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Local-global principles for hermitian spaces over semi-global fields

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Abstract

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This dissertation studies the Hasse principle for projective homogenous spaces under unitary groups over semi-global fields. Let K be a complete discrete valued field with residue field k and F the function field of a curve over K. Let $A \in {}_2Br(F)$ be a central simple algebra with an involution σ of any kind and $F_0 = F^\sigma$. Let h be an hermitian space over (A, σ) and $G = SU(A, \sigma, h)$ if σ is of first kind and $G = U(A, \sigma, h)$ if σ is of second kind. Suppose that $\text{char}(k) \neq 2$ and one of the following holds;

a) $\text{ind}(A) \leq 4$;

b) For every finite extension ℓ/k, every element in ${}_2Br(\ell)$ has index at most 2.

Then we prove that projective homogeneous spaces under G over F_0 satisfy a local-global principle for rational points with respect to discrete valuations of F. The proof implements patching techniques of Harbater, Hartmann and Krashen. Furthermore, we shall prove a Springer-type theorem for isotropy of hermitian spaces over odd degree extensions of function fields of p-adic curves.

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Contents

Chapter 1

Preliminaries

1.1 Central Simple Algebras

We refer to readers to [10] for further information on central simple algebras.

Let K be a field. Let A be a finite dimensional associative algebra over K with center $Z(A)$. We say that A is central if $K = Z(A)$. Furthermore, A is simple if it has no non-trivial two-sided ideals. By a theorem of Wedderburn, it is known that every central simple algebra A is isomorphic to $M_n(D)$, where D is a central division algebra over K and D and n are uniquely determined upto isomorphism. We say that a central simple algebra is *split* (over K) if $A \cong M_n(K)$.

The tensor product $A \otimes_K B$ of K-central simple algebras A and B is a K-central simple algebra. We define an equivalence relation on the set of all central simple algebras over K by; $A \sim B$ if there exist positive integers m_1, m_2 such that $A \otimes_K M_{m_1}(K) \cong B \otimes_K M_{m_2}(K)$. Then the tensor product gives a a well defined operation on the set of equivalence classes of K-central simple algebras and it is a group under this operation. This group is called the *Brauer group* of K, denoted by $Br(K)$. The Brauer group is an abelian group. The identity element of this group is the class of all matrix rings over K. The inverse of $[A]$ is given by $[A^{op}]$, the opposite ring of A. We denote the n-torsion subgroup of $Br(K)$ by ${}_nBr(K)$.

Let $A = M_m(D)$ be a K-central simple algebra with an underlying division algebra D. The dimension of a K-central simple algebra A over K is always a square, say n^2. The degree of A over K is given by n; $deg(A) = \sqrt{dim_K(A)}$. The index of A is given by the degree of the underlying division algebra D over K; $ind(A) = deg(D)$. The period or exponent of A is given by the order of $[A]$ in $Br(K)$; $per(A) = ord([A])$. An interesting fact is that $per(A)|ind(A)$ and they always share the same prime factors ([10], Proposition 4.5.13).

Example. Let K be a field of characteristic not 2. A central simple of algebra of degree 2 over K is called a *quaternion algebra*. For $a, b \in K^*$, let $(a, b)_K$ denote the quaternion algebra over K generated by $\{1, i, j, ij\}$ with the relations $i^2 = a, j^2 = b, ij = -ji$. Furthermore, every central simple algebra of index 2 is Brauer equivalent to a quaternion algebra over the underlying field.

Let R be a commutative ring. Let A be an R-algebra and A^{op} the opposite ring of A. Then A is a left $A \otimes A^{op}$-module via the action $(a \otimes b^o)x = axb$ for all $a, b, x \in A$. We say that A is an *Azumaya algebra* over R if A is central over R, finitely generated R-module and projective as left module over $A \otimes A^{op}$. Two Azumaya algebras A and B are *Brauer equivalent* if there exist finitely generated faithful projective modules P and Q over R such that $A \otimes_R \mathrm{End}_R(P) \cong B \otimes_R \mathrm{End}_R(Q)$. We may construct the Brauer group $\mathrm{Br}(R)$, an abelian group whose underlying set is the set of all equivalence classes of Azumaya algebras over R under Brauer equivalence. As above, the group operation is given by tensoring two algebras over R, and the identity element is given by the class of $\mathrm{End}_R(P)$, where P is finitely generated projective R-module.

Theorem 1.1.1. *([3], 6.5) Let R be a complete local ring with residue field k. Then* *the canonical map* $Br(R) \cong Br(k)$ *is an isomorphism.*

Let R be a regular local ring and K its field of fractions. We say that a central simple algebra B over K is *unramified* on R is there is an Azumaya algebra A over R with $B \simeq A \otimes_R K$. If B is not unramified then we say that A is *ramified*.

Let $\mathscr{X}$ be a regular integral scheme with the field of fractions K and B a central simple algebra over K. Let $x \in \mathscr{X}$ be a point. We say that B is *unramified* (resp. *ramified*) at x if it is unramified (resp. ramified) on the local ring at x.

If ν is a discrete valuation on a field K and B is a central simple algebra over K we say that B is *unramified* (resp. *ramified*) at ν if it is unramified (resp. ramified) on the valuation ring at ν.

1.2 Involutions and Hermitian Forms

Let A be a ring with identity. An involution σ on A is given by a map $\sigma : A \longrightarrow A$ such that $\sigma(xy) = \sigma(y)\sigma(x)$, $\sigma(x+y) = \sigma(x)+\sigma(y)$ and $\sigma(\sigma(x)) = x$ for all $x, y \in A$. Suppose that A is a central simple algebra over a field K with an involution σ. One may consider $K^\sigma = \{\alpha \in K | \sigma(\alpha) = \alpha\}$. Since σ has order 2, $[K : K^\sigma] \leq 2$. If $[K : K^\sigma] = 1$, σ is called an involution of the *first* kind. If $[K : K^\sigma] = 2$, σ is called an involution of the *second kind*.

Let (A, σ) be a central simple algebra over a field K of degree d with an involution σ. Let $A^\sigma = \{x \in A | \sigma(x) = x\}$. An involution σ of first kind on A is called *orthogonal* if $\dim_K(A^\sigma) = \dfrac{d(d+1)}{2}$ and *symplectic* if $\dim_K(A^\sigma) = \dfrac{d(d-1)}{2}$. An involution σ on A of second kind is called *unitary* involution.

Example. Let K be a field and $A = (a, b)$ a quaternion algebra over K with $a, b \in K^*$. Let $i, j \in A$ be the standard generators of A with $i^2 = a$, $j^2 = b$ and $ij = -ji$. Then there is a unique involution σ on A with $\sigma(i) = -i$ and $\sigma(j) = -j$. Then σ is a symplectic involution and it is called the canonical involution on A.

Remark. Notice that if σ is an involution of the first kind, then A has period 2, since an involution defines an isomorphsim $A \cong A^{op}$.

Let K be a field of characteristic not 2. Let A be a central simple over K. Let V be a finitely generated module over A. Then $A = M_m(D)$ for some division algebra D. Therefore $V = (D^m)^s$ for some positive integer s. The reduced dimension $\mathrm{rdim}_A(V)$ of V over A is defined to be $\dfrac{\dim_K(V)}{\deg(A)} = s \cdot \mathrm{ind}(A)$.

Let (A, σ) be a central simple algebra over a field K with an involution σ. Let V be a finitely generated right A-module and $\varepsilon = \pm 1$. A map $h : V \times V \longrightarrow A$ is called an ε-hermitian form over (A, σ) if $h(x + x', y) = h(x, y) + h(x', y)$, $h(x, y + y') = h(x, y) + h(x, y')$ and $h(xa, yb) = \sigma(a)h(x, y)b$, $h(x, y) = \varepsilon\sigma(h(y, x))$ for all $x, y, x', y' \in V$, $a, b \in A$. If $\varepsilon = -1$, then h is called a skew hermitian form, else it is simply a hermitian form.

Consider the dual space of V, $V^* = \mathrm{Hom}_A(V, A)$. The dual space can be viewed as a right A-module given by $(f * a)(x) = \sigma(a)f(x)$ for all $f \in V^*, x \in V, a \in A$. Then a hermitian form h induces a right A-module homomorphism $\tilde{h} : V \longrightarrow V^*$ given by $\tilde{h}(x)(y) = h(x, y)$. If such a map is an isomorphsim, then h is referred to as a hermitian space. The rank of h is given by $\mathrm{Rank}(h) = \dfrac{\dim_K(V)}{\mathrm{ind}(A)\deg(A)} = s$.

Suppose now that (D, σ) is a central division algebra over K ($\mathrm{char}(K) \neq 2$) with an involution σ and V is a finite dimensional right vector space over D. Then $V \cong D^n$ for some positive integer n. Let $\varepsilon = \pm 1$. Suppose that if $\epsilon = -1$, then $\mathrm{ind}(D) \geq 2$. If h is an ε-hermitian form on D, then there exist $a_i \in D^*$ such that $\sigma(a_i) = \varepsilon a_i$ and $h(x, y) = \sum_{i=1}^{n} \sigma(x_i)a_i y_i$ for all $(x_i), (y_i) \in D^n$. We write $h = \langle a_1, \cdots a_n \rangle$, with

$\mathrm{Rank}(h) = \dim_D(V) = n$.

Example. If $A = K$, σ is the identity map, and $\varepsilon = 1$, then a hermitian form h is a symmetric bilinear form and $q_h(x) = h(x,x)$ for all $x \in V$ is a quadratic form. Conversely, let $q : V \longrightarrow K$ be any quadratic form. Consider the associated symmetric bilinear form $b_q(x,y) = \frac{1}{2}(q(x+y) - q(x) - q(y))$. Then b_q is a hermitian form over K.

Let A be a central simple algebra over a field K. Let V be a finitely generated A-module. Let $E = \mathrm{End}_A(V)$. Let $\varepsilon = \pm 1$ and (V,h) be a ε-hermitian form on (A,σ). Define the adjoint involution of h given by ad_h, which satisfies the relation $h(x, f(y)) = h(\mathrm{ad}_h(f)(x), y)$ for all $x, y \in V$ and $f \in E$. Suppose that $\mathrm{rdim}(V) = 2r$ and ad_h is orthogonal. Then define the determinant of h given by $\det(h) = \mathrm{Nrd}_{\mathrm{End}_A(V)/K}(f) \in K^*/K^{*2}$ for $f \in \mathrm{End}_A(V)$ such that $\mathrm{ad}_h(f) = -f$. This definition is independent of the choice of f. Define the discriminant of h by $\mathrm{disc}(h) = (-1)^r \det(h)$. For example, if (A,σ,h) is a division algebra with ad_h an orthogonal form and $h = \langle a_1, a_2 \cdots a_{2t}\rangle$ then $\det(h) = \mathrm{Nrd}_{A/K}(a_1 a_2 \cdots a_{2t})$ and $\mathrm{disc}(h) = (-1)^{t\deg(A)}\mathrm{Nrd}_{A/K}(a_1 a_2 \cdots a_{2t})$, both in K^*/K^{*2}.

An ε-hermitian form (V,h) on (A,σ) is called *isotropic* if there exists a non-zero vector $x \in V$ such that $h(x,x) = 0$. It is called anisotropic if

$$h(x,x) = 0$$

iff $x = 0$. A subspace $W \subset V$ is called totally isotropic if $h(W,W) = 0$.

Suppose that B is a central simple algebra over a field K with an involution τ. Then τ is said to be isotropic if there exists $b \in B$ such that $\tau(b)b = 0$. A right ideal I of B is totally isotropic if $\tau(b')b = 0$ for all $b' \in I$ and $b \in B$. Suppose that

$B = \mathrm{End}_D V$ and $\tau = \mathrm{ad}_h$, the adjoint involution of h. Then h is isotropic if and only if τ is. If $A = D$ is a division algebra, then a subspace $W \subset V$ is isotropic if and only if the ideal $I = \mathrm{Hom}_D(V, W)$ is an isotropic ideal of B.

Example. Let A be a central simple algebra over a field K with an involution σ. Let V be a finitely generated right A-module. Consider the ε-hermitian space $(V \oplus V^*, \mathbb{H})$ over (A, σ), where

$$\mathbb{H}((x, f), (y, g)) = f(x) + \varepsilon\sigma(g(y))$$

for all $f, g \in V^*$ and $x, y \in V$. Then the subspaces of $\mathbb{H}$ given by $V \oplus 0$ and $0 \oplus V^*$ are totally isotropic. The space $(V \oplus V^*, \mathbb{H})$ is called the *hyperbolic space* on V.

Suppose that (V_1, h_1) and (V_2, h_2) are two ε-hermitian spaces over a K-central simple algebra (A, σ). Define the orthogonal sum of V_1 and V_2 to be the ε-hermitian space $(V_1 \oplus V_2, h_1 \perp h_2)$, where

$$(h_1 \perp h_2)((u_1, v_1), (u_2, v_2)) = h_1(u_1, u_2) + h_2(v_1, v_2)$$

for all $u_1, u_2, v_1, v_2 \in V_1, V_2$. Orthogonal sum of hyperbolic spaces is a hyperbolic spaces. Under the operation orthogonal sum, the set of isomorphism classes of ε-hermitian forms form an abelian monoid. We can construct the Grothendieck group $KU^\varepsilon(A, \sigma)$ from this abelian monoid, making it an abelian group. The Witt group $W^\varepsilon(A, \sigma)$ is the quotient of the Grothendieck group $KU^\varepsilon(A, \sigma)$ by the subgroup of hyperbolic spaces in $KU^\varepsilon(A, \sigma)$. If $A = D$ is a division algebra, then every ε-hermitian form h can be written uniquely as an orthogonal sum $h = h_{an} \perp h_{hyp}$, where h_{an} is anisotropic and h_{hyp} is hyperbolic. Two hermitian spaces h_1 and h_2 over (D, σ) are said to be Witt equivalent if their anisotropic parts are isomorphic. Two hermitian spaces are Witt equivalent if and only if they represent the same element in $W^\varepsilon(D, \sigma)$. The group operation is given by orthogonal sum; $[h_1] \perp [h_2] = [h_1 \perp h_2]$. The identity element is given by the class pf hyperbolic plane $\mathbb{H}$ and the inverse is given by $-[h] = [-h]$.

Let $A = K$, $\sigma = id$ and $\epsilon = -1$. Then every ϵ-hermitian space over (A, σ) is hyperbolic and two ϵ-hermitian spaces are isometric if and only they have same dimensions. For the rest of the thesis we assume that if $\epsilon = -1$, then $\operatorname{ind}(A) \geq 2$.

