiv r^2 \pmod{4m}$.

First we work out an example for $m = 2$ in some detail, then list one example for each m. In Section 3.1.4, we explain how to find representatives of $Q_{\Delta,r}/\Gamma_0(m)$ using a method of Gross, Kohen, and Zagier.

Consider the case $X = A_1^{24}, \Delta = -7, r = 1$. Note here that $m = 2$. Using the method of Section 3.1.4, we compute that $Q_{-7,1}/\Gamma_0(2) = \{Q_1, Q_2\}$ and that

$Q_{-7,-1}/\Gamma_0(2) = \{-Q_1, -Q_2\}$, where the quadratic forms Q, their Heenger points α_Q, and their generalized genus characters $\chi_\Delta(Q)$ are given in Table 3.1. We also include the value of j_2 at each Heegner point. Using the table, the divisor of $\Psi^X_{-7,1}$ is given by:

Table 3.1: Quadratic Forms Needed for $m = 2$, $\Delta = -7, r = 1$ Case

quadratic form $= Q$	α_Q	$\chi_\Delta(Q)$	$j_2(\alpha_Q)$
$Q_1 = [2, 1, 1]$	$\alpha_1 = \frac{-1+\sqrt{-7}}{4}$	1	$\gamma_1 := \frac{1+45\sqrt{-7}}{2}$
$Q_2 = [-2, 1, -1]$	$\alpha_2 = \frac{1+\sqrt{-7}}{4}$	-1	$\gamma_2 := \frac{1-45\sqrt{-7}}{2}$
$-Q_2$	α_2	1	γ_2
$-Q_1$	α_1	-1	γ_1

$$(-\alpha_1 + \alpha_2) - (\alpha_1 - \alpha_2) = 2\alpha_2 - 2\alpha_1.$$

Therefore,

$$\Psi^X_{-7,1}(\tau) = \frac{(j_2(\tau) - \gamma_2)^2}{(j_2(\tau) - \gamma_1)^2}.$$

Similarly, for each value of m corresponding to a pure A-type case, we demonstrate in Table 3.2 how to write $\Psi_{\Delta,r}(\tau, \widehat{H}^{(m)})$ as a rational function in j_m for some nice choice of Δ, r. In all the examples we consider,

$$\Psi^X_{\Delta,r}(\tau) = \frac{(j_m(\tau) - \gamma_2)^2}{(j_m(\tau) - \gamma_1)^2}$$

for some $\gamma_1, \gamma_2 \in \mathcal{O}_{\mathbb{Q}(\sqrt{\Delta})}$. Note that $\Psi^X_{\Delta,r}$ will not always be a rational function of this particular form - we always picked Δ with class number 1.

Table 3.2: Examples

m	Δ	r	γ_1	γ_2
2	-7	1	$\frac{1+45\sqrt{-7}}{2}$	$\frac{1-45\sqrt{-7}}{2}$
3	-11	1	$17+8\sqrt{-11}$	$17-8\sqrt{-11}$
4	-7	3	$\frac{-15+3\sqrt{-7}}{2}$	$\frac{-15-3\sqrt{-7}}{2}$
5	-11	3	$-3+2\sqrt{-11}$	$-3-2\sqrt{-11}$
7	-19	3	$\frac{3+3\sqrt{-19}}{2}$	$\frac{3-3\sqrt{-19}}{2}$
9	-11	5	$-1+\sqrt{-11}$	$-1-\sqrt{-11}$
13	-43	3	$\frac{7+\sqrt{-43}}{2}$	$\frac{7-\sqrt{-43}}{2}$
25	-19	9	$\frac{\sqrt{-19}}{2}$	$\frac{-\sqrt{-19}}{2}$

3.1.4 Computing the Elements in $Q_{\Delta,r}/\Gamma_0(m)$

In this section, we explain how to compute $Q_{\Delta,r}/\Gamma_0(m)$, following [GKZ87].

Let $Q^0_{\Delta,r}$ be the subset of primitive forms. Then we have a $\Gamma_0(m)$-invariant bijection of sets

$$Q_{\Delta,r} = \bigcup_{\ell^2 \mid \Delta} \left(\bigcup_{h \in S(\ell)} \ell Q^0_{\Delta/\ell^2,h} \right),$$

where $S(\ell) := \{j \in \mathbb{Z}/2m\mathbb{Z} : j^2 \equiv \Delta/\ell^2 \pmod{4m}, \ell j \equiv r \pmod{2m}\}$. Since we pick Δ to be a fundamental discriminant, the only possible prime we need to worry about is $\ell = 2$. In our examples, we always choose Δ, r such that $S(2) = \emptyset$. In this case, we just need to work with $Q^0_{\Delta,r}$.

Now, let $n := \left(m, r, \frac{r^2-\Delta}{4m}\right)$. Then for $Q = [mc, b, a] \in Q^0_{\Delta,r}$, define $n_1 := (m, b, a), n_2 := (m, b, c)$, which are coprime and have product n. We have the following result:

Lemma 3.1.2. *(Section 1.1 of [GKZ87]) Define* n *as above and fix a decomposition* $n = n_1 n_2$ *with* n_1, n_2 *positive and relatively prime. Then there is a 1:1 correspondence between the* $\Gamma_0(m)$*-equivalence classes of forms* $[cm, b, a] \in Q^0_{\Delta,r}$ *satisfying* $(m, b, a) = n_1, (m, b, c) = n_2$ *and the* $\mathrm{SL}_2(\mathbb{Z})$ *equivalence classes of forms in* Q^0_Δ *given by* $Q = [mc, b, a] \mapsto \tilde{Q} = [cm_1, b, am_2]$*, where* $m_1 \cdot m_2$ *is any decomposition of* m *into coprime positive factors satisfying* $(n_1, m_2) = (n_2, m_1) = 1$*. In particular,* $|Q^0_{\Delta,r}/\Gamma_0(m)| = 2^v |Q^0_\Delta/\mathrm{SL}_2(\mathbb{Z})|$*, where* v *is the number of prime factors of* n*.*

Note that $|Q^0_\Delta/\mathrm{SL}_2(\mathbb{Z})|$ equals $2h(\Delta)$ for $\Delta < 0$, where the factor of 2 arises because Q^0_Δ also contains negative semi-definite forms.

