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a thesis submitted to the Faculty of Emory College of Arts and Sciences
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Abstract

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This paper looks at one of the inferences studied in Inferential Erotetic Logic (IEL): *erotetic implication* (EI). EI is a type of inference that enables the logical formulation of one question based on another. There are various formalizations of this derivation, and this analysis uses the formalized definition proposed by Jared Millson. While numerous efforts have been made to further understand and refine EI, it currently lacks a simple and efficient decision procedure. In other words, there is no straightforward way to determine whether one practical question logically follows from another. This paper proposes one such decision procedure. We begin with a short background and history of EI. Then, we introduce our test — which we call the Erotetic Matrix (EM) Test — and its rules. Finally, we will prove that the EM Test is both sound and complete, after which we will introduce areas for further research.

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1 Introduction: What Is Erotetic Implication?

Known as **Inferential Erotetic Logic (IEL)**, the logic of questions begins with the observation that there are similarities between arriving at a conclusion and arriving at a question. This suggests that questions can be treated much like conclusions in an argument. Suppose, for example, you were given the following statements:

> In his twenty years working at the Krusty Krab, SpongeBob has never made
> a mistake. But today, he left his spatula at home.

Given these assertions, one may ask: "in what room did SpongeBob put his spatula?" In this instance, the above statements act as premises, while the question acts like a conclusion.

While the intuition behind it may feel natural, IEL remains a relatively new field in philosophy. The term "erotetic logic" was coined only seventy years ago, in 1955, by Mary and Arthur Prior. Since then, many philosophers have attempted to critically examine situations in which questions may be entailed (such as Sylvain Bromberger in his 1966 paper, "Why-Questions"),[1] but it was not until the publication of Nuel Belnap and Thomas Steel's 1976 book, *The Logic of Questions and Answers*, that IEL would receive a logical formalization. In this book, Belnap and Steel argue that a question can be defined in terms of the set of its **direct answers**, which they define as "piece[s] of language that completely, but just completely, answer [a] question." [2] To illustrate, suppose SpongeBob's house consisted only of a bedroom, kitchen, and living room (he lives in a pineapple so this is quite plausible). He may then ask, "is my spatula in the kitchen?" — the answer to which is either "the spatula is in the kitchen," or "the spatula is not in the kitchen." Thus, under the definition proposed by Belnap and Steel, the question would be formalized as such:

[1] Sylvain Bromberger, "Why-Questions," in Mind and Cosmos: Essays in Contemporary Science and Philosophy, ed. Robert Garland Colodny (University of Pittsburgh Press, 1966), 86–111.

[2] Nuel D. Belnap and Thomas B. Steel, *The Logic of Questions and Answers* (New Haven, CT: Yale University Press, 1976), 3.

$$Q = ?\{k, \neg k\}$$

Here, each answer to the question is symbolised as either k ("the spatula is in the kitchen") or $\neg k$ ("the spatula is not in the kitchen"). The question itself is denoted as Q, and the notation $?\{\}$ signifies the *set* of answers to Q. Note that we express the set of direct answers to Q as dQ. In this case, the set of direct answers to Q would be formalized as follows:

$$dQ = \{k, \neg k\}$$

While Belnap and Steel laid the groundwork for erotetic logic, they believed that formalizing inferential relations among questions was impossible. In their words, "absolutely the wrong thing is to think [erotetic logic] is a logic in the sense of a deductive system, since one would thus be driven to the pointless task of inventing an inferential scheme in which questions, or interrogatives, could serve as premises and conclusions." [3]

Yet, the field of IEL has expanded considerably since Belnap and Steel's foundational work. In 1995, Polish philosopher Andrzej Wiśniewski published a book titled *The Posing of Questions: Logical Foundations of Erotetic Inferences*. In this book, he suggests that inferential relations between questions *do* exist, and actually proposes two such inferences (now known as **erotetic inferences**): **erotetic evocation (EE)** and **erotetic implication (EI)**.[4] The former allows for the derivation of a question based on a set of premises, and the latter involves inferring one question from another.

In 2018, Jared Millson would publish *A Cut-Free Sequent Calculus for Defeasible Erotetic Inferences* — a paper in which he proposes a sequent calculus for both erotetic evocation and erotetic implication. He also provides succinct, formalized definitions of both inferences, which will serve as the foundation for this paper.[5] Our analysis, however, will focus solely on erotetic implication.

We begin our discussion on erotetic implication by taking another look at SpongeBob's question regarding the location of his spatula. Recall that SpongeBob's house only has three rooms: the bedroom, the kitchen, and the living room. This information is *background knowledge* that limits the number of possible answers to the question "in which room did I leave my spatula?" to three. This means that we can formulate this question (which we will call Q_1) as:

$$Q_1 = ?\{b, k, l\}$$

[3] Belnap and Steel, *The Logic of Questions and Answers*, 1.

[4] Andrzej Wiśniewski, *The Posing of Questions*, Synthese Library 252 (Dordecht: Kluwer Academic Publishers, 1995).

[5] Jared Millson, "A Cut-Free Sequent Calculus for Defeasible Erotetic Inferences," *Studia Logica*, November 2018, https://doi.org/10.1007/s11225-018-9839-z.

Another important piece of background knowledge is that the spatula cannot be in all three rooms at once. This means that, for example, if the spatula is in the kitchen, it cannot be in the living room. Here, we introduce the idea of a **background set** — a set of propositions assumed to be true that provides the context in which a question can entail another. We denote the background set as X. For SpongeBob, X is defined in the following way:

$$X = \{b \vee (k \vee l), \quad b \rightarrow (\neg k \wedge \neg l), \quad k \rightarrow (\neg b \wedge \neg l), \quad l \rightarrow (\neg k \wedge \neg b)\}$$

The first of these propositions tells us that the spatula is either in the bedroom, the kitchen, or the living room; and the rest tell us that if the spatula is in one room, it cannot be in the others. Note that Q_1 (the *implying* question, $?\{b, k, l\}$), when combined with the background set X, will entail the question "is my spatula in the kitchen?" — which we will define as Q_2 (the *implied* question, $?\{k, \neg k\}$).