Let (K, ν) be a discrete valued field with valuation ring R_ν, maximal ideal $\mathfrak{m}_\nu$ and residue field $k(\nu) = R_\nu/\mathfrak{m}_\nu$, $\operatorname{char}(k(\nu)) \neq 2$. Let $(\hat{R_\nu}, \hat{\mathfrak{m}_\nu})$ be the completion of $(R_\nu/\mathfrak{m}_\nu)$ and $K_\nu = \operatorname{Frac}(\hat{R_\nu})$. Let $\hat{\nu}$ be the extension of ν to K_ν. We have $k(\hat{\nu}) = \hat{R_\nu}/\hat{\mathfrak{m}_\nu} = k(\nu)$. Let D be a finite-dimensional division algebra over K with an involution σ such that $Z(D)^\sigma = K$. Suppose that $D \otimes_K K_\nu$ is a division algebra over K_ν. Then the valuation $\hat{\nu}$ on K_ν extends to a unique valuation ν' on $Z(D \otimes K_\nu)$ such that;

$$\nu'(x) = \frac{1}{[Z(D \otimes K_\nu) : K_\nu]} \nu(N_{Z(D\otimes K_\nu)/K_\nu}(x))$$

for all $x \in Z(D \otimes_K K_\nu)^*$. Furthermore, ν' extends to a valuation w on $D \otimes_K K_\nu$ given by;

$$w(x) = \frac{1}{\operatorname{ind}(D \otimes K_\nu)} \nu'(\operatorname{Nrd}_{D\otimes K_\nu/Z(D\otimes K_\nu)}(x))$$

for all $x \in (D \otimes_K K_\nu)^*$. The restriction of w to D is a valuation given by $w(x) = \dfrac{1}{\operatorname{ind}(D)} \nu(\operatorname{Nrd}_{D/Z(D)}(x))$. Let t_D be a parameter of (D, w) ([27], 13.2). Then we can choose $\pi_D \in D$ such that $\sigma(\pi_D) = \pm\pi_D$ and $w(\pi_D) \equiv w(t_D) \bmod 2w(D^*)$ ([19], 2.7). Let $\Lambda_w = \{x \in D^* \mid w(x) \geq 0\} \cup \{0\}$ be the valuation ring of w and $\mathfrak{m}_w = \{x \in D^* \mid w(x) > 0\} \cup \{0\}$ be the maximal ideal of Λ. Let $\mathfrak{D}_w = \Lambda_w/\mathfrak{m}_w$ be the division algebra over the residue field $k(v)$. Let $q_w : \Lambda_w \to \mathfrak{D}_w$ be the quotient map. Then we an involution σ_w on $\mathfrak{D}_w$ given by $\sigma_w(q_w(x))) = q_w(\sigma(x))$ for all $x \in \Lambda_w$ ([27], 13.2).

Suppose that (V, h) is a ε-hermitian space over a division algebra (D, σ) for $\varepsilon \in \{-1, 1\}$. There is an orthogonal basis of V such that h has a diagonal form $h = \langle a_1, \cdots a_m\rangle$, where $a_i \in D$ and $\sigma(a_i) = \varepsilon a_i$ for all i. If $w(a_i) = 0$ for all i, then

$q_w(h) = \langle q_w(a_1), \cdots q_w(a_m)\rangle$ is a ε-hermitian space over $(\mathfrak{D}_w, \sigma(w))$. Up to isometry, we may assume that any hermitian space h over (D, σ) has diagonal entries with w-value either 0 or $w(t_D)$ ([19], 2.20).

Proposition 1.2.1. *([19], 3.27, 3.29) Let (V, h) be a ε-hermitian space over a division algebra (D, σ) for $\varepsilon \in \{-1, 1\}$. Let $\pi_D \in D$ be as above. Suppose that $\sigma(\pi_D) = \varepsilon'\pi_D$ for $\varepsilon' \in \{-1, 1\}$. Then there is a unique decomposition $h_{K_\nu} = h_1 \perp h_2\pi_D$, where h_1 is a ε-hermitian form over $(D \otimes K_\nu, \sigma \otimes Id_{K_\nu})$ and h_2 is a $\varepsilon\varepsilon'$-hermitian form over $(D \otimes K_\nu, Int(\pi_D) \cdot \sigma \otimes Id_{K_\nu})$. Each diagonal entry of h_1 and h_2 has w-value 0. Furthermore, the following are equivalent;*

a) h is isotropic.

b) h_1 or h_2 is isotropic.

c) $q_w(h_1)$ or $q_w(h_2)$ is isotropic.

1.3 Linear Algebraic Groups

Let F be a field. An algebraic group G is an affine variety over F along with a group structure that is compatible with the variety structure such that the multiplication map $m(x, y) = xy$ and the inverse map $i(x) = x^{-1}$ are morphisms of varieties.

Remark. Let **Groups** be the category of groups and group homomorphisms. Let $\mathbf{Algebras}_F$ be the category of unital associative commutative algebras over F and F-algebra homomorphisms. A variety G over F is an algebraic group over F if its functor of points is from $\mathbf{Algebras}_F^{op}$ to **Groups**. A morphism between two algebraic groups is a natural transformation of their functor points.

Example.

1. The general linear group over a field F is given by a functor $GL_n : \mathbf{Algebras}_F^{op} \longrightarrow \mathbf{Groups}$ such that $GL_n(L)$ is the set of invertible $n \times n$ matrices with entries in L.

2. The multiplicative group over a field F is given by a functor $\mathbb{G}_m : \mathbf{Algebras}_F^{op} \longrightarrow \mathbf{Groups}$ such that $\mathbb{G}_m(L) = L^*$.

For the rest of this dissertation, a variety over F will be a geometrically reduced separated scheme of finite type over F. Let $L|F$ be a field extension and X be a variety over F. Then define $X_L = X \times_{\mathrm{Spec}(F)} \mathrm{Spec}(L)$ as the scalar extension of X to L. Denote $X_{K_{\mathrm{sep}}}$ by X_{sep}. Define $X(L) = \mathrm{Hom}_{\mathrm{Spec}(F)}(\mathrm{Spec}(L), X)$ as the L-points of X.

A connected linear algebraic group G over F is rational if its function field F(G) is a purely transcendental extension of F.

Example.

1. The general linear group GL_n over F is a rational linear conncected group over F since it is open in $\mathbb{A}_F^{n^2}$. We can say the same about the algebraic group PGL_n, the projective linear group over F. For any central simple algebra A over F of degree n, the algebraic groups $GL_n(A)$ and $PGL_n(A)$ are also rational connected linear groups over F.

2. Let F be a field of characteristic not 2. Let L be a quadratic field extension of F. Let A be a central division algebra over L and σ be an involution on A of the second kind such that $L^\sigma = F$. Let V be a finitely generated right A-module. Let $h : V \times V \longrightarrow A$ be an ε-hermitian form for $\varepsilon = \pm 1$. The unitary group of A is defined to be $U(A, \sigma, h) = \{f \in \mathrm{End}_A(V)^* | h(f(x), f(y)) = h(x, y)\}$. Let ad_h be the adjoint involution of h in $\mathrm{End}_A(V)$. Let $U\mathrm{End}_A(V), \mathrm{ad}_h) = \{f \in \mathrm{End}_A(V)^* | f \circ \mathrm{ad}_h(f) = \mathrm{Id}_V\}$. Then $U(A, \sigma, h) \cong U(\mathrm{End}_A(V), \mathrm{ad}_h)$. By ([17],

23A), $U(A,\sigma,h)$ is a connected linear algebraic group. By Cayley parametrization ([8], Lemma 5), it is a rational group as well.

3. Let F be a field of characteristic not 2. Let A be a central simple algebra over F. Let σ be an involution on A of the first kind. Let V be a finitely generated right A-module. Let $h : V \times V \longrightarrow A$ be an ε-hermitian form for $\varepsilon = \pm 1$. The special unitary group of is defined to be $SU(A,\sigma,h) = \{f \in \mathrm{End}_A(V)^* | h(f(x), f(y)) = h(x,y), \det(f) = 1\}$. By ([17], 23A), $SU(A,\sigma,h)$ is a connected linear algebraic group. By Cayley parametrization ([8], Lemma 5), it is a rational group as well.

1.4 Projective Homogeneous Spaces

Definition. Let G be an algebraic group over K and X an algebraic variety over K. We say that X is a homogeneous space under G if G acts on X on the left and $G(L)$ acts on $X(L)$ transitively for all associative K-algebras L.

Remark. The above definition is equivalent to the surjectivity of the map $\phi : G(L) \times X(L) \longrightarrow X(L) \times X(L)$, where $\phi(g,x) = (x, gx)$ for all $g \in G(L)$ and $x \in X(L)$. A homogeneous space X is said to be a principal homogeneous space under G if the map ϕ is a bijection for all associative K-algebras L.

Definition. Let G be an algebraic group over K and X an algebraic variety over K. We say that X is a projective homogeneous space under G if X is a homogeneous space under G and a projective variety over K.

Let F be an arbitrary field, $\mathrm{char}(F) \neq 2$. Let A be a central simple algebra whose center L is a field extension of F. Let σ be an involution on A such that $L^\sigma = F$. Suppose that V is a finitely generated A-module and $h : V \times V \longrightarrow A$ is a ε-hermitian form over (A,σ) with $\varepsilon \in \{-1, 1\}$. Let $G(A,\sigma,h) = SU(A,\sigma,h)$ if σ is an involution of the first kind, else let $G(A,\sigma,h) = U(A,\sigma,h)$. Then $G(A,\sigma,h)$ is a connected rational linear group with rank n such that the reduced dimension of V is $n+1$ if σ is unitary, $2n+1$ if $A = F$ and $\dim(V)$ is odd, $2n$ otherwise.

Let $0 < n_1 < \cdots n_r = n$ be an increasing sequence of positive integers. For every field extension L of F, define $X(n_1, n_2, \cdots n_r)(L)$ as the set;

$\{(W_1, \cdots W_r) | 0 \subsetneq W_1 \subset \cdots \subset W_r, W_i$ is a totally isotropic subspace of $V \otimes L, \mathrm{rdim}_{A_L} W_i = n_i, \forall 1 \leq i \leq r\}$

Alternatively, we may define $X(n_1, n_2, \cdots n_r)(L)$ as;

$\{(I_1, \cdots I_r) | 0 \subsetneq I_1 \subset \cdots \subset I_r, I_i$ is a totally isotropic ideal of $\mathrm{End}_{A_L} V \otimes L, \mathrm{rdim}_{A_L} I_i = n_i, \forall 1 \leq i \leq r\}$.

If $r = 1$, we write $X(n_1) := X_{n_1}$.

Lemma 1.4.1. *([20],[21], sec.5 and sec 9) Let $L|F$ be a field extension and $n_1, \cdots n_r$ be an increasing sequence of positive integers as above. Then;*

a) $X(n_1, n_2, \cdots n_r)(L) \neq \emptyset$ iff $X_{n_r}(L) \neq \emptyset$ and ind$(A_L) | \gcd(n_1, \cdots n_r)$.

b) $X^{\varepsilon}(n_1, n_2, \cdots n_r)(L) \neq \emptyset$ iff $X^{\varepsilon}_{n_r}(L) \neq \emptyset$ and ind$(A_L) | \gcd(n_1, \cdots n_r)$.

Example.

1. A generalized Severi-Brauer variety over $SB_r(A)$ of A over F is described as follows;

$$SB_r(A)(L) = \{I | I \text{ is a right ideal of } A_L, \mathrm{rdim}_{A_L}(I) = r\}$$

   for all field extensions $L|F$. $SB_r(A)$ is a projective homogeneous space under $PGL_n(A)$, where $PGL_n(A)$ acts on $SB_r(A)$ by left multiplication. The set of projective homogeneous space under $PGL_n(A)$ is given by $\{(X(n_1, \cdots n_r) | 0 < n_1 \cdots < n_r < n\}$, where for all field extensions $L|F$, $Y(n_1, \cdots n_r)(L) = \{(I_1, \cdots I_r) \in \prod_{i=1}^{r} SB_{n_i}(A)(L) | 0 \subset I_1 \cdots I_r\}$.

2. Consider the group $U(A,\sigma,h)$. The set of projective homogeneous spaces under this group is given by $\{(X(n_1,\cdots n_r)|0 \leq n_1 \cdots \leq n_r \leq \lfloor n/2 \rfloor\}$ ([21], 9.1).

3. Consider the group $SO_{2n+1}(q)$. Let X_q be the projective homogeneous space given by Proj $\left(\frac{\mathrm{Sym}(V)^*}{(q)}\right)$. Let $L|F$ be a field extension. Then q_L is isotropic over L iff $X_q(L) \neq \emptyset$. The set of projective homogeneous spaces under this group is given by $\{(X(n_1,\cdots n_r)|0 \leq n_1 \cdots \leq n_r \leq n\}$ ([20], 5.II).

4. Consider the group $SU(A,\sigma,h)$. Let ad_h be orthogonal: either σ is orthogonal and h is hermitian, or σ is symplectic and h is skew-hermitian. Let $\mathrm{disc}(h) \neq 1$. Then the set of projective homogeneous spaces under this group is given by $\{(X(n_1,\cdots n_r)|0 \leq n_1 \cdots \leq n_r < n\}$ ([20] 5.III).

5. Continuing from the last example, let $\mathrm{disc}(h) = 1$, $r = 1$, $n_1 = n$. Then X has two connected components, say X^+ and X^-. We may define;

$$X^+(n_1, n_2 \cdots n_r) = \{(I_1,\cdots I_r) \in (X(n_1,\cdots n_r)(L)|I_r \in X_n^+(L)\}$$

$$X^-(n_1, n_2 \cdots n_r) = \{(I_1,\cdots I_r) \in (X(n_1,\cdots n_r)(L)|I_r \in X_n^-(L)\}$$

Therefore the set of projective homogeneous spaces under the group $SU(A,\sigma,h) = \{(X(n_1,\cdots n_r)|0 \leq n_1 \cdots \leq n_r < n\} \cup X_n^+ \cup X_n^- \cup \{(X^\varepsilon(n_1,\cdots n_r)|0 \leq n_1 \cdots \leq n_{r-1} < n-1, n_r = n, r > 1, \varepsilon = \pm 1\}$ ([20] 5.IV).

1.5 Morita Equivalence

Let F be a field and $A = M_m(D)$ a central simple algebra over F for a central division algebra D over F. Suppose that A has an involution σ. Then, by ([17], Th. 3.1, Rem. 3.11, Rem. 3.20), D has an involution τ of the same kind as σ. Further there exists a ε'- hermitian space (D^m, g) over (D,τ), $\varepsilon' = \pm 1$, such that $\sigma = \mathrm{ad}_g$.

Let V be a right A-module and $h : V \times V \longrightarrow A$ be ε-hermitian space, $\varepsilon = \pm 1$. Let $V_0 = V \otimes_A D^m$. Then V_0 is a right D-module and there an $\varepsilon\varepsilon'$-hermitian space on V_0 with $h_0(x \otimes a, y \otimes b) = g(a, h(x,y)b)$ for a $x, y \in V$ and $a, b \in D^n$. By Morita equivalence, this correspondence is an equivalence of categories between the category of hermitian forms over (A, σ) and the category of hermitian forms over (D, τ).

Remark.

a) $\mathrm{rdim}_A(V) = \mathrm{rdim}_D(V_0)$. We see this via the following calculation;

$$\mathrm{rdim}_D(V_0) = \frac{\dim_K(V_0)}{\deg(D)} = \frac{\dim_K(V \otimes_A D^m)}{\deg(D)} = \frac{m \cdot \dim_K(V)\dim_K(D)}{\dim_K(A)\deg(D)}$$

$$= \frac{\dim_K(V)}{m \cdot \deg(D)} = \frac{\dim_K(V)}{\deg(A)} = \mathrm{rdim}_A(V)$$

b) $\mathrm{Rank}(h) = \mathrm{Rank}(h_0)$. By definition of the rank of an ε-hermitian space, one sees that;

$$\mathrm{Rank}(h) = \frac{\mathrm{rdim}(V)}{\mathrm{ind}(A)} = \frac{\mathrm{rdim}(V_0)}{\mathrm{ind}(D)} = \mathrm{Rank}(h_0)$$

Lemma 1.5.1. *([16], Chapter 1, 9.3.5)*

a) h is isotropic iff h_0 is isotropic.

b) h is hyperbolic iff h_0 is hyperbolic

Let X be the projective homogeneous space under $G(A, \sigma, h)$ and X_0 be the space under $G(D, \tau, h_0)$.

