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Renormalization Group Solution of the Chutes&Ladder Model

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An abstract of
a thesis submitted to the Faculty of Emory College of Arts and Sciences
of Emory University in partial fulfillment
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L. Ball, S. Boettcher

April 16, 2014

Abstract

We analyze a semi-infinite one-dimensional random walk process with a biased motion that is incremental in one direction and long-range in the other. On a network with a fixed hierarchy of long-range jumps we find with exact renormalization group calculations that there is a dynamical transition between a localized adsorption phase and an anomalous diffusion phase in which the mean-square displacement exponent depends non-universally on the bias of the system. We compare these results with similar findings of unconventional phase behavior in hierarchical networks, as well as with related systems involving Levy-distributed backjumps.

1 Introduction

The variety of real networks found in biology, engineering, social sciences, and communication provides a need for new ideas to explore and classify the full range of critical phenomena that emerge as a result of the complex geometry[1]. Recently, networks with a hierarchical organization of its sites have received considerable attention due to the exotic phase transitions that can be observed in such structures for well-known equilibrium models such as percolation[2] and Ising ferromagnets[3]. However, here we are concerned with non-equilibrium systems, for transport processes are a flow through time towards equilibrium. Such complex dynamics can be studied on a designed structure that is intricate enough for interesting results, yet simple enough to reveal analytic insights. The network we are working with is labeled the Hanoi Network 3 (HN3) and is described in Section 2[4]. Initial numerical simulations upon this network for the system studied here showed non-universal scaling behavior. We performed a renormalization group analysis to study this system and determine explicitly its scaling behavior.

1.1 Introduction to scaling behavior and universality

A power law is a relationship that exhibits scale invariance, that is, given a function $f(x) = ax^k$, scaling the argument by a constant factor c causes only a proportionate scaling of the function itself by a factor of c^k. Power laws with particular scaling exponents are therefore equivalent up to constant factors, and thus can be defined by their particular scaling exponent. The scaling exponents of systems with phase transitions are referred to as the critical exponents of the system. Systems with the same critical exponent display identical scaling behavior as they approach criticality. Diverse systems can therefore be shown, via renormalization group theory, to share the same fundamental dynamics. The phenomenon of sharing dynamics is referred to as universality. Those systems with precisely the same critical exponents are said to belong to the same universality class. Each member in a particular universality class exhibits identical critical behavior, often unifying very different physical phenomena. For example, the universality class to which the Ising model belongs not only applies to magnetic spins, but also to boiling fluids, and even the Higgs mechanism that gives all particles their masses [3]. Almost all physical systems with phase transitions belong to a relatively small collection of universality classes, which are specified by the critical exponents.

1.2 Background of Renormalization Group

Physics often wishes to relate the present theories in an attempt to find a more general unifying theory that can be applied to broader scales. Often the search for a final theory is impeded by singularities that arise at length scales far from observation[5]. The renormalization group, as developed by Kenneth Wilson, serves as the primary mathematical tool used in physics to connect these theories at different length scales.

In statistical mechanics, phase transitions are necessarily connected with singularities which require infinite systems to deal with. In 1937, all statistical mechanics theories of thermodynamics, which were mean field theories, failed near critical points [6]. Any theory describing critical points must take into account all scales of length because changes in a system occur at widely varying size scales near a critical point or phase transition. For example, when water is brought near its critical point fluctuations in density develop at all possible scales and extend over an indefinite range. The distinction between gas and liquid disappears, for there are drops of liquid interspersed with bubbles of gas of all sizes, varying from single molecules up to the size of the system. For a theory to describe water near its critical point, it must take into account all of these possible length scales [7]. The renormalization group is a tool for dealing with such problems that involve many scales of length. Mean field theories have no manifestation of infinities in their descriptions of phase transitions, and so the RG method was developed to supplement mean field theories, and unites "a breaking of internal symmetries with a proper description of spatial infinities," allowing for a sufficient tool with which to describe and understand these types of problems [5]. The RG is a tool used to make a problem as simple as possible, but not simpler [8]. It is not an exact nor completely controlled process, and should be regarded as a largely conceptual framework, relying fundamentally only on scaling, which can be adapted to the particular problem at hand. John Cardy explains that "all renormalization group studies have in common the idea of re-expressing the parameters which define a problem in terms of some other, perhaps simpler set, while keeping unchanged those physical aspects of a problem which are of interest" [9].