It may be intuitive from this information to see why the question "is my spatula in the kitchen?" (Q_2) follows from the question "is my spatula in the bedroom, kitchen, or living room?" (Q_1). The key insight here is that *answering the second question would give the inquirer information relevant to the first.* That is, if SpongeBob can answer the second question in the affirmative, then he's answered his first question, too. On the other hand, if SpongeBob checks his kitchen and realizes that the spatula is not there, he will know that the spatula is either in the bedroom or living room. In this way, he's narrowed the search space of the first question. Thus, SpongeBob's first question, Q_1, has implied his second, Q_2, against the background set of information X. Note that this implication is captured with the notation $Im(Q_1, X, Q_2)$.

From this, it follows that every direct answer to the second question must imply a proper subset of answers to the first. However, there is another criterion that must be considered in order for the logical relation to be satisfied. That is, the second question must be, in some way, relevant to the first question. If SpongeBob knows that his spatula is either in the kitchen, the bedroom, or the living room, it would be illogical for him to ask whether the spatula has been left in the Krusty Krab. These definitions can be formalized as such:

$$\forall A \in dQ_1 (X \cup \{A\}) \Vert\!\models dQ_2 \tag{1}$$

$$\forall B \in dQ_2 \, \exists \Psi \subset dQ_1 (X \cup \{B\}) \models \Psi \tag{2}$$

Condition 1 captures the relevance criterion of erotetic implication. It says that every answer to question 1, when combined with the background set X, entails the set of answers to question 2. This means that if the first question has an answer, the second one must also have an answer. In more technical terms, if the first question is **sound**, the second one must be sound as well.

Because the condition requires the entailment of a *set* of answers, we must introduce **multi-conclusion logic**, which allows us to express instances of sets of premises that have more than one possible conclusion. In classical logic, for example, we may say that $p \wedge q \models p$, and that $p \wedge q \models q$. In multi-conclusion logic, however, we can express both of these at once by writing $p \wedge q \,||\!\!= p, q$. Importantly, in multi-conclusion logic, *implying a set of premises is the same thing as implying their disjunct*. In other words, entailing a set of premises {p, q} is equivalent to entailing $p \vee q$. Thus, Condition 1 states that every direct answer A to Q_1, when combined with the background set X, entails the disjunct of all direct answers to Q_2.

Condition 2 captures the idea that answering question 2 will provide some information that will narrow the set of possible answers to question 1. It states that every direct answer B to Q_2 entails a **proper subset** of direct answers Ψ to Q_1. It is important that the answers to question 2 imply a *proper* subset of answers to question 1, because if each answer to the second question implied the set of all answers to the first, it would not provide any novel information. Moreover, without this stipulation, any binary question would imply any other binary question. The reason for this is explained in the following example.

Consider two binary questions where $dQ_1 = \{p, \neg p\}$ and $dQ_2 = \{q, \neg q\}$. Q_2 should not be implied by Q_1, as answering Q_2 would not give us any information about Q_1 whatsoever. However, if there were no stipulation that dQ_2 must imply a proper subset of answers to dQ_1, Condition 2 would be satisfied, because anything implies $p \vee \neg p$.

Together, Condition 1 and Condition 2 define **general erotetic implication**.

There is a special case of general erotetic implication called **regular erotetic implication**. The idea behind this type of implication is not simply that answering the second question would give the inquirer some novel information about the first, but answering the second question would directly answer it. A question is *regularly* implied from another iff answering the second, no matter what answer it is, will give you a direct answer to the first. Note that this does not mean *every* answer to the first question must be entailed by an answer to the second. Rather, it means that every answer to the second question implies an answer to the first, even if all of the answers to the second question imply the same answer in the first. The conditions for regular erotetic implication are as follows:

$$\forall A \in dQ_1 (X \cup \{A\}) \,||\!\!= dQ_2 \tag{3}$$

$$\forall B \in dQ_2 \; \exists A \in dQ_1 (X \cup \{B\}) \models A \quad (4)$$

Note that Condition 3 (the first condition for regular erotetic implication) is the same as Condition 1 (the first condition for general erotetic implication). Condition 2 guarantees that every direct answer to the second question implies at least one direct answer to the first.

These conditions define the two types of EI. However, they do not give us any way of identifying pairs of questions that entail one another. In other words, there currently exists no simple **decision procedure** for EI. That is, there is no efficient, mechanical method analogous to truth trees or truth tables that allows the inquirer to test whether one question is implied from another.

However, because both conditions for both types of EI are deductive, one could, in theory, create a truth tree (or similar test) for each implication. Recall SpongeBob's spatula predicament. Q_1 had three possible direct answers, and Q_2 had two. In order to test Condition 1, you would need three distinct truth trees, testing whether each answer to Q_1 implied the disjunction of answers to Q_2. In order to test Condition 2, however, you would need *twelve*. The reason for this is that Condition 2 requires each answer to Q_2 to imply a proper subset of answers to Q_1, which would mean that you would need to test whether each answer to Q_2 entails at least one of the subsets of direct answers to Q_1, of which there are six. Thus, you would need fifteen distinct truth trees to help SpongeBob find his spatula.

Before proceeding, however, we take a moment to explain the notation that will be used. To represent conjunctions, we use both the traditional symbol, $\wedge$, as well as juxtapositions. That is, $p \wedge q$ is equivalent to pq. Moreover, negations may be represented as either $\neg p$ or $\bar{p}$ for compactness. Finally, we use the symbols $\bigwedge$ and $\bigvee$ to represent the conjunction or disjunction of *sets*, respectively. For example, $\neg \bigvee dQ_1$ represents the negated disjunct of direct answers to a question denoted as Question 1.

We now move forward with discussing some essential background elements of our proposed decision procedure, specifically truth trees and disjunctive normal form.

2 Disjunctive Normal Form And The Tree Test

The rationale behind this test, which we will call the **Erotetic Matrix (EM) Test**, is analogous to the rationale of a truth tree. By definition, an argument is valid iff there is no situation in which all of its premises are true and its conclusion is false. Thus, in order for an argument to be *invalid*, it must be possible for all of its premises to be consistent with the negation of its conclusion. A truth tree tests this by listing out all of the premises of an argument and the negation of its conclusion. The premises are then broken into smaller pieces until they are reduced to their atomic formulas. At this point, it would be apparent on the tree whether there is a way of making all premises true and the negation of the conclusion true. If this is not possible, the argument is valid.