Lemma 1.5.2. *([14], 16.10)* $X(n_1, \cdots n_r) \cong X_0(n_1, \cdots n_r)$.

It is sufficient to show that $X(n_1, \cdots n_r) \neq \emptyset \iff X_0(n_1, \cdots n_r) \neq \emptyset$ since Morita equivalence preserves reduced dimension and isotropy ([16], Chapter 1, 9.3.5).

Lemma 1.5.3. *Let $rdim(V) = 2n$ and ad_h be orthogonal with $disc(h) = 1$, $n_{r-1} < n - 1$ (if $r > 1$) and $n_r = n$. If $ind(A_L) | \gcd(n_1, \cdots n_r)$, then $X^\varepsilon(n_1, \cdots n_r)(L) \neq \emptyset$ iff $X_0^\varepsilon(n_1, \cdots n_r)(L) \neq \emptyset$, for $\varepsilon = \pm 1$.*

Proof. By (1.4.1), it is sufficient to show that $X_n^\varepsilon(L) \neq \emptyset \iff (X_0)_n^\varepsilon(L) \neq \emptyset$. This is true by the definition of X_n^ε ([20], p.577). □

Lemma 1.5.4. *Let rdim*$(V) = 2n$ *and* ad_h *be orthogonal with disc*$(h) = 1$, $n_{r-1} < n-1$ *(if* $r > 1$*) and* $n_r = n$. *Let* $X^\varepsilon = X^\varepsilon(n_1, \cdots, n_r)$ *for* $\varepsilon = \pm 1$. *Then* $X^\varepsilon(L) \neq \emptyset$ *iff* A_L *is split and* h_L *is hyperbolic.*

Proof. Let A_L be split and h_L be hyperbolic. Let h_L be Morita equivalent to a quadratic form q. Let X_0^ε be the corresponding projective homogeneous spaces under $SO_{2n}(q)$. We can conclude that $X^\varepsilon(L) \neq \emptyset$, since q has Witt-index n. Furthermore, $\mathrm{ind}(A_L)|\,\mathrm{gcd}(n_1, \cdots n_r)$ trivially. By (1.4.1), $X_0^\varepsilon(L) \neq \emptyset$ and so $X^\varepsilon(L) \neq \emptyset$ (1.5.3). Conversely, suppose that $X^\varepsilon \neq \emptyset$. Let $W^\varepsilon \in X^\varepsilon(L)$. Since there is a totally isotropic subspace of reduced dimension n, which coincides with the Witt index of h, h_L must be hyperbolic. By Witt's extension theorem ([6], Ch.4, no.3, th.1), there exists $f \in U(A, \sigma, h)$ such that $f(W^+) = W^-$. This must mean that $f \notin SU(A, \sigma, h)$, since any element $g \in SU(A, \sigma, h)$ preserves the sign of W^ε. By ([15], 2.6, lem.1.a), A_L is split. □

Chapter 2

Hasse Principle

2.1 Introduction

Let K be a complete discrete valuation field with a residue field k of good characteristic. Let F be the function field of a smooth, projective, geometrically integral curve $\mathfrak{X}_0$ over K. Such fields have been referred to as semi-global fields. Let Ω be the set of all places of F. For all $\nu \in \Omega$, let F_ν be the completion of F at ν. Let G be a connected linear algebraic group over F. Let X be a projective homogeneous space over G under F. The Hasse Principle is said to hold for X if

$$\prod X(F_\nu) \neq \emptyset \implies X(F) \neq \emptyset.$$

A fair amount of progress has been made due to the patching techniques of Harbater, Hartmann and Krashen. In ([26]), Reddy and Suresh have shown if A is a central simple F-algebra of degree coprime to $\mathrm{char}(k)$, then the Hasse principle holds for every projective homogeneous space under $\mathrm{PGL}_1(A)$. Furthermore, Harbater and Hartmann have shown that if k is algebraically closed and of characteristic zero, then the Hasse principle holds projective homogeneous spaces under connected rational groups.

Let G be any connected linear algebraic group over F. We say that G is of classical type if every factor of the simply connected cover $\tilde{G}$ of the semi-simplification of $G/\mathrm{Rad}(G)$ is of classical type. Suppose $p \neq 2$. It was proved in ([22]) that a quadratic form q over F of rank at least 3 is isotropic over F if and only if q is isotropic over F_ν for all $\nu \in \Omega$. Let A be a central simple algebra over F with an involution σ of either kind. If σ is of the second kind, then assume that $\mathrm{ind}(A) \leq 2$. Let h be an hermitian form over (A, σ). Then Wu ([32]) proved the Hasse principle holds for projective homogeneous spaced under the unitary groups of (A, σ).

Thus, the Hasse principle holds for classical groups of type B_n, C_n, and D_n. A paper of Parimala and Suresh ([24]) show that it holds for groups of type ${}^1A_n$ and ${}^2A_n$, with some restrictions on the characteristic of k.

We now focus on unitary groups of (A, σ), for a central simple algebra A over F satisfying certain conditions.

Let K be a complete discrete valued ring with residue field k and F the function field of a curve over K. Let Ω be set of discrete valuations of F. Let F_ν be the completion of F at the place ν. Let X be a projective homogeneous variety under a connected linear algebraic group G. Then, the Hasse principle is said to hold for X with respect to Ω if $\prod X(F_\nu) \neq \emptyset$ implies that $X(F) \neq \emptyset$.

Let $A \in {}_2\mathrm{Br}(F)$ be a central simple algebra with an involution σ. Let $F^\sigma = F_0$. Let h be a hermitian form over (A, σ) and $G = SU(A, \sigma, h)$ if σ is an involution of the first kind or $G = U(A, \sigma, h)$ otherwise. By Cayley parametrization, G is a connected linear algebraic group. Suppose that $\mathrm{char}(k) \neq 2$ and $\mathrm{ind}(A) \leq 4$. The aim of this section is to show that the Hasse principle holds for any projective homogeneous space X under G over F_0.

2.2 Division Algebras with an involution of the first kind over two dimensional local fields

Let $(R, \mathfrak{m})$ be a 2-dimensional complete regular ring with maximal ideal (π, δ), field of fractions F and residue field k. Suppose that $\text{char}(k) \neq 2$. Let $D \in {}_2\text{Br}(F)$ be a division algebra over F which is unramified on R except possibly at $\langle \pi \rangle$ or $\langle \delta \rangle$. In this section we show that if $\text{ind}(D) = 4$, then D is a tensor product of two quaternion algebras with some properties. Suppose that for any central simple algebra $A \in {}_2\text{Br}(k)$, $\text{ind}(A) \leq 2$. Then we show that $\text{ind}(D) \leq 8$. Further we show that if $\text{ind}(D) = 8$, then D is isomorphic to a tensor product of three quaternion algebras with some properties.

Lemma 2.2.1. *Let $A \in {}_2Br(k)$ be a central division algebra over k. If $A \otimes_k k(\sqrt{a})$ has index at most 2 for some $a \in k^*$, then $A = (a, c) \otimes (b, d) \in Br(k)$ for some $b, c, d, \in k^*$.*

Proof. If $\deg(A) = 1$ or 2, then it is immediate. Suppose that $\deg(A) \geq 4$. Suppose $A \otimes_k k(\sqrt{a})$ has index at most 2 for some $a \in k^*$. Then $\deg(A) = \text{ind}(A) = 4$. We may identify $K = k(\sqrt{a})$ as a subfield of A. Let A_1 be the commutant of K in A. Then A_1 is a quaternion algebra over K ([30, Theorem 5.4]). Since $A \in {}_2\text{Br}(k)$, A admits an involution ([30, Chapter 8, Theorem 8.4]) and the non-trivial automorphism of K/k can be extended to an involution σ on A ([30, Chapter 8, Theorem 10.1]). Since $\sigma(K) = K$, the restriction of σ to A_1 is an involution of second kind. Thus, by a theorem of Albert ([2, Chapter 10, Theorem 21]), $A_1 = K \otimes Q_1$ for some quaternion algebra. Let Q_2 be the commutant of Q_1 in A. Then $K \subset Q_2$ and by a similar argument as above, $A = Q_2 \otimes Q_1$. Since $K \subset Q_2$, $Q_2 = (a, c)$ for some $c \in k^*$. Since Q_1 is a quaternion algebra, $Q_1 = (b, d)$ for some $b, d \in k^*$. Hence $A = (a, c) \otimes (b, d)$. □

Lemma 2.2.2. *Let $A \in {}_2Br(k)$ be a central division algebra over k. If $A \otimes_k k(\sqrt{a}, \sqrt{b})$*

is a matrix algebra for some $a, b \in k^*$, *then* $A = (a, c) \otimes (b, d) \in Br(k)$ *for some* $c, d, \in k^*$.

Proof. If $\deg(A) = 1$ or 2, then it is immediate. Suppose that $\deg(A) \geq 4$. Suppose $A \otimes_k k(\sqrt{a}, \sqrt{b})$ is a matrix algebra for some $a, b \in k^*$. Then $\deg(A) = \text{ind}(A) = 4$ and $K = k(\sqrt{a}, \sqrt{b})$ is a maximal subfield of A. Again, we may show that the commutant A_1 of K in A is a quaternion algebra and $A_1 = K \otimes Q_1$ for some quaternion algebra Q_1. Then as above we have $A = Q_2 \otimes Q_1$ with $K = k(\sqrt{a}) \subset Q_2$. Since $k(\sqrt{a}, \sqrt{b})$ is a maximal subfield of A, it follows that $k(\sqrt{b}) \subset Q_1$. Hence $Q_2 = (a, c)$ and $Q_1 = (b, d)$ for some $c, d \in k^*$. Thus $A = (a, c) \otimes (b, d)$. □

Lemma 2.2.3. *Let* R *be a complete regular local ring with residue field* k *and field of fractions* F. *Suppose that char*$(k) \neq 2$. *Let* Γ_0 *be an Azumaya algebra over* R *and* $D_0 = \Gamma_0 \otimes_R F \in {}_2Br(F)$. *Let* $u \in R$ *be a unit. If ind*$(D_0 \otimes_F (F(\sqrt{u}))) \leq 2$. *Then there exists* $v, w, t \in R^*$ *such that* $D_0 = (u, v) \otimes (w, t) \in Br(F)$.

Proof. Suppose that $\text{ind}(D_0 \otimes_F (F(\sqrt{u}))) \leq 2$. Let $\mathfrak{D}_0 = \Gamma_0 \otimes_R k$. Since $\text{ind}(D_0 \otimes F(\sqrt{u}) \leq 2$, $\mathfrak{D}_0 \otimes k(\sqrt{\overline{u}}) \leq 2$ ([26, Lemma 1.1]), where $\overline{u}$ is the image of u in k. Thus there exist $b, c, d \in k^*$ such that $\mathfrak{D}_0 = (\overline{u}, c) \otimes (b, d)$ (cf. 2.2.1). Let $v, w, t \in R^*$ be the lifts of $b, c, d \in k^*$. Since R is a complete regular local ring, $\text{Br}(R) \cong \text{Br}(k)$ ([3], 6.5). Hence $D_0 = (u, v) \otimes (w, t) \in Br(F)$. □

Lemma 2.2.4. *Let* R *be a complete regular local ring with residue field* k *and field of fractions* F. *Suppose that char*$(k) \neq 2$. *Let* Γ_0 *be an Azumaya algebra over* R *with* $D_0 = \Gamma_0 \otimes_R F \in {}_2Br(F)$. *Let* $u, v \in R$ *be units. If* $D_0 \otimes_F (F(\sqrt{u}, \sqrt{v}))$ *is a matrix algebra, then there exists* $w, t \in R$ *units such that* $D_0 = (u, w) \otimes (v, t) \in Br(F)$.

Proof. Let $\mathfrak{D}_0 = \Gamma_0 \otimes_R k$. Since $D_0 \otimes_F (F(\sqrt{u}, \sqrt{v}))$ is a matrix algebra, $\mathfrak{D}_0 \otimes k(\sqrt{\overline{u}}, \sqrt{\overline{v}})$ is a matrix algebra. Hence $\mathfrak{D}_0 = (\overline{u}, c) \otimes (\overline{v}, d) \in Br(k)$ (cf. 2.2.2). Let $w, t \in R^*$ be lifts of $c, d \in k^*$. Since R is a complete regular local ring, $D_0 = (u, w) \otimes (v, t) \in \text{Br}(F)$. □

Lemma 2.2.5. *Let R be a two dimensional complete regular local ring with residue field k, maximal ideal (π, δ) and field of fractions F. Suppose that char$(k) \neq 2$. Let Γ_0 be an Azumaya algebra on R and $D_0 = \Gamma_0 \otimes F$. Let $u, v \in R$ units.*
i) If $D = D_0 \otimes (u, \pi) \otimes (v, \delta)$, then ind$(D) =$ ind$(D_0 \otimes F(\sqrt{u}, \sqrt{v}))[F(\sqrt{u}, \sqrt{v}) : F]$.
ii) If $D = D_0 \otimes (u\pi, v\delta)$, then ind$(D) = 2$ ind(D_0).

Proof. Let $\kappa(\pi)$ be the residue field at π. Then $\kappa(\pi)$ is the field of fractions of the complete discrete valuation ring $R/(\pi)$ and the the image $\overline{\delta}$ of δ in $\kappa(\pi)$ is a parameter.

Suppose $D = D_0 \otimes (u, \pi) \otimes (v, \delta)$. Since $D_0 \otimes (v, \delta)$ is unramified at π, by ([26, Lemma 2.1]), we have $\text{ind}(D \otimes F_\pi) = \text{ind}((D_0 \otimes (v, \delta) \otimes F_\pi(\sqrt{u}))[F_\pi(\sqrt{u}] : F_\pi]$. Since $\Gamma_0 \otimes \kappa(\pi)$ is unramified on $R/(\delta)$, by ([26, Lemma 2.1]), we have $\text{ind}(\Gamma_0 \otimes (\overline{v}, \overline{\delta}) \otimes \kappa(\pi)(\sqrt{\overline{u}})) = \text{ind}(\Gamma_0 \otimes \kappa(\pi)(\sqrt{\overline{u}}, \sqrt{\overline{v}}))[\kappa(\pi)(\sqrt{\overline{u}}, \sqrt{\overline{v}}) : \kappa(\pi)(\sqrt{\overline{u}})]$. Since F_π is complete, we have

$$ind((D_0 \otimes (v, \delta) \otimes F_\pi(\sqrt{u})) = ind(D_0 \otimes F_\pi(\sqrt{u}, \sqrt{v}))[F_\pi(\sqrt{u}, \sqrt{v}) : F_\pi(\sqrt{u})].$$

Hence $\text{ind}(D \otimes F_\pi) = \text{ind}((D_0 \otimes F_\pi(\sqrt{u}, \sqrt{v}))[F_\pi(\sqrt{u}, \sqrt{v}] : F_\pi]$. By ([26, Lemma 2.4]), we have $\text{ind}(D) = \text{ind}(D \otimes F_\pi)$, $\text{ind}(D_0 \otimes F_\pi(\sqrt{u}, \sqrt{v})) = \text{ind}((D_0 \otimes F_\pi(\sqrt{u}, \sqrt{v})))$ and $[F_\pi(\sqrt{u}, \sqrt{v}) : F_\pi] = [F(\sqrt{u}, \sqrt{v}) : F]$. Thus $\text{ind}(D) = \text{ind}(D_0 \otimes F(\sqrt{u}, \sqrt{v}))[F(\sqrt{u}, \sqrt{v}) : F]$.