In our examples, we always choose Δ, r such that $n = 1$, so that $|Q^0_{\Delta,r}/\Gamma_0(m)| = |Q^0_\Delta/\mathrm{SL}_2(\mathbb{Z})| = 2h(\Delta)$, where $h(\Delta)$ is the class number of $\mathbb{Q}(\sqrt{\Delta})$. The theory of reduced forms allows us to easily compute $Q^0_\Delta/\mathrm{SL}_2(\mathbb{Z})$.

3.2 p-adic Properties of the Logarithmic Derivative

3.2.1 p-adic Modular Forms

For each $i \in \mathbb{N}$, let $f_i = \sum a_i(n) q^n$ be a modular form of weight k_i with $a_i(n) \in \mathbb{Q}$. If for each n, the $a_i(n)$ converge p-adically to $a(n) \in \mathbb{Q}_p$, then $f := \sum a(n) q^n$ is called a p-adic modular form. For $p \neq 2$, we define the weight space

$$W := \varprojlim_t \mathbb{Z}/\phi(p^t)\mathbb{Z} = \mathbb{Z}_p \times \mathbb{Z}/(p-1)\mathbb{Z}.$$

For $p = 2$, we define

$$W := \varprojlim_t \mathbb{Z}/2^{t-2}\mathbb{Z} = \mathbb{Z}_2.$$

Then the k_i converge to an element $k \in W$, which we call the weight of f. We identify integers by their image in $\mathbb{Z}_p \times \{0\}$.

3.2.2 Proof of Theorem 1.2.3

Proof of Theorem 1.2.3: By Theorem 1.2.1, $\Psi^X_{\Delta,r}(\tau)$ is a meromorphic modular function, so that $\Theta(\Psi^X_{\Delta,r}(\tau))$ is a weight 2 meromorphic modular form on $\Gamma_0(m)$. Thus, the logarithmic derivative

$$\frac{\Theta(\Psi^X_{\Delta,r}(\tau))}{\Psi^X_{\Delta,r}(\tau)}$$

is a weight 2 meromorphic modular form on $\Gamma_0(m)$ whose poles are simple and are supported on Heegner points of discriminant Δ. $\square$

3.2.3 Proofs of Theorem 1.2.4 and Corollary 1.2.5

Proof of Theorem 1.2.4: We show that if (Δ, r) is an admissible pair and p is inert or ramified in $\mathbb{Q}(\sqrt{\Delta})$, that

$$L_{\Delta,r} := \frac{\Theta(\Psi^X_{\Delta,r}(\tau))}{\Psi^X_{\Delta,r}(\tau)}$$

is a p-adic modular form of weight 2. Say L has poles at $\alpha_1, \ldots, \alpha_n$, all of which are CM points of discriminant Δ. For each α_i, there is some zero β_i of E_{p-1} such that $j(\tau) - j(\alpha_i) \equiv j(\tau) - j(\beta_i)$ (see Theorem 1 of [KZ98]). Then let

$$\mathcal{E} := E_{p-1} \prod_i \frac{(j(\tau) - j(\alpha_i))}{(j(\tau) - j(\beta_i))}.$$

This has weight $p - 1$, is congruent to 1 modulo p, has zeros at $\alpha_1, \ldots, \alpha_n$, and has no poles. Let $f_t := f\mathcal{E}^{(p^t)}$. Then $L_t \equiv L \pmod{p^t}$ and is a modular form of weight $k_t = 2 + (p-1)p^t \equiv 2 \pmod{\phi(p^{t+1})}$, so L is a p-adic modular form of weight 2. $\square$

Proof of Corollary 1.2.5: This corollary follows directly for the coefficients of any p-adic modular form using the following beautiful result, proven by Serre [Ser76] using the theory of Galois representations.

Lemma 3.2.1 (Serre [Ser76] Theorem 4.7 (I))**.** *Let K be a number field and $\mathcal{O}_K$ the ring of integers of K. Suppose $f(\tau) = \sum_{n\geq 0} a_n q^n \in \mathcal{O}_K[[q]]$ is a modular form of integer weight $k \geq 1$ on a congruence subgroup. For any prime p, let $\mathfrak{p}$ be a prime above p in $\mathcal{O}_K$. Let $m \geq 1$. Then there exists a positive constant α_m such that*

$$\#\left\{n \leq X \colon a_n \not\equiv 0 \pmod{\mathfrak{p}^m}\right\} = O\left(\frac{X}{(\log X)^{\alpha_m}}\right).$$

$\square$

Chapter 4

Proof of Second Result

4.1 Proof of Theorem 1.2.1

Following the notation in Section 2.3.4, we define $H_g^{[\mu]}$ and $F_g^{[\nu]}$ as follows for $g \in M_{23}$:

$$H_g^{[\mu]}(\tau) := -2R^{[\mu]}_{\Gamma,\psi,1/2} = -2q^{\mu} - 2\sum_{\nu} c_{\Gamma,\psi,1/2}(\mu,-\nu)q^{-\nu}, \tag{4.1.1}$$

$$F_g^{[\nu]}(\tau) := -2R^{[\nu]}_{\Gamma,\overline{\psi},3/2} = -2q^{\nu} - 2\sum_{\mu} c_{\Gamma,\overline{\psi},3/2}(\nu,-\mu)q^{-\mu}. \tag{4.1.2}$$

Recall that $\mu,\nu < 0$ satisfy $\mu \in \mathbb{Z} - 1/8$ and $\nu \in \mathbb{Z} + 1/8$. Since we're focusing on $g \in M_{23}$, we have that ρ_g is trivial and hence $\psi = \epsilon^{-3}$. Note that $H_g^{[-1/8]}(\tau)$ is the McKay-Thompson series in Mathieu moonshine, but we will work with the $F_g^{[\nu]}$'s instead of the $H_g^{[\mu]}$'s. We define $S_g^{[\nu]}$ to be the shadow of $F_g^{[\nu]}$. An explicit expression for it can be found using Theorem 2.3.1.