The EM Test uses the same principles used in truth trees, but compresses all of the information into one table. Before we introduce the rationale behind the EM Test, however, we must introduce the concept of **disjunctive normal form**.

Translating formulas in propositional logic to disjunctive normal form relies on the insight that all logical connectives (in propositional logic) have an equivalent form that uses only negations and disjuncts. For example, $p \rightarrow q$ is equivalent to $\neg p \vee q$. Moreover, because of De Morgan's laws and the distribution rule, all formulas in classical propositional logic can be converted into a form where the main connective is a disjunct.

For example, $(p \vee q) \rightarrow r$ is equivalent to $\neg(p \vee q) \vee r$ by material implication. Then, because of De Morgan's law, we can write this formula as $(\bar{p} \wedge \bar{q}) \vee r$ (recall that $\bar{p}$ is equivalent to $\neg p$). Finally, this formula can be turned into $(\bar{p} \wedge r) \vee (\bar{q} \wedge r)$ because of the distribution rule. For compactness, we change the syntax slightly and write the formula as $\bar{p}r \vee \bar{q}r$.

When a formula is written in disjunctive normal form, the only time negations appear is next to atomic statements. Moreover, only atomic statements can be part of a conjunction. Also note that we do not use parentheses in disjunctive form when presenting formulae. Rather, formulae in disjunctive normal form are presented as the disjunction of conjunctions. This means that no conjunction can be a main connective over a disjunct.

Importantly, *each disjunct represents a way of making its original formula true*. In the example above, $\bar{p}r \vee \bar{q}r$, the falsity of p and the truth of r (as well as the falsity of q and the truth of r) are each sufficient for the truth of $(p \vee q) \rightarrow r$. Note that each disjunct represents an open path on a truth tree since both contain atomic formulas that are sufficient for the truth of their original premises. This means that we can easily use truth trees to identify disjunctive normal form by conjoining all of the atomic propositions (and negated ones) on each open branch, and then creating a disjunction of all of these conjunctions.

Multiple premises can be translated into disjunctive normal form simply by conjoining them and converting the result accordingly. In the next section, we will show how this conversion forms the basis of the EM Test.

3 Conceptualizing The Erotetic Matrix Test

In its essence, the EM Test is a way of simultaneously creating a truth tree for each direct answer to both the implied and implying question. Because there are two conditions that must be satisfied for both general and regular EI, there are two tests (one for each condition). Note that there is no separate test for general and regular EI.

The process begins by testing whether Conditions 1 and 3 (the first condition for each type of erotetic implication) hold. There is only one test because both conditions are the same. Remember that these conditions state that every answer A in dQ_1, when combined with the background set X, implies the set of answers to dQ_2. Also recall that implying a set of answers is the same as implying their disjunct.

The first step in testing Conditions 1 and 3 involves taking each answer to Q_1 (separately) and conjoining it with the background set X. The conjunction is then converted into disjunctive normal form. Each disjunct in the new formula now represents a way of making that answer and the background set true. We then make a table (known as an Erotetic Matrix), with each column headed by one direct answer to Q_1. This will be called a **main column**. Each main column is divided into **sub-columns**. These sub-columns are each headed by one of the disjuncts found by converting the answer heading its main column (and the background set) into disjunctive normal form.

Consider the following:

Let $dQ_1 = \{p, q\}$, $X = \{r \vee s, r \rightarrow p, s \rightarrow q\}$, $dQ_2 = \{r, s\}$

In disjunctive normal form, $\bigwedge\{p\} \bigcup X$ is $pr\bar{s} \vee pq\bar{r}s \vee pqr \vee pqs$, and $\bigwedge\{q\} \bigcup X$ is $pqr\bar{s} \vee q\bar{r}s \vee pqr \vee pqs$. So, the columns of the Erotetic Matrix will be formed as such:

p				q			
$pr\bar{s}$	$pq\bar{r}s$	pqr	pqs	$pqr\bar{s}$	$q\bar{r}s$	pqr	pqs

Essentially, all we have done is listed out the ways of making each answer to Q_1 and the background set true. Note that each main column is separated by a double line, while each sub-column is separated by a single line.

The test for Conditions 1 and 3 is relatively simple because there is only one **main row**. This row is found by creating a disjunct of the answers to Q_2 and negating them (which will create a conjunction due to De

Morgan's laws). Building upon the previous example, the corresponding table is structured as follows:

	p				q			
	$pr\bar{s}$	$pq\bar{r}s$	pqr	pqs	$pqr\bar{s}$	$q\bar{r}s$	pqr	pqs
$\bar{r}\bar{s}$								

If the answers to Q_2 are not atomic, the formulae are converted to disjunctive normal form, and the main row is then split into **sub-rows** representing different ways of making the negated disjuncts of the answers to Q_2 true. We provide an example of an EM Test with sub-rows later in this paper. For the sake of simplicity, no sub-rows are included in this first example.

Note that the intersection between a sub-column and a sub-row is called a **sub-cell**, and the intersection of a main column and a main row is called a **main cell**. Moreover, if, in a cell, there is an inconsistency between the formula heading the (sub)column and the formula heading the (sub)row, that cell is marked with an × and is considered **closed**. The key insight here is that *if all sub-columns in a main column are inconsistent with the negation of the disjunction of answers to Q_2, the answer heading the main column entails that disjunct.* Since the disjunct heading the main row is the set of direct answers to Q_2, the answer heading the main column entails the set of answers to Q_2, as required by Conditions 1 and 3. So, if all sub-columns in all main rows of the table are inconsistent with the negation of the disjuncts of Q_2, Condition 1 and Condition 3 hold, by their very definitions. Note that the test for Conditions 1 and 3 is known as Test 1 for general and regular EI, respectively.

Looking at the previous Erotetic Matrix, we can see that the main row *is* inconsistent with every sub-column, so Test 1 is **passed**.

	p				q			
	$pr\bar{s}$	$pq\bar{r}s$	pqr	pqs	$pqr\bar{s}$	$q\bar{r}s$	pqr	pqs
$\bar{r}\bar{s}$	×	×	×	×	×	×	×	×

Setting up the test for Conditions 2 and 4 (known as Test 2 for general and regular EI, respectively) is quite similar. This time, the direct answers to Q_2 are conjoined with X and converted into disjunctive normal form. A table is then created with each main column representing an answer B to Q_2, and each sub-column representing a way of making B (combined with the background set X) true.