The general application of the RG is as follows: (1) There is a flow through space of all dynamic equations involved (could be of Hamiltonians, master equations, etc.) with "coupling constants" as coordinates. (2) At each level of renormalization, the "coupling constants" are related back to the previous set of constants. The critical point of a system is the fixed point towards which these constants flow. (3) These fixed points define particular universality classes. (4) The RG transformation that describes the flow can typically be linearized about the fixed point. The eigenvalues of this linearization describe the how the system flows towards the critical point. The eigenvalues can therefore by used to determine the critical exponent of the power law.

The foundation for the RG began in 1944 when Onsager computed exactly the partition function and thermodynamic properties of a simple ferromagnet - a model which became known as the Ising model - and found that the explicit properties disagreed completely from the mean field theory predictions used at that time [10]. Next, at King's College School, the mean field theory critical indices were found to be wrong, further supporting the need for a revised theory [5]. Patashnskii and Pokrovsky then begin looking at correlations in fluctuations at different scales. Following this, Ben Widom realized significant scaling properties of critical phenomena, but had not determined their origin. In 1966, Kadanoff suggested a theory that fully described scaling behavior, and incorporated universality, but still was not complete[5]. In 1971, Wilson finally produced a complete theory of RG [10].

In Wilson's theory he considers all possible couplings, instead of guessing which couplings to use. The scale change then produces a closed algebra of couplings. His theory also considers a succession of renormalizations, unlike the previous theories that would consider just one[5]. After many renormalizations you reach a fixed point, where the couplings stop changing. Each fixed point can be considered to be its own separate physical theory, which gave rise to universality classes. This theory has proven to be a powerful tool in physics, having treated the following broad range of problems[10]:

1. The KAM (Kolmogorov-Arnold-Moser) theory of Hamiltonian stability
2. The constructive theory of Euclidean Fields
3. Universality theory of the critical point in statistical mechanics
4. Onset of chaotic motion in dynamical systems
5. The convergence of Fourier series on a circle
6. The theory of the Fermi surface in Fermi liquids
7. The theory of polymers in solutions and in melts
8. Derivation of the Navier-Stoker equations for hydrodynamics
9. The fluctuations of membranes and interfaces
10. The existence and properties of 'critical phases'
11. Phenomena in random systems, fluid percolation, electron localization, etc.
12. The Kondo problem for magnetic impurities in nonmagnetic metals

Wilson earned the Nobel Prize in Physics in 1982 for this brilliant "...theory of critical phenomena in connection with phase transitions" [8].

We will use this RG theory to determine the critical scaling exponent d_w that defines how the time scale of the system changes with the mean square displacement. There is assumed to be a relation between the length scale, denoted L, and the time scale, denoted T of the form

$$L \sim T^{d_w}, \tag{1}$$

where

$$T' = \lambda T \quad L' = 2L. \tag{2}$$

The λ in equation (2) is determined by the largest eigenvalue of the Jacobian of the recursion equations, because this eigenvalue gives the time dependence of the equations flowing into the fixed point. Solving equations (1) and (2) for the exponent gives

$$d_w \sim \frac{\ln \lambda}{\ln 2} \tag{3}$$

, which is the scaling behavior we are looking for in this system.

1.3 Motivation

First and foremost, our study here serves as a simple example of the unusual - and often non-universal – scaling behavior for dynamic processes on complex networks. However, our model also provides a sense of what might happen in an ordinary, one-dimensional lattice with an incremental bias to walk one direction and back-jumps in the opposite direction, drawn randomly from a Levy-flight distribution., which is a mix of long trajectories and short random movements. Among other applications, the Levy-Flight is considered to be a model of animal behavior particularly when hunting[11]. This could also be extended, for example, to model the behavior of directional transport of kinesin proteins interrupted with finite failure rate that leads to dissociation off the filament to reset the process.