Suppose $D = D_0 \otimes (u\pi, v\delta)$. Then as above, we have $\text{ind}(D) = \text{ind}(D_0 \otimes F(\sqrt{\delta}))[F(\sqrt{\delta}) : F]$. Since D_0 is unramified at δ, we have $\text{ind}(D_0 \otimes F_\delta(\sqrt{\delta})) = \text{ind}(D_0 \otimes F_\delta)$. Once again, by ([26, Lemma 2.4]), we have $\text{ind}(D_0 \otimes F(\sqrt{\delta})) = \text{ind}(D_0 \otimes F_\delta(\sqrt{\delta})) = \text{ind}(D_0 \otimes F_\delta) = \text{ind}(D_0)$. Since $[F(\sqrt{\delta}) : F] = 2$, we have $\text{ind}(D) = 2\ \text{ind}(D_0)$. □

We recall the following.

Lemma 2.2.6. *([32, Lemma 3.6]) Let R be a two dimensional complete regular local ring with residue field k, maximal ideal (π, δ) and field of fractions F. Suppose that* *char(k) $\neq 2$. Let D be quaternion division algebra over F which is unramified on R except possibly at (π) and (δ). Then D is isomorphic to one of the following:*

i) (u, w)

ii) $(u, v\pi)$ *or* $(u, v\delta)$

iii) $(u, v\pi\delta)$

iv) $(u\pi, v\delta)$

for some units $u, v, w, \in R$.

Proposition 2.2.7. *Let R be a two dimensional complete regular local ring with residue field k, maximal ideal (π, δ) and field of fractions F. Suppose that char(k) $\neq 2$. Let $D \in {}_2Br(F)$ be a division algebra over F which is unramified on R except possibly at (π) and (δ). Suppose that ind(D) $= 4$. Then D is isomorphic to one of the following:*

i) $(u, w) \otimes (v, t)$

ii) $(u, w) \otimes (v, t\pi)$ *or* $(u, w) \otimes (v, t\delta)$

iii) $(u, v) \otimes (w, t\pi\delta)$

iv) $(u, w\pi) \otimes (v, t\delta)$

v) $(u, v) \otimes (w\pi, t\delta)$

for some units $u, v, w, t \in R$.

Proof. Suppose that D is unramified on R. Let Γ be an Azumaya algebra on R with $\Gamma \otimes F \simeq D$ ([3, Theorem 7.4]). Since $\mathrm{ind}(D) = 4$, we have $\mathrm{ind}(\Gamma \otimes k) = 4$. Hence $\Gamma \otimes k = (a, b) \otimes (c, d)$ for some $a, b, c, d \in k^*$. Let $u, w, v, t \in R$ be lifts of a, b, c, d. Since R is complete, we have $\Gamma = (u, w) \otimes (v, t) \in \mathrm{Br}(R)$ and hence $D = (u, w) \otimes (v, t) \in \mathrm{Br}(F)$. Since $\deg(D) = 4$, $D \simeq (u, w) \otimes (v, t)$.

Suppose that D is ramified only at $\langle\pi\rangle$. Then, by Saltman's classification ([32], Proposition 3.5), we have $D = D_0 \otimes (u, \pi)$, where D_0 is unramified on R and $u \in R$ a unit which is not a square. Since $D = D_0 \otimes (u, \pi) \otimes (1, \delta)$, by (2.2.5(i)), we have $\mathrm{ind}(D) = 2\ \mathrm{ind}(D_0 \otimes F(\sqrt{u}))$. Since $\mathrm{ind}(D) = 4$, we have $\mathrm{ind}(D_0 \otimes F(\sqrt{u})) = 2$.

Hence, by (2.2.3), we have $D_0 = (u, w) \otimes (v, t)$ for some $u, v, w, t \in R$ units. We have $D = D_0 \otimes (u, \pi) = (u, w) \otimes (v, t) \otimes (u, \pi) = (u, w\pi) \otimes (v, t)$. Since $\text{ind}(D) = 4$, $D \simeq (u, w\pi) \otimes (v, t)$. Similarly if D is ramified only at $\langle \delta \rangle$, then $D \simeq (u, w\delta) \otimes (v, t)$.

Suppose that D is ramified both at $\langle \pi \rangle$ and $\langle \delta \rangle$. Then by ([29, Theorem 2.1] & [28, Theorem 1.2]), we have $D = D_0 \otimes (u, \pi) \otimes (v, \delta)$ or $D = D_0 \otimes (w\pi, t\delta)$ for some $u, v \in R - R^2$ units, $w, t \in R$ units and D_0 unramified on R.

Suppose $D = D_0 \otimes (u, \pi) \otimes (v, \delta)$. Suppose that $uv \in R$ is a square. Then $D = D_0 \otimes (u, \pi\delta)$. By (2.2.5(i)), we have $\text{ind}(D) = 2\ \text{ind}(D_0 \otimes F(\sqrt{u}))$ and hence $\text{ind}(D_0 \otimes F(\sqrt{u})) = 2$. Thus, by (2.2.3), we have $D_0 = (u, w) \otimes (v, t)$ for some $u, v, w, t \in R$ units. In particular $D \simeq (v, t) \otimes (u, w\pi\delta)$.

Suppose that uv not a square in R. By (2.2.5(i)), we have $\text{ind}(D) = 4\ \text{ind}(D_0 \otimes F(\sqrt{u}))$ and hence $\text{ind}(D_0 \otimes F(\sqrt{u})) = 1$. Hence $D_0 \otimes F(\sqrt{u}, \sqrt{v})$ is a matrix algebra. Thus, by (2.2.4), we have $D_0 = (u, w) \otimes (v, t)$ for some units $w, t \in R$. In particular $D \simeq (u, w\pi) \otimes (v, t\delta)$.

Suppose $D = D_0 + (w\pi, t\delta)$. By (2.2.5(ii)), we have $\text{ind}(D) = 2\ \text{ind}(D_0)$ and hence $\text{ind}(D_0) = 2$. Thus $D_0 = (u, v)$ and $D \simeq (u, v) \otimes (w\pi, t\delta)$. □

Proposition 2.2.8. *Let R be a two dimensional complete regular local ring with residue field k, maximal ideal (π, δ) and field of fractions F. Suppose that char$(k) \neq 2$ and every central simple algebra in ${}_2Br(k)$ has index at most 2. Let $D \in {}_2Br(F)$ be a division algebra over F which is unramified on R except possibly at (π) and (δ). Then ind$(D) \leq 8$. Further if ind$(D) = 8$, then $D \simeq (w, t) \otimes (u, \pi) \otimes (v, \delta)$ for some units $u, v, w, t \in R$.*

Proof. Suppose that D is unramified on R. Let Γ be an Azumaya algebra on R with $\Gamma \otimes F \simeq D$ ([3, Theorem 7.4]). By the assumption on k, $\text{ind}(\Gamma \otimes k) \leq 2$. Since R is complete, we have $\text{ind}(D) \leq 2$ ([3, Theorem 6.5]).

Suppose that D is ramified only at $\langle \pi \rangle$. Then, by Saltman's classification ([32], Proposition 3.5), we have $D = D_0 \otimes (u, \pi)$, where D_0 is unramified on R and $u \in R$

a unit which is not a square. Since D_0 is unramified on R, by the assumption on k, $\operatorname{ind}(D_0) \leq 2$ and hence $\operatorname{ind}(D) \leq 4$. Similarly if D is ramified only at $\langle \delta \rangle$, then $\operatorname{ind}(D) \leq 4$.

Suppose that D is ramified both at $\langle \pi \rangle$ and $\langle \delta \rangle$. Then by ([29, Theorem 2.1] & [28, Theorem 1.2]), we have $D = D_0 \otimes (u, \pi) \otimes (v, \delta)$ or $D = D_0 \otimes (w\pi, t\delta)$ for some $u, v \in R - R^2$ units, $w, t \in R$ units and D_0 unramified on R. Since D_0 is unramified on R, by the assumption on k, $\operatorname{ind}(D_0) \leq 2$. In particular $\operatorname{ind}(D) \leq 8$.

Suppose $\operatorname{ind}(D) = 8$. Then $D = D_0 \otimes (u, \pi) \otimes (v, \delta)$ for some units $u, v \in R$ units and D_0 unramified on R. Once again by the assumption on k, we have $D_0 = (w, t)$ for some units $w, t \in R$. Hence $D \simeq (w, t) \otimes (u, \pi) \otimes (v, \delta)$. □

2.3 2-torsion division algebras with an involution of the second kind over two dimensional local fields

Let R_0 be a 2-dimensional complete regular local ring with maximal ideal $\mathfrak{m}_0 = (\pi_0, \delta_0)$ and residue field k_0. Suppose that $\operatorname{char}(k_0) \neq 2$. Let F_0 be the field of fractions of R_0 and let $F = F_0(\sqrt{\lambda})$ be an extension of degree 2, with λ a unit in R_0 or $\lambda = w\pi_0$ for some unit $w \in R_0$. Let $D \in {}_2\mathrm{Br}(F)$ be a division algebra with F/F_0-involution σ. In this section we show that if $\operatorname{ind}(D) = 4$, then D is a tensor product of two quaternion algebras with some properties. Suppose that for any central simple algebra $A \in {}_2\mathrm{Br}(k)$, $\operatorname{ind}(A) \leq 2$. Then we show that $\operatorname{ind}(D) \leq 8$. Further we show that if $\operatorname{ind}(D) = 8$, then D is isomorphic to a tensor product of three quaternion algebras with some properties.

Let R be the integral closure of R_0 in F. By the assumption on λ, R is a 2-dimensional regular local ring with maximal ideal $\mathfrak{m} = (\pi, \delta)$ ([23, Theorem 3.1, 3.2]), where; if λ is a unit in R_0, then $\pi = \pi_0$ and $\delta = \delta_0$ and if $\lambda = w\pi_0$, then $\pi = \sqrt{\lambda}$ and $\delta = \delta_0$.

Proposition 2.3.1. *Let R and F be as above. Let $D \in {}_2Br(F)$ with an F/F_0 involution. Suppose that D is unramified on R except possibly at (π) and (δ). If $\lambda = w\pi_0$ for some unit $w \in R_0$, then $D = (D_0 \otimes (v_0, \delta_0)) \otimes_{F_0} F$ for some $D_0 \in {}_2Br(F_0)$ which is unramified on R_0 and $v_0 \in R_0$ a unit.*

Proof. Since F/F_0 is ramified at π_0, by ([24, Lemma 6.3]), D is unramified at π. Hence, by([32, Proposition 3.5]), $D = D' \otimes (v, \delta)$ for some unit $v \in R$ and D' unramified on R. Let Γ' be an Azumaya R-algebra with $\Gamma' \otimes F \simeq D'$. Since $R/\mathfrak{m} \simeq R_0/\mathfrak{m}_0$ and R is complete, there exists an Azumaya R_0-algebra Γ_0 with $\Gamma' \simeq \Gamma_0 \otimes R$. Let $D_0 = \Gamma_0 \otimes F_0$. Since $R/\mathfrak{m} \simeq R_0/\mathfrak{m}_0$, we have $v = v_0 v_1^2$ for some $v_1 \in R_0$ a unit. Since $\delta = \delta_0$, we have $D = (D_0 \otimes (v_0, \delta_0)) \otimes F$ as required. □

For the rest of the section, we assume that $\lambda \in R_0$ is a unit. In particular $\pi = \pi_0$ and $\delta = \delta_0$ and F/F_0 is unramified on R_0. Let τ denote the non trivial automorphism of F/F_0.

Proposition 2.3.2. *Let Γ' be an Azumaya R-algebra and $D' = \Gamma_0 \otimes F$. Let $D = D' \otimes (u, \pi) \otimes (v, \delta)$ or $D_0 \otimes (u\pi, v\delta)$. If D has a F/F_0-involution, then D', (u, π) and (v, δ) or $(u\pi, v\delta)$ have F/F_0-involution.*

Proof. Suppose $D = D' \otimes (u, \pi) \otimes (v, \delta)$ has a F/F_0-involution. Since $\mathrm{cores}_{F/F_0}(D) = 0$, by ([24, Lemma 6.4]), $\mathrm{cores}_{F_\delta/F_{0\delta_0}}(D' \otimes (u, \pi)) = 0$ and $\mathrm{cores}_{F_\delta/F_{0\delta_0}}(v, \delta) = 0$. Since D' is unramified on R, $\mathrm{cores}_{F/F_0}(D')$ is unramified on R_0. Since $\pi = \pi_0 \in R_0$ and $\delta = \delta_0 \in R_0$, $\mathrm{cores}_{F/F_0}(u, \pi) = (N_{F/F_0}(u), \pi) \otimes F_0$ and $\mathrm{cores}_{F/F_0}(v, \delta) = (N_{F/F_0}(v), \delta_0) \otimes F_0$. In particular $\mathrm{cores}_{F/F_0}(D' \otimes (u, \pi))$ and $\mathrm{cores}_{F/F_0}(v, \delta)$ are unramified on R_0 except possibly at (π_0) and (δ_0). Hence, by ([26, Proposition 2.4]), $\mathrm{cores}_{F/F_0}(D' \otimes (u, \pi)) = 0$ and $\mathrm{cores}_{F/F_0}(v, \delta) = 0$. The same argument implies that $\mathrm{cores}_{F/F_0}(D') = 0$ and $\mathrm{cores}_{F/F_0}(u, \pi) = 0$. Hence D_0, (u, π) and (v, δ) have F/F_0-involutions.

The case $D = D_0 \otimes (u\pi, v\delta)$ is similar. $\square$

Lemma 2.3.3. *Let $D_1 = (u, \pi)$ (resp. $(u\pi, v\delta)$, (u, v)) for some units $u, v \in R$. If D_1 has a F/F_0 involution, then $D_1 = (u_0, \pi)$ (resp. $(u_0\pi, v_0\delta)$, (u_0, v_0)) for some $u_0, v_0 \in R_0$ units.*

Proof. Suppose $D_1 = (u, \pi)$ for some $u \in R$ unit. Since $\pi = \pi_0 \in R_0$, we have $\mathrm{cores}_{F/F_0}(u, \pi) = (N_{F/F_0}(u), \pi)$. Since D_1 has a F/F_0-involution, we have $(N_{F/F_0}(u), \pi) = 0 \in Br(F_0)$. Since the residue of $(N_{F/F_0}(u), \pi)$ at π_0 is the image of $N_{F/F_0}(u)$ in $\kappa(\pi_0)^*/\kappa(\pi_0)^*$, the image of $N_{F/F_0}(u)$ is a square in $\kappa(\pi_0)$. Since R_0 is a complete local ring with π_0 a regular prime, $N_{F/F_0}(u)$ is a square in R_0. Hence, replacing u be a square times u, we assume that $N_{F/F_0}(u) = 1$. Thus $u = \theta\tau(\theta)^{-1}$ for some $\theta \in R$. We have $u\tau(\theta)^2 = \theta\tau(\theta) = u_0 \in R_0$ and $(u, \pi) = (u_0, \pi)$.