Beginning with regular erotetic implication, Condition 4 (the second condition for regular erotetic implication) states that every direct answer to Q_2 implies at least one direct answer to Q_1. This means that the EM Test for Condition 4, unlike the EM Test for Conditions 1 and 3, must have more than one row (the reason being that the second condition for regular erotetic implication captures the relationship between the direct answers of Q_1 and Q_2, not the relationship between the direct answers to Q_1 and the *disjunct* of answers to Q_2).

Each main row is headed by the negation of one of the answers to Q_1. If an answer is not atomic, it is put into disjunctive normal form and divided into sub-rows that represent the different ways of making it true (or rather, because it is negated, false). Thus, each intersection of sub-rows and sub-column represents a way of making an answer B to Q_2, the background set X, and the negation of an answer A to Q_1 all true together.

Using our same example, we convert the answers to Q_2 into disjunctive normal form as follows: $\bigwedge\{r\} \cup X$ is translated as $pr\bar{s} \vee pqr \vee pqrs$, and $\bigwedge\{s\} \cup X$ is translated as $q\bar{r}s \vee pqrs \vee pqs$. Our table would be represented as such:

	r			s		
	$pr\bar{s}$	pqr	$pqrs$	$q\bar{r}s$	$pqrs$	pqs
$\bar{p}$	×	×	×		×	×
$\bar{q}$		×	×	×	×	×

Remember that each main row and main column is separated with a double line, and each main sub-row and sub-column is separated with a single line. In this example, there are no sub-rows, so we use only double lines to separate the rows.

If every sub-cell in a main cell closes, this would mean that there is no way of making the answer to Q_2 true and the answer to Q_1 false at the same time. In other words, by the definition of entailment, the answer B, when combined with the background set X, entails A. Recall that Condition 4 (the second condition of regular erotetic implication) states that every direct answer to Q_2 must entail at least one direct answer to Q_1. So, if every main column has a main cell that closes, Condition 4 holds. Our example is a therefore a case of regular erotetic implication.

Now consider the following example:

Let $dQ_1 = \{p \wedge q, p \wedge \bar{q}\}$, $X = \{p\}$, $dQ_2 = \{q, \bar{q}\}$

Note that the answers to the second question, when turned into a disjunct, forms a **tautology**, meaning that it is entailed by everything. Therefore, Test 1 is passed trivially.

The table for Test 2 would be structured as follows:

		q	$\bar{q}$
		pq	$p\bar{q}$
$\neg(p \wedge q)$	$\bar{p}$	$\times$	$\times$
	$\bar{q}$	$\times$	
$\neg(p \wedge \bar{q})$	$\bar{p}$	$\times$	$\times$
	q		$\times$

In this example, we see that each negated answer to Q_1 is separated into sub-rows representing the disjuncts of $\neg(p \wedge q)$ and $\neg(p \wedge \bar{q})$, respectively. Recall that these disjuncts are ways of making their original formulae true. In this case, the disjuncts represent ways of making the *negation* of each answer to Q_1 true.

To illustrate how this table should be read, consider the main column headed by q. There is only one "sub-column," pq, which represents the conjunction of q with the background set X. This "sub-column" is inconsistent with both of the disjuncts that represent ways of making $\neg(p \wedge q)$ true. This means that there is an answer to Q_2 (q) that entails an answer to Q_1 ($p \wedge q$) when combined with the background set X. The same reasoning can be used to show that $\bar{q}$ entails $p \wedge \bar{q}$. Therefore, each answer to Q_2 implies at least one answer to Q_1. Test 2 is thus passed and the implication is regular.

In general erotetic implication, the first condition (Condition 1) is the same as the first condition for regular erotetic implication (Condition 3). Hence, passing Test 1 for regular erotetic implication passes Test 1 for general erotetic implication. However, the second condition for general erotetic implication (Condition 2) differs slightly from the second condition for regular erotetic implication (Condition 4). This second condition for general erotetic implication states that every answer B to Q_2 entails a *proper subset* of answers to Q_1. This means that, although each answer to Q_2 may not entail a single answer to Q_1, it may entail a disjunct of answers to Q_1, so long as that disjunct is not comprised of all answers to Q_1.

Test 2 for general erotetic implication is set up in the same way that Test 2 for regular erotetic implication is set up. Indeed, we read them both off of the same table. If Test 2 for regular erotetic implication fails (that is, not every main column has a main cell that closes completely), we move on to determine if the test passes for general erotetic implication. Test 2 for general erotetic implication requires that the remaining open columns be **cleared**.

When testing for clearance of a main column, we are looking to see if the answer that heads it entails a disjunct of at least two of the answers to Q_1. Remember that any answer to Q_2 cannot entail a disjunct of all of the answers Q_1 because it must entail a *proper* subset of answers.[6] In the EM Test, the basic concept of clearance is that for at least one proper subset of answers to Q_1 (i.e., at least two main rows), there is no situation where there is an open sub-cell for both or all of them at the same time in any given column. If this happens, it would mean that the answer heading the main column entails a proper subset of answers to Q_1. Hence, by definition, the second condition of general erotetic implication is satisfied. The following example shows a case of general erotetic implication.

Let $dQ_1 = \{p, q, r\}$, $X = \{p \vee q \vee r\}$, $dQ_2 = \{p, \bar{p}\}$

Since $\bigwedge dQ_2$ is inconsistent, Test 1 is passed trivially as the disjunction of all the answers to Q_2 will be entailed by everything. Test 2 will be structured as such:

	p			$\bar{p}$	
	p	pq	pr	$\bar{p}q$	$\bar{p}r$
$\bar{p}$	×	×	×		
$\bar{q}$		×		×	
$\bar{r}$			×		×

Here, we can see that the main column headed by p contains a main cell that closes entirely. However, the main column headed by $\bar{p}$ has no main cell that closes. It does, however, have a *pair* of main rows, headed by $\bar{q}$ and $\bar{r}$ where every row is closed by a sub-column, and vice versa. Hence, the column is cleared. So, Test 2 is passed and the implication is general. Note that determining clearance simply involves looking to see if every sub-column closes for at least one of the main rows in the n-tuple of main rows being considered.