Previous renormalization group studies of random walks on this particular network have led to distinct universal behavior. However, as suggested in the numerical simulations, the asymptotic behavior of our model exhibits anomalous behavior with non-universal exponents. Anomalous diffusion is a diffusion process with a nonlinear relationship to time, in contrast to a typical diffusion process, in which the mean square displacement (MSD) of a particle is a linear function of time.[12] Anomalous diffusion has been shown to describe many different physical scenarios, for example protein diffusion within cells, diffusion through porous media, telomeres in the nucleus of cells, and other biological system including heartbeat intervals and in DNA sequences [13]. This model can help us to better understand the behavior of such systems with anomalous diffusion.

2 Network Design

The network we are discussing in this paper consist of a simple geometric backbone, a one-dimensional line of $N = 2^k$ sites $\left(0 \leq n \leq 2^k,\, k \to \infty\right)$. Each site

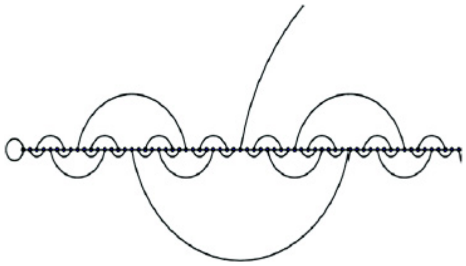

Figure 1: Depiction of HN3 on a semi-infinite line. The leftmost site here is $n = 0$.

on the one-dimensional lattice backbone is connected to its nearest neighbor. To generate the small-world hierarchy in these graphs, consider parameterizing any integer n (except for zero) *uniquely* in terms of two other integers (i, j), $i \geq 0$, via

$$n = 2^i (2j + 1) . \tag{4}$$

Here, i denotes the level in the hierarchy whereas j labels consecutive sites within each hierarchy. For instance, $i = 0$ refers to all odd integers, $i = 1$ to all integers once divisible by 2 (i.e., 2, 6, 10,...), and so on. In these networks, aside from the backbone, each site is also connected with (one or both) of its nearest neighbors *within* the hierarchy. We obtain the 3-regular network HN3 by connecting first all nearest neighbors along the backbone, but in addition also 1 to 3, 5 to 7, 9 to 11, etc, for $i = 0$, next 2 to 6, 10 to 14, etc, for $i = 1$, and 4 to 12, 20 to 28, etc, for $i = 2$, and so on, as depicted in Figure 1.

3 Renormalization Group Analysis

We will study biased random walks on HN3. All walks are controlled by the parameter p, which is the probability of a walker to jump off the lattice in a long range jump back towards the origin. The walker will move forward along the backbone with probability $(1 - p)$ if there is a long range jump available, and with probability 1 if there is no such option. These probabilities are displayed in Figure 2. If p were set to zero we would return to a simple one-dimensional walk.

In order to have a finite set of closed equations to work with, we closed the lattice, as shown in Figure 3, by connecting the 8th site to the 16th site, instead of the 24th site as it would have otherwise been by the rules of construction of the network. This small change does not affect the long term behavior of the system, for after renormalization it leaves no effect on the asymptotic behavior.

Here we demonstrate the RG process used to determine the recursion equations by starting on a 16 site network and renormalizing to an 8 site network, revealing the self-similar probability coefficients. We begin with time dependent probability of a walker to be at a given sight at time $t + 1$ in terms of probabilities of the system at time t. The master equations for this system are given

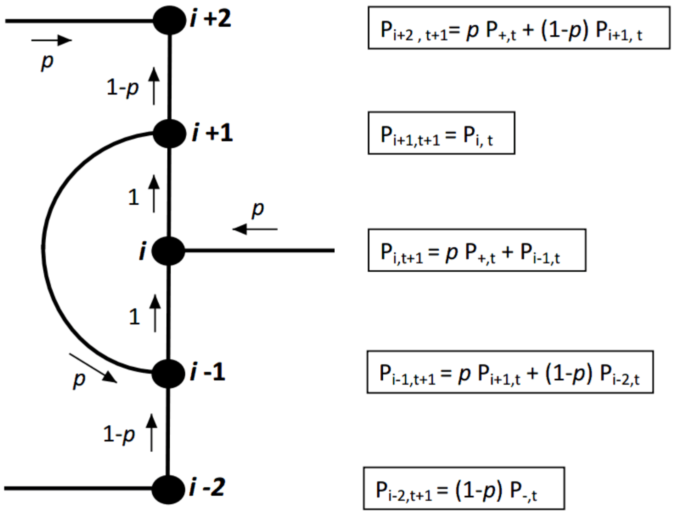

Figure 2: Sample of system showing hopping probabilities at each site and the corresponding equations describing the probability of the site being occupied at time $t + 1$.