To prove Theorem 1.2.1, we will need to study the effect of Atkin-Lehner operators on $F_g^{[\nu]}$ and $S_g^{[\nu]}$. Let u be an exact divisor of n_g, so that $(n_g/u, u) = 1$. Then let W_u

be a determinant 1 matrix of the form

$$\frac{1}{\sqrt{u}}\begin{pmatrix} au & b \\ cn_g & du \end{pmatrix},$$

for some $a, b, c, d \in \mathbb{Z}$. The following result is proven for $H_g^{[-1/8]}$ and its shadow in [CD12], and works more generally:

Proposition 4.1.1. For $g \in M_{23}$, let u be an exact divisor of n_g. Then the q-expansion of $F_g^{[\nu]}|W_u$ is supported on $q^{k+u/8}$ with $k \in \mathbb{Z}$ and the q-expansion of $S_g^{[\nu]} \mid W_u$ is supported on $q^{k-u/8}$ with $k \in \mathbb{Z}$.

4.1.1 Proof that f_g are Modular

Lemma 4.1.2. We have that $F_g^{[\nu]}$ is modular for all $\nu < 0$ satisfying $\nu \in \mathbb{Z} + \frac{1}{8}$ and all $g \in M_{23}$.

Proof: We know that $S_g^{[\nu]}$ is a cusp form of weight 1/2 on $\Gamma_0(n_g)$ with multiplier system ϵ^{-3}, and so

$$\eta^3 S_g^{[\mu]} \in S_2(\Gamma_0(n_g)).$$

It suffices to show that $\eta^3 S_g^{[\mu]} = 0$. Assume that $\eta^3 S_g^{[\mu]} \neq 0$.

If $g \in \{1, 2, 3, 4, 5, 6, 7ab, 8\}$, then $\dim S_2(\Gamma_0(n_g)) = 0$, so we have a contradiction.

Next, if $g \in \{11ab, 14ab, 15ab\}$, we have that $\dim S_2(\Gamma_0(n_g)) = 1$. Using `sage`, we can see that all nonzero cuspforms are eigenforms of W_{n_g} and have a zero of order 1 at $i\infty$. Therefore, we have that

$$ord_{i\infty}(\eta^3 S_g^{[\nu]} \mid W_{n_g}) = ord_{i\infty}(\eta^3 S_g^{[\nu]}) = 1.$$

However, using Proposition 4.1.1 and Table 4.1 we see that

$$ord_{i\infty}(\eta^3 S_g^{[\nu]} \mid W_{n_g}) \geq 2.$$

This is a contradiction.

Table 4.1: Computing Order of Vanishing for $g \in \{11ab, 14ab, 15ab\}$

M_{23} Conj. Class g	$ord_{i\infty}(\eta^3 \mid W_{n_g})$	$ord_{i\infty}(S_g^{[\nu]} \mid W_{n_g})$	$ord_{i\infty}(\eta^3 S_g^{[\nu]} \mid W_{n_g})$
$11ab$	$11/8$	$\geq 5/8$	≥ 2
$14ab$	$14/8$	$\geq 2/8$	≥ 2
$15ab$	$15/8$	$\geq 1/8$	≥ 2

Lastly, if $g \in \{23ab\}$, we have that $\dim S_2(\Gamma_0(n_g)) = 2$. using `sage`, we can see that all nonzero cuspforms are eigenforms of W_N with eigenvalue -1 and have a zero of order 1 or 2 at $i\infty$. Therefore, we have that

$$ord_{i\infty}(\eta^3 S_g^{[\nu]} \mid W_N) = ord_{i\infty}(\eta^3 S_g^{[\nu]}) \leq 2.$$

However, $ord_{i\infty}(\eta^3 \mid W_N) = 23/8$ and $ord_{i\infty}(S_g^{[\nu]} \mid W_N) \geq 1/8$ by Proposition 4.1.1. Therefore, $ord_{i\infty}((\eta^3 S_g^{[\nu]}) \mid W_N) \geq 3$. This is a contradiction. $\square$

Corollary 4.1.3. We have that

$$f_g(\tau) := \frac{F_g^{[-7/8]}(\tau)}{2\eta^3(\tau)} \in M_0^!(\Gamma_0(n_g)).$$

4.1.2 Proof that f_g are Hauptmoduln

For $g \in \{1a, 2a, 3a, 4a, 5a, 6a, 7ab, 8a\}$, we have that $\Gamma_0(n_g) = \Gamma_{\hat{g}}$, which has genus zero. However, for $g \in \{11ab, 14ab, 15ab, 23ab\}$, we have that $\Gamma_0(n_g) \subsetneq \Gamma_{\hat{g}} = \Gamma_0(n_g)+$

n_g, and so we need to do a bit more work to show that the f_g are invariant under the genus zero group $\Gamma_{\hat{g}}$.

Proposition 4.1.4. For $g \in M_{23}$, let $u \neq 1$ be an exact divisor of n_g. If $u \leq 8$, then $ord_{i\infty}(f_g \mid W_u) \geq 0$. If $9 \leq u \leq 16$, then $ord_{i\infty}(f \mid W_u) \geq -1$. If $17 \leq u \leq 24$, then $ord_{i\infty}(f \mid W_u) \geq -2$.

Proof: This follows from Lemma 4.1.1 by breaking up f_g into $F_g^{[-7/8]}$ and η^3, and considering the action of W_u on each separately. We know that $ord_{i\infty}(\eta^3 \mid W_u) = u/8$ and that $ord_{i\infty}(F_g^{[-7/8]} \mid W_u)$ must be greater than or equal to the smallest nonnegative number in the appropriate arithmetic progression. $\square$

Lemma 4.1.5. Let $g \in \{11ab, 14ab, 15ab, 23ab\}$. Then f_g is invariant under W_{n_g}.

Proof: We have that $f_g \in M_0^!(\Gamma_0(n_g))$ from Corollary 4.1.3.