There are two additional considerations for clearance, the first of which involves inconsistency. If there exists an n-tuple of answers to Q_1 that is inconsistent (that is, there is an n-tuple of main rows is inconsistent), each column automatically clears and Test 2 shows a case of general erotetic implication. The reason for this is that any disjunction of inconsistent formulas is entailed by everything, so each column entails a proper subset of answers to Q_1 (that is, it entails a disjunction consisting of a proper subset of answers to Q_1).

Consider the following example:

Let $dQ_1 = \{p, \bar{p}, q\}$, $X = \{\varnothing\}$, $dQ_2 = \{s, \bar{s}\}$

[6] This means that, if Q_1 contains two answers or less, the implication must be regular (if there is an implication, at all).

Test 1 is passed trivially because Q_2 is binary. Test 2, however, would look like this:

	s	$\bar{s}$
$\bar{p}$		
p		
$\bar{q}$		

In this example, there is no main column that closes. However, p and $\bar{p}$ are inconsistent with one another. This means that a disjunct of the two answers (which would be a subset of the answers to Q_1) would form a tautology, and thus be entailed by everything. So, Test 2 automatically passes as a case of general erotetic implication. This is why our definition of clearance (formalized below) specifies that a column clears if there is an inconsistent n-tuple of main rows.

Here, we must reiterate the importance of p and $\bar{p}$ not exhausting the complete set of answers to Q_1, as required by Condition 2 of general EI.

The second consideration for clearance involves sub-rows. Situations like these complicate our analysis, because one can no longer simply look at an Erotetic Matrix to see if every sub-column closes for at least one main row in the n-tuple of main rows being considered. Instead, we must look at every sub-row in every possible n-tuple of answers to Q_1. The key here is that one need only look at the sub-rows in an n-tuple of main rows that are *consistent* with one another. The reason for this is similar to the reasoning presented in the previous paragraph: because each sub-row shows a way of making the negation of the answer it represents true, if there are two or more sub-rows that are inconsistent with one another, then the answer they represent is a tautology and will be entailed by everything. However, if these inconsistent sub-rows belong to different main rows, it would simply be the case that there is at least one way of making the disjunction of the n-tuple of answers that the sub-rows represent true (due to the fact that there is at least one way of making the disjunct of their negated answers inconsistent). Thus, we only need to consider the consistent sub-rows and make sure that each of these sub-rows is inconsistent with at least one sub-column in the main column being considered. If this is the case, it would mean that there is no possible way of making B (the answer that heads the main column), alongside the background set X and the disjunct of the negated answers in consideration, true. Hence, B alongside the background set X must entail the disjunct of answers to Q_1 in consideration. Consider the following example:

Let $dQ_1 = \{p \wedge q, p \wedge \bar{q}, \bar{p} \wedge q\}$, $X = \{p \vee q\}$, $dQ_2 = \{p, q\}$

Test 1 would be constructed as such:

	$p \wedge q$	$p \wedge \bar{q}$	$\bar{p} \wedge q$
	pq	p	q
$\bar{p}\bar{q}$	×	×	×

Because all sub-columns are closed, Test 1 is passed.

Test 2 can now be created:

		p		q	
		p	pq	pq	q
$\neg(p \wedge q)$	$\bar{p}$	×	×	$\bot$	$\bot$
	$\bar{q}$	$\bot$	$\bot$	×	×
$\neg(p \wedge \bar{q})$	$\bar{p}$	×	×		×
	q	$\bot$	$\bot$		
$\neg(\bar{p} \wedge q)$	p			$\bot$	$\bot$
	$\bar{q}$		×	×	×

Test 2 is passed because each column clears. The main column p is cleared by the first and second main rows. The sub-rows for q and $\bar{q}$ are inconsistent, and we mark this inconsistency with $\bot$ in all cells that are in the intersection of the sub-row and the column headed by p. Considering the *remaining*, consistent sub-rows (both of which are $\bar{p}$), every sub-row is closed by a sub-column of p, and every sub-column of p is closed at some sub-row. The first and third main rows clear the q column in a similar way.

We now present two examples that fail the EM Test. The first shows a failure of Test 1, and the second shows a failure of Test 2.

3.1 Example 1 (Test 1 Failure)

Let $dQ_1 = \{p \to q, \neg(p \to q)\}$, $X = \varnothing$, $dQ_2 = \{\bar{p}, p \wedge \bar{q}\}$

Test 1

$\neg(\bar{p} \vee (p \wedge \bar{q})) \iff pq$, so pq is the row for Test 1

The first column does not close, so Test 1 fails and there is no implication.

	$p \to q$		$\neg(p \to q)$
	$\bar{p}$	q	$p\bar{q}$
pq	×		×

3.2 Example 2 (Test 2 Failure)

$$\text{Let } dQ_1 = \{p, q, r\}, X = \{(p \vee q \vee r) \to (s \vee \bar{s})\}, dQ_2 = \{s, \bar{s}\}$$

Test 1

Test 1 is passed trivially because the answers to Q_2 are inconsistent.

Test 2

	s			$\bar{s}$		
	sp	sq	sr	$\bar{s}p$	$\bar{s}q$	$\bar{s}r$
$\bar{p}$	×			×		
$\bar{q}$		×			×	
$\bar{r}$			×			×

Test 2 is failed because no column is closed, and neither column has a pair of rows that will clear. Since a triple of rows is the whole table, no column clears.

In the following section, we provide the formalized steps of the EM Test, after which we will show that the Test is sound and complete for both regular and general erotetic implication.

4 Rules For The Erotetic Matrix Test

The following presents the Erotetic Matrix Test for $Im\{Q_1, X, Q_2\}$, where Q_1 is the implying question and Q_2 is the implied question.

Q_1 passes the test for entailing Q_2 given X *if and only if* it passes both Test 1 and Test 2. Whether the entailment is general or regular depends on how Test 2 is passed.

As a restatement of terminology introduced earlier in the paper, erotetic matrices are composed of main rows and main columns. These intersect in main cells. Where main rows and columns are broken into sub-rows and sub-columns, the intersections are sub-cells. For convenience, a column or row without sub-columns or sub-rows also counts as a sub-row in the criteria below.