by

$$
\begin{aligned}
P_{0,t+1} &= 1 \\
P_{1,t+1} &= P_{0,t} + pP_{3,t} \\
P_{2,t+1} &= P_{1,t} + pP_{6,t} \\
P_{3,t+1} &= P_{2,t} \\
P_{4,t+1} &= (1-p)P_{3,t} + pP_{12,t} \\
P_{5,t+1} &= P_{4,t} + pP_{7,t} \\
P_{6,t+1} &= P_{5,t} \\
P_{7,t+1} &= (1-p)P_{6,t} \\
P_{8,t+1} &= (1-p)P_{7,t} + pP_{16,t} \\
P_{9,t+1} &= P_{8,t} + pP_{11,t} \\
P_{10,t+1} &= P_{9,t} + pP_{14,t} \\
P_{11,t+1} &= P_{10,t} \\
P_{12,t+1} &= (1-p)P_{11,t} \\
P_{13,t+1} &= (1 - P_{4,t} + pP_{7,t} \\
P_{14,t+1} &= P_{13,t} \\
P_{15,t+1} &= (1-p)P_{14,t} \\
P_{16,t+1} &= (1-p)P_{15,t}
\end{aligned}
\tag{5}
$$

We introduce the generating function

$$P_l(z) = \sum_{t=0}^{\infty} P_{l,t} z^t, \tag{6}$$

in order to eliminate time dependence with this Laplace transform. We then have the following master equations in terms of z and p, where z represents time such that as z goes to 1, time goes to infinity:

$$\begin{aligned}
\tilde{P_0} &= 1 \\
\tilde{P_1} &= z\tilde{P_0} + zp\tilde{P_3} \\
\tilde{P_2} &= z\tilde{P_1} + zp\tilde{P_6} \\
\tilde{P_3} &= z\tilde{P_2} \\
\tilde{P_4} &= z(1-p)\tilde{P_3} + zp\tilde{P_{12}} \\
\tilde{P_5} &= z\tilde{P_4} + zp\tilde{P_7} \\
\tilde{P_6} &= z\tilde{P_5} \\
\tilde{P_7} &= z(1-p)\tilde{P_6} \\
\tilde{P_8} &= z(1-p)\tilde{P_7} + zp\tilde{P_{16}} \\
\tilde{P_9} &= z\tilde{P_8} + zp\tilde{P_{11}} \\
\tilde{P_{10}} &= z\tilde{P_9} + zp\tilde{P_{14}} \\
\tilde{P_{11}} &= z\tilde{P_{10}} \\
\tilde{P_{12}} &= z(1-p)\tilde{P_{11}} \\
\tilde{P_{13}} &= z(1-p)\tilde{P_{12}} + zp\tilde{P_{15}} \\
\tilde{P_{14}} &= z\tilde{P_{13}} \\
\tilde{P_{15}} &= z(1-p)\tilde{P_{14}} \\
\tilde{P_{16}} &= z(1-p)\tilde{P_{15}}
\end{aligned} \tag{7}$$

We then introduce generalized hopping parameters,

$$\begin{array}{cccc} a = z & b = zp & c = z & d = zp, \\ e = z(1-p) & f = z(1-p) & g = z & k = z(1-p). \end{array} \tag{8}$$

These parameters were determined to be the minimum necessary parameters through analysis of the RG. We began with the maximum possible parameters, and looked for those that renormalized identically to find the minimal set.