For $g \in \{11ab, 23ab\}$, there is only one exact divisor u of n_g which is greater than 1, and so we split up f_g into $f_g^+ + f_g^-$, where f_g^+ is invariant under W_u and f_g^- is anti-invariant.

Using `sage`, we see $\dim S_2(\Gamma_0(11)) = 1$ and all nonzero elements are anti-invariant under W_{11}. Let $f^{(11)}$ be such an element. We also see that $\dim S_2(\Gamma_0(23)) = 2$, and all nonzero elements are anti-invariant under W_{23}. Let $f^{(23)}$ be such an element satisfying $ord_{i\infty}(f^{(23)}) = 2$. Then $f_g^- \cdot f^{(n_g)} \in M_2(\Gamma_0(n_g) + n_g)$, which has dimension zero because n_g is prime. So $f_g^- = 0$ and hence f_g is invariant under W_{n_g}.

For $g \in \{14ab, 15ab\}$ there are three exact divisors $u > 1$ to consider. We break up f_g into $f_g^{+++} + f_g^{+--} + f_g^{-+-} + f_g^{--+}$ where the plus and minuses correspond to the exact divisors in order from largest to smallest, so that f_{14}^{--+} is anti-invariant under W_{14} and W_7, but invariant under W_2. The same method as before shows that

$f_g^{--+} = 0$. Now, we have that

$$\begin{aligned}
f_{14ab} \mid W_2 &= f_{14ab}^{+++} - f_{14ab}^{+--} - f_{14ab}^{-+-} \\
f_{14ab} \mid W_7 &= f_{14ab}^{+++} - f_{14ab}^{+--} + f_{14ab}^{-+-} \\
f_{15ab} \mid W_3 &= f_{15ab}^{+++} - f_{15ab}^{+--} - f_{15ab}^{-+-} \\
f_{15ab} \mid W_5 &= f_{15ab}^{+++} - f_{15ab}^{+--} + f_{15ab}^{-+-}.
\end{aligned}$$

Moreover, by Proposition 4.1.4, we know that $ord_{i\infty}(f_g \mid W_u) \geq 0$ when $1 < u \leq 8$, so we have that

$$\begin{aligned}
\mathrm{Princ}(f_g^{+++}) &= \mathrm{Princ}(f_g^{+--}) + \mathrm{Princ}(f_g^{-+-}) \\
\mathrm{Princ}(f_g^{+++}) &= \mathrm{Princ}(f_g^{+--}) - \mathrm{Princ}(f_g^{-+-}).
\end{aligned}$$

Therefore, we have that f_g^{-+-} has no principal part, so f_g^{-+-} is a modular function with no poles, and hence must be constant. But since it is anti-invariant under some Atkin-Lehner operators, it must then be zero.

So f_g^{-+-} and f_g^{--+} are zero, and hence f_g is invariant under W_{n_g}. □

This completes the proof of Theorem 1.2.1.

4.2 Proof of Theorem 1.2.2

We will use the strategy described in Section 2.2.3 to prove Theorem 1.2.2. In particular, we define

$$T_\chi(\tau) := \frac{2}{|M_{23}|} \sum_{g \in M_{23}} \chi(g) f_g(\tau) = \sum_k m_\chi(k) q^k, \tag{4.2.1}$$

where $m_\chi(k)$, once proven to be a nonnegative integer, will be the multiplicity of the irreducible representation V_χ in the graded component V_k of our moonshine module V. Note that we're summing over all elements of M_{23}.

Remark 11. If we didn't multiply by two, we'd still have that all $m_\chi(k)$ are nonnegative integers for $k \neq 0$. However, some of the constant terms of the T_χ would be half-integers, as can be seen in Appendix C.

4.2.1 Proof that m_χ are Integral

First, we show that there is a virtual M_{23} module V, which is equivalent to proving that the coefficients $m_\chi(n)$ are nonnegative. Since our f_g agree with the McKay-Thompson series of monstrous moonshine up to the constant terms, we can take advantage of work that's already been done on those functions.

Lemma 4.2.1. The virtual modules V_0, V_1, V_2, V_3, V_5 exist.

Proof: It suffices to show that $m_\chi(n)$ is integral for all $\chi \in \widehat{M_{23}}$ and $n \in \{0, 1, 2, 3, 5\}$. This is a straightforward computation using Definition 4.2.1 and our identification of the f_g with monstrous moonshine functions. See Appendix C for the coefficients. $\square$

Using the replication formulas as described in Section 2.4, we can define the other V_n recursively.

We define

$$\begin{aligned}
V_{4k} &= V_{2k+1} \bigoplus \wedge^2(V_k) \bigoplus_{1\le j<k} V_j \otimes V_{2k-j},\\
V_{4k+1} &= V_{2k+3} \bigoplus (-V_2 \otimes V_{2k}) \bigoplus S^2(V_{2k}) \bigoplus \wedge^2(V_{k+1}) \bigoplus (\oplus_{1\le j\le k} V_j \otimes V_{2k-j+2})\\
&\quad \bigoplus (\oplus_{1\le j<k}(S^2(V_j) - \wedge^2(V_j))V_{4k-4j}) \bigoplus (\oplus_{1\le j<2k}(-1)^j V_j V_{4k-j}),\\
V_{4k+2} &= V_{2k+2} \bigoplus (\oplus_{1\le j\le k} V_j V_{2k-j+1}),\\
V_{4k+3} &= V_{2k+4} \bigoplus (-V_2 \otimes V_{2k+1}) \bigoplus (-\wedge^2 (V_{2k+1})) \bigoplus (\oplus_{1\le j\le k+1} V_j \otimes V_{2k-j+3})\\
&\quad \bigoplus (\oplus_{1\le j\le k}(S^2(V_j) - \wedge^2(V_j)) \otimes V_{4k-4j+2}) \bigoplus (\oplus_{1\le j\le 2k}(-1)^j V_j \otimes V_{4k-j+2}).
\end{aligned}$$

Therefore, all V_n are virtual modules with $2c_g(n) = \mathrm{Tr}(g \mid V_n)$

4.2.2 Estimation Tools

This section describes some estimates that we'll need in Section 4.2.3 to prove that the $m_\chi(n)$ are nonnegative.