4.1 Erotetic Matrix Test Conditions

4.1.1 Test 1 (For Conditions 1 And 3)

1. A table is set up where each main column is headed by a formula Φ, where $\Phi \in dQ_1$. For each $\Phi \in dQ_1$, elements of the set $\{\Phi\} \cup X$ are conjoined (denoted $\bigwedge\{\Phi\} \cup X$) and the resulting formula put into disjunctive normal form, $\Phi_1 \vee \Phi_2 \vee \cdots \vee \Phi_n$. Each sub-column is headed by one of the disjuncts, Φ_i.

2. The table has one main row headed by $\neg \bigvee dQ_2$. If the elements of dQ_2 are atomic, negating their disjunction will result in a conjunction of atomic propositions, Ψ. If the elements of dQ_2 are not atomic, $\neg \bigvee dQ_2$ is put into disjunctive normal form and each sub-row is headed by a disjunct Ψ_i.

3. If $\neg \bigvee dQ_2$ is inconsistent, Test 1 is passed.

4. A sub-cell, $\Psi_i \between \Phi_i$, is closed and marked by an "×" iff Ψ_i and Φ_i are inconsistent.

5. A main cell, $\Psi \between \Phi$, closes iff each sub-cell, $\Psi_i \between \Phi_i$, closes.

6. Test 1 is passed iff for each main column, Φ, there is a main cell, $\Psi \between \Phi$, that closes.

   Note that because there is only one main row, Test 1 is passed iff *every* main cell is closed.

4.1.2 Test 2 (For Conditions 2 and 4)

1. A table is set up where each main column is headed by a formula Φ, where $\Phi \in dQ_2$. For each $\Phi \in dQ_2$, elements of the set $\{\Phi\} \cup X$ are conjoined and $\bigwedge\{\Phi\} \cup X$ put into disjunctive normal form, $\Phi_1 \vee \Phi_2 \vee \cdots \vee \Phi_n$. Each sub-column is headed by one of the disjuncts, Φ_i.

2. Each main row of the table is headed by a formula of the form $\neg\Psi$, where $\Psi \in dQ_1$. If Ψ is not atomic, $\neg\Psi$ is put into disjunctive normal form, and each sub-row is headed by a disjunct, Ψ_i.

3. A sub-cell, $\Psi_i \between \Phi_i$, is closed and marked by an "×" iff Ψ_i and Φ_i are inconsistent.

4. A main cell, $\Psi \between \Phi$, closes iff either each sub-cell, $\Psi_i \between \Phi_i$, closes, or there is a pair of sub-rows, $\Psi_i \between \Psi_j$, that are inconsistent.

5. Test 2 is passed and the entailment is regular iff for each main column, Φ, there is a main cell, $\Psi \between \Phi$, that closes.

6. Test 2 is passed and the entailment is general iff all columns either have a main cell that closes or are cleared. Note that to be general, the entailment must not be regular.

   A main column, Φ, is *cleared* iff either there is an inconsistent n-tuple of main rows where n is less than the total number of main rows, or there is a consistent n-tuple of main rows, Ψ^n, where n is less than the total number of main rows, and for every sub-row, Ψ_i^n, that is consistent with the other sub-rows in Ψ^n, there is a sub-column, Φ_i, such that $\Psi_i^n \times \Phi_i$ closes; and for every Φ_i, there is a sub-row, Ψ_i^n, such that $\Psi_i^n \times \Phi_i$ closes.

4.2 Mechanics For The Erotetic Matrix Test

To put Φ or $\bigwedge\{\Phi\} \cup \{X\}$ into disjunctive normal form:

1. Begin a tree test with either Φ or Φ along with each element of X as the initial list.

2. Finish the tree in the standard way.

3. For each open path on the tree, conjoin all of the atomic formulae or negated atomic formulae.

4. Create a disjunction of all conjunctions created in Step 3.

5. Eliminate all inconsistent disjuncts from the formula.

To test whether a main column, Φ, clears, consider each n-tuple of main rows, Ψ^n, where n is less than the total number of main rows, and apply the following test:

1. If n = 2 and the formulae heading the rows in Ψ^n contradict one another, then move to the next pair of main rows. If no further pairs remain, the column Φ does not clear and the test for general implication fails.

2. If the formulae heading any pair of sub-rows in Ψ^n contradict one another, ignore these sub-rows.

3. Considering the remaining sub-rows of Ψ^n, column Φ is cleared iff for every sub-column Φ_n, there is a sub-row Ψ_i^n such that $\Psi_i^n \times \Phi_i$ closes; and for every sub-row Ψ_i^n, there is a Φ_n such that $\Psi_i^n \times \Phi_i$ closes.

4. If column Φ is not cleared by Ψ^n, move to the next n-tuple of rows and repeat steps 1 – 3.

5. If column Φ is not cleared after all n-tuples have been tested, the test for general implication fails and there is no implication between Q_1 and Q_2.

6. If column Φ is cleared, move to the next main column without a closed main cell and repeat steps 1 – 4.

We will now show that the EM Test is both sound and complete.

5 Soundness And Completeness Of The Erotetic Matrix Test

To prove the soundness and completeness of the EM Test, we must show that it is sound and complete for both regular *and* general erotetic implication.

5.1 Regular Erotetic Implication: Soundness And Completeness

In the following two sections, we will demonstrate how the EM Test is both sound and complete relative to inferences of **regular** erotetic implication (noted as $Im_R\{Q_1, X, Q_2\}$). In the first section, we will show the Test to be sound; in the second section, we will show it to be complete.

5.1.1 Regular Erotetic Implication: Soundness

By definition of soundness: if the EM Test is passed, then the inference $Im_R\{Q_1, X, Q_2\}$ holds.

1. Assume for conditional proof that the EM Test is passed. In other words, both Test 1 and Test 2 are passed.

2. This means that, in Test 1, every main cell closes. Hence, every sub-cell in Test 1 closes.

3. In Test 2, it must be the case that every main column and main row has a (main) cell that closes. If a main cell closes, all sub-cells must be closed, as well.