Inserting these parameters in equations 8 gives

$$
\begin{aligned}
\tilde{P_0} &= 1 \\
\tilde{P_1} &= a\tilde{P_0} + b\tilde{P_3} \\
\tilde{P_2} &= c\tilde{P_1} + d\tilde{P_6} \\
\tilde{P_3} &= g\tilde{P_2} \\
\tilde{P_4} &= e\tilde{P_3} + d\tilde{P_{12}} \\
\tilde{P_5} &= a\tilde{P_4} + b\tilde{P_7} \\
\tilde{P_6} &= c\tilde{P_5} \\
\tilde{P_7} &= f\tilde{P_6} \\
\tilde{P_8} &= e\tilde{P_7} + d\tilde{P_{16}} \\
\tilde{P_9} &= a\tilde{P_8} + b\tilde{P_{11}} \\
\tilde{P_{10}} &= c\tilde{P_9} + d\tilde{P_{14}} \\
\tilde{P_{11}} &= g\tilde{P_{10}} \\
\tilde{P_{12}} &= e\tilde{P_{11}} \\
\tilde{P_{13}} &= k\tilde{P_{12}} + b\tilde{P_{15}} \\
\tilde{P_{14}} &= c\tilde{P_{13}} \\
\tilde{P_{15}} &= f\tilde{P_{14}} \\
\tilde{P_{16}} &= e\tilde{P_{15}}
\end{aligned}
\tag{9}
$$

This system of equations is depicted in Figure 3, and is considered to be the master equations determining the system's behavior.

A single step of the RG involves solving the master equations (10) for P_l with odd values of l, and then eliminating them from every other equation, which would have even values of l. We are then left with the equations:

$$
\begin{aligned}
\tilde{P_0} &= 1 \\
\tilde{P_2} &= \frac{ac}{1-bcg}\tilde{P_1} + \frac{d}{1-bcg}\tilde{P_6} \\
\tilde{P_4} &= eg\tilde{P_3} + d\tilde{P_{12}} \\
\tilde{P_6} &= \frac{ac}{1-bcf}\tilde{P_5} \\
\tilde{P_8} &= ef\tilde{P_7} + d\tilde{P_{16}} \\
\tilde{P_{10}} &= \frac{ac}{1-bcg}\tilde{P_9} + \frac{d}{1-bcg}\tilde{P_{14}} \\
\tilde{P_{12}} &= eg\tilde{P_{11}} \\
\tilde{P_{14}} &= \frac{ck}{1-bcf}\tilde{P_{13}} \\
\tilde{P_{16}} &= ef\tilde{P_{15}}
\end{aligned}
\tag{10}
$$

Comparing the coefficients of these equations to the coefficients of equations

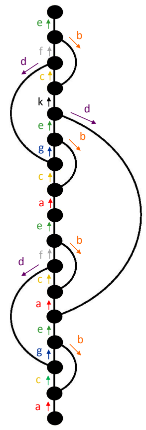

Figure 3: Definition of renormalizable hopping parameters for the biased walk along HN3. During each RG-step, every second site is eliminated algebraically and a new set of equations result which are identical in form to the previous set. Comparing the hopping parameters in these equations before and after each step leads to the RG-flow equations 11. Note that $d = zp$ does not renormalize.

(11) gives us the RG recursion equations describing the flow of the system:

$$
\begin{aligned}
a' &= \frac{ac}{1-bcg}, \\
b' &= \frac{zp}{1-bcg}, \\
c' &= eg, \\
e' &= ef, \\
f' &= \frac{ck}{1-bcf}, \\
g' &= \frac{ac}{1-bcf}, \\
k' &= \frac{ck}{1-bcg}.
\end{aligned} \tag{11}
$$

We verified the recursion equations by beginning with graphs of 64 sites and confirmed that the recursion equations maintained their self-similarity at each RG step, and therefore closed. Each renormalization gave a self-similar expression, and we determined that the seven unique parameters given in the 16 site graph shown in Figure 3 sufficiently explain the behavior of the system [14].