First, we state two approximations for the I-Bessel function $I_1(x)$. See Appendix A for the definition of this function. From these very precise approximations we derive much simpler approximations which will do for our purposes.

Lemma 4.2.1 (Abramowitz and Stegun, pg 378)**.** *If* $|x| \le 3.75$ *and* $t = x/3.75$*, then*

$$\begin{aligned}
\frac{I_1(x)}{x} =& \frac{1}{2} + .87890594t^2 + .51498869t^4 + .15084934t^6 + .02658733t^8\\
&+ .00301532t^{10} + .00032411t^{12} + \epsilon,
\end{aligned}$$

where $|\epsilon| < 8 \times 10^{-9}$.

Corollary 4.2.2. For $|x| \le 3.75$, we have that $.4 \le \frac{I_1(x)}{x} \le 2.1$.

Lemma 4.2.2 (Abramowitz and Stegun, pg 378)**.** *If* $x \geq 3.75$ *and* $t = 3.75/x$*, then*

$$\frac{\sqrt{x} I_1(x)}{e^x} = .39894228 - .03988024t - .00362018t^2 + .00163801t^3 - .01031555t^4$$
$$+ .02282967t^5 - .02895312t^6 + .01787654t^7 - .00420059t^8 + \epsilon,$$

where $|\epsilon| < 2.2 \times 10^{-7}$

Corollary 4.2.3. If $x \geq 3.75$, then $.3 \leq \frac{\sqrt{x} I_1(x)}{e^x} \leq .4$.

Lemma 4.2.3 (Weil [Wei48])**.** *We have that* $|K(a, b; m)| \leq \tau(m)\sqrt{\gcd(a, b, m)}\sqrt{m}$*.*

We will also make use of the function $d(n)$, which denotes the number of positive divisors of n. For any $\epsilon > 0$, there exists C_ϵ such that $d(n) \leq C_\epsilon n^\epsilon$. For our estimates, it will suffice to use $\epsilon = \frac{1}{4}$.

Lemma 4.2.4 (See pg 27 of [Gan16])**.** *Let* $C_{1/4} = 8.55$*. Then* $d(n) \leq C_{1/4} n^{1/4}$.

We will also use the following straightforward result connecting the divisor function to the Riemann-zeta function $\zeta(s)$, which can be obtained by expanding

$$\zeta^2(s) = \left(\sum \frac{1}{n^s}\right)^2.$$

Lemma 4.2.5. *For* $s > 1$*, we have that*

$$\sum_{n=1}^{\infty} \frac{d(n)}{n^s} = \zeta^2(s).$$

Lastly, we will need estimates on sums of powers, which can be proved by estimating the sums by integrals.

Lemma 4.2.6. *If* $r > -1$*, then*

$$\sum_{x=1}^{n} x^r \leq \frac{(n+1)^{r+1}}{r+1}$$

4.2.3 Proof that m_χ are Nonnegative

Now, we show that the $m_\chi(n) \geq 0$. Using Definition 2.2.2 and the triangle inequality, we have that

$$m_\chi(k) \geq \frac{2}{|M_{23}|}\left(c_{1a}(k)\chi(1) - \sum_{\substack{g\in M_{23}\\ g\neq 1a}} |c_g(k)|\cdot|\chi(g)|\right). \tag{4.2.2}$$

Our strategy will therefore be to give a lower bound on $c_{1a}(k)$ and an upper bound on $|c_g(k)|$ for $g \neq 1a$.

Recall from Section 2.3 that

$$c_g(k) = \frac{2\pi}{\sqrt{k}}\sum_{b>0}\frac{1}{n_h b} I_1\left(\frac{4\pi\sqrt{k}}{n_g b}\right) K(k,1,n_g b).$$

Let

$$P_g(b;k) := \frac{2\pi}{n_g b\sqrt{k}} I_1\left(\frac{4\pi\sqrt{k}}{n_g b}\right) K(k,1,n_g b),$$

so that $c_g(k) = \sum P_g(b;k)$. Define

$$b_0(g) = \begin{cases} 2, & g = 1a \\ 1, & \text{otherwise} \end{cases}$$

and

$$L = \frac{4\pi\sqrt{k}}{3.75 n_g}.$$

Then for $g = 1a$, we'll show that

$$c_1(k) \geq P_{1a}(1;k) - \sum_{b_0(g)\leq b<L} P_{1a}(b;k) - \sum_{b\geq L} P_{1a}(b;k).$$

For $g \neq 1a$, we'll show that

$$|c_h(k)| \leq \sum_{b_0(g) \leq b < L} |P_g(b;k)| + \sum_{b \geq L} |P_g(b;k)|.$$

Splitting up the sum at L allows us to use Corollary 4.2.2 and Corollary 4.2.3. Note that L is not an integer. For by $b \geq L$ we simply mean all integers b satisfying that condition.

First, we will estimate $P_{1a}(1;k)$. Recall that

$$P_{1a}(1;k) = \frac{2\pi}{\sqrt{k}} I_1(4\pi\sqrt{k}) K(k,1,1).$$

By Corollary 4.2.3, we have that

$$I_1(4\pi\sqrt{k}) \geq \frac{.3}{2\sqrt{\pi}} \frac{e^{4\pi\sqrt{k}}}{k^{1/4}}.$$

We also know that $K(k,1,1) = 1$. Therefore, we have that

$$P_{1a}(1;k) \geq .5 \frac{e^{4\pi\sqrt{k}}}{k^{3/4}}. \tag{4.2.3}$$

Next, we will estimate

$$\sum_{b_0(g) \leq b < L} |P_g(b;k)| = \frac{2\pi}{\sqrt{k}} \sum_{b_0(g) \leq b < L} \frac{1}{n_g b} I_1\left(\frac{4\pi\sqrt{k}}{n_g b}\right) |K(k,1,n_g b)|.$$