Suppose $D_1 = (u\pi, v\delta)$. As above, by taking the residues at π and δ, we see that $N_{F/F_0}(u)$ and $N_{F/F_0}(v)$ are squares. Hence as above, we can replace u and v by u_0 and v_0 for some $u_0, v_0 \in R_0$ units.

Suppose that $D_1 = (u, v)$ for some $u, v \in R$. Since D_1 has an F/F_0-involution and D_1 is unramified on R, $D_1 = D_0 \otimes F$ for some quaternion algebra D_0 over F_0 which is unramified on R_0 ([3, Theorem 7.4]). In particular $D_0 = (u_0, v_0)$ for some units $u_0, v_0 \in R_0$ $\square$

Corollary 2.3.4. *Let $D \in {}_2Br(F)$ with an F/F_0-involution. Suppose that D is unramified on R except possibly at (π) and (δ) and $ind(D) = 2$. Then one of the following holds*

i) D is unramified on R

ii) $D \simeq (u_0, u_1\pi_0)$ or $D \simeq (v_0, v_1\delta_0)$

iii) $D \simeq (u_0, u_1\pi_0\delta_0)$

iv) $D \simeq (u_0\pi_0, v_0\delta_0)$

for some units $w_i, u_i, v_i \in R_0$

Proof. Follows from (2.2.6), (2.3.2) and (2.3.3). □

Corollary 2.3.5. *Let $D \in {}_2Br(F)$ with an F/F_0-involution. Suppose that D is unramified on R except possibly at (π) and (δ) and ind$(D) = 4$. Then one of the following holds*

i) D is unramified on R

ii) $D \simeq (w_0, w_1) \otimes (u_0, u_1\pi_0)$ or $D \simeq (w_0, w_1) \otimes (v_0, v_1\delta_0)$

iii) $D \simeq (w_0, w_1) \otimes (u_0, u_1\pi_0\delta_0)$

iv) $D \simeq (u_0, u_1\pi_0) \otimes (v_0, v_1\delta_0)$

v) $D \simeq (w_0, w_1) \otimes (u_0\pi_0, v_0\delta_0)$

for some units $w_i, u_i, v_i \in R_0$

Proof. Follows from (2.2.7), (2.3.2) and (2.3.3). □

Corollary 2.3.6. *Let $D \in {}_2Br(F)$ with an F/F_0-involution. Suppose that D is unramified on R except possibly at (π) and (δ) and every element of ${}_2Br(k)$ has index at most 2. If ind$(D) = 8$, then Then $D \simeq (w_0, w_1) \otimes (u_0, \pi_0) \otimes (v_0, \delta_0)$ for some units $w_0, w_1, u_0, v_0 \in R_0$.*

Proof. By the assumptions on k and D, by (2.2.8), $D \simeq (w_0, w_1) \otimes (u_0, \pi) \otimes (v_0, \delta)$ for some units $w_0, w_1, u_0, v_0 \in R$. Since D has a F/F_0-involution, by (2.3.2), (w_0, w_1), (u_0, π) and (v_0, δ) have F/F_0-involutions. As in the proof of (2.3.5), we can assume $w_0, w_1, u_0, v_0 \in R_0$. □

2.4 Maximal Orders

Definition. Let R be a Noetherian integral domain with field of fractions K. Let A be a finite dimensional associative algebra over K. A subring Γ of A is called an R-order in A if Γ is finitely generated as an R-submodule and $K\Gamma = A$.

Let (K,ν) be a discrete valued field with valuation ring R_ν and residue field $k(\nu)$. Let K_ν be the completion of K at ν. Let D be a finite-dimensional central division algebra over K with an involution σ. If $D\otimes K_\nu$ is a division algebra, then the valuation ν extends to a unique valuation w on D such that $w(\sigma(x)) = w(x)$ for all $X \in D$.

Lemma 2.4.1. *Suppose that R and (K,ν) are as above. Suppose that $D \otimes K_\nu$ is division. Then there exists a unique maximal R_ν-order Γ in D. Furthermore, Γ is identical to the following sets;*

1. *the valuation ring $R_w = \{x \in D | w(x) \geq 0\}$,*

2. *$N = \{x \in D | Nrd_{D/K}(x) \in R_\nu\}$,*

3. *the integral closure S of R_ν in D.*

Proof. [32] □

Let R be a complete regular local ring with residue field k, (π,δ) maximal ideal and field of fractions F. Suppose that $\mathrm{char}(k) \neq 2$. Let $D \in {}_2\mathrm{Br}(F)$ be a division algebra which is unramified on R except possibly at (π) and (δ). By (([32], Proposition 3.5)), we know that $D = D_0 \otimes D_1$ for some $D_0 \in {}_2\mathrm{Br}(F)$ which is unramified on R and D_1 is $(u, v\pi)$ or $(u, v\delta)$ or $(u, w\pi) \otimes (v, t\delta)$ or $(u, v\pi\delta)$ or $(u\pi, v\delta)$ for some units $u, v \in R$. If $D \simeq D_0 \otimes D_1$, then in this section we show that there is a maximal R-order with some properties.

For an integral domain R and $a, b \in R$ non zero elements, let $R(a,b)$ be the R-algebra generated by i, j with $i^2 = a$, $j^2 = b$ and $ij = -ji$. Suppose that $2 \in R$ is a unit. Then $R(a,b)$ is a R-order in the quaternion algebra (a,b) over F. Further note that if $a, b \in R$ are units, then $R(a,b)$ is an Azumaya R-algebra.

We would like to find suitable maximal orders for division algebras of the form $D_0 \otimes D_1$ where D_0 is unramified over R and D_1 is given by;

- $D_1 = (u, \pi)$
- $D_1 = (v, \delta)$
- $D_1 = (u, \pi\delta)$
- $D_1 = (\pi, \delta)$

The following propositions construct a suitable maximal R_ν-order for the case $D_1 = (u, \pi)$, but the same proof may be used for all the cases listed above.

Proposition 2.4.2. *Let R be a complete discrete valuation ring with residue field k and field of fractions F. Suppose that char$(k) \neq 2$. Let Γ_0 be an Azumaya algebra over R and $D_0 = \Gamma_0 \otimes_R K$. Let $u \in R$ be a unit and $\pi \in R$ a parameter. If $D = D_0 \otimes (u, \pi)$ is a division algebra, then $\Gamma = \Gamma_0 \otimes_R R(u, \pi)$ is the maximal R-order of D.*

Proof. Suppose that D is division. Let $d = \deg(D)$ and $d_0 = \deg(D_0)$. Then $d = 2d_0$.

There is a discrete valuation on D given by $\nu_D(z) = \nu(Nrd_D(z))$ ([27, 139]). Furthermore $\Gamma' = \{z \in D^* \mid \nu_D(z) \geq 0\} \cup \{0\} = \{z \in D \mid z \text{ is integral over } R\}$ is the unique maximal R-order of D ([27, Theorem 12.8]).

Since Γ_0 and $R(u, \pi)$ are finitely generated R-modules, Γ is a finitely generated R-module and hence every element of Γ is integral over R. Hence $\Gamma \subseteq \Gamma'$. We now show that $\Gamma' \subseteq \Gamma$.

Let $i, j \in (u, \pi)$ be the standard generators with $i^2 = u$, $j^2 = \pi$ and $ij = -ji$. Let $D_1 = D_0 \otimes F(i) \subset D$ and $\Gamma_1 = \Gamma_0 \otimes R[i]$. Then $D = D_1 + D_1 j$ and $\Gamma = \Gamma_1 \oplus \Gamma_1 j$. Since D_0 is unramified, D_1 is unramified. Since D is a division algebra, D_1 is a division algebra. Hence Γ_1 is the maximal $R[i]$-order in D_1. Since $F(i)/F$ is an unramified extension and D_0 is unramified on R, π is a parameter in D_1. Therefore, $\nu_D(z)$ is a multiple of $2d_0$ for all $z \in D_1$ ([27, 139]). Since $j^2 = \pi$ and $Nrd_D(j) = \pi^{d_0}$, we have $\nu_D(j) = d_0$.

Let $z \in \Gamma'$. Then $z = z_1 + z_2 j$ for some $z_1, z_2 \in D_1$. Suppose that $\nu_D(z_1) = \nu_D(z_2 j)$. Then $\nu_D(z_1) - \nu_D(z_2) = \nu_D(j) = d_0$. This is a contradiction, since $\nu_D(z_1)$ and $\nu_D(z_2)$ are multiple of $2d_0$. Hence $\nu_D(z_1) \neq \nu_D(z_2 j)$. Then $\nu_D(z) = \min\{\nu_D(z_1), \nu_D(z_2 j)\} \geq 0$. In particular $\nu_D(z_1) \geq 0$ and hence $z_1 \in \Gamma_1$. Since $\nu_D(z_2 j) = \nu_D(z_2) + \nu_D(j) = \nu_D(z_2) + d_0$, we have $\nu_D(z_2) \geq -d_0$. Since $\nu_D(z_2)$ is a multiple of $2d_0$, it follows that $\nu_D(z_2) \geq 0$ and hence $z_2 \in \Gamma_1$. Thus $z \in \Gamma$. □

Proposition 2.4.3. *Let R be a discrete valuation ring with residue field k, field of fractions F and $\hat{F}$ the completion of F. Suppose that char$(k) \neq 2$. Let Γ_0 be an Azumaya algebra over R and $D_0 = \Gamma_0 \otimes_R F$. Let $u \in R$ be a unit and $\pi \in R$ a parameter. If $(D_0 \otimes_F (u, \pi)) \otimes_F \hat{F}$ is a divison algebra, then $\Gamma_0 \otimes_R R(u, \pi)$ is the maximal R-order of $D_0 \otimes_F (u, \pi)$.*

Proof. Let $\hat{R}$ be the completion of R. Let $\hat{\Gamma}_0 = \Gamma_0 \otimes \hat{R}$. Then $\hat{\Gamma}_0$ is an Azumaya algebra over $\hat{R}$. Since $(D_0 \otimes_F (u, \pi)) \otimes_F \hat{F}$ is a divison algebra, by (2.4.2), $\hat{\Gamma}_0 \otimes \hat{R}(u, \pi)$ is a maximal $\hat{R}$-order of $(D_0 \otimes_F (u, \pi)) \otimes_F \hat{F}$. Thus, by ([27], Theorem 11.5), $\Gamma_0 \otimes_R R(u, \pi)$ is the maximal R-order of $D_0 \otimes_F (u, \pi)$. □

Corollary 2.4.4. *Let R be a two dimensional complete regular local ring with residue field k, field of fractions F and maximal ideal $\mathfrak{m} = (\pi, \delta)$. For units $u, v \in R$, let D_1 and Γ_1 be one of the following:*

i) $D_1 = (u, v)$, $\Gamma_1 = R(u, v)$

ii) $D_1 = (u, \pi)$, $\Gamma_1 = R(u, \pi)$

iii) $D_1 = (\pi, \delta)$, $\Gamma_1 = R(\pi, \delta)$

iv) $D_1 = (u, \pi\delta)$, $\Gamma_1 = R(u, \pi\delta)$

v) $D_1 = (u, \pi) \otimes (v, \delta)$, $\Gamma_1 = R(u, \pi) \otimes R(v, \delta)$.

Let Γ_0 be an Azumaya algebra over R and $D_0 = \Gamma_0 \otimes_R F$. If $D_0 \otimes_F D_1$ is a division algebra, then $\Gamma = \Gamma_0 \otimes_R \Gamma_1$ is a maximal R-order of $D_0 \otimes_F D_1$.

Proof. An order of a Noetherian integrally closed domain is maximal if and only if it is reflexive and its localization at all height one prime ideals are maximal orders ([27], Theorem 11.4). Since Γ is a finitely generated free module, it is reflexive. Furthermore, R is a regular local ring, hence it is Noetherian and integrally closed. We only need to show that Γ_P is a maximal R_P-order for all height one prime ideals P.

Suppose that $D = D_0 \otimes_F D_1$ is a division algebra. Let P be a height one prime ideal of R. Suppose $P \neq \langle\pi\rangle, \langle\delta\rangle$. Since $u, v \in R$ are units, u, v, π, δ are units in R_P and hence $\Gamma_1 \otimes R_P$ is an Azumaya R_P-algebra. In particular $\Gamma \otimes R_P$ is an Azumaya R_P-algebra. Hence Γ_P is a maximal R_P-order of D. Suppose that $P = \langle\pi\rangle, \langle\delta\rangle$.

i) Since $u, v \in R$ are units, $(\Gamma_1)_P$ is an Azumaya algebra over R_P. Hence Γ_P is a maximal R_P-order on D.

ii) If $P \neq \langle\pi\rangle)$, then Γ_1 is an Azumaya R_P-algebra and hence Γ_P is a maximal R_P-order on D. If $P = \langle\pi\rangle$, then Γ_P is a maximal R_P-order on D by (2.4.3).

iii), iv) If $P = \langle\pi\rangle$ or $P = \langle\delta\rangle$, then Γ_P is a maximal R_P-order on D by (2.4.3).

v) Suppose $P = \langle\pi\rangle$. Let $\Gamma_0' = \Gamma_0 \otimes R_P(v, \delta)$. Since v, δ are units in R_P, $R_P(v, \delta)$ is an Azumaya R_P-algebra. Since $D = (D_0 \otimes (v, \delta)) \otimes (u, \pi)$ and $\Gamma = (\Gamma_0 \otimes R_P(v, \delta)) \otimes R_P(u, \pi)$, by (2.4.3), Γ_P is a maximal R_P-order on D. If $P = \langle\delta\rangle$, a similar argument holds.

□

2.5 A local global principle for hermitian forms over two dimensional local fields

Let R_0 be a 2-dimensional complete regular local ring with maximal ideal $\mathfrak{m}_0 = (\pi_0, \delta_0)$ and residue field k_0. Suppose that $\mathrm{char}(k_0) \neq 2$. Let F_0 be the field of fractions of

R_0 and let $F = F_0(\sqrt{\lambda})$ be an extension of degree at most 2, with λ a unit in R_0 or a unit times π_0. Let $D \in {}_2\mathrm{Br}(F)$ be a division algebra with F/F_0-involution σ and h an hermitian form over (D, σ). In this section, under some assumptions on D, F_0 and h, we prove that if $h \otimes F_\pi$ or $h \otimes F_\delta$ is isotropic, then h is isotropic.

Let R be the integral closure of R_0 in F. By the assumption on λ, R is a 2-dimensional regular local ring with maximal ideal (π, δ) ([23, 3.1, 3.2]), where; if λ is a unit in R_0, then $\pi = \pi_0$ and $\delta = \delta_0$ and if λ is a unit times π_0, then $\pi = \sqrt{\lambda}$ and $\delta = \delta_0$.

Let $G(D, \sigma, h) = SU(D, \sigma, h)$ if $F = F_0$ and $G(D, \sigma, h) = U(A, \sigma, h)$ if $[F : F_0] = 2$.

We begin with the following, which is proved by Wu ([32, Corollary 3.12]) for D a quaternion algebra.