As was described in Section 1.2, the most important information we look to gain from the RG comes from the system's fixed point, for it is the behavior near this point that we are analyzing. Now that we have determined the RG-flow equations, we can find this fixed point of the system, where the evolution of the recursion equations stabilizes. We solve for this by setting each primed parameter in the set {a', b', c',e',f',g',k'} equal to the corresponding parameter in the set {a, b, c, e, f, g, k}. Say the parameters {a, b, c, e, f, g, k} represent the kth RG-step, then the set of parameters {a', b', c',e',f',g',k'} represent the $k+1$ step, therefore setting the respective parameters equal and solving the system of equations (11) gives us the point at which the system stabilizes. However, when we solved the equations (11) we were unable to eliminate all of the parameters, and therefore know that we were missing a condition within our set of master equations. In order to find this missing relation between the hopping parameters, we numerically analyzed the recursion equations by evolving them for $z = 1$ and various values of p, and discovered the conservation law $b = pa$. Adding this to the set of master equations, we were then able to solve for the fixed point of

$$\begin{aligned} a &\rightarrow \frac{z(pz-1)}{pz^2+pz-1} \\ b &\rightarrow \frac{pz(pz-1)}{pz^2+pz-1} \\ c &\rightarrow \frac{pz^2+pz-1}{pz-1} \\ e &\rightarrow \frac{-(pz^2+pz-1)}{z} \\ f &\rightarrow 1 \\ g &\rightarrow \frac{z}{1-pz} \\ k &\rightarrow \frac{-(pz-1)^2}{pz^2+pz-1} \end{aligned} \tag{12}$$

This means that these recursion equations converge for $k \rightarrow \infty$ towards this fixed point 12, which characterizes the dynamics of the system in the infinite-time limit. This corresponds to the limit as $z \rightarrow 1$.

The next step is to linearize the recursion equations (11) describing the RG-flow about the fixed point. We do this by taking the Jacobian J of these recursion equations, which is the matrix consisting of the first order derivatives of each recursion equations with respect to each parameter, shown in equation (13).

$$J = \begin{pmatrix} \frac{c}{1-bcg} & \frac{ac^2g}{(bcg-1)^2} & \frac{a}{(bcg-1)^2} & 0 & 0 & \frac{ac^2b}{(bcg-1)^2} & 0 \\ 0 & \frac{zpcg}{(bcg-1)^2} & \frac{zpbg}{(bcg-1)^2} & 0 & 0 & \frac{zpcb}{(bcg-1)^2} & 0 \\ 0 & 0 & 0 & g & 0 & e & 0 \\ 0 & 0 & 0 & f & e & 0 & 0 \\ 0 & \frac{kc^2f}{(bcf-1)^2} & \frac{k}{(bcf-1)^2} & 0 & \frac{kc^2b}{(bcf-1)^2} & 0 & \frac{c}{1-bcf} \\ \frac{c}{1-bcf} & \frac{ac^2f}{(bcf-1)^2} & \frac{a}{(bcf-1)^2} & 0 & \frac{bc^2a}{(bcf-1)^2} & 0 & 0 \\ 0 & \frac{kc^2g}{(bcg-1)^2} & \frac{k}{(bcg-1)^2} & 0 & 0 & \frac{kc^2b}{(bcg-1)^2} & \frac{c}{1-bcg} \end{pmatrix} \tag{13}$$

Since we are concerned with the infinite-time limit, we set the parameters equal to their respective fixed point values, and set $z = 1$, leaving p as our only variable in the matrix, shown in equation (14).

$$J = \begin{pmatrix} 1 & \frac{1}{1-2p} & \frac{(p-1)^3}{(2p-1)^3} & 0 & 0 & \frac{p(p-1)^2}{(2p-1)^2} & 0 \\ 0 & \frac{p}{1-2p} & -\frac{(p-1)^2p^2}{(2p-1)^3} & 0 & 0 & \frac{(p-1)^2p^2}{(2p-1)^2} & 0 \\ 0 & 0 & 0 & -\frac{1}{p-1} & 0 & 1-2p & 0 \\ 0 & 0 & 0 & 1 & 1-2p & 0 & 0 \\ 0 & -\frac{2p-1}{(p-1)^2} & -\frac{1}{2p-1} & 0 & -\frac{p}{p-1} & 0 & -\frac{2p-1}{(p-1)^2} \\ -\frac{2p-1}{(p-1)^2} & \frac{2p-1}{(p-1)^3} & \frac{1}{(p-1)(2p-1)} & 0 & \frac{p}{(p-1)^2} & 0 & 0 \\ 0 & \frac{p-1}{2p-1} & -\frac{(p-1)^4}{(2p-1)^3} & 0 & 0 & -\frac{(p-1)^3p}{(2p-1)^2} & 1 \end{pmatrix} \tag{14}$$