Since $c < L = \frac{4\pi\sqrt{k}}{3.75 n_g}$, we have that $\frac{4\pi\sqrt{k}}{n_g b} > 3.75$. So we can apply Corollary 4.2.3 to get that

$$I_1\left(\frac{4\pi\sqrt{k}}{n_g b}\right) \leq \frac{.4\sqrt{n_g b}}{2\sqrt{\pi}} \frac{e^{4\pi\sqrt{k}/(n_g b)}}{k^{1/4}}.$$

We can also use Lemma 4.2.3 and Lemma 4.2.4 to get

$$|K(k,1,n_g b)| \leq d(n_g b)\sqrt{n_g b} \leq 8.55 b^{3/4}.$$

Putting this together, we get that

$$\sum_{b_0(g)\leq b<L} |P_g(b;k)| \leq 3.42 n_g^{1/4}\sqrt{\pi}\frac{e^{4\pi\sqrt{k}/(b_0(g)n_g)}}{k^{3/4}} \sum_{b_0(g)\leq b<L} b^{1/4}.$$

Using Lemma 4.2.6, we have that

$$\sum_{b_0(g)\leq b<L} b^{1/4} \leq \frac{L^{5/4}}{5/4} = \frac{4}{5}\left(\frac{4\pi}{3.75 n_g}\right)^{5/4} k^{5/8}$$

so we get that

$$\sum_{b_0(g)\leq b<L} |P_{1a}(b;k)| \leq \frac{22}{n_g}\frac{e^{4\pi\sqrt{k}/(b_0(g)n_g)}}{k^{1/8}}. \tag{4.2.4}$$

Note that either $b_0(g) = 2$ or $n_g \geq 2$, so we have that the exponent is at most $2\pi\sqrt{k}$, making this sum grow more slowly than our main term $P_{1a}(1;k)$, which has exponent $4\pi\sqrt{k}$.

Lastly, we will estimate

$$\sum_{b\geq L} |P_g(b,k)| = \frac{2\pi}{\sqrt{k}} \sum_{b\geq L} \frac{1}{n_g b} I_1\left(\frac{4\pi\sqrt{k}}{n_g b}\right) |K(k,1,n_g b)|.$$

Since

$$b \geq L = \frac{4\pi\sqrt{k}}{3.75 n_g},$$

we have that

$$0 < \frac{4\pi\sqrt{k}}{n_g b} \leq 3.75.$$

So we can apply Corollary 4.2.2 to get that

$$I_1\left(\frac{4\pi\sqrt{k}}{n_g b}\right) \leq \frac{8.4\pi\sqrt{k}}{n_g b}.$$

We can also use Lemma 4.2.3 to get

$$|K(k,1,n_g b)| \leq d(n_g b)\sqrt{n_g b}.$$

Putting this together and using $d(ab) \geq d(a)d(b)$ and Lemma 4.2.5, we get that

$$\sum_{b\geq L} |P_g(b,k)| \leq \frac{16.8\pi^2 d(n_g)}{n_g^{3/2}} \sum_{b\geq L} \frac{d(b)}{b^{3/2}} \leq \frac{16.8\pi^2 d(n_g)}{n_g^{3/2}}\zeta(3/2).$$

Note that $d(n_g) \leq 4$. Therefore, we have that

$$\sum_{b\geq L} |P_g(b,k)| \leq \frac{1733}{n_g^{3/2}} \tag{4.2.5}$$

So using (4.2.3), (4.2.4), and (4.2.5), we can now find estimates for our $c_g(k)$.

We have that

$$c_1(k) \geq .5\frac{e^{4\pi\sqrt{k}}}{k^{3/4}} - 22\frac{e^{2\pi\sqrt{k}}}{k^{1/8}}$$

and that for $g \neq 1a$, we have that

$$c_g(k) \leq \frac{22}{n_g}\frac{e^{4\pi\sqrt{k}/(2n_g)}}{k^{1/8}} + \frac{1733}{n_g^{3/2}}.$$

Plugging these into (4.2.2) along with the information from the character table in

Appendix B, we get that $m_\chi(k) > 0$ for all $k \geq k_0(\chi)$ where

$$k_0(\chi) = \begin{cases} 4 & \chi = \chi_1, \chi_2 \\ 3 & \chi_3, \chi_4, \chi_5, \chi_6, \chi_9 \\ 2 & \text{otherwise} \end{cases}$$

Therefore, it just remains to check that $m_\chi(k)$ is nonnegative for $k < k_0(\chi) \leq 4$. These values are given in Appendix C. This completes the proof of Theorem 1.2.2.

Appendix A

Definitions of Special Functions

We define the Jacobi theta functions $\theta_i(\tau, z)$ as follows for $q := e(\tau)$ and $y := e(z)$.

$$\begin{aligned}
\theta_2(\tau, z) &:= q^{1/8} y^{1/2} \prod_{n=1}^{\infty} (1 - q^n)(1 + yq^n)(1 + y^{-1}q^{n-1}) \\
\theta_3(\tau, z) &:= \prod_{n=1}^{\infty} (1 - q^n)(1 + yq^{n-1/2})(1 + y^{-1}q^{n-1/2}) \\
\theta_4(\tau, z) &:= \prod_{n=1}^{\infty} (1 - q^n)(1 - yq^{n-1/2})(1 - y^{-1}q^{n-1/2})
\end{aligned}$$

We use them to define weight zero index $m-1$ weak Jacobi forms $\varphi_1^{(m)}$ as follows.

Let

$$\begin{aligned}
\varphi_1^{(2)} &:= 4(f_2^2 + f_3^2 + f_4^2),\\
\varphi_1^{(3)} &:= 2(f_2^2 f_3^2 + f_3^2 f_4^2 + f_4^2 f_2^2),\\
\varphi_1^{(4)} &:= 4 f_2^2 f_3^2 f_4^2,\\
\varphi_1^{(5)} &:= \frac{1}{4}\left(\varphi_1^{(4)}\varphi_1^{(2)} - (\varphi_1^{(3)})^2\right)\\
\varphi_1^{(7)} &:= \varphi_1^{(3)}\varphi_1^{(5)} - (\varphi_1^{(4)})^2\\
\varphi_1^{(9)} &:= \varphi_1^{(3)}\varphi_1^{(7)} - (\varphi_1^{(5)})^2\\
\varphi_1^{(13)} &:= \varphi_1^{(5)}\varphi_1^{(9)} - 2(\varphi_1^{(7)})^2
\end{aligned}$$

where $f_i(\tau, z) := \theta_i(\tau, z)/\theta_i(\tau, 0)$ for $i = 2, 3, 4$.