4. Now suppose for reductio that the inference $Im_R\{Q_1, X, Q_2\}$ does *not* hold (the inference is not a case of regular erotetic implication).

5. This means that either Condition 3 or Condition 4 does not hold.

6. **Case 1 (Condition 3):**

   (a) Condition 3 for regular erotetic implication states that $\forall A \in dQ_1(X \cup \{A\}) \models dQ_2$. So for the Condition to fail, it must be the case that $\exists A \in dQ_1(X \cup \{A\}) \not\models dQ_2$.

(b) Therefore, $\exists A \in dQ_1(X \cup \{A\})$ is consistent with $\neg \bigvee dQ_2$.

(c) Condition 3 is evaluated with Test 1 for regular erotetic implication.

(d) Note that in Test 1, $\neg \bigvee dQ_2$ represents the one (and only) main row of the EM, and each $\{A\} \in dQ_1$ represents each main column. Recall that the intersection between a main column and a main row is called a main cell.

(e) Because we have assumed the passage of Test 1, it must be the case that every sub-cell in all main cells close.

(f) By definition, all open sub-cells indicate a possible way in which $(X \cup \{A\})$ can be consistent with $\neg \bigvee dQ_2$.

(g) Because we have assumed the closure of all sub-cells in line 6e, it must be the case that every $(X \cup \{A\}) \in dQ_1$ is *inconsistent* with $\neg \bigvee dQ_2$.

(h) Since a multi-conclusion argument is valid iff the premises are inconsistent with the negation of the disjunction of conclusions, the inconsistency of $(X \cup \{A\})$ with $\neg \bigvee dQ_2$ shows that the inference $\forall A \in dQ_1(X \cup \{A\}) \vDash dQ_2$ is valid.

(i) Therefore, the inference $Im_R\{Q_1, X, Q_2\}$ *holds*.

(j) But this is inconsistent with our earlier assumption in line 4, so Case 1 leads to a contradiction.

7. **Case 2 (Condition 4):**

(a) Condition 4 for regular erotetic implication states that $\forall B \in dQ_2\ \exists A \in dQ_1(X \cup \{B\}) \vDash A$. So for the condition to fail, it must be the case that $\exists B \in dQ_2\ \forall A \in dQ_1(X \cup \{B\}) \nvDash A$.

(b) Therefore, $\exists B \in dQ_2(X \cup \{B\})$ must be consistent with each negated answer in dQ_1.

(c) Condition 4 is evaluated with Test 2 for regular erotetic implication.

(d) In Test 2, each answer to dQ_2 is represented by a main column, with each sub-column representing a way of making that answer (along with the formulae in X) true.

(e) We have assumed that Test 2 is passed, which means that for every main column, there is a main cell that closes (which, by definition, means that all sub-cells in that main cell close).

(f) This would mean either that each answer to Q_2 is consistent with the negation of one answer to Q_1 or that the answers to Q_1 are inconsistent with each other.

(g) In either case, it would mean that every $\{B\} \in dQ_2(X \cup \{B\})$ entails at least one $A \in dQ_1$

(h) So, $\forall B \in dQ_2\ \exists A \in dQ_1(X \cup \{B\}) \vDash A$ is a valid inference.

(i) Therefore, the inference $Im_R\{Q_1, X, Q_2\}$ *holds*.

(j) But this is inconsistent with our earlier assumption in line 4, so Case 2 also leads to a contradiction.

8. So, both Condition 3 and Condition 4 hold.

9. Hence, the inference $IM_R\{Q_1, X, Q_2\}$ *does* hold and our assumption in line 4 is false.

Therefore, we have shown that if the EM Test is passed, the inference $IM_R\{Q_1, X, Q_2\}$ holds. In other words, the EM Test is sound relative to cases of regular erotetic implication. QED

5.1.2 Regular Erotetic Implication: Completeness

By definition of completeness: if the inference $Im_R\{Q_1, X, Q_2\}$ holds, the EM Test will pass.

1. Assume for conditional proof that $Im_R\{Q_1, X, Q_2\}$ holds

2. Therefore, both EM Test conditions hold. Recall that this means the inference satisfies *both* Condition 3 and Condition 4:

   (a) $\forall A \in dQ_1 (X \cup \{A\}) \vDash dQ_2$

   (b) $\forall B \in dQ_2 \; \exists A \in dQ_1 (X \cup \{B\}) \vDash A$

3. Now suppose for reductio that the EM Test for the inference in line 1 *fails*.

4. This means that either Test 1 or Test 2 fails.

5. **Case 1 (Test 1)**:

   (a) Test 1 is used to evaluate Condition 3.

   (b) In order for Test 1 to fail, there must be at least one main column with an open main cell. In other words, at least one main column has at least one open sub-cell.

   (c) Recall that there is only one main row in Test 1, represented by $\neg \bigvee dQ_2$.

   (d) Because we have assumed the failure of Test 1, the negation of the disjuncts in dQ_2 must be consistent with one another.

   (e) Also recall that each main column in Test 1 represents an answer $A \in dQ_1$ (combined with the background set X), with each sub-column representing a way of making $(X \cup \{A\})$ true.

(f) This means that if at least one sub-cell is open in any main column, $(X \cup \{A\})$ is consistent with $\neg \bigvee dQ_2$.

(g) In order for Condition 3 to hold, the premises of dQ_1 (combined with the background set), represented by $(X \cup \{A\})$, must be *inconsistent* with the disjunct of the answers to dQ_2 , $\neg \bigvee dQ_2$.

(h) But because we have already assumed the failure of Test 1, we know from line 5f that $X \cup \{A\}$ must be *consistent* with $\neg \bigvee dQ_2$.

(i) Therefore, Condition 3 must fail.

(j) However, this is inconsistent with our assumption in line 2, so Case 1 leads to a contradiction.

6. **Case 2 (Test 2):**

(a) Test 2 is used to evaluate Condition 4.

(b) Each main column in Test 2 represents $B \in dQ_2$ (combined with the background set X), with each sub-column representing a way of making $(X \cup \{B\})$ true, and each main row represents $\{\neg A\} \in dQ_1$, with each sub-row representing one way of making $\neg A$ true.

(c) Because we have assumed the failure of Test 2, there must be at least one main column with no main cell that closes.