Proposition 2.5.1. *Let F_0 and F be as above. Let $D \in {}_2Br(F)$ and σ an F/F_0-involution. Suppose that D is a division algebra which unramified on R expect possibly at (π) and (δ). Let d =deg(D), e_0 the ramification index of D at π and e_1 the ramification index of D at δ. Suppose that there exists a maximal R-order Γ of D and $\pi_D, \delta_D \in \Gamma$ such that $\sigma(\pi_D) = \pm\pi_D$, $\sigma(\delta_D) = \pm\pi_D$ and $\pi_D\delta_D = \pm\delta_D\pi_D$ and Nrd$(\pi_D) = \theta_0\pi^{\frac{d}{e_0}}$ and Nrd$(\delta_D) = \theta_1\delta^{\frac{d}{e_1}}$ for some units $\theta_0, \theta_1 \in R$. Let $h = \langle a_1, \cdots, a_n\rangle$ be an hermitian form over (D, σ). Suppose that for $1 \leq i \leq n$, $a_i \in \Gamma$ and Nrd(a_i) is a product of a unit in R, a power of π and a power of δ. If $h \otimes F_\pi$ or $h \otimes F_\delta$ is isotropic, then h is isotropic.*

Proof. Follows from ([32, Corollary 3.3]). ∎

As a consequence we have the following (cf. [32, Corollary 3.12])

Proposition 2.5.2. *Let F_0 and F be as above. Let $D \in {}_2Br(F)$ and σ an F/F_0-involution. Suppose that D is a division algebra which is unramified on R expect possibly at (π) and (δ). Let d =deg(D), e_0 the ramification index of D at π and e_1 the ramification index of D at δ. Suppose that there exists a maximal R-order Γ of D and $\pi_D, \delta_D \in \Gamma$ such that $\sigma(\pi_D) = \pm\pi_D$, $\sigma(\delta_D) = \pm\pi_D$, $\pi_D\delta_D = \pm\delta_D\pi_D$, Nrd$(\pi_D) = \theta_0\pi^{\frac{d}{e_0}}$ and Nrd$(\delta_D) = \theta_1\delta^{\frac{d}{e_1}}$ for some units $\theta_0, \theta_1 \in R$. Let $h = \langle a_1, \cdots, a_n\rangle$ be an hermitian form over (D, σ). Suppose that for $1 \le i \le n$, $a_i \in \Gamma$ and Nrd(a_i) is a product of a unit in R, a power of π and a power of δ. Let X be a projective homogeneous space under $G(D, \sigma, h)$. If $X(F_{0\pi}) \neq \emptyset$ or $X(F_{0\delta}) \neq \emptyset$, then $X(F_0) \neq \emptyset$.*

Proof. First note that $\operatorname{ind}(D) = \operatorname{ind}(D \otimes F_\pi)$ is proved in ([26, 2.4]) only under the assumption that F conatins a primitive d^{th} root of unity. However that proof uses only the assumption that F contains a primitive r^{rh} root of unity for $r = \operatorname{per}(D)$ in $\operatorname{Br}(F)$. Since the period of D divides 2 by our assumption on D, we have $\operatorname{ind}(D) = \operatorname{ind}(D \otimes F_\pi)$. Suppose that $X(F_{0\pi}) \neq \emptyset$ or $X(F_{0\delta}) \neq \emptyset$. Using, (2.5.1), the rest of the proof of ([32, Corollary 3.12]) can be applied here to show that $X(F_0) \neq \emptyset$. □

We fix the following.

Notation 2.5.3. *Let R_0 be a 2-dimensional complete regular local ring with maximal ideal $\mathfrak{m}_0 = (\pi_0, \delta_0)$ and residue field k_0. Suppose that char$(k_0) \neq 2$. Let F_0 be the field of fractions of R_0 and let $F = F_0(\sqrt{\lambda})$ be an extension of degree at most 2, with λ a unit in R_0 or a unit times π_0. Let R be the integral closure of R_0 in F. By the assumption on λ, R is a 2-dimensional regular local ring with maximal ideal (π, δ) ([23, 3.1, 3.2]), where; if λ is a unit in R_0, then $\pi = \pi_0$ and $\delta = \delta_0$ and if λ is a unit times π_0, then $\pi = \sqrt{\lambda}$ and $\delta = \delta_0$. Let $u_i, v_i \in R_0$ be units and D_1, Γ_1 and σ_1 denote one of the following:*

i) $D_1 = F_0$, $\Gamma_1 = R$, $\sigma_1 = id$

ii) $D_1 = (u_0, u_1\pi_0)$, $\Gamma_1 = R(u_0, u_1\pi_0)$, σ_1 *the canonical involution* *iii)* $D_1 = (u_0\pi_0, v_0\delta_0)$, $\Gamma_1 = R(u_0\pi_0, v_0\delta_0)$, σ_1 *the canonical involution*

iv) $D_1 = (u_0, u_1\pi_0) \otimes (v_0, v_1\delta_0)$, $\Gamma_1 = R(u_0, u_1\pi_0) \otimes R(v_0, v_1\delta_0)$, σ_1 *the tensor product of the canonical involutions*

Let Γ_0 *be an Azumaya algebra over* R *with an* R/R_0*- involution* $\tilde{\sigma}_1$*. Let* $D_0 \simeq \Gamma_0 \otimes F$ *and* $\sigma_0 = \tilde{\sigma_0} \otimes 1$*. Let* $D \simeq D_0 \otimes D_1$*,* $\Gamma = \Gamma_0 \otimes \Gamma_1$ *and* $\sigma = \sigma_0 \otimes \sigma_1$*. Then* σ *is a* F/F_0*-involution on* D *and* $\sigma(\Gamma) = \Gamma$*. Let* d_i *denote the degree of* D_i*. The following table gives a choice of* $\pi_D, \delta_D \in \Gamma$ *and some of their properties.*

D	π_D	δ_D	Nrd(π_D)	Nrd(δ_D)	$\sigma(\pi_D)$	$\sigma(\delta_D)$	$\sigma(\pi_D\delta_D)$
D_0	π_0	δ_0	$\pi_0^{d_0}$	$\delta_0^{d_0}$	π_D	δ_D	$\pi_D\delta_D$
$D_0 \otimes (u_0, u_1\pi_0)$	$1 \otimes j$	$1 \otimes \delta$	$(u_1\pi_0)^{d_0}$	$\delta_0^{2d_0}$	$-\pi_D$	δ_D	$-\pi D\delta_D$
$D_0 \otimes (u_0\pi_0, v_0\delta_0)$	$1 \otimes i$	$1 \otimes j$	$(u_0\pi_0)^{d_0}$	$(v_0\delta_0)^{d_0}$	$-\pi_D$	$-\delta_D$	$-\pi_D\delta_D$
$D_0 \otimes (u_0, u_1\pi_0)\otimes (v_0, v_1\delta_0)$	$1 \otimes j_1 \otimes 1$	$1 \otimes 1 \otimes j_2$	$(u_1\pi_0)^{2d_0}$	$(v_1\delta_0)^{2d_0}$	$-\pi_D$	$-\delta_D$	$\pi_D\delta_D$

Corollary 2.5.4. *Let* F*,* D*,* σ *and* Γ *be as in (2.5.3). Suppose that* D *is a division algebra. Let* $h = \langle a_1, \cdots, a_n\rangle$ *be a hermitian form over* (D, σ) *with* $a_i \in \Gamma$*. Suppose Nrd(*a_i*) is a unit times a power of* π *and a power of* δ*. Let* X *be a projective homogeneous space under* $G(D, \sigma, h)$ *over* F_0*. If* $X(F_{0\pi_0}) \neq \emptyset$ *or* $X(F_{0\delta}) \neq \emptyset$*, then* $X(F_0) \neq \emptyset$*.*

Proof. By (2.4.4), Γ is a maximal R-order of D. Let e_0 be the ramification index of D at π and e_1 be the ramification index of D at δ. If D_1 as in (2.5.3(i)), then $e_0 = e_1 = 1$. If D_1 is as in (2.5.3(ii)), then $e_0 = 2$ and $e_1 = 1$. If D_1 as in (2.5.3(iii) or (iv)), then $e_0 = e_1 = 2$. Let π_D and δ_D be as in (2.5.3). Then π_D and δ_D satisfy the assumptions of (2.5.2). Hence, by (2.5.2), $X(F_0) \neq \emptyset$. □

2.6 Behavior under blowups

Let R_0, R, F_0, F, $\mathfrak{m}_0 = (\pi_0, \delta_0)$, $\mathfrak{m} = (\pi, \delta)$, A, σ, h and $G(A, \sigma, h)$ be as in (§2.5). Let X be a projective homogeneous variety under $G(A, \sigma, h)$ over F. Suppose that $X(F_\nu) \neq \emptyset$ for all divisorial discrete valuations ν of F. Under some assumptions on D, in this section we prove that there exists a sequence of blowups $\mathscr{Y}$ of Spec(R) such that $X(F_P) \neq \emptyset$ for all closed points P of $\mathscr{Y}$.

Let $\mathscr{X}_0 = Proj(R_0[x, y]/(\pi_0 x - \delta_0 y))$. Let Q_1 and Q_2 be the closed points of $\mathscr{X}_0$ given by the homogeneous ideals (π_0, δ_0, y) and (π_0, δ_0, x). Let τ be the nontrivial automorphism of F/F_0 if $F \neq F_0$ and let τ be the identity if $F = F_0$.

We begin with the following.

Lemma 2.6.1. *Let $a, b \in R_0$ be nonzero and square free. Suppose that the support of a and b is at most π_0 and δ_0 and have no common factors. Then for any closed point $P \in \mathscr{X}_0$, there exist $a', b', \pi', \delta' \in \mathcal{O}_P$ such that the maximal ideal at P is generated by π' and δ', a' and b' are square free, have no common factors, the support is at most (π') or (δ') and $(a, b) \otimes F_{0P} = (a', b')$ and $R_0(a, b) \subset \hat{\mathcal{O}}_P(a', b')$.*

Proof. Suppose a is a unit R_0. Then $b = v_0$ or $v_0\pi_0$ or $v_0\delta_0$ or $v_0\pi_0\delta_0$ for some unit $v_0 \in R_0$. If $b = v_0$ or $v_0\pi_0$ or $v_0\delta_0$, then it is easy to see that $a' = a$ and $b' = b$ have the required properties. Suppose $b = v_0\pi_0\delta_0$. Suppose $P \neq Q_1$, Q_2. Then the maximal ideal at P is given by (π_0, δ') with $\pi_0 = w'\delta_0$ for some unit w' in $\mathcal{O}_P$. We have $(a, b) = (a, v_0\pi_0\delta_0) = (a, v_0 w')$. In this case it is easy to see that $a' = a$ and $b' = v_0 w'$ have the required property. Suppose $P = Q_1$. Then the maximal ideal at P is given by (t, δ_0) with $\pi_0 = t\delta_0$. We have $(a, b) = (a, v_0 t\delta_0)$ and $a' = a$, $b' = v_0 t\delta_0$ have the required properties. The case $P = Q_2$ is similar.

Suppose neither a nor b is a unit in R_0. Then by the assumption on a, b, we have $\{a, b\} = \{u_0\pi_0, v_0\delta_0\}$ for some units $u_0, v_0 \in R_0$. Suppose $P = Q_1$. Since the maximal ideal of $\hat{\mathcal{O}}_{Q_1}$ is given by (t, δ_0) with $\pi_0 = t\delta_0$, we have $(a, b) \otimes F_{0Q_1} =$

$(u_0\pi_0, v_0\delta_0) \otimes F_{0Q_1} = (u_0 t\delta, v_0\delta) \otimes F_{0Q_1} \simeq (-u_0v_0t, v_0\delta_0)$. It is easy to see that $a' = -u_0v_0t$ and $b' = v_0\delta_0$ have the required properties. The case $P = Q_2$ is similar. Suppose $P \neq Q_1, Q_2$. Since the maximal ideal at P is given by (π_0, δ') with $\pi_0 = w'\delta_0$ for some unit w' in $\mathcal{O}_P$ and $(a, b) = (u_0\pi_0, v_0\delta_0) = (u_0\pi_0, v_0w'\pi_0) = (u_0\pi_0, v_0w'u_0)$, $a' = u_0\pi_0$ and $b' = v_0w'u_0$ have the required properties. □

Lemma 2.6.2. *Suppose $D = (u_0, u_1\pi_0) \otimes (v_0, v_1\delta_0)$ is a division algebra for some units $u_i, v_i \in R_0$. Let σ be the tensor product of canonical involutions on $(u_0, u_1\pi_0)$, $(v_0, v_1\delta_0)$. Then for $P = Q_1$ or Q_2, there exist an isomorphism $\phi_P : D \otimes F_{0P} \to (u'_0, u'_1\pi') \otimes (v'_0, v'_1\delta')$ for some u'_i, v'_i units in the local ring $\mathcal{O}_P$ at P and the maximal ideal of $\mathcal{O}_P$ is given by (π', δ') and $\theta_P \in \mathcal{O}_P(u'_0, u'_1\pi') \otimes \mathcal{O}_P(v'_0, v'_1\delta')$ such that $\phi(R_0(u_0, u_1\pi) \otimes R_0(v_0, v_1\delta)) \subset \mathcal{O}_P(u'_0, u'_1\pi') \otimes \mathcal{O}_P(v'_0, v'_1\delta')$ and $int(\theta_P)\sigma' = \phi_P\sigma\phi_P^{-1}$ and the support of $Nrd(\theta_P)$ at most (π') and (δ').*

Proof. Since Q_1 is the closed point given by the homogeneous ideal (π_0, δ_0, y), the maximal ideal of $\mathcal{O}_{Q_1}$ is given by (t, δ_0) with $\pi_0 = t\delta_0$. Thus we have $D = (u_0, u_1\pi_0) \otimes (v_0, v_1\delta_0) = (u_0, u_1t\delta_0) \otimes (v_0, v_1\delta_0) \simeq (u_0, u_1v_1^{-1}t) \otimes (u_0v_0, v_1\delta_0)$.

Let $i_1, j_1 \in (u_0, u_1\pi_0)$, $i_2, j_2 \in (v_0, v_1\delta_0)$, $i_3, j_3 \in (u_0, u_1v_1^{-1}t)$ and $i_4, j_4 \in (u_0v_0, v_1\delta_0)$ be the standard generators. Then we have an isomorphism $\phi_P : (u_0, u_1\pi_0) \otimes (v_0, v_1\delta_0) \to (u_0, u_1v_1^{-1}t) \otimes (u_0v_0, v_1\delta_0)$ given by $\phi(i_1 \otimes 1) = i_3 \otimes 1$, $\phi(j_1 \otimes 1) = j_3 \otimes j_4$, $\phi(1 \otimes i_2) = u_0^{-1}(i_3 \otimes i_4)$ and $\phi(1 \otimes j_2) = 1 \otimes j_4$. Since u_0 is a unit in R_0, $\phi(R_0(u_0, u_1\pi_0) \otimes R_0(v_0, v_1\delta_0)) \subset \mathcal{O}_{Q_i}(u_0, u_1v_1^{-1}t) \otimes \mathcal{O}_{Q_i}(u_0v_0, v_1\delta_0)$. Let $\theta_{Q_1} = i_3 \otimes j_4$. Then it is easy to see that θ_{Q_1} has the required properties. A similar computation gives the required θ_{Q_2}. □

Lemma 2.6.3. *Suppose $D = (u_0, u_1\pi_0) \otimes (v_0, v_1\delta_0)$ is a division algebra for some units $u_i, v_i \in R_0$. Let σ be the tensor product of canonical involutions on $(u_0, u_1\pi_0)$, $(v_0, v_1\delta_0)$. Let $P \in \mathscr{X}_0$ be a closed point not equal to Q_1 or Q_2. Then there exists an isomorphism $\phi_P : D \otimes F_{0P} \simeq (u'_0, u'_1) \otimes (v'_0, v'_1\pi')$ for some u'_0, v'_0, u'_1, v'_1 units in*

$\mathcal{O}_P$ and $m_P = (\pi', \delta')$ such that $\phi_P(R_0(u,\pi) \otimes R_0(v,\delta)) \subset \mathcal{O}_P(u_0', u_1') \otimes \mathcal{O}_P(v_0', v_1'\pi)$. Further if σ' is the tensor product of canonical involutions on (u_0', u_1') and $(v_0', v_1'\pi')$, then there exists $\theta_P \in \mathcal{O}_P(u_0', u_1') \otimes \mathcal{O}_P(v_0', v_1'\pi)$ such that $int(\theta_P)\sigma' = \phi_P \sigma \phi_P^{-1}$ and the support of $Nrd(\theta_P)$ at most (π') and (δ').