For the remaining positive integers m with $m \leq 25$, we define $\varphi_1^{(m)}$ recursively. For $(12, m-1) = 1$ and $m > 5$ we set

$$\varphi_1^{(m)} = (12, m-5)\varphi_1^{(m-4)}\varphi_1^{(5)} + (12, m-3)\varphi_1^{(m-2)}\varphi_1^{(3)} - 2(12, m-4)\varphi_1^{(m-3)}\varphi_1^{(4)}.$$

For $(12, m-1) = 2$ and $m > 10$ we set

$$\varphi_1^{(m)} = \frac{1}{2}\left((12, m-5)\varphi_1^{(m-4)}\varphi_1^{(5)} + (12, m-3)\varphi_1^{(m-2)}\varphi_1^{(3)} - 2(12, m-4)\varphi_1^{(m-3)}\varphi_1^{(4)}\right).$$

For $(12, m-1) = 3$ and $m > 9$, we set

$$\varphi_1^{(m)} = \frac{2}{3}(12, m-4)\varphi_1^{(m-3)}\varphi_1^{(4)} + \frac{1}{3}(12, m-7)\varphi_1^{(m-6)}\varphi_1^{(7)} - (12, m-5)\varphi_1^{(m-4)}\varphi_1^{(5)}.$$

For $(12, m-1) = 4$ and $m > 16$ we set

$$\varphi_1^{(m)} = \frac{1}{4}\left((12, m-13)\varphi_1^{(m-12)}\varphi_1^{(13)} + (12, m-5)\varphi_1^{(m-4)}\varphi_1^{(5)} - (12, m-9)\varphi_1^{(m-8)}\varphi_1^{(9)}\right).$$

For $(12, m-1) = 6$ and $m > 18$ we set

$$\varphi_1^{(m)} = \frac{1}{3}(12, m-4)\varphi_1^{(m-3)}\varphi_1^{(4)} + \frac{1}{6}(12, m-7)\varphi_1^{(m-6)}\varphi_1^{(7)} - \frac{1}{2}(12, m-5)\varphi_1^{(m-4)}\varphi_1^{(5)}.$$

For $m = 25$, we set

$$\varphi_1^{(25)} = \frac{1}{2}\varphi_1^{(21)}\varphi_1^{(5)} - \varphi_1^{(19)}\varphi_1^{(7)} + \frac{1}{2}(\varphi_1^{(13)})^2.$$

See the appendix of [CDH14a] for more information on the space of weight zero Jacobi forms.

We use two versions of an Appell-Lerch sum. The first is the generalized Appell-Lerch sum $\mu_{m,0}$, defined as in [CDH14a]. It is given by

$$\mu_{m,0}(\tau, z) := -\sum_{k \in \mathbb{Z}} q^{mk^2} y^{2mk} \frac{1 + yq^k}{1 - yq^k},$$

and is the holomorphic part of a weight 1 index m "real-analytic Jacobi form".

Zwegers [Zwe02] uses a slightly different version of the Appell-Lerch sum. He first defines the theta function

$$\vartheta(z, \tau) := \sum_{\nu \in 1/2 + \mathbb{Z}} q^{\nu^2/2} y^{\nu} e(\nu/2).$$

Then he defines

$$\mu(u, v; \tau) := \frac{e(u/2)}{\vartheta(v; \tau)} \sum_{n \in \mathbb{Z}} \frac{(-1)^n q^{(n^2+n)/2} e(nv)}{1 - q^n e(u)}.$$

This is completed to a "real-analytic Jacobi form" $\tilde{\mu}(u, v; \tau)$ of weight $1/2$ by letting

$$\tilde{\mu}(u, v; \tau) := \mu(u, v; \tau) + \frac{i}{2} R(u - v; \tau),$$

where

$$R(z,\tau) := \sum_{\nu \in 1/2+\mathbb{Z}} \left\{ \operatorname{sgn}(\nu) - E(\nu + a)\sqrt{2t} \right\} (-1)^{\nu - 1/2} q^{-\nu^2/2} y^{-\nu},$$

$t := \Im(\tau)$, $a := \frac{\Im(u)}{\Im(\tau)}$, and $E(z) := 2\int_0^z e^{-\pi u^2} du$.

The Dedekind eta-function, denoted by $\eta(\tau)$, is a holomorphic function on the upper half-plane defined by the infinite product

$$\eta(\tau) = q^{1/24} \prod_{n>0} (1 - q^n).$$

It is a modular form of weight 1/2 for the $\mathrm{SL}_2(\mathbb{Z})$ with multiplier $\epsilon : \mathrm{SL}_2(\mathbb{Z}) \to \mathbb{C}$ so that $\eta(\gamma\tau)\epsilon(\gamma)j(\gamma,\tau)^{1/4} = \eta(\tau)$ for all $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$, where $j(\gamma,\tau) = (c\tau)+d)^{-2}$. The multiplier system ϵ may be described as

$$\epsilon \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) = \begin{cases} e(-b/24), & c = 0, d = 1 \\ e(-(a+d)/24c + s(d,x)/2 + 1/8), & c > 0 \end{cases}$$

where

$$s(d,c) = \sum_{m=1}^{c-1} (d/c)((md/c))$$

and $((x))$ is 0 for $x \in \mathbb{Z}$ and $x - \lfloor x \rfloor - 1/2$ otherwise.

The modified Bessel function of the first kind is denoted $I_\alpha(x)$ and may be defined by the power series expression

$$I_\alpha(z) = \sum_{n \geq 0} \frac{1}{\Gamma(m + \alpha + 1)m!} \left(\frac{z}{2}\right)^{2m+\alpha}.$$

This converges absolutely and locally uniformaly in z so long as z avoids the negative reals.