(d) Thus, some answer $B \in dQ_2$ combined with the background set X is consistent with *every* $\neg A \in dQ_1$ and no two sub-rows in any $\neg A \in dQ_1$ are inconsistent with one another.

(e) In order for Condition 4 to hold, however, every answer to dQ_2 (combined with the background set), represented by $(X \cup \{B\})$, must be *inconsistent* with at least one $\neg A \in dQ_1$ (or at least two sub-rows in any $\neg A \in dQ_1$ must be inconsistent with each other.)

(f) Therefore, Condition 4 must fail.

(g) However, this is inconsistent with our assumption in line 2, so Case 2 leads to a contradiction.

7. So, both Test 1 and Test 2 are passed, and it therefore cannot be true that $Im_R\{Q_1, X, Q_2\}$ holds but does not pass the EM Test.

8. Hence, the EM Test *does* pass and our assumption in line 3 is false.

Therefore, we have shown that if the inference $IM_R\{Q_1, X, Q_2\}$ holds, the EM Test is passed. In other words, the EM Test is complete for cases of regular erotetic implication. QED

5.2 General Erotetic Implication: Soundness And Completeness

In the following two sections, we will demonstrate how the EM Test is both sound and complete for general erotetic implication (noted as $Im_G\{Q_1, X, Q_2\}$). In the first section, we will show the Test to be sound; in the second section, we will show it to be complete.

5.2.1 General Erotetic Implication: Soundness

By definition of soundness: if the EM Test is passed, then the inference $Im_G\{Q_1, X, Q_2\}$ holds.

1. Assume for conditional proof that the EM Test is passed. In other words, both Test 1 and Test 2 are passed.
2. In order for the inference $Im_G\{Q_1, X, Q_2\}$ to hold, *both* Condition 1 and Condition 2 must hold:
   (a) $\forall A \in dQ_1(X \cup \{A\}) \vDash dQ_2$
   (b) $\forall B \in dQ_2 \; \exists \Psi \subset dQ_1 \; (X \cup \{B\}) \vDash \Psi$
3. Test 1 is used to evaluate Condition 1 for $Im_G\{Q_1, X, Q_2\}$.
4. Because Condition 1 is the same for both general and regular EI, we defer to the soundness proof for Condition 3 in Section 5.1.1 line 6 to show that Condition 1 must hold if Test 1 is passed.
5. Note that we must assume the *failure* of Test 2 for $Im_R\{Q_1, X, Q_2\}$ (otherwise, it would show an inference of regular — and not general — EI).
6. This means that, if Test 2 is passed, it must not be the case that each main column has a main cell that closes. So, it must be the case that each main column has a main cell that *clears*.
7. By the definition of clearance presented in Section 4.1.2 line 6, we know that each $B \in dQ_2$ (combined with the background set X) is inconsistent with at least one n-tuple (but not all) of $\neg A \in dQ_1$. In other words, every answer B to dQ_2, when combined with the background set X, entails a proper subset, Ψ, of answers A to dQ_1.
8. Thus, Condition 2 must hold if Test 2 is passed.
9. Because both Condition 1 and Condition 2 hold, the inference $Im_G\{Q_1, X, Q_2\}$ must hold.

Hence, have shown that, if the EM Test is passed, the inference $Im_G\{Q_1, X, Q_2\}$ must hold. QED

5.2.2 General Erotetic Implication: Completeness

By definition of completeness: if the inference $Im_G\{Q_1, X, Q_2\}$ holds, then the EM Test is passed.

1. Assume for conditional proof that the inference $Im_G\{Q_1, X, Q_2\}$ holds.
2. This means that both Conditions 1 and 2 must hold.
3. Test 1 is used to evaluate Condition 1 for $Im_G\{Q_1, X, Q_2\}$.
4. Because Condition 1 is the same for both general and regular EI, we defer to the proof in Section 5.1.2 line 5 to show that Test 1 must pass if Condition 1 holds.
5. Assume for reductio that Test 2 does not pass.
   (a) This would mean that, for at least one main column representing $(X \cup \{B\})$, no main cell closes *and* no main cell clears.
   (b) If there is a main column that does not close, this would mean that (for the column that does not close) $(X \cup \{B\})$ is consistent with every $\neg A \in dQ_1$.
   (c) So, there is at least one main answer B to dQ_2 such that $(X \cup \{B\})$ does not entail any A in dQ_2. In other words, Condition 2 for regular EI does not hold. But this is inconsistent with our assumption in 1, so we have a contradiction.
   (d) We also know that, for the main column with no main cell that closes, no main cell clears, either. This would mean that (for the column that does not clear) $(X \cup \{B\})$ is consistent with every possible proper subset, Ψ, of answers A to dQ_1.
   (e) Therefore, there is at least one main answer B to dQ_2 such that $(X \cup \{B\})$ does not entail any proper subset of answers to dQ_1. In other words, Condition 2 for general EI does not hold. But this is inconsistent with our assumption in 1, so we have a contradiction.
6. Because of this contradiction, Test 2 is passed, and our assumption in 5 must be false.
7. Hence, both Test 1 and Test 2 must pass.

Thus, we have shown that if the inference $Im_G\{Q_1, X, Q_2\}$ holds, then the EM Test is passed. In other words, the EM Test is complete relative to cases of general erotetic implication. QED

6 Areas For Further Research

It should be noted that the EM Test can be generalized outside of IEL. Because the EM Test is essentially a condensed tree test, it can be used as a decision procedure for most, if not all, other cases of multi-conclusion logic. Moreover, erotetic evocation (mentioned earlier in this paper) has similar conditions to those of erotetic implication. Therefore, it is likely that a similar analysis to the one presented in this essay can be used to create an EM Test for erotetic evocation.

Beyond its theoretical implications, it is apparent that the EM Test is decidable. While we haven't proven the decidability of the EM Test in this paper, the rules set out in Section 4 are an algorithm, and since the tables have a finite number of main cells, it is clear that they will terminate. Therefore, the EM Test should be able to be implemented as a computer program, which would allow it to handle questions with much larger sets of direct answers. Turning the EM Test into a program may also aid in the development of more sophisticated large language models capable of generating logically sound and coherent questions. The intersection between logic and artificial intelligence is undoubtedly robust, and the EM Test presents a promising contribution to this area.