Proof. Since P is a closed point not equal to Q_1 or Q_2, the maximal ideal at P is given by (π_0, δ') and $\delta_0 = w'\pi_0$ for some unit w' in $\mathcal{O}_P$. Thus we have $D = (u_0, u_1\pi_0) \otimes (v_0, v_1\delta_0) = (u_0, u_1\pi_0) \otimes (v_0, v_1 w'\pi_0) \simeq (v_0, v_1 w' u_1^{-1}) \otimes (u_0 v_0, u_1\pi_0)$.

Let $i_1, j_1 \in (u_0, u_1\pi)$, $i_2, j_2 \in (v_0, v_1\delta_0)$, $i_3, j_3 \in (v_0, v_1 w' u_1^{-1})$ and $i_4, j_4 \in (u_0 v_0, u_1\pi_0)$ be the standard generators. Then we have an isomorphism $\phi_P : (u_0, u_1\pi_)) \otimes (v_0, v_1\delta_0) \to (v_0, v_1 w' u_1^{-1}) \otimes (u_0 v_0, u_1\pi_0)$ given by $\phi(i_1 \otimes 1) = v_0^{-1}(i_3 \otimes i_4)$, $\phi(j_1 \otimes 1) = 1 \otimes j_4$, $\phi(1 \otimes i_2) = (i_3 \otimes 1)$ and $\phi(1 \otimes j_2) = j_3 \otimes j_4$. Since $v_0 \in R_0$ is a unit, we have $\phi_P(R_0(u_0, u_1\pi) \otimes R_0(v_0, v_1\delta)) \subset \mathcal{O}_P(v_0, v_1 w' u_1^{-1}) \otimes \mathcal{O}_P(u_0 v_0, u_1\pi)$. Let $\theta_P = i_3 \otimes j_4$. Then θ_P has the required properties. □

We record the following theorem from ([13, Proposition 5.8]).

Theorem 2.6.4. *Let T be a complete discrete valuation ring with residue field k and field of fractions K. Let F be the function field of a smooth projective geometrically integral curve over K and $\mathscr{X}$ a model of F with X_0 its closed fibre. Let Y be a variety over F. Suppose that $Y(F_\nu) \neq \emptyset$ for all divisorial discrete valuations ν of F. For every irreducible component C of X_0, there exists a nonempty proper open subset U of C such that $Y(F_U) \neq \emptyset$. In particular there exists a finite subset $\mathscr{P}$ of closed points of X_0 such that $Y(F_P) \neq \emptyset$ for all $P \in X_0 \setminus \mathscr{P}$.*

The following two result are extracted from ([32, §4]).

Lemma 2.6.5. *Let F_0 and F be as above. Let $D \in Br(F)$ be a quaternion division algebra over F with a F/F_0-involution σ. Then there exists a sequence of blowups $\mathscr{X}_0 \to SpecR_0$ such that, the integral closure $\mathscr{X}$ of $\mathscr{X}_0$ in F is regular and $ram_{\mathscr{X}}(D)$ is a union of regular curves with normal crossings. Further for every closed point P*

of $\mathscr{X}_0$ with $D \otimes F_{0P}$ division, $D \otimes F_{0P}$ is as in (2.2.6 or 2.3.4) and not of the type (2.2.6(iii) or 2.3.4(iii)).

Proof. There exists a sequence of blowups $\mathscr{Y}_0 \to Spec(R_0)$ such that the integral closure $\mathscr{Y}$ of $\mathscr{Y}_0$ is regular and $\mathrm{ram}_{\mathscr{Y}}(D)$ is a union of regular curves with normal crossings (cf. [24, Corollary 11.3]). Let P be a closed point of $\mathscr{Y}_0$. Since the integral closure of $\mathscr{Y}_0$ in F is regular, the maximal ideal at P is generated by (π_P, δ_P) and $F = F_0(\sqrt{\lambda_P})$ for some $\lambda_P = u$ or $u\pi_P$ for some unit u at P.

Suppose that $D \otimes F_{0P}$ is division. In particular $F \otimes F_{0P}$ is a field. Let R_P be the integral closure of $\mathcal{O}_P$ in F. Then R_P is a regular two dimensional local ring with maximal ideal (π'_P, δ'_P) with $\delta'_P = \delta_P$, $\pi'_P = \pi_P$ if λ_P is a unit in $\mathcal{O}_P$ and $\pi'_P = \sqrt{\lambda_P}$ if λ_P is not a unit in $\mathcal{O}_P$. Further D is unramified on $\mathcal{O}_P$ except possibly at (π'_P) or (δ'_P). In particular $D \otimes F_{0P}$ is as in (2.2.6 or 2.3.4).

Suppose $D \otimes F_{0P}$ as in (2.2.6(iii) or 2.3.4(iii)). Note that there are only finitely many such closed points.

Suppose $F \otimes F_{0P}/F_{0P}$ is ramified. Then, by (2.3.1), we can assume that $D \otimes F_{0P}$ is not of type (2.3.5(iii)).

Suppose that $F \otimes F_{0P}/F_{0P}$ is unramified. Let $\mathscr{X}_P \to Spec(\mathcal{O}_P)$ be the simple blow up. Then, it is easy to see that for every closed point Q of $\mathscr{X}_P$, $D \otimes F_{0Q}$ is not of type (2.3.5(iii) or 2.3.4(iii)) (cf. [32, Lemma 4.1]). □

Proposition 2.6.6. *Let F_0 and F be as above. Let D be a central simple algebra over F with a F/F_0-involution σ and h an hermitian form over (D, σ). Let X be a projective homogeneous variety under $G(D, \sigma, h)$ over F_0. Suppose that $ind(D) \leq 2$. If that $X(F_{0\nu}) \neq \emptyset$ for all divisorial discrete valuations ν of F_0, then there exists a sequence of blowups $\mathscr{Y} \to Spec(R_0)$ such that for every closed point P of $\mathscr{Y}$, $X(F_{0P}) \neq \emptyset$.*

Proof. By Morita equivalence ([17, Theorem 3.1,3.11,3.20] & [16, Chapter 1, 9.3.5]), we assume that D is division. If $D = F$, then $X(F) \neq \emptyset$ ([32, Corollary 3.12]) and hence any blowup of Spec(R) has the required property.

Suppose ind(D) = 2. Without loss of generality we assume that σ is the canonical involution. Then using (2.6.5), we get a sequence of blowups $\mathscr{Y}_0$ of Spec(R_0) such that for every closed point P of $\mathscr{Y}$ with $D \otimes F_{0P}$ division, $D \otimes F_{0P}$ is as in (2.2.6 or 2.3.4) and not of the type (2.2.6(iii) or 2.3.4(iii)).

Suppose that $D \otimes F_{0P}$ is not division. Then $X(F_{0P}) \neq \emptyset$ ([9, Theorem 3.1]).

Suppose that $D \otimes F_{0P}$ is division. Then $D \otimes F_P = (a_p, b_P)$ for some a_P, b_P as in (2.6.1). In particular $\mathscr{O}_P(a_P, b_P) \otimes R$ is a maximal $\mathscr{O}_P$-order of $D \otimes F_{0P}$. We have $h \otimes F_{0P} = \langle a_{1P}, \cdots, a_{nP} \rangle$ for some $a_{iP} \in \hat{\mathscr{O}}_P(a, b) \otimes R$.

Let Y_0 be the special fibre of $\mathscr{Y}_0$. By (2.6.4), there exists a finite subset $\mathscr{P}$ of Y_0 such that $X(F_P) \neq \emptyset$ for all $P \notin \mathscr{P}_0$. Thus replacing R by $\hat{\mathscr{O}}_P$, we assume that $D = (a, b)$ for some a, b as in (2.6.1) and $h = \langle a_1, \cdots, a_n \rangle$ for some $a_i \in R(a, b)$.

By ([32, Lemma 4.2]), there exists a sequence of blowups $\mathscr{Y}_1 \to Spec(R_0)$ such that the support of Nrd(a_i) is a union of regular curves with normal crossings and for every closed point Q of $\mathscr{Y}_1$ with $D \otimes F_{0Q}$ is division, $D \otimes F_{0Q}$ is not of the form ([32, Lemma 3.6(5)]).

Let Q be a closed point of $\mathscr{Y}_1$. If $D \otimes F_{0Q}$ is a matrix algebra, then by ([12, Corollary 4.7]) and Morita equivalence, $X(F_{0Q}) \neq \emptyset$. Suppose $D \otimes F_{0Q}$ is a division algebra. Then, by (2.6.1), $R(a, b)$ is contained in the corresponding maximal order $\hat{\mathscr{O}}_Q(a_Q, b_Q)$. Since $h = \langle a_1, \cdots, a_n \rangle$ with $a_i \in R(a, b)$ with support of Nrd(a_i) is a union of regular curves with normal crossings, by (2.5.4), $X(F_{0Q}) \neq \emptyset$. □

Lemma 2.6.7. *Let F_0 and F be as above. Let $D \in {}_2Br(F)$ be a central simple algebra over F with a F/F_0-involution σ and h an hermitian form over (D, σ). Suppose that ind(D) = 4. Then there exists a sequence of blowups $\mathscr{X}_0 \to SpecR_0$ such that, the integral closure $\mathscr{X}$ of $\mathscr{X}_0$ in F is regular and $ram_{\mathscr{X}}(D)$ is a union of regular curves with normal crossings. Further for every closed point P of $\mathscr{X}_0$ with ind($D \otimes F_{0P}$) = 4,*

$D \otimes F_{0P}$ is as in (2.2.7 or 2.3.5) and not of the type (2.2.7(iii) or 2.3.5(iii)).

Proof. There exists a sequence of blowups $\mathscr{Y}_0 \to Spec(R_0)$ such that the integral closure $\mathscr{Y}$ of $\mathscr{Y}_0$ is regular and $\mathrm{ram}_{\mathscr{Y}}(D)$ is a union of regular curves with normal crossings (cf. [24, Corollary 11.3]).

Let P be a closed point of $\mathscr{Y}_0$. In particular if $\mathrm{ind}(D \otimes F_{0P}) = 4$, then $D \otimes F_{0P}$ is as in (2.2.7 or 2.3.5). Suppose $D \otimes F_{0P} = (u_P, v_P) \otimes (w_P, w'_P \pi_P \delta_P)$ as in (2.2.7(iii) or 2.3.5(iii)) for some units $u_P, v_P, w_P, w'_P \in \mathcal{O}_P$. Note that there are only finitely many such closed points.

Suppose $F \otimes F_{0P}/F_{0P}$ is ramified. Then, by (2.3.1), we can assume that $D \otimes F_{0P}$ is not of type (2.3.5(iii)).

Suppose that $F \otimes F_{0P}/F_{0P}$ is unramified. Let $\mathscr{X}_P \to Spec(\mathcal{O}_P)$ be the simple blow up. Then, it is easy to see that for every closed point Q of $\mathscr{X}_P$, $D \otimes F_{0Q}$ is not of type (2.2.7(iii) or 2.3.5(iii)). □

Proposition 2.6.8. *Let F_0 and F be as above. Let $D \in {}_2Br(F)$ be a central simple algebra over F with a F/F_0-involution σ and h an hermitian form over (D, σ). Let X be a projective homogeneous variety under $G(D, \sigma, h)$ over F_0. Suppose that $ind(D) \leq 4$. If $X(F_{0\nu}) \neq \emptyset$ for all divisorial discrete valuations ν of F_0, then there exists a sequence of blowups $\mathscr{Y} \to Spec(R_0)$ such that for every closed point P of $\mathscr{Y}$, $X(F_{0P}) \neq \emptyset$.*

Proof. By (2.6.6), we assume that $\mathrm{ind}(D) = 4$. As in the proof of (2.6.6), we assume that D is division as in (2.2.7 or 2.3.5) and not of the type (2.2.7(iii) or 2.3.5(iii)). Let Γ be the maximal R-order of D as in (2.4.4) and write $h = \langle a_1, \cdots, a_n \rangle$ with $a_i \in \Gamma$.

Let $\mathscr{Y}_1 \to Spec(R_0)$ be a sequence of blowups such that the support of $\mathrm{Nrd}(a_i)$ is a union of regular curves with normal crossings. Further replacing $\mathscr{Y}$ by a sequence of blow ups (2.6.7), we assume that for every closed point P of $\mathscr{Y}$, $D \otimes F_{0P}$ is not of the form (2.2.7(iii) or 2.3.5(iii)). Once again we have a finite set of closed points $\mathscr{P}_1$ of $\mathscr{Y}_1$ such that $X(F_{0P}) \neq \emptyset$ for all $P \notin \mathscr{P}$.

Let $P \in \mathscr{P}_1$ be a closed point. If $\mathrm{ind}(D \otimes F_{0P}) \leq 2$, then by (2.6.6), there exists a sequence of blowups $\mathscr{X}_P$ of $\mathrm{Spec}(\mathcal{O}_P)$ such that for every closed point Q of $\mathscr{X}_P$, $X(F_{0Q}) \neq \emptyset$.

Suppose $\mathrm{ind}(D \otimes F_P) = 4$.

Suppose that D is not of type (2.2.7(iv) or 2.3.5(iv)). Then $D \simeq (u_0, w_0) \otimes (a, b)$ for some units $u, v, \in R_0$ and $a, b \in R_0$ as in (2.6.1). Then, by the choice, we have $\Gamma = R(u_0, v_0) \otimes R(a, b)$. By (2.6.1), $(a, b) \otimes F_{0P} \simeq (a_P, b_P)$ for some $a_P, b_P \in \mathcal{O}_P$ as in (2.6.1) and $R_0(a, b) \subset \hat{\mathcal{O}}_P(a_P, b_P)$. In particular $\Gamma \subset \hat{\mathcal{O}}_P(u_0, v_0) \otimes \hat{\mathcal{O}}_P(a_P, b_P)$. Since $a_i \in \Gamma$ and $D \otimes F_{0P}$ is not of type (2.2.7(iii)), by (2.5.4), $X(F_{0P}) \neq \emptyset$.

Suppose that D is of type (2.2.7(iv) or 2.3.5(iv)). Then $D \simeq (u_0, u_1\pi_0) \otimes (v_0, v_1\delta_0)$ for some units $u_i, v_i \in R_0$.