We then look at the eigenvalue of this Jacobian, and take λ to be equal to the largest of these eigenvalues,

$$\lambda = \frac{2-3p}{1-2p,},$$

for the largest eigenvalue dictates the behavior in the long-time limit. This λ is related to the mean square displacement exponent d_w by equation (3), which gives the exponent of

$$d_w = \frac{ln2}{ln\frac{2-3p}{1-2p}}, \quad (0 \leq p \leq \frac{1}{2}). \tag{15}$$

For $p > \frac{1}{2}$, the walker will stay confined near the origin, because long-range jumps backwards along the network will be taken frequently. This means that as space rescales by 2 (eliminating every odd site), the rescaling of time is dependent on the chosen p, for any given value of $p < \frac{1}{2}$. This is why this system exhibits non-universal behavior, for the long-term behavior of the system is dependent upon the micoscopic detail of the value of p.

We compare the extrapolation of the numerical data obtained for the mean square displacement for the walk at different values of p in Figure 4, and find the simulation to be consistent with our exact RG-result.

4 Conclusion

In this project I have learned how to use the powerful tools of the dynamic renormalization group in order to dissect the biased random walk problem on the hierarchical network HN3. Our study serves as a simple example of the unusual scaling behavior for dynamic processes on complex networks. Previous renormalization group studies of random walks on this particular network have led to distinct universal behavior [4]. However, as suggested in the numerical simulations, we have found a dynamical transition between a localized adsorption phase and an anomalous diffusion phase in which the mean-square displacement exponent depends non-universally on the bias p of the walker to jump back down the lattice at a long-range jump. We hope this model can provide insight into the phenomenon of anomalous diffusion on complex networks, since the causes of anomalous diffusion are often not fully understood and are thus a topic of ongoing research interest[12]. Additionally, we hope that these results can also be useful in a more direct application of this system as it compares to similar models, such as the behavior of kinesin proteins. We also hope

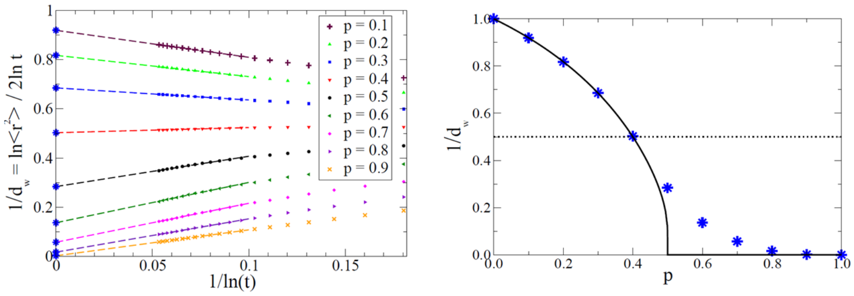

Figure 4: Extrapolation of the data obtained for the mean square displacement (left) for the walk at different values of the back-jump probability p from simulations run up to a temporal cutoff at $t = 2^{27}$. We used simply a linear fit to the data set for each value of p deep in the asymptotic regime for large t, as indicated by the fitted lines, data for all smaller t were ignored. Large corrections to linear behavior are apparent for larger p, suggesting large errors. The fitted values for $1/d_w$ of the extrapolation for $t \to \infty$ are marked as blue stars at the intercept. These extrapolated values (blue stars) for the exponent d_w are shown plotted as a function of p (right). The line corresponds to the exact RG-result from Eq. 15. For $0 \leq p \leq \frac{1}{2}$, the numerical results fit the exact result within errors. However, near and above the transition $p = \frac{1}{2}$ large errors are observed, as expected from the imperfect linear fits above.

that having a system which contains both a regime of a universal exponent, as well as a regime of non-universality can provide insight into the universality hypothesis of critical phenomena.

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