The Klooserman sum $K(a, b; m)$ is defined as

$$K(a, b; m) = \sum_{h} e\left(\frac{ah + bh^{*}}{n}\right),$$

where h runs through a complete set of residues prime to n and h^{*} is defined by $hh^{*} \equiv 1 \pmod{n}$.

Appendix B

Character Table of M_{23}

The Matheiu group M_{23} has 17 conjugacy classes and $2^7 \cdot 3^2 \cdot 5 \cdot 7 \cdot 11 \cdot 23 = 10200960$ elements. Table B.1 gives the number of elements in each conjugacy class. Table B.2 gives the full character table of M_{23}, and uses the following:

$$\begin{aligned} A &:= \frac{-1+\sqrt{-7}}{2} \\ B &:= \frac{-1+\sqrt{-11}}{2} \\ C &:= \frac{-1+\sqrt{-15}}{2} \\ D &:= \frac{-1+\sqrt{-23}}{2}. \end{aligned}$$

Both tables are used in the computations of Section 4.2.3.

Conjugacy Class	Number of Elements
$1a$	1
$2a$	3795
$3a$	56672
$4a$	318780
$5a$	680064
$6a$	860080
$7a$	728640
$7b$	728640
$8a$	1275120
$11a$	927360
$11b$	927360
$14a$	728640
$14b$	728640
$15a$	680064
$15b$	680064
$23a$	443520
$23b$	443520

Table B.1: Conjugacy Classes of M_{23}

	$1a$	$2a$	$3a$	$4a$	$5a$	$6a$	$7a$	$7b$	$8a$	$11a$	$11b$	$14a$	$14b$	$15a$	$15b$	$23a$	$23b$
χ_1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
χ_2	22	6	4	2	2	0	1	1	0	0	0	−1	−1	−1	−1	−1	−1
χ_3	45	−3	0	1	0	0	A	$\overline{A}$	−1	1	1	$-A$	$-\overline{A}$	0	0	−1	−1
χ_4	45	−3	0	1	0	0	$\overline{A}$	A	−1	1	1	$-\overline{A}$	$-A$	0	0	−1	−1
χ_5	230	22	5	2	0	1	−1	−1	0	−1	−1	1	1	0	0	0	0
χ_6	231	7	6	−1	1	−2	0	0	−1	0	0	0	0	1	1	1	1
χ_7	231	7	−3	−1	1	1	0	0	−1	0	0	0	0	C	$\overline{C}$	1	1
χ_8	231	7	−3	−1	1	1	0	0	−1	0	0	0	0	$\overline{C}$	C	1	1
χ_9	253	13	1	1	−2	1	1	1	−1	0	0	−1	−1	1	1	0	0
χ_{10}	770	−14	5	−2	0	1	0	0	0	0	0	0	0	0	0	D	$\overline{D}$
χ_{11}	770	−14	5	−2	0	1	0	0	0	0	0	0	0	0	0	$\overline{D}$	D
χ_{12}	896	0	−4	0	1	0	0	0	0	B	$\overline{B}$	0	0	1	1	−1	−1
χ_{13}	896	0	−4	0	1	0	0	0	0	$\overline{B}$	B	0	0	1	1	−1	−1
χ_{14}	990	−18	0	2	0	0	A	$\overline{A}$	0	0	0	A	$\overline{A}$	0	0	1	1
χ_{15}	990	−18	0	2	0	0	$\overline{A}$	A	0	0	0	$\overline{A}$	A	0	0	1	1
χ_{16}	1035	27	0	−1	0	0	−1	−1	1	1	1	−1	−1	0	0	0	0
χ_{17}	2024	8	−1	0	−1	−1	1	1	0	0	0	1	1	−1	−1	0	0

Table B.2: Character Table of M_{23}

Appendix C

Coefficients of T_χ

We have the following:

$$
\begin{aligned}
T_{\chi_1} &= 2q^{-1} + 6 + 18 * q + 36 * q^2 + 236 * q^3 + 4088 * q^4 + 65746 * q^5 + O(q^6)\\
T_{\chi_2} &= 4 * q + 72 * q^2 + 3722 * q^3 + 87108 * q^4 + 1437888 * q^5 + O(q^6)\\
T_{\chi_3} = T_{\chi_4} &= 1 + 8 * q + 216 * q^2 + 7644 * q^3 + 178836 * q^4 + 2939568 * q^5 + O(q^6)\\
T_{\chi_5} &= 16 * q + 920 * q^2 + 39110 * q^3 + 912160 * q^4 + 15028196 * q^5 + O(q^6)\\
T_{\chi_6} &= 14 * q + 964 * q^2 + 39200 * q^3 + 916784 * q^4 + 15091598 * q^5 + O(q^6)\\
T_{\chi_7} = T_{\chi_8} &= 8 * q + 966 * q^2 + 39206 * q^3 + 916628 * q^4 + 15091674 * q^5 + O(q^6)\\
T_{\chi_9} &= 12 * q + 1044 * q^2 + 42976 * q^3 + 1003784 * q^4 + 16529676 * q^5 + O(q^6)\\
T_{\chi_{10}} = T_{\chi_{11}} &= 28 * q + 3262 * q^2 + 130356 * q^3 + 3057014 * q^4 + 50300306 * q^5 + O(q^6)\\
T_{\chi_{12}} = T_{\chi_{13}} &= 32 * q + 3776 * q^2 + 151826 * q^3 + 3556504 * q^4 + 58533616 * q^6 + O(q^6)\\
T_{\chi_{14}} = T_{\chi_{15}} &= 36 * q + 4196 * q^2 + 167598 * q^3 + 3930356 * q^4 + 64671968 * q^5 + O(q^6)\\
T_{\chi_{16}} &= 48 * q + 4328 * q^2 + 175644 * q^3 + 4107408 * q^4 + 67617996 * q^5 + O(q^6)\\
T_{\chi_{17}} &= 78 * q + 8520 * q^2 + 343052 * q^3 + 8033772 * q^4 + 132224398 * q^5 + O(q^6).
\end{aligned}
$$

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