Suppose P is a nodal point of $\mathscr{Y}_1$. Then, by (2.6.2), there exists an isomorphism $\phi_P : D \otimes F_{0P} \to (u_0', w_1'\pi_P) \otimes (v_0', v_1'\delta_P)$ and $\theta_P \in \hat{\mathcal{O}}_P(u_0', w_1'\pi_P) \otimes \hat{\mathcal{O}}_P(v_0', v_1'\delta_P)$ such that $\phi_P(R_0(u_0, u_1\pi_0) \otimes R_0(v_0, v_1\pi_0)) \subset \hat{\mathcal{O}}_P(u_0', w_1'\pi_P) \otimes \hat{\mathcal{O}}_P(v_0', v_1'\delta_P)$ and $int(\theta_P)\sigma = \phi_P^{-1}\sigma'\phi_P$, where σ' is the product of the canonical involutions on the right hand side. Let h' be the hermitian form on $((u_0', w_1'\pi_P) \otimes (v_0', v_1'\delta_P)), \phi_P\sigma\phi_P^{-1})$ which is the image of h under ϕ_P. Since $h = \langle a_1, \cdots, a_n\rangle$, we have $h_1 = \langle \phi_P(a_1), \cdots, \phi_P(a_n)\rangle$. Let $h' = \theta_P h_1$. Then h' is an hermitian form with respect σ'. Let X' be the projective homogeneous variety under $G((u_0', w_1'\pi_P) \otimes (v_0', v_1'\delta_P), \sigma', h')$ associated to X. Then $X(F_{0P}) \neq \emptyset$ if and only if $X'(F_{0P}) \neq \emptyset$. Since $\phi_P(a_i), \theta_P \in \hat{\mathcal{O}}_P(u_0', w_1'\pi_P) \otimes \hat{\mathcal{O}}_P(v_0', v_1'\delta_P)$, by (2.5.4), $X'(F_{0P}) \neq \emptyset$ and hence $X(F_{0P}) \neq \emptyset$.

If P is a non-nodal point, then using (2.6.3), we get $X(F_{0P}) \neq \emptyset$ as above. $\square$

Proposition 2.6.9. *Let k, F_0 and F be as above. Suppose that for finite extension ℓ/k, every element in ${}_2Br(\ell)$ has index at most 2. Let $D \in {}_2Br(F)$ be a central simple algebra over F with a F/F_0-involution σ and h an hermitian form over (D, σ). Let*

X be a projective homogeneous variety under $G(D,\sigma,h)$ over F_0. If $X(F_{0\nu}) \neq \emptyset$ for all divisorial discrete valuations ν of F_0, then there exists a sequence of blowups $\mathscr{Y} \to Spec(R_0)$ such that for every closed point P of $\mathscr{Y}$, $X(F_{0P}) \neq \emptyset$.

Proof. By (2.6.8), we assume that $\text{ind}(D) > 4$. As in the proof of (2.6.6), we assume that D is unramified on R_0 except possibly at (π_0) and (δ_0). Then, by (2.2.8, 2.3.6), $\text{ind}(D) = 8$ and $D \simeq (u_0, u_1) \otimes (v_0, v_1\pi_0) \otimes (w_0, w_1\delta_0) \otimes F$ for some units $u_, v_i, w_i \in R_0$. Without loss of generality we assume that σ is the tensor product of canonical involutions on (u_0, u_1), $(v_0, v_1\pi)$ and $(w_0, w_1\delta)$ and τ. Let $\Gamma = R(u_0, u_1) \otimes R(v_0, v_1\pi) \otimes R(w_0, w_1\delta)$. Then Γ is a maximal R-order of D. We have $h = \langle a_1, \cdots, a_n \rangle$ for some $a_i \in \Gamma$.

Let $\mathscr{X}_0 \to Spec(R_0)$ be a sequence of blowups such that the integral closure of $\mathscr{Y}_0$ in F is regular and the support of $Nrd(a_i)$ and $\text{ram}_{\mathscr{Y}}(D)$ is a union of regular curves with normal crossings. By (2.6.4), there exists a finite subset $\mathscr{P}$ of Y_0 such that $X(F_{0P}) \neq \emptyset$ for all $P \notin \mathscr{P}_0$.

Let $P \in \mathscr{P}_0$. If $\text{ind}(D \otimes F_{0P}) \leq 4$, then by (2.6.8), there exists a sequence of blowups $\mathscr{X}_P$ of $\text{Spec}(\mathcal{O}_P)$ such that for every closed point Q of $\mathscr{X}_P$, $X(F_{0Q}) \neq \emptyset$.

Suppose $\text{ind}(D \otimes F_{0P}) > 4$. Then as above we have $\text{ind}(D \otimes F_{0P}) = 8$. Arguing as in the proof of (2.6.8), we get that $X(F_{0P}) \neq \emptyset$. □

2.7 Main theorem

In this section we prove the main theorems.

Theorem 2.7.1. *Let K be a complete discretely valued field with valuation ring T and residue field k. Suppose that $char(k) \neq 2$. Let F be the function field of a smooth projective geometrically integral curve over K. Let $A \in {}_2Br(F)$ be a central simple algebra over F with an involution σ of any kind, $F_0 = F^\sigma$ and h a hermitian form over (A, σ). Suppose that $ind(A) \leq 4$. Let $G = SU(A, \sigma, h)$ if σ is first kind or*

$U(A,\sigma,h)$ if σ is of second kind. Let X be a projective homogeneous variety under G over F_0. If $X(F_{0\nu}) \neq \emptyset$ for all divisorial discrete valuations ν of F, then $X(F_0) \neq \emptyset$.

Proof. Since F is the function field of a curve over K, F_0 is also the function field of a curve over K. Let $\mathscr{X}_0 \to Spec(R_0)$ be a regular proper model of F_0 with the closed fibre X_0 a union of regular curves with normal crossings ([1]). Let $\eta \in X_0$ be a generic point. Then η gives a divisorial discrete valuation ν of F. Since $X(F_{0\eta}) \neq \emptyset$, by ([13, 5.8]), there exists a nonempty open set U_η of the closure of η in X_0 such that $X(F_{0U_\eta}) \neq \emptyset$. By shrinking U_η, we assume that U_η does not contain any singular points of X_0.

Let $\mathscr{P} = X_0 \setminus \cup_\eta U_\eta$. Then $\mathscr{P}$ is a finite set of closed points of X_0 containing all the singular points of X_0. Let P be a closed point of X_0. Suppose $P \notin \mathscr{P}$. Then $P \in U_\eta$ for some η. Since $F_{0U_\eta} \subset F_{0P}$, $X(F_{0P}) \neq \emptyset$.

Let $P \in \mathscr{P}$. Since $\mathrm{ind}(A) \leq 4$ and $A \in {}_2Br(F)$, by (2.6.8), there exists a sequence of blowups $\mathscr{X}_P$ of $\mathrm{Spec}(\mathcal{O}_P)$ such that $X(F_{0Q}) \neq \emptyset$ for all closed points of $\mathscr{X}_P$. Thus replacing $\mathscr{X}$ by these finitely many sequences of blowups at all $P \in \mathscr{P}$, we assume that $X(F_{0Q}) \neq \emptyset$ for all closed points Q of X_0. Since for any generic point η of X_0, $X(F_{0\eta}) \neq \emptyset$, we have $X(F_{0x}) \neq \emptyset$ for all points $x \in X_0$. Since G is a connected rational group ([8, Lemma 5]), by ([11, Theorem 3.7]), we have $X(F) \neq \emptyset$. □

Theorem 2.7.2. *Let K be a complete discretely valued field with valuation ring T and residue field k. Suppose that $char(k) \neq 2$. Let F be the function field of a smooth projective geometrically integral curve over K. Let A be a central simple algebra over F with an σ of any kind, $F_0 = F^\sigma$ and h a hermitian form over (A,σ). Suppose that for every finite extension ℓ/k, every element in ${}_2Br(\ell)$ has index at most 2. Let $G = SU(A,\sigma,h)$ if σ is first kind or $U(A,\sigma,h)$ if σ is of second kind. Let X be a projective homogeneous variety under G over F_0. If $X(F_{0\nu}) \neq \emptyset$ for all divisorial discrete valuations ν of F_0, then $X(F_0) \neq \emptyset$.*

Proof. Using (2.6.9), the proof is similar to the proof of (2.7.1). □

Corollary 2.7.3. *Let K be a complete discretely valued field with valuation ring T and residue field k. Suppose that k is a global field, local field or a C_2-field with char(k) $\neq 2$. Let F be the function field of a smooth projective geometrically integral curve over K. Let $A \in {}_2Br(F)$ be a central simple algebra over F with an involution σ of any kind and h a hermitian form over (A, σ). Let $F_0 = F^\sigma$, $G = SU(A, \sigma, h)$ if σ is first kind and $G = U(A, \sigma, h)$ if σ is of second kind. Let X be a projective homogeneous variety under G over F_0. If $X(F_{0\nu}) \neq \emptyset$ for all divisorial discrete valuations ν of F_0, then $X(F_0) \neq \emptyset$.*

Proof. Suppose k is a global field or a local field or a C_2-field. Then any finite extension ℓ of k is also same type and hence every element in ${}_2Br(k)$ is of index at most 2 ([30, Chapter 10, 2.3(vi)], [30, Chapter 10, 2.2(i)], [18, Theorem 4.8]). Hence the corollary follows from (2.7.2). □

Chapter 3

Springer's problem for odd degree extensions

Let F be a field of characteristic not 2. Let q be a quadratic form over F. Let M be an odd degree extension of F. By a theorem of Springer ([31]), if q_M is isotropic, then q is isotropic.

Let A be a central simple algebra over F with an involution σ. Let $h : V \times V \longrightarrow A$ be an ε-hermitian form over (A, σ) for $\varepsilon = \pm 1$. Let M be an odd degree extension of F^σ. One can ask if the isotropy of h_M implies the isotropy of h?

The following are some results to this question.

Bayer-Fluckiger and Lenstra ([4]) have proved that if h_M is hyperbolic, then h is hyperbolic. Parimala, Sridharan and Suresh ([25]) have proved that if A is a quaternion algebra and σ is of the first kind, if h_M is isotropic, then h is isotropic. However, they also show that this is not true in general if $ind(A)$ is odd and σ of the second kind. Let $B = \mathrm{End}_A(V)$ and let τ be the adjoint involution of h. Black and Queguiner Mathieu ([5]) proved that when $\deg(B) = 12$ and τ is orthogonal, if τ_M is isotropic, then τ is isotropic. Furthermore, the same holds when B has period 2, $\deg(B) = 6$, and τ is unitary.

3.1 Complete discretely valued fields

In this section we prove an analogue of Springer's theorem for hermitian forms over complete discretely valued fields with residue fields local fields or function fields of curves over local fields. We begin by recalling the following

Lemma 3.1.1. *([32], Lemma 5.1) Let (F, ν) be a complete discrete valued field with residue field κ, with char$(\kappa) \neq 2$. Let L/F be an extension of degree at most 2 with residue field κ_L. Let M be an odd degree extension of F with residue field κ_M. Suppose that for for every period 2 central division algebra B over κ_L with a κ_L/κ involution τ and ε-hermitian form g over (B, τ) if g_{κ_M} is isotropic, then g is isotropic. Let D be a central division algebra over L with period 2. Let σ be an involution on D. Let h be an ε'-hermitian form over D, $\varepsilon' = \pm 1$. If h_M is isotropic, then h is isotropic.*

Corollary 3.1.2. *Let (F, ν) be a complete discrete valued field with residue field κ, with char$(\kappa) \neq 2$. Suppose that κ is a non-dyadic local field or a function field of a curve over a non-dyadic local field. Let L/F be an extension of degree at most 2 and A a central simple algebra over L of period 2 with a L/F involution. Let h be an ε-hermitian form over (A, σ), $\varepsilon = \pm 1$. Let X be a projective homogeneous space under $G(A, \sigma, h)$. If $X(M) \neq \emptyset$ for some odd degree extension M/F, then $X(F) \neq \emptyset$.*

Proof. By Morita equivalence, we may assume that A is division. By the description of X and by induction on the Witt index of h, it is enough to show that if h_M is isotropic for some odd degree extension M/F, then h is isotropic.

Let M/F be an odd degree extension. Suppose that h_M isotropic. Let κ_M be the residue field of M. Then κ_M/κ is an odd degree extension. Let B be a central division algebra over κ_L of period 2 with a κ_L/κ-involution τ. Let g be ε'-hermitian form over (B, τ). Suppose that g_{κ_M} is isotropic.

If κ is a non-dyadic local field, then by ([32], Lemma 5.6), g is isotropic. If κ is a function field of a curve a non-dyadic local field, then by ([32], Theorem 5.8), g is isotropic. Hence, by (3.1.1), h is isotropic. $\square$

3.2 Applications

In this section we prove an analogue of Springer's theorem for hermitian forms over semiglobal fields with residue field a local field.

Let L be a field (char$(L) \neq 2$) and M be an odd degree extension of L. For any discrete valuation ν on L, let R_ν be the valuation ring and $\mathfrak{p}_\nu$ be the maximal ideal of this ring. Let $\hat{R}_\nu$ be the completion of R_ν and L_ν its field of fractions. Let S be the integral closure of R in M and β_i be the prime ideals of S over the ideals $\mathfrak{p}_\nu$ in R_ν, where $1 \leq i \leq n$. Let $\hat{S}_i$ be the completion of S at β_i and M_i be its field of fractions. Then, by ([7], p.15), $M \otimes_L L_\nu \cong \prod_{i=1}^n M_i$. Furthermore, since $[M : L]$ is odd, then $[M : L] = [M \otimes_L L_\nu : L_\nu] = \sum_{i=1}^n [M_i : L_\nu]$ implies that at least one of the terms in the sum $\sum_{i=1}^n [M_i : L_\nu]$ must be odd.

Theorem 3.2.1. *Let K be a complete discretely valued field with residue field k. Suppose that k is a nondyadic local field. Let F be the function field of a smooth projective geometrically integral curve over K. Let $A \in {}_2Br(F)$ be a central simple algebra over F with an involution σ and h a hermitian form over (A, σ). Let X be a projective homogenous space under $G(A, \sigma, h)$ over F^σ. If $X(M) \neq \emptyset$ for some odd degree extension M/F^σ, then $X(F^\sigma) \neq \emptyset$.*

Proof. Let M/F^σ be an extension of odd degree. Suppose $X(M) \neq \emptyset$.

Let ν be a divisorial discrete valuation of F^σ. Then the residue field $\kappa(\nu)$ is either a finite extension of K or a function field of a curve over a finite extension of k ([22, Theorem 8.1]).

Since $M \otimes F_\nu^\sigma \cong \prod_{i=1}^n M_i$ and $X(M) \neq \emptyset$, $X(M_i) \neq \emptyset$ for all i. Since M/F^σ is an odd degree extension, M_i/F_ν^σ is an odd degree extension. Hence, by (3.1.2), $X(F_\nu^\sigma) \neq \emptyset$. Hence, by ([11], 3.7), $X(F^\sigma) \neq \emptyset$. $\square$

Corollary 3.2.2. *Let K be a complete discretely valued field with residue field k. Suppose that k is a nondyadic local field. Let F be the function field of a smooth projective geometrically integral curve over K. Let $D \in {}_2Br(F)$ be a central division algebra over F with an involution σ and h a hermitian form over (A, σ). If h_M is isotropic for some odd degree extension M/F^σ, then h is isotropic.*

Proof. Let M/F^σ be an extension of odd degree. Suppose h_M is isotropic.

Let d =deg(D) and $X = X_d$ be the projective homogeneous space under $G(D, \sigma, h)$ given by the totally isotropic subspaces of reduced dimension d.

Since per(D) is at most 2 and M/F^σ is an odd degree extension, $D \otimes M$ is a division algebra. Since h_M is a isotropic, $X(M) \neq \emptyset$. Hence, by (3.2.1), $X(F^\sigma) \neq \emptyset$. In particular h is isotropic over F^σ. □

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